On the Notion of Substitution†

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> *...on ne dit pas par exemple :* « Ta cousine n'a pas encore voyagé en Afrique »*, mais :* « Elle n'y a encore pas voyagé, ta cousine, en Afrique »*, on commence par énoncer les signes grammaticaux abstraits, le* « résumé algébrique » *de la pensée, puis on emplit cette forme vide avec des désignations de choses et de faits précis.*
>
> Raymond Queneau

Abstract

We consider a concept of substitutive structure, called "logos", in order to study simple substitution, independently of formal or programming languages. We provide a definition of simultaneous substitution in an arbitrary logos and use it to prove a completeness theorem expressing that the equational properties of the usual substitution can be proved from the logos axioms only.

Keywords: substitution, variables, logos

1 Introduction

We will investigate here the notion of substitution in an abstract way, independently of formal or programming languages. We will not define it explicitly, but axiomatize it instead. The idea is to single out the essential features of the operation of performing a substitution of a piece of text for a symbol, in a text, in order to define a concept of substitutive structure, hereafter called logos, that deserves to be studied for its own sake. Thus, although the substitution will not be left at an informal level but made explicit, nevertheless it will not be explicitly defined since it will be considered as a primitive notion, much like the epsilon relation in set theory.

A logos is a set of terms with an infinite supply of variables. Nothing particular is assumed concerning the internal structure of these terms except that they may contain at most a finite number of variables. A variable by itself is a term supposed to contain no other variable than itself.

An operation —intended to formalize substitution— transforms a variable, and a pair of terms into a term, in such a way that the set of variables in the resulting term can be determined from the variables in the input. Furthermore, substitution axioms indicate what happens when the first term of the pair is the variable, or doesn't contain the variable, how to change the variables —which is similar to α-conversion—

†Part of the material of this paper —mainly the proof of the completeness theorem— is already contained in [3].

and how to compute two successive substitutions.

After the introduction and a description of the axioms, we give some examples and properties that follow from the axioms, then we proceed to define a notion of simultaneous substitution and prove a completeness theorem that makes precise and justifies the intuition that formulas true for the usual substitution can be proved from the logos axioms only.

The study of the abstract notion of substitution was initiated in the twenties by Curry as a part of what he called "prelogic" and culminated in the fifties (see for example [4], [5] and [6]). On the other hand, the idea to work with explicit substitution, which is widespread nowadays, originated mainly in relation with the problem of bound variables in the frameworks of the λ-calculus and computing systems. The main differences between these approaches and the one scrutinised here is, on the one hand, that we are not concerned with the notion of bound variable, and, on the other hand, that we rely on axiomatisation and not on definitional or computational aspects. Thus our notion of substitution is explicit, but only implicitely defined[1].

2 The logos concept

A **logos** $\mathcal{L}$ is a structure $\langle \mathrm{Ter}_{\mathcal{L}}, V_{\mathcal{L}}, \mathrm{vl}_{\mathcal{L}}, \mathsf{sub}_{\mathcal{L}} \rangle$, satisfying the following conditions:

- $\mathrm{Ter}_{\mathcal{L}}$ is a non-empty class whose elements are called **terms** and will be denoted by the letters M, N, $P, \ldots$;
- $V_{\mathcal{L}}$ is an infinite subclass of $\mathrm{Ter}_{\mathcal{L}}$ of which elements will be called **variables** and will be denoted by the letters x, y, $z, \ldots$;
- $\mathrm{vl}_{\mathcal{L}}$ is a function from $\mathrm{Ter}_{\mathcal{L}}$ to the finite subsets of $V_{\mathcal{L}}$, and $\mathrm{vl}_{\mathcal{L}}(M)$ is the **set of variables of** M;
- $\mathsf{sub}_{\mathcal{L}}$ is a function from $V_{\mathcal{L}} \times \mathrm{Ter}_{\mathcal{L}} \times \mathrm{Ter}_{\mathcal{L}}$ to $\mathrm{Ter}_{\mathcal{L}}$, and $\mathsf{sub}_{\mathcal{L}}(\langle x, M, N \rangle)$ is the result of the **substitution** of N for x in M; it will be denoted by $M[x := N]$.

The subscript $\mathcal{L}$ will generally be omitted if the context allows it.
In short:

$$\mathcal{L} = \langle \mathrm{Ter}, V, \mathrm{vl}, \mathsf{sub} \rangle;$$
$$V \subseteq \mathrm{Ter}, \text{ and} |V| \text{ is infinite};$$
$$\mathrm{vl} : \mathrm{Ter} \to \mathcal{P}_{\omega}(V); \mathsf{sub} : V \times \mathrm{Ter} \times \mathrm{Ter} \to \mathrm{Ter}.$$

Moreover, $\mathcal{L}$ must verify the:

Axioms

VARIABLES

1. $\mathrm{vl}(x) = \{x\}$.
2. If $x \in \mathrm{vl}(M)$, then $\mathrm{vl}(M[x := N]) = (\mathrm{vl}(M) \setminus \{x\}) \cup \mathrm{vl}(N)$.

[1] The logoi were introduced some time ago in [1], where an argument connecting substitution and category theory was presented. The structures discussed here are the same except that we will limit ourselves to logoi without types and that the axioms are slightly different.

SUBSTITUTION

3. If $y \notin \mathrm{vl}(M)$, then $M[x := y][y := x] = M$.
4. If $x \neq y$ and $x \notin \mathrm{vl}(P)$, then $M[x := N][y := P] = M[y := P][x := N[y := P]]$.
5. $x[x := M] = M$.
6. If $x \notin \mathrm{vl}(M)$, then $M[x := N] = M$.

Remarks. The concept of logos can be extended to structures with type restrictions: see [1].

Axiom 1 is so deeply rooted in the structure of the logoi and will be so often used in the sequel that we will generally omit mentioning it.

Axiom 3 allows the change of variables. Its condition may be weakened, by the proof of proposition 3.1.2, to $y \notin \mathrm{vl}(M) \setminus \{x\}$.

The following example shows that axiom 3 is independent from the other axioms. Consider an infinite set of variables and two non variable symbols f and g and call "terms" expressions of the form Hx, where H is a possibly empty sequence of fs and/or gs, and let x be its sole variable, i.e. $\mathrm{vl}(Hx) = \{x\}$. Let $\gamma(H)$ denote the result of replacing the fs in H by g, and define $Hz[x := H'y]$ as $\gamma(H)H'y$ if $z = x$, and Hz otherwise. Then all the axioms are verified, except 3, because $fx[x := y][y := x] = gx \neq fx$.

Axiom 4, which is the fundamental axiom of substitution, does not depend on M. However one may weaken it, to a form easier to check, by adding the condition $x \in \mathrm{vl}(M)$ and $y \in \mathrm{vl}(M) \cup \mathrm{vl}(N)$. Indeed, in case $x \notin \mathrm{vl}(M)$ and $x \notin \mathrm{vl}(P)$, we have $x \notin \mathrm{vl}(M[y := P])$, by axiom 2. Therefore, by axiom 6, $M[x := N][y := P] = M[y := P] = M[y := P][x := N[y := P]]$. And in case $x \in \mathrm{vl}(M)$, $y \notin \mathrm{vl}(M)$ and $y \notin \mathrm{vl}(N)$, we have $y \notin \mathrm{vl}(M[x := N])$, by axiom 2. Therefore, by axiom 6, $M[x := N][y := P] = M[x := N] = M[y := P][x := N[y := P]]$.

The requirement that the variables of a logos form an infinite set, but that the variables in a term be finite in number, is made to guarantee that there is a sufficiently large amount of fresh variables outside a term or a given finite set of terms. This requirement is not easily axiomatisable in a simple and appealing way because it involves the concept of finite set. If one wants to axiomatise a notion of logos without resorting to this concept, one can reinforce it by adding axioms to the effect that, for each $M_1, \ldots, M_n$, there exists a variable x such that $x \notin \mathrm{vl}(M_1) \cup \ldots \cup \mathrm{vl}(M_n)$. Taking axiom 1 into account, this entails that, for each $M_1, \ldots, M_n$ and m, there are distinct variables $x_1, \ldots, x_m$ such that $x_1, \ldots, x_m \notin U = \mathrm{vl}(M_1) \cup \ldots \cup \mathrm{vl}(M_n)$; because if $x_1, \ldots, x_{m-1} \notin U$, then there exists a variable $x_m \notin U \cup \{x_1, \ldots, x_{m-1}\} = U \cup \mathrm{vl}(x_1) \cup \ldots \cup \mathrm{vl}(x_{m-1})$. Another interesting possibility arises when one simply drops the requirement that the set of variables be infinite, without imposing at the same time the existence of fresh variables. However, we will not go along these paths here as they don't concern the most familiar examples of logoi; they would also affect the other axioms and complicate significantly the end of the paper.

It follows easily from axioms 1 and 2 that either there is, for each $n > 0$, a term with exactly n distinct variables or that every term has no more than one variable. Combined with the fact that a logos may have "constants", i.e. terms without variables, we are thus left with four major alternatives concerning the cardinalities of the $\mathrm{vl}(M)$. It was shown in [1] that a logos with types in which the terms have exactly one variable is essentially a category.

2.1 Examples

Logoi of substitution. It will hardly be a surprise that the terms of a language with ordinary substitution form a logos. One starts with an infinite set of variables and a set of symbols for n-ary operations ϕ^n. The terms are defined by:

- variables are terms;
- if $M_1, \ldots, M_n \in \mathrm{Ter}$, then $\phi^n(M_1, \ldots, M_n) \in \mathrm{Ter}$.

One defines also vl and `sub` inductively:

- $\mathrm{vl}(x) = \{x\}$;
- $\mathrm{vl}(\phi^n(M_1, \ldots, M_n)) = \mathrm{vl}(M_1) \cup \ldots \cup \mathrm{vl}(M_n)$;
- $x[x := M] = M$;
- $y[x := M] = y$, if $y \neq x$;
- $\phi^n(M_1, \ldots, M_n)\ [x := N] = \phi^n(M_1[x := N], \ldots, M_n[x := N])$.
  This includes the case of the constants: $\phi^0[x := N] = \phi^0$.

Set abstracts. The following example illustrates generalisations of the logos of substitution when terms can contain bound variables.

In addition to the binary relation symbol ϵ, the parentheses, and the usual logical symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$, there is an abstract formation operator $\{\,|\quad\}$ and a countably infinite set of variables. Terms are expressions, made up from these symbols, defined inductively (together with the notion of formula) as follows:

- variables are terms;
- if P and Q are terms $P \,\epsilon\, Q$ is an atomic formula;
- if A, B are formulas and x is a variable, then $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$, $\forall x A$ and $\exists x A$ are formulas and $\{\, x \mid A \,\}$ is a term.

The variable binding operators are $\forall$, $\exists$ and $\{\quad|\quad\}$. If the bound variables are "eliminated" by a method like Bourbaki's squares, then we can define $Q[x := P]$ (simultaneously with $A[x := P]$) in the natural inductive way.

We can of course restrict the collection of terms to any subcollection closed under substitution (see for example [2]).

The completeness result below entails that we can handle such cases as if the terms had the same structure as in the first example.

Logoi of finite sequences and trees. The set of finite sequences of elements of a set with an infinite subset of "variables", endowed with the evident substitution operation, is a logos.

This generalises in a natural way to trees with finite branches and leafs labelled with constants (non variables) and a finite set of variables. In these last logoi a variable can have infinitely many "occurrences" in a term.

The logoi of finite sets. Another example of logos arises when the finite subsets of an infinite set are considered as terms, the variables being the unit subsets. $\mathrm{vl}(M)$ is then nothing else than the set of singletons included in M. $M[x := N]$ is $(M \setminus x) \cup N$, if $x \in \mathrm{vl}(M)$ and M if $x \notin \mathrm{vl}(M)$.

Sets. Let A be an infinite set of atoms. The variables are the elements of A, and the other terms are the hereditarily finite well-founded sets over A. This is a logos if one defines inductively $\mathrm{vl}(x) = \{x\}$, $x[x := N] = N$, $y[x := N] = y$ for $y \neq x$; and, if M is not a variable, $\mathrm{vl}(M) = \bigcup\{\, \mathrm{vl}(N) \mid N \in M \,\}$, and $M[x := N] = \{\, P[x := N] \mid P \in M \,\}$.

This example generalises to non well-founded sets, if one replaces e.g. the foundation axiom by the antifoundation axiom called AFA —extended so as to allow the existence of atoms—, thus opening the way to non well-founded languages.

3 Simultaneous substitution

PROPOSITION 3.1
1. Renaming the variables: If $y \notin \mathrm{vl}(M)$, $M[x := N] = M[x := y][y := N]$.
2. $M[x := x] = M$.

PROOF.
1. Let us suppose that $y \notin \mathrm{vl}(M)$ and let $z \notin \mathrm{vl}(M) \cup \mathrm{vl}(N) \cup \{x, y\}$. We have, on the one hand,

$$\begin{aligned} M[x := N] &= M[x := z][z := x][x := N] && \text{(by axiom 3)} \\ &= M[x := z][x := N][z := N] && \text{(by axioms 4 and 5)} \\ &= M[x := z][z := N]. && \text{(by axioms 2 and 6)} \end{aligned}$$

And, on the other hand,

$$\begin{aligned} M[x := y][y := N] &= M[x := z][z := x][x := y][y := N] && \text{(by axiom 3)} \\ &= M[x := z][x := y][z := y][y := N] && \text{(by axioms 4 and 5)} \\ &= M[x := z][z := y][y := N] && \text{(by axioms 2 and 6)} \\ &= M[x := z][y := N][z := N] && \text{(by axioms 4 and 5)} \\ &= M[x := z][z := N]. && \text{(by axioms 2 and 6)} \end{aligned}$$

2. Even though one deduces immediately $M[x := x] = M$ from 1 and axiom 3, one can still prove it more simply as follows. If $y \notin \mathrm{vl}(M)$ and $y \neq x$, then

$$\begin{aligned} M[x := x] &= M[x := y][y := x][x := x] && \text{(by axiom 3)} \\ &= M[x := y][x := x][y := x] && \text{(by axioms 4 and 5)} \\ &= M[x := y][y := x] && \text{(by axioms 2 and 6)} \\ &= M. && \text{(by axiom 3)} \blacksquare \end{aligned}$$

The sequence of terms $N_1, \ldots, N_n$ is **free** for the sequence of variables $x_1, \ldots, x_n$ if and only if these variables are distinct and $x_j \notin \mathrm{vl}(N_i)$, if $1 \leq i < j \leq n$. This definition is intended to provide conditions for a sequence of substitutions $[x_1 := N_1] \ldots [x_n := N_n]$ to be considered as a simultaneous substitution of the N_is for the x_is.

The sequence of variables $x'_1, \ldots, x'_n$ is **suitable** for the sequence $x_1, \ldots, x_n$, M, $N_1, \ldots, N_n$ if and only if they are not in $\mathrm{vl}(M)$; and $N_1, \ldots, N_n$ is free for $x'_1, \ldots, x'_n$, and $x'_1, \ldots, x'_n$ is free for $x_1, \ldots, x_n$. The variables in a suitable sequence are intended to be used as fresh variables enabling simultaneous substitution, even when a sequence of terms is not free for the corresponding sequence of variables.

We observe that (for $1 \leq i_1 < \ldots < i_k \leq n$) if $N_1, \ldots, N_n$ is free for $x_1, \ldots, x_n$, then $N_{i_1}, \ldots, N_{i_k}$ is free for $x_{i_1}, \ldots, x_{i_k}$ as well; and if $x_1, \ldots, x_n$ is suitable for $x'_1, \ldots, x'_n, M, N_1, \ldots, N_n$, then $x_{i_1}, \ldots, x_{i_k}$ is suitable for $x'_{i_1}, \ldots, x'_{i_k}, M, N_{i_1}, \ldots, N_{i_k}$.

We also note that if $x'_1, \ldots, x'_n$ is free for $x_1, \ldots, x_n$, then $x_i \neq x'_j$, if $1 \leq j < i \leq n$.

For convenience, let us define $\mathrm{vl}_{xM}(N)$ as being $\mathrm{vl}(N)$ if $x \in \mathrm{vl}(M)$ and otherwise the empty set.

Proposition 3.2

1. If $N_1, \ldots, N_n$ is free for $x_1, \ldots, x_n$, then
$\mathrm{vl}(M[x_1 := N_1] \ldots [x_n := N_n]) =$
$(\mathrm{vl}(M) \setminus \{x_1, \ldots, x_n\}) \cup \mathrm{vl}_{x_1 M}(N_1) \cup \ldots \cup \mathrm{vl}_{x_n M}(N_n)$.

In particular, $\mathrm{vl}(M[x := N]) = (\mathrm{vl}(M) \setminus \{x\}) \cup \mathrm{vl}_{xM}(N)$.

2. Generalisation of proposition 3.1.1: if $x'_1, \ldots, x'_n$ is suitable for $x_1, \ldots, x_n, M, N_1, \ldots, N_n$, and if $N_1, \ldots, N_n$ is free for $x_1, \ldots, x_n$, then

$$M[x_1 := x'_1] \ldots [x_n := x'_n][x'_1 := N_1] \ldots [x'_n := N_n] = M[x_1 := N_1] \ldots [x_n := N_n].$$

Proof. The two parts of the proposition are established by induction on n.

1. We suppose that $N_1, \ldots, N_{n+1}$ is free for $x_1, \ldots, x_{n+1}$, which means that $N_1, \ldots, N_n$ is free for $x_1, \ldots, x_n$ and that $x_{n+1} \notin \mathrm{vl}(N_1) \cup \ldots \cup \mathrm{vl}(N_n) \cup \{x_1, \ldots, x_n\}$. Thus, by inductive hypothesis, $x_{n+1} \in \mathrm{vl}(M[x_1 := N_1] \ldots [x_n := N_n])$ if and only if $x_{n+1} \in \mathrm{vl}(M)$; hence $\mathrm{vl}_{x_{n+1} M[x_1:=N_1]\ldots[x_n:=N_n]}(N_{n+1}) = \mathrm{vl}_{x_{n+1}M}(N_{n+1})$.

We thus obtain:

$\mathrm{vl}(M[x_1 := N_1] \ldots [x_n := N_n][x_{n+1} := N_{n+1}])$
$= (\mathrm{vl}(M[x_1 := N_1] \ldots [x_n := N_n]) \setminus \{x_{n+1}\}) \cup \mathrm{vl}_{x_{n+1}M}(N_{n+1})$ (by axioms 2 and 6)
$= (((\mathrm{vl}(M) \setminus \{x_1, \ldots, x_n\}) \cup \mathrm{vl}_{x_1 M}(N_1) \cup \ldots \cup \mathrm{vl}_{x_n M}(N_n))$
$\setminus \{x_{n+1}\}) \cup \mathrm{vl}_{x_{n+1}M}(N_{n+1})$ (by inductive hypothesis)
$= (\mathrm{vl}(M) \setminus \{x_1, \ldots, x_{n+1}\}) \cup \mathrm{vl}_{x_1 M}(N_1) \cup \ldots \cup \mathrm{vl}_{x_{n+1}M}(N_{n+1})$.

2. We have:

$M[x_1 := x'_1] \ldots [x_n := x'_n][x_{n+1} := x'_{n+1}][x'_1 := N_1]$
$= M[x_1 := x'_1][x'_1 := N_1][x_2 := x'_2] \ldots [x_{n+1} := x'_{n+1}]$ (by axioms 4 and 6)
$= M[x_1 := N_1][x_2 := x'_2] \ldots [x_{n+1} := x'_{n+1}]$. (by proposition 3.1.1)

If $2 \leq i \leq n+1$, then, by 1,
$x'_i \notin (\mathrm{vl}(M) \setminus \{x_1\}) \cup \mathrm{vl}_{x_1 M}(N_1) = \mathrm{vl}(M[x_1 := N_1])$. Hence $x'_2, \ldots, x'_{n+1}$ is suitable for $x_2, \ldots, x_{n+1}, M[x_1 := N_1], N_2, \ldots, N_n$, and, by inductive hypothesis,

$M[x_1 := N_1][x_2 := x'_2] \ldots [x_{n+1} := x'_{n+1}][x'_2 := N_2] \ldots [x'_{n+1} := N_{n+1}] =$
$M[x_1 := N_1][x_2 := N_2] \ldots [x_{n+1} := N_{n+1}]$. ∎

We now show that the **simultaneous substitution** is definable in every logos. Then we establish the properties that we will use in the proof of the completeness theorem. In order to define simultaneous substitution properly, we first show the

Lemma 3.3

If both sequences of variables $x'_1, \ldots, x'_n$ and $x''_1, \ldots, x''_n$ are suitable for $x_1, \ldots, x_n$, M, $N_1, \ldots, N_n$, then

$M[x_1 := x'_1] \dots [x_n := x'_n][x'_1 := N_1] \dots [x'_n := N_n] =$
$M[x_1 := x''_1] \dots [x_n := x''_n][x''_1 := N_1] \dots [x''_n := N_n].$

Proof. Let us choose a sequence of variables $x^*_1, \dots, x^*_n$ that is suitable for $x_1, \dots, x_n$, M, $N_1, \dots, N_n$ and distinct from the $x'_1, \dots, x'_n, x''_1, \dots, x''_n$.
It is clear that

- $x^*_1, \dots, x^*_n$ is suitable for $x_1, \dots, x_n$, M, $x'_1, \dots, x'_n$, by axiom 1;
- $x'_1, \dots, x'_n$ is suitable for $x^*_1, \dots, x^*_n$, $M[x_1 := x^*_1] \dots [x_n := x^*_n]$, $N_1, \dots, N_n$, because, if $1 \le i \le n$, $x'_i \notin \mathrm{vl}(M[x_1 := x^*_1] \dots [x_n := x^*_n]) \subseteq \mathrm{vl}(M) \cup \{x^*_1, \dots, x^*_n\}$, by proposition 3.2.1.

Therefore, by proposition 3.2.2:
$M[x_1 := x'_1] \dots [x_n := x'_n][x'_1 := N_1] \dots [x'_n := N_n] =$
$M[x_1 := x^*_1] \dots [x_n := x^*_n][x^*_1 := x'_1] \dots [x^*_n := x'_n]$
$[x'_1 := N_1] \dots [x'_n := N_n] =$
$M[x_1 := x^*_1] \dots [x_n := x^*_n][x^*_1 := N_1] \dots [x^*_n := N_n].$
In the same way,
$M[x_1 := x''_1] \dots [x_n := x''_n][x''_1 := N_1] \dots [x''_n := N_n] =$
$M[x_1 := x^*_1] \dots [x_n := x^*_n][x^*_1 := N_1] \dots [x^*_n := N_n].$ ■

We are now in a position to give the definition of simultaneous substitution.

If $n \ge 1$ and the sequence of variables $x'_1, \dots, x'_n$ is suitable for $x_1, \dots, x_n$, M, $N_1, \dots, N_n$,

$$M[x_1 := N_1, \dots, x_n := N_n]$$

is, by definition, the term $M[x_1 := x'_1] \dots [x_n := x'_n][x'_1 := N_1] \dots [x'_n := N_n]$.

This definition is *legitimate* because, on the one hand, it is always possible to choose a sequence of variables $x'_1, \dots, x'_n$ that is suitable and whose variables are outside a given finite set; and, on the other hand, lemma 3.3 shows that the definition does not depend on the particular choice of the sequence $x'_1, \dots, x'_n$.

Let us notice that

- if $n = 0$, $M[\,] = M$;
- if $n = 1$, then, since any (sequence of one) term is clearly free for any (sequence of one) variable, it follows from proposition 3.1.1 that the simultaneous substitution of N for x is nothing else than the previous $[x := N]$. Therefore, our notational conventions are fortunately consistent.

If a number n is specified or can be inferred from the context, then the notation $\vec{X}$ may be used to denote $X_1, \dots, X_n$; and the notation $[\vec{x} := \vec{N}]$ may be used to denote $[x_1 := N_1, \dots, x_n := N_n]$.

Proposition 3.4
1. If the sequence of terms $N_1, \dots, N_n$ is free for $x_1, \dots, x_n$, then

$$M[\vec{x} := \vec{N}] = M[x_1 := N_1] \dots [x_n := N_n].$$

2. $\mathrm{vl}(M[\vec{x} := \vec{N}]) = \mathrm{vl}(M) \setminus \{x_1, \dots, x_n\} \cup \mathrm{vl}_{x_1 M}(N_1) \cup \dots \cup \mathrm{vl}_{x_n M}(N_n)$.

3. $M[\vec{x} := \vec{x}] = M$.
4. If $y \notin \mathrm{vl}(M) \setminus \{x_1, \ldots, x_n\}$, then

$$M[\vec{x} := \vec{N}][y := P] = M[x_1 := N_1[y := P], \ldots, x_n := N_n[y := P]].$$

PROOF.
1. This statement is a reformulation of proposition 3.2.2.

2. If $x'_1, \ldots, x'_n$ is suitable for $x_1, \ldots, x_n$, M, $N_1, \ldots, N_n$ then:
$\mathrm{vl}(M[\vec{x} := \vec{N}]) = \mathrm{vl}(M[\vec{x} := \vec{x}'][\vec{x}' := \vec{N}])$ (by 1)
$= (\mathrm{vl}(M[\vec{x} := \vec{x}']) \setminus \{x'_1, \ldots, x'_n\}) \cup \mathrm{vl}_{x'_1 M[\vec{x}:=\vec{x}']}(N_1) \cup \ldots \cup \mathrm{vl}_{x'_n M[\vec{x}:=\vec{x}']}(N_n)$.
(by proposition 3.2)
However:
$\mathrm{vl}(M[\vec{x} := \vec{x}']) = (\mathrm{vl}(M) \setminus \{x_1, \ldots, x_n\}) \cup \mathrm{vl}_{x_1 M}(x'_1) \cup \ldots \cup \mathrm{vl}_{x_n M}(x'_n)$.
(by proposition 3.2)
Thus $\mathrm{vl}_{x'_i M[\vec{x}:=\vec{x}']}(N_i) = \mathrm{vl}_{x_i M}(N_i)$, if $1 \leq i \leq n$.
Consequently,
$\mathrm{vl}(M[\vec{x} := \vec{N}]) = \mathrm{vl}(M[\vec{x} := \vec{x}'][\vec{x}' := \vec{N}]) =$
$(((\mathrm{vl}(M) \setminus \{x_1, \ldots, x_n\}) \cup \mathrm{vl}_{x_1 M}(x'_1) \cup \ldots \cup \mathrm{vl}_{x_n M}(x'_n))$
$\setminus\{x'_1, \ldots, x'_n\}) \cup \mathrm{vl}_{x_1 M}(N_1) \cup \ldots \cup \mathrm{vl}_{x_n M}(N_n) =$
$(\mathrm{vl}(M) \setminus \{x_1, \ldots, x_n\}) \cup \mathrm{vl}_{x_1 M}(N_1) \cup \ldots \cup \mathrm{vl}_{x_n M}(N_n)$.

3. Since $x_1, \ldots, x_n$ is free for $x_1, \ldots, x_n$ if they are distinct, we have, by 1 and proposition 3.1.2, $M[\vec{x} := \vec{x}] = M[x_1 := x_1] \ldots [x_n := x_n] = M$.

4. Let $x'_1, \ldots, x'_n$ be a sequence of variables suitable for $x_1, \ldots, x_n$, M, $N_1, \ldots, N_n$ and for $x_1, \ldots, x_n$, M, $N_1[y := P], \ldots, N_n[y := P]$; and whose variables are distinct from y and not in $\mathrm{vl}(P)$. By hypothesis and proposition 3.2.1, $y \notin \mathrm{vl}(M[x_1 := x'_1] \ldots [x_n := x'_n])$. Hence we have:
$M[\vec{x} := \vec{N}][y := P] =$
$M[x_1 := x'_1] \ldots [x_n := x'_n][x'_1 := N_1] \ldots [x'_n := N_n][y := P] =$
$M[x_1 := x'_1] \ldots [x_n := x'_n][y := P][x'_1 := N_1[y := P]] \ldots [x'_n := N_n[y := P]]$
(by axiom 4)
$= M[x_1 := x'_1] \ldots [x_n := x'_n][x'_1 := N_1[y := P]] \ldots [x'_n := N_n[y := P]]$ (by axiom 6)
$= M[x_1 := N_1[y := P], \ldots, x_n := N_n[y := P]]$. ∎

3.1 The intuitive simultaneous substitution

The result of the intuitive simultaneous substitution in a substitution logos of $N_1, \ldots, N_n$ for $x_1, \ldots, x_n$ in M, denoted here by $M[N_1/x_1, \ldots, N_n/x_n]$, can be defined as follows:

- $x_i[\vec{N}/\vec{x}] = N_i$, if $1 \leq i \leq n$;
- $x[\vec{N}/\vec{x}] = x$, if $x \notin \{x_1, \ldots, x_n\}$;
- $\phi^m(M_1, \ldots, M_m)[\vec{N}/\vec{x}] = \phi^m(M_1[\vec{N}/\vec{x}], \ldots, M_m[\vec{N}/\vec{x}])$.

PROPOSITION 3.5
In the substitution logoi, one has $M[\vec{N}/\vec{x}] = M[\vec{x} := \vec{N}]$.

PROOF. By induction on the length of the terms:

- $x_i[\vec{N}/\vec{x}] = N_i = x_i[\vec{x} := \vec{N}]$, if $1 \leq i \leq n$, by axiom 5;
- $x[\vec{N}/\vec{x}] = x = x[\vec{x} := \vec{N}]$, if $x \notin \{x_1, \ldots, x_n\}$, by axiom 6;
- Let $x'_1, \ldots, x'_n$ be suitable for $x_1, \ldots, x_n, \phi^m(M_1, \ldots, M_m), \vec{N}$. Then it is also suitable for $\vec{x}, M_i, \vec{N}$ if $1 \leq i \leq m$.
  Hence:
  $\phi^m(M_1, \ldots, M_m)[\vec{N}/\vec{x}] = \phi^m(M_1[\vec{N}/\vec{x}], \ldots, M_m[\vec{N}/\vec{x}]) =$
  $\phi^m(M_1[\vec{x} := \vec{N}], \ldots, M_m[\vec{x} := \vec{N}])$ (by inductive hypothesis)
  $= \phi^m(M_1[x_1 := x'_1] \ldots [x_n := x'_n][x'_1 := N_1] \ldots [x'_n := N_n], \ldots,$
  $M_m[x_1 := x'_1] \ldots [x_n := x'_n][x'_1 := N_1] \ldots [x'_n := N_n]) =$
  $\phi^m(M_1, \ldots, M_m)[\vec{x} := \vec{N}]$. ∎

4 Completeness

4.1 The logos language

The **language of the logoi** contains a denumerable set of **variables**, a ternary operation symbol Sub, the binary relation symbol $\in_{\mathrm{vl}}$ and the usual logical connectives $=, \neg, \wedge, \vee, \rightarrow \leftrightarrow, \forall, \exists$.

The concepts of **term** and of **formula** are defined as usual. Terms of the form $\mathsf{Sub}(S, T, R)$ will be denoted by $T\,(S := R)$.

A valuation α of the variables of the language to the terms of a logos $\mathcal{L}$ extends naturally by induction to the terms of the language by interpreting $T\,(S := R)$ as the result of an unconditioned substitution:

$\alpha(T\,(S := R)) = \alpha(T)[\alpha(S) := \alpha(R)]$, if $\alpha(S)$ is a variable; and
$\alpha(T\,(S := R)) = \alpha(T)$ else.

The satisfaction of the atomic formulas can now be defined as:

$\mathcal{L} \models_\alpha T \in_{\mathrm{vl}} R$ if and only if $\alpha(T) \in \mathrm{vl}(\alpha(R))$,
$\mathcal{L} \models_\alpha T = R$ if and only if $\alpha(T) = \alpha(R)$.

A formula is **valid** in a logos if, interpreting the logical connectives as usual, it is satisfied by any valuation in that logos.

We will actually consider only some kind of terms and of formulas, namely the "simple terms", the "conditions" and the "conditional equations", which we define as follows:

- a **condition** is a formula in which the symbol Sub does not occur and where the quantifiers are restricted to the formulas of the form $v \in_{\mathrm{vl}} v$ —conditions are thus built up from atomic formulas of the form $v = w$, $v \in_{\mathrm{vl}} w$, according to the syntax of propositional logic and the rule: if A is a condition, then $\forall v\,(v \in_{\mathrm{vl}} v \rightarrow A)$ and $\exists v\,(v \in_{\mathrm{vl}} v \wedge A)$ are also conditions;
- a **simple term** is either a variable or a term of the form $T\,(v := R)$, where v is a variable and T, R simple terms—thus every simple term has the form $u\sigma$, for a

variable u and a suffix constituted of a, possibly empty, sequence σ of $(v := R)$, with simple terms R;

- an **equation** is an expression of the form $T = S$, where T and S are simple terms;
- a **conditional equation** is a formula of the form $C \to E$, where C is a condition and E an equation.

This definition of conditional equation is intended to express the notion of *essential* property of substitution, and contrasts it with the contingent or the accidental ones. For example, the axioms of substitution are all conditional equations, but a statement concerning the number of terms or the maximum number of variables in the terms is not a conditional equation. The completeness theorem, which states that the axioms of logoi entail exactly the conditional equations valid in the substitution logoi, enables us, for example, when we are working in first order logic, to assume simply that the terms form a logos, without worrying about their internal structure.

The sentence "v is a variable" is expressed by the condition $v \in_{\mathrm{vl}} v$. A sentence like "w has exactly three variables distinct from u and v" can also be expressed as a condition. On the other hand, formulas like "$\mathsf{Sub}(u, w, v) = \mathsf{Sub}(u, w, v')$" cannot be expressed as a condition, as will be seen in proposition 5.1.2. It will also be a consequence of the completeness theorem that the formula $\mathsf{Sub}(v, u, w) = u \to v \notin_{\mathrm{vl}} u \vee v = w$, which is valid in the substitution logoi but not in every logos, cannot be expressed as a conditional equation.

4.2 The completeness theorem

Theorem 4.1
A conditional equation is valid in every logos if and only if it is valid in every substitution logos.

Proof. We take the proof from [3]. A bijective function F between a finite set of terms of the logos $\mathcal{S}$ and a finite set of terms of the logos $\mathcal{L}$ is a **partial isomorphism** between $\mathcal{S}$ and $\mathcal{L}$ if and only if

- any variable of a term in the domain of F is in the domain, and any variable of a term in the range of F is in the range;
- $M \in \mathrm{vl}_{\mathcal{S}}(N)$ if and only if $F(M) \in \mathrm{vl}_{\mathcal{L}}(F(N))$ for M, N in the domain of F.

We see that

- for M in the domain of F, M is a variable if and only if $F(M)$ is a variable, because $M \in \mathrm{vl}_{\mathcal{S}}(M)$ iff $F(M) \in \mathrm{vl}_{\mathcal{L}}(F(M))$;
- if x is a variable of $\mathcal{S}$ not in the domain of F and y is a variable of $\mathcal{L}$ not in the range of F, then the extension of F obtained by associating x with y is still a partial isomorphism between $\mathcal{S}$ and $\mathcal{L}$, because $x \in \mathrm{vl}_{\mathcal{S}}(x)$, $y \in \mathrm{vl}_{\mathcal{L}}(y)$, and for all M in the domain of F, $x \notin \mathrm{vl}_{\mathcal{S}}(M)$, $y \notin \mathrm{vl}_{\mathcal{L}}(F(M))$, $M \notin \mathrm{vl}_{\mathcal{S}}(x)$ and $F(M) \notin \mathrm{vl}_{\mathcal{L}}(y)$.

Therefore, a back-and-forth argument shows that, if C is a condition then

$$\mathcal{S} \models_{\alpha} C \text{ if and only if} \mathcal{L} \models_{\beta} C,$$

if there is a partial isomorphism F whose domain includes the images under α of the free variables in C and such that $\beta(v) = F(\alpha(v))$, for v free in C.

Let now X be a finite set of terms of a logos $\mathcal{L}$ including all the variables belonging to a term in X. To each term M of X that is not a variable, we injectively associate a functional symbol ϕ_M of a substitution logos $\mathcal{S}$ whose arity is the number of variables of M, and we define a bijection $\nu_{\ldots}$ between the variables in X and a finite set of variables of $\mathcal{S}$. Let us moreover, for convenience, fix a strict order $\prec$ on the variables in the range of ν. The function F of range X defined by

- $F(\nu_x) = x$;
- $F(\phi_M(\nu_{x_1}, \ldots, \nu_{x_n})) = M$,
  where $\nu_{x_1} \prec \ldots \prec \nu_{x_n}$ and $\mathrm{vl}_{\mathcal{L}}(M) = \{x_1, \ldots, x_n\}$,

is a partial isomorphism between $\mathcal{S}$ and $\mathcal{L}$.

The crucial fact that the terms in a substitution logos have a unique construction from the functional symbols and the variables, allows one to extend F by induction to a function $\widehat{F}$ defined for all terms in $\mathcal{S}$ containing the ϕ_M and the variables in the range of ν:

$$\widehat{F}(\phi_M(M_1, \ldots, M_n)) = M[x_1 := \widehat{F}(M_1), \ldots, x_n := \widehat{F}(M_n)],$$

where $F(\phi_M(\nu_{x_1}, \ldots, \nu_{x_n})) = M$.

We note that this agrees with the definition of $F(\phi_M(\nu_{x_1}, \ldots, \nu_{x_n}))$, by proposition 3.4.3, and that the domain of $\widehat{F}$ is closed under substitution.

One shows by induction on the terms of $\mathcal{S}$ that:

$$\widehat{F}(M[\nu_y := N]) = \widehat{F}(M)[y := \widehat{F}(N)]$$

as follows:

- $\widehat{F}(\nu_y[\nu_y := N]) = \widehat{F}(N) = y[y := \widehat{F}(N)] = \widehat{F}(\nu_y)[y := \widehat{F}(N)]$ (by axiom 5);
- $\widehat{F}(\nu_x[\nu_y := N]) = \widehat{F}(\nu_x) = \widehat{F}(\nu_x)[y := \widehat{F}(N)]$, if $x \neq y$ (by axioms 1 and 6);
- if $F(\phi_P(\nu_{x_1}, \ldots, \nu_{x_n})) = P$, then:
  $\widehat{F}(\phi_P(M_1, \ldots, M_n)[\nu_y := N]) =$
  $\widehat{F}(\phi_P(M_1[\nu_y := N], \ldots, M_n[\nu_y := N])) =$
  $P[x_1 := \widehat{F}(M_1[\nu_y := N]), \ldots, x_n := \widehat{F}(M_n[\nu_y := N])] =$
  $P[x_1 := \widehat{F}(M_1)[y := \widehat{F}(N)], \ldots, x_n := \widehat{F}(M_n)[y := \widehat{F}(N)]]$
  (by inductive hypothesis)
  $= P[x_1 := \widehat{F}(M_1), \ldots, x_n := \widehat{F}(M_n)][y := \widehat{F}(N)]$
  (by proposition 3.4.4, since $\mathrm{vl}(P) = \{x_1, \ldots, x_n\}$)
  $= \widehat{F}(\phi_P(M_1, \ldots, M_n))[y := \widehat{F}(N)]$.

Let us suppose finally that $C \to E$ is valid in $\mathcal{S}$ and let β be a valuation to the terms of a logos $\mathcal{L}$ that makes C true. Let F be —as above— a partial isomorphism between a substitution logos $\mathcal{S}$ and $\mathcal{L}$, whose range includes the range of β. Let also α be a valuation in $\mathcal{S}$ such that $F(\alpha(v)) = \beta(v)$, for the variables v occurring free in

$C \to E$. It was seen that C is true in $\mathcal{S}$ for the valuation α. Since $C \to E$ is valid in $\mathcal{S}$ by assumption, E is true in $\mathcal{S}$ for α and thus it remains to show that E is true in $\mathcal{L}$ for β.
This is done by showing inductively that $\widehat{F}(\alpha(T)) = \beta(T)$, for all simple terms "generated" by the free variables of $C \to E$, as follows. We suppose that $\alpha(v)$ is a variable of $\mathcal{S}$, the other situation being trivial[2].
We have, by inductive hypothesis:

$$\widehat{F}(\alpha(T\,(v := R)) = \widehat{F}(\alpha(T)[\alpha(v) := \alpha(R)]) = \widehat{F}(\alpha(T)[\nu_{\beta(v)} := \alpha(R)]) =$$
$$\beta(T)[\beta(v) := \beta(R)] = \beta(T\,(v := R)).$$ ∎

Corollary 4.2
Let LC be the union of a denumerable set of *variables* and a denumerable set of *constants*, and let LC* be the logos of the finite sequences of elements of LC. A conditional equation is valid in every logos if and only if it is valid in LC*.

Proof. Let's assume that $C \to R = T$ is valid in LC*, and that α is a valuation in a substitution logos making C true. Let us associate to each function symbol ϕ^n occurring in a term of the image under α of a variable occurring free in $C \to E$ a distinct constant $\phi^{n\,\mathtt{pol}}$ in L —its "polish notation"— and let us further translate each of these terms in the logos LC* as follows:

- $x^{\mathtt{pol}} \equiv x$;
- $\phi^n(N_1, \ldots, N_m)^{\mathtt{pol}} \equiv \phi^{n\,\mathtt{pol}} N_1{}^{\mathtt{pol}} \ldots N_m{}^{\mathtt{pol}}$,

where we have supposed w.l.o.g. that the variables in the domain of $\ldots^{\mathtt{pol}}$ belong to the substitution logos.

The valuation associating $\alpha(v)^{\mathtt{pol}}$ to each v free in $C \to E$ makes C true in LC*. By assumption, it follows that it also makes $R = T$ true in LC*. Thus $R = T$ is true in the substitution logos relatively to the valuation α. Hence it is valid in every logos, by the theorem. ∎

This result would be false if LC comprised only variables, as it happens e.g. on my computer where the replacement of any character in a text is permitted. Indeed, in that case the "commutative property", "*v is the sole variable in u and u'*" $\to$ $u\,(v := u') = u'\,(v := u)$, is valid in this logos, but not in every substitution logos. However, we have:

Corollary 4.3
Let L^* be the logos of the finite sequences of elements of a denumerable set of *variables* L. A conditional equation without quantifiers is valid in every logos if and only if it is valid in L^*.

Proof. Let us assume that $C \to R = T$ is valid in L^*, and that C is true with respect to the valuation α in LC*. We associate to each constant a occurring in a term $\alpha(v)$, for v a free variable in $C \to R = T$, a distinct variable x_a in L. We then translate these $\alpha(v)$ into L^* by replacing the a by x_a.

A valuation, associating the translation of $\alpha(v)$ to v, makes C true in L^*, because C is quantifier-free. It follows that it also makes $R = T$ true in L^*. Hence $R = T$

[2] It is at this point that we need the restriction to the simple terms.

is true in LC* relatively to α. Therefore $C \to R = T$ is valid in every logos, by corollary 4.2. ∎

5 Properties of substitution

PROPOSITION 5.1
1. The formula $v \in_{\mathrm{vl}} T$ is equivalent to conditions, if T is a simple term.
2. Formulas of the form $T = S$, or even $v = T$, are generally not equivalent to conditions.

PROOF.
1. Using proposition 3.2.1, we can prove this by induction on the length of simple terms because $v \in_{\mathrm{vl}} T\,(w := S)$ is equivalent to
$(v \in_{\mathrm{vl}} T \wedge \neg v = w) \vee (v \in_{\mathrm{vl}} S \wedge w \in_{\mathrm{vl}} T)$.

2. The first half of the proof of the theorem shows that if a condition is satisfied in a logos, then it is satisfied in a substitution logos.

Therefore $v\,(w := w') = v\,(w := w'')$ is not equivalent to a condition because the formula $v\,(w := w') = v\,(w := w'') \wedge w \in_{\mathrm{vl}} v \wedge \neg w' = w''$, though never satisfiable in a substitution logos, is satisfied in the logos of finite sets by a valuation α such that $\alpha(v) = \{x, y\}$, $\alpha(w) = \{x\}$, $\alpha(w') = \{x\}$ and $\alpha(w'') = \{x, y\}$, where $x \neq y$.

Even $v = v'\,(w := w')$ is not equivalent to a condition because the formula $v = v'\,(w := w') \wedge v \in_{\mathrm{vl}} v \wedge \neg v' \in_{\mathrm{vl}} v'$, which is not satisfiable in any substitution logos, is again satisfied in the logos of finite sets by a valuation α verifying $\alpha(v) = \{x\}$, $\alpha(v') = \{x, y\}$, $\alpha(w) = \{y\}$ and $\alpha(w') = \{x\}$, for $x \neq y$. ∎

As a consequence of proposition 5.1.1, we note that the notions "$\vec{N}$ is free for $\vec{x}$" and "$\vec{x'}$ is suitable for $\vec{x}, M, \vec{N}$" are expressible as conditions —for any simple terms $M, \vec{N}$. Therefore many properties of the simultaneous substitution can be translated as conditional equations and thus, by proposition 3.5, easily checked via the substitution logoi.

5.1 Substitutive properties

We conclude by mentioning two remarkable kinds of properties. The first one is constituted by the properties of the substitution *suffix* alone. They can be expressed by conditional equations $C \to v\sigma = v\tau$ such that v doesn't occur free in C, nor in σ, τ. Axiom 4, for example, can be written as such a suffix equation:

$$\begin{gathered} v \in_{\mathrm{vl}} v \wedge v' \in_{\mathrm{vl}} v' \wedge v \neq v' \wedge v \notin_{\mathrm{vl}} w' \to \\ u[v := w][v' := w'] = u[v' := w'][v := w[v' := w']] \end{gathered}$$

However, when naturally translated in the logos language, the three other axioms of substitution are not equations of this sort:

$$\begin{gathered} v \in_{\mathrm{vl}} v \wedge w \in_{\mathrm{vl}} w \wedge w \notin_{\mathrm{vl}} u \to u[v := w][w := v] = u, \\ v \in_{\mathrm{vl}} v \to v[v := w] = w, \\ v \in_{\mathrm{vl}} v \wedge v \notin_{\mathrm{vl}} u \to u[v := w] = u. \end{gathered}$$

In the expression of suitability, the properties involving simultaneous substitution, that, at first sight, seem to concern the suffix only, contain hidden conditions to the effect that some variables are not in the term in which the substitutions are performed. In order to take such properties into account, we relax the restriction that v cannot occur free in C, and we thus define a **substitutive equation** as a conditional equation of the form

$$D \wedge w_1 \notin v \wedge \ldots \wedge w_n \notin v \rightarrow v\sigma = v\tau,$$

where the free occurrences of v in the condition are all indicated, i.e. v doesn't occur free in D, σ, τ, and v is not one of the variables $w_1, \ldots, w_n$.

Thus, since $D \wedge w_1 \notin v \wedge \ldots \wedge w_n \notin v$ doesn't mention v, except for the specification of fresh variables, a substitutive equation may also be seen as a description of a property of the substitution suffix —a "substitutive property"— in a somewhat generalized sense. Of the four substitution axioms, the third is now the only one that is not a substitutive property.

Our last proposition states that a substitutive property holds if and only if it holds for the variables.

Proposition 5.2
A substitutive equation $C \rightarrow v\sigma = v\tau$ is valid in every logos if and only if the conditional equation $C \wedge v \in_{\mathrm{vl}} v \rightarrow v\sigma = v\tau$ is valid in every logos.

Proof. By the theorem, this follows from the fact that the result is true for the substitution logoi: such a condition on v is obviously verified by the subterms of a term verifying it. ∎

5.1.1 Examples

Using proposition 5.2, the following substitutive properties are easily seen to hold in every logos:

- $M[x := y][y := x] = M[y := x]$ —this property of the suffix entails axiom 3 (with axiom 6).
- $M[x_1 := N_1, \ldots, x_n := N_n] = M[x_{\pi(1)} := N_{\pi(1)}, \ldots, x_{\pi(n)} := N_{\pi(n)}]$, for every permutation π of $\{1, \ldots, n\}$.
- If $y_1, \ldots, y_m \notin \mathrm{vl}(M)$, then

  $$M[x_1 := N_1, \ldots, x_n := N_n, y_1 := P_1, \ldots, y_m := P_m] = M[x_1 := N_1, \ldots, x_n := N_n].$$

- A sequence of substitutions reduces to a simultaneous substitution. Let $M\sigma = M[x_1 := N_1] \ldots [x_n := N_n]$ and put $H_n = N_n$ and $H_j = N_j[x_{j+1} := N_{j+1}] \ldots [x_n := N_n]$, if $j < n$. Then

  $$M\sigma = M[x_{i_1} := H_{i_1}, \ldots, x_{i_k} := H_{i_k}],$$

  for $1 \le i_1 < \ldots < i_k \le n$, $x_{i_p} \neq x_j$ if $j < i_p$, and $\{x_{i_1}, \ldots, x_{i_k}\} = \{x_1, \ldots, x_n\}$.

References

[1] M. Crabbé, "Prelogic of logoi", *Studia Logica* 35 (1976), 219–226.

[2] M. Crabbé, "The Hauptsatz for stratified comprehension: a semantic proof", *Mathematical Logic Quarterly*, 40 (1994): 481–489.

[3] M. Crabbé, "Une axiomatisation de la substitution", *Comptes rendus de l'Académie des sciences de Paris* (2004).

[4] H. B. Curry, "On the definition of substitution, replacement and allied notions in an abstract formal system", *Revue philosophique de Louvain* 50 (1952), 251–269.

[5] H. B. Curry & R. Feys, *Combinatory Logic I*, North-Holland, 1958.

[6] H. B. Curry, *Foundations of Mathematical Logic*, MacGraw-Hill, 1963.

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