INTRODUCTORY CLASSICAL MECHANICS

* WITH PROBLEMS AND SOLUTIONS *

David Morin

Contents

Preface

This textbook has grown out of the first-semester honors freshman physics course that has been taught at Harvard University during recent years. The book is essentially two books in one. Roughly half of it follows the form of a normal textbook, consisting of text, along with exercises suitable for homework assignments. The other half takes the form of a problem book, with all sorts of problems (with solutions) of varying degrees of difficulty. If you have been searching for a supply of practice problems to work on, this should keep you busy for a while.

A brief outline of the book is as follows. Chapter 1 covers statics. Most of it will look familiar, but it has some fun problems. In Chapter 2, we learn about forces and how to apply $F = ma$. There's a bit of math here needed for solving some simple differential equations. Chapter 3 deals with oscillations and coupled oscillators. Again, there's a fair amount of math here needed for solving linear differential equations, but there's no way to avoid it. Chapter 4 deals with conservation of energy and momentum. You've probably seen much of this before, but again, it has lots of neat problems.

In Chapter 5, we introduce the Lagrangian method, which will undoubtedly be new to you. It looks rather formidable at first, but it's really not all that rough. There are difficult concepts at the heart of the subject, but the nice thing is that the technique is easy to apply. The situation here analogous to taking a derivative in calculus; there are substantive concepts on which the theory rests, but the act of taking a derivative is fairly straightforward.

Chapter 6 deals with central forces, Kepler's Laws, and such things. Chapter 7 covers the easier type of angular momentum situations, ones where the direction of the angular momentum is fixed. Chapter 8 covers the more difficult type, ones where the direction changes. Gyroscopes, spinning tops, and other fun and perplexing objects fall into this category. Chapter 9 deals with accelerated frames of reference and fictitious forces.

Chapters 10 through 13 cover relativity. Chapter 10 deals with relativistic kinematics – abstract particles flying through space and time. In Chapter 11, we discuss relativistic dynamics – energy, momentum, force, etc. Chapter 12 introduces the important concept of "4-vectors". The material in this chapter could alternatively be put in the previous two, but for various reasons I thought it best to create a separate chapter for it. Chapter 13 covers a few topics from general relativity. It's not possible for one chapter to do this subject justice, of course, so we'll just look at some basic (but still very interesting) examples.

The appendices contain various useful things. Indeed, Appendices B and C, which cover dimensional analysis and limiting cases, are the first parts of this book you should read.

Throughout the book, I have included many "remarks". These are written in a slightly smaller font than the surrounding text. They begin with a small-capital "REMARK", and they end with a shamrock (♣). The purpose of these remarks is to say something that needs to be said, without disrupting the overall flow of the argument. In some sense these are "extra" thoughts, but they are invariably useful in understanding what is going on. They are usually more informal than the rest of the text. I reserve the right to occasionally use them to babble about things I find interesting, but which you may find a bit tangential. For the most part, however, the remarks address issues and questions that arise naturally in the course of the discussion.

At the end of the solutions to many problems, the obvious thing to do is to check limiting cases.[1] I have written these in a smaller font, but I have not always bothered to start them with a "REMARK" and end them with a "♣", because they are not "extra" thoughts. Checking limiting cases of your answer is something you should *always* do.

For your reading pleasure (I hope), I have included many limericks scattered throughout the text. I suppose that they may be viewed as educational, but they certainly don't represent any deep insight I have on the teaching of physics. I have written them solely for the purpose of lightening things up. Some are funny. Some are stupid. But at least they're all physically accurate (give or take).

A word on the problems. Some are easy, but many are quite difficult. I think you'll find them quite interesting, but don't get discouraged if you have trouble solving them. Some are designed to be brooded over for hours. Or days, or weeks, or months (as I can attest to). I have chosen to write them up for two reasons: (1) Students invariably want extra practice problems, with solutions, to work on, and (2) I find them rather fun.

The problems are marked with a number of asterisks. Harder problems earn more asterisks (on a scale from zero to four). You may, of course, disagree with my judgment of difficulty; but I think an arbitrary weighting scheme is better than none at all.

Just to warn you, even if you understand the material in the text backwards and forwards, the four-star (and many of the three-star) problems will still be very challenging. But that's how it should be. My goal was to create an unreachable upper bound on the number (and difficulty) of problems, since it would be an unfortunate circumstance, indeed, if you were left twiddling your thumbs, having run out of problems to solve. I hope I have succeeded.

For the problems you choose to work on, be careful not to look at the solution too soon. There is nothing wrong with putting a problem aside for a while and coming back to it later. Indeed, this is probably the best way to approach things. If you head to the solution at the first sign of not being able to solve a problem, then

[1] This topic is discussed in Appendix C.

you have wasted the problem.

REMARK: This gives me an opportunity for my first remark (and first limerick, too). One thing many people don't realize is that you need to know more than the correct way(s) to do a problem; you also need to be familiar with a lot of *incorrect* ways to do it. Otherwise, when you come upon a new problem, there may be a number of decent-looking approaches to take, and you won't be able to immediately weed out the poor ones. Struggling a bit with a problem invariably leads you down some wrong paths, and this is an essential part of learning. To understand something, you not only have to know what's right about the right things; you also have to know what's wrong about the wrong things. Learning takes a serious amount of effort, many wrong turns, and a lot of sweat. Alas, there are no short-cuts to understanding physics.

The ad said, For one little fee,
You can skip all that course-work ennui.
So send your tuition,
For boundless fruition!
Get your mail-order physics degree! ♣

Note: The problems with included solutions are called "Problems". The problems without included solutions are called "Exercises". There is no fundamental difference between the two, except for the existence of written-up solutions.

I hope you enjoy the book!

— David Morin

Chapter 1

Statics

Before reading any of the text of this book, you should read Appendices B and C. The material discussed there (dimensional analysis, checking limiting cases, etc.) is extremely important. It's fairly safe to say that an understanding of these topics is absolutely necessary for an understanding of physics. And they make the subject a lot more fun, too!

For many of you, the material in this chapter will be mainly review. As such, the text here will be relatively short. This is an "extra" chapter. Its main purpose is that it provides me with an excuse to give you some cool statics problems. Try as many as you like, but don't go overboard and spend too much time with them; more important and relevant material will soon be at hand.

1.1 Balancing forces

This chapter deals with "static" situations, that is, ones where all the objects are motionless. If an object is motionless, then $F = ma$ tells us that the total force acting on it must be zero. (The converse is not true, of course. The total force can be zero with a constant non-zero velocity. But we'll deal only with static problems here). The whole goal in a statics problem is to find out what the various forces have to be so that there is zero net force acting on each object (and zero net torque, too; but that's the topic of the next section). Since a force is a vector, this goal involves breaking the force up into its components. You can pick cartesian coordinates, polar coordinates, or perhaps another set. (It is usually clear from the problem which system will make your calculations easiest.) Once this is done, you simply demand that the total force in each direction is zero.

There are many different types of forces in the world, most of which are large-scale effects of complicated things going on at smaller scales. For example, the tension in a rope comes about from the chemical bonds that hold the molecules in the rope together. In doing a mechanics problem, there is of course no need to analyze all the details of the forces taking place in the rope at the molecular scale.

You simply call whatever force there is a "tension" and get on with the problem.

Four types of forces come up repeatedly when doing problems:

Tension

Tension is a general name for a force that a rope, stick, etc., exerts when it is pulled on. Every piece of the rope feels a tension force in both directions, except the end point, which feels a tension on one side and a force on the other side from whatever object is attached to the end.

In some cases, the tension may vary along the rope. (The "Rope wrapped around pole" example at the end of this section is an example of this.) In other cases, the tension must be the same everywhere. For example, in a hanging massless rope, or in a massless rope hanging over a frictionless pulley, the tension must be the same at all points, because otherwise there would be a net force on at least one tiny piece, and then $F = ma$ would give an infinite acceleration for this tiny piece.

Normal force

This is the force perpendicular to a surface that a surface applies to an object. The total force applied by a surface is usually a combination of the normal force and the friction force (see below). But for "frictionless" surfaces such as greasy ones or ice, only the normal force exists. The normal force comes about because the surface actually compresses a tiny bit and acts like a very rigid spring; the surface gets squeezed until the restoring force equals the force the object applies.

REMARK: Technically, the only difference between a "normal force" and a "tension" is the direction of the force. Both situations can be modeled by a spring. In the case of a normal force, the spring (a plane, a stick, or whatever) is compressed, and the force on the given object is directed away from the spring. In the case of a tension, the spring is stretched, and the force on the given object is directed toward the spring. Things like sticks can provide both normal forces and tensions. But a rope, for example, has a hard time providing a normal force. ♣

Friction

Friction is the force parallel to a surface that a surface applies to an object. Some surfaces, such as sandpaper, have a great deal of friction. Some, such as greasy ones, have essentially no friction. There are two types of friction, called "kinetic" friction and "static" friction.

Kinetic friction (which we won't deal with in this chapter) deals with two objects moving relative to each other. It is usually a good approximation to say that the kinetic friction between two objects is proportional to the normal force between them. We call the constant of proportionality μ_k (called the "coefficient of kinetic friction"), where μ_k depends on the two surfaces involved. Thus, $F = \mu_k N$. The direction of the force is opposite to the motion.

Static friction deals with two objects at rest relative to each other. In the static case, all we can say prior to solving a problem is that the static friction force has a *maximum* value equal to $F_{\text{max}} = \mu_s N$ (where μ_s is the "coefficient of static friction"). In a given problem, it is most likely less than this. For example, if a block of large mass M sits on a surface with coefficient of friction μ_s, and you give the block a tiny push to the right (tiny enough so that it doesn't move), then the friction force is of course not $\mu_s N = \mu_s Mg$ to the left. Such a force would send the block sailing off to the left. The true friction force is simply equal and opposite to the tiny force you apply. What the coefficient μ_s tells you is that if you apply a force larger than $\mu_s Mg$ (the maximum friction force), then the block will end up moving to the right.

Gravity

Consider two point objects, with masses M and m, separated by a distance R. Newton's law for the gravitational force says that the force between these objects is attractive and has magnitude $F = GMm/R^2$, where $G = 6.67 \cdot 10^{-11}$ $\text{m}^3/(\text{kg} \cdot \text{s}^2)$. As we will show in Chapter 4, the same law applies to spheres. That is, a sphere may be treated like a point mass located at its center. Therefore, an object on the surface of the earth feels a gravitational force equal to

$$F = m\left(\frac{GM}{R^2}\right) \equiv mg, \tag{1.1}$$

where M is the mass of the earth, and R is its radius. This equation defines g. Plugging in the numerical values, we obtain (as you can check) $g \approx 9.8$ m/s^2. Every object on the surface of the earth feels a force of mg downward. If the object is not accelerating, then there must also be other forces present (normal forces, etc.) to make the total force zero.

Example (Block on plane): A block of mass M rests on a plane inclined at angle θ (see Fig. 1.1). You apply a horizontal force of $F = Mg$ to the block, as shown.

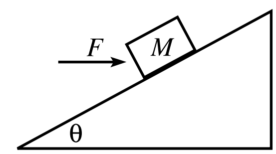


Figure 1.1

(a) Assume that the friction force between the block and plane is large enough to keep the block still. What are the normal and friction forces (call them N and F_f) that the plane exerts on the block?

(b) Let the coefficient of static friction be μ. For what range of angles θ will the block remain still?

Solution:

(a) We will break the forces up into components parallel and perpendicular to the plane ($\hat{x}$ and $\hat{y}$ coordinates would work just as well). The forces are $F = Mg$, F_f, N, and the weight Mg (see Fig. 1.2). Balancing the forces parallel and perpendicular to the plane gives, respectively (with upward along the plane taken to be positive),

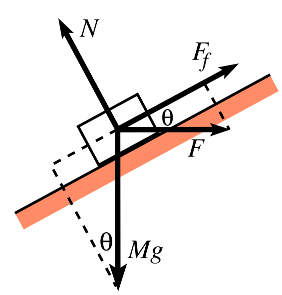


Figure 1.2

$$\begin{aligned} F_f &= Mg\sin\theta - Mg\cos\theta, \qquad \text{and} \\ N &= Mg\cos\theta + Mg\sin\theta. \end{aligned} \tag{1.2}$$

Remarks: Note that if $\tan\theta < 1$, then F_f is positive; and if $\tan\theta < 1$, then F_f is negative. F_f ranges from $-Mg$ to Mg, as θ ranges from 0 to $\pi/2$ (these limiting cases are fairly obvious). You can show that N is maximum when $\tan\theta = 1$, in which case $F_f = 0$ and $N = \sqrt{2}Mg$. ♣

(b) The coefficient μ tells us that $|F_f| \le \mu N$. So we have, from eqs. (1.2),

$$Mg|\sin\theta - \cos\theta| \le \mu Mg(\cos\theta + \sin\theta). \tag{1.3}$$

The absolute value here signifies that we must consider two cases.

- If $\tan\theta \ge 1$, then eq. (1.3) becomes

$$\begin{aligned} \sin\theta - \cos\theta &\le \mu(\cos\theta + \sin\theta) \\ \Longrightarrow \quad \tan\theta &\le \frac{1+\mu}{1-\mu}. \end{aligned} \tag{1.4}$$

- If $\tan\theta \le 1$, then eq. (1.3) becomes

$$\begin{aligned} -\sin\theta + \cos\theta &\le \mu(\cos\theta + \sin\theta) \\ \Longrightarrow \quad \tan\theta &\ge \frac{1-\mu}{1+\mu}. \end{aligned} \tag{1.5}$$

Putting these two ranges for θ together, we have

$$\frac{1-\mu}{1+\mu} \le \tan\theta \le \frac{1+\mu}{1-\mu}. \tag{1.6}$$

Remarks: For very small μ, these bounds both approach 1. (This makes sense. If there is little friction, then the components along the plane of the horizontal and vertical Mg forces must nearly cancel.) A special value for μ is 1. From eq. (1.6), we see that $\mu = 1$ is the cutoff value that allows θ to reach 0 and $\pi/2$. ♣

Let's now do an example involving a rope in which the tension varies with position. We'll need to consider differential pieces of the rope to solve this problem.

Example (Rope wrapped around pole): A rope wraps an angle θ around a pole. You grab one end and pull with a tension T_0. The other end is attached to a large object, say, a boat. If the coefficient of static friction between the rope and the pole is μ, what is the largest force the rope can exert on the boat, if the rope is to not slip around the pole?

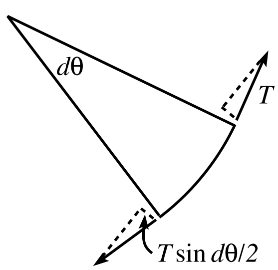


Figure 1.3

Solution: Consider a small piece of the rope. Let this piece subtend and angle $d\theta$, and let the tension in it be T. From Fig. 1.3, we see that the sum of the radially inward components of the tensions at the two ends of this piece is $2T\sin(d\theta/2)$. Therefore, this equals the (radially outward) normal force that the pole exerts on the rope. Note that for small $d\theta$, we may (using $\sin\epsilon \approx \epsilon$) write this normal force as $N_{d\theta} = T\,d\theta$.

The maximum friction force on this little piece of rope is $\mu N_{d\theta} = \mu T\,d\theta$. This friction force is what gives rise to the difference in tension between the two ends of the little piece. In other words, the tension, as a function of θ, satisfies

$$\begin{aligned} T(\theta + d\theta) &\le T(\theta) + \mu N_{d\theta} \\ \Longrightarrow \quad dT &\le \mu T\,d\theta \end{aligned}$$

$$
\begin{aligned}
\Longrightarrow \quad \int \frac{dT}{T} &\leq \int \mu\, d\theta \\
\Longrightarrow \quad \ln T &\leq \mu\theta + C \\
\Longrightarrow \quad T &\leq T_0 e^{\mu\theta},
\end{aligned} \tag{1.7}
$$

where we have used the fact that $T = T_0$ when $\theta = 0$.

This exponential behavior here is quite strong (as exponential behaviors tend to be). If we let $\mu = 1$, then just a quarter turn around the pole produces a factor of $e^{\pi/2} \approx 5$. One full revolution yields a factor of $e^{2\pi} \approx 530$, and two full revolutions yield a factor of $e^{4\pi} \approx 300,000$. Needless to say, the limiting factor in such a case is not your strength, but rather the structural integrity of the pole around which the rope winds.

1.2 Balancing torques

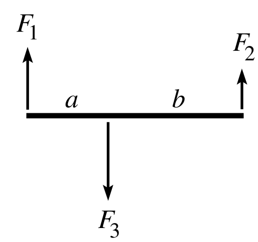


Figure 1.4

In addition to balancing forces in a statics problem, we must also balance torques. We'll have much more to say about torques in Chapters 7 and 8, but we'll need one important fact here.

Consider the situation in Fig. 1.4, where three forces are applied perpendicularly to a stick, which is assumed to remain motionless. F_1 and F_2 are the forces at the ends, and F_3 is the force in the interior. (We have, of course, $F_3 = F_1 + F_2$, because the stick is at rest.)

Claim 1.1 *If the system is motionless, then $F_3 a = F_2(a + b)$. (In other words, the torques around the left end cancel. And you can show that they cancel around any other point, too.)*

We'll prove this claim in another way using angular momentum, in Chapter 7, but let's give a short proof here.

Proof: We'll make one reasonable assumption, which is that the correct relationship between the forces and distances is of the form

$$F_3 f(a) = F_2 f(a + b), \tag{1.8}$$

where $f(x)$ is a function to be determined.[1] Applying this assumption with the roles of "left" and "right" reversed, we have

$$F_3 f(b) = F_1 f(a + b) \tag{1.9}$$

Adding these two equations, and using $F_3 = F_1 + F_2$, gives

$$f(a) + f(b) = f(a + b). \tag{1.10}$$

This implies that $f(x)$ is a linear function, $f(x) = Ax$, as was to be shown.[2] The constant A is irrelevant, since it cancels out in eq. (1.8).) ■

[1] We're simply assuming linearity in F. That is, two forces of F applied at a point should be the same as a force of $2F$ applied at that point. You can't really argue with that.

[2] Another proof of this Claim is given in Problem 11.

The quantities F_3a, $F_2(a+b)$, F_3b, etc., are of course just the torques around various pivot points. Note that dividing eq. (1.8) by eq. (1.9) gives $F_1f(a) = F_2f(b)$, and hence $F_1a = F_2b$, which says that the torques cancel around the point where F_3 is applied. You can easily show that the torques cancel around any arbitrary pivot point.

When adding up all the torques in a given physical setup, it is of course required that you use the same pivot point when calculating each torque.

In the case where the forces aren't perpendicular to the stick, the claim applies to the components of the forces perpendicular to the stick. (This is fairly obvious. The components parallel to the stick won't have any effect on rotating the stick around the pivot point.) Therefore, referring to the figures shown below, we have

$$F_1 a \sin\theta_a = F_2 b \sin\theta_b. \tag{1.11}$$

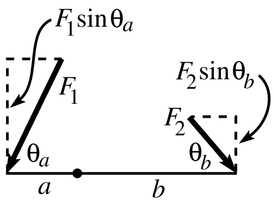


Figure 1.5

This equation can be viewed in two ways:

- $(F_1\sin\theta_a)a = (F_2\sin\theta_b)b$. In other words, we effectively have smaller forces acting on the given "lever-arms". (See Fig. 1.5.)
- $F_1(a\sin\theta_a) = F_2(b\sin\theta_b)$ In other words, we effectively have the given forces acting on smaller "lever-arms". (See Fig. 1.6.)

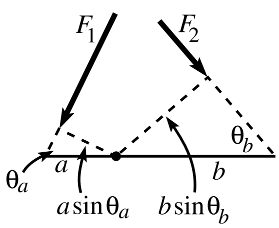


Figure 1.6

Claim 1.1 shows that even if you apply a just a tiny force, you can balance the torque due to a very large force, provided that you make your lever-arm sufficiently long. This fact led a well-know mathematician of long ago to claim that he could move the earth if given a long enough lever-arm.

One morning while eating my Wheaties,
I felt the earth move 'neath my feeties.
The cause for alarm
Was a long lever-arm,
At the end of which grinned Archimedes!

One handy fact that is often used is that the torque, due to gravity, on a stick of mass M is the same as the torque due to a point-mass M located at the center of the stick. (The truth of this statement relies on the fact that torque is a linear function of distance to the pivot point.) More generally, the torque on an object due to gravity may be treated simply as the torque due to a force Mg located at the center of mass.

We'll have much more to say about torque in Chapters 7 and 8, but for now we'll simply use the fact that in a statics problem, the torques around any given point must balance.

Example (Leaning ladder): A ladder leans against a frictionless wall. If the coefficient of friction with the ground is μ, what is the smallest angle the ladder can make with the ground, and not slip?

Solution: As shown in Fig. 1.7, we have three unknown forces: the friction force

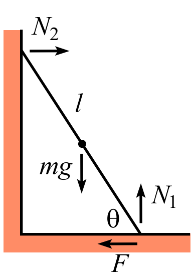


Figure 1.7

F, and the normal forces N_1 and N_2. And we have three equations that will allow us to solve for these three forces: $\Sigma F_{\text{vert}} = 0$, $\Sigma F_{\text{horiz}} = 0$, and $\Sigma\tau = 0$.

Looking at the vertical forces, we see that $N_1 = mg$. And then looking at the horizontal forces, we see that $N_2 = F$. So we have quickly reduced the unknowns from three to one.

We will now use $\Sigma\tau = 0$ to find N_2. But first we must pick the "pivot" point, around which we will calculate the torques. Any stationary point will work fine, but certain choices make the calculations easier than others. The best choice for the pivot is generally the point at which the most forces act, because then the $\Sigma\tau = 0$ equation will have the smallest number of terms in it (because a force provides no torque around the point where it acts, since the lever-arm is zero).

So in this problem, the best choice for pivot is the bottom end of the ladder.[3] Balancing the torques due to gravity and N_2, we have

$$N_2 \ell \sin\theta = mg(\ell/2)\cos\theta \qquad \Longrightarrow \qquad N_2 = \frac{mg}{2\tan\theta}. \tag{1.12}$$

This, then, also equals the friction force F. The condition $F \le \mu N_2 = \mu mg$ therefore becomes

$$\frac{mg}{2\tan\theta} \le \mu mg \qquad \Longrightarrow \qquad \tan\theta \ge \frac{1}{2\mu}\,. \tag{1.13}$$

Remarks: The factor of 1/2 in the above answer comes from the fact that the ladder behaves like a point mass located half way up itself. You can quickly show that the answer for the analogous problem, but now with a massless ladder and a person standing a fraction f up along it, is $\tan\theta \ge f/\mu$.

Note that the total force exerted on the ladder by the floor points up at an angle given by $\tan\beta = N_1/F = (mg)/(mg/2\tan\theta) = 2\tan\theta$. We see that this force does *not* point along the ladder. There is simply no reason why it should. But there *is* a nice reason why it should point upwards with twice the slop of the ladder. This is the direction which causes the lines of the three forces on the ladder to be collinear, as shown in Fig. 1.8.

This collinearity is a neat little theorem for statics problems involving three forces. The proof is simple. If the three lines weren't collinear, then one force would produce a nonzero torque around the intersection point of the other two lines of force.[4] ♣

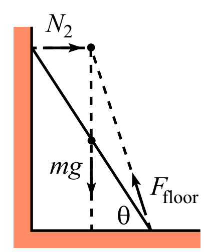


Figure 1.8

That's about all there is to statics problems. All you have to do is balance the forces and torques. To be sure, this sometimes requires a bit of cleverness. There are all sorts of tricks to be picked up by doing problems, so you may as well tackle a few....

[3]But you should verify that other choices of pivot, for example, the middle or top of the ladder, give the same result.

[4]The one exception to this reasoning is when no two of the lines intersect; that is, when all three lines are parallel. Equilibrium is certainly possible in such a scenario. (Of course, you can hang on to our collinearity theorem if you consider the parallel lines to meet at infinity.)

1.3 Problems

Section 1.1: Balancing forces

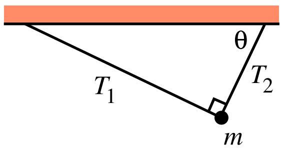


Figure 1.9

1. **Hanging mass**

   A mass m, held up by two strings, hangs from a ceiling (see Fig. 1.9). The strings form a right angle. In terms of the angle θ shown, what is the tension in each string?

2. **Block on a plane**

   A block sits on a plane inclined at an angle θ. Assume that the friction force is large enough to keep the block still. For what θ is the horizontal component of the normal force maximum?

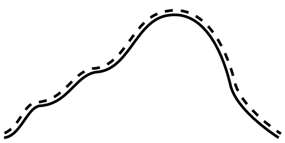


Figure 1.10

3. **Motionless chain** $*$

   A frictionless surface is in the shape of a function which has its endpoints at the same height but is otherwise arbitrary. A chain of uniform mass per unit length rests on this surface (from end to end; see Fig. 1.10). Show that the chain will not move.

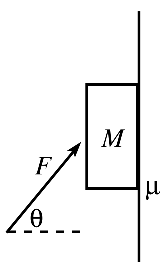


Figure 1.11

4. **Keeping the book up**

   A book of mass M is positioned against a vertical wall. The coefficient of friction between the book and the wall is μ. You wish to keep the book from falling by pushing on it with a force F applied at an angle θ to the horizontal ($-\pi/2 < \theta < \pi/2$). (See Fig. 1.11.) For a given θ, what is the minimum F required? What is the limiting value for θ for which there exists an F which will keep the book up?

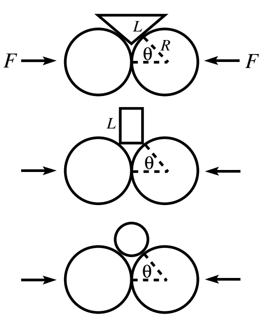


Figure 1.12

5. **Objects between circles** $**$

   Each of the following planar objects is placed, as shown in Fig. 1.12, between two frictionless circles of radius R. The mass density of each object is σ, and the radii to the points of contact make an angle θ with the horizontal. For each case, find the horizontal force that must be applied to the circles to keep them together. For what θ is this force maximum or minimum?

   (a) An isosceles triangle with common side length L.

   (b) A rectangle with height L.

   (c) A circle.

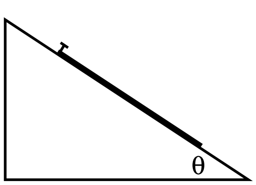


Figure 1.13

6. **Hanging rope** $*$

   (a) A rope with length ℓ and mass density ρ per unit length is suspended from one end. Find the tension along the rope.

   (b) The same rope now lies on a plane inclined at an angle θ (see Fig. 1.13). The top end is nailed to the plane. The coefficient of friction is μ. What is the tension at the top of the rope? (Assume the setup is obtained by initially having the rope lie without any tension on a horizontal plane, and then tilting the plane up to an angle θ.)

7. **Supporting a disc** $***$

   (a) A disc of mass M and radius R is held up by a massless string, as shown in Fig. 1.14. The surface of the disc is frictionless. What is the tension in the string? What is the normal force per unit length the string applies to the disc?

   (b) Let there now be friction between the disc and the string, with coefficient μ. What is the smallest possible tension in the string at its lowest point?

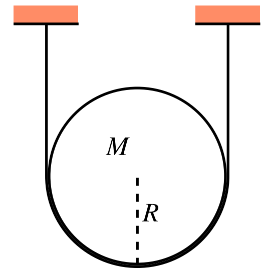


Figure 1.14

8. **Hanging chain** $****$

   (a) A chain of uniform mass density per unit length hangs between two walls. Find the shape of the chain. (Except for an arbitrary additive constant, the function describing the shape should contain one unknown constant.)

   (b) The unknown constant in your answer depends on the horizontal distance d between the walls, the vertical distance h between the support points, and the length ℓ of the chain (see Fig. 1.15). Find an equation involving these given quantities that determines the unknown constant.

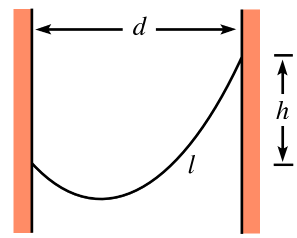


Figure 1.15

9. **Hanging gently** $**$

   A chain hangs between two supports located at the same height, a distance $2d$ apart (see Fig. 1.16). How long should the chain be in order to minimize the magnitude of the force on the supports?

   You may use the fact that the height of the hanging chain is of the form $y(x) = (1/\alpha)\cosh(\alpha x) + a$. You will eventually have to solve an equation numerically in this problem.

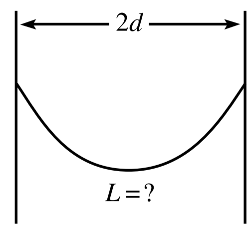


Figure 1.16

10. **Mountain Climber** $****$

    A mountain climber wishes to climb up a frictionless conical mountain. He wants to do this by throwing a lasso (a rope with a loop) over the top and climbing up along the rope. (Assume the mountain climber is of negligible height, so that the rope lies along the mountain; see Fig. 1.17.) At the bottom of the mountain are two stores. One sells "cheap" lassos (made of a segment of rope tied to loop of rope of *fixed* length). The other sells "deluxe" lassos (made of one piece of rope with a loop of *variable* length; the loop's length may change without any friction of the rope with itself). See Fig. 1.18.

    When viewed from the side, this conical mountain has an angle α at its peak. For what angles α can the climber climb up along the mountain if he uses:

    (a) a "cheap" lasso and loops it once around the top of the mountain?

    (b) a "deluxe" lasso and loops it once around the top of the mountain?

    (c) a "cheap" lasso and loops it N times around the top of the mountain? (Assume no friction of the rope with itself.)

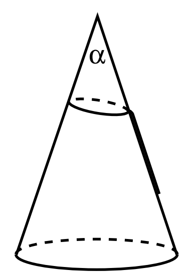


Figure 1.17

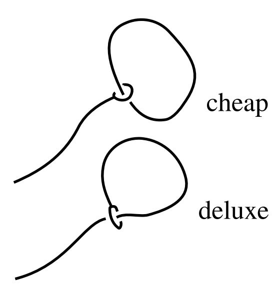


Figure 1.18

(d) a "deluxe" lasso and loops it N times around the top of the mountain? (Assume no friction of the rope with itself.)

Section 1.2: Balancing torques

11. **Equality of torques** $**$

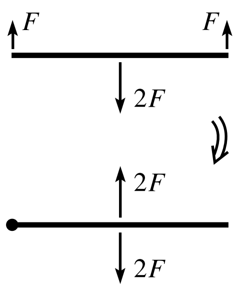


Figure 1.19

This problem gives another way to demonstrate Claim 1.1, using an inductive argument. We'll get you started, and then you can do the general case.

Consider the situation where forces F are applied upward at the ends of a stick of length ℓ, and a force $2F$ is applied downward at the midpoint (see Fig. 1.19). The stick will not rotate (by symmetry), and it will not translate (because the net force is zero). We may consider the stick to have a pivot at the left end, if we wish. If we then erase the force F on the right end and replace it with a force $2F$ at the middle, then the two $2F$ forces in the middle will cancel, so the stick will remain still. (There will now be a different force applied at the pivot, namely zero, but the purpose of the pivot is to simply apply whatever force is necessary to keep the end still.) Therefore, we see that a force F applied at a distance ℓ from a pivot is 'equivalent' to a force $2F$ applied at a distance $\ell/2$.

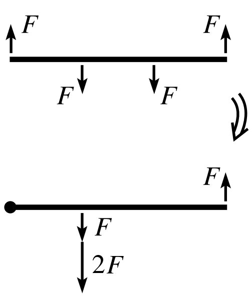


Figure 1.20

Now consider the situation where forces F are applied upward at the ends, and forces F are applied downward at the $\ell/3$ and $2\ell/3$ marks (see Fig. 1.20). The stick will not rotate (by symmetry), and it will not translate (because the net force is zero). Consider the stick to have a pivot at the left end. From the above paragraph, the force F at $2\ell/3$ is equivalent to a force $2F$ at $\ell/3$. Making this replacement, we have the situation shown in Fig. 1.20, with a force $3F$ at the $\ell/3$ mark. Therefore, we see that a force F applied at a distance ℓ is equivalent to a force $3F$ applied at a distance $\ell/3$.

Use induction to show that a force F applied at a distance ℓ is equivalent to a force nF applied at a distance ℓ/n, and then argue why this demonstrates Claim 1.1.

12. **Find the force** $*$

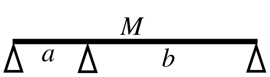


Figure 1.21

A stick of mass M is held up by supports at each end. Each support clearly provides a force of $Mg/2$. Now put another support somewhere in the middle (say, at a distance a from one support, and b from the other; see Fig. 1.21). What forces do the three supports now provide? Can you solve this?

13. **Leaning sticks** $*$

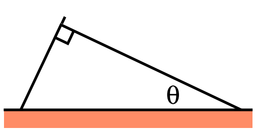


Figure 1.22

One stick leans on another as shown in Fig. 1.22. A right angle is formed where they meet, and the right stick makes an angle θ with the horizontal. The left stick extends infinitesimally beyond the end of the right stick. The coefficient of friction between the two sticks is μ. The sticks have the same mass density per unit length and are both hinged at the ground. What is the minimum angle θ for which the sticks do not fall?

14. **Supporting a ladder** $*$

A ladder of length L and mass M has its bottom end attached to the ground by a pivot. It makes an angle θ with the horizontal, and is held up by a person of total length ℓ who is attached to the ground by a pivot at his feet (see Fig. 1.23). Assume that the person has zero mass, for simplicity. The person and the ladder are perpendicular to each other. Find the force that the person applies to the ladder.

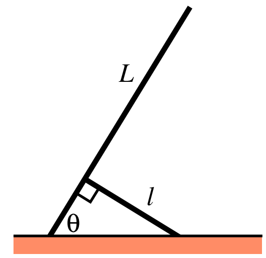


Figure 1.23

15. **Stick on a circle** $*$

A stick of mass per unit length ρ rests on a circle of radius R (see Fig. 1.24). The stick makes an angle θ with the horizontal. The stick is tangent to the circle at its upper end. Friction exists at all points of contact in this problem. Assume that all of these friction forces are large enough to keep the system still. Find the friction force between the ground and the circle.

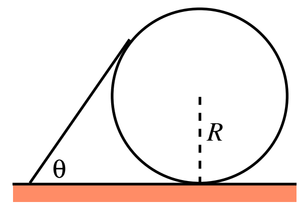


Figure 1.24

16. **Leaning sticks and circles** $**$

A large number of sticks (of mass per unit length ρ) and circles (of radius R) lean on each other, as shown in Fig. 1.25. Each stick makes an angle θ with the ground. Each stick is tangent to a circle at its upper end. The sticks are hinged to the ground, and every other surface is *frictionless* (unlike in the previous problem). In the limit of a very large number of sticks and circles, what is the normal force between a stick and the circle it rests on, very far to the right? (Assume that the last circle is glued to the floor, to keep it from moving.)

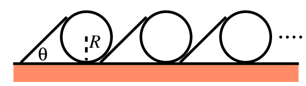


Figure 1.25

17. **Balancing the stick** $**$

Given a semi-infinite stick (that is, one that goes off to infinity in one direction), find how its density should depend on position so that it has the following property: If the stick is cut at an arbitrary location, the remaining semi-infinite piece will balance on a support located a distance b from the end (see Fig. 1.26).

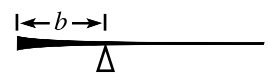


Figure 1.26

18. **The spool** $**$

A spool consists of an axis of radius r and an outside circle of radius R which rolls on the ground. A thread which is wrapped around the axis is pulled with a tension T (see Fig. 1.27).

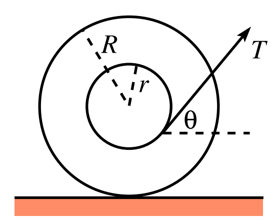


Figure 1.27

(a) Given R and r, what angle, θ, should the thread make with the horizontal so that the spool does not move. Assume there is large enough friction between the spool and ground so that the spool doesn't slip.

(b) Given R, r, and a coefficient of friction μ between the spool and ground, what is the largest T can be (assuming the spool doesn't move)?

(c) Given R and μ, what should r be so that the upper bound on T found in part (b) is as small as possible (assuming the spool doesn't move)? What is the resulting value of T?

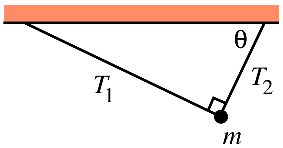


Figure 1.28

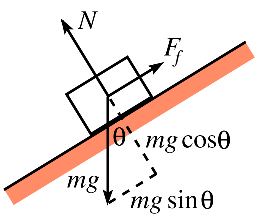


Figure 1.29

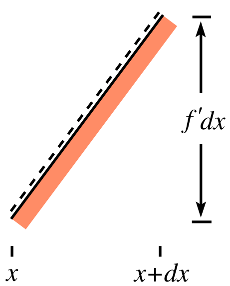


Figure 1.30

1.4 Solutions

1. **Hanging mass**

   Balancing the horizontal and vertical force components on the mass gives (see Fig. 1.28)

$$\begin{aligned} T_1 \sin\theta &= T_2 \cos\theta, \\ T_1 \cos\theta + T_2 \sin\theta &= mg. \end{aligned} \tag{1.14}$$

   The solution to these equations is

$$T_1 = mg\cos\theta, \qquad \text{and} \qquad T_2 = mg\sin\theta. \tag{1.15}$$

   As a double-check, these have the correct limits when $\theta \to 0$ or $\theta \to \pi/2$.

2. **Block on a plane**

   The component of the block's weight perpendicular to the plane is $mg\cos\theta$ (see Fig. 1.29). The normal force is therefore $N = mg\cos\theta$. The horizontal component of this is $mg\cos\theta\sin\theta$. To maximize this, we can either take a derivative or we can write it as $(1/2)mg\sin 2\theta$, from which it is clear that the maximum occurs at $\theta = \pi/4$. (The maximum is $mg/2$.)

3. **Motionless chain**

   Let the curve run from $x = a$ to $x = b$. Consider a little piece of the chain between x and $x + dx$ (see Fig. 1.30). The length of this piece is $\sqrt{1 + f'^2}\,dx$. Therefore, its mass is $\rho\sqrt{1 + f'^2}\,dx$, where ρ is the mass per unit length. The component of gravity along the curve is $-gf'/\sqrt{1 + f'^2}$ (with positive taken to be to the right). So the total force, F, along the curve is

$$\begin{aligned} F &= \int_a^b \left(\frac{-gf'}{\sqrt{1+f'^2}} \right) \left(\rho\sqrt{1+f'^2}\,dx \right) \\ &= -\rho g \int_a^b f'\,dx \\ &= -g\rho\big(f(a) - f(b)\big) \\ &= 0. \end{aligned} \tag{1.16}$$

4. **Keeping the book up**

   The normal force on the wall is $F\cos\theta$. So the friction force holding the book up is at most $\mu F\cos\theta$. The other vertical forces on the book are $-Mg$ and the vertical component of F, which is $F\sin\theta$. If the book is to stay up, we must have

$$\mu F\cos\theta + F\sin\theta - Mg > 0. \tag{1.17}$$

   So F must satisfy

$$F > \frac{Mg}{\mu\cos\theta + \sin\theta}. \tag{1.18}$$

   There is no F that satisfies this if the right-hand-side is infinite. This occurs when

$$\tan\theta = -\mu. \tag{1.19}$$

   So if θ is more negative than this, then it is impossible to keep the book up.

5. **Objects between circles**

(a) Let N be the normal force. The goal in this problem is to find the horizontal component of N, which is $N\cos\theta$.

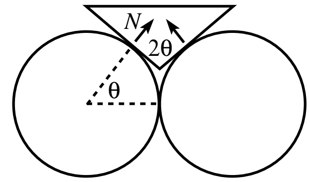


Figure 1.31

The upward force on the triangle from the normal forces is $2N\sin\theta$ (see Fig. 1.31). This must equal the weight of the triangle, which is σ times the area. Since the bottom angle of the isosceles triangle is 2θ, the top side of the triangle has length $2L\sin\theta$, and the altitude to that side is $L\cos\theta$. So the area of the triangle is $L^2\sin\theta\cos\theta$. The mass is therefore $\sigma L^2\sin\theta\cos\theta$. Equating the weight with the upward normal force gives $N = g\sigma L^2\cos\theta/2$, independent of R. The horizontal component is therefore

$$N\cos\theta = \frac{g\sigma L^2\cos^2\theta}{2}. \tag{1.20}$$

This is 0 at $\theta = \pi/2$, and it grows as θ decreases to $\theta = 0$ (even though the triangle is getting smaller). It has the interesting property of approaching the finite number $g\sigma L^2/2$, as $\theta \to 0$.

(b) From Fig. 1.32, the base of the rectangle has length $2R(1-\cos\theta)$. The mass is therefore $\sigma 2RL(1-\cos\theta)$. Equating the weight with the upward normal force, $2N\sin\theta$, gives $N = g\sigma LR(1-\cos\theta)/\sin\theta$. The horizontal component is therefore

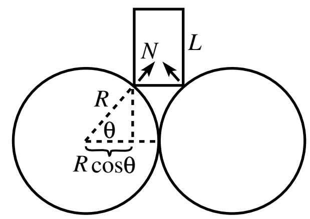


Figure 1.32

$$N\cos\theta = \frac{g\sigma LR(1-\cos\theta)\cos\theta}{\sin\theta}. \tag{1.21}$$

This is 0 at both $\theta = 0$ and $\theta = \pi/2$. Taking the derivative to find where it reaches a maximum, we find (using $\sin^2\theta = 1-\cos^2\theta$),

$$\cos^3\theta - 2\cos\theta + 1 = 0. \tag{1.22}$$

An obvious root of this equation is $\cos\theta = 1$ (which we know is not the maximum). Dividing through by the factor $(\cos\theta - 1)$ gives

$$\cos^2\theta + \cos\theta - 1 = 0. \tag{1.23}$$

The roots of this are

$$\cos\theta = \frac{-1\pm\sqrt{5}}{2}. \tag{1.24}$$

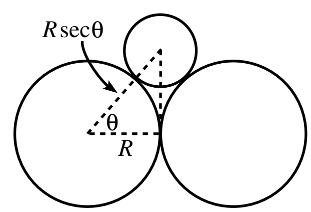


Figure 1.33

We must choose the plus sign, since $|\cos\theta| \leq 1$. This root is the golden ratio, $\cos\theta \approx 0.618\ldots$! The angle θ is $\approx 51.8°$.

(c) From Fig. 1.33, the length AB is $R\sec\theta$, so the radius of the top circle is $R(\sec\theta - 1)$. The mass is therefore $\sigma\pi R^2(\sec\theta-1)^2$. Equating the weight with the upward normal force, $2N\sin\theta$, gives $N = g\sigma\pi R^2(\sec\theta-1)^2/2\sin\theta$. The horizontal component is therefore

$$N\cos\theta = \frac{g\sigma\pi R^2}{2}\frac{\cos\theta}{\sin\theta}\left(\frac{1}{\cos\theta}-1\right)^2. \tag{1.25}$$

This is 0 at $\theta = 0$ (from using $\cos\theta \approx 1-\theta^2/2$ for small θ, so $1/\cos\theta \approx 1+\theta^2/2$; so there is an extra θ in the numerator in the small θ limit). For $\theta \to \pi/2$, it behaves like $1/\cos\theta$, which goes to infinity. (In this limit, N points almost vertically, but its magnitude is so large that the horizontal component still approaches infinity.)

6. **Hanging rope**

   (a) Consider a small piece of the rope between y and $y + dy$ ($0 \le y \le \ell$). The forces on the piece are $T(y + dy)$ upward, $T(y)$ downward, and the weight of the piece (which can be written as $\rho g\, dy$) downward. If the rope is still, then we have $T(y + dy) = T(y) + \rho g\, dy$. Expanding this to first order in dy gives $T'(y) = \rho g$. The tension in the bottom of the rope is 0, so integrating from $y = 0$ up to a position y gives

   $$T(y) = \rho g y. \tag{1.26}$$

   As a double-check, at the top end we have $T(\ell) = \rho g \ell$, which is just the weight of the whole rope, as it should be.

   Of course, one can simply write down the correct answer $T(y) = \rho g y$ by demanding that the tension at a given point accounts for the weight of all the rope below it.

   (b) Let z be the coordinate along the plane ($0 \le z \le \ell$). Consider a small piece of the rope between z and $z + dz$. Balancing the forces on the rope along the plane gives $T(z + dz) + F_f(z)dz = T(z) + \rho g \sin\theta\, dz$, where upward is taken to be the positive direction for the friction force $F_f dz$ (where $F_f(z)$ is the friction force per unit length). Expanding this to first order in dz gives

   $$T'(z) = \rho g \sin\theta - F_f(z). \tag{1.27}$$

   The largest the friction force on a small piece can be is $\mu N dz$, where N is the normal force per unit length (namely $N = \rho g \cos\theta$). But it may not need to be this large, depending on θ and μ. F_f will not be so large that it makes the right-hand-side of eq. (1.27) negative. There are two cases to consider.

   - If $\rho g \sin\theta < \mu N$ (i.e., if $\tan\theta < \mu$), then $F_f(z)$ will simply be equal to $\rho g \sin\theta$ (i.e., the friction of each little piece accounts for its weight; so $T'(z) = 0$ everywhere, and so $T(z) = 0$ everywhere).
   - If $\rho g \sin\theta > \mu N$ (i.e., if $\tan\theta > \mu$), then $F_f(z) = \mu N = \mu \rho g \cos\theta$. Therefore,

     $$T'(z) = \rho g \sin\theta - \mu \rho g \cos\theta. \tag{1.28}$$

     Using $T(0) = 0$, this gives

     $$T(\ell) = \rho g \ell \sin\theta - \mu \rho g \ell \cos\theta \equiv \rho g (y_0 - \mu x_0), \tag{1.29}$$

     where x_0 and y_0 are the width and height of the rope. In the limit $x_0 = 0$ (i.e., a vertical rope), we get the answer from part (a).

   The angle $\theta_0 = \arctan(\mu)$ is the minimum angle of inclination for which there is any force on the nail at the top end.

7. **Supporting a disc**

   (a) The force down on the disc is Mg, and the force up is $2T$. These forces must balance, so

   $$T = \frac{Mg}{2}. \tag{1.30}$$

   We can find the normal force per unit length the string applies to the disc in two ways.

   First method: Let $N d\theta$ be the normal force on an arc which subtends an angle θ. (So N/R is the desired normal force per unit arclength.) The tension in the string is uniform, so N is a constant, independent of θ. The upward component of this force is $N d\theta \cos\theta$ (where θ is measured from the vertical, i.e., $-\pi/2 \le \theta \le \pi/2$). The total upward force must be Mg, so we require

$$\int_{-\pi/2}^{\pi/2} N \cos\theta \, d\theta = Mg. \tag{1.31}$$

The integral on the left is $2N$, so $N = Mg/2$. The normal force per unit length, N/R, is $Mg/2R$.

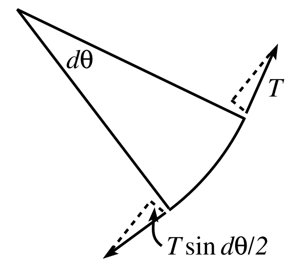


Figure 1.34

Second method: Consider the normal force, $N d\theta$, on a small arc of the circle which subtends and angle $d\theta$. The tension forces on each end of the small piece of string here almost cancel, but they don't exactly, due to the fact that they point in different directions (see Fig. 1.34). Their non-zero sum is what gives the normal force. It's easy to see that the two forces have a sum equal to $2T\sin(d\theta/2)$ (directed radially inward). Since $d\theta$ is small, we may approximate this as $N d\theta = T d\theta$. Hence, $N = T$. The normal force per unit arclength, N/R, is therefore T/R. And since $T = Mg/2$, this equals $Mg/2R$.

(b) Let $T(\theta)$ be the tension, as a function of θ, for $-\pi/2 \le \theta \le \pi/2$. ($T$ will depend on θ now, since there is a tangential force from the friction.) Let $N(\theta)d\theta$ be the normal force, as a function of θ, on an arc which subtends an angle $d\theta$. Then from the second solution above, we have (the existence of friction doesn't affect this equality)

$$T(\theta) = N(\theta). \tag{1.32}$$

Let $F_f(\theta)d\theta$ be the friction force that this little piece of string applies to the disc between θ and $\theta + d\theta$. Balancing the forces on this little piece of (massless) string, we have

$$T(\theta + d\theta) = T(\theta) + F_f(\theta)d\theta. \tag{1.33}$$

(This holds for $\theta > 0$. There would be a minus sign in front of F_f for $\theta < 0$. Since the tension is symmetric around $\theta = 0$, we'll only bother with the $\theta > 0$ case.) Writing $T(\theta + d\theta) \approx T(\theta) + T'(\theta)d\theta$, we find

$$T'(\theta) = F_f(\theta). \tag{1.34}$$

Since the goal is to find the minimum value for $T(0)$, and since we know that $T(\pi/2)$ must be equal to the constant $Mg/2$ (because the tension in the string above the disc is $Mg/2$, from part (a)), we want to look at the case where T' (which equals F_f) is as large as possible. But by the definition of static friction, we have $F_f(\theta)d\theta \le \mu N(\theta)d\theta = \mu T(\theta)d\theta$ (where the second equality comes from eq. (1.32)). Therefore, $F_f \le \mu T$. So eq. (1.34) becomes

$$T'(\theta) \le \mu T(\theta). \tag{1.35}$$

Separating variables and integrating from the bottom of the rope up to an angle θ gives $\ln\big((T(\theta)/T(0)\big) \le \mu\theta$. Exponentiating gives

$$T(\theta) \le T(0)e^{\mu\theta}. \tag{1.36}$$

Letting $\theta = \pi/2$, and noting that T equals $Mg/2$ when $\theta = \pi/2$, yields $Mg/2 \le T(0)e^{\mu\pi/2}$. So we finally have

$$T(0) \ge \frac{Mg}{2}e^{-\mu\pi/2}. \tag{1.37}$$

This minimum value for $T(0)$ goes to Mg/2 as $\mu \to 0$, as it should. And it goes to zero as $\mu \to \infty$, as it should (imagine a very sticky surface, so that the friction force from the rope near $\theta = \pi/2$ accounts for essentially all the weight).

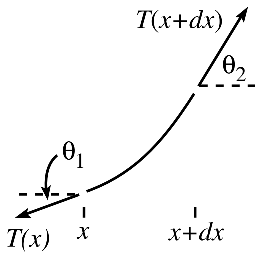


Figure 1.35

8. **Hanging chain**

(a) Let the chain be described by the function $Y(x)$. Let the tension in the chain be described by the function $T(x)$. Consider a small piece of the chain, with endpoints having coordinates x and $x + dx$ (see Fig. 1.35). Let the tension at x pull downward at an angle θ_1 with respect to the horizontal. Let the tension at $x + dx$ pull upward at an angle θ_2 with respect to the horizontal. (So $\cos\theta_1 = 1/\sqrt{1 + (Y'(x))^2}$, and $\cos\theta_2 = 1/\sqrt{1 + (Y'(x + dx))^2} \approx 1/\sqrt{1 + (Y'(x) + Y''(x)dx)^2}$.) Balancing the horizontal and vertical forces on the small piece of chain gives

$$\begin{aligned} T(x + dx)\cos\theta_2 &= T(x)\cos\theta_1, \\ T(x + dx)\sin\theta_2 &= T(x)\sin\theta_1 + \frac{g\rho}{\cos\theta_1}\,dx, \end{aligned} \tag{1.38}$$

where ρ is the mass per unit length. The second term on the right above is the weight of the small piece, since $dx/\cos\theta_1$ is its length. (The second of these equations is valid only for x on the right side of the minimum, i.e., where $Y'(x) > 0$. When $Y'(x) < 0$, there should be a minus sign in front of the second term on the right.)

We somehow have to solve these two differential equations for the two unknown functions, $Y(x)$ and $T(x)$. (The angles θ_1 and θ_2 depend on $Y(x)$). There are various ways to do this. Here is one way, broken down into three steps.

- Squaring and adding eqs. (1.38) gives

$$\big(T(x + dx)\big)^2 = \big(T(x)\big)^2 + 2T(x)g\rho\tan\theta_1\,dx + \mathcal{O}(dx^2). \tag{1.39}$$

Writing $T(x + dx) \approx T(x) + T'(x)dx$, and using $\tan\theta_1 = Y'$, we find (neglecting higher order terms in dx)

$$T' = g\rho Y', \tag{1.40}$$

and so

$$T(x) = g\rho Y(x) + C. \tag{1.41}$$

- Now let's see what we can extract from the first equation in (1.38). Expanding things to first order gives (all the functions are evaluated at x, which we won't bother writing, for the sake of neatness)

$$(T + T'dx)\frac{1}{\sqrt{1 + (Y' + Y''dx)^2}} = T\frac{1}{\sqrt{1 + Y'^2}}. \tag{1.42}$$

Expanding the first square root gives (to first order in dx)

$$(T + T'dx)\frac{1}{\sqrt{1 + Y'^2}}\left(1 - \frac{Y'Y''dx}{1 + Y'^2}\right) = T\frac{1}{\sqrt{1 + Y'^2}}. \tag{1.43}$$

To first order in dx this yields

$$\frac{T'}{T} = \frac{Y'Y''}{1 + Y'^2}. \tag{1.44}$$

Integrating both sides yields

$$\ln T + c = \frac{1}{2}\ln(1 + Y'^2), \tag{1.45}$$

where c is a constant of integration. Exponentiation then gives

$$b^2T^2 = 1 + Y'^2, \tag{1.46}$$

where $b \equiv e^c$.

- We may now combine eq. (1.46) with eq. (1.40) to solve for T. Eliminating Y' gives $b^2T^2 = 1 + T'^2/(g\rho)^2$. Solving for T' and separating variables yields

$$g\rho \int dx = \int \frac{dT}{\sqrt{b^2T^2 - 1}}. \tag{1.47}$$

(We took the positive square-root because we are looking at x on the right side, for which $T' > 0$.)
The integral on the left is $g\rho(x - a)$, for some constant a. The integral on the right equals $(1/b)\ln(bT + \sqrt{b^2T^2 - 1})$. So we find (with $\alpha \equiv bg\rho$)

$$T(x) = \frac{g\rho}{2\alpha}\left(e^{\alpha(x-a)} + e^{-\alpha(x-a)}\right) \equiv \frac{g\rho}{\alpha}\cosh\big(\alpha(x - a)\big). \tag{1.48}$$

Using eq. (1.41) to find Y, we have

$$Y(x) = \frac{1}{\alpha}\cosh\big(\alpha(x - a)\big) + B, \tag{1.49}$$

where B is some constant (which is rather meaningless; it just depends on where you choose the $y = 0$ point). This is valid when $Y'(x) > 0$, that is, when $x > a$. If $Y'(x) < 0$, then there is a minus sign in the second of eqs. (1.38), but the result turns out to be the same.

We may eliminate the need for a if we pick the $x = 0$ point to be at the minimum of the chain. Then $Y'(0) = 0$ implies $a = 0$. So we finally have

$$Y(x) = \frac{1}{\alpha}\cosh(\alpha x) + B. \tag{1.50}$$

This is the shape of the chain.

(b) The constant α may be determined by the locations of the endpoints and the length of the chain. If one hangs a chain between two points, then the given information is (1) the horizontal distance, d, between the two points, (2) the vertical distance, h, between the two points, and (3) the length, ℓ, of the chain (see Fig. 1.36). Note that one does *not* easily know the horizontal distances between the ends and the minimum point (which we have chosen as the $x = 0$ point). If $h = 0$, then these distances are of course $d/2$; but otherwise, they are not obvious.

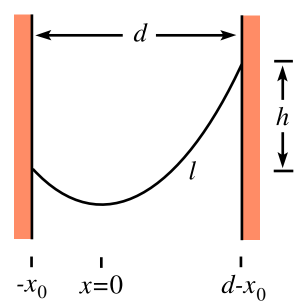


Figure 1.36

If we let the left endpoint be located at $x = -x_0$, then the right endpoint is at $x = d - x_0$. We now have two unknowns, x_0 and α. Our two conditions are

$$Y(d - x_0) - Y(-x_0) = h \tag{1.51}$$

(we take the right end to be higher than the left end, without loss of generality), and the condition that the length equals ℓ, which takes the form (using eq.

(1.50))

$$\begin{aligned} \ell &= \int_{-x_0}^{d-x_0} \sqrt{1+Y'^2}\,dx \\ &= \frac{1}{\alpha}\sinh(\alpha x)\Big|_{-x_0}^{d-x_0}. \end{aligned} \tag{1.52}$$

If $h = 0$, the limits are simply $\pm d/2$, so we may (numerically) solve for b, using only this equation, $\alpha\ell/2 = \sinh(\alpha d/2)$. If $h \neq 0$, one has to (numerically) solve two equations for two unknowns. Writing out eqs. (1.51) and (1.52) explicitly, we have

$$\begin{aligned} \cosh\big(\alpha(d-x_0)\big) - \cosh(-\alpha x_0) &= \alpha h, \\ \sinh\big(\alpha(d-x_0)\big) - \sinh(-\alpha x_0) &= \alpha \ell. \end{aligned} \tag{1.53}$$

If we take the difference of the squares of these two equations, and use the hyperbolic identities $\cosh^2 x - \sinh^2 x = 1$ and $\cosh x \cosh y - \sinh x \sinh y = \cosh(x-y)$, we obtain

$$2\cosh(\alpha d) - 2 = \alpha^2(\ell^2 - h^2), \tag{1.54}$$

which determines α. (This can be rewritten as $2\sinh(\alpha d/2) = \alpha\sqrt{\ell^2 - h^2}$, if desired.)

There are various limits one can check here. If $\ell^2 = d^2 + h^2$ (i.e., the chain forms a straight line), then we have $2\cosh(\alpha d) - 2 = \alpha^2 d^2$; the solution to this is $\alpha = 0$, which does indeed correspond to a straight line. Also, if ℓ is much larger than both d and h, then the solution is a very large α, which corresponds to a 'droopy' chain.

9. **Hanging gently**

We need to calculate the length of the chain, to get its mass. Then we need to find the slope at the support, to break the force there into its components.

The slope as a function of x is

$$y' = \frac{d}{dx}\left(\frac{1}{\alpha}\cosh(\alpha x) + a\right) = \sinh(\alpha x). \tag{1.55}$$

The total length is therefore

$$\begin{aligned} L &= \int_{-d}^{d} \sqrt{1+y'^2}\,dx \\ &= \int_{-d}^{d} \cosh(\alpha x) \\ &= \frac{2}{\alpha}\sinh(\alpha d). \end{aligned} \tag{1.56}$$

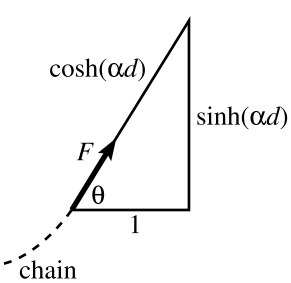


Figure 1.37

The weight of the rope is $W = \rho L g$, where ρ is the mass per unit length. Each support applies a vertical force of $W/2$. This must equal $F\sin\theta$, where F is the support force, and θ is the angle it makes with the horizontal. Since $\tan\theta = y'(d) = \sinh(\alpha d)$, we have $\sin\theta = \tanh(\alpha d)$ (from Fig. 1.37). Therefore,

$$\begin{aligned} F &= \frac{W}{2}\frac{1}{\sin\theta} \\ &= \frac{\rho g \sinh(\alpha d)}{\alpha}\frac{1}{\tanh(\alpha d)} \\ &= \frac{\rho g}{\alpha}\cosh(\alpha d). \end{aligned} \tag{1.57}$$

Taking the derivative of this (as a function of α), and setting it equal to zero gives

$$\tanh(\alpha d) = \frac{1}{\alpha d}. \tag{1.58}$$

This must be solved numerically. The result is

$$\alpha d \approx 1.1997 \equiv \eta. \tag{1.59}$$

The shape of the chain that requires the minimum F is therefore

$$y(x) \approx \frac{d}{\eta}\cosh\left(\frac{\eta x}{d}\right) + a. \tag{1.60}$$

From eqs. (1.56) and (1.59), the length is

$$L = \frac{2d}{\eta}\sinh(\eta) \approx (2.52)d. \tag{1.61}$$

To get an idea of what the chain looks like, we can calculate the ratio of the height, h, to d.

$$\begin{aligned} \frac{h}{d} &= \frac{y(d) - y(0)}{d} \\ &= \frac{\cosh(\eta) - 1}{\eta} \\ &\approx 0.675. \end{aligned} \tag{1.62}$$

We can also calculate the angle of the rope at the supports; we find $\theta \approx 56.5°$.

Remark: One can also ask what shape the chain should take in order to minimize the horizontal or vertical component of F.

The vertical component F_y is just the weight, so we clearly want the shortest possible chain, namely a horizontal one (which requires an infinite F.) This corresponds to $\alpha = 0$.

The horizontal component is $F_x = F\cos\theta$. Since $\cos\theta = 1/\cosh(\alpha d)$, eq. (1.57) gives $F_x = \rho g/\alpha$. This goes to zero as $\alpha \to \infty$, which corresponds to a chain of infinite length. ♣

10. **Mountain Climber**

(a) We will take advantage of the fact that a cone is 'flat', in the sense that you can make one out of a piece of paper, without crumpling the paper.

Cut the cone along a straight line emanating from the peak and passing through the knot of the lasso, and roll the cone flat onto a plane. Call the resulting figure, a sector of a circle, S. (See Fig. 1.38.)

If the cone is very sharp, then S will look like a thin 'pie piece'. If the cone is very wide, with a shallow slope, then S will look like a pie with a piece taken out of it. Points on the straight-line boundaries of the sector S are identified with each other. Let P be the location of the lasso's knot. Then P appears on

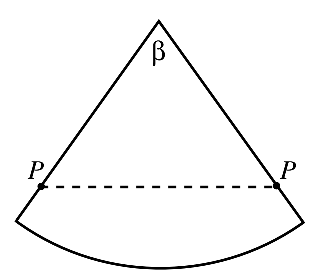


Figure 1.38

each straight-line boundary, at equal distances from the tip of S. Let β be the angle of the sector S.

The key to the problem is realizing that the path of the lasso's loop must be a straight line on S, as shown in Fig. 1.38. (The rope will take the shortest distance between two points since there is no friction, and rolling the cone onto a plane does not change distances.) Such a straight line between the two identified points P is possible if and only if the sector S is smaller than a semicircle, i.e., $\beta < 180°$.

Let C denote a cross sectional circle, at a distance d (measured along the cone) from the top of the mountain, and let μ equal the ratio of the circumference of C to d. Then a semicircular S implies that $\mu = \pi$. This then implies that the radius of C is equal to $d/2$. Therefore, $\alpha/2 = \sin^{-1}(1/2)$. So we find that if the climber is to be able to climb up along the mountain, then

$$\alpha < 60°. \tag{1.63}$$

Having $\alpha < 60°$ guarantees that there is a loop around the cone of shorter length than the distance straight to the peak and back.

Remark: When viewed from the side, the rope should appear perpendicular to the side of the mountain at the point opposite the lasso's knot. A common mistake is to assume that this implies $\alpha < 90°$. This is not the case, because the loop does not lie in a plane. Lying in a plane, after all, would imply an elliptical loop; but the loop must certainly have a discontinuous change in slope where the knot is. (For planar, triangular mountains, the answer to the problem would be $\alpha < 90°$.) ♣

(b) Use the same strategy. Roll the cone onto a plane. If the mountain very steep, the climber's position can fall by means of the loop growing larger; if the mountain has a shallow slope, the climber's position can fall by means of the loop growing smaller. The only situation in which the climber will not fall is the one where the change in the position of the knot along the mountain is exactly compensated by the change in length of the loop.

In terms of the sector S in a plane, the condition is that if we move P a distance ℓ up (down) along the mountain, the distance between the identified points P decreases (increases) by ℓ. We must therefore have $2\sin(\beta/2) = 1$. So $\beta = 60°$, and hence μ (defined in part (a)) is equal to $\pi/3$. This corresponds to

$$\alpha = 2\sin^{-1}(1/6) \approx 19°. \tag{1.64}$$

We see that there is exactly one angle for which the climber can climb up along the mountain.

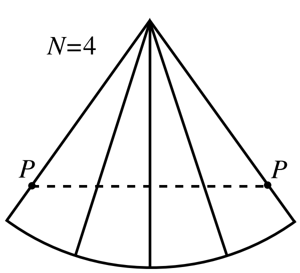


Figure 1.39

Remark: Another way to see that β equals 60° is to note that the three directions of rope emanating from the knot all have the same tension, since the deluxe lasso is one continuous piece of rope. Therefore they must have 120° angles between themselves. This implies that $\beta = 60°$. ♣

(c) Roll the cone N times onto a plane, as shown in Fig. 1.39 for $N = 4$. The resulting figure S_N is a sector of a circle divided into N equal sectors, each representing a copy of the cone. S_N must be smaller than a semicircle, so we must have $\mu < \pi/N$. Therefore,

$$\alpha < 2\sin^{-1}\left(\frac{1}{2N}\right). \tag{1.65}$$

(d) Roll the cone N times onto a plane. From the reasoning in part (b), we must have $N\beta = 60°$. Therefore,

$$\alpha = 2\sin^{-1}\left(\frac{1}{6N}\right). \tag{1.66}$$

11. **Equality of torques**

The pattern is clear, so let's prove it by induction. Assume that we have shown that a force F applied at a distance d is equivalent to a force kF applied at a distance d/k, for all integers k up to $n-1$. We now want to show that the statement holds for $k = n$.

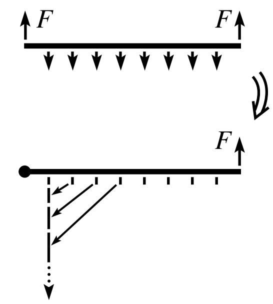


Figure 1.40

Consider the situation in Fig. 1.40. Forces F are applied at the ends of a stick, and forces $2F/(n-1)$ are applied at the $j\ell/n$ marks (for $1 \le j \le n-1$). The stick will not rotate (by symmetry), and it will not translate (because the net force is zero). Consider the stick to have a pivot at the left end. Replacing the interior forces by their 'equivalent' ones at the ℓ/n mark (see Fig. 1.40) gives a total force there equal to

$$\frac{2F}{n-1}\Big(1+2+3+\cdots+(n-1)\Big) = \frac{2F}{n-1}\left(\frac{n(n-1)}{2}\right) = nF. \tag{1.67}$$

We therefore see that a force F applied at a distance ℓ is equivalent to a force nF applied at a distance ℓ/n, as was to be shown.

It is now clear that the Claim 1.1 holds. To be explicit, consider a tiny distance ϵ (small compared to a). Then a force F_3 at a distance a is equivalent to a force $F_3(a/\epsilon)$ at a distance ϵ. (Actually, our reasoning above only works if a/ϵ is an integer, but since a/ϵ is very large, we can just pick the closest integer to it, and there will be a negligible error.) But a force $F_3(a/\epsilon)$ at a distance ϵ is equivalent to a force $F_3(a/\epsilon)(\epsilon/(a+b)) = F_3a/(a+b)$ at a distance $(a+b)$. Since this 'equivalent' force at the distance $(a+b)$ cancels the force F_2 there (since the stick is motionless), we have $F_3a/(a+b) = F_2$, which proves the claim.

12. **Find the force**

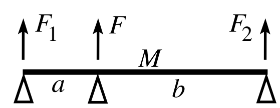


Figure 1.41

In Fig. 1.41, let the supports at the ends exert forces F_1 and F_2, and let the support in the interior exert a force F. Then

$$F_1 + F_2 + F = Mg. \tag{1.68}$$

Balancing torques around the left and right ends gives, respectively,

$$\begin{aligned} Fa + F_2(a+b) &= Mg\frac{a+b}{2}, \\ Fb + F_1(a+b) &= Mg\frac{a+b}{2}. \end{aligned} \tag{1.69}$$

We have used the fact that the mass of the stick can be treated as a point mass at its center. Note that the equation for balancing the torques around the center-of-mass is redundant; it is obtained by taking the difference between the two previous equations. (Balancing torques around the middle pivot also takes the form of a linear combination of these equations.)

It appears as though we have three equations and three unknowns, but we really have only two equations, because the sum of eqs. (1.69) gives eq. (1.68). So, since we have two equations and three unknowns, the system is underdetermined. Solving eqs. (1.69) for F_1 and F_2 in terms of F, we see that any forces of the form

$$(F_1, F, F_2) = \left(\frac{Mg}{2} - \frac{Fb}{a+b}, F, \frac{Mg}{2} - \frac{Fa}{a+b}\right) \tag{1.70}$$

are possible. In retrospect, it is obvious that the forces are not determined. By changing the height of the new support an infinitesimal distance, one can make F be anything from 0 up to $Mg(a+b)/2b$, which is when the stick comes off the left support (assuming $b \geq a$).

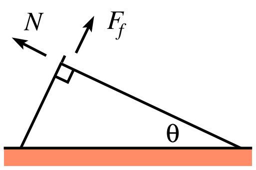


Figure 1.42

13. **Leaning sticks**

Let M_l be the mass of the left stick, and let M_r be the mass of the right stick. Then $M_l/M_r = \tan\theta$ (see Fig. 1.42). Let N be the normal force between the sticks, and let F_f be the friction force between the sticks. (So F_f has a maximum value of μN.) Balancing the torques on the left stick (around the contact point with the ground) gives

$$N = \frac{M_l g}{2}\sin\theta. \tag{1.71}$$

Balancing the torques on the right stick (around the contact point with the ground) gives

$$F_f = \frac{M_r g}{2}\cos\theta. \tag{1.72}$$

The condition $F_f \leq \mu N$ becomes

$$M_r\cos\theta \leq \mu M_l \sin\theta. \tag{1.73}$$

Using $M_l/M_r = \tan\theta$, this becomes

$$\tan^2\theta \geq \frac{1}{\mu}. \tag{1.74}$$

This answer checks in the two extremes: In the limit $\mu = 0$, we see that θ must be very close to $\pi/2$, which makes sense. In the limit $\mu = \infty$ (that is, we have sticky sticks), we see that θ can be anything above a very small lower bound, which also makes sense.

14. **Supporting a ladder**

Let F be the desired force. Note that F must be directed along the person (that is, perpendicular to the ladder), because otherwise there would be a net torque on the person relative to his pivot. This would result in an infinite acceleration of the (massless) person. (If the person has mass m, you can easily show that he must apply an additional force of $mg\sin\theta/2$ down along the ladder.)

Look at torques around the pivot point of the ladder. The gravitational force on the ladder provides a torque of $Mg(L/2)\cos\theta$ (tending to turn it clockwise). The force F provides a torque of $F(\ell/\tan\theta)$ (tending to turn it counterclockwise). Equating these torques gives

$$F = \frac{MgL}{2\ell}\sin\theta. \tag{1.75}$$

REMARKS: This F goes to zero as $\theta \to 0$, as it should.[5]

[5]For $\theta \to 0$, we need to lengthen the ladder with a massless extension, because the person will have to be very far to the right if the sticks are to be perpendicular.

F grows to the constant $MgL/2\ell$, as θ increases to $\pi/2$ (which isn't entirely obvious). So if you ever find yourself lifting up a ladder in the (strange) manner where you keep yourself perpendicular to it, you will find that you must apply a larger force, the higher the ladder goes. (However, in the special case where the ladder is exactly vertical, no force is required. You can see that our above calculations are not valid in this case, because we made a division by $\cos\theta$, which is zero when $\theta = \pi/2$.)

The normal force at the pivot of the person (that is, the vertical component of F, if the person is massless) is equal to $MgL\sin\theta\cos\theta/2\ell$. This has a maximum value of $MgL/4\ell$ at $\theta = \pi/4$. ♣

15. **Stick on a circle**

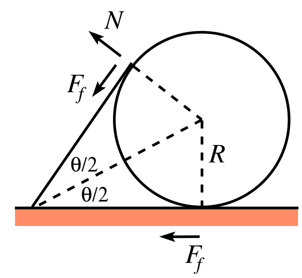


Figure 1.43

Let N be the normal force between the stick and the circle, and let F_f be the friction force between the ground and the circle (see Fig. 1.43). Then we immediately see that the friction force between the stick and the circle is also F_f (since the torques from the two friction forces on the circle must cancel).

Looking at torques on the stick, around the point of contact with the ground, we have $Mg\cos\theta(L/2) = NL$ (since the mass of the stick is effectively all located at its center, as far as torques are concerned), where M is the mass of the stick and L is its length. So $N = (Mg/2)\cos\theta$. Balancing the horizontal forces on the circle gives $N\sin\theta = F_f + F_f\cos\theta$. So we have

$$F_f = \frac{N\sin\theta}{1+\cos\theta} = \frac{Mg}{2}\frac{\sin\theta\cos\theta}{1+\cos\theta}. \quad (1.76)$$

But $M = \rho L$, and from the figure we have $L = R/\tan(\theta/2)$. Using the identity $\tan(\theta/2) = \sin\theta/(1+\cos\theta)$, we finally obtain

$$F_f = \frac{1}{2}\rho g R\cos\theta. \quad (1.77)$$

In the limit $\theta \to \pi/2$, F_f approaches 0, which makes sense. In the limit $\theta \to 0$ (i.e., a very long stick), the friction force approaches the constant $\rho g R/2$, which isn't so obvious.

16. **Leaning sticks and circles**

Let s_i be the ith stick, and let c_i be the ith circle.

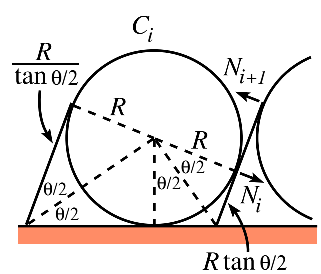


Figure 1.44

The normal forces c_i feels from s_i and from s_{i+1} are equal, because these two forces provide the only horizontal forces on the frictionless circle, so they must cancel. Let N_i be this normal force.

Look at the torques on s_{i+1} (around the hinge on the ground). The torques come from N_i, N_{i+1}, and the weight of s_i. From Fig. 1.44, we see that N_i acts at a point which is a distance $R\tan(\theta/2)$ away from the hinge. Since the stick has a length $R/\tan(\theta/2)$, this point is a fraction $\tan^2(\theta/2)$ up along the stick. Therefore, balancing the torques on s_{i+1} gives

$$\frac{1}{2}Mg\cos\theta + N_i\tan^2\frac{\theta}{2} = N_{i+1}. \quad (1.78)$$

N_0 is by definition 0, so we have $N_1 = (Mg/2)\cos\theta$ (as in the previous problem). Successively using eq. (1.78), we see that N_2 equals $(Mg/2)\cos\theta\big(1+\tan^2(\theta/2)\big)$, and N_3 equals $(Mg/2)\cos\theta\big(1+\tan^2(\theta/2)+\tan^4(\theta/2)\big)$, and so on. In general,

$$N_i = \frac{Mg}{2}\cos\theta\left(1+\tan^2\frac{\theta}{2}+\tan^4\frac{\theta}{2}+\cdots+\tan^{2(i-1)}\frac{\theta}{2}\right). \quad (1.79)$$

In the limit $i \to \infty$, we may write the infinite sum in closed form as

$$\lim_{i\to\infty} N_i \equiv N_\infty = \frac{Mg}{2}\frac{\cos\theta}{1-\tan^2(\theta/2)}. \tag{1.80}$$

(This is the solution to eq. (1.78), with $N_i = N_{i+1}$, so if a limit exists, it must be this.)
Using $M = \rho L = \rho R/\tan(\theta/2)$, we may write N_∞ as

$$N_\infty = \frac{\rho R g}{2}\frac{1}{\tan(\theta/2)}\frac{\cos\theta}{1-\tan^2(\theta/2)}. \tag{1.81}$$

The identity $\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2)$ may be used to put this in the form

$$N_\infty = \frac{\rho R g}{2}\frac{\cos^3(\theta/2)}{\sin(\theta/2)}. \tag{1.82}$$

This blows up for $\theta \to 0$, which is obvious (N_∞ approaches half the weight of a stick in this limit). And it approaches the constant $\rho Rg/4$ for $\theta \to \pi/2$, which is not at all obvious.

Note that the horizontal force that must be applied to the last circle far to the right is $N_\infty \sin\theta = \rho Rg\cos^4(\theta/2)$. This ranges from ρRg at $\theta = 0$, to $\rho Rg/4$ at $\theta = \pi/2$.

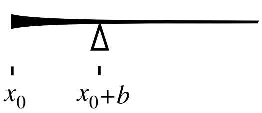


Figure 1.45

17. **Balancing the stick**

Let the stick go off to infinity in the positive x direction. Let it be cut at $x = x_0$, so the pivot point is at $x = x_0 + b$ (see Fig. 1.45). Let the density be $\rho(x)$. Then the condition that the torques around $x_0 + b$ cancel is

$$\int_{x_0}^{x_0+b} \rho(x)\big((x_0+b)-x\big)dx = \int_{x_0+b}^{\infty} \rho(x)\big(x-(x_0+b)\big)dx. \tag{1.83}$$

Combining the two integrals gives

$$I \equiv \int_{x_0}^{\infty} \rho(x)\big((x_0+b)-x\big)dx = 0. \tag{1.84}$$

We want this to equal 0 for all x_0, so the derivative of I with respect to x_0 must be 0. I depends on x_0 through both the limits of integration and the integrand. In taking the derivative, the former dependence requires finding the value of the integrand at the limits, while the latter dependence requires taking the derivative of the integrand w.r.t x_0, and then integrating. We obtain (using the fact that there is zero contribution from the ∞ limit)

$$0 = \frac{dI}{dx_0} = -b\rho(x_0) + \int_{x_0}^{\infty} \rho(x)dx. \tag{1.85}$$

Taking the derivative of this equation with respect to x_0 gives

$$b\rho'(x_0) = -\rho(x_0). \tag{1.86}$$

The solution to this is (rewriting the arbitrary x_0 as x)

$$\rho(x) = Ae^{-x/b}. \tag{1.87}$$

This falls off quickly if b is very small, which makes sense. And it falls off slowly if b is very large. Note that the density at the pivot is $1/e$ times the density at the end. And $1 - 1/e \approx 63\%$ of the mass is contained between the end and the pivot.

18. **The spool**

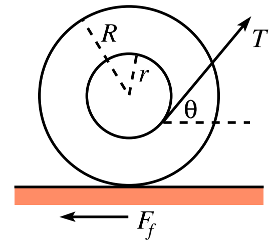


Figure 1.46

(a) Let F_f be the friction force the ground provides. Balancing horizontal forces gives (from Fig. 1.46)

$$T\cos\theta = F_f. \tag{1.88}$$

Balancing torques around the center of the circles gives

$$Tr = F_f R. \tag{1.89}$$

These two equations imply

$$\cos\theta = \frac{r}{R}. \tag{1.90}$$

(b) The normal force from the ground is

$$N = Mg - T\sin\theta. \tag{1.91}$$

The friction force, eq. (1.88), is $F_f = T\cos\theta$. So the statement $F_f \le \mu N$ becomes $T\cos\theta \le \mu(Mg - T\sin\theta)$. Therefore,

$$T \le \frac{\mu Mg}{\cos\theta + \mu\sin\theta}, \tag{1.92}$$

where θ is given by eq. (1.90).

(c) The maximum value of T is given in (1.92). This depends on θ, which in turn depends on r. We want to find the r which minimizes this maximum T.

The θ which maximizes the denominator in eq. (1.92) is easily found to be given by $\tan\theta = \mu$. The value of T for this θ is

$$T = \frac{\mu Mg}{\sqrt{1+\mu^2}} = Mg\sin\theta. \tag{1.93}$$

To find the corresponding r, we can use eq. (1.90) to write $\tan\theta = \sqrt{R^2 - r^2}/r$. The equality $\tan\theta = \mu$ then yields

$$r = \frac{R}{\sqrt{1+\mu^2}}. \tag{1.94}$$

This is the r which yields the smallest upper bound on T.

Remarks: In the limit $\mu = 0$, we have $\theta = 0$, $T = 0$, and $r = R$. In the limit $\mu = \infty$, we have $\theta = \pi/2$, $T = Mg$, and $r = 0$.

We can also ask the question: What should r be so that the upper bound on T found in part (b) is as large as possible? We then want to make the denominator in eq. (1.92) as small as possible. If $\mu < 1$, this is achieved at $\theta = \pi/2$ (with $r = 0$ and $T = Mg$). If $\mu > 1$, this is achieved at $\theta = 0$ (with $r = R$ and $T = \mu Mg$). These answers make sense. ♣

Chapter 2

Using $F = ma$

The general goal of classical mechanics is to determine what happens to a given set of objects in a given physical situation. In order to figure this out, we need to know what makes objects move the way they do. There are two main ways of going about this task. One way (the more advanced one) involves the *Lagrangian* method, which we will discuss in Chapter 5. The other (which you are undoubtedly familiar with) involves Newton's laws, which will be the subject of the present chapter.

It should be noted that each of these two methods is perfectly sufficient for solving any problem. They both produce the same information in the end (that is, the same equations), but they are based on vastly different principles. We'll talk more about this is Chapter 5.

2.1 Newton's Laws

Newton published his three laws in 1687 in his *Principia Mathematica*. The laws are quite intuitive (although it seems a bit strange to attach the adjective "intuitive" to a set of statements that took millennia for humans to write down). They may be formulated as follows (although there are other possible variations).

- **First Law:** A body moves with constant velocity (which may be zero) unless acted on by a force.
- **Second Law:** The time rate of change of the momentum of a body equals the force acting on the body.
- **Third Law:** The forces two bodies apply to each other are equal in magnitude and opposite in direction.

We could discuss for days on end the degree to which these statements are physical laws, and the degree to which they are definitions. Sir Arthur Eddington once made the unflattering comment that the first law essentially says that "every particle continues in its state of rest or uniform motion in a straight line except insofar that it doesn't." Although Newton's laws may seem somewhat vacuous at first glance, there is actually a bit more content to them than Eddington's statement implies. Let's look at each in turn. The discussion will be brief, because we have to save time for other things in this book that we really *do* want to discuss for days on end.

First Law

One thing this law does is give a definition of zero force.

Another thing it does is give a definition of an *inertial frame* (which is defined simply as a reference frame in which the first law holds). Since the term 'velocity' is used, we have to state what frame of reference we are measuring the velocity with respect to. The first law does *not* hold in an arbitrary frame. For example, it fails in the frame of a spinning turntable.[1] Intuitively, an inertial frame is one that moves at constant speed. But this is ambiguous, because you have to say what the frame is moving at constant speed *with respect to*. At any rate, an inertial frame is simply defined as the special type of frame where the first law holds.

So, what we have now are two intertwined definitions of "force" and "inertial frame". Not much physical content here. But, however sparse in content the law is, it still holds for *all* particles. So if we have a frame where one free particle moves with constant velocity, then if we replace it with another particle, it will likewise move with constant velocity. This is a statement with content.

Second Law

One thing this law does is give a definition of non-zero force. Momentum is defined[2] to be $m\mathbf{v}$. If m is constant, then this law says that

$$\mathbf{F} = m\mathbf{a}, \tag{2.1}$$

where $\mathbf{a} \equiv d\mathbf{v}/dt$. This law holds only in an inertial frame (which was defined by the first law).

> For things moving free or at rest,
> Observe what the first law does best.
> It defines a key frame,
> "Inertial" by name,
> Where the second law then is expressed.

So far, the second law merely gives a definition of $\mathbf{F}$. But the meaningful statement arises when we invoke the fact that the law holds for *all* particles. If the same force (for example, the same spring stretched by the same amount) acts on two particles, with masses m_1 and m_2, then their accelerations must be related by

$$\frac{a_1}{a_2} = \frac{m_2}{m_1}. \tag{2.2}$$

[1] It is, however, possible to fudge things so that Newton's laws hold in such a frame, but we'll save this discussion for Chapter 9.

[2] We're doing everything nonrelativistically here, of course. Chapter 11 gives the relativistic modification of the $m\mathbf{v}$ expression.

This relation holds regardless of what the common force is. Therefore, once you've used one force to find the relative masses of two objects, then you know what the ratio of their a's will be when they are subjected to any other force.

Of course, we haven't really defined *mass* yet. But eq. (2.2) gives an experimental method for determining an object's mass in terms of a standard (say, 1 kg) mass. All you have to do is compare its acceleration with that of the standard mass (when acted on by the same force).

There is also another piece of substance in this law, in that it says $\mathbf{F} = m\mathbf{a}$, instead of, say, $\mathbf{F} = m\mathbf{v}$ or $\mathbf{F} = m\, d^3\mathbf{x}/dt^3$. This issue is related to the first law. $\mathbf{F} = m\mathbf{v}$ is certainly not viable, because the first law says that it is possible to have a velocity without a force. And $\mathbf{F} = md^3\mathbf{x}/dt^3$ would make the first law incorrect, because it would then be true that a particle moves with constant acceleration (instead of constant velocity) unless acted on by a force.

Note that $\mathbf{F} = m\mathbf{a}$ is a vector equation, so it is really three equations in one. In cartesian coordinates, it says that $F_x = ma_x$, $F_y = ma_y$, and $F_z = ma_z$.

Third Law

This law essentially postulates that momentum is conserved, (that is, not dependent on time). To see this, note that

$$\begin{aligned} \frac{d\mathbf{p}}{dt} &= \frac{d(m_1\mathbf{v}_1 + m_2\mathbf{v}_2)}{dt} \\ &= m_1\mathbf{a}_1 + m_2\mathbf{a}_2 \\ &= \mathbf{F}_1 + \mathbf{F}_2, \end{aligned} \tag{2.3}$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are the forces acting on m_1 and m_2, respectively. Therefore, momentum conservation (that is, $d\mathbf{p}/dt = 0$) is equivalent to Newton's third law (that is, $\mathbf{F}_1 = -\mathbf{F}_2$.)

There isn't much left to be defined in this law, so the statement of the third law is one of pure content. It says that if you have two isolated particles, then their accelerations are opposite in direction and inversely proportional to their masses.

This third law cannot be a definition, because it's actually not always valid. It only holds for forces of the "pushing" and "pulling" type. It fails for the magnetic force, for example.

2.2 Free-body diagrams

The law that allows us to be quantitative is the second law. Given a force, we can apply $\mathbf{F} = m\mathbf{a}$ to find the acceleration. And knowing the acceleration, we should be able to determine the behavior of a given object (that is, where it is and how fast it is moving). This process sometimes takes a bit of work, but there are two basic types of situations that commonly arise.

- In many problems, all you are given is a physical situation (for example, a block resting on a plane, strings connecting masses, etc.), and it is up to you to find all the forces acting on all the objects. These forces generally point in various directions, so it is easy to lose track of them. It therefore proves useful to isolate the objects and draw all the forces acting on each of them. This is the subject of the present section.

- In other problems, you are *given* the force, $F(x)$, as a function of position (we'll just work in one dimension here), and the task immediately becomes the mathematical one of solving the $F(x) = ma \equiv m\ddot{x}$ equation. These *differential equations* can be difficult (or impossible) to solve exactly. They are the subject of Section 2.3.

Let's now consider the first of these two types of problems, where we are presented with a physical situation, and where we must determine all the forces involved. The term *free-body diagram* is used to denote a diagram with all the forces drawn on all the objects. After drawing such a diagram, we simply write down all the $F = ma$ equations it implies. The result will be a system of linear equations in various unknown forces and accelerations, for which we must then solve. This procedure is best understood through an example.

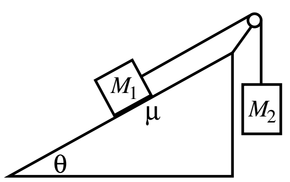


Figure 2.1

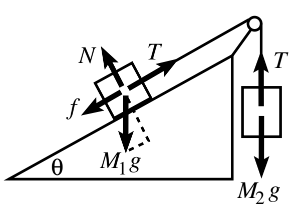


Figure 2.2

Example (A plane and masses): Mass M_1 is held on a plane with inclination angle θ, and mass M_2 hangs over the side. The two masses are connected by a massless string which runs over a massless pulley (see Fig. 2.1). The coefficient of friction (assume the kinetic and static coefficients are equal) between M_1 and the plane is μ. Mass M_1 is released. Assuming that M_2 is sufficiently large so that M_1 gets pulled up the plane, what is the acceleration of the system? What is the tension in the string?

Solution: The first thing to do is draw all the forces on the two masses. These are shown in Fig. 2.2. The forces on M_2 are gravity and the tension. The forces on M_1 are gravity, friction, the tension, and the normal force. Note that the friction force points down the plane, since we are assuming that M_1 moves up the plane.

We now simply have to write down all the $F = ma$ equations. When dealing with M_1, we could break things up into horizontal and vertical components, but it is much cleaner to use the components tangential and normal to the plane. These two components of $\mathbf{F} = m\mathbf{a}$, along with the vertical $F = ma$ for M_2, give

$$\begin{aligned} T - f - M_1 g \sin\theta &= M_1 a, \\ N - M_1 g \cos\theta &= 0, \\ M_2 g - T &= M_2 a, \end{aligned} \qquad (2.4)$$

where we have used the fact that the two masses accelerate at the same rate (and we have defined the positive direction for M_2 to be downward). We have also used the fact that tension is the same at both ends of the string, because otherwise there would be a net force on some part of the string which would then have to undergo infinite acceleration, since it is massless.

There are four unknowns: T, a, N, and f. Fortunately, we have a fourth equation, namely $f = \mu N$. Therefore, the second equation above gives $f = \mu M_1 g \cos\theta$. The first equation then becomes $T - \mu M_1 g \cos\theta - M_1 g \sin\theta = M_1 a$. This may be combined with the third equation to give

$$a = \frac{g(M_2 - \mu M_1 \cos\theta - M_1 \sin\theta)}{M_1 + M_2}, \qquad \text{and} \qquad T = \frac{M_1 M_2 g}{M_1 + M_2}(1 + \mu\cos\theta + \sin\theta). \tag{2.5}$$

Note that we must have $M_2 > M_1(\mu\cos\theta + \sin\theta)$ in order for M_1 to move upward. This is clear from looking at the forces tangential to the plane.

REMARK: If we had instead assumed that M_1 was sufficiently large so that it slides down the plane, then the friction force would point up the plane, and we would have found

$$a = \frac{g(M_2 + \mu M_1 \cos\theta - M_1 \sin\theta)}{M_1 + M_2}, \qquad \text{and} \qquad T = \frac{M_1 M_2 g}{M_1 + M_2}(1 - \mu\cos\theta + \sin\theta). \tag{2.6}$$

In order for M_1 to move downward (i.e., $a < 0$), we must have $M_2 < M_1(\sin\theta - \mu\cos\theta)$. Therefore, $M_1(\sin\theta - \mu\cos\theta) < M_2 < M_1(\sin\theta + \mu\cos\theta)$ is the range of M_2 for which the system doesn't move. ♣

In problems like the one above, it is clear what things you should pick as the objects on which you're going to draw forces. But in other problems, where there are various different subsystems you can choose, you must be careful to include all the relevant forces on a given subsystem. Which subsystems you want to pick depends on what quantities you're trying to find. Consider the following example.

Example (Platform and pulley): A person stands on a platform-and-pulley system, as shown in Fig. 2.3. The masses of the platform, person, and pulley are M, m, and μ, respectively.[3] The rope is massless. Let the person pull up on the rope so that she has acceleration a upwards.

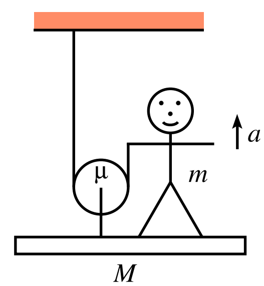


Figure 2.3

(a) What is the tension in the rope?

(b) What is the normal force between the person and the platform? What is the tension in the rod connecting the pulley to the platform?

Solution:

(a) To find the tension in the rope, we simply want to let our subsystem be the whole system. If we imagine putting the system in a black box (to emphasize the fact that we don't care about any internal forces within the system), then the forces we see "protruding" from the box are the three weights (Mg, mg, and μg) downward, and the tension T upward. Applying $F = ma$ to the whole system gives

$$T - (M+m+\mu)g = (M+m+\mu)a \qquad \Longrightarrow \qquad T = (M+m+\mu)(g+a). \tag{2.7}$$

(b) To find the normal force, N, between the person and the platform, and also the tension, f, in the rod connecting the pulley to the platform, it is not sufficient to consider the system as a whole. We must consider subsystems.

[3]Assume that the pulley's mass is concentrated at its center, so we don't have to worry about any rotational dynamics (the subject of Chapter 7).

- Let's apply $F = ma$ to the person. The forces acting on the person are gravity, the normal force from the platform, and the tension from the rope (pulling downward at her hand). Therefore, we have

$$N - T - mg = ma. \tag{2.8}$$

- Now apply $F = ma$ to the platform. The forces acting on the platform are gravity, the normal force from the person, and the force upwards from the rod. Therefore, we have

$$f - N - Mg = Ma. \tag{2.9}$$

- Now apply $F = ma$ to the pulley. The forces acting on the pulley are gravity, the force downward from the rod, and *twice* the tension in the rope (since it pulls up on both sides). Therefore, we have

$$2T - f - \mu g = \mu a. \tag{2.10}$$

Note that if we add up the three previous equations, we obtain the $F = ma$ equation in eq. (2.7), as should be the case, since the whole system is the sum of the three previous subsystems. Eqs. (2.8) – (2.10) are three equations in the three unknowns (T, N, and f). Their sum yields the T in (2.7), and then eqs. (2.8) and (2.10) give, respectively (as you can show),

$$N = (M + 2m + \mu)(g + a), \qquad \text{and} \qquad f = (2M + 2m + \mu)(g + a). \tag{2.11}$$

Remark: Of course, you could also obtain these results by considering subsystems different from the ones we chose above (for example, you might choose the pulley-plus-platform, etc.). But no matter how you choose to break up the system, you will need to produce three independent $F = ma$ statements in order to solve for the three unknowns (T, N, and f).

In problems like this one, it is easy to make a mistake by forgetting to include one of the forces, such as the second T in eq. (2.10). The safest thing to do, therefore, is to isolate each subsystem, draw a box around it, and then write down all the forces that "protrude" from the box. Fig. 2.4 shows the free-body diagram for the subsystem of the pulley. ♣

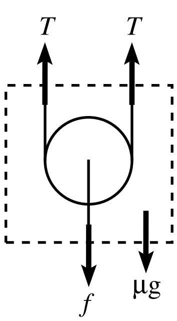


Figure 2.4

2.3 Solving differential equations

Let's now consider the type of problem where we are *given* the force $F(x)$ as a function of position (we'll just work in one dimension here), and where our task is to solve the $F(x) = ma \equiv m\ddot{x}$ differential equation, to find the position, $x(t)$, as a function of time. In the present section, we will develop a few techniques for solving differential equations. The ability to apply these techniques dramatically increases the number of problems we can solve.

In general, the force F is a function of the position x, the speed $\dot{x}$, and the time t. (Of course, it could be a function of d^2x/dt^2, d^3x/dt^3, etc., but these cases don't arise much, so we won't worry about them.) We therefore want to solve the differential equation,

$$m\ddot{x} = F(x, \dot{x}, t). \tag{2.12}$$

In general, this equation cannot be solved exactly for $x(t)$.[4] But for most of the problems we will deal with, it can be solved. The problems we'll encounter will often fall into one of three special cases, namely, where F is a function of t only, or x only, or $v \equiv \dot{x}$ only. In all of these cases, we must invoke the given initial conditions, $x_0 \equiv x(t_0)$ and $v_0 \equiv v(t_0)$, to obtain our final solutions. These initial conditions will appear in the limits of the integrals in the following discussion.[5]

Note: You may want to just skim the following page and a half, and then refer back to it, as needed. Don't try to memorize all the different steps. We present them only for completeness. The whole point here can basically be summarized by saying that sometimes you want to write $\ddot{x}$ as dv/dt, and sometimes you want to write it as $v\,dv/dx$ (see eq. (2.16)). Then you "simply" separate variables and integrate. We'll go through the three special cases, and then we'll do some examples.

- *F is a function of t only: $F = F(t)$.*

  Since $a = d^2x/dt^2$, we simply have to integrate $F = ma$ twice to obtain $x(t)$. Let's do this in a very systematic way, just to get used to the general procedure. First, write $F = ma$ as

$$m\frac{dv}{dt} = F(t). \tag{2.13}$$

  Then separate variables and integrate both sides to obtain[6]

$$m\int_{v_0}^{v(t)} dv' = \int_{t_0}^{t} F(t')dt'. \tag{2.14}$$

  (Primes have been put on the integration variables so that we don't confuse them with the limits of integration.) This yields v as a function of t, $v(t)$. Then separate variables in $dx/dt = v(t)$ and integrate to obtain

$$\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')dt'. \tag{2.15}$$

  This yields x as a function of t, $x(t)$. This procedure may seem like a cumbersome way to simply integrate something twice. That's because it is. But the technique proves more useful in the following case.

[4]You can always solve for $x(t)$ *numerically*, to any desired accuracy. This is discussed in Appendix D.

[5]It is no coincidence that we need *two* initial conditions to completely specify the solution to our *second*-order $F = m\ddot{x}$ differential equation. It is a general result (which we'll just accept here) that the solution to an nth-order differential equation has n free parameters, which must then be determined from the initial conditions.

[6]If you haven't seen such a thing before, the act of multiplying both sides by the infinitesimal quantity dt' might make you feel a bit uneasy. But it is in fact quite legal. If you wish, you can imagine working with the small (but not infinitesimal) quantities Δv and Δt, for which it is certainly legal to multiply both sides through by Δt. Then you can take a discrete sum over many Δt intervals, and then finally take the limit $\Delta t \to 0$, which results in eq. (2.14)

- *F is a function of x only: $F = F(x)$.*

  We will use
  $$a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx} \tag{2.16}$$
  to write $F = ma$ as
  $$mv\frac{dv}{dx} = F(x). \tag{2.17}$$
  Now separate variables and integrate both sides to obtain
  $$m\int_{v_0}^{v(x)} v'\,dv' = \int_{x_0}^{x} F(x')dx'. \tag{2.18}$$
  The left side will contain the square of $v(x)$. Taking a square root, this gives v as a function of x, $v(x)$. Separate variables in $dx/dt = v(x)$ to obtain
  $$\int_{x_0}^{x(t)} \frac{dx'}{v(x')} = \int_{t_0}^{t} dt'. \tag{2.19}$$
  This gives t as a function of x, and hence (in principle) x as a function of t, $x(t)$. The unfortunate thing about this case is that the integral in eq. (2.19) might not be doable. And even if it is, it might not be possible to invert $t(x)$ to produce $x(t)$.

- *F is a function of v only: $F = F(v)$.*

  Write $F = ma$ as
  $$m\frac{dv}{dt} = F(v). \tag{2.20}$$
  Separate variables and integrate both sides to obtain
  $$m\int_{v_0}^{v(t)} \frac{dv'}{F(v')} = \int_{t_0}^{t} dt'. \tag{2.21}$$
  This yields t as a function of v, and hence (in principle) v as a function of t, $v(t)$. Integrate $dx/dt = v(t)$ to obtain $x(t)$ from
  $$\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')dt'. \tag{2.22}$$
  Note: If you want to find v as a function of x, $v(x)$, you should write a as $v(dv/dx)$ and integrate
  $$m\int_{v_0}^{v(x)} \frac{v'\,dv'}{F(v')} = \int_{x_0}^{x} dx'. \tag{2.23}$$
  You may then obtain $x(t)$ from eq. (2.19), if desired.

When dealing with the initial conditions, we have chosen to put them in the limits of integration above. If you wish, you can perform the integrals without any limits, and just tack on a constant of integration to your result. The constant is then determined from the initial conditions.

Again, as mentioned above, you do *not* have to memorize the above three procedures, because there are variations, depending on what you want to solve for. All you have to remember is that $\ddot{x}$ can be written as either dv/dt or $v\,dv/dx$. One of these will get the job done (the one that makes only two out of the three variables, t,x,v, appear in your differential equation). And then be prepared to separate variables and integrate as many times as needed.

a is dv by dt.
Is this useful? There's no guarantee.
If it leads to "Oh, heck!"'s,
Take dv by dx,
And then write down its product with v.

Example 1 (Gravitational force): A particle of mass m is subject to a constant force $F = -mg$. The particle starts at rest at height h. Because this constant force falls into all of the above three categories, we should be able to solve for $y(t)$ in two ways.

(a) Find $y(t)$ by writing a as dv/dt.

(b) Find $y(t)$ by writing a as $v\,dv/dy$.

Solution:

(a) $F = ma$ gives $dv/dt = -g$. Integrating this yields $v = -gt + C$, where C is a constant of integration. The initial condition $v(0) = 0$ says that $C = 0$. Hence, $dy/dt = -gt$. Integrating this and using $y(0) = h$ gives

$$y = h - \frac{1}{2}gt^2. \tag{2.24}$$

(b) $F = ma$ gives $v\,dv/dy = -g$. Separating variables and integrating gives $v^2/2 = -gy + C$. The initial condition $v(0) = 0$ yields $v^2/2 = -gy + gh$. Therefore, $v \equiv dy/dt = -\sqrt{2g(h-y)}$ (we have chosen the negative square root, because the particle is falling). Separating variables gives

$$\int \frac{dy}{\sqrt{h-y}} = -\sqrt{2g}\int dt. \tag{2.25}$$

This yields $2\sqrt{h-y} = \sqrt{2g}\,t$, where we have used the initial condition $y(0) = h$. Hence, $y = h - gt^2/2$, in agreement with part (a). The solution in part (a) was clearly the simpler one.

Example 2 (Dropped ball): A beach-ball is dropped from rest at height h. Assume[7] that the drag force from the air is $F_d = -\beta v$. Find the velocity and height as a function of time.

Solution: For simplicity in future formulas, let's write the drag force as $F_d = -\beta v \equiv -m\alpha v$. Taking upward to be the positive y direction, the force on the ball is

$$F = -mg - m\alpha v. \tag{2.26}$$

Note that v is negative here, since the ball is falling, so the drag force points upward, as it should. Writing $F = m\,dv/dt$, and separating variables, gives

$$\int_0^{v(t)} \frac{dv'}{g + \alpha v'} = -\int_0^t dt'. \tag{2.27}$$

The integration yields $\ln(1 + \alpha v/g) = -\alpha t$. Exponentiation then gives

$$v(t) = -\frac{g}{\alpha}\left(1 - e^{-\alpha t}\right). \tag{2.28}$$

Integrating $dy/dt \equiv v(t)$ to obtain $y(t)$ yields

$$\int_h^{y(t)} dy' = -\frac{g}{\alpha}\int_0^t \left(1 - e^{-\alpha t'}\right) dt'. \tag{2.29}$$

Therefore,

$$y(t) = h - \frac{g}{\alpha}\left(t - \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\right). \tag{2.30}$$

Remarks:

(a) Let's look at some limiting cases. If t is very small (more precisely, if $\alpha t \ll 1$), then we can use $e^{-x} \approx 1 - x + x^2/2$ to make approximations to leading order in t. You can show that eq. (2.28) gives $v(t) \approx -gt$ (as it should, since the drag force is negligible at the start, so the ball is essentially in free fall). And eq. (2.30) gives $y(t) \approx h - gt^2/2$, as expected.

We may also look at large t. In this case, $e^{-\alpha t}$ is essentially 0, so eq. (2.28) gives $v(t) \approx -g/\alpha$. (This is the terminal velocity. Its value makes sense, because it is the velocity for which the force $-mg - m\alpha v$ vanishes.) And eq. (2.30) gives $y(t) \approx h - (g/\alpha)t + g/\alpha^2$. Interestingly, we see that for large t, g/α^2 is the distance our ball lags behind another ball which started out already at the terminal velocity, g/α.

(b) The speed of the ball obtained in eq. (2.28) depends on α, which was defined in the coefficient of the drag force, $F_d = -m\alpha v$. We explicitly wrote the m here just to make all of our formulas look a little nicer, but it should *not* be inferred that the speed of the ball is independent of m. The coefficient $m\alpha$ depends (in some complicated way) on the cross-sectional area, A, of the ball. Therefore, $\alpha \propto A/m$. Two balls of the same size, one made of lead and one made of styrofoam, will have the same A but different m's. Hence, their α's will be different, and they will fall at different rates.

For heavy objects in a thin medium such as air, α is small, and so the drag effects are not very noticeable over short distances. Heavy objects fall at roughly the same rate. If the air were a bit thicker, different objects would fall at noticeably different rates, and maybe it would have taken Galileo a bit longer to come to his conclusions.

[7] The drag force is roughly proportional to v as long as the speed is fairly slow (up to, say, 50 m/s, but this depends on various things). For larger speeds, the drag force is roughly proportional to v^2.

What would you have thought, Galileo,
If instead you dropped cows and did say, "Oh!
To lessen the sound
Of the moos from the ground,
They should fall not through air, but through mayo!" ♣

2.4 Projectile motion

Consider a ball thrown through the air (not necessarily vertically). We will neglect air resistance in the following discussion.

Let x and y be the horizontal and vertical positions, respectively. The force in the x-direction is $F_x = 0$, and the force in the y-direction is $F_y = -mg$. So $\mathbf{F} = m\mathbf{a}$ gives

$$\ddot{x} = 0, \qquad \text{and} \qquad \ddot{y} = -g. \tag{2.31}$$

Note that these two equations are "decoupled". That is, there is no mention of y in the equation for $\ddot{x}$, and vice-versa. The motions in the x- and y-directions are therefore completely independent.

REMARK: The classic demonstration of the independence of the x- and y-motions is the following. Fire a bullet horizontally (or, preferably, just imagine firing a bullet horizontally), and at the same time drop a bullet from the height of the gun. Which bullet will hit the ground first? (Neglect air resistance, and the curvature of the earth, etc.) The answer is that they will hit the ground at the same time, because the effect of gravity on the two y-motions is exactly the same, independent of what is going on in the x-direction. ♣

If the initial position and velocity are (X, Y) and (V_x, V_y), then we can easily integrate eqs. (2.31) to obtain

$$\begin{aligned} \dot{x}(t) &= V_x, \\ \dot{y}(t) &= V_y - gt. \end{aligned} \tag{2.32}$$

Integrating again gives

$$\begin{aligned} x(t) &= X + V_x t, \\ y(t) &= Y + V_y t - \frac{1}{2}gt^2. \end{aligned} \tag{2.33}$$

These equations for the speeds and positions are all you need to solve a projectile problem.

Example (Throwing a ball):

(a) For a given initial speed, at what inclination angle should a ball be thrown so that it travels the maximum distance along the ground? Assume that the ground is horizontal, and that the ball is released from ground level.

(b) What is the optimal angle if the ground is sloped upward at an angle β (or downward, if β is negative)?

Solution:

(a) Let the inclination angle be θ, and let the initial speed be V. Then the horizontal speed is (always) $V_x = V\cos\theta$, and the initial vertical speed is $V_y = V\sin\theta$.

Let d be the horizontal distance traveled, and let t be the time in the air. Then the vertical speed is zero at time $t/2$ (since the ball is moving horizontally at its highest point), so the second of eqs. (2.32) gives $V_y = g(t/2)$. Hence[8], $t = 2V_y/g$. The first of eqs. (2.33) gives $d = V_x t$. Using $t = 2V_y/g$ here gives

$$d = \frac{2V_xV_y}{g} = \frac{V^2}{g}(2\sin\theta\cos\theta) = \frac{V^2}{g}\sin 2\theta. \tag{2.34}$$

The $\sin 2\theta$ factor has a maximum at

$$\theta = \frac{\pi}{4}. \tag{2.35}$$

The maximum distance traveled along the ground is then $d = V^2/g$.

Remarks: For $\theta = \pi/4$, you can show that the maximum height achieved is $V^2/4g$. This may be compared to the maximum height of $V^2/2g$ (as you can show) if the ball is thrown straight up.

Note that any possible distance you might wish to find in this problem must be proportional to V^2/g, by dimensional analysis. The only question is what the numerical factor is. ♣

(b) If the ground is sloped at an angle β, then the equation for the line of the ground is

$$y = (\tan\beta)x. \tag{2.36}$$

The path of the ball is given in terms of t by

$$x = (V\cos\theta)t, \qquad \text{and} \qquad y = (V\sin\theta)t - \frac{1}{2}gt^2. \tag{2.37}$$

We may now solve for the t which makes $y = (\tan\beta)x$ (since this gives the place where the path of the ball intersects the line of the ground). Using eqs. (2.37), we find that $y = (\tan\beta)x$ when

$$t = \frac{2V}{g}(\sin\theta - \tan\beta\cos\theta). \tag{2.38}$$

(There is, of course, also the solution $t = 0$.) Plugging this into our expression for x in eq. (2.37) gives

$$x = \frac{2V^2}{g}(\sin\theta\cos\theta - \tan\beta\cos^2\theta). \tag{2.39}$$

We must now maximize this value for x (which is the same as maximizing the distance along the slope). Taking the derivative with respect to θ gives (with the help of the double-angle formulas, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$) $\tan\beta = -\cot 2\theta \equiv -\tan(\pi/2 - 2\theta)$. Therefore, $\beta = -(\pi/2 - 2\theta)$, so we have

$$\theta = \frac{1}{2}\left(\beta + \frac{\pi}{2}\right). \tag{2.40}$$

[8] Alternatively, the time of flight can be found from the second of eqs. (2.33), which says that the ball returns to height zero when $V_y t = gt^2/2$.

In other words, the throwing angle should bisect the angle between the ground and the vertical.

REMARKS: For $\beta = \pi/2$, we have $\theta = \pi/2$, as should be the case. For $\beta = 0$, we have $\theta = \pi/4$, as found in part (a). And for $\beta = -\pi/2$, we have $\theta = 0$, which makes sense. Substituting the value of θ from eq. (2.40) into eq. (2.39), you can show (after a bit of algebra) that the maximum distance traveled along the tilted ground is

$$d = \frac{x}{\cos\beta} = \frac{V^2/g}{1+\sin\beta}\,. \tag{2.41}$$

This checks in the various limits for β. ♣

2.5 Motion in a plane, polar coordinates

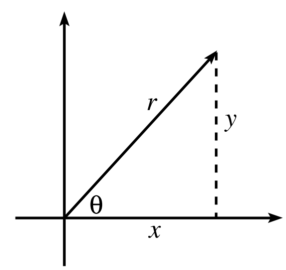


Figure 2.5

When dealing with problems where the motion lies in a plane, it is often convenient to work with polar coordinates, r and θ. These are related to the cartesian coordinates by (see Fig. 2.5)

$$x = r\cos\theta, \qquad \text{and} \qquad y = r\sin\theta. \tag{2.42}$$

Depending on the problem, either cartesian or polar coordinates will be easier to use. It is usually clear from the setup which is better. For the cases where you want to use polar coordinates, you need to know what Newton's second law looks like in terms of them. Therefore, the goal of the present section will be to determine what $\mathbf{F} = m\mathbf{a} \equiv m\ddot{\mathbf{r}}$ looks like when written in terms of polar coordinates.

At a given position $\mathbf{r}$ in the plane, the basis vectors in polar coordinates are $\hat{\mathbf{r}}$, which is a unit vector pointing in the radial direction; and $\hat{\boldsymbol{\theta}}$, which is a unit vector pointing in the counterclockwise tangential direction. In polar coords, a general vector may therefore be written as

$$\mathbf{r} = r\hat{\mathbf{r}}. \tag{2.43}$$

Note that the directions of the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ basis vectors depend, of course, on $\mathbf{r}$.

Since the goal of this section is to find $\ddot{\mathbf{r}}$, we must, in view of eq. (2.43), get a handle on the time derivative of $\hat{\mathbf{r}}$ (and we'll eventually need the derivative of $\hat{\boldsymbol{\theta}}$, too). In contrast with the (fixed) cartesian basis vectors ($\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$), the polar basis vectors ($\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$) change as a point moves around in the plane.

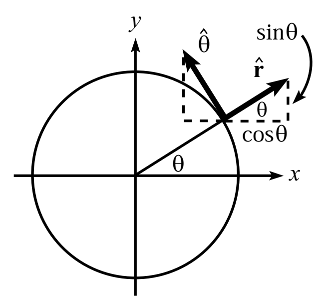


Figure 2.6

We may find $\dot{\hat{\mathbf{r}}}$ and $\dot{\hat{\boldsymbol{\theta}}}$ in the following way. In terms of the cartesian basis, Fig. 2.6 shows that

$$\begin{aligned} \hat{\mathbf{r}} &= \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}, \\ \hat{\boldsymbol{\theta}} &= -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}. \end{aligned} \tag{2.44}$$

Taking the time derivative of these equations gives

$$\begin{aligned} \dot{\hat{\mathbf{r}}} &= -\sin\theta\dot{\theta}\hat{\mathbf{x}} + \cos\theta\dot{\theta}\hat{\mathbf{y}}, \\ \dot{\hat{\boldsymbol{\theta}}} &= -\cos\theta\dot{\theta}\hat{\mathbf{x}} - \sin\theta\dot{\theta}\hat{\mathbf{y}}. \end{aligned} \tag{2.45}$$

Using eqs. (2.44), we then arrive at the nice clean expressions,

$$\dot{\hat{\mathbf{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}}, \qquad \text{and} \qquad \dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\mathbf{r}}. \tag{2.46}$$

These relations are fairly evident from viewing what happens to the basis vectors as $\mathbf{r}$ moves a tiny distance in the tangential direction. (Note that the basis vectors do not change as $\mathbf{r}$ moves in the radial direction.)

We may now start differentiating eq. (2.43). One derivative gives (yes, the product rule works fine here)

$$\begin{aligned} \dot{\mathbf{r}} &= \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}} \\ &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}. \end{aligned} \tag{2.47}$$

This is quite clear, because $\dot{r}$ is the speed in the radial direction, and $r\dot{\theta}$ is the speed in the tangential direction (which is often written as ωr, where $\omega \equiv \dot{\theta}$ is the angular speed, or 'angular frequency').[9]

Differentiating eq. (2.47) then gives

$$\begin{aligned} \ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{\mathbf{r}}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\dot{\hat{\boldsymbol{\theta}}} \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}(\dot{\theta}\hat{\boldsymbol{\theta}}) + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}(-\dot{\theta}\hat{\mathbf{r}}) \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}. \end{aligned} \tag{2.48}$$

Finally, equating $m\ddot{\mathbf{r}}$ with $\mathbf{F} \equiv F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}}$ gives the radial and tangential forces as

$$\begin{aligned} F_r &= m(\ddot{r} - r\dot{\theta}^2), \\ F_\theta &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}). \end{aligned} \tag{2.49}$$

Exercise 7 gives a slightly different derivation of these equations.

Let's look at each of the four terms on the right-hand sides of eqs. (2.49).

- The $m\ddot{r}$ term is quite intuitive. For radial motion, it simply states that $F = ma$ along the radial direction.

- The $mr\ddot{\theta}$ term is also quite intuitive. For circular motion, it states that $F = ma$ along the tangential direction.

- The $-mr\dot{\theta}^2$ term is also fairly clear. For circular motion, it says that the radial force is $-m(r\dot{\theta})^2/r = -mv^2/r$, which is the familiar term that causes the centripetal acceleration.

- The $2m\dot{r}\dot{\theta}$ term is not so obvious. It is called the *Coriolis* force. There are various ways to look at this term. One is that it exists in order to keep the angular momentum conserved. We'll have much more to say about this in Chapter 9.

---

[9] For $r\dot{\theta}$ to be the tangential speed, we must of course measure θ in radians and not degrees. Then $r\theta$ is by definition the distance along the circumference; so $r\dot{\theta}$ is the speed along the circumference.

Example (Circular pendulum): A mass hangs from a string of length ℓ. Conditions have been set up so that the mass swings around in a horizontal circle, with the string making an angle of θ with the vertical (see Fig. 2.7). What is the angular frequency, ω, of this motion?

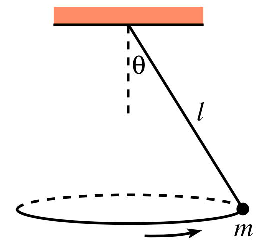


Figure 2.7

Solution: The mass travels in a circle, so the horizontal radial force must be $F_r = mr\dot{\theta}^2 \equiv mr\omega^2$ (with $r = \ell \sin\theta$), directed radially inward. The forces on the mass are the tension in the string, T, and gravity, mg (see Fig. 2.8). There is no acceleration in the vertical direction, so $F = ma$ in the vertical and radial directions gives, respectively,

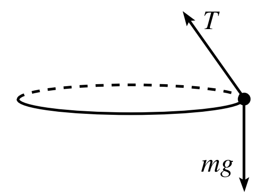


Figure 2.8

$$\begin{aligned} T\cos\theta &= mg, \\ T\sin\theta &= m(\ell\sin\theta)\omega^2. \end{aligned} \tag{2.50}$$

Solving for ω gives

$$\omega = \sqrt{\frac{g}{\ell\cos\theta}}. \tag{2.51}$$

Note that if $\theta \approx 0$, then $\omega \approx \sqrt{g/\ell}$, which equals the frequency of a plane pendulum of length ℓ. And if $\theta \approx 90^\circ$, then $\omega \to \infty$, as it should.

2.6 Exercises

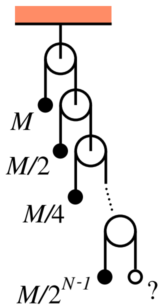


Figure 2.9

Section 2.2: Free-body diagrams

1. **A Peculiar Atwood's Machine**

   Consider an Atwood's machine (see Fig. 2.9) consisting of N masses, M, $M/2$, $M/4$, ..., $M/2^{N-1}$. (All the pulleys and strings are massless, as usual.)

   (a) Put a mass $M/2^{N-1}$ at the free end of the bottom string. What are the accelerations of all the masses?

   (b) Remove the mass $M/2^{N-1}$ (which was arbitrarily small, for very large N) that was attached in part (a). What are the accelerations of all the masses, now that you've removed this infinitesimal piece?

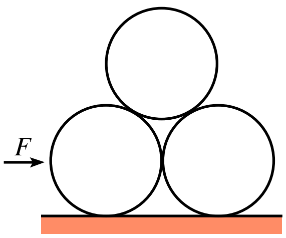


Figure 2.10

2. **Accelerated Cylinders** $**$

   Three identical cylinders are arranged in a triangle as shown in Fig. 2.10, with the bottom two lying on the ground. The ground and the cylinders are frictionless.

   You apply a constant horizontal force (directed to the right) on the left cylinder. Let a be the acceleration you give to the system. For what range of a will all three cylinders remain in contact with each other?

Section 2.3: Solving differential equations

3. $-bv^2$ **force** $*$

   A particle of mass m is subject to a force $F(v) = -bv^2$. The initial position is 0, and the initial speed is v_0. Find $x(t)$.

4. $-kx$ **force** $**$

   A particle of mass m is subject to a force $F(x) = -kx$. The initial position is 0, and the initial speed is v_0. Find $x(t)$.

5. kx **force** $**$

   A particle of mass m is subject to a force $F(x) = kx$. The initial position is 0, and the initial speed is v_0. Find $x(t)$.

Section 2.4: Projectile motion

6. **Newton's apple** $*$

   Newton is tired of apples falling on his head, so he decides to throw a rock at one of the larger and more formidable-looking apples positioned directly above his favorite sitting spot. Forgetting all about his work on gravitation (along with general common sense), he aims the rock directly at the apple (see Fig. 2.11). To his surprise, however, the apple falls from the tree just as he

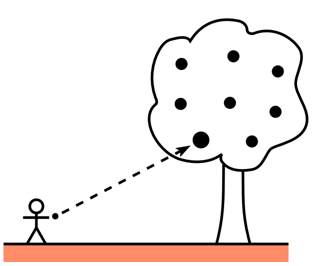


Figure 2.11

releases the rock. Show that the rock will hit the apple.[10]

Section 2.5: Motion in a plane, polar coordinates

7. **Derivation of F_r and F_θ** $**$

   In cartesian coords, a general vector takes the form

$$\begin{aligned} \mathbf{r} &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \\ &= r\cos\theta\hat{\mathbf{x}} + r\sin\theta\hat{\mathbf{y}}. \end{aligned} \tag{2.52}$$

   Derive eqs. (2.49) by taking two derivatives of this expression for $\mathbf{r}$, and then using eqs. (2.44) to show that the result may be written in the form of eq. (2.48). (Unlike $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, the vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ do not change with time.)

8. **A force $F_\theta = 2\dot{r}\dot{\theta}$** $**$

   Consider a particle that feels an angular force only, of the form $F_\theta = 2m\dot{r}\dot{\theta}$. (As in Problem 19, there's nothing all that physical about this force; it simply makes the $F = ma$ equations solvable.) Show that the trajectory takes the form of an exponential spiral, that is, $r = Ae^{\theta}$.

9. **A force $F_\theta = 3\dot{r}\dot{\theta}$** $**$

   Consider a particle that feels an angular force only, of the form $F_\theta = 3m\dot{r}\dot{\theta}$. (As in the previous problem, we're solving this problem simply because we can. And it's good practice.) Show that $\dot{r} = \sqrt{Ar^4 + B}$. Also, show that the particle reaches $r = \infty$ in a finite time.

[10] This problem suggests a way in which William Tell and his son might survive their ordeal if they were plopped down on a planet with an unknown gravitational constant (provided the son isn't too short or g isn't too big).

2.7 Problems

Section 2.2: Free-body diagrams

1. **Sliding down a plane** $**$

   (a) A block slides down a frictionless plane from the point $(0, y)$ to the point $(b, 0)$, where b is given. For what value of y does the journey take the shortest time? What is this time?

   (b) Answer the same questions in the case where there is a coefficient of kinetic friction, μ, between the block and the plane.

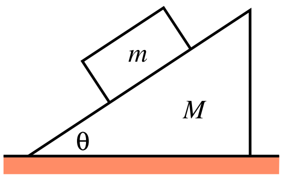


Figure 2.12

2. **Sliding plane** $***$

   A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ (see Fig. 2.12). The plane rests on a frictionless horizontal surface. The block is released. What is the horizontal acceleration of the plane?

3. **Sliding sideways on plane** $***$

   A block is placed on a plane inclined at an angle θ. The coefficient of friction between the block and the plane is $\mu = \tan\theta$. The block is given a kick so that it initially moves with speed V horizontally along the plane (that is, in the direction perpendicular to the direction pointing straight down the plane). What is the speed of the block after a very long time?

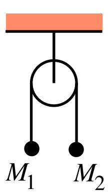


Figure 2.13

4. **Atwood's machine** $**$

   (a) A massless pulley hangs from a fixed support. A string connecting two masses, M_1 and M_2, hangs over the pulley (see Fig. 2.13). Find the accelerations of the masses.

   (b) Consider now the double-pulley system with masses M_1, M_2, and M_3 (see Fig. 2.14). Find the accelerations of the masses.

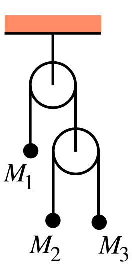


Figure 2.14

5. **Infinite Atwood's machine** $***$

   Consider the infinite Atwood's machine shown in Fig. 2.15. A string passes over each pulley, with one end attached to a mass and the other end attached to another pulley. All the masses are equal to M, and all the pulleys and strings are massless.

   The masses are held fixed and then simultaneously released. What is the acceleration of the top mass?

   (You may define this infinite system as follows. Consider it to be made of N pulleys, with a non-zero mass replacing what would have been the $(N + 1)$st pulley. Then take the limit as $N \to \infty$. It is not necessary, however, to use this exact definition.)

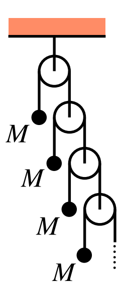


Figure 2.15

6. **Line of Pulleys** $*$

$N+2$ equal masses hang from a system of pulleys, as shown in Fig. 2.16. What is the acceleration of the masses at the end of the string?

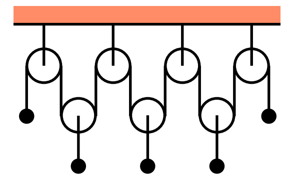


Figure 2.16

7. **Ring of Pulleys** $**$

Consider the system of pulleys shown in Fig. 2.17. The string (which is a loop with no ends) hangs over N fixed pulleys. N masses, m_1, m_2, $\ldots$, m_N, are attached to N pulleys which hang on the string. Find the acceleration of each mass.

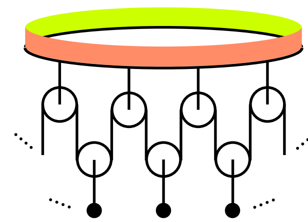


Figure 2.17

Section 2.3: Solving differential equations

8. **Exponential force**

A particle of mass m is subject to a force $F(t) = me^{-bt}$. The initial position and speed are 0. Find $x(t)$.

9. **Falling chain** $**$

   (a) A chain of length ℓ is held on a frictionless horizontal table, with a length y_0 hanging over the edge. The chain is released. As a function of time, find the length that hangs over the edge. (Don't bother with t after the chain loses contact with the table.) Also, find the speed of the chain right when it loses contact with the table.

   (b) Do the same problem, but now let there be a coefficient of friction μ between the chain and the table. (Assume that the chain initially hangs far enough over the edge so that it will indeed fall when released.)

10. **Ball thrown upward** $***$

A beach-ball is thrown upward with initial speed v_0. Assume that the drag force is $F = -m\alpha v$. What is the speed of the ball, v_f, when it hits the ground? (An implicit equation is sufficient.) Does the ball spend more time or less time in the air than it would if it were thrown in vacuum?

11. **Falling pencil** $**$

Consider a pencil, standing upright on its tip and then falling over. Since we haven't yet talked about torques and moments of inertia (the subject of Chapter 7), we will idealize the pencil as a mass m sitting at the end of a massless rod of length ℓ.[11]

   (a) Assume that the rod has initial (small) angle θ_0 and initial angular speed ω_0. The angle will eventually become large, but while it is small (so that $\sin\theta \approx \theta$), determine θ as a function of time.

[11]It actually involves only a trivial modification to do the problem correctly with torques. But the point-mass version will be quite sufficient for the present purposes.

(b) You might think that it would be possible (theoretically, at least) to make the pencil balance for an arbitrarily long time, by making the initial θ_0 and ω_0 sufficiently small.

It turns out that due to Heisenberg's uncertainty principle (which puts a constraint on how well we can know the position and momentum of a particle), it is impossible to balance a pencil all that long. The point is that you can't be sure that the pencil is initially both at the top *and* at rest. The goal of this problem is to be quantitative about this. (The time limit is sure to surprise you.)

Without getting into quantum mechanics, let's just say that the uncertainty principle says (up to factors of order 1) that $\Delta x \Delta p \geq \hbar$ (where $\hbar = 1.06 \cdot 10^{-34}$ Js is Planck's constant). We'll take this to mean that the initial conditions satisfy $(\ell\theta_0)(m\ell\omega_0) \geq \hbar$.

With this condition, find the maximum time it can take your solution in part (a) to become of order 1. In other words, determine (roughly) the maximum time the pencil can stay up. (Assume $m = 0.01$ kg, and $\ell = 0.1$ m.)

Section 2.4: Projectile motion

12. **Throwing a ball from a cliff** **

A ball is thrown from the edge of a cliff of height h. At what inclination angle should it be thrown so that it travels a maximum horizontal distance? Assume that the ground below the cliff is level.

13. **Redirected horizontal motion** *

A ball falls from height h. It bounces off a surface at height y (with no loss in speed). The surface is inclined at 45°, so that the ball bounces off horizontally. What should y be so that the ball travels a maximum horizontal distance?

14. **Redirected general motion** *

A ball falls from height h. It bounces off a surface at height y (with no loss in speed). The surface is inclined so that the ball bounces off at an angle of θ with respect to the horizontal. What should y and θ be so that the ball travels a maximum horizontal distance?

15. **Maximum length of trajectory** ***

A ball is thrown at speed v from zero height on level ground. Let θ_0 be the angle at which the ball should be thrown so that the distance traveled *through the air* is maximum. Show that θ_0 satisfies

$$1 = \sin\theta_0 \ln\left(\frac{1+\sin\theta_0}{\cos\theta_0}\right).$$

(The solution is found numerically to be $\theta_0 \approx 56.5°$.)

16. **Maximum area under trajectory** $*$

    A ball is thrown at speed v from zero height on level ground. At what angle should the ball be thrown so that the area under the trajectory is maximum?

17. **Bouncing ball** $*$

    A ball is thrown straight upward so that it reaches a height h. It falls down and bounces repeatedly. After each bounce, it returns to a certain fraction f of its previous height. Find the total distance traveled, and also the total time, before it comes to rest. What is its average speed?

Section 2.5: Motion in a plane, polar coordinates

18. **Free particle** $**$

    Consider a free particle in a plane. Using cartesian coordinates, it is trivial to show that the particle moves in a straight line. The task of this problem is to demonstrate this result in the much more cumbersome way, using eqs. (2.49). More precisely, show that $\cos\theta = r_0/r$ for a free particle, where r_0 is the radius at closest approach to the origin, and θ is measured with respect to this radius.

19. **A force $F_\theta = \dot{r}\dot{\theta}$** $**$

    Consider a particle that feels an angular force only, of the form $F_\theta = m\dot{r}\dot{\theta}$. (There's nothing all that physical about this force; it simply makes the $F = ma$ equations solvable.) Show that $\dot{r} = \sqrt{A \ln r + B}$, where A and B are constants of integration, determined by the initial conditions.

2.8 Solutions

1. **Sliding down a plane**

   (a) Let θ be the angle the plane makes with the horizontal. The component of gravity along the plane in $g \sin\theta$. The acceleration in the horizontal direction is then $a_x = g \sin\theta \cos\theta$. Since the horizontal distance is fixed, we simply want to maximize a_x. So $\theta = \pi/4$, and hence $y = b$.

   The time is obtained from $a_x t^2/2 = b$, with $a_x = g \sin\theta \cos\theta = g/2$. Therefore, $t = 2\sqrt{b/g}$.

   (b) The normal force on the plane is $Mg \cos\theta$, so the friction force is $\mu Mg \cos\theta$. The acceleration along the plane is therefore $g(\sin\theta - \mu \cos\theta)$, and so the acceleration in the horizontal direction is $a_x = g(\sin\theta - \mu\cos\theta)\cos\theta$. We want to maximize this. Setting the derivative equal to zero gives

$$(\cos^2\theta - \sin^2\theta) + 2\mu \sin\theta \cos\theta = 0, \qquad \Longrightarrow \qquad \tan 2\theta = \frac{-1}{\mu}. \tag{2.53}$$

   For $\mu \to 0$, this reduces to the answer in part (a). For $\mu \to \infty$, we obtain $\theta \approx \pi/2$, which makes sense.

   To find t, we need to find a_x. Using $\sin 2\theta = 1/\sqrt{1+\mu^2}$ and $\cos 2\theta = -\mu/\sqrt{1+\mu^2}$, we have

$$\begin{aligned} a_x &= g \sin\theta \cos\theta - \mu g \cos^2\theta \\ &= \frac{g \sin 2\theta}{2} - \frac{\mu g(1+\cos 2\theta)}{2} \\ &= \frac{g}{2}(\sqrt{1+\mu^2} - \mu). \end{aligned} \tag{2.54}$$

   (For $\mu \to \infty$, this behaves like $a_x \approx g/(4\mu)$.) Therefore, $a_x t^2/2 = b$ gives

$$\begin{aligned} t = \sqrt{\frac{2b}{a_x}} &= \frac{2\sqrt{b/g}}{\sqrt{\sqrt{1+\mu^2} - \mu}} \\ &= 2\sqrt{b/g}\sqrt{\sqrt{1+\mu^2} + \mu}. \end{aligned} \tag{2.55}$$

   (For $\mu \to \infty$, this behaves like $t \approx 2\sqrt{2\mu b/g}$.)

2. **Sliding plane**

   Let F be the normal force between the block and the plane. Then the various $F = ma$ equations (vertical and horizontal for the block, and horizontal for the plane) are

$$\begin{aligned} mg - F\cos\theta &= ma_y \\ F\sin\theta &= ma_x \\ F\sin\theta &= MA_x. \end{aligned} \tag{2.56}$$

   (We've chosen positive a_x and a_y to be leftward and downward, respectively, and positive A_x to be rightward.)

   There are four unknowns here (a_x,a_y,A_x,F), so we need one more equation. This last equation is the constraint that the block remains in contact with the plane. The horizontal distance between the block and its starting point on the plane is

$(a_x + A_x)t^2/2$, and the vertical distance is $a_y t^2/2$. The ratio of these distances must equal $\tan\theta$ if the block is to remain on the plane. Therefore, we must have

$$\frac{a_y}{a_x + A_x} = \tan\theta. \tag{2.57}$$

Using eqs. (2.56), this becomes

$$\frac{g - \frac{F}{m}\cos\theta}{\frac{F}{m}\sin\theta + \frac{F}{M}\sin\theta} = \tan\theta$$

$$\Longrightarrow \qquad F = g\left(\sin\theta\tan\theta\left(\frac{1}{m} + \frac{1}{M}\right) + \frac{\cos\theta}{m}\right)^{-1}. \tag{2.58}$$

The third of eqs. (2.56) then yields A_x, which may be written as

$$A_x = \frac{F\sin\theta}{M} = \frac{mg\tan\theta}{M(1+\tan^2\theta) + m\tan^2\theta}. \tag{2.59}$$

REMARKS: For given M and m, the angle θ_0 which maximizes A_x is found to be

$$\tan\theta_0 = \sqrt{\frac{M}{M+m}}. \tag{2.60}$$

If $M \ll m$, then $\theta_0 \approx 0$. If $M \gg m$, then $\theta_0 \approx \pi/4$.

In the limit $M \ll m$, we have $A_x \approx g/\tan\theta$. This makes sense, because m falls essentially straight down, and the plane gets squeezed out to the right.

In the limit $M \gg m$, we have $A_x \approx g(m/M)\tan\theta/(1+\tan^2\theta) = g(m/M)\sin\theta\cos\theta$. This is more transparent if we instead look at $a_x = (M/m)A_x \approx g\sin\theta\cos\theta$. Since the plane is essentially at rest in this limit, this value of a_x implies that the acceleration of m *along* the plane is essentially equal to $a_x/\cos\theta \approx g\sin\theta$, as expected. ♣

3. **Sliding sideways on plane**

The normal force from the plane is $N = mg\cos\theta$, so the friction force on the block is $\mu N = (\tan\theta)N = mg\sin\theta$. This force acts in the direction opposite to the motion. The block also feels the gravitational force of $mg\sin\theta$ pointing down the plane.

Since the magnitudes of the friction and gravitational forces are equal, the acceleration along the direction of motion equals the negative of the acceleration in the direction down the plane. Therefore, in a small increment of time, the speed that the block loses along its direction of motion exactly equals the speed that it gains in the direction down the plane. Letting v be the speed of the block, and letting v_y be the component of the velocity in the direction down the plane, we therefore have

$$v + v_y = C, \tag{2.61}$$

where C is a constant. C is given by its initial value, which is $V + 0 = V$. The final value of C is $V_f + V_f = 2V_f$ (where V_f is the final speed of the block), since the block is essentially moving straight down the plane after a very long time. Therefore,

$$2V_f = V \qquad \Longrightarrow \qquad V_f = V/2. \tag{2.62}$$

4. **Atwood's machine**

   (a) Let T be the tension in the string. Let a be the acceleration of M_2 (with downward taken to be positive). Then $-a$ is the acceleration of M_1. So we have

$$\begin{aligned} M_1 g - T &= M_1(-a), \\ M_2 g - T &= M_2 a. \end{aligned} \tag{2.63}$$

   Subtracting the two equations yields

$$a = g\frac{M_2 - M_1}{M_1 + M_2}. \tag{2.64}$$

   As a double-check, this has the correct limits when $M_2 \gg M_1$, $M_2 \ll M_1$, and $M_2 = M_1$, namely $a \approx g$, $a \approx -g$, and $a = 0$, respectively.

   We may also solve for the tension, $T = 2M_1M_2/(M_1 + M_2)$. If $M_1 = M_2 \equiv M$, then $T = Mg$, as it should. If $M_1 \ll M_2$, then $T \approx 2M_1 g$, as it should (because then the net upward force on M_1 is $M_1 g$, so its acceleration equals g upwards, as it must, since M_2 is essentially in free-fall).

   (b) The key here is that since the pulleys are massless, there can be no net force on them, so the tension in the bottom string must be half of that in the top string. Let these be $T/2$ and T, respectively. Let a_p be the acceleration of the bottom pulley, and let a be the acceleration of M_3 relative to the bottom pulley (with downward taken to be positive). Then we have

$$\begin{aligned} M_1 g - T &= M_1(-a_p), \\ M_2 g - \frac{T}{2} &= M_2(a_p - a), \\ M_3 g - \frac{T}{2} &= M_3(a_p + a). \end{aligned} \tag{2.65}$$

   Solving for a_p and a gives

$$a_p = g\frac{4M_2M_3 - M_1(M_2 + M_3)}{4M_2M_3 + M_1(M_2 + M_3)}, \qquad a = g\frac{2M_1(M_3 - M_2)}{4M_2M_3 + M_1(M_2 + M_3)}. \tag{2.66}$$

   The accelerations of M_2 and M_3, namely $a_p - a$ and $a_p + a$, are

$$\begin{aligned} a_p - a &= g\frac{4M_2M_3 + M_1(M_2 - 3M_3)}{4M_2M_3 + M_1(M_2 + M_3)}, \\ a_p + a &= g\frac{4M_2M_3 + M_1(M_3 - 3M_2)}{4M_2M_3 + M_1(M_2 + M_3)}. \end{aligned} \tag{2.67}$$

You can check various limits. One nice one is where M_2 is much less than both M_1 and M_3. The accelerations of M_1, M_2, and M_3 are then

$$-a_p = g, \qquad a_p - a = -3g, \qquad a_p + a = g, \tag{2.68}$$

(with downward taken to be positive).

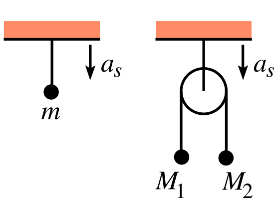


Figure 2.18

5. **Infinite Atwood's machine**

First Solution: Consider the following auxiliary problem.

Problem: Two set-ups are shown in Fig. 2.18. The first contains a hanging mass m. The second contains a hanging pulley, over which two masses, M_1 and M_2, hang.

Let both supports have acceleration a_s downward. What should m be, in terms of M_1 and M_2, so that the tension in the top string is the same in both cases?

Answer: In the first case, we have

$$mg - T = ma_s. \tag{2.69}$$

In the second case, let a be the acceleration of M_2 relative to the support (with downward taken to be positive). Then we have

$$\begin{aligned} M_1 g - \frac{T}{2} &= M_1(a_s - a), \\ M_2 g - \frac{T}{2} &= M_2(a_s + a). \end{aligned} \tag{2.70}$$

Note that if we define $g' \equiv g - a_s$, then we may write these three equations as

$$\begin{aligned} mg' &= T, \\ M_1 g' &= \frac{T}{2} - M_1 a, \\ M_2 g' &= \frac{T}{2} + M_2 a. \end{aligned} \tag{2.71}$$

The last two give $4M_1M_2g' = (M_1 + M_2)T$. The first equation then gives

$$m = \frac{4M_1M_2}{M_1 + M_2}. \tag{2.72}$$

Note that the value of a_s is irrelevant. (We effectively have a fixed support in a world where the acceleration from gravity is g'.) This problem shows that the two-mass system in the second case may be equivalently treated as a mass m, as far as the upper string is concerned. ■

Now let's look at our infinite Atwood machine. Start at the bottom. (Assume that the system has N pulleys, where $N \to \infty$.) Let the bottom mass be x. Then the above problem shows that the bottom two masses, M and x, may be treated as an effective mass $f(x)$, where

$$f(x) = \frac{4x}{1 + (x/M)}. \tag{2.73}$$

We may then treat the combination of the mass $f(x)$ and the next M as an effective mass $f(f(x))$. These iterations may be repeated, until we finally have a mass M and a mass $f^{(N-1)}(x)$ hanging over the top pulley.

We must determine the behavior of $f^N(x)$, as $N \to \infty$. The behavior is obvious by looking at a plot of $f(x)$ (which we'll let you draw). (Note that $x = 3M$ is a fixed point of f, i.e., $f(3M) = 3M$.) It is clear that no matter what x we start with, the iterations approach $3M$ (unless, of course, $x = 0$). So our infinite Atwood machine is equivalent to (as far as the top mass is concerned) just the two masses M and $3M$.

We then easily find that the acceleration of the top mass equals (net downward force)/(total mass) $= 2Mg/(4M) = g/2$.

As far as the support is concerned, the whole apparatus is equivalent to a mass $3M$. So $3Mg$ is the weight the support holds up.

Second Solution: If the gravity in the world were multiplied by a factor η, then the tension in all the strings would likewise be multiplied by η. (The only way to make a tension, i.e., a force, is to multiply a mass times g.) Conversely, if we put the apparatus on another planet and discover that all the tensions are multiplied by η, then we know the gravity there must be ηg.

Let the tension in the string above the first pulley be T. Then the tension in the string above the second pulley is $T/2$ (since the pulleys are massless). Let the acceleration of the second pulley be a_{p2}. Then the second pulley effectively lives in a world where the gravity is $g - a_{p2}$. If we imagine holding the string above the second pulley and accelerating downward at a_{p2} (so that our hand is at the origin of the new world), then we really haven't changed anything, so the tension in this string in the new world is still $T/2$.

But in this infinite setup, the system of all the pulleys except the top one is the same as the original system of all the pulleys. Therefore, by the arguments in the first paragraph, we must have

$$\frac{T}{g} = \frac{T/2}{g - a_{p2}}. \tag{2.74}$$

Hence, $a_{p2} = g/2$. (Likewise, the relative acceleration of the second and third pulleys is $g/4$, etc.) But a_{p2} is also the acceleration of the top mass. So our answer is $g/2$.

Note that $T = 0$ also makes eq. (2.74) true. But this corresponds to putting a mass of zero at the end of a finite pulley system.

6. **Line of Pulleys**

Let m be the common mass, and let T be the tension in the string. Let a be the acceleration of the end masses, and let a' be the acceleration of the other masses (with downward taken to be positive). Then we have

$$\begin{aligned} T - mg &= ma, \\ 2T - mg &= ma'. \end{aligned} \tag{2.75}$$

The string has fixed length, therefore

$$N(2a') + a + a = 0. \tag{2.76}$$

Eliminating T from eqs. (2.75) gives $a' = 2a + g$. Combining this with eq. (2.76) then gives

$$a = \frac{-g}{2 + \frac{1}{N}}. \tag{2.77}$$

For $N = 0$ we have $a = 0$. For $N = 1$ we have $a = -g/3$. For larger N, a increases in magnitude until it equals $-g/2$ for $N \to \infty$.

7. **Ring of Pulleys**

Let T be the tension in the string. Then $F = ma$ for m_i gives

$$2T - m_i g = m_i a_i, \tag{2.78}$$

with upward taken to be positive.

But the string has a fixed length. Therefore, the sum of all the displacements of the masses is zero. Hence,

$$a_1 + a_2 + \cdots + a_N = 0. \tag{2.79}$$

If we divide eq. (2.78) by m_i, and then add the N such equations together, we then obtain

$$2T\left(\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_N}\right) - Ng = 0. \tag{2.80}$$

Substituting this value for T into (2.78) gives

$$a_i = g\left(\frac{N}{m_i\left(\frac{1}{m_1}+\frac{1}{m_2}+\cdots+\frac{1}{m_N}\right)} - 1\right). \tag{2.81}$$

If all the masses are equal, then all $a_i = 0$. If $m_k = 0$ (and all the others are not 0), then $a_k = (N-1)g$, and all the other $a_i = -g$.

8. **Exponential force**

We are given $\ddot{x} = e^{-bt}$. Integrating this w.r.t. time gives $v(t) = -e^{-bt}/b + A$. Integrating again gives $x(t) = e^{-bt}/b^2 + At + B$. The initial conditions are $0 = v(0) = -1/b + A$ and $0 = x(0) = 1/b^2 + B$. Therefore,

$$x(t) = \frac{e^{-bt}}{b^2} + \frac{t}{b} - \frac{1}{b^2}. \tag{2.82}$$

For $t \to \infty$, the speed is $v \to 1/b$. The particle eventually lags a distance $1/b^2$ behind a particle that starts at the same position but with speed $v = 1/b$.

9. **Falling chain**

(a) Let $y(t)$ be the length hanging over the edge at time t. Let the density of the chain be ρ. Then the total mass is $M = \rho\ell$, and the mass hanging over the edge is ρy. The downward force on the chain (that isn't countered by a normal force from the table) is therefore $(\rho y)g$, so $F = ma$ gives

$$\rho g y = \rho \ell \ddot{y} \qquad \Longrightarrow \qquad \ddot{y} = \frac{g}{\ell}y. \tag{2.83}$$

The solution to this equation is

$$y(t) = Ae^{\alpha t} + Be^{-\alpha t}, \qquad \text{where } \alpha \equiv \sqrt{\frac{g}{\ell}}. \tag{2.84}$$

Taking the derivative of this to obtain $\dot{y}(t)$, and using the given information that $\dot{y}(0) = 0$, we find $A = B$. Using $y(0) = y_0$, we then find $A = B = y_0/2$. So the length that hangs over the edge is

$$y(t) = \frac{y_0}{2}\left(e^{\alpha t} + e^{-\alpha t}\right) \equiv y_0 \cosh(\alpha t). \tag{2.85}$$

And the speed is

$$\dot{y}(t) = \frac{\alpha y_0}{2}\left(e^{\alpha t} - e^{-\alpha t}\right) \equiv \alpha y_0 \sinh(\alpha t). \tag{2.86}$$

The time T that satisfies $y(T) = \ell$ is given by $\ell = y_0 \cosh(\alpha T)$. Using $\sinh x = \sqrt{\cosh^2 x - 1}$, we find that the speed of the chain right when it loses contact with the table is

$$\dot{y}(T) = \alpha y_0 \sinh(\alpha T) = \alpha\sqrt{\ell^2 - y_0^2} \equiv \sqrt{g\ell}\sqrt{1-\eta_0^2}, \tag{2.87}$$

where $\eta_0 \equiv y_0/\ell$ is the initial fraction hanging over the edge. If $\eta_0 \approx 0$, then the speed at T is $\sqrt{g\ell}$ (which is clear, since the center-of-mass falls a distance $\ell/2$) Also, you can show that T goes to infinity logarithmically as $\eta_0 \to 0$.

(b) The normal force on the table is $g\rho(\ell - y)$, so the friction force opposing gravity is $\mu g\rho(\ell - y)$. Therefore, $F = ma$ gives

$$\rho g y - \mu g \rho(\ell - y) = \rho \ell \ddot{y}. \tag{2.88}$$

This equation is valid only if the gravitational force is greater than the friction force (i.e., the left-hand side is positive), otherwise the chain just sits there. The left-hand side is positive if $y > \mu\ell/(1+\mu)$. Let us define a new variable $z \equiv y - \mu\ell/(1+\mu)$. (So our ending point, $y = \ell$, corresponds to $z = \ell/(1+\mu)$.) Then eq. (2.88) becomes

$$\ddot{z} = z\frac{g}{\ell}(1+\mu). \tag{2.89}$$

At this point, we can either repeat all the steps in part (a), with slightly different variables, or we can just realize that we now have the exact came problem, with the only change being that ℓ has turned into $\ell/(1+\mu)$. So we have

$$\begin{aligned} z(t) &= z_0 \cosh(\alpha' t), \qquad \text{where } \alpha' \equiv \sqrt{\frac{g(1+\mu)}{\ell}}, \\ \Longrightarrow \quad y(t) &= \left(y_0 - \frac{\mu\ell}{1+\mu}\right)\cosh(\alpha' t) + \frac{\mu\ell}{1+\mu}. \end{aligned} \tag{2.90}$$

And the final speed is

$$\dot{y}(T') = \dot{z}(T') = \alpha' z_0 \sinh(\alpha T') = \alpha' \sqrt{\frac{\ell^2}{(1+\mu)^2} - z_0^2} \equiv \sqrt{\frac{g\ell}{1+\mu}}\sqrt{1 - \eta_0'^2}, \tag{2.91}$$

where $\eta_0' \equiv z_0/[\ell/(1+\mu)]$ is the initial 'excess fraction'. That is, it is the excess length above the minimal length, $\mu\ell/(1+\mu)$, divided by the maximum possible excess length, $\ell/(1+\mu)$. If $\eta_0' \approx 0$, then the speed at T' is $\sqrt{g\ell/(1+\mu)}$.

10. **Ball thrown upward**

Let's take upward to be the positive direction. Then on the way up, the force is $F = -mg - m\alpha v$. (v is positive here, so the drag force points downward, as it should.)

The first thing we must do is find the maximum height, h, the ball reaches. You can use the technique in eqs. (2.21) and (2.22) to solve for $v(t)$ and then $y(t)$. But it is much simpler to use eq. (2.23) to solve for $v(y)$, and to then take advantage of the fact that we know the speed of the ball at the top, namely zero. Eq. (2.23) gives

$$\int_{v_0}^{0} \frac{v\,dv}{g + \alpha v} = -\int_0^h dy. \tag{2.92}$$

Write $v/(g+\alpha v)$ as $[1 - g/(g+\alpha v)]/\alpha$ and integrate to obtain

$$\frac{v_0}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 + \frac{\alpha v_0}{g}\right) = h. \tag{2.93}$$

On the way down, the force is again $F = -mg - m\alpha v$. (v is negative here, so the drag force points upward, as it should.) If v_f is the final speed (we'll take v_f to be a positive number, so that the final velocity is $-v_f$), then eq. (2.23) gives

$$\int_0^{-v_f} \frac{v\,dv}{g + \alpha v} = -\int_h^0 dy. \tag{2.94}$$

This gives

$$-\frac{v_f}{\alpha} - \frac{g}{\alpha^2}\ln\left(1 - \frac{\alpha v_f}{g}\right) = h. \tag{2.95}$$

(This is the same as eq. (2.93), with v_0 replaced by $-v_f$.)

Equating the expressions for h in eqs. (2.93) and (2.95) gives an implicit equation for v_f in terms of v_0,

$$v_0 + v_f = \frac{g}{\alpha}\ln\left(\frac{g + \alpha v_0}{g - \alpha v_f}\right). \tag{2.96}$$

REMARKS: Let's find approximate values for h in eqs. (2.93) and (2.95), in the limit of small α (which is the same as large g). More precisely, let's look at the limit $\alpha v_0/g \ll 1$. Using $\ln(1+x) = x - x^2/2 + x^3/3 - \cdots$, we find

$$h \approx \frac{v_0^2}{2g} - \frac{\alpha v_0^3}{3g^2} \qquad \text{and} \qquad h \approx \frac{v_f^2}{2g} + \frac{\alpha v_f^3}{3g^2}. \tag{2.97}$$

The leading terms, $v^2/2g$, are as expected.

For small α, we may find v_f in terms of v_0. Equating the two expressions for h in eq. (2.97) (and using the fact that $v_f \approx v_0$) gives

$$\begin{aligned}
v_0^2 - v_f^2 &\approx \frac{2\alpha}{3g}(v_0^3 + v_f^3) \\
\Longrightarrow (v_0 + v_f)(v_0 - v_f) &\approx \frac{2\alpha}{3g}(v_0^3 + v_f^3) \\
\Longrightarrow 2v_0(v_0 - v_f) &\approx \frac{4\alpha}{3g}v_0^3 \\
\Longrightarrow v_0 - v_f &\approx \frac{2\alpha}{3g}v_0^2 \\
\Longrightarrow v_f &\approx v_0 - \frac{2\alpha}{3g}v_0^2.
\end{aligned} \tag{2.98}$$

We may also make approximations for large α (or small g). In this case, eq. (2.93) gives $h \approx v_0/\alpha$. And eq. (2.95) gives $v_f \approx g/\alpha$ (because the log term must be a very large negative number, in order to yield a positive h). There is no way to relate v_f and h is this case, because the ball quickly reaches the terminal velocity of g/α, independent of h. ♣

Let's now find the time it takes for the ball to go up and to go down. If T_1 is the time of the upward path, then integrating eq. (2.21), with $F = -mg - m\alpha v$, from the start to the apex gives

$$T_1 = \frac{1}{\alpha}\ln\left(1 + \frac{\alpha v_0}{g}\right). \tag{2.99}$$

Likewise, the time T_2 for the downward path is

$$T_2 = -\frac{1}{\alpha}\ln\left(1 - \frac{\alpha v_f}{g}\right). \tag{2.100}$$

Therefore,

$$T_1 + T_2 = \frac{1}{\alpha}\ln\left(\frac{g + \alpha v_0}{g - \alpha v_f}\right). \tag{2.101}$$

Using eq. (2.96), we have

$$T_1 + T_2 = \frac{v_0 + v_f}{g}\,. \tag{2.102}$$

This is shorter than the time in vacuum, namely $2v_0/g$, because $v_f < v_0$.

REMARKS: The fact that the time here is shorter than the time in vacuum isn't all that obvious. On one hand, the ball doesn't travel as far in air as it would in vacuum (so you might think that $T_1 + T_2 < 2v_0/g$). But on the other hand, the ball moves slower in air (so you might think that $T_1 + T_2 > 2v_0/g$). It isn't obvious which effect wins, without doing a calculation.

For any α, you can easily use eq. (2.99) to show that $T_1 < v_0/g$. T_2 is harder to get a handle on, since it is given in terms of v_f. But in the limit of large α, the ball quickly reaches terminal velocity, so we have $T_2 \approx h/v_f \approx (v_0/\alpha)/(g/\alpha) = v_0/g$. Interestingly, this is the same as the downward (and upward) time for a ball thrown in vacuum.

The very simple form of eq. (2.102) suggests that there should be a cleaner way to calculate the total time of flight. And indeed, if we integrate $m\,dv/dt = -mg - m\alpha v$ with respect to time on the way up, we obtain $-v_0 = -gT_1 - \alpha h$ (because $\int v\,dt = h$). Likewise, if we integrate $m\,dv/dt = -mg - m\alpha v$ with respect to time on the way down, we obtain $-v_f = -gT_2 + \alpha h$ (because $\int v\,dt = -h$). Adding these two results gives eq. (2.102). This procedure only worked, of course, because the drag force was proportional to v. ♣

11. **Falling pencil**

(a) The component of gravity in the tangential direction is $mg\sin\theta \approx mg\theta$. Therefore, the tangential $F = ma$ equation is $mg\theta = m\ell\ddot{\theta}$, which simplifies to $\ddot{\theta} = (g/\ell)\theta$. The general solution to this equation is

$$\theta(t) = Ae^{t\sqrt{g/\ell}} + Be^{-t\sqrt{g/\ell}}. \tag{2.103}$$

The constants A and B are found from the initial conditions,

$$\begin{aligned} \theta(0) = \theta_0 \quad &\Longrightarrow \quad \theta_0 = A + B, \\ \dot{\theta}(0) = \omega_0 \quad &\Longrightarrow \quad \omega_0 = \sqrt{g/\ell}(A - B). \end{aligned} \tag{2.104}$$

Solving for A and B, and then plugging into eq. (2.103) gives

$$\theta(t) = \frac{1}{2}\left(\theta_0 + \omega_0\sqrt{\ell/g}\right)e^{t\sqrt{g/\ell}} + \frac{1}{2}\left(\theta_0 - \omega_0\sqrt{\ell/g}\right)e^{-t\sqrt{g/\ell}}. \tag{2.105}$$

(b) The constants A and B will turn out to be small (they will each be of order $\sqrt{\hbar}$). Therefore, by the time the positive exponential has increased enough to make θ of order 1, the negative exponential will have become negligible. We will therefore ignore the latter term from here on. In other words,

$$\theta(t) \approx \frac{1}{2}\left(\theta_0 + \omega_0\sqrt{\ell/g}\right)e^{t\sqrt{g/\ell}}. \tag{2.106}$$

The goal is to keep θ small for as long as possible. Hence, we want to minimize the coefficient of the exponential, subject to the uncertainty-principle constraint, $(\ell\theta_0)(m\ell\omega_0) \ge \hbar$. This constraint gives $\omega_0 \ge \hbar/(m\ell^2\theta_0)$. Hence,

$$\theta(t) \ge \frac{1}{2}\left(\theta_0 + \frac{\hbar\sqrt{\ell/g}}{m\ell^2\theta_0}\right)e^{t\sqrt{g/\ell}}. \tag{2.107}$$

Taking the derivative with respect to θ_0 to minimize the coefficient, we find that the minimum value occurs at

$$\theta_0 = \sqrt{\frac{\hbar}{m\ell^2}\sqrt{\frac{\ell}{g}}}\,. \tag{2.108}$$

Substituting this back into eq. (2.107) gives

$$\theta(t) \geq \sqrt{\frac{\hbar}{m\ell^2}\sqrt{\frac{\ell}{g}}}\, e^{t\sqrt{g/\ell}}. \tag{2.109}$$

Setting $\theta = 1$, and then solving for t gives

$$t \leq \sqrt{\frac{\ell}{g}} \ln\left(\frac{m^{1/2}\ell^{3/4}g^{1/4}}{\hbar^{1/2}}\right). \tag{2.110}$$

With the given values, $m = 0.01\,\text{kg}$ and $\ell = 0.1\,\text{m}$, along with $g = 10\,\text{m/s}^2$ and $\hbar = 1.06 \cdot 10^{-34}\,\text{Js}$, we obtain

$$t \leq (0.1\,\text{s})\ln(3 \cdot 10^{15}) \approx 3.5\,\text{s}. \tag{2.111}$$

No matter how clever you are, no matter how much money you spend on the newest, cutting-edge pencil-balancing equipment, you can never get a pencil to balance for more than about four seconds.

REMARKS: This smallness of this answer is quite amazing. It is remarkable that a quantum effect on a macroscopic object produces an everyday value for a time scale. Basically, what is going on here is that the fast exponential growth of θ (which gives rise to the log in the final result for t) wins out over the smallness of $\hbar$, and produces a result for t of order 1. When push comes to shove, exponential effects always win.

The above value for t depends strongly on ℓ and g, through the $\sqrt{\ell/g}$ term. But the dependence on m, ℓ, and g in the log term is very weak. If m were increased by a factor of 1000, for example, the result for t would increase by only about 10%. Note that this implies that any factors of order 1 that we neglected throughout this problem are completely irrelevant. They will appear in the argument of the log term, and will thus have negligible effect. ♣

12. **Throwing a ball from a cliff**

Let the angle be θ, and let the speed be v. Then the horizontal speed is $v_x = v\cos\theta$, and the initial vertical speed is $v_y = v\sin\theta$.

The time it takes for the ball to hit the ground is given by $(v\sin\theta)t - gt^2/2 = -h$. Therefore,

$$t = \frac{v}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \alpha}\right), \qquad \text{where } \alpha \equiv \frac{2gh}{v^2}. \tag{2.112}$$

(The other solution for t corresponds to the ball being thrown backwards down through the cliff.) The horizontal distance traveled is $d = (v\cos\theta)t$, so

$$d = \frac{v^2}{g}\cos\theta\left(\sin\theta + \sqrt{\sin^2\theta + \alpha}\right). \tag{2.113}$$

We must maximize this function $d(\theta)$. Taking the derivative, multiplying through by $\sqrt{\sin^2\theta + \alpha}$, and setting the result equal to zero, gives

$$(\cos^2\theta - \sin^2\theta)\sqrt{\sin^2\theta + \alpha} = \sin\theta\big(\alpha - (\cos^2\theta - \sin^2\theta)\big). \tag{2.114}$$

Squaring, simplifying, and using $\cos^2\theta = 1 - \sin^2\theta$, gives

$$\sin\theta = \frac{1}{\sqrt{2+\alpha}} \equiv \frac{1}{\sqrt{2 + 2gh/v^2}}. \tag{2.115}$$

This is the optimal angle. Plugging this into eq. (2.113) gives a maximum distance of

$$d_{\max} = \frac{v^2}{g}\sqrt{1+\alpha} \equiv \frac{v^2}{g}\sqrt{1+\frac{2gh}{v^2}}. \tag{2.116}$$

If $h = 0$, then we obtain the result of the example in Section 2.4. If $h \to \infty$ or $v \to 0$, then $\theta \approx 0$, which makes sense.

13. **Redirected horizontal motion**

First Solution: Let v be the speed right after the bounce (which is the same as the speed right before the bounce). Integrating $mv\,dv/dy = -mg$ gives $mv^2/2 = mg(h-y)$ (where the constant of integration has been chosen so that $v = 0$ when $y = h$). This is simply the conservation-of-energy statement. So we have

$$v = \sqrt{2g(h-y)}. \tag{2.117}$$

The vertical speed is zero right after the bounce, so the time it takes to hit the ground is given by $gt^2/2 = y$. Hence $t = \sqrt{2y/g}$. So the horizontal distance, d, traveled is

$$d = vt = 2\sqrt{y(h-y)}. \tag{2.118}$$

Taking a derivative, we see that this function of y is maximum at

$$y = \frac{h}{2}. \tag{2.119}$$

The corresponding value of d is $d_{\max} = h$.

Second Solution: Assume that the greatest distance, d_0, is obtained when $y = y_0$ (and let the speed at y_0 be v_0).

Consider the situation where the ball falls all the way down to $y = 0$ and then bounces up at an angle such that when it reaches the height y_0, it is traveling horizontally. When it reaches the height y_0, the ball will have speed v_0 (by conservation of energy, which will be introduced in Chapter 4, but which you're all familiar with anyway), so it will travel a horizontal distance d_0 from this point. The total horizontal distance traveled is therefore $2d_0$.

So to maximize d_0, we simply have to maximize the horizontal distance in this new situation. From the example in Section 2.4, we want the ball to leave the ground at a 45° angle. Since it leaves the ground with speed $\sqrt{2gh}$, one can easily show that such a ball will be traveling horizontally at a height $y = h/2$, and it will travel a distance $2d_0 = 2h$. Hence, $y_0 = h/2$, and $d_0 = h$.

14. **Redirected general motion**

First Solution: We will use the results of Problem 12, namely eqs. (2.116) and (2.115), which say that an object projected from a height y at speed v travels a maximum distance of

$$d = \frac{v^2}{g}\sqrt{1+\frac{2gy}{v^2}}, \tag{2.120}$$

and the optimal angle yielding this distance is

$$\sin\theta = \frac{1}{\sqrt{2+2gh/v^2}}. \tag{2.121}$$

In the problem at hand, the object is dropped from a height h, so conservation of energy (or integration of $mv\,dv/dy = -mg$) says that the speed at height y is

$$v = \sqrt{2g(h-y)}. \tag{2.122}$$

Plugging this into eq. (2.120) shows that the maximum horizontal distance, as a function of y, is

$$d_{\text{max}}(y) = 2\sqrt{h(h-y)}. \tag{2.123}$$

This is clearly maximum at $y = 0$, in which case the distance is $d_{\text{max}} = 2h$. Eq. (2.121) then gives the associated optimal angle as $\theta = 45°$.

Second Solution: Assume that the greatest distance, d_0, is obtained when $y = y_0 \neq 0$ and $\theta = \theta_0$ (and let the speed at y_0 be v_0). We will show that this cannot be the case. We will do this by explicitly constructing a situation yielding a greater distance.

Consider the situation where the ball falls all the way down to $y = 0$ and then bounces up at an angle such that when it reaches the height y_0, it is traveling at an angle θ_0 with respect to the horizontal. When it reaches the height y_0, the ball will have speed v_0 (by conservation of energy), so it will travel a horizontal distance d_0 from this point. But the ball traveled a nonzero horizontal distance on its way up to the height y_0. We have therefore constructed a situation yielding a distance greater than d_0. Hence, the optimal setup cannot have $y_0 \neq 0$. Therefore, the maximum distance must be obtained when $y = 0$ (in which case the example in Section 2.4 says that the optimal angle is $\theta = 45°$).

If you want the ball to go even further, simply dig a (wide enough) hole in the ground and have the ball bounce from the bottom of the hole.

15. **Maximum length of trajectory**

The coordinates are given by $x = (v\cos\theta)t$ and $y = (v\sin\theta)t - gt^2/2$. Eliminating t gives

$$y = (\tan\theta)x - \frac{gx^2}{2v^2\cos^2\theta}. \tag{2.124}$$

The length of the arc is twice the length up to the maximum. The maximum occurs at $t = (v/g)\sin\theta$, and hence $x = (v^2/g)\sin\theta\cos\theta$. So the length of the arc is

$$\begin{aligned} L &= 2\int_0^{(v^2/g)\sin\theta\cos\theta} \sqrt{1+(dy/dx)^2}\,dx \\ &= 2\int_0^{(v^2/g)\sin\theta\cos\theta} \sqrt{1+(\tan\theta - gx/v^2\cos^2\theta)^2}\,dx. \end{aligned} \tag{2.125}$$

Letting $z \equiv \tan\theta - gx/v^2\cos^2\theta$, we obtain

$$\begin{aligned} L &= \frac{2v^2\cos^2\theta}{g}\int_0^{\tan\theta} \sqrt{1+z^2}\,dz \\ &= \frac{2v^2\cos^2\theta}{g}\frac{1}{2}\left(z\sqrt{1+z^2} + \ln\left(z+\sqrt{1+z^2}\right)\right)\Big|_0^{\tan\theta} \\ &= \frac{v^2}{g}\left(\sin\theta + \cos^2\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right)\right). \end{aligned} \tag{2.126}$$

(As a double-check, some special cases are $L = 0$ at $\theta = 0$, and $L = v^2/g$ at $\theta = 90°$, as one can explicitly verify.) Taking the derivative to find the maximum, we have

$$0 = \cos\theta - 2\cos\theta\sin\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right) + \cos^2\theta\frac{\cos\theta}{1+\sin\theta}\frac{\cos^2\theta + (1+\sin\theta)\sin\theta}{\cos^2\theta} \tag{2.127}$$

This reduces to

$$1 = \sin\theta\ln\left(\frac{1+\sin\theta}{\cos\theta}\right), \tag{2.128}$$

as was to be shown.

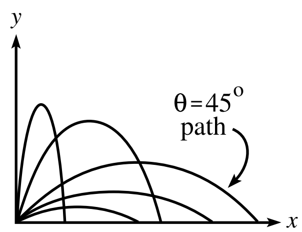


Figure 2.19

REMARK: The possible trajectories are shown in Fig. 2.19. Since it is well-known that $\theta = 45°$ provides the maximum *horizontal* distance, it is clear from the figure that the θ_0 yielding the arc of maximum length must satisfy $\theta_0 \geq 45°$. The exact angle, however, requires the above detailed calculation. ♣

16. **Maximum area under trajectory**

The coordinates are given by $x = (v\cos\theta)t$ and $y = (v\sin\theta)t - gt^2/2$. The time in the air is $T = 2(v\sin\theta)/g$. The area under the trajectory is

$$\begin{aligned} A &= \int_0^{x_{\max}} y\,dx \\ &= \int_0^{2v\sin\theta/g} \Big((v\sin\theta)t - gt^2/2\Big)v\cos\theta\,dt \\ &= \frac{2v^4}{3g^2}\sin^3\theta\cos\theta. \end{aligned} \tag{2.129}$$

Taking a derivative, we find that the maximum occurs when $\tan\theta = \sqrt{3}$, i.e.,

$$\theta = 60°. \tag{2.130}$$

The maximum area is then $A_{\max} = \sqrt{3}v^4/8g^2$. (By dimensional analysis, we know that it has to be proportional to v^4/g^2.)

17. **Bouncing ball**

The ball travels $2h$ during the first up-and-down journey. It travels $2hf$ during the second, then $2hf^2$ during the third, and so on. Therefore, the total distance traveled is

$$\begin{aligned} D &= 2h(1 + f + f^2 + f^3 + \cdots) \\ &= \frac{2h}{1-f}. \end{aligned} \tag{2.131}$$

The time it takes to fall down during the first up-and-down is obtained from $h = gt^2/2$. So the time for the first up-and-down is $2t = 2\sqrt{2h/g}$. The time for the second up-and-down will likewise be $2\sqrt{2(hf)/g}$. Each successive time decreases by a factor of $\sqrt{f}$. The total time is therefore

$$\begin{aligned} T &= 2\sqrt{\frac{2h}{g}}(1 + f^{1/2} + f^1 + f^{3/2} + \cdots) \\ &= 2\sqrt{\frac{2h}{g}}\frac{1}{1-\sqrt{f}}. \end{aligned} \tag{2.132}$$

(Note that if f is exactly equal to 1, then the summations of the above series' are not valid.)

The average speed

$$\frac{D}{T} = \frac{\sqrt{gh/2}}{1+\sqrt{f}}. \tag{2.133}$$

Remark: For $f \approx 1$, the average speed is roughly half of the average speed for $f \approx 0$. This may seem somewhat counterintuitive, because in the $f \approx 0$ case the ball slows down far sooner than in the $f \approx 1$ case. But the point is that the $f \approx 0$ case consists of essentially only one bounce, and the average speed for that bounce is the largest of any bounce. Both D and T are smaller for $f \approx 0$ than for $f \approx 1$; but T is smaller by a larger factor. ♣

18. **Free particle**

For zero force, eqs. (2.49) yield

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2, \\ -2\dot{r}\dot{\theta} &= r\ddot{\theta}. \end{aligned} \tag{2.134}$$

Separating variables in the second equation and then integrating gives

$$-\int \frac{2\dot{r}}{r} = \int \frac{\ddot{\theta}}{\dot{\theta}} \quad \Longrightarrow \quad -2\ln r = \ln\dot{\theta} + C \quad \Longrightarrow \quad \dot{\theta} = \frac{D}{r^2}, \tag{2.135}$$

where $D = e^{-C}$ is a constant of integration, determined by the initial conditions.[12] Substituting this value of $\dot{\theta}$ into the first of eqs. (2.140), and then multiplying both sides by $\dot{r}$ and integrating, gives

$$\ddot{r} = r\left(\frac{D}{r^2}\right)^2 \quad \Longrightarrow \quad \int \ddot{r}\dot{r} = D^2\int\frac{\dot{r}}{r^3} \quad \Longrightarrow \quad \frac{\dot{r}^2}{2} = -\frac{D^2}{2r^2} + E. \tag{2.136}$$

Therefore,

$$\dot{r} = V\sqrt{1 - \frac{r_0^2}{r^2}}, \tag{2.137}$$

where $V \equiv D/r_0$. The constant E was chosen so that $\dot{r} = 0$ when $r = r_0$. Separating variables and integrating gives

$$\int \frac{r\dot{r}}{\sqrt{r^2 - r_0^2}} = V \quad \Longrightarrow \quad \sqrt{r^2 - r_0^2} = Vt \quad \Longrightarrow \quad r = \sqrt{r_0^2 + (Vt)^2}, \tag{2.138}$$

where the constant of integration is zero, because $r = r_0$ when $t = 0$. Using this value for r in the $\dot{\theta} = D/r^2 \equiv Vr_0/r^2$ result from eq. (2.135) gives

$$\int d\theta = \int \frac{Vr_0\,dt}{r_0^2 + (Vt)^2} \quad \Longrightarrow \quad \theta = \tan^{-1}\left(\frac{Vt}{r_0}\right) \quad \Longrightarrow \quad \cos\theta = \frac{r_0}{\sqrt{r_0^2 + (Vt)^2}}. \tag{2.139}$$

Combining this with the result for r from eq. (2.137) finally gives $\cos\theta = r_0/r$, as desired.

[12] The statement that $r^2\dot{\theta}$ is a constant is simply the statement of conservation of angular momentum, because $r^2\dot{\theta} = r(r\dot{\theta}) = rv_\theta$. Much more on this in Chapters 6 and 7.

19. **A force** $F_\theta = \dot{r}\dot{\theta}$

With the given force, eqs. (2.49) yield

$$\begin{aligned} 0 &= m(\ddot{r} - r\dot{\theta}^2), \\ m\dot{r}\dot{\theta} &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}). \end{aligned} \tag{2.140}$$

The second of these equations gives $-\dot{r}\dot{\theta} = r\ddot{\theta}$. Therefore,

$$-\int \frac{\dot{r}}{r} = \int \frac{\ddot{\theta}}{\dot{\theta}} \quad \Longrightarrow \quad -\ln r = \ln \dot{\theta} + C \quad \Longrightarrow \quad \dot{\theta} = \frac{D}{r}, \tag{2.141}$$

where $D = e^{-C}$ is a constant of integration, determined by the initial conditions. Substituting this value of $\dot{\theta}$ into the first of eqs. (2.140), and then multiplying both sides by $\dot{r}$ and integrating, gives

$$\ddot{r} = r\left(\frac{D}{r}\right)^2 \quad \Longrightarrow \quad \int \ddot{r}\dot{r} = D^2 \int \frac{\dot{r}}{r} \quad \Longrightarrow \quad \frac{\dot{r}^2}{2} = D^2 \ln r + E. \tag{2.142}$$

Therefore,

$$\dot{r} = \sqrt{A \ln r + B}\,, \tag{2.143}$$

where $A = 2D^2$ and $B = 2E$.

Chapter 3

Oscillations

In this chapter we will discuss oscillatory motion. The simplest examples of such a motion are a swinging pendulum and a mass attached to the end of a spring, but it is possible to make a system more complicated by introducing a damping force and/or an external driving force. We will study all of these cases.

We are interested in oscillatory motion for two reasons. First, we study it because we *can* study it. This is one of the few systems in physics where we can solve for the motion exactly. (There's nothing wrong with looking under the lamppost every now and then.) Second, such systems are ubiquitous in physics, for reasons that will become clear in Section 4.2. If there was ever a type of physical system worthy of study, this is it.

We'll jump right into some math at the beginning of this chapter. Then we'll show how the math is applied to the physics.

3.1 Linear differential equations

A *linear differential equation* is one in which x and its time derivatives enter only through their first powers. An example is $3\ddot{x}+7\dot{x}+x=0$. An example of a nonlinear differential equation is $3\ddot{x}+7\dot{x}^2+x=0$.

If the right-hand side of the equation is zero, then we use the term *homogeneous* differential equation. If the right-hand side is some function of t (as in the case of $3\ddot{x}-4\dot{x}=9t^2-5$), then we use the term *inhomogeneous* differential equation. The goal of this chapter will be to learn how to solve these two types of equations. Linear differential equations come up again and again in physics, so we had better find a systematic method of solving them.

The techniques that we will need are best learned through examples, so let's solve a few differential equations, starting with some simple ones. Throughout this chapter, x will be understood to be a function of t. Hence, a dot will denote time differentiation.

Example 1 ($\dot{x}=ax$): This is a very simple differential equation. There are (at least) two ways to solve it.

First method: Separate variables to obtain $dx/x = a\,dt$, and then integrate to obtain $\ln x = at + c$. Exponentiate to obtain

$$x = Ae^{at}, \tag{3.1}$$

where $A \equiv e^c$ is a constant factor. A is determined by the value of x at, say, $t = 0$.

Second method: Guess an exponential solution, that is, one of the form $x = Ae^{\alpha t}$. Substitution then immediately gives $\alpha = a$. Hence, the solution is $x = Ae^{at}$. Note that we can't solve for A, due to the fact that the equation is homogeneous and linear in x. (Translation: the A cancels out.) A is determined from the initial condition.

This method may seem a bit silly. And somewhat cheap. But as we will see below, guessing these exponential functions (or sums of them) is actually the most general thing we can try, so the method is indeed quite general.

REMARK: Using this method, you may be concerned that although you have found one solution to the equation, you might have missed another one. But the general theory of differential equations says that a first-order linear equation has only one independent solution (we'll just accept this fact here). So if you find one solution, you know that you've found the whole thing. ♣

Example 2 ($\ddot{x} = ax$): If a is negative, then this equation describes the oscillatory motion of, say, a spring (about which we'll have much more to say later). If a is positive, then it describes exponentially growing or decaying motion. There are (at least) three ways to solve this equation.

First method: You can use the separation-of-variables method of Section 2.3 here, because our system is one where the force depends on only the position x. But this method is rather cumbersome (as you found if you did Exercise 2.5). It will certainly work, but in the case where our equation is a *linear* function of x, there is a much simpler method.

Second method: As in the first example above, guess a solution of the form $x(t) = Ae^{\alpha t}$, and then find out what α must be. (Again, we can't solve for A, since it cancels out.) Plugging $Ae^{\alpha t}$ into $\ddot{x} = ax$ gives $\alpha = \pm\sqrt{a}$. We have therefore found two solutions. The most general solution is an arbitrary linear combination of these,

$$x(t) = Ae^{\sqrt{a}\,t} + Be^{-\sqrt{a}\,t}. \tag{3.2}$$

A and B are determined from the initial conditions.

VERY IMPORTANT REMARK: The fact that the sum of two different solutions is again a solution to our equation (as you should check right now) is a monumentally important property of *linear* differential equations. This property does *not* hold for nonlinear differential equations, for example $\ddot{x}^2 = x$, because the act of squaring after adding the two solutions produces a cross-term which destroys the equality (as you should again check right now).

This property is called the *principle of superposition*. That is, superimposing two solutions yields another solution. This quality makes theories in physics that are governed by linear equations *much* easier to deal with than those that are governed by nonlinear ones. General Relativity, for example, is permeated with nonlinear equations, and solutions to most General Relativity systems are extremely difficult to come by.

For equations with one main condition
(Those linear), we give you permission
To take your solutions,
With firm resolutions,
And add them in superposition. ♣

Let's say a little more about the solution in eq. (3.2).

If a is negative, then let's define $a \equiv -\omega^2$, where ω is a real number. The solution now becomes $x(t) = Ae^{i\omega t} + Be^{-i\omega t}$. Using $e^{i\theta} = \cos\theta + i\sin\theta$, this can be written in terms of trig functions, if desired. Various ways of writing the solution are:

$$\begin{aligned} x(t) &= Ae^{i\omega t} + Be^{-i\omega t} \\ x(t) &= C\cos\omega t + D\sin\omega t, \\ x(t) &= E\cos(\omega t + \phi_1), \\ x(t) &= F\sin(\omega t + \phi_2). \end{aligned} \tag{3.3}$$

The various constants are related to each other; for example, $C = E\cos\phi_1$, and $D = -E\sin\phi_1$. Note that there are two free parameters in each of the above expressions for $x(t)$. These parameters are determined by the initial conditions (say, the position and speed at $t = 0$). Depending on the specifics of a given problem, one of the above forms will work better than the others.

If a is positive, then let's define $a \equiv \omega^2$, where ω is a real number. The solution now becomes $x(t) = Ae^{\omega t} + Be^{-\omega t}$. Using $e^{\theta} = \cosh\theta + \sinh\theta$, this can be written in terms of hyperbolic trig functions, if desired. Various ways of writing the solution are:

$$\begin{aligned} x(t) &= Ae^{\omega t} + Be^{-\omega t} \\ x(t) &= C\cosh\omega t + D\sinh\omega t, \\ x(t) &= E\cosh(\omega t + \phi_1), \\ x(t) &= F\sinh(\omega t + \phi_2). \end{aligned} \tag{3.4}$$

Again, the various constants are related to each other. If you're unfamiliar with the hyperbolic trig functions, a few facts are listed in Appendix A.

Remarks: Although the solution in eq. (3.2) is completely correct for both signs of a, it is generally more illuminating to write the negative-a solutions in either the trig form or the $e^{\pm i\omega t}$ exponential form where the i's are explicit.

Again, you may be concerned that although you have found two solutions to the equation, you might have missed others. But the general theory of differential equations says that our second-order linear equation has only two independent solutions. Therefore, having found two independent solutions, we know we've found them all. ♣

The usefulness of this method of guessing exponential solutions cannot be overemphasized. It may seem somewhat restrictive, but it works. The examples in the remainder of this chapter should convince you of this.

This is our method, essential,
For equations we solve, differential.
It gets the job done,
And it's even quite fun.
We just try a routine exponential.

Example 3 ($\ddot{x}+2\gamma\dot{x}+ax=0$): This will be our last mathematical example, then we'll get into some physics. As we will see later, this example pertains to a damped harmonic oscillator. We have put a factor of 2 in the coefficient of $\dot{x}$ here in order to make some later formulas look nicer.

Note that the force in this example, which is $-2\gamma\dot{x}-ax$ (times m), depends on both v and x, so our methods of Section 2.3 don't apply. This leaves us with only our method of guessing an exponential solution, $Ae^{\alpha t}$. Plugging this into the given equation, and cancelling the nonzero factor of $Ae^{\alpha t}$, yields

$$\alpha^2 + 2\gamma\alpha + a = 0. \tag{3.5}$$

The solutions for α are

$$-\gamma \pm \sqrt{\gamma^2 - a}. \tag{3.6}$$

Call these α_1 and α_2. Then the general solution to our equation is

$$\begin{aligned} x(t) &= Ae^{\alpha_1 t} + Be^{\alpha_2 t} \\ &= e^{-\gamma t}\left(Ae^{t\sqrt{\gamma^2-a}} + Be^{-t\sqrt{\gamma^2-a}}\right). \end{aligned} \tag{3.7}$$

(Well, our method of trying $Ae^{\alpha t}$ doesn't look so trivial anymore...)

If $\gamma^2 - a < 0$, then we can write our answer in terms of sines and cosines, so we have oscillatory motion which decreases in time due to the $e^{-\gamma t}$ factor (or it increases, if $\gamma < 0$, but this is rarely physical). If $\gamma^2 - a > 0$, then we have exponential motion. More on these different possibilities in Section 3.2.2.

In general, if we have a homogeneous linear differential equation of the form

$$\frac{d^n x}{dt^n} + c_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + c_1\frac{dx}{dt} + c_0 x = 0, \tag{3.8}$$

then our strategy is to guess an exponential solution, $x(t) = Ae^{\alpha t}$, and then (in theory) solve the resulting nth order equation (namely $\alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_1\alpha + c_0 = 0$), for α, to obtain the solutions $\alpha_1, \ldots, \alpha_n$. The general solution for $x(t)$ is then

$$x(t) = A_1e^{\alpha_1 t} + A_2e^{\alpha_2 t} + \cdots + A_ne^{\alpha_n t}, \tag{3.9}$$

where the A_i are determined by the initial conditions. In practice, however, we will rarely encounter differential equations of degree higher than 2. (Note: if some of the α_i happen to be equal, then eq. (3.9) is not valid. We will encounter such a situation in Section 3.2.2.)

3.2 Oscillatory motion

3.2.1 Simple harmonic motion

Let's now do some real live physical problems. We'll start with simple harmonic motion. This is the motion undergone by a particle subject to a force $F(x) = -kx$.

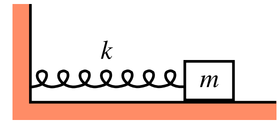


Figure 3.1

The classic system that undergoes simple harmonic motion is a mass attached to a spring (see Fig. 3.1). A typical spring has a force of the form $F(x) = -kx$, where x is the displacement from equilibrium. (This is "Hooke's law", and it holds as long as the spring isn't stretched too far; eventually this expression breaks down for any real spring.) Hence, $F = ma$ gives $-kx = m\ddot{x}$, or

$$\ddot{x} + \omega^2 x = 0, \qquad \text{where} \ \ \omega \equiv \sqrt{\frac{k}{m}}\,. \tag{3.10}$$

From Example 2 in the previous section, the solution to this may be written as in eq. (3.3),

$$x(t) = A\cos(\omega t + \phi). \tag{3.11}$$

The system therefore oscillates back and forth forever in time.

Remark: The constants A and ϕ are determined by the initial conditions. If, for example, $x(0) = 0$ and $\dot{x}(0) = v$, then we must have $0 = A\cos\phi$ and $v = -A\omega\sin\phi$. Hence, $\phi = \pi/2$, and $A = -v/\omega$. Therefore, the solution is $x(t) = -(v/\omega)\cos(\omega t + \pi/2)$. This looks a little nicer as $x(t) = (v/\omega)\sin(\omega t)$. So, given these initial conditions, we may have arrived at this result a little quicker if we had chosen the "sin" solution in eq. (3.3). ♣

Example (Simple pendulum): Another classic system that undergoes (approximate) simple harmonic motion is the simple pendulum, that is, a mass that hangs on a massless string and swings in a vertical plane.

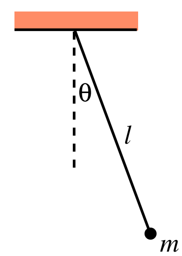


Figure 3.2

Let ℓ be the length of the string. Let θ be the angle the string makes with the vertical (see Fig. 3.2). Then the gravitational force on the mass in the tangential direction is $-mg\sin\theta$. So $F = ma$ in the tangential direction gives

$$-mg\sin\theta = m(\ell\ddot{\theta}) \tag{3.12}$$

(The tension in the string exactly cancels the radial component of gravity, so the radial $F = ma$ serves only to tell us the tension, which is of no use to us here.) We will now enter the realm of approximations and assume that the amplitude of the oscillations is small. (Without this approximation, the problem cannot be solved exactly.) This allows us to write $\sin\theta \approx \theta$, which gives

$$\ddot{\theta} + \omega^2\theta = 0, \qquad \text{where} \ \ \omega \equiv \sqrt{\frac{g}{\ell}}\,. \tag{3.13}$$

Therefore,

$$\theta(t) = A\cos(\omega t + \phi), \tag{3.14}$$

where A and ϕ are determined from the initial conditions.

The true motion is arbitrarily close to this, for sufficiently small amplitudes. Exercise 1 deals with the higher-order corrections to the motion in the case where the amplitude is not small.

3.2.2 Damped harmonic motion

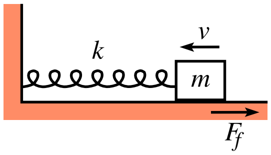


Figure 3.3

Consider a mass m attached to the end of a spring which has a spring constant k. Let the mass be subject to a drag force proportional to its velocity, $F_f = -bv$ (see Fig. 3.3). What is the position as a function of time?

The force on the mass is $F = -b\dot{x} - kx$. So $F = m\ddot{x}$ gives

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0, \tag{3.15}$$

where $2\gamma \equiv b/m$, and $\omega \equiv \sqrt{k/m}$. But this is exactly the equation we solved in Example 3 in the previous section (with $a \to \omega^2$). Now, however, we have the physical restrictions that $\gamma > 0$ and $\omega^2 > 0$. Letting $\Omega^2 \equiv \gamma^2 - \omega^2$, for simplicity, we may write the solution in eq. (3.7) as

$$x(t) = e^{-\gamma t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right), \qquad \text{where} \;\; \Omega^2 \equiv \gamma^2 - \omega^2. \tag{3.16}$$

There are three cases to consider.

Case 1: Underdamping ($\Omega^2 < 0$)

In this case, $\omega > \gamma$. Since Ω is imaginary, let us define $\Omega \equiv i\tilde{\omega}$ (so $\tilde{\omega} = \sqrt{\omega^2 - \gamma^2}$). Eq. (3.16) then gives

$$\begin{aligned} x(t) &= e^{-\gamma t}\left(Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}\right) \\ &\equiv e^{-\gamma t} C\cos(\tilde{\omega}t + \phi). \end{aligned} \tag{3.17}$$

These two forms are equivalent. Depending on the circumstances of the problem, one form works better than the other. (Or perhaps one of the other forms in eq. (3.3) will be the most useful one, to be multiplied by the $e^{-\gamma t}$ factor.) The constants are related by $A + B = C\cos\phi$ and $A - B = iC\sin\phi$. In a physical problem, $x(t)$ is real, so we must have $A^* = B$ (where the star denotes complex conjugation). The two constants A and B, or C and ϕ, are determined from the initial conditions.

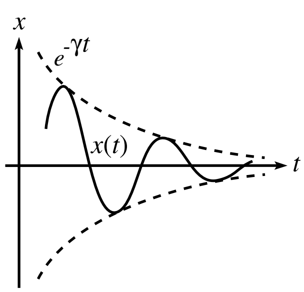


Figure 3.4

The cosine form makes it apparent that the motion is harmonic motion whose amplitude decreases in time because of the $e^{-\gamma t}$ factor. A plot of such motion is shown in Fig. 3.4. Note that the frequency of the motion, $\tilde{\omega} = \sqrt{\omega^2 - \gamma^2}$, is less than the natural frequency, ω, of the undamped oscillator.

REMARKS: If γ is very small, then $\tilde{\omega} \approx \omega$, which makes sense, because we almost have an undamped oscillator. If γ is very close to ω, then $\tilde{\omega} \approx 0$. So the oscillations are very slow. Of course, for very small $\tilde{\omega}$ it's hard to even tell that the oscillations exist, since they will damp out on a time scale of order $1/\gamma$, which will be short compared to the long time scale of the oscillations, $1/\tilde{\omega}$. ♣

Case 2: Overdamping ($\Omega^2 > 0$)

In this case, $\omega < \gamma$. Ω is real (and taken to be positive), so eq. (3.16) gives

$$x(t) = Ae^{-(\gamma-\Omega)t} + Be^{-(\gamma+\Omega)t}. \tag{3.18}$$

There is no oscillatory motion in this case (see Fig. 3.5). Note that $\gamma > \Omega \equiv \sqrt{\gamma^2 - \omega^2}$, so both of the exponents are negative. The motion therefore goes to zero for large t. (This had better be the case. A real spring is not going to have the motion go off to infinity. If we had obtained a positive exponent somehow, we'd know we had made a mistake.)

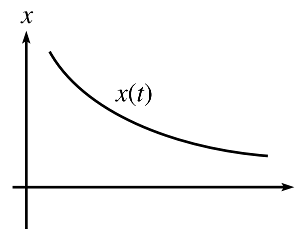


Figure 3.5

Remarks: If γ is just slightly larger than ω, then $\Omega \approx 0$, so the two terms in (3.18) are roughly equal, and we essentially have exponential decay, according to $e^{-\gamma t}$. If $\gamma \gg \omega$ (that is, strong damping), then $\Omega \approx \gamma$, so the first term in (3.18) dominates, and we essentially have exponential decay according to $e^{-(\gamma-\Omega)t}$. We can be somewhat quantitative about this by approximating Ω as $\Omega \equiv \sqrt{\gamma^2 - \omega^2} = \gamma\sqrt{1 - \omega^2/\gamma^2} \approx \gamma(1 - \omega^2/2\gamma^2)$. Therefore, the exponential behavior goes like $e^{-\omega^2 t/2\gamma}$. This is slow decay (that it, slow compared to $t \sim 1/\omega$), which makes sense if the damping is very strong. The mass ever so slowly creeps back to the origin, as for a weak spring immersed in molasses. ♣

Case 3: Critical damping ($\Omega^2 = 0$)

In this case, $\gamma = \omega$. Eq. (3.15) therefore becomes $\ddot{x} + 2\gamma\dot{x} + \gamma^2 x = 0$. In this special case, we have to be careful in solving our differential equation. The solution in eq. (3.16) is not valid, because in the procedure leading to eq. (3.7), the roots α_1 and α_2 are equal (to $-\gamma$), so we have really found only one solution, $e^{-\gamma t}$. We'll just invoke here the result from the theory of differential equations which says that in this special case, the other solution is of the form $te^{-\gamma t}$.

Remark: You should check explicitly that $te^{-\gamma t}$ solves the equation $\ddot{x}+2\gamma\dot{x}+\gamma^2 x = 0$. Or if you want to, you can derive it in the spirit of Problem 1. In the more general case where there are n identical roots in the procedure leading to eq. (3.9) (call them all α), the n independent solutions to the differential equation are $t^k e^{\alpha t}$, for $0 \le k \le (n-1)$. But more often than not, there are no repeated roots, so you don't have to worry about this. ♣

Our solution is therefore of the form

$$x(t) = e^{-\gamma t}(A + Bt). \tag{3.19}$$

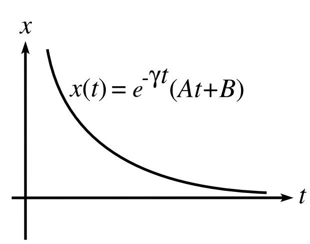


Figure 3.6

The exponential factor eventually wins out over the Bt term, of course, so the motion goes to zero for large t (see Fig. 3.6).

If we are given a spring with a fixed ω, and if we look at the system at different values of γ, then critical damping (when $\gamma = \omega$) is the case where the motion converges to zero in the quickest way (which is like $e^{-\omega t}$). This is true because in the underdamped case ($\gamma < \omega$), the envelope of the oscillatory motion goes like $e^{-\gamma t}$, which goes to zero slower than $e^{-\omega t}$, since $\gamma < \omega$. And in the overdamped case ($\gamma > \omega$), the dominant piece is the $e^{-(\gamma-\Omega)t}$ term. And as you can verify, if $\gamma > \omega$ then $\gamma - \Omega \equiv \gamma - \sqrt{\gamma^2 - \omega^2} < \omega$, so this motion also goes to zero slower than $e^{-\omega t}$.

Critical damping is very important in many real systems, such as screen doors and automobile shock absorbers, where the goal is to have the system head to zero (without overshooting and bouncing around) as fast as possible.

3.2.3 Driven (and damped) harmonic motion

Mathematical prelude

Before we examine driven harmonic motion, we have to learn how to solve a new type of differential equation. How can we solve something of the form

$$\ddot{x} + 2\gamma\dot{x} + ax = C_0 e^{i\omega_0 t}, \tag{3.20}$$

where γ, a, and ω_0 are given quantities? This is an inhomogeneous differential equation, due to the term on the right-hand side. It's not very physical, since the right-hand side is complex, but we're doing math now. Equations of this sort will come up again and again, and fortunately there is a nice, easy (although sometimes messy) method for solving them. As usual, the method is to make a reasonable guess, plug it in, and see what condition comes out.

Since we have the $e^{i\omega_0 t}$ sitting on the right side, let's try a solution of the form $x(t) = Ae^{i\omega_0 t}$. (A will depend on ω_0, among other things, as we will see.) Plugging this into eq. (3.20), and cancelling the non-zero factor of $e^{i\omega_0 t}$, we obtain

$$(-\omega_0^2)A + 2\gamma(i\omega_0)A + aA = C_0. \tag{3.21}$$

Solving for A, we find our solution for x to be

$$x(t) = \left(\frac{C_0}{-\omega_0^2 + 2i\gamma\omega_0 + a}\right) e^{i\omega_0 t}. \tag{3.22}$$

Note the differences between this technique and the one in Example 3 in Section 3.1. In that example, the goal was to determine what the α in $x(t) = Ae^{\alpha t}$ had to be. And there was no way to solve for A; the initial conditions determined A. But in the present technique, the ω_0 in $x(t) = Ae^{i\omega_0 t}$ is a given quantity, and the goal is to solve for A in terms of the given constants. Therefore, in the solution in eq. (3.22), there are *no free constants* to be determined by initial conditions. We've found one particular solution, and we're stuck with it. (The term *particular solution* is what people use for eq. (3.22).)

With no freedom to adjust the solution in eq. (3.22), how can we satisfy an arbitrary set of initial conditions? Fortunately, eq. (3.22) does not represent the most general solution to eq. (3.20). The most general solution is the sum of our particular solution in eq. (3.22), *plus* the "homogeneous" solution we found in eq. (3.7). This is obvious, because the solution in eq. (3.7) was explicitly constructed to yield zero when plugged into the left-hand side of eq. (3.20). Therefore, tacking it onto our particular solution won't change the equality in eq. (3.20), because the left side is linear. The principle of superposition has saved the day.

The complete solution to eq. (3.20) is therefore

$$x(t) = e^{-\gamma t}\left(Ae^{t\sqrt{\gamma^2-a}} + Be^{-t\sqrt{\gamma^2-a}}\right) + \left(\frac{C_0}{-\omega_0^2 + 2i\gamma\omega_0 + a}\right) e^{i\omega_0 t}, \tag{3.23}$$

where A and B are determined by the initial conditions.

With superposition in mind, it is clear what the strategy should be if we have a slightly more general equation to solve, for example,

$$\ddot{x} + 2\gamma\dot{x} + ax = C_1 e^{i\omega_1 t} + C_2 e^{i\omega_2 t}. \tag{3.24}$$

Simply solve the equation with only the first term on the right. Then solve the equation with only the second term on the right. Then add the two solutions. And then add on the homogeneous solution from eq. (3.7). We are able to apply the principle of superposition because the left-hand side of our equation is linear.

Finally, let's look at the case where we have many such terms on the right-hand side, for example,

$$\ddot{x} + 2\gamma\dot{x} + ax = \sum_{n=1}^{N} C_n e^{i\omega_n t}. \tag{3.25}$$

We simply have to solve N different equations, each with just one of the N terms on the right-hand side. Then add up all the solutions, then add on the homogeneous solution from eq. (3.7). If N is infinite, that's fine. You'll just have to add up an infinite number of solutions. This is the principle of superposition at its best.

REMARK: The previous paragraph, combined with a basic result from Fourier analysis, allows us to solve (in principle) any equation of the form

$$\ddot{x} + 2\gamma\dot{x} + ax = f(t). \tag{3.26}$$

Fourier analysis says that any (nice enough) function, $f(t)$, may be decomposed into its Fourier components,

$$f(t) = \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega. \tag{3.27}$$

In this continuous sum, the functions $g(\omega)$ take the place of the coefficients C_n in eq. (3.25). So, if $S_\omega(t)$ is the solution for $x(t)$ when there is only the term $e^{i\omega t}$ on the right-hand side of eq. (3.26) (that is, $S_\omega(t)$ is the solution given in eq. 3.22)), then the complete particular solution to (3.26) is

$$x(t) = \int_{-\infty}^{\infty} g(\omega) S_\omega(t)\, d\omega. \tag{3.28}$$

Finding the coefficients $g(\omega)$ is the hard part (or, rather, the messy part), but we won't get into that here. We won't do anything with Fourier analysis in this book, but we just wanted to let you know that it *is* possible to solve (3.26) for any function $f(t)$. Most of the functions we'll consider will be nice functions like $\cos\omega_0 t$, for which the Fourier decomposition is simply the finite sum, $\cos\omega_0 t = \frac{1}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t})$. ♣

Let's now do a physical example.

Example (Driven spring): Consider a spring with spring constant k. A mass m at the end of the spring is subject to a friction force proportional to its velocity, $F_f = -bv$. The mass is also subject to a driving force, $F_d(t) = F_d \cos\omega_d t$ (see Fig. 3.7). What is its position as a function of time?

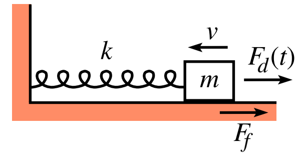


Figure 3.7

Solution: The force on the mass is $F(x, \dot{x}, t) = -b\dot{x} - kx + F_d \cos\omega_d t$. So $F = ma$ gives

$$\begin{aligned}\ddot{x} + 2\gamma\dot{x} + \omega^2 x &= F\cos\omega_d t \\ &= \frac{F}{2}\left(e^{i\omega_d t} + e^{-i\omega_d t}\right).\end{aligned} \tag{3.29}$$

where $2\gamma \equiv b/m$, $\omega \equiv \sqrt{k/m}$, and $F \equiv F_d/m$. Using eq. (3.22) and the technique of adding solutions mentioned after eq. (3.24), our particular solution is

$$x_p(t) = \left(\frac{F/2}{-\omega_d^2 + 2i\gamma\omega_d + \omega^2}\right)e^{i\omega_d t} + \left(\frac{F/2}{-\omega_d^2 - 2i\gamma\omega_d + \omega^2}\right)e^{-i\omega_d t}. \tag{3.30}$$

The complete solution is the sum of this particular solution and the homogeneous solution from eq. (3.16).

Let's simplify eq. (3.30) a bit. Getting the i's out of the denominators, and turning the exponentials into sines and cosines, we find (after a little work)

$$x_p(t) = \left(\frac{F(\omega^2 - \omega_d^2)}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}\right)\cos\omega_d t + \left(\frac{2F\gamma\omega_d}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}\right)\sin\omega_d t. \tag{3.31}$$

Note that this is real, as it must be, if it is to describe the position of a particle.

Remarks: If you wish, you can solve eq. (3.29) simply by taking the real part of the solution to eq. (3.20), that is, the $x(t)$ in eq. (3.22). This is true because if we take the real part of eq. (3.20), we obtain

$$\frac{d^2}{dt^2}\big(\mathrm{Re}(x)\big) + 2\gamma\frac{d}{dt}\big(\mathrm{Re}(x)\big) + a\big(\mathrm{Re}(x)\big) = \mathrm{Re}\left(C_0 e^{i\omega_0 t}\right) = C_0\cos(\omega_0 t) \tag{3.32}$$

In other words, if x satisfies eq. (3.20) with a $C_0 e^{i\omega_0 t}$ on the right side, then $\mathrm{Re}(x)$ satisfies it with a $C_0\cos(\omega_0 t)$ on the right.

At any rate, it is clear that (with $C_0 = F$) the real part of eq. (3.22) does indeed give the result in eq. (3.31), because in eq. (3.30) we just took half of a quantity plus its complex conjugate, which is the real part.

If you don't like using complex numbers, another way of solving eq. (3.29) is to keep it in the form with the $\cos\omega_d t$ on the right, and then simply guess a solution of the form $A\cos\omega_d t + B\sin\omega_d t$, and solve for A and B. (This is the task of Problem 5.) The result will be eq. (3.31). ♣

We can simplify eq. (3.31) a bit further. If we define

$$R \equiv \sqrt{(\omega^2 - \omega_d^2)^2 + (2\gamma\omega_d)^2}, \tag{3.33}$$

then we may rewrite eq. (3.31) as

$$\begin{aligned}x_p(t) &= \frac{F}{R}\left(\frac{(\omega^2 - \omega_d^2)}{R}\cos\omega_d t + \frac{2\gamma\omega_d}{R}\sin\omega_d t\right) \\ &\equiv \frac{F}{R}\cos(\omega_d t - \phi),\end{aligned} \tag{3.34}$$

where ϕ is defined by

$$\cos\phi = \frac{\omega^2 - \omega_d^2}{R}, \quad \sin\phi = \frac{2\gamma\omega_d}{R} \qquad \Longrightarrow \qquad \tan\phi = \frac{2\gamma\omega_d}{\omega^2 - \omega_d^2}. \tag{3.35}$$

(Note that $0 < \phi < \pi$, since $\sin\phi$ is positive.)

Recalling the homogeneous solution in eq. (3.16), we may write the complete solution to eq. (3.29) as

$$x(t) = \frac{F}{R}\cos(\omega_d t - \phi) + e^{-\gamma t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right). \tag{3.36}$$

The constants A and B are determined by the initial conditions. Note that if there is any damping at all in the system (that is, $\gamma > 0$), then the homogeneous part of the solution goes to zero for large t, and we are left with only the particular solution. In other words, the system approaches a definite $x(t)$, namely $x_p(t)$, independent of the initial conditions.

REMARK: The amplitude of the solution in eq. (3.34) is proportional to $1/R = [(\omega^2 - \omega_d^2)^2 + (2\gamma\omega_d)^2]^{-1/2}$. Given ω_d and γ, this is maximum when $\omega = \omega_d$. Given ω and γ, it is maximum when $\omega_d = \sqrt{\omega^2 - 2\gamma^2}$ (as you can show); in the case of weak damping (that is, $\gamma \ll \omega$), the maximum is achieved when $\omega_d \approx \omega$. The term *resonance* is used to describe this situation where the amplitude of oscillations is as large as possible. Note that, using eq. (3.35), the phase angle ϕ equals $\pi/2$ when $\omega_d \approx \omega$. Hence, the motion of the particle lags the driving force by a quarter of a cycle at resonance. ♣

3.3 Coupled oscillators

Mathematical prelude

In the previous sections, we have dealt with only one function of time, $x(t)$. What if we have two functions of time, say $x(t)$ and $y(t)$, which are related by a pair of "coupled" differential equations? For example,

$$\begin{aligned} 2\ddot{x} + \omega^2(5x - 3y) &= 0, \\ 2\ddot{y} + \omega^2(5y - 3x) &= 0. \end{aligned} \tag{3.37}$$

We'll assume $\omega^2 > 0$ here, but this isn't necessary. We call these equations "coupled" because there are x's and y's in both of them, and it is not immediately obvious how to separate them to solve for x and y. There are (at least) two methods of solving these equations.

First method: Sometimes it is easy, as in this case, to find certain linear combinations of the given equations for which nice things happen. Taking the sum, we find

$$(\ddot{x} + \ddot{y}) + \omega^2(x + y) = 0. \tag{3.38}$$

This equation involves x and y only in the combination of their sum, $x + y$. With $z \equiv x + y$, it is just our old friend, $\ddot{z} + \omega^2 z = 0$. The solution is

$$x + y = A_1\cos(\omega t + \phi_1), \tag{3.39}$$

where A_1 and ϕ_1 are determined by initial conditions. (It could also be written as a sum of exponentials, or the sum of a sine and cosine, of course.) We may also take the difference, to find

$$(\ddot{x} - \ddot{y}) + 4\omega^2(x - y) = 0. \tag{3.40}$$

This equation involves x and y only in the combination of their difference, $x - y$. The solution is

$$x - y = A_2 \cos(2\omega t + \phi_2), \tag{3.41}$$

Taking the sum and difference of eqs. (3.39) and (3.41), we find

$$\begin{aligned} x(t) &= B_1 \cos(\omega t + \phi_1) + B_2 \cos(2\omega t + \phi_2), \\ y(t) &= B_1 \cos(\omega t + \phi_1) - B_2 \cos(2\omega t + \phi_2), \end{aligned} \tag{3.42}$$

where the B_i's are half of the A_i's.

The strategy of this solution was simply to fiddle around and try to form differential equations that involve only one combination of the variables, such as eqs. (3.38) and (3.40). Then you just label this combination with a new letter, "z", if you wish, and write down the obvious solution for z, as in eqs. (3.39) and (3.41).

We've managed to solve our equations for x and y. However, the more interesting thing that we've done is produce the equations (3.39) and (3.41). The combinations $(x + y)$ and $(x - y)$ are called the *normal coordinates* of the system. These are the combinations that oscillate with one pure frequency. The motion of x and y will, in general, look rather complicated, and it may be difficult to tell that the motion is really made up of just the two frequencies in eq. (3.42). But if you plot the values of $(x + y)$ and $(x - y)$ as time goes by, for *any* motion of the system, then you will find nice sinusoidal graphs, even if x and y are each behaving in a rather unpleasant manner.

Second method: In the event that it is not easy to guess which linear combinations of eqs. (3.37) will yield equations involving just one combination of x and y (the $x + y$ and $x - y$ above), there is a fail-proof method for solving for x and y. In the spirit of Section 3.1, let us try a solution of the form $x = Ae^{i\alpha t}$ and $y = Be^{i\alpha t}$, which we will write (for convenience) as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\alpha t}. \tag{3.43}$$

It is not obvious that there should exist solutions for x and y which have the same t dependence, but let's try it and see what happens. Note that we've explicitly put the i in the exponent, but there's no loss of generality here. If α happens to be imaginary, then the exponent is real. It's personal preference whether or not you put the i in.

Plugging our guess into eqs. (3.37), and dividing through by $e^{i\omega t}$, we find

$$\begin{aligned} 2A(-\alpha^2) + 5A\omega^2 - 3B\omega^2 &= 0, \\ 2B(-\alpha^2) + 5B\omega^2 - 3A\omega^2 &= 0, \end{aligned} \tag{3.44}$$

or equivalently, in matrix form,

$$\begin{pmatrix} -2\alpha^2 + 5\omega^2 & -3\omega^2 \\ -3\omega^2 & -2\alpha^2 + 5\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.45}$$

This homogeneous equation for A and B has a nontrivial solution (that is, one where A and B aren't both 0) only if the matrix is *not* invertible. This is true because if it were invertible, then we could just multiply through by the inverse to obtain $(A, B) = (0, 0)$.

When is a matrix invertible? There is a straightforward (although tedious) method for finding the inverse of a matrix. It involves taking cofactors, taking a transpose, and dividing by the determinant. The step that concerns us here is the division by the determinant. The inverse will exist if and only if the determinant is not zero. So we see that eq. (3.45) has a nontrivial solution only if the determinant is zero. Since we seek a nontrivial solution, we must demand that

$$\begin{aligned} 0 &= \begin{vmatrix} -2\alpha^2 + 5\omega^2 & -3\omega^2 \\ -3\omega^2 & -2\alpha^2 + 5\omega^2 \end{vmatrix} \\ &= 4\alpha^4 - 20\alpha^2\omega^2 + 16\omega^4. \end{aligned} \tag{3.46}$$

The roots of this equation are $\alpha = \pm\omega$ and $\alpha = \pm 2\omega$. We have therefore found four types of solutions. If $\alpha = \pm\omega$, then we can plug this back into eq. (3.45) to obtain $A = B$. (Both equations give this same result. This was essentially the point of setting the determinant equal to 0.) If $\alpha = \pm 2\omega$, then eq. (3.45) gives $A = -B$. (Again, the equations are redundant.) Note that we cannot solve specifically for A and B, but only for their ratio. Adding up our four solutions, we see that x and y take the general form (written in vector form for the sake of simplicity and bookkeeping),

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega t} + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega t} \\ &+ A_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2i\omega t} + A_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2i\omega t}. \end{aligned} \tag{3.47}$$

The four A_i are determined from the initial conditions.

We can rewrite eq. (3.47) in a somewhat cleaner form. If the coordinates x and y describe the positions of particles, they must be real. Therefore, A_1 and A_2 must be complex conjugates, and likewise for A_3 and A_4. If we then define some ϕ's and B's via $A_2^* = A_1 \equiv (B_1/2)e^{i\phi_1}$ and $A_4^* = A_3 \equiv (B_2/2)e^{i\phi_2}$, we may rewrite our solution in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = B_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega t + \phi_1) + B_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2\omega t + \phi_2), \tag{3.48}$$

where the B_i and ϕ_i are real (and are determined from the initial conditions). We have therefore reproduced the result in eq. (3.42).

It is clear from eq. (3.48) that the combinations $x+y$ and $x-y$ (the normal coordinates) oscillate with the pure frequencies, ω and 2ω, respectively.

It is also clear that if $B_2 = 0$, then $x = y$ at all times, and they both oscillate with frequency ω. And if $B_1 = 0$, then $x = -y$ at all times, and they both oscillate with frequency 2ω. These two pure-frequency motions are called the *normal modes*. They are labeled by the vectors $(1,1)$ and $(1,-1)$. The significance of normal modes will become clear in the following example.

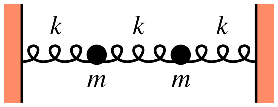


Figure 3.8

Example (Two masses, three springs): Consider two masses, m, connected to each other and to two walls by three springs, as shown in Fig. 3.8. The three springs have the same spring constant k. Find the positions of the masses as functions of time. What are the normal coordinates? What are the normal modes?

Solution: Let $x_1(t)$ and $x_2(t)$ be the positions of the left and right masses, respectively, relative to their equilibrium positions. The middle spring is stretched a distance $x_2 - x_1$. Therefore, the force on the left mass is $-kx_1 + k(x_2 - x_1)$, and the force on the right mass is $-kx_2 - k(x_2 - x_1)$. (It's easy to make a mistake on the sign of the second term in these expressions. You can double check the sign by, say, looking at the force when x_2 is very big.) Therefore, $F = ma$ on each mass gives (with $\omega^2 = k/m$)

$$\begin{aligned} \ddot{x}_1 + 2\omega^2 x_1 - \omega^2 x_2 &= 0, \\ \ddot{x}_2 + 2\omega^2 x_2 - \omega^2 x_1 &= 0. \end{aligned} \tag{3.49}$$

These are rather friendly coupled equations, and we can see that the sum and difference are the useful combinations to take. The sum gives

$$(\ddot{x}_1 + \ddot{x}_2) + \omega^2(x_1 + x_2) = 0, \tag{3.50}$$

and the difference gives

$$(\ddot{x}_1 - \ddot{x}_2) + 3\omega^2(x_1 - x_2) = 0. \tag{3.51}$$

The solutions to these equations are the normal coordinates,

$$\begin{aligned} x_1 + x_2 &= A_+ \cos(\omega t + \phi_+), \\ x_1 - x_2 &= A_- \cos(\sqrt{3}\omega t + \phi_-). \end{aligned} \tag{3.52}$$

Taking the sum and difference of these normal coordinates, we have

$$\begin{aligned} x_1(t) &= B_+ \cos(\omega t + \phi_+) + B_- \cos(\sqrt{3}\omega t + \phi_-), \\ x_2(t) &= B_+ \cos(\omega t + \phi_+) - B_- \cos(\sqrt{3}\omega t + \phi_-), \end{aligned} \tag{3.53}$$

where the B's are half of the A's.

Remark: We may also derive eqs. (3.53) by using the determinant method. Letting $x_1 = Ae^{i\alpha t}$ and $x_2 = Be^{i\alpha t}$, we see that for there to be a nontrivial solution for A and B, we must have

$$\begin{aligned} 0 &= \begin{vmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -\alpha^2 + 2\omega^2 \end{vmatrix} \\ &= \alpha^4 - 4\alpha^2\omega^2 + 3\omega^4. \end{aligned} \tag{3.54}$$

The roots of this equation are $\alpha = \pm\omega$ and $\alpha = \pm\sqrt{3}\omega$. If $\alpha = \pm\omega$, then eq. (3.49) yields $A = B$. If $\alpha = \pm\sqrt{3}\omega$, then eq. (3.49) yields $A = -B$. The solutions for x_1 and x_2 therefore take the general form

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega t} + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega t} \\ &\quad + A_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\sqrt{3}i\omega t} + A_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-\sqrt{3}i\omega t} \\ &\Longrightarrow B_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega t + \phi_+) + B_- \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{3}\omega t + \phi_-). \qquad (3.55)\end{aligned}$$

This is equivalent to eq. (3.53). ♣

The normal modes are obtained by setting either B_- or B_+ equal to zero in eq. (3.53). Therefore, they are $(1, 1)$ and $(1, -1)$. How do we visualize these? The mode $(1, 1)$ oscillates with frequency ω. In this case (where $B_- = 0$), we have $x_1(t) = x_2(t)$, at all times. So the masses simply oscillate back and forth in the same manner, as shown in Fig. 3.9. It is clear that such motion has frequency ω, because as far as the masses are concerned, the middle spring is not there, so each mass moves under the influence of just one spring, and hence with frequency ω.

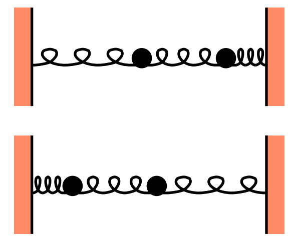


Figure 3.9

The mode $(1, -1)$ oscillates with frequency $\sqrt{3}\omega$. In this case (where $B_+ = 0$), we have $x_1(t) = -x_2(t)$, at all times. So the masses oscillate back and forth with opposite displacements, as shown in Fig. 3.10. It is clear that this mode should have a frequency larger than that of the other mode, because the middle spring is being stretched, so the masses feel a larger force. But it takes a little thought to show that the frequency is $\sqrt{3}\omega$.[1]

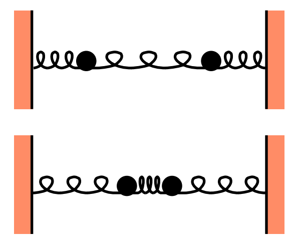


Figure 3.10

Remark: The normal mode $(1, 1)$ above is associated with the normal coordinate $x_1 + x_2$; they both involve the frequency ω. However, this association is *not* due to the fact that the coefficients of both x_1 and x_2 in this normal coordinate are equal to 1. Rather, it is due to the fact that the *other* normal mode (namely $(x_1, x_2) \propto (1, -1)$) gives no contribution to the sum $x_1 + x_2$.

There are a few too many 1's floating around in the above example, so it's hard to see what results are meaningful and what results are coincidence. The following example should clear things up. Let's say we solved a problem using the determinant method, and we found the solution to be

$$\begin{pmatrix} x \\ y \end{pmatrix} = B_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cos(\omega_1 t + \phi_1) + B_2 \begin{pmatrix} 1 \\ -5 \end{pmatrix} \cos(\omega_2 t + \phi_2). \qquad (3.56)$$

Then $5x + y$ is the normal coordinate associated with the normal mode $(3, 2)$, which has frequency ω_1. (This is true because there is no $\cos(\omega_2 t + \phi_2)$ dependence in the combination $5x + y$.) And similarly, $2x - 3y$ is the normal coordinate associated with the normal

[1]If you want to obtain this $\sqrt{3}\omega$ result without going through all of the above work, just note that the center of the middle spring doesn't move. Therefore, it acts like two "half springs", each with spring constant $2k$ (verify this). Hence, each mass is effectively attached to a "k" spring and a "$2k$" spring, yielding a total effective spring constant of $3k$. Thus the $\sqrt{3}$.

mode $(1, -5)$, which has frequency ω_2 (because there is no $\cos(\omega_1 t + \phi_1)$ dependence in the combination $2x - 3y$). ♣

ANOTHER REMARK: Note the difference between the types of differential equations we solved in Section 2.3 of the previous chapter, and the types we solved in this chapter. The former dealt with forces that did not have to be linear in x or $\dot{x}$, but which had to depend on only x, or only $\dot{x}$, or only t. The latter dealt with forces that could depend on all three of these quantities, but which had to be linear in x and $\dot{x}$. ♣

3.4 Exercises

Section 3.2: Oscillatory motion

1. **Corrections to the pendulum** ∗∗∗

   (a) For small oscillations, the period of a pendulum is approximately $T \approx 2\pi\sqrt{\ell/g}$, independent of amplitude, θ_0. For finite oscillations, show that the exact expression for T is

   $$T = \sqrt{\frac{8\ell}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}\,. \tag{3.57}$$

   (b) Let's now find an approximation to this value of T. It's more convenient to deal with quantities that go to 0 as $\theta \to 0$, so make use of the identity $\cos\phi = 1 - 2\sin^2(\phi/2)$ to write T in terms of sines. Then make the change of variables, $\sin x \equiv \sin(\theta/2)/\sin(\theta_0/2)$. Finally, expand your integrand judiciously in powers of (the fairly small quantity) θ_0, and perform the integrals to show

   $$T \approx 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \frac{\theta_0^2}{16} + \cdots\right). \tag{3.58}$$

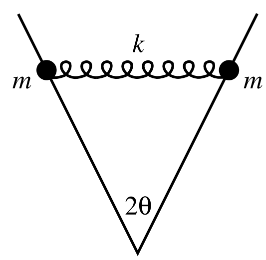


Figure 3.11

2. **Angled rails**

   Two particles of mass m are constrained to move along two rails which make an angle of 2θ with respect to each other, as shown in Fig. 3.11. They are connected by a spring with spring constant k. What is the frequency of oscillations for the motion where the spring remains parallel to its equilibrium position?

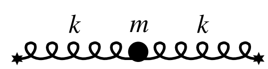


Figure 3.12

3. **Springs all over** ∗∗

   (a) A mass m is attached to two springs which have equilibrium lengths equal to zero. The other ends of the springs are fixed at two points (see Fig. 3.12). The spring constants are the same. The mass rests at its equilibrium position and is then given a kick in an arbitrary direction. Describe the resulting motion. (Ignore gravity.)

   (b) A mass m is attached to a number of springs which have equilibrium lengths equal to zero. The other ends of the springs are fixed at various points in space (see Fig. 3.13). The spring constants are all the same. The mass rests at its equilibrium position and is then given a kick in an arbitrary direction. Describe the resulting motion. (Ignore gravity.)

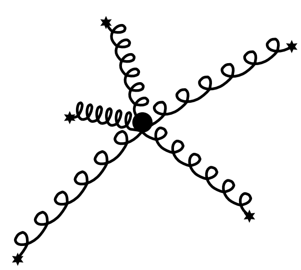


Figure 3.13

Section 3.3: Coupled oscillators

4. **Springs between walls** ∗∗

   Four identical springs and three identical masses lie between two walls (see Fig. 3.14). Find the normal modes.

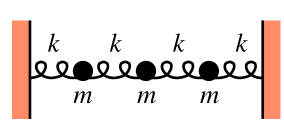


Figure 3.14

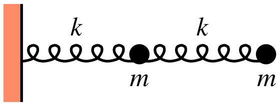


Figure 3.15

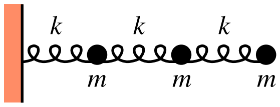


Figure 3.16

5. **Springs and one wall** ∗∗

   (a) Two identical springs and two identical masses are attached to a wall as shown in Fig. 3.15. Find the normal modes.

   (b) Three identical springs and three identical masses are attached to a wall as shown in Fig. 3.16. Find the normal modes.

3.5 Problems

Section 3.1: Linear differential equations

1. **A limiting case** $*$

   Consider the equation $\ddot{x} = ax$. If $a = 0$, then the solution to $\ddot{x} = 0$ is of course $x(t) = C + Dt$. Show that in the limit $a \to 0$, eq. (3.2) reduces to this form. *Note:* $a \to 0$ is a very sloppy way of saying what we mean. What is the precise mathematical condition we should write?

Section 3.2: Oscillatory motion

2. **Exponential force**

   A particle of mass m is subject to a force $F(t) = me^{-bt}$. The initial position and speed are 0. Find $x(t)$.

   (This problem was already given in Chapter 2, but solve it here in the spirit of Section 3.2.3.)

3. **Average tension** $**$

   Is the average (over time) tension in the string of a pendulum larger or smaller than mg? How much so? (As usual, assume that the angular amplitude, A, is small.)

4. **Through the circle** $**$

   A very large plane (consider it to be infinite), of mass density σ (per area), has a hole of radius R cut in it. A particle initially sits in the center of the circle, and is then given a tiny kick perpendicular to the plane. Assume that the only force acting on the particle is the gravitational force from the plane. Find the frequency of small oscillations (that is, where the amplitude is small compared to R).

5. **Driven oscillator** $*$

   Derive eq. (3.31), by guessing a solution of the form $x(t) = A\cos\omega_d t + B\sin\omega_d t$ in eq. (3.29).

Section 3.3: Coupled oscillators

6. **Springs on a circle** $****$

   (a) Two identical masses are constrained to move on a circle. Two identical springs connect the masses and wrap around a circle (see Fig. 3.17). Find the normal modes.

   (b) Three identical masses are constrained to move on a circle. Three identical springs connect the masses and wrap around a circle (see Fig. 3.18). Find the normal modes.

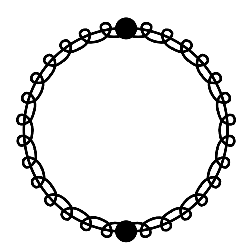


Figure 3.17

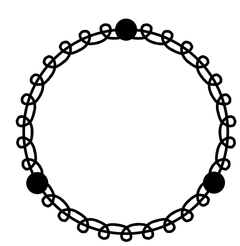


Figure 3.18

(c) How about the general case with N identical masses and N identical springs?

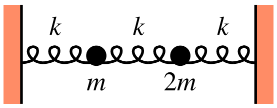


Figure 3.19

7. **Unequal masses** $**$

Three identical springs and two masses, m and $2m$, lie between two walls as shown in Fig. 3.19. Find the normal modes.

3.6 Solutions

1. **A limiting case**

   The statement $a \to 0$ is nonsensical, because a has units of [time]$^{-2}$, and the number 0 has no units. The proper statement is that eq. (3.2) reduces to $x(t) = C + Dt$ when t satisfies $t \ll 1/\sqrt{a}$. Both sides of this relation have units of time. Under this condition, $\sqrt{a}\,t \ll 1$, so we may write $e^{\pm\sqrt{a}\,t}$ approximately as $1 \pm \sqrt{a}\,t$. Therefore, eq. (3.2) becomes

$$\begin{aligned} x(t) &\approx A(1+\sqrt{a}\,t) + B(1-\sqrt{a}\,t) \\ &= (A+B) + \sqrt{a}(A-B)t \\ &\equiv C + Dt \end{aligned} \tag{3.59}$$

   If C and D happen to be of order 1 in the units chosen, then A and B must be roughly negatives of each other, and both of order $1/\sqrt{a}$.

   If a is small but nonzero, then t will eventually become large enough so that the linear form in eq. (3.59) is not valid.

2. **Exponential force**

   Guess a particular solution to $\ddot{x} = e^{-bt}$ of the form $x(t) = Ce^{-bt}$. Then $C = 1/b^2$. The solution to the homogeneous equation $\ddot{x} = 0$ is $x(t) = At + B$. Therefore, the complete solution for x is $x(t) = e^{-bt}/b^2 + At + B$. The initial conditions are $0 = v(0) = -1/b + A$, and $0 = x(0) = 1/b^2 + B$. Therefore,

$$x(t) = \frac{e^{-bt}}{b^2} + \frac{t}{b} - \frac{1}{b^2}. \tag{3.60}$$

3. **Average tension**

   Let the length of the pendulum be ℓ. We know that the angle, θ, depends on time according to

$$\theta(t) = A\cos(\omega t), \tag{3.61}$$

   where $\omega = \sqrt{g/\ell}$, and A is small. The tension, T, in the string must account for the radial component of gravity, $mg\cos\theta$, plus the centripetal acceleration, $m\ell\dot{\theta}^2$. Using eq. (3.61), this gives

$$T = mg\cos\Big(A\cos(\omega t)\Big) + m\ell\Big(-\omega A\sin(\omega t)\Big)^2. \tag{3.62}$$

   Using the small-angle approximation $\cos\alpha \approx 1 - \alpha^2/2$, we have (since A is small)

$$\begin{aligned} T &\approx mg\left(1 - \frac{1}{2}A^2\cos^2(\omega t)\right) + m\ell\omega^2 A^2 \sin^2(\omega t) \\ &= mg + mgA^2\left(\sin^2(\omega t) - \frac{1}{2}\cos^2(\omega t)\right). \end{aligned} \tag{3.63}$$

   The average value of $\sin^2\theta$ and $\cos^2\theta$ over one period is 1/2, so the average value for T is

$$\overline{T} = mg + \frac{1}{2}mgA^2, \tag{3.64}$$

   which is larger than mg, by $mgA^2/2$.

   Note that it is quite reasonable to expect $\overline{T} > mg$, because the average value of the vertical component of T equals mg (since the pendulum has no net rise or fall over a long period of time), and there is some positive contribution from the horizontal component of T.

4. **Through the circle**

By symmetry, only the component of the gravitational force perpendicular to the plane will survive. Let the particle's coordinate relative to the plane be z, and let its mass be m. Then a piece of mass dm at radius r on the plane will provide a force equal to $Gm(dm)/(r^2 + z^2)$. To get the component perpendicular to the plane, we must multiply this by $z/\sqrt{r^2 + z^2}$. So the total force on the particle is

$$\begin{aligned} F(z) &= -\int_R^\infty \frac{\sigma G m z 2\pi r\, dr}{(r^2 + z^2)^{3/2}} \\ &= 2\pi\sigma G m z (r^2 + z^2)^{-1/2}\Big|_{r=R}^{r=\infty} \\ &= -\frac{2\pi\sigma G m z}{\sqrt{R^2 + z^2}} \\ &\approx -\frac{2\pi\sigma G m z}{R}, \end{aligned} \tag{3.65}$$

where we have used $z \ll R$. Therefore, $F = ma$ gives

$$\ddot{z} + \frac{2\pi\sigma G}{R} z = 0. \tag{3.66}$$

The frequency of small oscillations is then

$$\omega = \sqrt{\frac{2\pi\sigma G}{R}}. \tag{3.67}$$

REMARK: For everyday values of R, this is a rather small number, because G is so small. Let's roughly determine its size. If the sheet has thickness d, and it is made out of a material with density ρ (per volume), then $\sigma = \rho d$. Hence, $\omega = \sqrt{2\pi\rho d G/R}$.

In the above analysis, we assumed the sheet was infinitely thin. In practice, we would need d to be much smaller than the amplitude of the motion. But this amplitude needs to be much smaller than R, in order for our approximations to hold. So we conclude that $d \ll R$. To get a rough upper bound on ω, let's pick $d/R = 1/10$; and let's make ρ be five times the density of water (i.e., $5000\,\text{kg/m}^3$). Then $\omega \approx 5 \cdot 10^{-4}\,\text{s}^{-1}$, which corresponds to a little more than one oscillation every 4 hours.

For an analogous system consisting of electrical charges, the frequency is much larger, since the electrical force is so much stronger than the gravitational force. ♣

5. **Driven oscillator**

Plugging $x(t) = A\cos\omega_d t + B\sin\omega_d t$ into eq. (3.29) yields

$$\begin{aligned} &-\omega_d^2 A\cos\omega_d t - \omega_d^2 B\sin\omega_d t \\ &\quad -2\gamma\omega_d A\sin\omega_d t + 2\gamma\omega_d B\cos\omega_d t \\ &\qquad +\omega^2 A\cos\omega_d t + \omega^2 B\sin\omega_d t = F\cos\omega_d t \end{aligned} \tag{3.68}$$

If this is to be true for all t, the coefficients of $\cos\omega_d t$ on both sides must be equal. Likewise for $\sin\omega_d t$. Hence,

$$\begin{aligned} -\omega_d^2 A + 2\gamma\omega_d B + \omega^2 A &= F, \\ -\omega_d^2 B - 2\gamma\omega_d A + \omega^2 B &= 0. \end{aligned} \tag{3.69}$$

Solving this system of equations for A and B gives

$$A = \frac{F(\omega^2 - \omega_d^2)}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}\,, \qquad B = \frac{2F\gamma\omega_d}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}\,, \tag{3.70}$$

in agreement with eq. (3.31).

6. **Springs on a circle**

(a) Pick two equilibrium positions (any diametrically opposite points will do). Let the distances of the masses from these points be x_1 and x_2 (measured counterclockwise). Then the equations of motion are

$$\begin{aligned} m\ddot{x}_1 &= -2k(x_1 - x_2), \\ m\ddot{x}_2 &= -2k(x_2 - x_1). \end{aligned} \tag{3.71}$$

The determinant method works here, but let's just do it the easy way. Adding these equations gives

$$\ddot{x}_1 + \ddot{x}_2 = 0. \tag{3.72}$$

Subtracting them equations gives

$$(\ddot{x}_1 - \ddot{x}_2) + 4\omega^2(x_1 - x_2) = 0. \tag{3.73}$$

The normal coordinates are therefore

$$\begin{aligned} x_1 + x_2 &= At + B, \qquad \text{and} \\ x_1 - x_2 &= C\cos(2\omega t + \phi). \end{aligned} \tag{3.74}$$

And the normal modes are

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}(At + B), \qquad \text{and} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= C\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(2\omega t + \phi). \end{aligned} \tag{3.75}$$

The first mode has frequency 0, and corresponds to the masses sliding around the circle, equally spaced, at constant speed.

(b) Pick three equilibrium positions (any three equally spaced points will do). Let the distances of the masses from these points be x_1, x_2, and x_3 (measured counterclockwise). Then the equations of motion are

$$\begin{aligned} m\ddot{x}_1 &= -k(x_1 - x_2) - k(x_1 - x_3), \\ m\ddot{x}_2 &= -k(x_2 - x_3) - k(x_2 - x_1), \\ m\ddot{x}_3 &= -k(x_3 - x_1) - k(x_3 - x_2). \end{aligned} \tag{3.76}$$

It's easy to see that the sum of these equations gives something nice, Also, differences between any two of the equations gives something useful. But let's use the determinant method to get some practice. Trying solutions proportional to $e^{i\alpha t}$ yields the determinant equation

$$\begin{vmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 & -\omega^2 \\ -\omega^2 & -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -\omega^2 & -\alpha^2 + 2\omega^2 \end{vmatrix} = 0. \tag{3.77}$$

One solution is $\alpha^2 = 0$. The other solution is the double root $\alpha^2 = 3\omega^2$.
The $\alpha = 0$ root corresponds to the vector $(1, 1, 1)$. So this normal mode is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (At + B). \tag{3.78}$$

This mode has frequency 0, and corresponds to the masses sliding around the circle, equally spaced, at constant speed.
The $\alpha^2 = 3\omega^2$ root corresponds to a two-dimensional subspace of normal modes. You can show that any vector of the form (a, b, c) with $a+b+c = 0$ is a normal mode with frequency $\sqrt{3}\omega$. We will arbitrarily pick the vectors $(0, 1, -1)$ and $(1, 0, -1)$ as basis vectors in this space. We can then write the normal modes as linear combinations of the vectors

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cos(\sqrt{3}\omega t + \phi_1), \qquad \text{and} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cos(\sqrt{3}\omega t + \phi_2). \end{aligned} \tag{3.79}$$

REMARKS: This is very similar to the example in section 3.3 with three springs and two masses oscillating between two walls. The way we've written these modes, the first one has the first mass stationary (so there could be a wall there, for all the other two masses know), and the second one has the second mass stationary.

The normal coordinates in this problem are $x_1+x_2+x_3$ (obtained by adding the three equations in (3.76)), $x_2 - x_3$ (obtained by subtracting the third eq. in (3.76) from the second), and $x_1 - x_3$ (obtained by subtracting the third eq. in (3.76) from the first). Actually, any combination of the form $ax_1 + bx_2 + cx_3$, with $a+b+c = 0$, is a normal mode (obtained by taking a times the first eq. in (3.76), etc.) ♣

(c) In part (b), what we were essentially doing, by setting the determinant in eq. (3.77) equal to 0, was finding the eigenvectors[2] of the matrix

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} = 3I - \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{3.80}$$

We haven't bothered writing the common factor ω^2, since this won't affect the eigenvectors. We'll let the reader show that for the general case of N springs and masses, the above matrix becomes the $N \times N$ matrix

$$3I - \begin{pmatrix} 1 & 1 & 0 & 0 & & 1 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & & 0 \\ 0 & 0 & 1 & 1 & & 0 \\ & \vdots & & & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \equiv 3I - M. \tag{3.81}$$

[2]An eigenvector, v, of a matrix, M, is a vector that gets taken into a multiple of itself when acted upon by M. That is, $Mv = \lambda v$, where λ is some number. You can prove for yourself that such a λ must satisfy det $|M - \lambda I| = 0$, where I is the identity matrix. We don't assume a knowledge of eigenvectors in this course, so don't worry about this problem.

In M, the three consecutive 1's keep shifting to the right, and they wrap around cyclicly.

Let's now be a little tricky. We can guess the eigenvectors and eigenvalues of M if we take a hint from its cyclic nature. A particular set of things that are rather cyclic are the Nth roots of 1. If η is an Nth root of 1, we leave it to you to show that $(1, \eta, \eta^2, \ldots, \eta^{N-1})$ is a eigenvector of M with eigenvalue $\eta^{-1}+1+\eta$. (This general method works for any matrix where the entries keep shifting to the right, and the entries don't even have to be equal.) The eigenvalues of the entire matrix in eq. (3.81) are therefore $3-(\eta^{-1}+1+\eta) = 2-\eta^{-1}-\eta$.

There are N different Nth roots of 1, namely $\eta_n = e^{2\pi in/N}$. So the N eigenvalues are

$$\lambda_n = 2 - \left(e^{-2\pi in/N} + e^{2\pi in/N}\right) = 2 - 2\cos(2\pi n/N). \tag{3.82}$$

The corresponding eigenvectors are

$$V_n = \left(1, \eta_n, \eta_n^2, \ldots, \eta_n^{N-1}\right). \tag{3.83}$$

The eigenvalues come in pairs. The numbers n and $N-n$ give the same value. This is fortunate, since we may then form real linear combinations of the two corresponding eigenvectors. The vectors

$$V_n^+ \equiv \frac{1}{2}(V_n + V_{N-n}) = \begin{pmatrix} 1 \\ \cos(2\pi n/N) \\ \cos(4\pi n/N) \\ \vdots \\ \cos(2(N-1)\pi n/N) \end{pmatrix} \tag{3.84}$$

and

$$V_n^- \equiv \frac{1}{2i}(V_n - V_{N-n}) = \begin{pmatrix} 0 \\ \sin(2\pi n/N) \\ \sin(4\pi n/N) \\ \vdots \\ \sin(2(N-1)\pi n/N) \end{pmatrix} \tag{3.85}$$

both have eigenvalue λ_n. The frequencies corresponding to these normal modes are

$$\omega_n = \sqrt{\lambda_n} = \sqrt{2 - 2\cos(2\pi n/N)}. \tag{3.86}$$

The only values for which the n's don't pair up are 0, and $N/2$ (if N is even).

Let's check our results for $N = 3$. If $n = 0$, we find $\lambda_0 = 0$, and $V_0 = (1, 1, 1)$. If $n = 1$, we find $\lambda_1 = 3$, and $V_1^+ = (1, -1/2, -1/2)$ and $V_1^- = (0, 1/2, -1/2)$. These two vectors span the same space we found in part (b).

7. **Unequal masses**

Let x_1 and x_2 be the positions of the left and right masses, respectively, relative to their equilibrium positions. The equations of motion are

$$\begin{aligned} \ddot{x}_1 + 2\omega^2 x_1 - \omega^2 x_2 &= 0, \\ 2\ddot{x}_2 + 2\omega^2 x_2 - \omega^2 x_1 &= 0. \end{aligned} \tag{3.87}$$

The appropriate linear combinations of these equations are not obvious, so we'll use the determinant method. Letting $x_1 = Ae^{i\alpha t}$ and $x_2 = Be^{i\alpha t}$, we see that for there to be a nontrivial solution for A and B, we must have

$$\begin{aligned} 0 &= \begin{vmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -2\alpha^2 + 2\omega^2 \end{vmatrix} \\ &= 2\alpha^4 - 6\alpha^2\omega^2 + 3\omega^4. \end{aligned} \tag{3.88}$$

The roots of this equation are

$$\alpha = \pm\omega\sqrt{\frac{3+\sqrt{3}}{2}} \equiv \pm\alpha_1, \qquad \text{and} \qquad \alpha = \pm\omega\sqrt{\frac{3-\sqrt{3}}{2}} \equiv \pm\alpha_2. \tag{3.89}$$

If $\alpha^2 = \alpha_1^2$, then the normal mode is proportional to $(\sqrt{3}+1, -1)$. If $\alpha^2 = \alpha_2^2$, then the normal mode is proportional to $(\sqrt{3}-1, 1)$. So the normal modes are

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \sqrt{3}+1 \\ -1 \end{pmatrix} \cos(\alpha_1 t + \phi_1), \qquad \text{and} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \sqrt{3}-1 \\ 1 \end{pmatrix} \cos(\alpha_2 t + \phi_2), \end{aligned} \tag{3.90}$$

Note that these two vectors are not orthogonal. There is no need for them to be.) You can easily show that the normal coordinates are $x_1 - (\sqrt{3}-1)x_2$, and $x_1 + (\sqrt{3}+1)x_2$, respectively.

Chapter 4

Conservation of Energy and Momentum

Conservation laws are extremely important in physics. They are an enormous help (both quantitatively and qualitatively) in figuring out what is going on in a physical system.

When we say that something is "conserved", we mean that it is constant over time. If a certain quantity is conserved, for example, while a ball rolls around on a hill, or while a group of particles interact, then the possible final motions are greatly restricted. If we can write down enough conserved quantities (which we are generally able to do), then we can restrict the final motions down to just one possibility, and so we've solved our problem. Conservation of energy and momentum are two of the main conservation laws in physics. A third, conservation of angular momentum, is discussed in Chapters 7 and 8.

It should be noted that it is not *necessary* to use conservation of energy and momentum when solving a problem. We will derive these conservation laws from Newton's laws. Therefore, if you felt like it, you could always simply start with first principles and use $F = ma$, etc. You would, however, soon grow weary of this approach. The point of conservation laws is that they make your life easier, and they provide a means for getting a good idea of the overall behavior of a given system.

4.1 Conservation of energy in 1-D

Consider a force, in one dimension, that depends on only position, that is, $F = F(x)$. If we write a as $v\,dv/dx$, then $F = ma$ becomes

$$mv\frac{dv}{dx} = F(x). \tag{4.1}$$

Separating variables and integrating gives $mv^2/2 = E + \int_{x_0}^{x} F(x')dx'$, where E is a constant of integration, dependent on the choice of x_0. (We're just following the procedure from Section 2.3 here.) Therefore, if we define the *potential energy*, $V(x)$, as

$$V(x) \equiv -\int_{x_0}^{x} F(x')dx', \tag{4.2}$$

then we may write

$$\frac{1}{2}mv^2 + V(x) = E. \tag{4.3}$$

We define the first term here to be the kinetic energy. This equation is true at any point in the particle's motion. Hence, the sum of the kinetic energy and potential energy is a constant.

Both E and $V(x)$ depend, of course, on the arbitrary choice of x_0 in eq. (4.2). E and $V(x)$ have no meaning by themselves. Only differences in E and $V(x)$ are relevant, because these differences are independent of the choice of x_0. For example, it makes no sense to say that the gravitational potential energy of an object at height y equals $-\int F\,dy = -\int(-mg)\,dy = mgy$. You have to say that mgy is the potential energy *with respect to the ground* (if your x_0 is at ground level). If you wanted to, you could say that the potential energy is $mgy + 7mg$ with respect to a point 7 meters below the ground. This is perfectly correct, although a little unconventional.[1]

If we take the difference between eq. (4.3) evaluated at two points, x_1 and x_2, then we obtain

$$\frac{1}{2}mv^2(x_2) - \frac{1}{2}mv^2(x_1) = V(x_1) - V(x_2) = \int_{x_1}^{x_2} F(x')dx'. \tag{4.4}$$

Here it is clear that only differences in energies matter. If we define the right-hand side of eq. (4.4) be the *work* done on the particle as it moves from x_1 to x_2, then we have produced the *work-energy theorem*,

Theorem 4.1 *The change in a particle's kinetic energy between points x_1 and x_2 is equal to the work done on the particle between x_1 and x_2.*

In Boston, lived Jack as did Jill,
Who gained mgh on a hill.
In their liquid pursuit,
Jill exclaimed with a hoot,
"I think we've just climbed a landfill!"

While noting, "Oh, this is just grand,"
Jack tripped on some trash in the sand.
He changed his potential
To kinetic, torrential,
But not before grabbing Jill's hand.

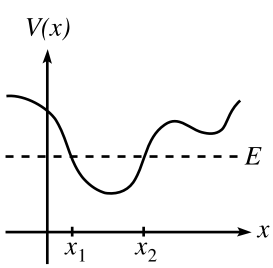


Figure 4.1

Having chosen a reference point for the energies, if we draw the $V(x)$ curve and also the constant E line (see Fig. 4.1), then the difference between them gives the kinetic energy. The places where $V(x) > E$ are the regions where the particle

[1] It gets to be a pain to keep repeating "with respect to the ground". Therefore, whenever

cannot go. The places where $V(x) = E$ are the "turning points", where the particle stops and changes direction. In the figure, the particle is trapped between x_1 and x_2, and oscillates back and forth. The potential $V(x)$ is extremely useful this way, because it makes clear the general properties of the motion.

REMARK: It may seem silly to introduce a specific x_0 as a reference point, since it is only eq. (4.4), which makes no mention of x_0, that has any meaning. (It's basically like taking the difference between 17 and 8 by first finding their sizes relative to 5, namely 12 and 3, and then subtracting 3 from 12 to obtain 9.) But it is extremely convenient to be able to label all positions with a definite number, $V(x)$, once and for all, and then take differences between the V's when needed. ♣

Note that eq. (4.2) implies

$$F(x) = -\frac{dV(x)}{dx}. \tag{4.5}$$

Given $V(x)$, it is easy to take its derivative to obtain $F(x)$. But given $F(x)$, it may be difficult (or impossible) to perform the integration in eq. (4.2) and write $V(x)$ in closed form. But this is not of much concern. The function $V(x)$ is well-defined, and if needed it can be computed numerically to any desired accuracy.

It is quite obvious that F needs to be a function of only x in order for the $V(x)$ in eq. (4.2) to be a well-defined function. If F depended on t or v, then $V(x_1) - V(x_2)$ would be path-dependent. That is, it would depend on *when* or *how quickly* the particle went from x_1 to x_2. Only for $F = F(x)$ is there no ambiguity in the result. (When dealing with more than one spatial dimension, however, there may be an ambiguity in V. This is the topic of Section 4.3.)

Example (Gravitational potential energy): Consider two point masses, M and m, separated by a distance r. Newton's law of gravitation says that the force between them is attractive and has magnitude GMm/r^2. The potential energy of the system at separation r, measured relative to separation r_0, is

$$V(r) - V(r_0) = -\int_{r_0}^{r} \frac{-GMm}{r'^2} dr' = \frac{-GMm}{r} + \frac{GMm}{r_0}. \tag{4.6}$$

A convenient choice for r_0 is ∞, since this makes the second term vanish. It will be understood from now on that this $r_0 = \infty$ reference point has been chosen. Therefore (see Fig. 4.2),

$$V(r) = \frac{-GMm}{r}. \tag{4.7}$$

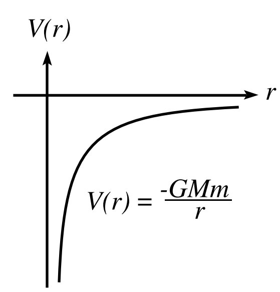


Figure 4.2

anyone talks about gravitational potential energy in an experiment at the surface of the earth, it is understood that the ground is the reference point. If, on the other hand, the experiment reaches out to distances far from the earth, then $r = \infty$ is understood to be the reference point, for reasons of convenience we will shortly see.

Example (Gravity near the earth): Here's a somewhat silly exercise. What is the gravitational potential energy of a mass m at height y, relative to the ground? We know, of course, that it is mgy, but let's do it the hard way. If M is the mass of the earth, and R is its radius (with $R \gg y$), then

$$\begin{aligned} V(R+y) - V(R) &= \frac{-GMm}{R+y} - \frac{-GMm}{R} \\ &= \frac{-GMm}{R}\left(\frac{1}{1+y/R} - 1\right) \\ &\approx \frac{-GMm}{R}\Big((1-y/R) - 1\Big) \\ &= \frac{GMmy}{R^2}, \end{aligned} \tag{4.8}$$

where we have used the Taylor series approximation for $1/(1+\epsilon)$ to obtain the third line. (We have also used the fact that a sphere can be treated like a point mass, as far as gravity is concerned. We'll prove this in Section 4.4.1.) But $g \equiv GM/R^2$, so the potential difference is mgy. We have, of course, simply gone around in circles here. We integrated in eq. (4.6), and then basically differentiated in eq. (4.8) by taking the difference between the forces. But it's good to check that everything works out.

You are encouraged to do Problem 11 at this point.

Remark: A good way to visualize a potential $V(x)$ is to imagine a ball sliding around in a valley or on a hill. For example, the potential of a typical spring is $V(x) = kx^2/2$ (which produces the Hooke's-law force, $F(x) = -dV/dx = -kx$), and we can get a decent idea of what's going on if we imagine a valley with height given by $y = x^2/2$. The gravitational potential of the ball is then $mgy = mgx^2/2$. Choosing $mg = k$ gives the desired potential. If we then look at the projection of the ball's motion on the x-axis, it seems like we have constructed a setup identical to the original spring.

However, although this analogy helps in visualizing the basic properties of the motion, the two setups are *not* the same. The details of this fact are left for Problem 5, but the following observation should convince you that they are indeed different. Let the ball be released from rest in both setups at a large value of x. Then the force, kx, due to the spring is very large. But the force in the x-direction on the particle in the valley is only a fraction of mg (namely $mg\sin\theta\cos\theta$, where θ is the angle of the tilted ground). ♣

4.2 Small Oscillations

Consider an object in one dimension, subject to the potential $V(x)$. Let the object initially be at rest at a local minimum of $V(x)$, and then let it be given a small kick so that it moves back and forth around the equilibrium point. What can we say about this motion? Is it harmonic? Does the frequency depend on the amplitude?

It turns out that for small amplitudes, the motion is indeed simple harmonic motion, and the frequency can easily be found, given $V(x)$. To see this, expand $V(x)$ in a Taylor series around the equilibrium point, x_0.

$$V(x) = V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2!}V''(x_0)(x-x_0)^2 + \frac{1}{3!}V'''(x_0)(x-x_0)^3 + \cdots. \tag{4.9}$$

We can simplify this greatly. $V(x_0)$ is an irrelevant additive constant. We can ignore it because only differences in energy matter (or equivalently, because $F = -dV/dx$). And $V'(x_0) = 0$, by definition of the equilibrium point. So that leaves us with the $V''(x_0)$ and higher order terms. For sufficiently small displacements, these higher order terms are negligible compared to the $V''(x_0)$ term, because they are suppressed by additional powers of $(x - x_0)$. So we are left with[2]

$$V(x) \approx \frac{1}{2}V''(x_0)(x - x_0)^2. \tag{4.10}$$

But this looks exactly like a Hooke's-law potential, $V(x) = (1/2)k(x - x_0)^2$, if we let $V''(x_0)$ be our "spring constant", k. The frequency of (small) oscillations, $\omega = \sqrt{k/m}$, is therefore

$$\omega = \sqrt{\frac{V''(x_0)}{m}}. \tag{4.11}$$

Example: A particle moves under the influence of the potential $V(x) = A/x^2 - B/x$. Find the frequency of small oscillations around the equilibrium point. (This potential is relevant to planetary motion, as we will see in Chapter 6.)

Solution: The first thing we must do is calculate the equilibrium point, x_0. We have

$$V'(x) = -\frac{2A}{x^3} + \frac{B}{x^2}. \tag{4.12}$$

So $V'(x) = 0$ when $x = 2A/B \equiv x_0$. The second derivative of $V(x)$ is

$$V''(x) = \frac{6A}{x^4} - \frac{2B}{x^3}. \tag{4.13}$$

Plugging in $x_0 = 2A/B$, we find

$$\omega = \sqrt{\frac{V''(x_0)}{m}} = \sqrt{\frac{B^4}{8mA^3}}. \tag{4.14}$$

Eq. (4.11) is an important result, because *any* function, $V(x)$, looks basically like a parabola (see Fig. 4.3) in a small enough region around a minimum (except for the special case when $V''(x_0) = 0$).

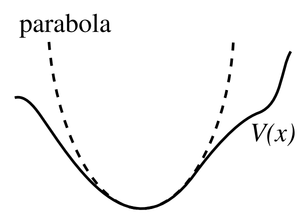


Figure 4.3

A potential may look quite erratic,
And its study may seem problematic.
But down near a min,
You can say with a grin,
"It behaves like a simple quadratic!"

---

[2]Even if $V'''(x_0)$ is much larger than $V''(x_0)$, we can always pick $(x - x_0)$ small enough so that the $V'''(x_0)$ term is negligible. The one case where this is not true is when $V''(x_0) = 0$. But the result in eq. (4.11) is still correct in this case. The frequency ω just happens to be zero.

4.3 Conservation of energy in 3-D

The concepts of work and potential energy in three dimensions are slightly more complicated than in one dimension, but the general ideas are the same. As in the 1-D case, we start with Newton's second law, which now takes the form $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ depends only on position, that is, $\mathbf{F} = \mathbf{F}(\mathbf{r})$. This vector equation is shorthand for three equations analogous to eq. (4.1), namely $mv_x(dv_x/dx) = F_x$, and likewise for y and z. Multiplying through by dx, etc., in these three equations, and then adding them together gives

$$F_x dx + F_y dy + F_z dz = m(v_x dv_x + v_y dv_y + v_z dv_z). \tag{4.15}$$

Integrating from the point (x_0, y_0, z_0) to the point (x, y, z) yields

$$E + \int_{x_0}^{x} F_x dx + \int_{y_0}^{y} F_y dy + \int_{z_0}^{z} F_z dz = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2}mv^2, \tag{4.16}$$

where E is a constant of integration.[3] Note that the integrations on the left-hand side depend on what path in 3-D space is chosen to go from (x_0, y_0, z_0) to (x, y, z). We will address this issue below.

With $d\mathbf{r} \equiv (dx, dy, dz)$, the left-hand side of eq. (4.15) is equal to $\mathbf{F} \cdot d\mathbf{r}$. Hence, eq. (4.16) may be written as

$$\frac{1}{2}mv^2 - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' = E. \tag{4.17}$$

Therefore, if we define the potential energy, $V(\mathbf{r})$, as

$$V(\mathbf{r}) \equiv -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}', \tag{4.18}$$

then we may write

$$\frac{1}{2}mv^2 + V(\mathbf{r}) = E. \tag{4.19}$$

In other words, the sum of the kinetic energy and potential energy is constant.

4.3.1 Conservative forces in 3-D

There is one complication that arises in 3-D that we didn't have to worry about in 1-D. The potential energy defined in eq. (4.18) may not be well defined. That is, it may be path-dependent. In 1-D, there is only one way to get from x_0 to x. But in 3-D, there is an infinite number of paths that go from $\mathbf{r}_0$ to $\mathbf{r}$. In order for the potential, $V(\mathbf{r})$, to have any meaning and to be of any use, it must be well-defined, that is, path-independent. A force for which this is the case is called a *conservative force*. Let us now see what types of forces are conservative.

[3] Technically, we should put primes on the integration variables, so that we don't confuse them with the limits of integration, but this gets too messy.

Theorem 4.2 *Given a force,* $\mathbf{F}(\mathbf{r})$*, a necessary and sufficient condition for the potential,*

$$V(\mathbf{r}) \equiv -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}', \tag{4.20}$$

to be well-defined (in other words, to be path independent) is that the curl of $\mathbf{F}$ *is zero (that is,* $\nabla \times \mathbf{F} = \mathbf{0}$*).*

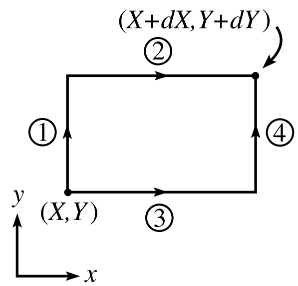


Figure 4.4

Proof: First, let us show that $\nabla \times \mathbf{F} = \mathbf{0}$ is a necessary condition for path independence. (That is, "If $V(\mathbf{r})$ is path independent, then $\nabla \times \mathbf{F} = \mathbf{0}$.")

Consider the infinitesimal rectangle shown in Fig. 4.4. (The rectangle lies in the x-y plane, so in the present analysis we will suppress the z-component of all coordinates, for convenience.) If the potential is path-independent, then the work done in going from (X, Y) to $(X + dX, Y + dY)$, namely the integral $\int \mathbf{F} \cdot d\mathbf{r}$, must be path-independent. In particular, the integral along the segments '1' and '2' must equal the integral along segments '3' and '4'. That is, $\int_1 F_y dy + \int_2 F_x dx = \int_3 F_x dx + \int_4 F_y dy$. Therefore, a necessary condition for path-independence is

$$\begin{aligned} \int_2 F_x dx - \int_3 F_x dx = \int_4 F_y dy - \int_1 F_y dy \qquad &\Longrightarrow \\ \int_X^{X+dX} \Big(F_x(x, Y + dY) - F_x(x, Y)\Big) dx & \\ = \int_Y^{Y+dY} \Big(F_y(X + dX, y) - F_y(X, y)\Big) dy.& \end{aligned} \tag{4.21}$$

Now,

$$F_x(x, Y + dY) - F_x(x, Y) \approx dY \frac{\partial F_x(x, y)}{\partial y}\bigg|_{(x,Y)} \approx dY \frac{\partial F_x(x, y)}{\partial y}\bigg|_{(X,Y)}. \tag{4.22}$$

The first approximation holds due to the definition of the partial derivative. The second approximation holds because our rectangle is small enough so that x is essentially equal to X. (Any errors due to this approximation will be second-order small, since we already have one factor of dY in our term.)

Similar treatment works for the F_y terms, so eq. (4.21) becomes

$$\int_X^{X+dX} dY \frac{\partial F_x(x, y)}{\partial y}\bigg|_{(X,Y)} dx = \int_Y^{Y+dY} dX \frac{\partial F_y(x, y)}{\partial x}\bigg|_{(X,Y)} dy. \tag{4.23}$$

The integrands here are constants, so we may quickly perform the integrals to obtain

$$dX dY \left(\frac{\partial F_x(x, y)}{\partial y} - \frac{\partial F_y(x, y)}{\partial x} \right) \bigg|_{(X,Y)} = 0. \tag{4.24}$$

Cancelling the $dXdY$ factor, and noting that (X, Y) is an arbitrary point, we see that if the potential is path-independent, then we must have

$$\frac{\partial F_x(x, y)}{\partial y} - \frac{\partial F_y(x, y)}{\partial x} = 0, \tag{4.25}$$

at any point (x, y).

The preceding analysis also works, of course, for little rectangles in the x-z and y-z planes, so we obtain two other similar conditions for the potential to be uniquely defined. All three conditions may be concisely written as

$$\nabla \times \mathbf{F} \equiv \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = 0. \tag{4.26}$$

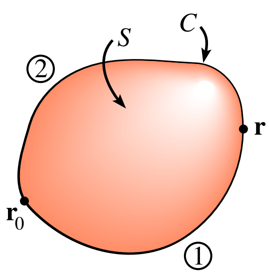


Figure 4.5

We have therefore shown that $\nabla \times \mathbf{F} = \mathbf{0}$ is a necessary condition for path-independence. Let us now show that it is sufficient. (That is, "If $\nabla \times \mathbf{F} = \mathbf{0}$, then $V(\mathbf{r})$ is path independent.")

The proof of sufficiency follows immediately from Stokes' theorem, which states that (see Fig. 4.5)

$$\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A}. \tag{4.27}$$

Here, C is an arbitrary closed curve, which we make pass through $\mathbf{r}_0$ and $\mathbf{r}$. S is an arbitrary surface which has C as its boundary. And $d\mathbf{A}$ has a magnitude equal to an infinitesimal piece of area on S and a direction defined to be orthogonal to S.

Therefore, if $\nabla \times \mathbf{F} = 0$ everywhere, then $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$ for any closed curve. But Fig. 4.5 shows that traversing the loop C entails traversing path '1' in the "forward" direction, and then traversing path '2' in the "backward" direction. Hence, $\int_1 \mathbf{F} \cdot d\mathbf{r} - \int_2 \mathbf{F} \cdot d\mathbf{r} = 0$. Therefore, any two paths give the same integral, as we wanted to show. ■

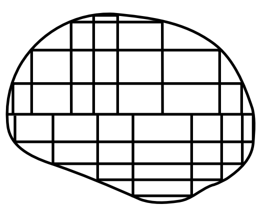


Figure 4.6

Remark: If you don't like invoking Stokes' theorem, then you can just back up a step and prove it from scratch. Here's the rough idea of the proof. For simplicity, pick a path confined to the x-y plane (the general case proceeds in the same manner). For the purposes of integration, any path can be approximated by a series of little segments parallel to the coordinate axes (see Fig. 4.6).

Now imagine integrating $\int \mathbf{F} \cdot d\mathbf{r}$ over every little rectangle in the figure (in a counter-clockwise direction). The result may be viewed in two ways: (1) From the above analysis (leading to eq. (4.24)), each integral gives the curl times the area of the rectangle. So whole integral gives $\int_S (\nabla \times \mathbf{F}) dA$. (2) Each interior line gets counted twice (in opposite directions) in the whole integration, so these contributions cancel. We are left with the integral over the edge segments, which gives $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$. ♣

Remark: Another way to show that $\nabla \times \mathbf{F} = 0$ is a necessary condition for path-independence (that is, "If $V(\mathbf{r})$ is well-defined, then $\nabla \times \mathbf{F} = 0$.") is the following. If $V(\mathbf{r})$ is well-defined, then it is legal to write down the differential form of eq. (4.18). This is

$$dV(\mathbf{r}) = -\mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \equiv -(F_x dx + F_y dy + F_z dz). \tag{4.28}$$

But another expression for dV is

$$dV(\mathbf{r}) = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz. \tag{4.29}$$

The previous two equations must be equivalent for arbitrary dx, dy, and dz. So we have

$$\begin{aligned} (F_x, F_y, F_z) &= -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right) \\ \Longrightarrow \quad \mathbf{F}(\mathbf{r}) &= -\nabla V(\mathbf{r}). \end{aligned} \tag{4.30}$$

In other words, the force is simply the gradient of the potential. Therefore,

$$\nabla \times \mathbf{F} = -\nabla \times \nabla V(\mathbf{r}) = 0, \tag{4.31}$$

because the curl of a gradient is identically zero (as you can explicitly verify). ♣

Example (Central force): A *central force* is defined to be a force that points radially, and whose magnitude depends on only r. That is, $\mathbf{F}(\mathbf{r}) = F(r)\hat{\mathbf{r}}$. Show that a central force is a conservative force, by explicitly showing that $\nabla \times \mathbf{F} = \mathbf{0}$.

Solution: $\mathbf{F}$ may be written as

$$\mathbf{F}(x, y, z) = F(r)\hat{\mathbf{r}} = F(r)\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right). \tag{4.32}$$

Note that (verify this)

$$\frac{\partial r}{\partial x} = \frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x} = \frac{x}{r}\,, \tag{4.33}$$

and similarly for y and z. The z component of $\nabla \times \mathbf{F}$ is therefore (writing F for $F(r)$, and F' for $dF(r)/dr$)

$$\begin{aligned}\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} &= \frac{\partial(yF/r)}{\partial x} - \frac{\partial(xF/r)}{\partial y} \\ &= \left(\frac{y}{r}F'\frac{\partial r}{\partial x} - yF\frac{1}{r^2}\frac{\partial r}{\partial x}\right) - \left(\frac{x}{r}F'\frac{\partial r}{\partial y} - xF\frac{1}{r^2}\frac{\partial r}{\partial y}\right) \\ &= \left(\frac{yxF'}{r^2} - \frac{yxF}{r^3}\right) - \left(\frac{xyF'}{r^2} - \frac{xyF}{r^3}\right) = 0. \end{aligned} \tag{4.34}$$

Likewise for the x- and y-components.

An example of a nonconservative force is friction. A friction force is certainly not a function of only position; its sign changes depending on which way the object is moving.

4.4 Gravity due to a sphere

4.4.1 Derivation via the potential energy

We know that the gravitational force on a point-mass m, located a distance r from a point-mass M, is given by Newton's law of gravitation,

$$F(r) = \frac{-GMm}{r^2}\,, \tag{4.35}$$

where the minus sign indicates an attractive force. What is the force if we replace the latter point mass by a sphere of radius R and mass M? The answer (as long as the sphere is spherically symmetric, that is, the density is a function of only r) is that it is still $-GMm/r^2$. A sphere acts just like a point mass, for the purposes of gravity. This is an extremely pleasing result, to say the least. If it were not the case, then the universe would be a far more complicated place than it is. In particular, the motion of planets and such things would be rather hard to describe.

To prove the above claim, it turns out to be much easier to calculate the potential energy due to a sphere, and to then take the derivative to obtain the force, rather than to calculate the force explicitly. So this is the route we will take. It will suffice to demonstrate the result for a thin spherical shell, because a sphere is the sum of many such shells.

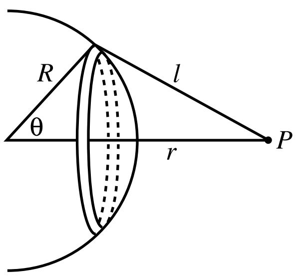


Figure 4.7

Our strategy in calculating the potential energy, at a point P, due to a spherical shell will be to slice the shell into rings as shown in Fig. 4.7. Let the radius of the shell be R, and let P be a distance r from the center of the shell. Let the ring make the angle θ as shown, and let P be a distance ℓ from the ring.

The length ℓ is a function of R, r, and θ. It may be found as follows. In Fig. 4.8, segment AB has length $R\sin\theta$, and segment BP has length $r - R\cos\theta$. So the length ℓ in triangle ABP is

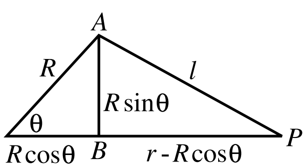


Figure 4.8

$$\ell = \sqrt{(R\sin\theta)^2 + (r - R\cos\theta)^2} = \sqrt{R^2 + r^2 - 2rR\cos\theta}. \tag{4.36}$$

This, of course, is just the law of cosines.

The area of a ring between θ and $\theta + d\theta$ is its width (which is $Rd\theta$) times its circumference (which is $2\pi R\sin\theta$). Letting $\sigma = M/(4\pi R^2)$ be the mass density of the shell, we see that the potential energy of a mass m at P due to a thin ring is $-Gm\sigma(Rd\theta)(2\pi R\sin\theta)/\ell$. This is true because the gravitational potential energy,

$$V(r) = \frac{-Gm_1m_2}{r}, \tag{4.37}$$

is a scalar quantity, so the contributions from little mass pieces simply add. The total potential energy at P is therefore

$$\begin{aligned} V(r) &= -\int_0^\pi \frac{2\pi\sigma G R^2 m\sin\theta d\theta}{\sqrt{R^2 + r^2 - 2rR\cos\theta}} \\ &= -\frac{2\pi\sigma G R m}{r}\sqrt{R^2 + r^2 - 2rR\cos\theta}\,\Big|_0^\pi. \end{aligned} \tag{4.38}$$

(As messy as the integral looks, the $\sin\theta$ in the numerator is what makes it doable.)

There are two cases to consider. If $r > R$, then we have

$$V(r) = -\frac{2\pi\sigma G R m}{r}\Big((r+R) - (r-R)\Big) = -\frac{G(4\pi\sigma R^2)m}{r} = -\frac{GMm}{r}, \tag{4.39}$$

which is the potential due to a point-mass M, as promised. If $r < R$, then we have

$$V(r) = -\frac{2\pi\sigma G R m}{r}\Big((r+R) - (R-r)\Big) = -\frac{G(4\pi\sigma R^2)m}{R} = -\frac{GMm}{R}, \tag{4.40}$$

which is independent of r.

Having found $V(r)$, we now simply have to take the negative of its gradient to obtain $F(r)$. The gradient is just $\hat{\mathbf{r}}(d/dr)$ here, because V is a function of only r, so we have

$$\begin{aligned} F(r) &= -\frac{GMm}{r^2}, \qquad \text{if } r > R, \\ F(r) &= 0, \qquad \text{if } r < R. \end{aligned} \tag{4.41}$$

These forces are directed radially, of course. A sphere is the sum of many spherical shells, so if P is outside a given sphere, then the force at P is $-GMm/r^2$, where M is the total mass of the sphere. The shells may have different mass densities (but each one must have uniform density), and this result will still hold.

Newton looked at the data, numerical,
And then observations, empirical.
He said, "But, of course,
We get the same force
From a point mass and something that's spherical!"

If P is inside a given sphere, then the only relevant material is the mass inside a concentric sphere through P, because all the shells outside this region give zero force, from the second equation in eq. (4.41). The material "outside" of P is, for the purposes of gravity, not there.

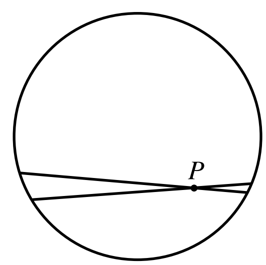


Figure 4.9

It is not obvious that the force inside a spherical shell is zero. Consider the point P in Fig. 4.9. A piece of mass, dm, on the right side of the shell gives a larger force on P than a piece of mass, dm, on the left side, due to the $1/r^2$ dependence. But there is more mass on the left side than the right side. These two effects happen to exactly cancel, as you can show in Problem 10.

Note that the gravitational force between two spheres is the same as if they are replaced by two point-masses. This follows from two applications of the result in eq. (4.41).

4.4.2 Tides

The tides on the earth are due to the fact that the gravitational field from a point mass (or a spherical object, in particular the moon or the sun) is not uniform. The direction of the force is not constant (the field lines converge to the source), and the magnitude is not constant (it falls off like $1/r^2$). As far as the earth goes, these effects cause the oceans to bulge around the earth, producing the observed tides.

The study of tides here is useful partly because tides are a very real phenomenon in the world, and partly because the following analysis gives us an excuse to make lots of neat mathematical approximations. Before considering the general case of tidal forces, let's look at two special cases.

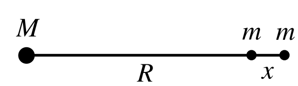


Figure 4.10

Longitudinal tidal force

In Fig. 4.10, two particles of mass m are located at points $(R, 0)$ and $(R + x, 0)$, with $x \ll R$. A planet of mass M is located at the origin. What is the difference between the gravitational forces acting on these two masses?

The difference in the forces is (using $x \ll R$ to make suitable approximations)

$$\frac{-GMm}{(R+x)^2} - \frac{-GMm}{R^2} \approx \frac{-GMm}{R^2+2Rx} + \frac{GMm}{R^2} = \frac{GMm}{R^2}\left(\frac{-1}{1+2x/R}+1\right)$$
$$\approx \frac{GMm}{R^2}\Big(-(1-2x/R)+1\Big) = \frac{2GMmx}{R^3}. \tag{4.42}$$

This is, of course, simply the derivative of the force, times x. This difference points along the line joining the masses, and its effect is to cause the separation between the masses to increase.

We see that this force difference is linear in the separation, x, and inversely proportional to the cube of the distance from the source. This *force difference* is the important quantity (as opposed to the force on each mass) when we are dealing with the *relative* motion of objects in free-fall around a given mass (for example, circular orbiting motion, or radial falling motion). This force difference is referred to as the "tidal force".

Consider two people, A and B, in radial free-fall toward a planet. Imagine that they are connected by a string, and enclosed in a windowless box. Neither can feel the gravitational force acting on him (for all they know, they are floating freely in space). But they will each feel a tension in the string equal to $T = GMmx/R^3$ (neglecting higher-order terms in x/R), pulling in opposite directions. The difference in these tension forces is $2T$, which exactly cancels the difference in gravitational force, thereby allowing their separation to remain fixed.

How do A and B view the situation? They will certainly feel the tension force. They will therefore conclude that there must be some other mysterious "tidal force" that opposes the tension, yielding a total net force of zero, as measured in their windowless box.

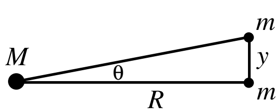


Figure 4.11

Transverse tidal force

In Fig. 4.11, two particles of mass m are located at points $(R, 0)$ and (R, y), with $y \ll R$. A planet of mass M is located at the origin. What is the difference between the gravitational forces acting on these two masses?

Both masses are the same distance R from the origin (up to second-order effects in y/R, using the Pythagorean theorem), so the magnitudes of the forces on them are essentially the same. The direction is the only thing that is different, to first order in y/R. The difference in the forces is the y-component of the force on the top mass. The magnitude of this component is

$$\frac{GMm}{R^2}\sin\theta \approx \frac{GMm}{R^2}\left(\frac{y}{R}\right) = \frac{GMmy}{R^3}. \tag{4.43}$$

This difference points along the line joining the masses, and its effect is to pull the masses together. As in the longitudinal case, the transverse tidal force is linear in the separation, y, and inversely proportional to the cube of the distance from the source.

General tidal force

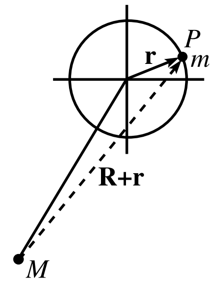


Figure 4.12

Let us now calculate the tidal force at an arbitrary point on a circle of radius r, centered at the origin. We'll calculate the tidal force relative to the origin. Let the source of the gravitational force be a mass M located at the vector $-\mathbf{R}$ (so that the vector from the source to a point P on the circle is $\mathbf{R}+\mathbf{r}$; see Fig. 4.12). As usual, assume $|\mathbf{r}| \ll |\mathbf{R}|$.

The attractive gravitational force may be written as $\mathbf{F}(\mathbf{x}) = -GMm\mathbf{x}/|\mathbf{x}|^3$, where $\mathbf{x}$ is the vector from the source to the point in question. (The cube is in the denominator because the vector in the numerator contains one power of the distance.) The desired difference between the force on a mass m at P and the force on a mass m at the origin is the tidal force,

$$\frac{\mathbf{F}_{\rm t}(\mathbf{r})}{GMm} = \frac{-(\mathbf{R}+\mathbf{r})}{|\mathbf{R}+\mathbf{r}|^3} - \frac{-\mathbf{R}}{|\mathbf{R}|^3}\,. \tag{4.44}$$

This is the exact expression for the tidal force; it doesn't get any more correct than this. However, it is completely useless.[4] Let us therefore make some approximations in eq. (4.44) and transform it into something technically incorrect (as approximations tend to be), but far more useful.

The first thing we have to do is rewrite the $|\mathbf{R}+\mathbf{r}|$ term. We have (using $r \ll R$ to ignore higher-order terms)

$$\begin{aligned}|\mathbf{R}+\mathbf{r}| &= \sqrt{(\mathbf{R}+\mathbf{r})\cdot(\mathbf{R}+\mathbf{r})} \\ &= \sqrt{R^2 + r^2 + 2\mathbf{R}\cdot\mathbf{r}} \\ &\approx R\sqrt{1 + 2\mathbf{R}\cdot\mathbf{r}/R^2} \\ &\approx R\left(1 + \frac{\mathbf{R}\cdot\mathbf{r}}{R^2}\right).\end{aligned} \tag{4.45}$$

Therefore (using $r \ll R$),

$$\begin{aligned}\frac{\mathbf{F}_{\rm t}(\mathbf{r})}{GMm} &\approx -\frac{\mathbf{R}+\mathbf{r}}{R^3(1+\mathbf{R}\cdot\mathbf{r}/R^2)^3} + \frac{\mathbf{R}}{R^3} \\ &\approx -\frac{\mathbf{R}+\mathbf{r}}{R^3(1+3\mathbf{R}\cdot\mathbf{r}/R^2)} + \frac{\mathbf{R}}{R^3} \\ &\approx -\frac{\mathbf{R}+\mathbf{r}}{R^3}\left(1 - \frac{3\mathbf{R}\cdot\mathbf{r}}{R^2}\right) + \frac{\mathbf{R}}{R^3}\,.\end{aligned} \tag{4.46}$$

Letting $\hat{\mathbf{R}} \equiv \mathbf{R}/R$, we finally have (using $r \ll R$)

$$\mathbf{F}_{\rm t}(\mathbf{r}) \approx \frac{GMm(3\hat{\mathbf{R}}(\hat{\mathbf{R}}\cdot\mathbf{r}) - \mathbf{r})}{R^3}\,. \tag{4.47}$$

If we let M lie on the negative x-axis, so that $\hat{\mathbf{R}} = \hat{\mathbf{x}}$, then $\hat{\mathbf{R}}\cdot\mathbf{r} = x$, and we see that the tidal force at the point $P = (x, y)$ may be written as

$$\mathbf{F}_{\rm t}(\mathbf{r}) \approx \frac{GMm(2x, -y)}{R^3}\,. \tag{4.48}$$

[4] This reminds me of a joke about two people lost in a hot-air balloon.

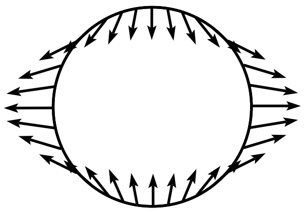


Figure 4.13

This reduces properly in the two special cases considered above. The tidal forces at various points on the circle are shown in Fig. 4.13.

If the earth were a rigid body, then these forces would be irrelevant. But the water in the oceans is free to slosh around. The water on the earth bulges toward the moon. As the earth rotates, the bulge moves around relative to the earth. This produces two high tides and two low tides per day. (It's not exactly two per day, because the moon moves around the earth. But this motion is fairly slow, taking about a month, so it's a decent approximation for the present purposes to think of the moon as motionless.)

Note that it is *not* the case, of course, that the moon *pushes* the water away on the far side of the earth. It pulls on that water, too; it just does so in a weaker manner than it pulls on the rigid part of the earth. Tides are a *comparative* effect.

REMARK: Consider two 1 kg masses on the earth separated by a distance of 1 m. An interesting fact is that the gravitational force from the sun on them is (much) larger than that from the moon. But the tidal force from the sun on them is (slightly) weaker than that from the moon. The ratio of the gravitational forces is

$$\frac{F_{\rm S}}{F_{\rm M}} = \left(\frac{GM_{\rm S}}{R_{\rm E,S}^2}\right) \Big/ \left(\frac{GM_{\rm M}}{R_{\rm E,M}^2}\right) = \frac{6\cdot 10^{-3}\,{\rm m/s^2}}{3.4\cdot 10^{-5}\,{\rm m/s^2}} \approx 175. \tag{4.49}$$

The ratio of the tidal forces is

$$\frac{F_{\rm t,S}}{F_{\rm t,M}} = \left(\frac{GM_{\rm S}}{R_{\rm E,S}^3}\right) \Big/ \left(\frac{GM_{\rm M}}{R_{\rm E,M}^3}\right) = \frac{4\cdot 10^{-14}\,{\rm s^{-2}}}{9\cdot 10^{-14}\,{\rm s^{-2}}} \approx 0.45. \tag{4.50}$$

The moon's tidal effect is roughly twice the sun's. ♣

4.5 Conservation of linear momentum

4.5.1 Conservation of p

Newton's third law says that for every force there is an equal and opposite force. More precisely, if $\mathbf{F}_{ab}$ is the force that particle a feels due to particle b, and $\mathbf{F}_{ba}$ is the force that particle b feels due to particle a, then $\mathbf{F}_{ba} = -\mathbf{F}_{ab}$, at any time.[5]

This law has important implications concerning momentum. Consider two particles that interact over a period of time. Assume they are isolated from outside forces. Because

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}\,, \tag{4.51}$$

we see that the total change in a particle's momentum equals the time integral of the force acting on it. That is,

$$\mathbf{p}(t_2) - \mathbf{p}(t_1) = \int_{t_1}^{t_2} \mathbf{F}\,dt. \tag{4.52}$$

[5] Some forces, such as magnetic forces from moving charges, do not satisfy the third law. But for any common "pushing" or "pulling" force (the type we will deal with), the third law holds.

(This integral is called the *impulse*.) Therefore, since $\mathbf{F}_{ba} = -\mathbf{F}_{ab}$ at all times, we have

$$\begin{aligned} \mathbf{p}_a(t_2) - \mathbf{p}_a(t_1) &= \int_{t_1}^{t_2} \mathbf{F}_{ab} dt \\ &= -\int_{t_1}^{t_2} \mathbf{F}_{ba} dt \;=\; -\Big(\mathbf{p}_b(t_2) - \mathbf{p}_b(t_1)\Big), \end{aligned} \tag{4.53}$$

and so

$$\mathbf{p}_a(t_2) + \mathbf{p}_b(t_2) = \mathbf{p}_a(t_1) + \mathbf{p}_b(t_1). \tag{4.54}$$

In other words, the total momentum of this isolated system is *conserved*. It does not depend on time. Note that eq. (4.54) is a vector equation, so it is really three equations, namely conservation of p_x, p_y, and p_z.

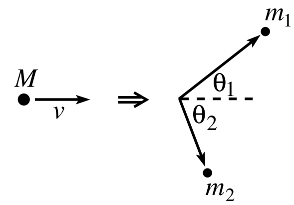


Figure 4.14

Example (Splitting mass): A mass M moves with speed V in the x-direction. It explodes into two pieces that go off at angles θ_1 and θ_2, as shown in Fig. 4.14. What are the magnitudes of the momenta of the two pieces?

Solution: Let $P \equiv MV$ be the initial momentum, and let p_1 and p_2 be the final momenta. Conservation of momentum in the x- and y-directions gives, respectively,

$$\begin{aligned} p_1 \cos\theta_1 + p_2 \cos\theta_2 &= P, \\ p_1 \sin\theta_1 - p_2 \sin\theta_2 &= 0. \end{aligned} \tag{4.55}$$

Solving for p_1 and p_2 (and using a trig addition formula) gives

$$p_1 = \frac{P\sin\theta_2}{\sin(\theta_1+\theta_2)}, \qquad \text{and} \qquad p_2 = \frac{P\sin\theta_1}{\sin(\theta_1+\theta_2)}. \tag{4.56}$$

Let's check a few limits. If $\theta_1 = \theta_2$, then $p_1 = p_2$, as it should. If, in addition, θ_1 and θ_2 are both small, then $p_1 = p_2 \approx P/2$, as they should. If, on the other hand, $\theta_1 = \theta_2 \approx 90^\circ$, then p_1 and p_2 are very large; the explosion must have provided a large amount of energy.

Note that we can't determine what the masses of the two pieces are, from the given information. To find these, we would need to know how much energy the explosion gave to the system.

Remark: Newton's third law makes a statement about forces. But force is defined in terms of momentum via $F = dp/dt$. So the third law essentially *postulates* conservation of momentum. (The "proof" above in eq. (4.53) is hardly a proof. It involves one simple integration.) So you might wonder if momentum conservation is something you can *prove*, or if it's something you have to *assume* (as we have basically done).

The difference between a postulate and a theorem is rather nebulous. One person's postulate might be another person's theorem, and vice-versa. You have to start *somewhere* in your assumptions. We chose to start with the third law. In the Lagrangian formalism in Chapter 5, the starting point is different, and momentum conservation is deduced as a consequence of translational invariance (as we will see). So it looks more like a theorem in that formalism.

But one thing is certain. Momentum conservation of two particles can *not* be proven from scratch for arbitrary forces, because it is not necessarily true. For example, if two charged particles interact in a certain way through the magnetic fields they produce, then the total momentum of the two particles might *not* be conserved. Where is the missing momentum? It is carried off in the electromagnetic field. The total momentum of the system *is* conserved, but the fact of the matter is that the system consists of the two particles *plus* the electromagnetic field.

For normal, everyday pushing and pulling forces, simple arguments can be made to justify conservation of momentum. But some forces (for example, the magnetic force) act in a sort of "sideways" manner. (A particle actually interacts with the field, and not the other particle.) Newton's third law does not necessarily hold for particles subject to such forces. ♣

Now let's look at momentum conservation for a system of many particles. As above, let $\mathbf{F}_{ij}$ be the force that particle i feels due to particle j. Then $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, at any time. Assume the particles are isolated from outside forces.

The change in the momentum of the ith particle, from t_1 to t_2 (we won't bother writing all the t's in the expressions below), is

$$\Delta \mathbf{p}_i = \int \left(\sum_j \mathbf{F}_{ij} \right) dt. \tag{4.57}$$

Therefore, the change in the total momentum of all the particles is

$$\Delta \mathbf{P} \equiv \sum_i \Delta \mathbf{p}_i = \int \left(\sum_i \sum_j \mathbf{F}_{ij} \right) dt. \tag{4.58}$$

But $\sum_i \sum_j \mathbf{F}_{ij} = 0$ at all times, because for every term $\mathbf{F}_{ab}$, there is a term $\mathbf{F}_{ba}$, and $\mathbf{F}_{ab} + \mathbf{F}_{ba} = 0$. (And also, $\mathbf{F}_{aa} = 0$.) Therefore, the total momentum of an isolated system of particles is conserved.

4.5.2 Rocket motion

The application of momentum conservation becomes a little more exciting when the mass, m, is allowed to vary. Such is the case with rockets, since most of their mass consists of fuel which is eventually ejected.

Let mass be ejected with speed u relative to the rocket,[6] at a rate dm/dt. We'll define the quantity dm to be negative; so during a time dt the mass dm gets *added* to the rocket's mass. (If you wanted, you could define dm to be positive, and then *subtract* it from the rocket's mass. Either way is fine.) Also, we'll define u to be positive; so the ejected particles *lose* a speed u relative to the rocket. It may sound silly, but the hardest thing about rocket motion is picking a sign for these quantities and sticking with it.

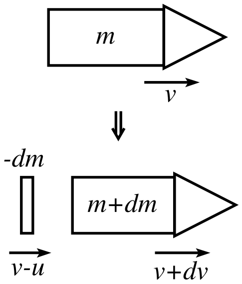


Figure 4.15

Consider a moment when the rocket has mass m and speed v. Then at a time dt later (see Fig. 4.15), the rocket has mass $m+dm$ and speed $v+dv$, while the exhaust

[6]Just to emphasize, u is the speed with respect to the rocket. It wouldn't make much sense to say "relative to the ground", because the rocket's engine spits out the matter relative to itself, and the engine has no way of knowing how fast the rocket is moving with respect to the ground.

has mass $(-dm)$ and speed $v - u$ (which may be positive or negative, depending on the relative size of v and u). There are no external forces, so the total momentum at each of these times must be equal. Therefore,

$$mv = (m + dm)(v + dv) + (-dm)(v - u). \tag{4.59}$$

Ignoring the second-order term yields $m\,dv = -u\,dm$. Dividing by m and integrating from t_1 to t_2 gives

$$\int dv = -\int u \frac{dm}{m} \qquad \Longrightarrow \qquad v_2 - v_1 = u \ln \frac{m_1}{m_2}. \tag{4.60}$$

For the case where the initial mass is M and the initial speed is 0, we have $v = u\ln(M/m)$. And if dm/dt happens to be constant (call it $-\eta$, where η is positive), then $v(t) = u\ln[M/(M - \eta t)]$.

The log in the result in eq. (4.60) is not very encouraging. If the mass of the metal of the rocket is m, and the mass of the fuel is $9m$, then the final speed is only $u \ln 10 \approx (2.3)u$. If the mass of the fuel is increased by a factor of 11 up to $99m$ (which is probably not even structurally possible, given the amount of metal required to hold it), then the final speed only doubles to $u \ln 100 = 2(u \ln 10) \approx (4.6)u$. How do you make a rocket go significantly faster? Exercise 8 deals with this question.

4.6 The CM frame

4.6.1 Definition

When talking about momentum, it is tacitly assumed that a certain frame of reference has been picked. After all, the velocities of the particles have to be measured with respect to some coordinate system. Any inertial (that is, non-accelerating) frame is as good as any other, but we will see that there is one particular reference frame that is generally advantageous to use.

Consider a frame S, and another frame S' which moves at constant velocity $\mathbf{u}$ with respect to S (see Fig. 4.16). Given a system of particles, the velocity of the ith particle in S is related to its velocity in S' by

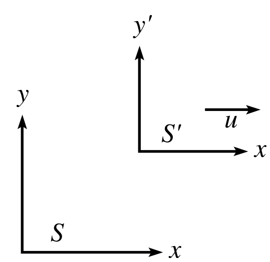


Figure 4.16

$$\mathbf{v} = \mathbf{v}' + \mathbf{u}. \tag{4.61}$$

It is then easy to see that if momentum is conserved during a collision in frame S', then it is also conserved in frame S. This is true because both the initial and final momenta of the system in S are increased by the same amount, $(\sum m_i)\mathbf{u}$, compared to what they are in S'.[7]

Let us therefore consider the unique frame in which the total momentum is zero. This is called the *center-of-mass frame*, or CM frame. If the total momentum is

[7] Alternatively, nowhere in our earlier derivation of momentum conservation did we say what frame we were using. We only assumed that the frame was not accelerating. If it were accelerating, then $\mathbf{F}$ would *not* equal $m\mathbf{a}$. We will see in Chapter 9 how $\mathbf{F} = m\mathbf{a}$ is modified in a non-inertial frame.

$\mathbf{P} \equiv \sum m_i \mathbf{v}_i$ in frame S, then it is easy to see that the CM frame, S', is the frame that moves with velocity

$$\mathbf{u} = \frac{\mathbf{P}}{\sum m_j} \equiv \frac{\sum m_i \mathbf{v}_i}{\sum m_j} \tag{4.62}$$

with respect to S. This is true because

$$\begin{aligned} \mathbf{P}' &= \sum m_i \mathbf{v}'_i \\ &= \sum m_i \left(\mathbf{v}_i - \frac{\mathbf{P}}{\sum m_j} \right) \\ &= \mathbf{P} - \mathbf{P} = \mathbf{0}, \end{aligned} \tag{4.63}$$

as desired. The CM frame is extremely useful. Physical processes are generally much more symmetrical in this frame, and this makes the results more transparent.

Remark: The CM frame could also be called the "zero-momentum" frame. The "CM" name is used because the center-of-mass of the particles (defined by $\mathbf{R}_{\text{CM}} \equiv \sum m_i \mathbf{r}_i / \sum m_j$, which is the location of the pivot upon which the particles would balance, if they were rigidly connected) does not move in this frame. This follows from

$$\frac{d\mathbf{R}_{\text{CM}}}{dt} = \frac{1}{\sum m_j} \sum m_i \frac{d\mathbf{r}_i}{dt} \propto \sum \mathbf{p}_i = 0. \tag{4.64}$$

The center-of-mass may therefore be chosen as the origin of the CM frame. ♣

The other frame which people generally work with is the *lab frame*. There is nothing at all special about this frame. It is simply the frame (assumed to be inertial) in which the conditions of the problem are given. Any inertial frame can be called the "lab frame". Part of the task of many problems is to switch back and forth between the lab and CM frames. For example, if the final answer is requested in the lab frame, then you may want to transform the given information from the lab frame into the CM frame where things are more obvious, and then transform back to the lab frame to give the answer.

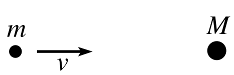


Figure 4.17

Example (Two masses in 1-D): A mass m with speed v approaches a stationary mass M (see Fig. 4.17). The masses bounce off each other without any loss in total energy. What are the final speeds of the particles? (Assume all motion takes place in 1-D.)

Solution: Doing this problem in the lab frame would require a potentially messy use of conservation of energy (see the example in Section 4.7.1). But if we work in the CM frame, things are much easier.

The total momentum in the lab frame is mv, so the CM frame moves to the right at speed $mv/(m + M) \equiv u$ with respect to the lab frame. Therefore, in the CM frame, the speeds of the two masses are

$$v_m = v - u = \frac{Mv}{m+M}, \qquad \text{and} \qquad v_M = -u = -\frac{mv}{m+M}. \tag{4.65}$$

These speeds are of course in the ratio M/m, and their difference is v.

In the CM frame, the two particles must simply reverse their velocities after the collision (provided they do indeed hit each other). This is true because the speeds must still be in the ratio M/m after the collision (so that the total momentum is still zero). Therefore, they must either both increase or both decrease. But if they do either of these, then energy is not conserved.[8]

If we now go back to the lab frame by adding the CM speed of $mv/(m+M)$ to the two new speeds of $-Mv/(m+M)$ and $mv/(m+M)$, we obtain final lab speeds of

$$v_m = \frac{(m-M)v}{m+M}\,, \qquad \text{and} \qquad v_M = \frac{2mv}{m+M}\,. \tag{4.66}$$

NOTE: If $m = M$, then the left mass stops, and the right mass picks up a speed of v. If $M \gg m$, then the left mass bounces back with speed $\approx v$, and the right mass hardly moves. If $m \gg M$, then the left mass keeps plowing along at speed $\approx v$, and the right mass picks up a speed of $\approx 2v$. This $2v$ is an interesting result (it is clear if you consider things in the frame of the heavy mass m) which leads to some neat effects, as in Problem 25.

4.6.2 Kinetic energy

Given a system of particles, the relationship between the total kinetic energy in two different frames is generally rather messy and unenlightening. But if one of the frames is the CM frame, then the relationship turns out to be quite nice.

Let S' be the CM frame, which moves at constant velocity $\mathbf{u}$ with respect to another frame S. Then the velocities in the two frames are related by

$$\mathbf{v} = \mathbf{v}' + \mathbf{u}. \tag{4.67}$$

The kinetic energy in the CM frame is

$$\text{KE}_{\text{CM}} = \frac{1}{2}\sum m_i|\mathbf{v}'_i|^2. \tag{4.68}$$

The kinetic energy in frame S is

$$\begin{aligned}
\text{KE}_{\text{S}} &= \frac{1}{2}\sum m_i|\mathbf{v}'_i + \mathbf{u}|^2 \\
&= \frac{1}{2}\sum m_i(\mathbf{v}'_i\cdot\mathbf{v}'_i + 2\mathbf{v}'_i\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{u}) \\
&= \frac{1}{2}\sum m_i|\mathbf{v}'_i|^2 + \mathbf{u}\cdot\Big(\sum m_i\mathbf{v}'_i\Big) + \frac{1}{2}|\mathbf{u}|^2\sum m_i \\
&= \text{KE}_{\text{CM}} + \frac{1}{2}M|\mathbf{u}|^2,
\end{aligned} \tag{4.69}$$

where M is the total mass of the system, and where we have used $\sum_i m_i\mathbf{v}'_i = 0$, by definition of the CM frame. Therefore, the KE in any frame equals the KE in the CM frame, plus the kinetic energy of the whole system treated like a point mass M located at the CM (which moves with velocity $\mathbf{u}$). An immediate corollary of this fact is that if the KE is conserved in a collision in one frame, then it is conserved in any other frame.

[8] So we *did* have to use conservation of energy, but in a far less messy way than the lab frame would have entailed.

4.7 Collisions

There are two basic types of collisions among particles, namely *elastic* ones (in which kinetic energy is conserved), and *inelastic* ones (in which kinetic energy is lost). In any collision, the total energy is conserved, but in inelastic collisions some of this energy goes into the form of heat (that is, relative motion of the atoms inside the particles) instead of showing up in the net translational motion of the particle.

We'll deal mainly with elastic collisions here (although some situations are inherently inelastic, as we'll discuss in Section 4.8). For inelastic collisions where it is stated that a certain fraction, say 20%, of the kinetic energy is lost, we need to make only a trivial modification to the following procedure.

To solve any elastic collision problem, we simply have to write down the conservation of energy and momentum equations, and then solve for whatever variables we want to find.

4.7.1 1-D motion

Let's first look at one-dimensional motion. To see the general procedure, we'll solve the example in Section 4.6.1 again.

Figure 4.18

Example (Two masses in 1-D, again): A mass m with speed v approaches a stationary mass M (see Fig. 4.18). The masses bounce off each other elastically. What are the final speeds of the particles? (Assume all motion takes place in 1-D.)

Solution: Let v' and V' be the final speeds of the masses. Then conservation of momentum and energy give, respectively,

$$\begin{aligned} mv + 0 &= mv' + MV', \\ \frac{1}{2}mv^2 + 0 &= \frac{1}{2}mv'^2 + \frac{1}{2}MV'^2. \end{aligned} \tag{4.70}$$

We must solve these two equations for the two unknowns v' and V'. Solving for V' in the first equation and substituting into the second gives

$$\begin{aligned} mv^2 &= mv'^2 + M\frac{m^2(v-v')^2}{M^2}, \\ \Longrightarrow \quad 0 &= (m+M)v'^2 - 2mvv' + (m-M)v^2, \\ \Longrightarrow \quad 0 &= \Big((m+M)v' - (m-M)v\Big)(v'-v). \end{aligned} \tag{4.71}$$

The $v' = v$ root is obvious, yet useless. $v' = v$ is of course a solution, because the initial conditions certainly satisfy conservation of energy and momentum with the initial conditions (how's that for a tautology). If you want, you can view $v' = v$ as the solution for the case where the particles miss each other. The fact that $v' = v$ is always a root can often save you a lot of quadratic-formula trouble.

The other root is the one we want. Putting this v' back into the first of eqs. (4.70) to obtain V' gives

$$v' = \frac{(m-M)v}{m+M}, \qquad \text{and} \qquad V' = \frac{2mv}{m+M}, \tag{4.72}$$

in agreement with eq. (4.66).

This solution was somewhat of a pain, because it involved quadratic equations. The following theorem is extremely useful because it offers a way to avoid the hassle of quadratic equations when dealing with 1-D elastic collisions.

Theorem 4.3 *In a 1-D elastic collision, the relative velocity of two particles after a collision is the negative of the relative velocity before the collision.*

Proof: Let the masses be m and M. Let v_i and V_i be the initial speeds. Let v_f and V_f be the final speeds. Conservation of momentum and energy give

$$\begin{aligned} mv_i + MV_i &= mv_f + MV_f \\ \frac{1}{2}mv_i^2 + \frac{1}{2}MV_i^2 &= \frac{1}{2}mv_f^2 + \frac{1}{2}MV_f^2. \end{aligned} \tag{4.73}$$

Rearranging these yields

$$\begin{aligned} m(v_i - v_f) &= M(V_f - V_i). \\ m(v_i^2 - v_f^2) &= M(V_f^2 - V_i^2) \end{aligned} \tag{4.74}$$

Dividing the second equation by the first gives $v_i + v_f = V_i + V_f$. Therefore,

$$v_i - V_i = -(v_f - V_f), \tag{4.75}$$

as was to be shown. Note that in taking the quotient of these two equations, we have lost the $v_f = v_i$ and $V_f = V_i$ solution. But, as stated in the above example, this is the trivial solution. ■

This is a splendid theorem. It has the quadratic energy-conservation statement built into it. Hence, using the theorem along with momentum conservation (both of which are linear statements) gives the same information as the standard combination of eqs. (4.73).

Note that the theorem is quite obvious in the CM frame (as we argued in the example in Section 4.6.1). Therefore, it is true in any frame, since it involves differences in velocities.

4.7.2 2-D motion

Let's now look at the more general case of two-dimensional motion. (3-D motion is just more of the same, so we'll confine ourselves to 2-D.) Everything is basically the same as in 1-D, except that there is one more momentum equation, and one more variable to solve for. This is best seen through an example.

Example (Billiards): A billiard ball with speed v approaches an identical stationary one. The balls bounce off each other elastically, in such a way that the incoming ball gets deflected by an angle θ (see Fig. 4.19). What are the final speeds

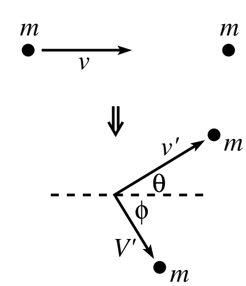


Figure 4.19

of the balls? What is the angle, ϕ, at which the stationary one is ejected?

Solution: Let v' and V' be the final speeds of the balls. Then conservation of p_x, p_y, and E give, respectively,

$$\begin{aligned} mv &= mv'\cos\theta + mV'\cos\phi, \\ mv'\sin\theta &= mV'\sin\phi, \\ \frac{1}{2}mv^2 &= \frac{1}{2}mv'^2 + \frac{1}{2}mV'^2. \end{aligned} \tag{4.76}$$

We must solve these three equations for the three unknowns v', V', and ϕ. There are various ways to do this. Here is one. Eliminate ϕ by adding the squares of the first two equations (after putting the $mv'\cos\theta$ on the left-hand side) to obtain

$$v^2 - 2vv'\cos\theta + v'^2 = V'^2. \tag{4.77}$$

Now eliminate V' by combining this with the third equation to obtain[9]

$$v' = v\cos\theta. \tag{4.78}$$

The third equation then implies

$$V' = v\sin\theta. \tag{4.79}$$

The second equation then gives $m(v\cos\theta)\sin\theta = m(v\sin\theta)\sin\phi$, which implies $\cos\theta = \sin\phi$, or

$$\phi = 90^\circ - \theta. \tag{4.80}$$

In other words, the balls bounce off at right angles with respect to each other. This fact is well known to pool players. Problem 20 gives another (cleaner) way to demonstrate this result.

4.8 Inherently inelastic processes

There is a nice class of problems where the system has inherently inelastic properties, even if it doesn't appear so at first glance. In such a problem, no matter how you try to set it up, there will be an inevitable mechanical energy loss that shows up in the form of heat. Total energy is conserved, of course; heat is simply another form of energy. But the point is that if you try to write down a bunch of $(1/2)mv^2$'s and conserve their sum, then you're going to get the wrong answer. The following example is the classic illustration of this type of problem.

Example (Sand on conveyor belt): Sand drops vertically at a rate σ kg/s onto a moving conveyor belt.

(a) What force must you apply to the belt in order to keep it moving at a constant speed v?

[9] Another solution is $v' = 0$. In this case, ϕ must equal zero, and θ is not well-defined. We simply have the 1-D motion of the example in section 4.6.1.

(b) How much kinetic energy does the sand gain per unit time?

(c) How much work do you do per unit time?

(d) How much energy is lost to heat per unit time?

Solution:

(a) Your force equals the rate of change of momentum. If we let m be the combined mass of conveyor belt plus sand on the belt, then

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} + \frac{dm}{dt}v = 0 + \sigma v, \tag{4.81}$$

where we have used the fact that v is constant.

(b) The kinetic energy gained per unit time is

$$\frac{d}{dt}\left(\frac{mv^2}{2}\right) = \frac{dm}{dt}\left(\frac{v^2}{2}\right) = \frac{\sigma v^2}{2}\,. \tag{4.82}$$

(c) The work done by your force per unit time is

$$\frac{d(\text{Work})}{dt} = \frac{F\,dx}{dt} = Fv = \sigma v^2, \tag{4.83}$$

where we have used eq. (4.81).

(d) If work is done at a rate σv^2, and kinetic energy is gained at a rate $\sigma v^2/2$, then the "missing" energy must be lost to heat at a rate $\sigma v^2 - \sigma v^2/2 = \sigma v^2/2$.

In this example, it turned out that exactly the same amount of energy was lost to heat as was converted into kinetic energy of the sand. There is an interesting and simple way to see why this is true. In the following explanation, we'll just deal with one particle of mass M that falls onto the conveyor belt, for simplicity.

In the lab frame, the mass simply gains a kinetic energy of $Mv^2/2$ (by the time it finally comes to rest with respect to the belt), because the belt moves at speed v.

Now look at things in the conveyor belt's frame of reference. In this frame, the mass comes flying in with an initial kinetic energy of $Mv^2/2$, and then it eventually slows down and comes to rest on the belt. Therefore, all of the $Mv^2/2$ energy is converted to heat. And since the heat is the same in both frames, this is the amount of heat in the lab frame, too.

We therefore see that in the lab frame, the equality of the heat loss and the gain in kinetic energy is a consequence of the obvious fact that the belt moves at the same rate with respect to the lab (namely, v) as the lab moves with respect to the belt (also v).

In the solution to the above example, we did not assume anything about the nature of the friction force between the belt and the sand. The loss of energy to heat is an unavoidable result. You might think that if the sand comes to rest on the belt very "gently" (over a long period of time), then you can avoid the heat loss. This is not the case. In that scenario, the smallness of the friction force is compensated by the fact that the force must act over a very large distance. Likewise, if the sand comes to rest on the belt very abruptly, then the largeness of the friction force is compensated by the smallness of the distance over which it acts. No matter how you set things up, the work done by the friction force is the same nonzero quantity.

Other problems of this sort are included in the problems for this chapter.

4.9 Exercises

Section 4.1: Conservation of energy in 1-D

1. **Heading to zero** $*$

   A particle moves toward $x = 0$ under the influence of a potential $V(x) = -A|x|^n$ (assume $n > 0$). The particle has barely enough energy to reach $x = 0$. For what values of n will it reach $x = 0$ in a finite time?

2. **Heading to infinity** $*$

   A particle moves away from the origin under the influence of a potential $V(x) = -A|x|^n$. For what values of n will it reach infinity in a finite time?

3. **Work in different frames** $**$

   A mass m is initially at rest. A constant force F (directed to the right) acts on it over a distance d. The increase in kinetic energy is therefore Fd.

   Consider the situation from the point of view of someone moving to the left at speed V. Show explicitly that this person measures an increase in kinetic energy equal to force times distance.

4. **Constant $\dot{x}$** $**$

   A bead, under the influence of gravity, slides along a frictionless wire whose height is given by the function $y(x)$. Assume that at position $(x, y) = (0, 0)$, the wire is tilted downward at an angle θ, and that the bead passes this point with a given speed V. What should the shape of the wire be (that is, what is y as a function of x) so that the horizontal speed remains constant at all times?

   Solve this exercise in the spirit of Problem 6, that is, by solving a differential equation. (You can write down the answer very quickly, of course, by simply noting that projectile motion has constant $\dot{x}$.)

5. **Beads on a hoop** $**$

   Two beads of mass m are positioned at the top of a frictionless hoop of mass M and radius R, which stands vertically on the ground. The beads are given tiny kicks, and they slide down the hoop, one to the right and one to the left. What is the smallest value of m/M for which the hoop will rise up off the ground, at some point during the motion?

6. **Tetherball** $****$

   A small ball is attached to a massless string of length L, the other end of which is attached to a very thin pole. The ball is thrown so that it initially travels in a horizontal circle, with the string making an angle θ_0 with the vertical.

   As time goes on, the string will wrap itself around the pole. Assume that (1) the pole is thin enough so that the length of string in the air decreases very slowly, so that the ball's motion may always be approximated as a circle, and

(2) the pole has enough friction so that the string does not slide on the pole, once it touches it.

Let the length of string in the air at a given instant be ℓ, and let the angle it makes with the vertical be θ.

(a) Show that

$$\ell \propto \frac{\cos\theta}{\sin^4\theta}\,. \tag{4.84}$$

(b) Show that the ball finally hits the pole at a distance

$$d = L\cos\theta_0\left(1 - \frac{\sin^2\theta_0}{2}\right). \tag{4.85}$$

below the top of the string.

(c) Show that the ratio of the ball's final speed to initial speed is

$$\frac{v_f}{v_i} = \sin\theta_0\,. \tag{4.86}$$

Section 4.4: Gravity due to a sphere

7. **Speedy travel** $**$

A straight tube is drilled between two points on the earth, as shown in Fig. 4.20. An object is dropped into the tube. What is the resulting motion? How long does it take to reach the other end? You may ignore friction, and you may assume (erroneously) that the density of the earth is constant.

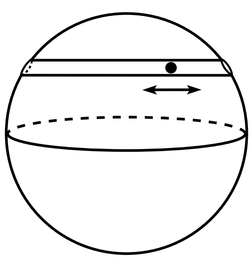


Figure 4.20

Section 4.5: Conservation of linear momentum

8. **Speedy rockets** $**$

Assume that it is impossible to build a structurally sound container that can hold fuel of more than, say, nine times its mass. It would then seem like the limit for the speed of a rocket is $u\ln 10$.

How can you make a rocket that goes faster than this?

9. **Maximum P and E of rocket** $*$

A rocket ejects its fuel at a constant rate per time. What is the mass of the rocket (including unused fuel) when its momentum is maximum? What is the mass when its energy is maximum?

Section 4.7: Collisions

10. **Maximum number of collisions** $**$

N balls are constrained to move in one-dimension. What is the maximum number of collisions they may have among themselves? (Assume the collisions are elastic.)

11. **Triangular room** $**$

A ball is thrown against a wall of a very long triangular room which has vertex angle θ. The initial direction of the ball is parallel to the angle bisector (see Fig. 4.21). How many bounces does the ball make?

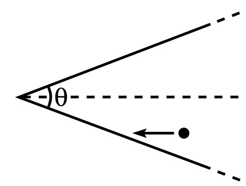


Figure 4.21

12. **Bouncing between rings** $**$

Two circular rings, in contact with each other, stand in a vertical plane (see Fig. 4.22). Each has radius R. A small ball, with mass m and negligible size, bounces elastically back and forth between the rings. (Assume that the rings are held in place, so that they always remain in contact with each other.) Assume that initial conditions have been set up so that the ball's motion forever lies in one parabola. Let this parabola hit the rings at an angle θ from the horizontal.

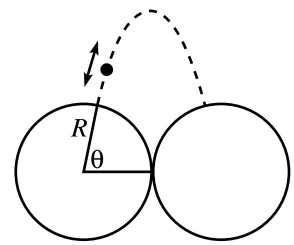


Figure 4.22

(a) Let $\Delta P_x(\theta)$ be the magnitude of the change in the horizontal component of the ball's momentum, at each bounce. For what angle θ is $\Delta P_x(\theta)$ maximum?

(b) Let S be the speed of the ball just before or after a bounce. And let $\overline{F}_x(\theta)$ be the average (over a long period of time) of the magnitude of the horizontal force needed to keep the rings in contact with each other (for example, the average tension in a rope holding the rings together). Consider the two limits: (1) $\theta \approx \epsilon$, and (2) $\theta \approx \pi/2 - \epsilon$, where ϵ is very small.

i. Derive approximate formulas for S, in these two limits.

ii. Derive approximate formulas for $\overline{F}_x(\theta)$, in these two limits.

Which of these two limits requires a larger $\overline{F}_x$?

13. **Bouncing between surfaces** $**$

Consider a more general case of the previous problem. Now let a ball bounce back and forth between a surface defined by $f(x)$ (for $x > 0$) and $f(-x)$ (for $x < 0$) (see Fig. 4.23). Again, assume that initial conditions have been set up so that the ball's motion forever lies in one parabola (the ball bounces back and forth between the contact points at $(x_0, f(x_0))$ and $(-x_0, f(x_0))$, for some x_0).

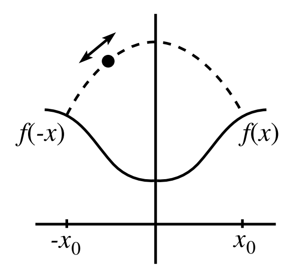


Figure 4.23

(a) Let $\Delta P_x(x_0)$ be the absolute value of the change in the horizontal component of the ball's momentum, at each bounce. For what function $f(x)$ is $\Delta P_x(x_0)$ independent of the contact position x_0?

(b) Let $\overline{F}_x(x_0)$ be the average of the magnitude of the horizontal force needed to keep the two halves of the surface together. For what function $f(x)$ is $\overline{F}_x(x_0)$ independent of the contact position x_0?

Section 4.8: Inherently inelastic processes

14. **Slowing down, speeding up** $*$

    A plate of mass M initially moves horizontally at speed v on a frictionless table. A mass m is dropped vertically onto it and soon comes to rest with respect to the plate. How much energy is required to bring the system back up to speed v?

15. **Downhill dustpan** $***$

    A dustpan slides down a plane inclined at angle θ. Dust is uniformly distributed on the plane, and the dustpan collects the dust in its path. After a long time, what is the acceleration of the dustpan? (Assume there is no friction between the dustpan and plane.)

4.10 Problems

Section 4.1: Conservation of energy in 1-D

1. **Exploding masses**

   A mass M moves with speed V. An explosion divides the mass in half, giving each half a speed v in the CM frame. Calculate the increase in kinetic energy in the lab frame. (Assume all motion is confined to one dimension.)

2. **Minimum length** $**$

   The shortest configuration of string joining three given points is the one shown in Fig. 4.24, where all three angles are $120°$.[10]

   Devise an experimental proof of this fact by attaching three equal masses to three string ends, and then attaching the other three ends together (as shown in Fig. 4.24), and using whatever other props you need.

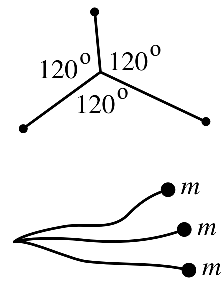


Figure 4.24

3. **Leaving the sphere** $*$

   A small particle rests on top of a frictionless sphere. The particle is given an infinitesimal kick and slides downward. At what point does it lose contact with the sphere?

4. **Pulling the pucks** $**$

   (a) A string of length 2ℓ connects two hockey pucks which lie on frictionless ice. A constant horizontal force, F, is applied to the midpoint of the string, perpendicular to it (see Fig. 4.25). How much kinetic energy is lost when the pucks collide, assuming they stick together?

   (b) The answer you obtained above should be very clean and nice. Find the slick solution (assuming you solved the problem the "normal" way, above) that makes it transparent why the answer is so nice. (This is a neat problem. Try not to peek at the solution.)

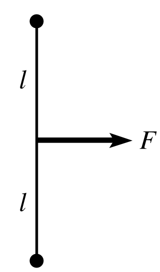


Figure 4.25

5. $V(x)$ **vs. a hill** $****$

   A bead, under the influence of gravity, slides along a frictionless wire whose height is given by the function $V(x)$ (see Fig. 4.26). What is the bead's horizontal acceleration, $\ddot{x}$?

   You should find that your $\ddot{x}$ is not the same as the $\ddot{x}$ for a particle moving in one dimension in the potential $mgV(x)$. If you grab hold of the wire, can you think of anything you can do with it to make the bead's $\ddot{x}$ equal the $\ddot{x}$ due to the 1-D potential $mgV(x)$?

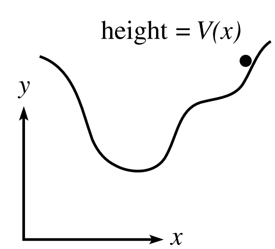


Figure 4.26

[10] If the three points form a triangle that has an angle greater than $120°$, then the vertex of the string lies at the point where that angle is. We won't worry about this case.

6. **Constant $\dot{y}$** $**$

   A bead, under the influence of gravity, slides along a frictionless wire whose height is given by the function $y(x)$. Assume that at position $(x, y) = (0, 0)$, the wire is vertical, and that the bead passes this point with a given speed V, downward. What should the shape of the wire be (that is, what is y as a function of x) so that the vertical speed remains V at all times?

7. **Winding spring** $****$

   A mass m is attached to one end of a spring of zero equilibrium length, the other end of which is fixed. The spring constant is K. Initial conditions are set up so that the mass moves around in a circle of radius L on a frictionless horizontal table. (By "zero equilibrium length", we mean that the equilibrium length is negligible compared to L.)

   At a given time, a vertical pole (of radius a, with $a \ll L$) is placed in the ground next to the center of the circle. The spring winds around the pole, and the mass eventually hits it. Assume that the pole is sticky, so that any part of the spring touching the pole does not slip. How long does it take the mass to hit the pole?

   (Work in the approximation where $a \ll L$.)

Section 4.2: Small Oscillations

8. **Small oscillations**

   A particle moves under the influence of the potential $V(x) = -Cx^n e^{-ax}$. Find the frequency of small oscillations around the equilibrium point.

9. **Hanging mass**

   A particle moves under the influence of the potential $V(x) = (k/2)x^2 + mgx$ (that is, it is a mass hanging from a spring). Find the frequency of small oscillations around the equilibrium point.

Section 4.4: Gravity due to a sphere

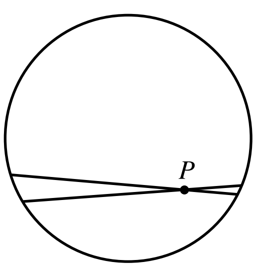


Figure 4.27

10. **Zero force inside a sphere** $**$

    Show that the gravitational force inside a spherical shell is zero by showing that the pieces of mass at the ends of the cones in Fig. 4.27 give canceling forces at point P.

11. **Escape velocity** $*$

    (a) Find the escape velocity (that is, the velocity above which a particle will escape to $r = \infty$) for a particle on a spherical planet of radius R and mass M. What is the numerical value for the earth? The moon? The sun?

    (b) Approximately how small must a spherical planet be in order for a human to be able to jump off? (Assume a density roughly equal to the earth's.)

12. **Ratio of potentials** $**$

Find the ratio of the gravitational potential energy at the corner of a cube (of uniform mass density) to that at the center of the cube. (*Hint:* There's a slick way that doesn't involve any messy integrals.)

Section 4.5: Conservation of linear momentum

13. **Snowball** $*$

A snowball is thrown against a wall. Where does its momentum go? Where does its energy go?

14. **Throwing at a car** $**$

For some odd reason, you decide to throw baseballs at a car (of mass M), which is free to move frictionlessly on the ground. You throw the balls at speed u, and at a mass rate of σ kg/s (assume the rate is continuous, for simplicity). If the car starts at rest, find its speed as a function of time, assuming that the balls bounce (elastically) directly backwards off the back window.

15. **Throwing at a car again** $**$

Do the previous problem, except now assume that the back window is open, so that the balls collect in the inside of the car.

16. **Chain on scale** $**$

A chain of length L and mass density σ is held such that it hangs vertically just above a scale. It is then released. What is the reading on the scale, as a function of the height of the top of the chain?

17. **Leaky bucket** $**$

At $t = 0$, a massless bucket contains a mass M of sand. It is connected to a wall by a massless spring with constant tension T (that is, independent of length). (See Fig. 4.28.) The ground is frictionless. The initial length of the spring is L. At later times, let x be the distance from the wall, and let m be the mass in the bucket.

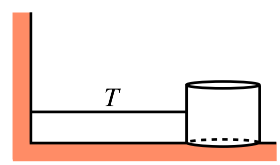


Figure 4.28

The bucket is released. On the way to the wall, the bucket leaks sand at a rate $dm/dx = M/L$ (so the rate is constant with respect to distance, not time; note that dx is negative, so dm is also).

(a) What is the kinetic energy of the (sand in the) bucket, as a function of distance to the wall? What is its maximum value?

(b) What is the momentum of the bucket, as a function of distance to the wall? What is its maximum value?

18. **Another leaky bucket** $***$

At $t = 0$, a massless bucket contains a mass M of sand. It is connected to a wall by a massless spring with constant tension T (that is, independent of length). The ground is frictionless. The initial length of the spring is L. At later times, let x be the distance from the wall, and let m be the mass in the bucket.

The bucket is released. On the way to the wall, the bucket leaks sand at a rate $dm/dt = -bM$ (so the rate is constant with respect to time, not distance; we've factored out an 'M' here, just to make the calculations a little neater).

(a) Find $v(t)$ and $x(t)$ (for the times when the bucket contains a nonzero amount of sand).

(b) What is the maximum value of the bucket's kinetic energy?

(c) What is the maximum value of the magnitude of the bucket's momentum?

(d) For what value of b does the bucket become empty right when it hits the wall?

19. **Yet another leaky bucket** $***$

At $t = 0$, a massless bucket contains a mass M of sand. It is connected to a wall by a massless spring with constant tension T (that is, independent of length). The ground is frictionless. The initial length of the spring is L. At later times, let x be the distance from the wall, and let m be the mass in the bucket.

The bucket is released. On the way to the wall, the bucket leaks sand at a rate proportional to its acceleration, that is, $dm/dt = b\ddot{x}$ (note that $\ddot{x}$ is negative, so dm is also).

(a) Find the mass as a function of time, $m(t)$.

(b) Find $v(t)$ and $x(t)$ (for the times when the bucket contains a nonzero amount of sand).
What is the speed right before all the sand leaves the bucket?

(c) What is the maximum value of the bucket's kinetic energy?

(d) What is the maximum value of the magnitude of the bucket's momentum?

(e) For what value of b does the bucket become empty right when it hits the wall?

Section 4.7: Collisions

20. **Right angle in billiards** $*$

A billiard ball collides elastically with an identical stationary one. Use the fact that the kinetic energy may be written as $m(\mathbf{v} \cdot \mathbf{v})/2$ to show that the angle between the resulting trajectories is 90°.

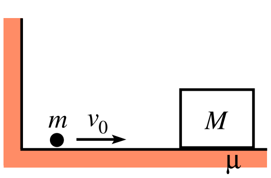


Figure 4.29

21. **Bouncing and recoiling** $***$

A ball of mass m and initial speed v_0 bounces between a fixed wall and a block of mass M (with $M \gg m$). (See Fig. 4.29.) M is initially at rest. Assume the ball bounces elastically and instantaneously. The coefficient of kinetic friction between the block and the ground is μ. There is no friction between the ball and the ground.

What is the speed of the ball after the nth bounce off the block? How far does the block eventually move? How much total time does the block spend moving?

(Work in the approximation $M \gg m$, and assume that μ is large enough so that the block comes to rest by the time the next bounce occurs.)

22. **Drag force on a sheet** ***

A sheet of mass M moves with speed V through a region of space that contains particles of mass m and speed v. There are n of these particles per unit volume. The sheet moves in the direction of its normal. Assume $m \ll M$, and assume the particles do not interact with each other.

(a) Assuming $v \ll V$, what is the drag force per unit area on the sheet?

(b) Assuming $v \gg V$, what is the drag force per unit area on the sheet?

(You may use the fact that the average speed of a particle in the x-direction is $v_x = v/\sqrt{3}$.)

23. **Drag force on a cylinder** **

A cylinder of mass M and radius R moves with speed V through a region of space that contains particles of mass m which are at rest. There are n of these particles per unit volume. The cylinder moves in a direction perpendicular to its axis. Assume $m \ll M$, and assume the particles do not interact with each other.

What is the drag force per unit length on the cylinder?

24. **Drag force on a sphere** **

A sphere of mass M and radius R moves with speed V through a region of space that contains particles of mass m which are at rest. There are n of these particles per unit volume. Assume $m \ll M$, and assume the particles do not interact with each other.

What is the drag force on the sphere?

25. **Basketball and tennis ball** **

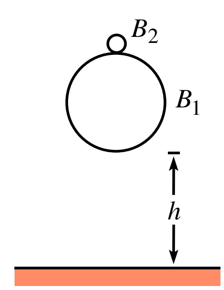


Figure 4.30

(a) A ball, B_2, with (very small) mass m_2 sits on top of another ball, B_1, with (very large) mass m_1 (see Fig. 4.30). The bottom of B_1 is at a height h above the ground, and the bottom of B_2 is at a height $h + d$ above the ground. The balls are dropped. To what height does the top ball bounce?

(You may work in the approximation where m_1 is much heavier than m_2. Assume that the balls bounce elastically. And assume, for the sake of having a nice clean problem, that the balls are initially separated by a small distance, and that the balls bounce instantaneously.)

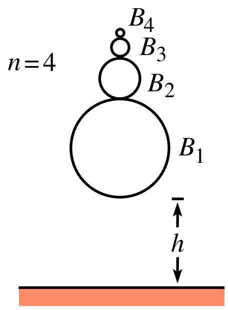


Figure 4.31

(b) n balls, $B_1, \ldots, B_n$, having masses $m_1, m_2, \ldots, m_n$ (with $m_1 \gg m_2 \gg \cdots \gg m_n$), sit in a vertical stack (see Fig. 4.31). The bottom of B_1 is at a height h above the ground, and the bottom of B_n is at a height $h + \ell$ above the ground. The balls are dropped. In terms of n, to what height does the top ball bounce?

(Work in the approximation where m_1 is much heavier than m_2, which is much heavier than m_3, etc., and assume that the balls bounce elastically. Also, make the "nice clean problem" assumptions as in part (a).)

If $h = 1$ meter, what is the minimum number of balls needed in order for the top one to bounce to a height of at least 1 kilometer?

Assume the balls still bounce elastically (which is not likely, in reality). Ignore wind resistance, etc., and assume that ℓ is negligible here.

Section 4.8: Inherently inelastic processes

26. **Colliding masses** $*$

A mass M, initially moving at speed v, collides and sticks to a mass m, initially at rest. Assume $M \gg m$. What are the final energies of the two masses, and how much energy is lost to heat, in:

(a) The lab frame?

(b) The frame in which M is initially at rest?

(Work in the approximation where $M \gg m$.)

27. **Pulling a chain** $**$

A chain of length L and mass density σ lies straight on a frictionless horizontal surface. You grab one end and pull it back along itself, in a parallel manner. Assume that you pull it at constant speed v. What force must you apply? What is the total work that you do, by the time the chain is straightened out? Is any energy lost to heat?

28. **Pulling a rope** $**$

A rope of length L and mass density σ lies in a heap on the floor. You grab an end and pull horizontally with constant force F. What is the position of the end of the rope, as a function of time (while it is unraveling)?

29. **Falling rope** $***$

(a) A rope of length L lies in a straight line on a frictionless table, except for a very small piece of it which hangs down through a hole in the table. The rope is released, and it slides down through the hole. What is the speed of the rope at the instant it loses contact with the table?

(b) A rope of length L lies in a heap on a table, except for a very small piece of it which hangs down through a hole in the table. The rope is released, and it unravels and slides down through the hole. What is the speed of the rope at the instant it loses contact with the table? (Assume the rope is greased, so that it has no friction with itself.)

30. **Raising the rope** $**$

A rope of length L and mass density σ lies in a heap on the floor. You grab one end of the rope and pull upward with a force such that the rope moves at constant speed v.

What is the total work you do, by the time the rope is completely off the floor? How much energy is lost to heat, if any?

31. **The raindrop** $****$

Assume that a cloud consists of tiny water droplets suspended in air (uniformly distributed, and at rest), and consider a raindrop falling through them. After a long time, what is the acceleration of the raindrop?

(Assume that when the raindrop hits a water droplet, the droplet's water gets added to the raindrop. Also, assume that the raindrop is spherical at all times. Ignore air resistance on the raindrop.)

4.11 Solutions

1. **Exploding masses**

   First Solution: In the CM frame, the increase in kinetic energy is

$$2\frac{1}{2}(M/2)v^2 = \frac{1}{2}Mv^2. \tag{4.87}$$

   This increase is due to the work done by the explosion, which is the same in any frame. So $Mv^2/2$ is the increase in kinetic energy in the lab frame, too.

   Using this reasoning, it is clear that $Mv^2/2$ is the answer even if the two pieces aren't constrained to move along the direction of the initial velocity.

   Second Solution: We can also do this problem in the lab frame. The final velocities in the lab frame are $V + v$ and $V - v$. The total kinetic energy is therefore

$$\frac{1}{2}(M/2)(V+v)^2 + \frac{1}{2}(M/2)(V-v)^2 = \frac{1}{2}(M/2)2(V^2+v^2). \tag{4.88}$$

   The initial KE was $MV^2/2$, so the increase is $Mv^2/2$.

   Remark: From the first solution, we know that $Mv^2/2$ is the answer even if the two pieces aren't constrained to move along the direction of the initial velocity. Let's check this explicitly. Let the x-y plane be the plane containing the initial and final velocities. Let the initial velocity lie along the x axis. If the new velocities make an angle θ with respect to the x-axis in the CM frame, then the final velocities in the lab frame are

$$\begin{aligned} \mathbf{V}_1 &= (V + v\cos\theta, v\sin\theta), \\ \mathbf{V}_2 &= (V - v\cos\theta, -v\sin\theta). \end{aligned} \tag{4.89}$$

   The total kinetic energy is therefore

$$\frac{1}{2}(M/2)V_1^2 + \frac{1}{2}(M/2)V_2^2 = \frac{1}{2}(M/2)2(V^2+v^2). \tag{4.90}$$

   The initial KE was $MV^2/2$, so the increase is $Mv^2/2$. ♣

2. **Minimum length**

   Cut three holes in a table, which represent the three given points. Drop the masses through the holes, and let the system reach its equilibrium position.

   The equilibrium position is the one with the lowest potential energy of the masses, i.e., the one with the most string hanging below the table, i.e., the one with the least string lying on the table. So this is the desired 'minimum length' configuration.

   What are the angles at the vertex of the string? The tensions in all three strings are mg. The (massless) vertex of the string is in equilibrium, so the net force on it must be zero. Therefore, each string must bisect the angle formed by the other two. This implies that the three strings must have 120° angles between them.

3. **Leaving the sphere**

   First Solution: Let R be the radius of the sphere, and let θ be the angle of the ball from the top of the sphere. The particle loses contact with the sphere when the normal force becomes zero, i.e., when the normal component of gravity is not large enough to account for the centripetal acceleration of the ball. Thus, the normal force becomes zero when

$$\frac{mv^2}{R} = mg\cos\theta. \tag{4.91}$$

By conservation of energy, we have $mv^2/2 = mgR(1-\cos\theta)$. Hence $v = \sqrt{2gR(1-\cos\theta)}$. Plugging this into eq. (4.91), we see that the particle leaves the sphere when

$$\cos\theta = \frac{2}{3}. \tag{4.92}$$

This corresponds to $\theta \approx 48.2°$.

Second Solution: Assume that the particle always stays in contact with the sphere, and find the point where the horizontal component of v starts to decrease (which it of course can't do). From above, the horizontal component, v_x, is $v_x = v\cos\theta = \sqrt{2gR(1-\cos\theta)}\,\cos\theta$. The derivative of this equals 0 when $\cos\theta = 2/3$, so this is where v_x starts to decrease (if the particle stays on the sphere). Since there is no force available to make v_x decrease, contact is lost when $\cos\theta = 2/3$.

4. **Pulling the pucks**

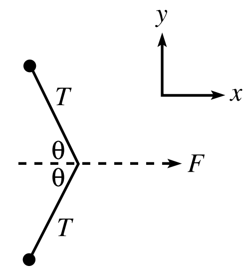


Figure 4.32

(a) Let θ be the angle shown in Fig. 4.32. The tension in the string is then $F/(2\cos\theta)$, because the force on the (massless) kink in the string must be zero. Consider the 'top' puck. The force in the y-direction is $-F\tan\theta/2$. The work done on this puck in the y-direction is therefore

$$\begin{aligned} W_y &= \int_\ell^0 \frac{-F\tan\theta}{2}dy \\ &= \int_{\pi/2}^0 \frac{-F\tan\theta}{2}d(\ell\sin\theta) \\ &= \int_{\pi/2}^0 \frac{-F\ell\sin\theta}{2}\,d\theta \\ &= \frac{F\ell\cos\theta}{2}\bigg|_{\pi/2}^0 \\ &= \frac{F\ell}{2}. \end{aligned} \tag{4.93}$$

The kinetic energy lost when the pucks stick together is twice this. Therefore,

$$\text{KE}_{\text{loss}} = F\ell. \tag{4.94}$$

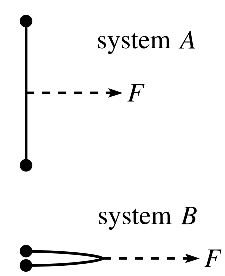


Figure 4.33

(b) Consider two systems, A and B (see Fig. 4.33). A is the original setup, while B starts with θ already at zero. Let the pucks in both systems start at $x = 0$. As the force F is applied, all the pucks will have the same $x(t)$, because the same force in the x-direction, $F/2$, is being applied to every puck at all times. After the collision, both systems look exactly the same.

Let the collision of the pucks occur at $x = d$. At this point, Fd work has been done on system A, while only $F(d-\ell)$ work has been done on system B. Since both systems have the same KE after the collision, the extra $F\ell$ of work done on system A must be what is lost in the collision.

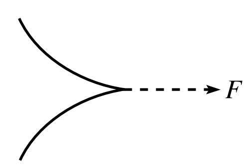


Figure 4.34

Remark: The reasoning in this second solution makes it clear that the same $F\ell$ result holds even if we have many masses, or if we have rope with a continuous mass distribution (so that the rope flops down, as in Fig. 4.34). The only requirement is that the mass be symmetrically distributed around the midpoint. Analyzing this more general setup along the lines of the first solution above would be extremely tedious, at best. ♣

5. $V(x)$ **vs. a hill**

The component of gravity along the wire is what causes the change in speed of the bead. That is,

$$-g\sin\theta = \frac{dv}{dt}\,, \tag{4.95}$$

where θ is given by

$$\tan\theta = V'(x) \qquad \Longrightarrow \qquad \sin\theta = \frac{V'}{\sqrt{1+V'^2}}\,, \qquad \cos\theta = \frac{1}{\sqrt{1+V'^2}}\,. \tag{4.96}$$

We are, however, not concerned with the rate of change of v, but rather with the rate of change of $\dot{x}$, where x is the horizontal position. In view of this, let us write v in terms of $\dot{x}$. Since $\dot{x} = v\cos\theta$, we have $v = \dot{x}/\cos\theta = \dot{x}\sqrt{1+V'^2}$. (Dots denote d/dt. Primes denote d/dx.) Therefore, eq. (4.95) becomes

$$\begin{aligned} \frac{-gV'}{\sqrt{1+V'^2}} &= \frac{d}{dt}\left(\dot{x}\sqrt{1+V'^2}\right) \\ &= \ddot{x}\sqrt{1+V'^2} + \frac{\dot{x}V'\frac{dV'}{dt}}{\sqrt{1+V'^2}}\,. \end{aligned} \tag{4.97}$$

Hence, $\ddot{x}$ is given by

$$\ddot{x} = \frac{-gV'}{1+V'^2} - \frac{\dot{x}V'\frac{dV'}{dt}}{1+V'^2}. \tag{4.98}$$

We'll simplify this in a moment, but first a remark.

Remark: A common incorrect solution to this problem is the following. The acceleration along the curve is $g\sin\theta = -g(V'/\sqrt{1+V'^2})$. Calculating the horizontal component of this acceleration brings in a factor of $\cos\theta = 1/\sqrt{1+V'^2}$. Therefore,

$$\ddot{x} = \frac{-gV'}{1+V'^2}\,. \tag{4.99}$$

But we have missed the second term in eq. (4.98). Where is the mistake? The error is that we forgot to take into account the possible change in the curve's slope. (Eq. (4.99) is true for straight lines.) We addressed only the acceleration due to a change in *speed.* We forgot about the acceleration due to a change in the *direction* of motion. (The term we missed is the one with dV'/dt.) Intuitively, if we have sharp enough bend in the wire, then $\dot{x}$ can change by an arbitrarily large amount. Clearly, eq. (4.99) must be incorrect, since it is bounded (by $g/2$, in fact). ♣

To simplify eq. (4.98), note that $V' \equiv dV/dx = (dV/dt)/(dx/dt) \equiv \dot{V}/\dot{x}$. Therefore,

$$\begin{aligned} \dot{x}V'\frac{dV'}{dt} &= \dot{x}V'\frac{d}{dt}\left(\frac{\dot{V}}{\dot{x}}\right) \\ &= \dot{x}V'\left(\frac{\dot{x}\ddot{V} - \dot{V}\ddot{x}}{\dot{x}^2}\right) \\ &= V'\ddot{V} - V'\ddot{x}\left(\frac{\dot{V}}{\dot{x}}\right) \\ &= V'\ddot{V} - V'^2\ddot{x}. \end{aligned} \tag{4.100}$$

Substituting this into eq. (4.98), we obtain the rather aesthetically pleasing result,

$$\ddot{x} = -(g + \ddot{V})V'. \tag{4.101}$$

Therefore, if the rate of change of the particle's vertical speed (that is, $\ddot{V}$) is not zero (and it usually isn't), then $\ddot{x}$ does not equal $-gV'$ which is the acceleration of a particle in the 1-D potential $mgV(x)$.

There is generally no way to construct a curve with height $y(x)$ which gives the same horizontal motion that a 1-D potential $V(x)$ gives, for all initial conditions. We would need $(g + \ddot{y})y' = V'$, for all x. But at a given x, the quantities V' and y' are fixed, whereas $\ddot{y}$ depends on the initial conditions. (For example, if there is a bend in the wire, then $\ddot{y}$ will be large if $\dot{y}$ is large. And $\dot{y}$ depends on how far the bead has fallen.)[11]

Eq. (4.101) holds the key to constructing a situation that does mimic a 1-D potential $V(x)$. All we have to do is get rid of the $\ddot{V}$ term. So here's what we do. We grab our $y = V(x)$ wire and then move it up (and/or down) in precisely the manner that makes the bead stay at the same height with respect to the ground (or move with constant vertical speed). This will make the $\ddot{V}$ term vanish, as desired. (Note that the vertical movement of the curve doesn't change the slope, V'.)

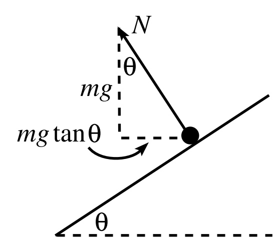


Figure 4.35

REMARK: Now that we've gone through all of the above calculations, here's a much simpler way to solve the problem. Things are much clearer if we examine the normal force, N, acting on the bead (see Fig. 4.35). The upward component is $N\cos\theta$. Therefore, the vertical $F = ma$ equation is

$$m\ddot{y} = -mg + N\cos\theta \quad \Longrightarrow \quad N\cos\theta = -mg - m\ddot{y}. \tag{4.102}$$

The horizontal force is $N\sin\theta$, which equals $(N\cos\theta)\tan\theta = (N\cos\theta)V'$. Hence, using eq. (4.102), the horizontal $F = ma$ equation is

$$m\ddot{x} = (-mg - m\ddot{y})\tan\theta \quad \Longrightarrow \quad \ddot{x} = -(g + \ddot{y})V', \tag{4.103}$$

which is equivalent to eq. (4.101). ♣

6. **Constant $\dot{y}$**

By conservation of energy, the bead's speed at any time is given by (note that y is negative here)

$$\frac{1}{2}mv^2 + mgy = \frac{1}{2}mV^2 \quad \Longrightarrow \quad v = \sqrt{V^2 - 2gy}. \tag{4.104}$$

The vertical component of the speed is $\dot{y} = v\sin\theta$, where $\tan\theta = y' \equiv dy/dx$ is the slope of the wire. Hence, $\sin\theta = y'/\sqrt{1 + y'^2}$, and the requirement that $\dot{y} = -V$ may be written as

$$\sqrt{V^2 - 2gy}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = -V. \tag{4.105}$$

Squaring both sides and solving for $y' \equiv dy/dx$ yields $dy/dx = -V/\sqrt{-2gy}$. Separating variables and integrating gives

$$\int \sqrt{-2gy}\, dy = -V \int dx \quad \Longrightarrow \quad \frac{(-2gy)^{3/2}}{3g} = Vx, \tag{4.106}$$

[11] But see Problem 6 for a particular setup where $\ddot{y} = 0$.

where the constant of integration has been set to zero, since $(x, y) = (0, 0)$ is a point on the curve. Therefore,

$$y = -\frac{(3gVx)^{2/3}}{2g}. \tag{4.107}$$

7. **Winding spring**

Let $\theta(t)$ be the angle through which the spring moves. Let $x(t)$ be the length of the unwrapped part of the spring. Let $v(t)$ be the speed of the mass. And let $k(t)$ be the spring constant of the unwrapped part of the spring. (The manner in which k changes will be derived below.)

Using the approximation $a \ll L$, we may say that the mass undergoes approximate circular motion. (This approximation will break down when x becomes of order a, but the time during which this is true is negligible compared to the total time. We will justify this at the end of the solution.) The instantaneous center of the circle is the point where the spring touches the pole. $F = ma$ along the instantaneous radial direction gives

$$\frac{mv^2}{x} = kx. \tag{4.108}$$

Using this value of v, the frequency of the circular motion is given by

$$\omega \equiv \frac{d\theta}{dt} = \frac{v}{x} = \sqrt{\frac{k}{m}}. \tag{4.109}$$

The spring constant, $k(t)$, of the unwrapped part of the spring is inversely proportional to its equilibrium length. (For example, if you cut a spring in half, the resulting springs have twice the original spring constant). All equilibrium lengths in this problem are infinitesimally small (compared to L), but the inverse relation between k and equilibrium length still holds. (If you want, you can think of the equilibrium length as a measure of the total number of spring atoms that remain in the unwrapped part.)

Note that the change in angle of the contact point on the pole equals the change in angle of the mass around the pole (which is θ.) Consider a small interval of time during which the unwrapped part of the spring stretches a small amount and moves through an angle $d\theta$. Then a length $a\,d\theta$ becomes wrapped on the pole. So the fractional decrease in the equilibrium length of the unwrapped part is (to first order in $d\theta$) equal to $(a\,d\theta)/x$. From the above paragraph, the new spring constant is therefore

$$k_{\text{new}} = \frac{k_{\text{old}}}{1 - \frac{a\,d\theta}{x}} \approx k_{\text{old}}\left(1 + \frac{a\,d\theta}{x}\right). \tag{4.110}$$

Therefore, $dk = ka\,d\theta/x$. Dividing by dt gives

$$\dot{k} = \frac{ka\omega}{x}. \tag{4.111}$$

The final equation we need is the one for energy conservation. At a given instant, consider the sum of the kinetic energy of the mass, and the potential energy of the unwrapped part of the spring. At a time dt later, a tiny bit of this energy will be stored in the newly-wrapped little piece. Letting primes denote quantities at this later time, conservation of energy gives

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}k'x'^2 + \frac{1}{2}m'v'^2 + \left(\frac{1}{2}kx^2\right)\left(\frac{a\,d\theta}{x}\right). \tag{4.112}$$

The last term is (to lowest order in $d\theta$) the energy stored in the newly-wrapped part, because $a\,d\theta$ is its length. Using eq. (4.108) to write the v's in terms of the x's, this becomes

$$kx^2 = k'x'^2 + \frac{1}{2}kxa\,d\theta. \tag{4.113}$$

In other words, $-(1/2)kxa\,d\theta = d(kx^2)$. Dividing by dt gives

$$\begin{aligned} -\frac{1}{2}kxa\omega &= \frac{d(kx^2)}{dt} \\ &= \dot{k}x^2 + 2kx\dot{x} \\ &= \left(\frac{ka\omega}{x}\right)x^2 + 2kx\dot{x}, \end{aligned} \tag{4.114}$$

where we have used eq. (4.111). Therefore,

$$\dot{x} = -\frac{3}{4}a\omega. \tag{4.115}$$

We must now solve the two couple differential equations, eqs. (4.111) and (4.115). Dividing the latter by the former gives

$$\frac{\dot{x}}{x} = -\frac{3}{4}\frac{\dot{k}}{k}. \tag{4.116}$$

Integrating and exponentiating gives

$$k = \frac{KL^{4/3}}{x^{4/3}}, \tag{4.117}$$

where the numerator is obtained from the initial conditions, $k = K$ and $x = L$. Plugging eq. (4.117) into eq. (4.115), and using $\omega = \sqrt{k/m}$, gives

$$x^{2/3}\dot{x} = -\frac{3aK^{1/2}L^{2/3}}{4m^{1/2}}. \tag{4.118}$$

Integrating, and using the initial condition $x = L$, gives

$$x^{5/3} = L^{5/3} - \left(\frac{5aK^{1/2}L^{2/3}}{4m^{1/2}}\right)t. \tag{4.119}$$

So, finally,

$$x(t) = L\left(1 - \frac{t}{T}\right)^{3/5}, \tag{4.120}$$

where

$$T = \frac{4}{5}\frac{L}{a}\sqrt{\frac{m}{K}} \tag{4.121}$$

is the time for which $x(t) = 0$ and the mass hits the pole.

REMARKS:

(a) Note that the angular momentum of the mass around the center of the pole is *not* conserved in this problem, because the force is not a central force.

(b) Integrating eq. (4.115) up to the point when the mass hit the pole gives $-L = -(3/4)a\theta$. But $a\theta$ is the total length wrapped around the pole, which we see is equal to $4L/3$.

(c) We may now justify the assumption of approximate circular motion used above. The exact radial $F = ma$ equation of motion is $-kx = m\ddot{x} - mv^2/x$. Therefore, our neglect of the $\ddot{x}$ term in eq. (4.108) is valid as long as $kx \gg m|\ddot{x}|$. Using our result for $x(t)$ in eq. (4.120), this condition is

$$kL\left(1 - \frac{t}{T}\right)^{3/5} \gg \frac{6}{25}\frac{mL}{T^2}\left(1 - \frac{t}{T}\right)^{-7/5}. \tag{4.122}$$

Using the above expression for T, this becomes

$$kL\left(1 - \frac{t}{T}\right)^{3/5} \gg \frac{3}{8}\frac{Ka^2}{L}\left(1 - \frac{t}{T}\right)^{-7/5}. \tag{4.123}$$

We can write k as a function of t if we wish, but there is no need, since $k \geq K$. Eq. (4.123) is valid up to t arbitrarily close to T, as long as a is sufficiently small compared to L. The solution for x in eq. (4.120) is therefore an approximate solution to the true equation of motion. ♣

8. **Small oscillations**

We will calculate the equilibrium point x_0, and then use $\omega = \sqrt{V''(x_0)/m}$. We have

$$V'(x) = -Ce^{-ax}x^{n-1}(n - ax). \tag{4.124}$$

So $V'(x) = 0$ when $x = n/a \equiv x_0$. The second derivative is

$$V''(x) = -Ce^{-ax}x^{n-2}\Big((n - 1 - ax)(n - ax) - ax\Big). \tag{4.125}$$

Plugging in $x_0 = n/a$ simplifies this a bit, and we find

$$\omega = \sqrt{\frac{V''(x_0)}{m}} = \sqrt{\frac{Ce^{-n}n^{n-1}}{ma^{n-2}}}. \tag{4.126}$$

9. **Hanging mass**

We will calculate the equilibrium point x_0, and then use $\omega = \sqrt{V''(x_0)/m}$. We have

$$V'(x) = kx + mg. \tag{4.127}$$

So $V'(x) = 0$ when $x = -mg/k \equiv x_0$. The second derivative is

$$V''(x) = k. \tag{4.128}$$

So we have

$$\omega = \sqrt{\frac{V''(x_0)}{m}} = \sqrt{\frac{k}{m}}. \tag{4.129}$$

This is independent of x_0, which is what we expect. The only effect of gravity is to change the equilibrium position. If x_r is the position relative to x_0, then the force is $-kx_r$, so it still looks like a regular spring. (This works, of course, only because the spring force in linear.)

10. **Zero force inside a sphere**

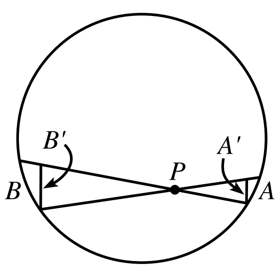


Figure 4.36

Let a be the distance from P to piece A, and let b be the distance from P to piece B (see Fig. 4.36). Draw the 'perpendicular' bases of the cones; call them A' and B'. The ratio of the areas of A' and B' is a^2/b^2.

The key point to realize here is that the angle between the planes of A and A' is the same as that between B and B'. (This is true because the chord between A and B meets the circle at equal angles at its ends. A circle is the only figure with this property for all chords.) So the ratio of the areas of A and B is also a^2/b^2. But the gravitational force decreases like $1/r^2$, so the forces at P due to A and B are equal in magnitude (and opposite in direction, of course).

11. **Escape velocity**

(a) The initial kinetic energy, $mv_{\text{esc}}^2/2$, must account for the gain in potential energy, GMm/R, out to infinity. Therefore,

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}. \tag{4.130}$$

In terms of the acceleration, $g = GM/R^2$, at the surface of the earth, v_{esc} may be written as $v_{\text{esc}} = \sqrt{2gR}$. Using $M = 4\pi\rho R^3/3$, we may also write it as $v_{\text{esc}} = \sqrt{8\pi G R^2 \rho/3}$.

We'll use the values of g in Appendix I to find the numerical values:

For the earth, $v_{\text{esc}} = \sqrt{2gR} = \sqrt{2(10\,\text{m/s}^2)(6.4 \cdot 10^6\,\text{m})} \approx 11,300\,\text{m/s}$.

For the moon, $v_{\text{esc}} = \sqrt{2gR} = \sqrt{2(1.62\,\text{m/s}^2)(1.74 \cdot 10^6\,\text{m})} \approx 2,370\,\text{m/s}$.

For the sun, $v_{\text{esc}} = \sqrt{2gR} = \sqrt{2(274\,\text{m/s}^2)(7 \cdot 10^8\,\text{m})} \approx 620,000\,\text{m/s}$.

Remark: Another reasonable question to ask is: what is the escape velocity from the sun for an object located where the earth is? (But imagine that the earth isn't there.) The answer is $\sqrt{2GM_S/R_{E,S}}$, where $R_{E,S}$ is the earth-sun distance. Numerically, this is $\sqrt{2(6.67 \cdot 10^{-11})(2 \cdot 10^{30})/(1.5 \cdot 10^{11})} \approx 42,200\,\text{m/s}$. ♣

(b) To get a rough answer, let's assume that the initial speed of the jump is the same on the small planet as it is on the earth (this probably isn't quite true, but it's close enough for the purposes here). A good jump on the earth is one meter. For this jump, $mv^2/2 = mg(1)$. Therefore, $v = \sqrt{2g} \approx \sqrt{20}$m/s. So we want $\sqrt{20} = \sqrt{8\pi G R^2 \rho/3}$. Using $\rho \approx 5500\,\text{kg/m}^3$, we find $R \approx 2.5\,\text{km}$. On such a planet, you should indeed tread lightly. And don't sneeze.

12. **Ratio of potentials**

Let ρ be the mass density of the cube. Let V_ℓ^{cor} be the potential energy at the corner of a cube of side ℓ. Let V_ℓ^{cen} be the potential energy at the center of a cube of side ℓ. By dimensional analysis,

$$V_\ell^{\text{cor}} \propto \frac{Q}{\ell} = \rho \ell^2. \tag{4.131}$$

Therefore,[12]

$$V_\ell^{\text{cor}} = 4V_{\ell/2}^{\text{cor}}\,. \tag{4.132}$$

But by superposition, we have

$$V_\ell^{\text{cen}} = 8V_{\ell/2}^{\text{cor}}\,, \tag{4.133}$$

because the center of the larger cube lies at a corner of the eight smaller cubes of which it is made. Therefore,

$$\frac{V_\ell^{\text{cor}}}{V_\ell^{\text{cen}}} = \frac{4V_{\ell/2}^{\text{cor}}}{8V_{\ell/2}^{\text{cor}}} = \frac{1}{2}\,. \tag{4.134}$$

[12] In other words, imagine expanding a cube with side $\ell/2$ to one with side ℓ. If we consider corresponding pieces of the two cubes, then the larger piece has $2^3 = 8$ times the mass of the smaller. But corresponding distances are twice as big in the large cube as in the small cube. Therefore, the larger piece contributes $8/2 = 4$ times as much to V_ℓ^{cor} as the smaller piece contributes to $V_{\ell/2}^{\text{cor}}$.

13. **Snowball**

All of the momentum of the snowball, mv, goes into the earth. So the earth translates (and rotates) a tiny bit faster (or slower, depending on which way the snowball was thrown).

Let M be the mass of the earth. Let V be the final speed of the earth (with respect to the original rest frame of the earth). Then $V \approx (m/M)v$, with $m \ll M$. The kinetic energy of the earth is therefore

$$\frac{1}{2}M\left(\frac{mv}{M}\right)^2 = \frac{1}{2}mv^2\left(\frac{m}{M}\right) \ll \frac{1}{2}mv^2. \tag{4.135}$$

(There is also a rotational kinetic-energy term of the same order of magnitude; but this doesn't matter.) So essentially none of the snowball's energy goes into the earth. It therefore all goes into the form of heat, which melts some of the snow.

This is a general result for a small object hitting a large object: The large object picks up all the momentum but essentially none of the energy.

14. **Throwing at a car** $**$

Let the speed of the car be $v(t)$. Consider the collision of a given ball (let it have mass ϵ) with the car. In the instantaneous rest frame of the car, the speed of the ball is $u - v$. In this frame, the ball reverses velocity when it bounces, so its change in momentum is (negative) $2\epsilon(u - v)$. The change in momentum is the same in the lab frame (since the two frames are related by a given speed at any instant). Therefore, in the lab frame the car gains a momentum of $2\epsilon(u - v)$, for each ball that hits it. The rate of change in momentum of the car (that is, the force) is thus

$$M\,dv/dt = dp/dt = 2\sigma'(u - v), \tag{4.136}$$

where σ' is the rate at which mass hits the car. σ' is related to the given σ by $\sigma' = \sigma(u - v)/u$, because although you throw the balls at speed u, the relative speed of the balls and the car is only $(u - v)$. We therefore arrive at

$$\begin{aligned} M\frac{dv}{dt} &= \frac{2(u-v)^2\sigma}{u} \\ \Longrightarrow \quad \int_0^v \frac{dv}{(u-v)^2} &= \int_0^t \frac{2\sigma}{Mu}\,dt \\ \Longrightarrow \quad \frac{1}{u-v} - \frac{1}{u} &= \frac{2\sigma t}{Mu} \\ \Longrightarrow \quad v(t) &= \frac{\left(\frac{2\sigma t}{M}\right)u}{1 + \frac{2\sigma t}{M}}\,. \end{aligned} \tag{4.137}$$

REMARK: The speed $v(t)$ may be integrated to obtain $x(t) = ut - u(M/2\sigma)\ln(1 + 2\sigma t/M)$. Therefore, even though its speed approaches u, the car will eventually be an arbitrarily large distance behind a ball with constant speed u (for example, pretend that your first ball missed the car and continued to travel forward at speed u). ♣

15. **Throwing at a car again** $**$

We'll carry over some results from the solution for the previous problem. The only change in the calculation of the force on the car is that since the balls don't bounce backwards, we don't pick up the factor of 2 in eq. (4.136). The force is therefore

$$m\frac{dv}{dt} = \frac{(u-v)^2\sigma}{u}, \tag{4.138}$$

where $m(t)$ is the mass of the car-plus-contents, as a function of time. The main difference between this problem and the previous one is that this mass m changes, because the balls are collecting inside the car. As in the previous problem, the rate at which the balls enter the car is $\sigma' = \sigma(u-v)/u$. Therefore,

$$\frac{dm}{dt} = \frac{(u-v)\sigma}{u}. \tag{4.139}$$

We must now solve the two preceding differential equations. Dividing eq. (4.138) by eq. (4.139), and separating variables, gives

$$\int_0^v \frac{dv}{u-v} = \int_M^m \frac{dm}{m} \qquad \Longrightarrow \qquad m = \frac{Mu}{u-v}. \tag{4.140}$$

(Note that $m \to \infty$ as $v \to u$, as it should.) Substituting this value for m into either of eqs. (4.138) or (4.139) gives

$$\begin{aligned} \int_0^v \frac{dv}{(u-v)^3} &= \int_0^t \frac{\sigma}{Mu^2}\,dt \\ \Longrightarrow \qquad \frac{1}{2(u-v)^2} - \frac{1}{2u^2} &= \frac{\sigma t}{Mu^2} \\ \Longrightarrow \qquad v(t) &= u\left(1 - \frac{1}{\sqrt{1+\frac{2\sigma t}{M}}}\right). \end{aligned} \tag{4.141}$$

REMARK: Note that if we erase the square-root sign here, we obtain the answer to the previous problem, eq. (4.137). For a given t, the $v(t)$ in eq. (4.141) is smaller than the $v(t)$ in eq. (4.137). It is clear that this should be the case, since the balls have less of an effect on $v(t)$, because (a) they now don't bounce back, and (b) the mass of the car-plus-contents is now larger. ♣

16. **Chain on scale**

Let y be the height of the chain. Let F be the desired force applied by the scale. The net force on the whole chain is $F - (\sigma L)g$ (with upward taken to be positive). The momentum of the chain is $(\sigma y)\dot{y}$. Equating the net force to the change in momentum gives

$$\begin{aligned} F - \sigma L g &= \frac{d(\sigma y \dot{y})}{dt} \\ &= \sigma y \ddot{y} + \sigma \dot{y}^2. \end{aligned} \tag{4.142}$$

The part of the chain that is still above the scale is in free-fall. Therefore, $\ddot{y} = -g$, and $\dot{y} = \sqrt{2g(L-y)}$, the usual result for a falling object. Putting these into eq. (4.142) gives

$$\begin{aligned} F &= \sigma L g - \sigma y g + 2\sigma(L-y)g \\ &= 3\sigma(L-y)g. \end{aligned} \tag{4.143}$$

This answer has the expected property of equaling zero when $y = L$, and also the interesting property of equaling $3(\sigma L)g$ right before the last bit touches the scale. Once the chain is completely on the scale, the reading will of course simply be the weight of the chain, namely $(\sigma L)g$.

17. **Leaky bucket**

(a) **First Solution:** The initial position is $x = L$. The given rate of leaking implies that the mass of the bucket at position x is $m = M(x/L)$. Therefore, $F = ma$ gives

$$-T = \frac{Mx}{L}\ddot{x} \qquad \Longrightarrow \qquad \frac{TL}{M} = -xv\frac{dv}{dx}\,. \tag{4.144}$$

Separating variables and integrating yields $C - (TL/M)\ln x = v^2/2$, which we may write as

$$B - \frac{TL}{M}\ln(x/L) = \frac{v^2}{2}. \tag{4.145}$$

(since it's much nicer to have dimensionless arguments in a log). The integration constant, B, must be 0, because $v = 0$ when $x = L$.

The kinetic energy at position x is therefore

$$E = \frac{mv^2}{2} = \left(\frac{Mx}{L}\right)\frac{v^2}{2} = -Tx\ln(x/L). \tag{4.146}$$

In terms of the fraction $z \equiv x/L$, we have $E = -TLz\ln z$. Setting $dE/dz = 0$ to find the maximum gives

$$z = \frac{1}{e} \qquad \Longrightarrow \qquad E_{\text{max}} = \frac{TL}{e}. \tag{4.147}$$

Note that E_{max} is independent of M. (This problem was just an excuse to give you an exercise where the answer contains an "e".)

Remark: We started this solution off by writing down $F = ma$ (where m is the mass of the bucket), and you may be wondering why we didn't use $F = dp/dt$ (where p is the momentum of the bucket). These are clearly different, because $dp/dt = d(mv)/dt = ma + (dm/dt)v$. We used $F = ma$, because at any instant, the mass m is what is being accelerated by the force F.

It is indeed true that $F = dp/dt$, if you let F be *total* force in the problem, and let p be the *total* momentum. The tension T is the only force in the problem, since we've assumed the ground to be frictionless. However, the total momentum consists of both the sand in the bucket and the sand that has leaked out and is sliding along the ground.[13] If you use $F = dp/dt$, with p being the total momentum, then you simply arrive at $F = ma$, as you should check.

See Appendix E for further discussion on the use of $F = ma$ and $F = dp/dt$.

♣

Second solution: Consider a small interval during which the bucket moves from x to $x + dx$ (where dx is negative). The bucket's kinetic energy changes by $(-T)dx$ (this is a positive quantity) due to the work done by the spring, and also changes by a fraction dx/x (this is a negative quantity) due to the leaking. Therefore, $dE = -T\,dx + E\,dx/x$, or

$$\frac{dE}{dx} = -T + \frac{E}{x}. \tag{4.148}$$

[13] If the ground had friction, we would have to worry about its effect on both the bucket and the sand outside, if we wanted to use $F = dp/dt$, where p is the total momentum.

In solving this differential equation, it is convenient to introduce the variable $y = E/x$. Then $E' = xy' + y$. So eq. (4.148) becomes $xy' = -T$, or

$$dy = \frac{-T dx}{x}. \tag{4.149}$$

Integrating gives $y = -T \ln x + C$, which we may write as

$$y = -T \ln(x/L) + B \tag{4.150}$$

(to have a dimensionless argument in the log). Writing this in terms of E, we find

$$E = -Tx \ln(x/L), \tag{4.151}$$

as in the first solution. The integration constant, B, must be 0, because $E = 0$ when $x = L$.

(b) From eq. (4.145) (with $B = 0$), we have $v = \sqrt{2TL/M}\,\sqrt{-\ln z}$ (where $z \equiv x/L$). Therefore,

$$p = mv = (Mz)v = \sqrt{2TLM}\sqrt{-z^2 \ln z}. \tag{4.152}$$

Setting $dp/dz = 0$ to find the maximum gives

$$z = \frac{1}{\sqrt{e}} \qquad \Longrightarrow \qquad p_{\text{max}} = \sqrt{\frac{TLM}{e}}\,. \tag{4.153}$$

We see that the location of p_{max} is independent of M,T,L, but its value is not.

REMARK: E_{max} occurs at a later time (that is, closer to the wall) than p_{max} does. This is because v matters more in $E = mv^2/2$ than it does in $p = mv$. As far as E is concerned, it is beneficial for the bucket to lose a little more mass if it means being able to pick up a little more speed (up to a certain point). ♣

18. **Another leaky bucket**

(a) The given rate of leaking implies that the mass of the bucket at time t is $M(1-bt)$, for $t < 1/b$. Therefore, $F = ma$ gives

$$-T = M(1-bt)\ddot{x} \qquad \Longrightarrow \qquad \frac{-T\,dt}{M(1-bt)} = dv. \tag{4.154}$$

Integration yields

$$v(t) = \frac{T}{bM} \ln(1-bt), \tag{4.155}$$

where the constant of integration is 0, because $v = 0$ when $t = 0$. This equation is valid for $t < 1/b$ (provided that the bucket hasn't hit the wall yet, of course). Integrating $v(t)$ to get $x(t)$ gives (using $\int \ln y = y \ln y - y$)

$$x(t) = L - \frac{T}{b^2M} - \frac{T}{b^2M}\Big((1-bt)\ln(1-bt) - (1-bt)\Big), \tag{4.156}$$

where the constant of integration has been chosen so that $x = L$ when $t = 0$.

(b) The mass at time t is $M(1-bt)$. Using eq. (4.155), the kinetic energy at time t is (with $z \equiv 1-bt$)

$$E = \frac{1}{2}mv^2 = \frac{1}{2}(Mz)v^2 = \frac{T^2}{2b^2M}z\ln^2 z. \tag{4.157}$$

Taking the derivative to find the maximum, we obtain

$$z = \frac{1}{e^2} \qquad \Longrightarrow \qquad E_{\text{max}} = \frac{2T^2}{e^2b^2M}. \tag{4.158}$$

(c) The mass at time t is $M(1-bt)$. Using eq. (4.155), the momentum at time t is (with $z \equiv 1-bt$)

$$p = mv = (Mz)v = \frac{T}{b}z\ln z. \tag{4.159}$$

Taking the derivative to find the maximum magnitude, we obtain

$$z = \frac{1}{e} \qquad \Longrightarrow \qquad |p|_{\text{max}} = \frac{T}{eb}. \tag{4.160}$$

(d) We want $M(1-bt)$ to become zero right when x reaches 0. So we want $x=0$ when $t=1/b$. Eq. (4.156) then gives

$$0 = L - \frac{T}{b^2M} \qquad \Longrightarrow \qquad b = \sqrt{\frac{T}{ML}}. \tag{4.161}$$

REMARK: This is the only combination of M,T,L that has units of t^{-1}. But we needed to do the calculation to show that the numerical factor is 1. b clearly should increase with T and decrease with L. The dependence on M is not as obvious (although if b increases with T then it must decrease with M, from dimensional analysis). ♣

19. **Yet another leaky bucket**

(a) $F=ma$ says that $-T = m\ddot{x}$. Combining this with the given $dm/dt = b\ddot{x}$ gives $mdm = -bTdt$. Integration yields $m^2/2 = C - bTt$. Since $m = M$ when $t=0$ we have $C = M^2/2$. Therefore,

$$m(t) = \sqrt{M^2 - 2bTt}\,. \tag{4.162}$$

This holds for $t < M^2/2bT$ (provided that the bucket hasn't hit the wall yet).

(b) The given equation $dm/dt = b\ddot{x} = bdv/dt$ integrates to $v = m/b + C$. Since $v=0$ when $t=0$ we have $C = -M/b$. Therefore,

$$v(t) = \frac{m-M}{b} = \frac{\sqrt{M^2-2bTt}}{b} - \frac{M}{b}\,. \tag{4.163}$$

At the instant just before all the sand leaves the bucket, we have $m=0$. Therefore, $v = -M/b$.

Integrating $v(t)$ to obtain $x(t)$, we find

$$x(t) = \frac{-(M^2-2bTt)^{3/2}}{3b^2T} - \frac{M}{b}t + L + \frac{M^3}{3b^2T}\,, \tag{4.164}$$

where the constant of integration has been chosen to satisfy $x = L$ when $t=0$.

(c) Using eq. (4.163), the kinetic energy at time t is

$$E = \frac{1}{2}mv^2 = \frac{1}{2b^2}m(m-M)^2. \tag{4.165}$$

Taking the derivative dE/dm to find the maximum, we obtain

$$m = \frac{M}{3} \qquad \Longrightarrow \qquad E_{\text{max}} = \frac{2M^3}{27b^2}\,. \tag{4.166}$$

(d) Using eq. (4.163), the momentum at time t is

$$p = mv = \frac{1}{b}m(m-M). \tag{4.167}$$

Taking the derivative to find the maximum, we obtain

$$m = \frac{M}{2} \qquad \Longrightarrow \qquad p_{\text{max}} = \frac{M^2}{4b}\,. \tag{4.168}$$

(e) We want $m(t) = \sqrt{M^2 - 2bTt}$ to become zero right when x reaches 0. So we want $x = 0$ when $t = M^2/2bT$. Eq. (4.164) then gives

$$0 = \frac{-M}{b}\left(\frac{M^2}{2bT}\right) + L + \frac{M^3}{3b^2T} \qquad \Longrightarrow \qquad b = \sqrt{\frac{M^3}{6TL}}\,. \tag{4.169}$$

20. **Right angle in billiards**

Let $\mathbf{v}$ be the initial velocity. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the final velocities. Conservation of momentum and energy give

$$\begin{aligned} m\mathbf{v} &= m\mathbf{v}_1 + m\mathbf{v}_2, \\ \frac{1}{2}m(\mathbf{v}\cdot\mathbf{v}) &= \frac{1}{2}m(\mathbf{v}_1\cdot\mathbf{v}_1) + \frac{1}{2}m(\mathbf{v}_2\cdot\mathbf{v}_2). \end{aligned} \tag{4.170}$$

Substituting the $\mathbf{v}$ from the first equation into the second, and using $(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{v}_1+\mathbf{v}_2) = \mathbf{v}_1\cdot\mathbf{v}_1 + 2\mathbf{v}_1\cdot\mathbf{v}_2 + \mathbf{v}_2\cdot\mathbf{v}_2$ gives

$$\mathbf{v}_1\cdot\mathbf{v}_2 = 0. \tag{4.171}$$

In other words, the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$ is 90°. (Or $\mathbf{v}_1 = \mathbf{0}$, which means the incoming mass stops because the collision is head-on. Or $\mathbf{v}_2 = \mathbf{0}$, which means the masses miss each other.)

21. **Bouncing and recoiling**

Let v_i be the speed of the ball after the ith bounce. Let V_i be the speed of the block right after the ith bounce. Then conservation of momentum gives

$$mv_i = MV_{i+1} - mv_{i+1}. \tag{4.172}$$

Theorem (4.3) says $v_i = V_{i+1} + v_{i+1}$. So we find (the usual result)

$$v_{i+1} = \frac{M-m}{M+m}v_i \equiv \frac{1-\epsilon}{1+\epsilon}v_i \approx (1-2\epsilon)v_i, \tag{4.173}$$

where $\epsilon \equiv m/M \ll 1$. (Likewise, $V_i \approx 2\epsilon v_i$, to leading order in ϵ.) So the speed of the ball after the nth bounce is

$$v_n = (1-2\epsilon)^n v_0 \approx e^{-2n\epsilon}v_0. \tag{4.174}$$

The total distance traveled by the block is easily obtained from energy conservation. Eventually, the ball has negligible energy, so all of its initial kinetic energy goes into heat from friction. Therefore, $mv_0^2/2 = F_f d = (\mu Mg)d$. So

$$d = \frac{mv_0^2}{2\mu Mg}. \tag{4.175}$$

To find the total time, we can add up the times, t_n, after each bounce. Since force times time is the change in momentum, we have $F_f t_n = MV_n$, and so $(\mu Mg)t_n = M(2\epsilon v_n) = 2M\epsilon e^{-2n\epsilon}v_0$. Therefore,

$$\begin{aligned} t = \sum_{n=1}^{\infty} t_n &= \frac{2\epsilon v_0}{\mu g}\sum_{n=1}^{\infty} e^{-2n\epsilon} \\ &= \frac{2\epsilon v_0}{\mu g}\frac{1}{1-e^{-2\epsilon}} \\ &\approx \frac{2\epsilon v_0}{\mu g}\frac{1}{1-(1-2\epsilon)} \\ &= \frac{v_0}{\mu g}. \end{aligned} \tag{4.176}$$

Note that this t is independent of the masses. Also, note that it is much larger than the result obtained in the case where the ball sticks to the block on the first hit (in which case the answer is $mv_0/(\mu Mg)$.

The calculation of d above can also be done by adding up the geometric series of the distances moved after each bounce.

22. **Drag force on a sheet**

(a) We will set $v = 0$ here. If the sheet hits a particle, then the particle acquires a speed essentially equal to $2V$ (by using Theorem (4.3), or by working in the frame of the heavy sheet), and hence a momentum of $2mV$. In time t, the sheet sweeps through a volume AVt, where A is the area of the sheet. Therefore, in time t, the sheet hits $AVtn$ particles. The sheet therefore loses momentum at a rate of $dP/dt = (AVn)(2mV)$. But $F = dP/dt$, so the force per unit area is

$$\frac{F}{A} = 2mnV^2 \equiv 2\rho V^2, \tag{4.177}$$

where ρ is the mass density of the particles. This depends quadratically on V.

(b) For $v \gg V$, particles now hit the sheet on both sides. Note that we can't set V exactly equal to zero here, because we would obtain a result of zero and miss the lowest-order effect. In solving this problem, we need only consider the particles' motions in the x-direction.

The particles in front of the sheet bounce off with speed $v_x + 2V$ (from Theorem (4.3), with the initial relative speed $v_x + V$). So the change in momentum of these particles is $m(2v_x + 2V)$. The rate at which the sheet hits them is $A(v_x + V)(n/2)$, from the reasoning in part (a). ($n/2$ is the relevant density, because half of the particles move to the right, and half move to the left.)

The particles in back of the sheet bounce off with speed $v_x - 2V$ (from Theorem (4.3), with the initial relative speed $v_x - V$). So the change in momentum of these particles is $m(2v_x - 2V)$. The rate at which the sheet hits them is $A(v_x - V)(n/2)$.

Therefore, the force slowing the sheet down is

$$\frac{F}{A} = \frac{1}{A}\frac{dP}{dt} = \Big((n/2)(v_x+V)\Big)\Big(m(2v_x+2V)\Big) - \Big((n/2)(v_x-V)\Big)\Big(m(2v_x-2V)\Big). \tag{4.178}$$

The leading term in this is

$$\frac{F}{A} = 4nmv_xV \equiv 4\rho v_x V, \tag{4.179}$$

where $v_x = v/\sqrt{3}$. This depends linearly on V.

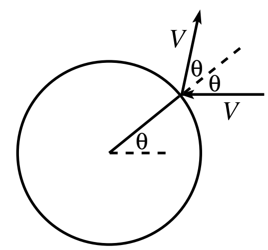


Figure 4.37

23. **Drag force on a cylinder**

Consider a particle which makes contact with the cylinder at an angle θ. In the frame of the (heavy) cylinder (see Fig. 4.37), the particle comes in with speed $-V$ and then bounces off with a horizontal speed $V\cos 2\theta$. So in the lab frame, the particle increases its horizontal momentum by $mV(1+\cos 2\theta)$.

The of area on the cylinder between θ and $\theta + d\theta$ sweeps out a volume at a rate of $(Rd\theta\cos\theta)V\ell$, where ℓ is the length of the cylinder. (The $\cos\theta$ factor gives the projection orthogonal to the direction of motion.)

The force per unit length on the cylinder (that is, the rate of change of momentum, per unit length) is therefore

$$\begin{aligned}
\frac{F}{\ell} &= \int_{-\pi/2}^{\pi/2}\Big(n(Rd\theta\,\cos\theta)V\Big)\Big(mV(1+\cos 2\theta)\Big) \\
&= 2nmRV^2\int_{-\pi/2}^{\pi/2}\cos\theta(1-\sin^2\theta)\,d\theta \\
&= 2nmRV^2\left(\sin\theta - \frac{1}{3}\sin^3\theta\right)\Bigg|_{-\pi/2}^{\pi/2} \\
&= \frac{8}{3}nmRV^2 \equiv \frac{8}{3}\rho RV^2.
\end{aligned} \tag{4.180}$$

Note that the force per cross-sectional area, $F/(2R\ell)$, equals $(4/3)\rho V^2$. This is less than the result for the sheet in the previous problem, as it should be, since the particles bounce off somewhat sideways from the cylinder.

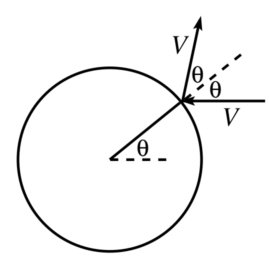


Figure 4.38

24. **Drag force on a sphere**

Consider a particle which makes contact with the sphere at an angle θ. In the frame of the (heavy) sphere (see Fig. 4.38), the particle comes in with speed $-V$ and then bounces off with a horizontal speed $V\cos 2\theta$. So in the lab frame, the particle increases its horizontal momentum by $mV(1+\cos 2\theta)$.

The of area on the sphere between θ and $\theta + d\theta$ (which is a ring of radius $R\sin\theta$) sweeps out a volume at a rate of $(2\pi R\sin\theta)(Rd\theta)\cos\theta\, V$. (The $\cos\theta$ factor gives the projection orthogonal to the direction of motion.)

The force on the sphere (that is, the rate of change of momentum) is therefore

$$\begin{aligned}
F &= \int_0^{\pi/2}(n2\pi R^2\sin\theta\cos\theta V)mV(1+\cos 2\theta)d\theta \\
&= 2\pi nmR^2V^2\int_0^{\pi/2}\sin\theta\cos\theta(1+\cos 2\theta)d\theta
\end{aligned}$$

$$\begin{aligned} &= 2\pi nmR^2V^2 \int_0^{\pi/2} \left(\frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{4} \right) d\theta \\ &= \pi nmR^2V^2 \equiv \rho\pi R^2V^2. \end{aligned} \tag{4.181}$$

Note that the force per cross-sectional area, $F/(\pi R^2)$, equals ρV^2. This is less than the results in the two previous problems, as it should be, since the particles bounce off in a more sideways manner from the sphere.

25. **Basketball and tennis ball**

(a) Just before B_1 hits the ground, both balls are moving downward with speed $v = \sqrt{2gh}$ (from $mv^2/2 = mgh$). Just after B_1 hits the ground, it moves upward with speed v, while B_2 is still moving downward with speed v. The relative speed is therefore $2v$. After the balls bounce off each other, the relative speed is still $2v$, from Theorem (4.3). Since the speed of B_1 stays essentially equal to v, the upward speed of B_2 is therefore $2v + v = 3v$. By conservation of energy, it will therefore rise to a height of $H = d + (3v)^2/(2g)$, or

$$H = d + 9h. \tag{4.182}$$

(b) Just before B_1 hits the ground, all of the balls are moving downward with speed $v = \sqrt{2gh}$.

We will inductively determine the speed of each ball after it bounces off the one below it. If B_i achieves a speed of v_i after bouncing off B_{i-1}, then what is the speed of B_{i+1} after it bounces off B_i? The relative speed of B_{i+1} and B_i (right before they bounce) is $v + v_i$. This is also the relative speed after they bounce. The final upward speed of B_{i+1} is therefore $(v + v_i) + v_i$, so

$$v_{i+1} = 2v_i + v. \tag{4.183}$$

Since $v_1 = v$, we obtain $v_2 = 3v$ (in agreement with part (a)), $v_3 = 7v$, $v_4 = 15v$, etc. In general,

$$v_n = (2^n - 1)v, \tag{4.184}$$

which is easily seen to satisfy eq. (4.183), with the initial value $v_1 = v$.

From conservation of energy, B_n will bounce to a height of $H = \ell + (2^n - 1)^2 v^2/(2g)$, or

$$H = \ell + (2^n - 1)^2 h. \tag{4.185}$$

If h is 1 meter, and we want this height to equal 1000 meters, then (assuming ℓ is not very large) we need $2^n - 1 > \sqrt{1000}$. Five balls won't quite do the trick, but six will, and in this case the height is almost four kilometers.

REMARK: Escape velocity from the earth is reached when $n = 14$. Of course, the elasticity assumption is absurd in this case, as is the notion that one may find 14 balls with the property that $m_1 \gg m_2 \gg \cdots \gg m_{14}$. ♣

26. **Colliding masses**

(a) The initial energy of M is $Mv^2/2$. By conservation of momentum, the final speed of the combined masses is $Mv/(M + m) \approx (1 - m/M)v$. The final energies are therefore

$$\begin{aligned} E_m &= \frac{1}{2}m\left(1 - \frac{m}{M}\right)^2 v^2 \approx \frac{1}{2}mv^2, \\ E_M &= \frac{1}{2}M\left(1 - \frac{m}{M}\right)^2 v^2 \approx \frac{1}{2}Mv^2 - mv^2. \end{aligned} \tag{4.186}$$

The missing energy, $mv^2/2$, is lost to heat.

(b) In this frame, m has initial speed v, so its initial energy is $mv^2/2$. By conservation of momentum, the final speed of the combined masses is $mv/(M+m) \approx (m/M)v$. The final energies are therefore

$$\begin{aligned} E_m &= \frac{1}{2}m\left(\frac{m}{M}\right)^2 v^2 = \left(\frac{m}{M}\right)^2 E \approx 0, \\ E_M &= \frac{1}{2}M\left(\frac{m}{M}\right)^2 v^2 = \left(\frac{m}{M}\right) E \approx 0. \end{aligned} \quad (4.187)$$

The missing energy, $mv^2/2$, is lost to heat, in agreement with part (a).

27. **Pulling a chain**

Let x be the distance your hand has moved. Then $x/2$ is the length of the moving part of the chain. The momentum of this part is therefore $p = (\sigma x/2)\dot{x}$. $F = dp/dt$ gives $F = \sigma\dot{x}^2/2 + \sigma x\ddot{x}$. Since v is assumed to be constant, the $\ddot{x}$ term vanishes. (The change in momentum here is due to additional mass acquiring speed v, not due to an increase in speed.) Hence,

$$F = \frac{\sigma v^2}{2}, \quad (4.188)$$

which is constant. Your hand applies this force over a distance $2L$, so the total work you do is

$$F(2L) = \sigma L v^2. \quad (4.189)$$

The total mass of the chain is σL, so the final kinetic energy of the chain is $(\sigma L)v^2/2$. This is only half of the work you did. Therefore, an energy of $(\sigma L)v^2/2$ is lost to heat.

Each atom goes abruptly from rest to speed v, and there is no way to avoid heat loss in such a process. This is quite clear when viewed in the reference frame of your hand. In that frame, the chain initially moves at speed v and eventually comes to rest, piece by piece.

28. **Pulling a rope**

Let x be the position of the end of the rope. Then the momentum of the rope is $(\sigma x)\dot{x}$. Integrating $F = dp/dt$, and using the fact that F is constant, gives $Ft = p = (\sigma x)\dot{x}$. Separating variables and integrating yields

$$\begin{aligned} \int_0^x \sigma x\, dx &= \int_0^t Ft\, dt \\ \Longrightarrow \qquad \frac{\sigma x^2}{2} &= \frac{Ft^2}{2} \\ \Longrightarrow \qquad x &= t\sqrt{F/\sigma}\,. \end{aligned} \quad (4.190)$$

The position therefore grows linearly with time. That is, the speed is constant, and it equals $\sqrt{F/\sigma}$.

Remark: Realistically, when you grab the rope, there is some small initial value of x (call it ϵ). The dx integral above now starts at ϵ instead of 0, and x takes the form $x = \sqrt{Ft^2/\sigma + \epsilon^2}$. If ϵ is very small, the speed very quickly approaches $\sqrt{F/\sigma}$.

Even if ϵ is not very small, the position becomes arbitrarily close to $t\sqrt{F/\sigma}$, as t becomes large. The "head-start" of ϵ will therefore not help you, in the long run. In retrospect, this has to be the case, since the phrase "if ϵ is not very small" makes no sense, because there is no natural length-scale in the problem which we can compare ϵ with. The only other length that can be formed is $t\sqrt{F/\sigma}$, so any effects of ϵ must arise through a series expansion in the dimensionless quantity $\epsilon/(t\sqrt{F/\sigma})$, which goes to zero for large t, independent of the value of ϵ. ♣

29. **Falling rope**

(a) **First Solution:** Let σ be the density of the rope. By conservation of energy, we may say that the rope's final kinetic energy, $(\sigma L)v^2/2$, equals its loss in potential energy, which is $(\sigma L)(L/2)g$ (because the center-of-mass falls a distance $L/2$). Therefore,

$$v = \sqrt{gL}\,. \tag{4.191}$$

This is the same speed as that obtained by an object that falls a distance $L/2$. If the initial piece hanging down through the hole is arbitrarily short, then the rope will, of course, take an arbitrarily long time to fall down. But the final speed will be always be (arbitrarily close to) $\sqrt{gL}$.

Second Solution: Let σ be the density of the rope. Let x be the length that hangs down through the hole. The $F = ma$ gives $(\sigma x)g = (\sigma L)\ddot{x}$. Therefore, $\ddot{x} = (g/L)x$, and the general solution for x is

$$x = Ae^{t\sqrt{g/L}} + Be^{-t\sqrt{g/L}}. \tag{4.192}$$

(If ϵ is the initial value for x, then $A = B = \epsilon/2$ will satisfy the initial conditions $x(0) = \epsilon$ and $\dot{x}(0) = 0$. But we won't need this information in what follows.)

Let t_f be the time for which $x(t_f) = L$. If ϵ is very small (more precisely, if $\epsilon \ll L$), then t_f will be very large. We may therefore neglect the negative-exponent term in eq. (4.192) for this t_f. We then have $L \approx Ae^{t_f\sqrt{g/L}}$. Hence,

$$\dot{x}(t_f) = \sqrt{g/L}\,Ae^{t_f\sqrt{g/L}} = \sqrt{g/L}\,(L) = \sqrt{gL}\,. \tag{4.193}$$

(b) Let σ be the density of the rope. Let x be the length that hangs down through the hole. Then the force on the rope is $(\sigma x)g$. The momentum of the rope is $(\sigma x)\dot{x}$. Therefore, $F = dp/dt$ gives

$$xg = x\ddot{x} + \dot{x}^2. \tag{4.194}$$

($F = ma$ would give the wrong equation, because it neglects the fact that m is changing. It therefore misses the last term in eq. (4.194).)

Since g is the only parameter in eq. (4.194), the solution for $x(t)$ can involve only g's and t's. (The other dimensionful quantities in this problem, L and σ, do not appear in 4.194, so they cannot appear in the solution.) By dimensional analysis, $x(t)$ must therefore be of the form $x(t) = bgt^2$, where b is a numerical constant. Plugging this into eq. (4.194) and dividing by g^2t^2 gives $b = 2b^2 + 4b^2$, and so $b = 1/6$. Our solution may therefore be written as

$$x(t) = \frac{1}{2}\left(\frac{g}{3}\right)t^2. \tag{4.195}$$

This is the equation for something that accelerates downward with acceleration $g' = g/3$. The time the rope takes to fall a distance L is $t = \sqrt{2L/g'} = \sqrt{6L/g}$. The final speed in then

$$v = g't = \sqrt{2Lg'} = \sqrt{2gL/3}\,. \tag{4.196}$$

This is less than the $\sqrt{gL}$ result from part (a). We therefore see that although the total time for the scenario in part (a) is very large, the final speed is still larger than that in the present scenario.

REMARKS: From eq. (4.196), we see that 1/3 of the available potential energy is lost to heat. This loss occurs during the abrupt motions that suddenly bring the atoms from zero to non-zero speed when they join the moving part of the rope. Using conservation of energy, therefore, is *not* a valid way to solve this problem.

You can show that the speed in part (a)'s scenario is smaller than the speed in part (b)'s scenario for x less than $2L/3$, but larger for x greater than $2L/3$. ♣

30. **Raising the rope**

Let y be the height of the top of the rope. Let $F(y)$ be the desired force applied by your hand. The net force on the moving part of the rope is $F - (\sigma y)g$, with upward taken to be positive. The momentum of the rope is $(\sigma y)\dot{y}$. Equating the net force to the change in momentum gives

$$\begin{aligned} F - \sigma y g &= \frac{d(\sigma y \dot{y})}{dt} \\ &= \sigma y \ddot{y} + \sigma \dot{y}^2. \end{aligned} \tag{4.197}$$

But $\ddot{y} = 0$, and $\dot{y} = v$. Therefore,

$$F = \sigma y g + \sigma v^2. \tag{4.198}$$

The work that you do is the integral of this force, from $y = 0$ to $y = L$. Hence,

$$W = \frac{\sigma L^2 g}{2} + \sigma L v^2. \tag{4.199}$$

The final potential energy of the rope is $(\sigma L)g(L/2)$, because the center-of-mass is at height $L/2$. This is the first term in eq. (4.199). The final kinetic energy is $(\sigma L)v^2/2$. Therefore, the missing energy $(\sigma L)v^2/2$ is converted into heat. (This is clear, when viewed in the reference frame of your hand. The whole rope is initially moving with speed v, and eventually it all comes to rest.)

31. **The raindrop**

Let ρ be the mass density of the raindrop, and let λ be the average mass density in space of the water droplets. Let $r(t)$, $M(t)$, and $v(t)$ be the radius, mass, and speed of the raindrop, respectively.

The mass of the raindrop is $M = (4/3)\pi r^3 \rho$. Therefore,

$$\dot{M} = 4\pi r^2 \dot{r} \rho = 3M\frac{\dot{r}}{r}. \tag{4.200}$$

Another expression for $\dot{M}$ is obtained by noting that the change in M is due to the acquisition of water droplets. The raindrop sweeps out volume at a rate given by its cross-sectional area times its velocity. Therefore,

$$\dot{M} = \pi r^2 v \lambda. \tag{4.201}$$

The force of Mg on the droplet equals the rate of change of its momentum, namely $dp/dt = d(Mv)/dt = \dot{M}v + M\dot{v}$. Therefore,

$$Mg = \dot{M}v + M\dot{v}. \tag{4.202}$$

We now have three equations involving the three unknowns, r, M, and v.

(**Note:** We *cannot* write down the naive conservation-of-energy equation, because mechanical energy is *not* conserved. The collisions between the raindrop and the droplets are completely inelastic. The raindrop will, in fact, heat up. See the remark at the end of the solution.)

The goal is to find $\dot{v}$ for large t. We will do this by first finding $\ddot{r}$ at large t. Eqs. (4.200) and (4.201) give

$$v = \frac{4\rho}{\lambda}\dot{r} \qquad \Longrightarrow \qquad \dot{v} = \frac{4\rho}{\lambda}\ddot{r}. \tag{4.203}$$

Plugging eqs. (4.200) and (4.203) into eq. (4.202) gives

$$Mg = \left(3M\frac{\dot{r}}{r}\right)\left(\frac{4\rho}{\lambda}\dot{r}\right) + M\left(\frac{4\rho}{\lambda}\ddot{r}\right). \tag{4.204}$$

Therefore,

$$\frac{g\lambda}{\rho}r = 12\dot{r}^2 + 4r\ddot{r}. \tag{4.205}$$

Given that the raindrop falls with constant acceleration at large times, we may write [14]

$$\ddot{r} \approx bg, \qquad \dot{r} \approx bgt, \qquad \text{and} \qquad r \approx \frac{1}{2}bgt^2, \tag{4.206}$$

for large t, where b is a numerical factor to be determined. Plugging eqs. (4.206) into eq. (4.205) gives

$$\left(\frac{g\lambda}{\rho}\right)\left(\frac{1}{2}bgt^2\right) = 12(bgt)^2 + 4\left(\frac{1}{2}bgt^2\right)bg. \tag{4.207}$$

Therefore, $b = \lambda/28\rho$. Hence, $\ddot{r} = g\lambda/28\rho$, and eq. (4.203) gives the acceleration of the raindrop at large t,

$$\dot{v} = \frac{g}{7}, \tag{4.208}$$

independent of ρ and λ.

REMARK: We can calculate how much mechanical energy is lost (and therefore how much the raindrop heats up) as a function of the height fallen.

The fact that v is proportional to $\dot{r}$ (shown in eq. (4.203)) means that the volume swept out by the raindrop is a cone. The center-of-mass of a cone is 1/4 of the way from the base to the apex. Therefore, if M is the mass of the raindrop after it has fallen a height h, then the loss in mechanical energy is

$$E_{\text{lost}} = Mg\frac{h}{4} - \frac{1}{2}Mv^2. \tag{4.209}$$

Using $v^2 = 2(g/7)h$, this becomes

$$\Delta E_{\text{int}} = E_{\text{lost}} = \frac{3}{28}Mgh, \tag{4.210}$$

where ΔE_{int} is the gain in internal thermal energy. The energy required to heat 1g of water by 1 degree is 1 calorie (= 4.18 Joules). Therefore, the energy required to heat 1 kg of water by 1 degree is $\approx$ 4200 J. In other words,

$$\Delta E_{\text{int}} = 4200\, M\, \Delta T, \tag{4.211}$$

[14] We may justify the constant-acceleration statement in the following way. For large t, let r be proportional to t^α. Then the left side of eq. (4.205) goes like t^α, while the right side goes like $t^{2\alpha-2}$. If these are to be equal, then we must have $\alpha = 2$. Hence, $r \propto t^2$, and $\ddot{r}$ is a constant (for large t).

where mks units are used, and T is measured in celsius. (We have assumed that the internal energy is uniformly distributed throughout the raindrop.) Eqs. (4.210) and (4.211) give the increase in temperature as a function of h,

$$4200\,\Delta T = \frac{3}{28} gh. \tag{4.212}$$

How far must the raindrop fall before it starts to boil? If we assume that the water droplets' temperature is near freezing, then the height through which the raindrop must fall to have $\Delta T = 100\,^\circ\mathrm{C}$ is found to be

$$h = 400\,\mathrm{km}. \tag{4.213}$$

We have, of course, idealized the problem. But needless to say, there is no need to worry about getting burned by the rain.

A typical value for h is 10 km, which would raise the temperature by two or three degrees. This effect, of course, is washed out by many other factors. ♣

Chapter 5

The Lagrangian Method

Consider the setup with a mass on the end of a spring. We can, of course, use $F = ma$ to write $m\ddot{x} = -kx$. The solutions to this are sinusoidal functions, as we well know. We can, however, solve this problem in another way which doesn't explicitly use $F = ma$. In many (in fact, probably most) physical situations, this new method is far superior to using $F = ma$. You will soon discover this for yourself when you tackle the problems for this chapter.

We will present our new method by first stating its rules (without any justification) and showing that they somehow end up magically giving the correct answer. We will then give the method proper justification.

5.1 The Euler-Lagrange equations

Here is the procedure. Form the following seemingly silly combination of the kinetic and potential energies (T and V, respectively),

$$\boxed{L \equiv T - V}. \tag{5.1}$$

This is called the *Lagrangian*. Yes, there is a minus sign in the definition (a plus sign would just give the total energy). In the problem of a mass on the end of a spring, $T = m\dot{x}^2/2$ and $V = kx^2/2$, so we have

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \tag{5.2}$$

Now write

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}}. \tag{5.3}$$

(Don't worry, we'll show you in a little while where this comes from.) This is called the *Euler-Lagrange (E-L) equation*. For the problem at hand, we have $\partial L/\partial \dot{x} = m\dot{x}$ and $\partial L/\partial x = -kx$, so eq. (5.3) gives

$$m\ddot{x} = -kx, \tag{5.4}$$

exactly the result obtained using $F = ma$. An equation such as eq. (5.4), which is derived from eq. (5.3), is called an *equation of motion*.[1]

If the problem involves more than one coordinate, as most problems do, we simply have to apply eq. (5.3) to each coordinate. We will obtain as many equations as there are coordinates.

At this point, you may be thinking, "That was a nice little trick, but we just got lucky here; the procedure won't work for a more general problem." Well, let's see. How about if we consider the more general problem of a particle moving in an arbitrary potential, $V(x)$. (We'll stick to one dimension for now.) Then the Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 - V(x). \tag{5.5}$$

The Euler-Lagrange equation, eq. (5.3) gives

$$m\ddot{x} = -\frac{dV}{dx}\,. \tag{5.6}$$

But $-dV/dx$ is simply the force on the particle. So we see that eqs. (5.1) and (5.3) together say exactly the same thing as $F = ma$ (when using a cartesian coordinate in one dimension).

Note that shifting the potential by a given constant clearly has no effect on the equation of motion, since eq. (5.3) involves only derivatives of V. This, of course, is the same as saying that only differences in energy are relevant, and not the actual values.

In the three-dimensional case, where the potential takes the form $V(x, y, z)$, it immediately follows that the three E-L equations may be combined into the vector statement,

$$m\ddot{\mathbf{x}} = -\nabla V. \tag{5.7}$$

In other words, $\mathbf{F} = m\mathbf{a}$.

Let's do one more example to convince you that there's really something non-trivial going on here.

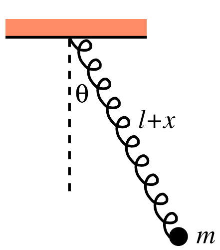


Figure 5.1

Example (Spring pendulum): Consider a pendulum made out of a spring with a mass m on the end (see Fig. 5.1). The spring is arranged to lie in a straight line. (We can do this by, say, wrapping the spring around a rigid massless rod.) The equilibrium length of the spring is ℓ. Let the spring have length $\ell + x(t)$, and let its angle with the vertical be $\theta(t)$. Find the equations of motions for x and θ.

Solution: The kinetic energy may be broken up into its radial and tangential parts, so we have

$$T = \frac{1}{2}m\Big(\dot{x}^2 + (\ell + x)^2\dot{\theta}^2\Big). \tag{5.8}$$

[1] The term "equation of motion" is slightly ambiguous. It is understood to refer to the second-order differential equation satisfied by x, and *not* the actual equation for x as a function of t, namely $x(t) = A\cos(\omega t + \phi)$ (which is obtained by integrating the equation of motion twice).

The potential energy comes from both gravity and the spring, so we have

$$V(x,\theta) = -mg(\ell + x)\cos\theta + \frac{1}{2}kx^2. \tag{5.9}$$

Therefore, the Lagrangian is

$$L = \frac{1}{2}m\Big(\dot{x}^2 + (\ell + x)^2\dot{\theta}^2\Big) + mg(\ell + x)\cos\theta - \frac{1}{2}kx^2. \tag{5.10}$$

There are two variables here, x and θ. The nice thing about the Lagrangian method is that you can simply use eq. (5.3) twice, once with x and once with θ. Hence, the two Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \qquad \Longrightarrow \qquad m\ddot{x} = m(\ell + x)\dot{\theta}^2 + mg\cos\theta - kx, \tag{5.11}$$

and

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \qquad &\Longrightarrow \qquad \frac{d}{dt}\Big(m(\ell + x)^2\dot{\theta}\Big) = -mg(\ell + x)\sin\theta \\ &\Longrightarrow \qquad m(\ell + x)\ddot{\theta} + 2m\dot{x}\dot{\theta} = -mg\sin\theta.\end{aligned} \tag{5.12}$$

Eq. (5.11) is simply the radial $F = ma$ equation (complete with the centripetal acceleration, $(\ell + x)\dot{\theta}^2$). The first line of eq. (5.12) is the statement that torque equals the rate of change of angular momentum (a topic of Chapter 7).[2]

After writing down the E-L equations, it is always best to double-check them by trying to identify them as $F = ma$ or $\tau = dL/dt$ equations. Sometimes, however, this identification is not obvious. For the times when everything is clear (that is, when you look at the E-L equations and say, "Oh . . . of course!"), it is usually clear only *after* you've derived the E-L equations. The Lagrangian method is generally the safer method to use.

The present example should convince you of the great utility of the Lagrangian method. Even if you've never heard of the terms "torque", "centripetal", "centrifugal", or "Coriolis", you can still get the correct equations by simply writing down the kinetic and potential energies, and then taking a few derivatives.

At this point it seems to be personal preference, and all academic, whether you use the Lagrangian method or the $F = ma$ method. The two methods produce the same equations. However, in problems involving more than one variable, it usually turns out to be *much* easier to write down T and V, as opposed to writing down all the forces. This is because T and V are nice and simple scalars. The forces, on the other hand, are vectors, and it's easy to get confused if they point in various directions. The Lagrangian method has the advantage that once you've written down $L = T - V$, you don't have to think anymore. All that remains to be done is to blindly take some derivatives. (Of course, you have to eventually *solve* the

[2] Alternatively, if you want to work in a rotating frame, then eq. (5.11) is the radial $F = ma$ equation, complete with the centrifugal force, $m(\ell + x)\dot{\theta}^2$. And the second line of eq. (5.12) is the tangential $F = ma$ equation, complete with the Coriolis force, $-2m\dot{x}\dot{\theta}$. But never mind about this now; we'll deal with rotating frames in Chapter 9.

resulting equations of motion, but you have to do that when using the $F = ma$ method, too.)

But ease-of-computation aside, is there any fundamental difference between the two methods? Is there any deep reasoning behind eq. (5.3)? Indeed, there is...

5.2 The principle of stationary action

Consider the quantity,

$$S \equiv \int_{t_1}^{t_2} L(x, \dot{x}, t)\, dt. \tag{5.13}$$

S is called the *action*. It is a number with the dimensions of (Energy) × (Time). S depends on L, and L in turn depends on the function $x(t)$ via eq. (5.1).[3] Given any function $x(t)$, we can produce the number S.

S is called a *functional*, and is sometimes denoted by $S[x(t)]$. It depends on the entire function $x(t)$, and not on just one input number, as a regular function $f(t)$ does. S can be thought of as a function of an infinite number of values, namely all the $x(t)$ for t ranging from t_1 to t_2. (If you don't like infinities, you can imagine breaking up the time interval into, say, a million pieces, and then replacing the integral by a discrete sum.)

Let us now pose the following question: Consider a function $x(t)$, for $t_1 \le t \le t_2$, which has its endpoints fixed (that is, $x(t_1) = x_1$ and $x(t_2) = x_2$), but is otherwise arbitrary. What function $x(t)$ yields a stationary value of S? (A stationary value is a local minimum, maximum, or saddle point.[4])

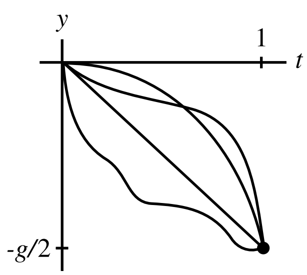


Figure 5.2

For example, consider a ball dropped from rest, and look at $y(t)$ for $0 \le t \le 1$. Assume that we know that $y(0) = 0$ and $y(1) = -g/2$. A number of possible functions are shown in Fig. 5.2, and each of these can (in theory) be plugged into eqs. (5.1) and (5.13) to generate S. Which one yields a stationary value of S? The following theorem will give us the answer.

Theorem 5.1 *If the function $x_0(t)$ yields a stationary value (that is, a local minimum, maximum, or saddle point) for S, then*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_0}\right) = \frac{\partial L}{\partial x_0}\,. \tag{5.14}$$

(It is understood that we are considering the class of functions whose endpoints are fixed. That is, $x(t_1) = x_1$ and $x(t_2) = x_2$.)

Proof: We will use the fact that if a certain function $x_0(t)$ yields a stationary value of S, then another function very close to $x_0(t)$ (with the same endpoint values) will yield essentially the same S, up to first order in any deviations. (This is actually the definition of a stationary value.) The analogy with regular functions is that if

[3] In some situations, the kinetic and potential energies in $L \equiv T - V$ may explicitly depend on time, so we have included the "t" in eq. (5.13).

[4] A saddle point is a point where there are no first-order changes in S, and where some of the second-order changes are positive and some are negative (like the middle of a saddle, of course).

$f(b)$ is a stationary value of f, then $f(b+\delta)$ differs from $f(b)$ only at second order in the small quantity δ. This is true because $f'(b) = 0$, so there is no first-order term in the Taylor series.

Assume that the function $x_0(t)$ yields a stationary value of S, and consider the function

$$x_a(t) \equiv x_0(t) + a\beta(t), \tag{5.15}$$

where $\beta(t)$ satisfies $\beta(t_1) = \beta(t_2) = 0$ (to keep the endpoints of the function the same), but is otherwise arbitrary.

The action $S[x_a(t)]$ is a function of a (the "t" is integrated out, so S is just a number, and it depends on a), and we demand that there be no change in S at first order in a. How does S depend on a? Using the chain rule, we have

$$\begin{aligned} \frac{d}{da}S[x_a(t)] &= \frac{d}{da}\int_{t_1}^{t_2} L\,dt \\ &= \int_{t_1}^{t_2} \frac{dL}{da}\,dt \\ &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x_a}\frac{\partial x_a}{\partial a} + \frac{\partial L}{\partial \dot{x}_a}\frac{\partial \dot{x}_a}{\partial a}\right) dt. \end{aligned} \tag{5.16}$$

In other words, a influences S through its effect on x, and also through its effect on $\dot{x}$. From eq. (5.15), we have

$$\frac{\partial x_a}{\partial a} = \beta, \qquad \text{and} \qquad \frac{\partial \dot{x}_a}{\partial a} = \dot{\beta}, \tag{5.17}$$

so eq. (5.16) becomes[5]

$$\frac{d}{da}S[x_a(t)] = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x_a}\beta + \frac{\partial L}{\partial \dot{x}_a}\dot{\beta}\right) dt. \tag{5.18}$$

Now comes the one sneaky part of the proof. (You will see this trick many times in your physics career.) We will integrate the second term by parts. Using

$$\int \frac{\partial L}{\partial \dot{x}_a}\dot{\beta}\,dt = \frac{\partial L}{\partial \dot{x}_a}\beta - \int \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a}\right)\beta\,dt, \tag{5.19}$$

eq. (5.18) becomes

$$\frac{d}{da}S[x_a(t)] = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x_a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a}\right)\beta\,dt + \left.\frac{\partial L}{\partial \dot{x}_a}\beta\right|_{t_1}^{t_2}. \tag{5.20}$$

But $\beta(t_1) = \beta(t_2) = 0$, so the boundary term vanishes. We now use the fact that $(d/da)S[x_a(t)]$ must be zero for *any* function $\beta(t)$ (assuming that $x_0(t)$ yields a

[5]Note that nowhere do we assume that x_a and $\dot{x}_a$ are independent variables. The partial derivatives in eq. (5.17) are very much related, in that one is the derivative of the other. The use of the chain rule in eq. (5.16) is still perfectly valid.

stationary value). The only way this can be true is if the quantity in parentheses above (evaluated at $a = 0$) is identically equal to zero, that is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_0}\right) = \frac{\partial L}{\partial x_0}. \qquad \blacksquare \tag{5.21}$$

This theorem implies then we may replace $F = ma$ by the following principle.

- **The Principle of Stationary-Action:**

  The path of a particle is the one that yields a stationary value of the action.

This principle is equivalent to $F = ma$ because Theorem 5.1 shows that if (and only if, as you can easily show) we have a stationary value of S, then the E-L equations hold. And the E-L equations are equivalent to $F = ma$ (as we showed for cartesian coordinates in Section 5.1 and which we'll prove for any coordinate system in Section 5.4). So "stationary-action" is equivalent to $F = ma$.

Given a classical mechanics problem, we can solve it with $F = ma$, or we can solve it with the E-L equations and the principle of stationary action (often called the principle of "minimal action", but see Remark 3, below). Either method will get the job done. But as mentioned at the end of Section 5.1, it is often easier to use the latter, because it avoids the use of force (and it's easy to get confused if you have forces pointing in all sorts of complicated directions).

It just stood there and did nothing, of course,
A harmless and still wooden horse.
But the minimal action
Was just a distraction.
The plan involved no use of force.

Let's return now to the example of a ball dropped from rest, mentioned above. You are encouraged to verify explicitly that the path obtained from eq. (5.21) (or equivalently, from $F = ma$), namely $y(t) = -gt^2/2$, yields an action that is smaller (the stationary point happens to be a minimum here) than the action obtained from, say, the path $y(t) = -gt/2$ (which also satisfies the endpoint conditions). Any other such path you choose will also yield an action larger than the action for $y(t) = -gt^2/2$.

The E-L equation, eq. (5.3), therefore doesn't just come out of the blue. It is a necessary consequence of requiring the action to have a stationary value.

REMARKS:

1. Admittedly, Theorem 5.1 simply shifts the burden of proof. We are now left with the task of justifying why we should want the action to have a stationary value. The good news is that there is a very solid reason for wanting this. The bad news is that the reason involves quantum mechanics, so we won't be able to discuss it properly here. Suffice it to say that a particle actually takes all possible paths in going from one place to another, and each path is associated with the complex number $e^{iS/\hbar}$ (where $\hbar = 1.05 \cdot 10^{-34}$ Js is *Planck's constant*). These complex numbers have absolute value

1 and are called "phases". It turns out that the phases from all possible paths must be added up to give the "amplitude" of going from one point to another. The absolute value of the amplitude must then be squared to obtain the probability.[6]

The basic point, then, is that at a non-stationary value of S the phases from different paths differ (greatly, since $\hbar$ is very small) from one another, which effectively leads to the addition of many random vectors in the complex plane. These end up cancelling each other, yielding a sum of essentially zero. There is therefore no contribution to the overall amplitude from non-stationary values of S. Hence, we do not observe the paths associated with these S's. At a stationary value of S, however, all the phases take on essentially the same value, thereby adding constructively instead of destructively. There is therefore a non-zero probability for the particle to take a path that yields a stationary value of S. So this is the path we observe.

2. Admittedly, again, the preceding remark simply shifts the burden of proof one step further. We must now justify why these phases $e^{iS/\hbar}$ should exist, and why the Lagrangian that appears in S should equal $T - V$. But here's where we're going to stop.

3. Our principle of stationary action is often referred to as the principle of "least" action. This is misleading. True, most of the time the stationary value turns out to be a minimum value, but it need not be, as we can see in the following example.

   Consider a harmonic oscillator. The Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \tag{5.22}$$

   Let $x_0(t)$ be a function which yields a stationary value of the action. Then we know that $x_0(t)$ satisfies the E-L equation, $m\ddot{x}_0 = -kx_0$.

   Consider a slight variation on this path, $x_0(t) + \xi(t)$, where $\xi(t)$ satisfies $\xi(t_1) = \xi(t_2) = 0$. With this new function, the action becomes

$$S_\xi = \int_{t_1}^{t_2} \left(\frac{m}{2}\left(\dot{x}_0^2 + 2\dot{x}_0\dot{\xi} + \dot{\xi}^2\right) - \frac{k}{2}\left(x_0^2 + 2x_0\xi + \xi^2\right)\right) dt. \tag{5.23}$$

   The two cross-terms add up to zero, because after integrating the $\dot{x}_0\dot{\xi}$ term by parts, their sum is

$$m\dot{x}_0\xi\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} (m\ddot{x}_0 + kx_0)\xi \, dt. \tag{5.24}$$

   The first term is zero, due to the boundary conditions on $\xi(t)$. The second term is zero, due to the E-L equation. (We've basically just reproduced the proof of Theorem 5.1 for the special case of the harmonic oscillator here.)

   The terms involving only x_0 give the stationary value of the action (call it S_0). To determine whether S_0 is a minimum, maximum, or saddle point, we must look at the difference,

$$\Delta S \equiv S_\xi - S_0 = \frac{1}{2}\int_{t_1}^{t_2} (m\dot{\xi}^2 - k\xi^2) \, dt. \tag{5.25}$$

[6]This is one of those remarks that is completely useless, since it's incomprehensible to those who haven't seen the topic before, and trivial to those who have. My apologies. But this and the following remarks are by no means necessary for an understanding of the material in this chapter. If you're interested in reading more about these quantum mechanics issues, you should take a look at Richard Feynman's book, *QED*. (Feynman was, after all, the one who thought of this idea.)

It is always possible to find a function ξ that makes ΔS positive. (Simply choose ξ to be small, but make it wiggle very fast, so that $\dot{\xi}$ is large.) Therefore, it is *never* the case that S_0 is a maximum. Note that this reasoning works for any potential, as long as it is a function of only position (that is, it contains no derivatives, as we always assume), not just a harmonic oscillator.

You might be tempted to use the same reasoning to say that it is also always possible to find a function ξ that makes ΔS negative, by making ξ large and $\dot{\xi}$ small. (If this were true, then we could put everything together to conclude that all stationary points are saddle points, for a harmonic oscillator.) This, however, is not always possible, due to the boundary conditions $\xi(t_1) = \xi(t_2) = 0$. If ξ is to change from zero to a large value, then $\dot{\xi}$ may also have to be large, if the time interval is short enough. Problem 6 deals quantitatively with this issue. For now, let's just say that in some cases S_0 is a minimum, and in some cases S_0 is a saddle point. "Least action", therefore, is a misnomer.

4. It is sometimes claimed that nature has a "purpose", in that it seeks to take the path that produces the minimum action. In view of the above remark, this is incorrect. In fact, nature does exactly the opposite. It takes *every* path, treating them all on equal footing. We simply end up seeing the path with a stationary action, due to the way the quantum mechanical phases add.

   It would be a harsh requirement, indeed, to demand that nature make a "global" decision (that is, to compare paths that are separated by large distances), and to choose the one with the smallest action. Instead, we see that everything takes place on a "local" scale. Nearby phases simply add, and everything works out automatically.

   When an archer shoots an arrow through the air, the aim is made possible by all the other arrows taking all the other nearby paths, each with essentially the same action. Likewise, when you walk down the street with a certain destination in mind, you're not alone.

   > When walking, I know that my aim
   > Is caused by the ghosts with my name.
   > And although I don't see
   > Where they walk next to me,
   > I know they're all there, just the same.

5. Consider a function, $f(x)$, of one variable (for ease of terminology). Let $f(b)$ be a local minimum of f. There are two basic properties of this minimum. The first is that $f(b)$ is smaller than all nearby values. The second is that the slope of f is zero at b. From the above remarks, we see that (when dealing with the action, S) the first property is completely irrelevant, and the second one is the whole point. In other words, saddle points (and maxima, although we showed above that these never exist for S) are just as good as minima, as far as the constructive addition of the $e^{iS/\hbar}$ phases is concerned.

6. Of course, given that classical mechanics is the approximate theory, while quantum mechanics is the (more) correct one, it is quite silly to justify the principle of stationary action by demonstrating its equivalence with $F = ma$. We should be doing it the other way around. However, because our intuition is based on $F = ma$, we'll assume that it's easier to start with $F = ma$ as the given fact, rather than calling upon the latent quantum-mechanics intuition hidden deep within all of us. Maybe someday.

   At any rate, in more advanced theories dealing with fundamental issues concerning the tiny building blocks of matter (where the action is of the same order of magnitude as $\hbar$), the approximate $F = ma$ theory is invalid, and you *have* to use the Lagrangian method. ♣

5.3 Forces of constraint

One nice thing about the Lagrangian method is that we are free to impose any given constraints at the beginning of the problem, thereby immediately reducing the number of variables. This is always done (perhaps without thinking) whenever a particle is constrained to move on a wire or surface, etc. Often we are not concerned with the exact nature of the forces doing the constraining, but only with the resulting motion, given that the constraints hold. By imposing the constraints at the outset, we can find this motion, but we can't say anything about the constraining forces.

If we want to determine these constraining forces, we must take a different approach. A major point, as we will show, is that we must not impose the constraints too soon. This, of course, leaves us with a larger number of variables to deal with, so the calculations are more cumbersome. But the benefit is that we are able to find the constraining forces.

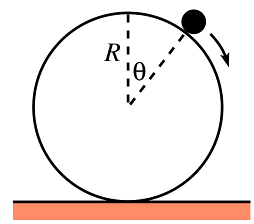


Figure 5.3

Consider the example of a particle sliding off a fixed frictionless sphere of radius R (see Fig. 5.3). Let's say that we are concerned only with finding the equation of motion for θ, and not the constraining forces. Then we can write everything in terms of θ, because we know that the radial distance, r, is constrained to be R. The kinetic energy is $mR^2\dot\theta^2/2$, and the potential energy (relative to the center of the sphere) is $mgr\cos\theta$. The Lagrangian is therefore

$$L = \frac{1}{2}mR^2\dot\theta^2 - mgR\cos\theta, \tag{5.26}$$

and the equation of motion, via eq. (5.3), is

$$\ddot\theta = (g/R)\sin\theta, \tag{5.27}$$

which is simply the tangential $F = ma$ statement.

Now let's say we want to find the constraining normal force that the sphere applies to the particle. To do this, let's solve the problem in a different way and write things in terms of both r and θ. Also (and here's the critical step), let's be really picky and say that r isn't *exactly* constrained to be R, because in the real world the particle actually pushes into the sphere a little bit. This may seem a bit silly, but it's really the whole point. The particle pushes in a (very tiny) distance until the sphere gets squashed enough to push back with the appropriate force to keep the particle from pressing in any more. (Just consider the sphere to be made of lots of little springs with very large spring constants.) The particle is therefore subject to a (very) steep potential due to the sphere. The constraining potential, $V(r)$, looks something like the plot in Fig. 5.4.

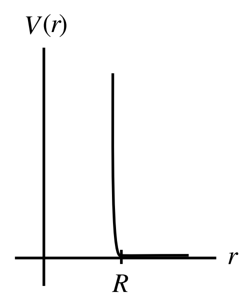


Figure 5.4

The *true* Lagrangian for the system is thus

$$L = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) - mgr\cos\theta - V(r). \tag{5.28}$$

(The $\dot{r}^2$ term in the kinetic energy will turn out to be insignificant.) The equations of motion obtained from varying θ and r are therefore

$$\begin{aligned} mr^2\ddot{\theta} &= mgr\sin\theta, \\ m\ddot{r} &= mr\dot{\theta}^2 - mg\cos\theta - V'(r). \end{aligned} \tag{5.29}$$

Having written down the equations of motion, we will *now* apply the constraint condition that $r = R$. This condition implies $\dot{r} = \ddot{r} = 0$. (Of course, r isn't *really* equal to R, but any differences are inconsequential from this point onward.) The first of eqs. (5.29) then simply gives eq. (5.27), while the second yields

$$-\frac{dV}{dr}\bigg|_{r=R} = mg\cos\theta - mR\dot{\theta}^2. \tag{5.30}$$

But $F \equiv -dV/dr$ is the constraint force applied in the r direction, which is precisely the force we are looking for. The normal force of constraint is therefore

$$F(\theta, \dot{\theta}) = mg\cos\theta - mR\dot{\theta}^2, \tag{5.31}$$

which is simply the radial $F = ma$ statement. Note that this result is valid only if $F(\theta, \dot{\theta}) > 0$, because the particle leaves the sphere (in which case r ceases to be equal to R) if the normal force becomes zero.

REMARKS:

1. What if we instead had (unwisely) chosen our coordinates to be x and y, instead of r and θ? Since the distance from the particle to the surface of the sphere is $\eta \equiv \sqrt{x^2 + y^2} - R$, we obtain a true Lagrangian equal to

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy - V(\eta). \tag{5.32}$$

The equations of motion are (using the chain rule)

$$m\ddot{x} = -\frac{dV}{d\eta}\frac{\partial\eta}{\partial x}, \qquad \text{and} \qquad m\ddot{y} = -mg - \frac{dV}{d\eta}\frac{\partial\eta}{\partial y}. \tag{5.33}$$

We now apply the constraint condition $\eta = 0$. Since $-dV/d\eta$ equals the constraint force F, you can show that the equations we end up with (namely, the two E-L equations and the constraint equation) are

$$m\ddot{x} = F\frac{x}{R}, \qquad m\ddot{y} = -mg + F\frac{y}{R}, \qquad \text{and} \qquad \sqrt{x^2 + y^2} - R = 0. \tag{5.34}$$

These three equations are (in principal) sufficient to determine the three unknowns ($\ddot{x}$, $\ddot{y}$, and F) as functions of the quantities x, $\dot{x}$, y, and $\dot{y}$.

2. You can see from eqs. (5.29) and (5.34) that the E-L equations end up taking the form,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} + F\frac{\partial \eta}{\partial q_i}, \tag{5.35}$$

for each coordinate, q_i. Here η is the constraint equation of the form $\eta = 0$. For the r and θ coordinates, $\eta = r - R$. And for the x and y coordinates, $\eta = \sqrt{x^2 + y^2} - R$. The set of E-L equations, combined with the $\eta = 0$ condition, gives you exactly the number of equations ($N + 1$ of them, where N is the number of coordinates) needed to determine all of the $N + 1$ unknowns (the $\ddot{q}_i$ and F), in terms of the q_i and $\dot{q}_i$.

3. When trying to determine the forces of constraint, you can simply start with eqs. (5.35), without bothering to write down $V(\eta)$, but you must be careful to make sure that η does indeed represent the distance the particle is from where it should be. In the r,θ coordinates, if someone gives you the constraint condition as $7(r-R)=0$, and if you use the left-hand side of this as the η in eq. (5.35), then you will get the wrong constraint force, off by a factor of 7. Likewise, in the x,y coordinates, writing the constraint as $y-\sqrt{R^2-x^2}=0$ would give you the wrong force.

   The best way to avoid this problem is, of course, to pick one of your variables as the distance the particle is from where it should be (or at least a linear function of the distance, as in the case of the "r" above). ♣

5.4 Change of coordinates

When L is written in terms of cartesian coordinates x,y,z, we showed in Section 5.1 that the Euler-Lagrange equations are equivalent to Newton's $\mathbf{F}=m\mathbf{a}$ equations (see eq. (5.7)). But what about the case when we use polar, spherical, or other coordinates? The equivalence of the E-L equations and $\mathbf{F}=m\mathbf{a}$ is not obvious. As far as trusting the E-L equations for such coordinates goes, you can achieve peace-of-mind in two ways. You can accept the principle of stationary action as something so beautiful and so profound that it simply has to work for any choice of coordinates. Or, you can take the more mundane road and show through a change of coordinates that if the E-L equations hold for one set of coordinates (and we know they *do* hold for cartesian coordinates), then they also hold for any other coordinates (of a certain form, described below). In this section, we will demonstrate the validity of the E-L equations through the explicit change of coordinates.[7]

Consider the set of coordinates,

$$x_i : \quad (x_1, x_2, \ldots, x_N). \tag{5.36}$$

For example, x_1,x_2,x_3 could be the cartesian x,y,z coordinates of one particle, and x_4,x_5,x_6 could be the r,θ,ϕ polar coordinates of a second particle, and so on. Assume that the E-L equations hold for these variables, that is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}, \qquad (1 \le i \le N). \tag{5.37}$$

We know that there is at least one set of variables for which this is true, namely the cartesian coordinates. Consider a new set of variables which are functions of the x_i and t,

$$q_i = q_i(x_1, x_2, \ldots, x_N; t). \tag{5.38}$$

We will restrict ourselves to the case where the q_i do not depend on the $\dot{x}_i$. (This is quite reasonable. If the coordinates depended on the velocities, then we wouldn't be able to label points in space with definite coordinates. We'd have to worry about how the particles were behaving when they were at the points. These would be

[7] This calculation is straightforward but a little messy, so you may want to skip this section and just settle for the "beautiful and profound" reasoning.

strange coordinates indeed.) Note that we can, in theory, invert eqs. (5.38) and express the x_i as functions of the q_i,

$$x_i = x_i(q_1, q_2, \ldots, q_N; t). \tag{5.39}$$

Claim 5.2 *If eq. (5.37) is true for the x_i coordinates, and if the x_i and q_i are related by eqs. (5.39), then eq. (5.37) is also true for the q_i coordinates. That is,*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = \frac{\partial L}{\partial q_m}, \qquad (1 \le m \le N). \tag{5.40}$$

Proof: We have

$$\frac{\partial L}{\partial \dot{q}_m} = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial \dot{q}_m}. \tag{5.41}$$

(Note that if the x_i depended on the $\dot{q}_i$, then we would have the additional term, $\sum(\partial L/\partial x_i)(\partial x_i/\partial \dot{q}_m)$, but we have excluded such dependence.) Let's rewrite the $\partial \dot{x}_i/\partial \dot{q}_m$ term. From eq. (5.39), we have

$$\dot{x}_i = \sum_{m=1}^{N} \frac{\partial x_i}{\partial q_m}\dot{q}_m + \frac{\partial x_i}{\partial t}. \tag{5.42}$$

Therefore,

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_m} = \frac{\partial x_i}{\partial q_m}. \tag{5.43}$$

Substituting this into eq. (5.41) and taking the time derivative of both sides gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = \sum_{i=1}^{N} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)\frac{\partial x_i}{\partial q_m} + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{x}_i}\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_m}\right). \tag{5.44}$$

In the second term here, it is legal to switch the order of the d/dt and $\partial/\partial q_m$ derivatives.

REMARK: Let's prove that this switching is legal, just is case you have your doubts.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_m}\right) &= \sum_{k=1}^{N} \frac{\partial}{\partial q_k}\left(\frac{\partial x_i}{\partial q_m}\right)\dot{q}_k + \frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial q_m}\right) \\ &= \frac{\partial}{\partial q_m}\left(\sum_{k=1}^{N} \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial x_i}{\partial t}\right) \\ &= \frac{\partial \dot{x}_i}{\partial q_m}, \end{aligned} \tag{5.45}$$

as was to be shown. ♣

In the first term on the right-hand side of eq. (5.44), we can use the given information in eq. (5.37) and rewrite the $(d/dt)(\partial L/\partial \dot{x}_i)$ term. We obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) &= \sum_{i=1}^{N} \frac{\partial L}{\partial x_i}\frac{\partial x_i}{\partial q_m} + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial q_m} \\ &= \frac{\partial L}{\partial q_m}, \end{aligned} \tag{5.46}$$

as was to be shown. ■

What we have shown is that if the Euler-Lagrange equations are true for a set of coordinates, x_i (and they *are* true for cartesian coordinates), then they are also true for any other set of coordinates, q_i, satisfying eq. (5.38). For those of you who look at the principle of stationary action with distrust (thinking that it might be a coordinate-dependent statement), this proof should put you at ease. The Euler-Lagrange equations are truly equivalent to $F = ma$ in any coordinates.

Note that the above proof did not in any way use the precise form of the Lagrangian. If L were equal to $T + V$, or $7T + \pi V^2/T$, or any other arbitrary function, our result would still be true: If eqs. (5.37) are true for one set of coordinates, then they are also true for any coordinates q_i satisfying eqs. (5.38). The point is that the only L for which the hypothesis is true at all (that is, for which eq. (5.37) holds) is $L = T - V$.

Remark: On one hand, it is quite amazing how little we assumed in proving the above claim. *Any* new coordinates of the very general form (5.38) satisfy the E-L equations, as long as the original coordinates do. If the E-L equations had, say, a factor of 5 on the right-hand side of eq. (5.37), then they would *not* hold in arbitrary coordinates. (To see this, just follow the proof through with the factor of 5.)

On the other hand, the claim is quite obvious, if you make an analogy with a function instead of a functional. Consider the function $f(z) = z^2$. This has a minimum at $z = 0$, consistent with the fact that $df/dz = 0$ at $z = 0$. Let's instead write f in terms of the variable defined by, say, $z = y^4$. Then $f(y) = y^8$, and f has a minimum at $y = 0$, consistent with the fact that df/dy equals zero at $y = 0$. So $f' = 0$ holds in both coordinates at the corresponding points $y = z = 0$. This is the (simplified) analog of the E-L equations holding in both coordinates. In both cases, the derivative equation describes where the stationary value occurs.

This change-of-variables result may be stated in a more geometrical (and friendly) way. If you plot a function and then stretch the horizontal axis in an arbitrary manner, a stationary value (that is, one where the slope is zero) will still be a stationary value after the stretching. (A picture is worth a dozen equations, it appears.)

As an example of an equation that does *not* hold for all coordinates, consider the preceding example, but with $f' = 1$ instead of $f' = 0$. In terms of z, $f' = 1$ when $z = 1/2$. But in terms of y, $f' = 1$ when $y = (1/8)^{1/7}$. The points $z = 1/2$ and $y = (1/8)^{1/7}$ are not the same point. In other words, $f' = 1$ is not a coordinate-independent statement. Most equations, of course, are coordinate dependent. The special thing about $f' = 0$ is that a stationary point is a stationary point no matter how you look at it.[8] ♣

[8] There is, however, one exception. A stationary point in one coordinate system might be located at a kink in another coordinate system, so that f' is not defined there. For example, if we had said that $z = y^{1/4}$, then $f(y) = y^{1/2}$, which has an undefined slope at $y = 0$. Basically, we've stretched (or shrunken) the horizontal axis by a factor of infinity at the relevant point. But let's not worry about this.

5.5 Conservation Laws

5.5.1 Cyclic coordinates

Consider the case where the Lagrangian does not depend on a certain coordinate q_k. Then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} = 0. \tag{5.47}$$

Therefore

$$\frac{\partial L}{\partial \dot{q}_k} = C, \tag{5.48}$$

where C is a constant, independent of time. In this case, we say that q_k is a *cyclic* coordinate, and that $\partial L/\partial \dot{q}_k$ is a *conserved* quantity (since it doesn't change with time).

If cartesian coordinates are used, then $\partial L/\partial \dot{x}_k$ is simply the momentum $m\dot{x}_k$, because $\dot{x}_k$ appears in only the $m\dot{x}_k^2/2$ term (we exclude cases where V depends on $\dot{x}_k$). We therefore call $\partial L/\partial \dot{q}_k$ the *generalized momentum* corresponding to the coordinate q_k. And in cases where $\partial L/\partial \dot{q}_k$ does not change with time, we call it a *conserved momentum*.

Note that a generalized momentum need not have the units of linear momentum, as the angular-momentum examples below show.

Example 1: Linear momentum

Consider a ball thrown through the air. Considering the full three dimensions, the Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz. \tag{5.49}$$

There is no x or y dependence here, so both $\partial L/\partial \dot{x} = m\dot{x}$ and $\partial L/\partial \dot{y} = m\dot{y}$ are constants, as we well know.

Example 2: Angular momentum in polar coordinates

Consider a potential which depends on only the distance to the source. (Examples are the gravitational and electrostatic potentials, which are proportional to $1/r$; and also the spring potential, which is proportional to $(r-a)^2$, where a is the equilibrium length.) In polar coordinates, the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r). \tag{5.50}$$

There is no θ dependence here, so $\partial L/\partial \dot{\theta} = mr^2\dot{\theta}$ is a constant. Since $r\dot{\theta}$ is the speed in the tangential direction, we see that our conserved quantity, $mr(r\dot{\theta})$, is the angular momentum. (Much more about angular momentum in Chapter 6.)

Example 3: Angular momentum in spherical coordinates

In spherical coordinates, consider a potential that depends on only r and θ. (Our convention for spherical coordinates will be that θ is the angle down from the north pole, and ϕ is the angle around the equator.) The Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - V(r, \theta). \tag{5.51}$$

There is no ϕ dependence here, so $\partial L/\partial\dot\phi = mr^2\sin^2\theta\dot\phi$ is a constant. Since $r\sin\theta\dot\phi$ is the speed in the tangential direction around the z-axis, and since $r\sin\theta$ is the distance from the z-axis, we see that our conserved quantity, $m(r\sin\theta)(r\sin\theta\dot\phi)$, is the angular momentum about the z-axis.

5.5.2 Energy conservation

We will now derive another conservation law, namely conservation of energy. The conservation of momentum or angular momentum above arose when the Lagrangian was independent of x, θ, or ϕ. Conservation of energy arises when the Lagrangian is independent of time. This conservation law is different from those in the above momenta examples, because t is not a coordinate which the stationary-action principle can be applied to, as x, θ, and ϕ are. (You can imagine varying these coordinates, which are functions of t. But it makes no sense to vary t.) Therefore, we're going to have to prove this conservation law in a different way.

Consider the quantity

$$E = \left(\sum_{i=1}^{N}\frac{\partial L}{\partial\dot q_i}\dot q_i\right) - L. \tag{5.52}$$

E will (usually) turn out to be the energy. We'll show this below. The motivation for this expression for E comes from the theory of Legendre transforms, but we won't get into that here. Let's just accept the definition in eq. (5.52) as given, and now we'll prove a nice little theorem about it.

Claim 5.3 *If L has no explicit time dependence (that is, $\partial L/\partial t = 0$), then E is conserved (that is, $dE/dt = 0$), assuming the motion obeys the E-L equations.*

Note that there is one partial derivative and one full derivative in this statement.

Proof: L is a function of the q_i, the $\dot q_i$, and possibly t. Making copious use of the chain rule, we have

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}\left(\sum_{i=1}^{N}\frac{\partial L}{\partial\dot q_i}\dot q_i\right) - \frac{dL}{dt}\\ &= \sum_{i=1}^{N}\left(\left(\frac{d}{dt}\frac{\partial L}{\partial\dot q_i}\right)\dot q_i + \frac{\partial L}{\partial\dot q_i}\ddot q_i\right) - \left(\sum_{i=1}^{N}\left(\frac{\partial L}{\partial q_i}\dot q_i + \frac{\partial L}{\partial\dot q_i}\ddot q_i\right) + \frac{\partial L}{\partial t}\right).\end{aligned} \tag{5.53}$$

There are five terms here. The second cancels with the fourth. And the first (after using the E-L equation, eq. (5.3), to rewrite it) cancels with the third. We therefore arrive at the simple result,

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}. \tag{5.54}$$

In the event that $\partial L/\partial t = 0$ (that is, there are no t's sitting on the paper when you write down L), which is invariably the case in the situations we consider (since we won't consider potentials that depend on time), then we have $dE/dt = 0$. ■

Not too many things are constant with respect to time, and the quantity E has units of energy, so it's a good bet that it is the energy. Let's show this in cartesian coordinates. (However, see the remark below.)

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z), \tag{5.55}$$

so eq. (5.52) gives

$$E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x, y, z), \tag{5.56}$$

which is, of course, the total energy.

To be sure, taking the kinetic energy T and subtracting the potential energy V to obtain L, and then using eq. (5.52) to produce $E = T + V$, seems like a rather convoluted way of arriving at $T + V$. But the point of all this is that we used the E-L equations to *prove* that E is conserved. Although we know very well from the $F = ma$ methods in Chapter 4 that the sum $T + V$ is conserved, it's not fair to assume that it is conserved in our new Lagrangian formalism. We have to show that this *follows* from the E-L equations.

REMARK: The quantity E in eq. (5.52) gives the energy of the system only if the entire system is represented by the Lagrangian. That is, the Lagrangian must represent a closed system (with no external forces). If the system is not closed, then Claim 5.3 is still perfectly valid for the E defined in eq. (5.54). But this E may simply not be the energy of the system. Problem 9 is a good example of such a situation.

A simple example is projectile motion in the x-y plane. The normal thing to do is to say that the particle moves under the influence of the potential $V(y) = mgy$. The Lagrangian for this closed system is $L = m(\dot{x}^2 + \dot{y})^2/2 - mgy$, and so eq. (5.52) gives $E = m(\dot{x}^2 + \dot{y})^2/2 + mgy$, which is indeed the energy of the particle. However, another way to do this problem is to consider the particle to be subject to an *external* gravitational force, which gives an acceleration of $-g$ in the y direction. If we assume that the mass starts at rest, then $\dot{y} = -gt$. The Lagrangian is therefore $L = m\dot{x}^2/2 + m(gt)^2/2$, and so eq. (5.52) gives $E = m\dot{x}^2/2 - m(gt)^2/2$. This is not the energy.

At any rate, most of the systems we will deal with are closed, so you can generally ignore this remark and assume that the E in eq. (5.52) gives the energy. ♣

5.6 Noether's Theorem

We now present one of the most beautiful and useful theorems in physics. It deals with two fundamental concepts, namely *symmetry* and *conserved quantities*. The theorem (due to Emmy Noether) may be stated as follows.

Theorem 5.4 (**Noether's Theorem**) *For each symmetry of the Lagrangian, there is a conserved quantity.*

By "symmetry", we mean that if the coordinates are changed by some small quantities, then the Lagrangian has no first-order change in these quantities. By "conserved quantity", we mean a quantity that does not change with time. The result in Section 5.5.1 for cyclic coordinates is a special case of this theorem.

Proof: Let the Lagrangian be invariant (to first order in the small number ϵ) under the change of coordinates,

$$q_i \longrightarrow q_i + \epsilon K_i(q). \tag{5.57}$$

Each $K_i(q)$ may be a function of all the q_i, which we collectively denote by the shorthand, q.

Example: Consider a mass on a spring (with zero equilibrium length), in the x-y plane. The Lagrangian $L = (m/2)(\dot{x}^2 + \dot{y}^2) - (k/2)(x^2 + y^2)$ is invariant under the change of coordinates, $x \to x + \epsilon y$, $y \to y - \epsilon x$, to first order in ϵ (as you can check). So in this case, $K_x = y$ and $K_y = -x$.

Of course, someone else might come along with $K_x = 5y$ and $K_y = -5x$, which is also a symmetry. And indeed, any factor can be taken out of ϵ and put into the K_i's without changing the quantity $\epsilon K_i(q)$ in eq. (5.57). Any such modification will simply bring an overall constant factor (and hence not change the property of being conserved) into the conserved quantity in eq. (5.60) below. It is therefore irrelevant. ♣

The fact that the Lagrangian does not change at first order in ϵ means that

$$\begin{aligned} 0 = \frac{dL}{d\epsilon} &= \sum_i \left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \epsilon} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \epsilon} \right) \\ &= \sum_i \left(\frac{\partial L}{\partial q_i} K_i + \frac{\partial L}{\partial \dot{q}_i} \dot{K}_i \right). \end{aligned} \tag{5.58}$$

Using the E-L equation, eq. (5.3), we may rewrite this as

$$\begin{aligned} 0 &= \sum_i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) K_i + \frac{\partial L}{\partial \dot{q}_i}\dot{K}_i \right) \\ &= \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i} K_i \right). \end{aligned} \tag{5.59}$$

Therefore, the quantity

$$P(q, \dot{q}) \equiv \sum_i \frac{\partial L}{\partial \dot{q}_i} K_i(q) \tag{5.60}$$

is conserved with respect to time. It is given the generic name of *conserved momentum.* (But it need not have the units of linear momentum). ■

As Noether most keenly observed
(And for which much acclaim is deserved),
We can easily see,
That for each symmetry,
A quantity must be conserved.

Example 1: In the mass-on-a-spring example mentioned above, with $L = (m/2)(\dot{x}^2 + \dot{y}^2) + (k/2)(x^2 + y^2)$, we found that $K_x = y$ and $K_y = -x$. So the conserved momentum is

$$P(x, y, \dot{x}, \dot{y}) = \frac{\partial L}{\partial \dot{x}} K_x + \frac{\partial L}{\partial \dot{y}} K_y = m(\dot{x}y - \dot{y}x). \tag{5.61}$$

This is simply the (negative of the) z-component of the angular momentum. (The angular momentum is conserved here because the potential $V(x, y) = x^2 + y^2 = r^2$ depends only on the distance from the origin; we'll discuss such potentials in Chapter 6).

Example 2: Consider a thrown ball. We have $L = (m/2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$. This is clearly invariant under translations in x, that is, $x \to x + \epsilon$; and also under translations in y, that is, $y \to y + \epsilon$. (Both x and y are cyclic coordinates.) Note that we only need invariance to first order in ϵ for Noether's theorem to hold, but this L is clearly invariant to all orders.

We therefore have two symmetries in our Lagrangian. The first has $K_x = 1$, $K_y = 0$, and $K_z = 0$. The second has $K_x = 0$, $K_y = 1$, and $K_z = 0$. Of course, the nonzero K's here may be chosen to be any constant, but we may as well pick them to be 1. Our two conserved momenta are

$$\begin{aligned} P_1(x, y, z, \dot{x}, \dot{y}, \dot{z}) &= \frac{\partial L}{\partial \dot{x}} K_x + \frac{\partial L}{\partial \dot{y}} K_y + \frac{\partial L}{\partial \dot{z}} K_z = m\dot{x}, \\ P_2(x, y, z, \dot{x}, \dot{y}, \dot{z}) &= \frac{\partial L}{\partial \dot{x}} K_x + \frac{\partial L}{\partial \dot{y}} K_y + \frac{\partial L}{\partial \dot{z}} K_z = m\dot{y}. \end{aligned} \tag{5.62}$$

These are simply the x- and y-components of the linear momentum, as we saw in Example 1 in Section 5.5.1.

Note that any combination of these momenta, say $3P_1 + 8P_2$, is also conserved. (In other words, $x \to x + 3\epsilon$, $y \to y + 8\epsilon$, and $z \to z$ is a symmetry of the Lagrangian.) But the above P_1 and P_2 are the simplest conserved momenta to choose as a "basis" for the infinite number of conserved momenta (which is how many you have, if there are two or more independent symmetries).

Example 3: Let $L = (m/2)(5\dot{x}^2 - 2\dot{x}\dot{y} + 2\dot{y}^2) + C(2x - y)$. This is clearly invariant under the transformation $x \to x + \epsilon$ and $y \to y + 2\epsilon$. Therefore, $K_x = 1$ and $K_y = 2$. So the conserved momentum is

$$P(x, y, \dot{x}, \dot{y}) = \frac{\partial L}{\partial \dot{x}} K_x + \frac{\partial L}{\partial \dot{y}} K_y = m(5\dot{x} - \dot{y})(1) + m(-\dot{x} + 2\dot{y})(2) = m(3\dot{x} + 3\dot{y}). \tag{5.63}$$

The overall factor of $3m$ doesn't matter, of course.

Remarks:

1. Note that in some cases the K_i's are functions of the coordinates, and in some cases they are not.
2. The cyclic-coordinate result in eq. (5.48) is a special case of Noether's theorem, for the following reason. If L doesn't depend on a certain coordinate q, then $q \longrightarrow q + \epsilon$ is obviously a symmetry. Hence $K = 1$, and eq. (5.60) reduces to eq. (5.48).

3. We use the word "symmetry" to describe the situation where the transformation in eq. (5.57) produces no first-order change in the Lagrangian. This is an appropriate choice of word, because the Lagrangian describes the system, and if the system essentially doesn't change when the coordinates are changed, then we would say that the system is symmetric. For example, if we have a setup that doesn't depend on θ, then we would say that the setup is symmetric under rotations. Rotate the system however you want, and it looks the same. The two most common applications of Noether's theorem are the conservation of angular momentum, which arises from symmetry under rotations; and conservation of linear momentum, which arises from symmetry under translations.

4. In simple systems, as in Example 2 above, it is quite obvious why the resulting P is conserved. But in more complicated systems, as in Example 3 above (which has an L of the type that arises in Atwood's machine problems; see the problems for this chapter), the resulting P might not have an obvious interpretation. But at least you know that it is conserved, and this will invariably help in solving a problem.

5. Although conserved quantities are extremely useful in studying a physical situation, it should be stressed that there is no more information contained in the them than there is in the E-L equations. Conserved quantities are simply the result of integrating the E-L equations. For example, if you write down the E-L equations for the third example above, and then add the "x" equation (which is $5m\ddot{x} - m\ddot{y} = 2C$) to twice the "$y$" equation (which is $-m\ddot{x} + 2m\ddot{y} = -C$), then you find $3m(\ddot{x} + \ddot{y}) = 0$, as desired.

   Of course, you might have to do some guesswork to find the proper combination of the E-L equations that gives a zero on the right-hand side. But you'd have to do some guesswork anyway, to find the symmetry for Noether's theorem. At any rate, a conserved quantity is useful because it is an integrated form of the E-L equations. It puts you one step closer to solving the problem, compared to where you would be if you started with the second-order E-L equations.

6. Does every system have a conserved momentum? Certainly not. The one-dimensional problem of a falling ball ($m\ddot{z} = -mg$) doesn't have one. And if you write down an arbitrary potential in 3-D, odds are that there won't be one. In a sense, things have to contrive nicely for there to be a conserved momentum. In some problems, you can just look at the physical system and see what the symmetry is, but in others (for example, in the Atwood's-machine problems for this chapter), the symmetry is not at all obvious.

7. By "conserved quantity", we mean a quantity that depends on (at most) the coordinates and their first derivatives (that is, not on their second derivatives). If we do not make this restriction, then it is trivial to construct quantities that do not vary with time. For example, in the third example above, the "x" E-L equation (which is $5m\ddot{x} - m\ddot{y} = 2C$) tells us that $5m\ddot{x} - m\ddot{y}$ has its time derivative equal to zero. Note that an equivalent way of excluding these trivial cases is to say that the value of a conserved quantity depends on initial conditions (that is, velocities and positions). The quantity $5m\ddot{x} - m\ddot{y}$ does not satisfy this criterion, because its value is always constrained to be $2C$. ♣

5.7 Small oscillations

In many physical systems, a particle undergoes small oscillations around an equilibrium point. In Section 4.2, we showed that the frequency of these small oscillations is

$$\omega = \sqrt{\frac{V''(x_0)}{m}}, \tag{5.64}$$

where $V(x)$ is the potential energy, and x_0 is the equilibrium point.

However, this result holds only for *one-dimensional* motion (we will see below why this is true). In more complicated systems, such as the one described below, it is necessary to use another procedure to obtain the frequency ω. This procedure is a fail-proof one, applicable in all situations. It is, however, a bit more involved than simply writing down eq. (5.64). So in 1-D problems, eq. (5.64) is what you want to use.

We'll demonstrate our fail-proof method through the following problem.

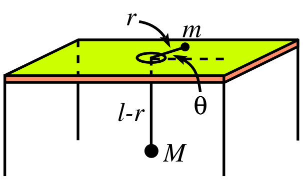


Figure 5.5

Problem:

A mass m is free to move on a frictionless table and is connected by a string, which passes through a hole in the table, to a mass M which hangs below (see Fig. 5.5). Assume that M moves in a vertical line only, and assume that the string always remains taut.

(a) Find the equations of motion for the variables r and θ shown in the figure.

(b) Under what condition does m undergo circular motion?

(c) What is the frequency of small oscillations (in the variable r) about this circular motion?

Solution:

(a) Let the string have length ℓ (this length won't matter). Then the Lagrangian (we'll call it '$\mathcal{L}$' here, and save 'L' for the angular momentum, which arises below) is

$$\mathcal{L} = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + Mg(\ell - r). \tag{5.65}$$

We've taken the table to be at height zero, for the purposes of potential energy, but any other value could be chosen, of course. The equations of motion obtained from varying θ and r are

$$\begin{aligned} \frac{d}{dt}(mr^2\dot{\theta}) &= 0, \\ (M+m)\ddot{r} &= mr\dot{\theta}^2 - Mg. \end{aligned} \tag{5.66}$$

The first equation says that angular momentum is conserved (much more about this in Chapter 6). The second equation says that the Mg gravitational force accounts for the acceleration of the two masses along the direction of the string, plus the centripetal acceleration of m.

(b) The first of eqs. (5.66) says that $mr^2\dot\theta = L$, where L is some constant (the angular momentum) which depends on the initial conditions. Plugging $\dot\theta = L/mr^2$ into the second of eqs. (5.66) gives

$$(M+m)\ddot r = \frac{L^2}{mr^3} - Mg. \tag{5.67}$$

Circular motion occurs when $\dot r = \ddot r = 0$. Therefore, the radius of the circular orbit is given by

$$r_0^3 = \frac{L^2}{Mmg}. \tag{5.68}$$

Remark: Note that since $L = mr^2\dot\theta$, eq. (5.68) is equivalent to

$$mr_0\dot\theta^2 = Mg, \tag{5.69}$$

which can be obtained by simply letting $\ddot r = 0$ in the second of eqs. (5.66). In other words, the gravitational force on M exactly accounts for the centripetal acceleration of m. Given r_0, this equation determines what $\dot\theta$ must be (in order to have circular motion), and vice versa. ♣

(c) To find the frequency of small oscillations about circular motion, we need to look at what happens to r if we perturb it slightly from its equilibrium value, r_0. Our fail-proof procedure is the following.

Let $r(t) \equiv r_0 + \delta(t)$ (where $\delta(t)$ is very small; more precisely, $\delta(t) \ll r_0$), and expand eq. (5.67) to first order in $\delta(t)$. Using

$$\frac{1}{r^3} \equiv \frac{1}{(r_0+\delta)^3} \approx \frac{1}{r_0^3 + 3r_0^2\delta} = \frac{1}{r_0^3(1+3\delta/r_0)} \approx \frac{1}{r_0^3}\left(1 - \frac{3\delta}{r_0}\right), \tag{5.70}$$

we have

$$(M+m)\ddot\delta = \frac{L^2}{mr_0^3}\left(1 - \frac{3\delta}{r_0}\right) - Mg. \tag{5.71}$$

The terms not involving δ on the right-hand side cancel, by the definition of r_0 (eq. (5.68)). (This cancellation will always occur in such a problem at this stage, due to the definition of the equilibrium point.) We are therefore left with

$$\ddot\delta + \left(\frac{3L^2}{(M+m)mr_0^4}\right)\delta = 0. \tag{5.72}$$

This is a nice simple-harmonic-oscillator equation in the variable δ. Therefore, the frequency of small oscillations about a circle of radius r_0 is

$$\omega = \sqrt{\frac{3L^2}{(M+m)mr_0^4}} = \sqrt{\frac{3M}{M+m}}\sqrt{\frac{g}{r_0}}\,, \tag{5.73}$$

where we have used eq. (5.68) to eliminate L in the second expression.

To sum up, the above frequency is the frequency of small oscillations in the variable r. In other words, if you plot r as a function of time (and ignore what θ is doing), then you will get a nice sinusoidal graph whose frequency is given by eq. (5.73). Note that this frequency need not have anything to do with the other relevant frequency in this problem, namely the frequency of circular motion, which is $\sqrt{Mg/mr_0}$, from eq. (5.69).

REMARKS: Let's look at some limits. For a given r_0, if $m \gg M$, then $\omega \approx \sqrt{3Mg/mr_0} \approx 0$. This makes sense (everything will be moving very slowly). Note that this frequency is equal to $\sqrt{3}$ times the frequency of circular motion, $\sqrt{Mg/mr_0}$, which isn't at all obvious.

For a given r_0, if $m \ll M$, then $\omega \approx \sqrt{3g/r_0}$, which isn't so obvious, either.

Note that the frequency of small oscillations is equal to the frequency of circular motion if $M = 2m$ (once again, not obvious). This condition is independent of r_0. ♣

The above procedure for finding the frequency of small oscillations may be summed up in three steps: (1) Find the equations of motion, (2) Find the equilibrium point, and (3) Let $x(t) \equiv x_0 + \delta(t)$ (where x_0 is the equilibrium point of the relevant variable), and expand one of the equations of motion (or a combination of them) to first order in δ, to obtain a simple-harmonic-oscillator equation for δ.

REMARK: Note that if you simply used the potential energy in the above problem (which is Mgr, up to a constant) in eq. (5.64), then you would obtain a frequency of zero, which is incorrect. You *can* use eq. (5.64) to find the frequency, if you instead use the "effective potential" for this problem, namely $L^2/(2mr^2)+Mgr$, and if you use the total mass, $M+m$, as the mass in eq. (5.64). (Check this.) The reason why this works will become clear in Chapter 6 when we introduce the effective potential.

In many problems, however, it is not obvious what "modified potential" should be used, or what mass should be used in eq. (5.64), so it is generally much safer to take a deep breath and go through an expansion similar to the one in part (c) above. ♣

Note that the one-dimensional result in eq. (5.64) is, of course, simply a special case of our above expansion procedure. We can repeat the derivation of Section 4.2 in the present language. In one dimension, we have $m\ddot{x} = -V'(x)$. Let x_0 be the equilibrium point (so that $V'(x_0) = 0$), and let $x \equiv x_0+\delta$. Expanding $m\ddot{x} = -V'(x)$ to first order in δ, we have $m\ddot{\delta} = -V'(x_0) - V''(x_0)\delta - \cdots$. Hence, $m\ddot{\delta} \approx -V''(x_0)\delta$, as desired.

5.8 Other applications

The formalism developed in Section 5.2 works for *any* function $L(x, \dot{x}, t)$. If our goal is to find the stationary points of $S \equiv \int L$, then eq. (5.14) holds, no matter what L is. There is no need for L to be equal to $T - V$, or indeed, to have anything to do with physics. And t need not have anything to do with time. All that is required is that the quantity x depend on the parameter t, and that L depend on only x, $\dot{x}$, and t (and not, for example, on $\ddot{x}$; see Exercise 2). The formalism is very general and quite powerful, as the following problem demonstrates.

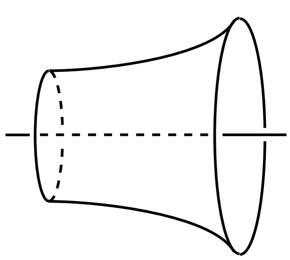


Figure 5.6

Example (Minimal surface of revolution): A surface of revolution has two given rings as its boundary (see Fig. 5.6). What should the shape of the surface be so that it has the minimum possible area?

(We'll present two solutions. A third is left for Problem 24.)

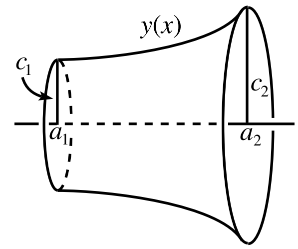


Figure 5.7

First solution: Let the surface be generated by rotating the curve $y = y(x)$ around the x-axis. The boundary conditions are $y(a_1) = c_1$ and $y(a_2) = c_2$ (see Fig. 5.7). Slicing the surface up into vertical rings, we see that the area is given by

$$A = \int_{a_1}^{a_2} 2\pi y\sqrt{1 + y'^2}\, dx. \tag{5.74}$$

The goal is to find the function $y(x)$ that minimizes this integral. We therefore have exactly the same situation as in Section 5.2, except that x is now the parameter (instead of t), and y in now the function (instead of x). Our "Lagrangian" is thus $L \propto y\sqrt{1 + y'^2}$. To minimize the integral A, we "simply" have to apply the E-L equation to this Lagrangian. This calculation, however, gets a bit tedious, so we've relegated it to Lemma 5.5 at the end of this section. For now we'll just use the result in eq. (5.84) which gives (with $f(y) = y$ here)

$$1 + y'^2 = By^2. \tag{5.75}$$

At this point we can cleverly guess (motivated by the fact that $1 + \sinh^2 z = \cosh^2 z$) that the solution is

$$y(x) = \frac{1}{b}\cosh b(x + d), \tag{5.76}$$

where $b = \sqrt{B}$, and d is a constant of integration. Or, we can separate variables to obtain (again with $b = \sqrt{B}$)

$$dx = \frac{dy}{\sqrt{(by)^2 - 1}}\,, \tag{5.77}$$

and then use the fact that the integral of $1/\sqrt{z^2 - 1}$ is $\cosh^{-1} z$, to obtain the same result.

Therefore, the answer to our problem is that $y(x)$ takes the form of eq. (5.76), with b and d determined by the boundary conditions,

$$c_1 = \frac{1}{b}\cosh b(a_1 + d), \qquad \text{and} \qquad c_2 = \frac{1}{b}\cosh b(a_2 + d). \tag{5.78}$$

In the symmetrical case where $c_1 = c_2$, we know that the minimum occurs in the middle, so we may choose $d = 0$ and $a_1 = -a_2$.

REMARK: Solutions for b and d exist only for certain ranges of the a's and c's. Basically, if $a_2 - a_1$ is too big, then there is no solution. In this case, the minimal "surface" turns out to be the two given circles, attached by a line (which isn't a nice two-dimensional surface). If you perform an experiment with soap bubbles (which want to minimize their area), and if you pull the rings too far apart, then the surface will break and disappear, as it tries to form the two circles. Problem 27 deals with this issue. ♣

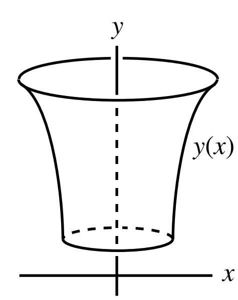


Figure 5.8

Second solution: Consider the surface to be generated by rotating the curve $y = y(x)$ around the y-axis, instead of the x-axis (see Fig. 5.8). The area is then given by

$$A = \int_{a_1}^{a_2} 2\pi x\sqrt{1 + y'^2}\, dx. \tag{5.79}$$

Note that the function $y(x)$ may be double-valued, so it may not really be a function. But it looks like a function locally, and all of our formalism deals with local variations.

Our "Lagrangian" is now $L \propto x\sqrt{1+y'^2}$, and the E-L equation is

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(\frac{xy'}{\sqrt{1+y'^2}}\right) = 0. \tag{5.80}$$

The nice thing about this way of solving the problem is the "0" on the right-hand side, which arises from the fact that L does not depend on y (that is, y is a cyclic coordinate). Therefore, $xy'/\sqrt{1+y'^2}$ is constant. If we define this constant to be $1/b$, then we may solve for y' and then separate variables to obtain

$$dy = \frac{dx}{\sqrt{(bx)^2 - 1}}\,. \tag{5.81}$$

This is identical to eq. (5.77), except with x and y interchanged. Therefore, we have

$$x(y) = \frac{1}{b}\cosh b(y+d). \tag{5.82}$$

This agrees with the result in the first solution, in view of the fact that we are now rotating the surface around the y-axis, so the roles of x and y are reversed.

Numerous other "extremum" problems are solvable with these general techniques. A few are presented in the problems for this chapter.

Let us now prove the following Lemma, which we invoked in the first solution above. This Lemma is very useful, because it is common to encounter problems where the quantity to be extremized depends on the arclength, $\sqrt{1+y'^2}$, and takes the form $\int f(y)\sqrt{1+y'^2}\,dx$.

We'll give two proofs. The first proof uses the Euler-Lagrange equation. The calculation here gets a bit messy, so it's a good idea to work through it once and for all; it's not something you'd want to repeat too often. The second proof makes use of a conserved quantity. And in contrast to the first proof, this method is exceedingly clean and simple. It actually *is* something you'd want to repeat quite often. (But we'll still do it once and for all.)

Lemma 5.5 *Let $f(y)$ be a given function of y. Then the function $y(x)$ that extremizes the integral,*

$$\int_{x_1}^{x_2} f(y)\sqrt{1+y'^2}\,dx, \tag{5.83}$$

satisfies the differential equation,

$$1 + y'^2 = Bf(y)^2, \tag{5.84}$$

where B is a constant of integration.[9]

First Proof: The goal is to find the function $y(x)$ that extremizes the integral in eq. (5.83). We therefore have exactly the same situation as in section 5.2, except

[9] B, along with one other constant of integration, will eventually be determined from the boundary conditions, once eq. (5.84) is integrated to solve for y.

with x in place of t, and y in place of x. Our "Lagrangian" is thus $L = f(y)\sqrt{1+y'^2}$, and the Euler-Lagrange equation is

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(f \cdot y' \cdot \frac{1}{\sqrt{1+y'^2}}\right) = f'\sqrt{1+y'^2}, \tag{5.85}$$

where $f' \equiv df/dy$. We must now perform some straightforward (albeit tedious) differentiations. Using the product rule on the three factors on the left-hand side, and making copious use of the chain rule, we obtain

$$\frac{f'y'^2}{\sqrt{1+y'^2}} + \frac{fy''}{\sqrt{1+y'^2}} - \frac{fy'^2y''}{(1+y'^2)^{3/2}} = f'\sqrt{1+y'^2}. \tag{5.86}$$

Multiplying through by $(1+y'^2)^{3/2}$ and simplifying gives

$$fy'' = f'(1+y'^2). \tag{5.87}$$

We have completed the first step of the solution, namely producing the Euler-Lagrange differential equation. Now we must integrate it. Eq. (5.87) happens to be integrable for arbitrary functions $f(y)$. If we multiply through by y' and rearrange, we obtain

$$\frac{y'y''}{1+y'^2} = \frac{f'y'}{f}\,. \tag{5.88}$$

Taking the dx integral of both sides gives $(1/2)\ln(1+y'^2) = \ln(f) + C$, where C is an integration constant. Exponentiation then gives (with $B \equiv e^{2C}$)

$$1 + y'^2 = Bf(y)^2, \tag{5.89}$$

as was to be shown. The next task would be to solve for y', and to then separate variables and integrate. But we would need to be given a specific function $f(y)$ to be able do this.

Second Proof: Note that our "Lagrangian", $L = f(y)\sqrt{1+y'^2}$, is independent of x. Therefore, in analogy with the conserved energy given in eq. (5.52), the quantity

$$E \equiv y'\frac{\partial L}{\partial y'} - L = \frac{-f(y)}{\sqrt{1+y'^2}} \tag{5.90}$$

is independent of x. Call it $1/\sqrt{B}$. Then we have easily reproduced eq. (5.89). ■

Important Remark: As demonstrated by the brevity of this second proof, it is highly advantageous to make use of a conserved quantity (for example, the E here, which arose from independence of x) whenever you can. ♣

5.9 Exercises

Section 5.2: The principle of stationary action

1. **Explicit minimization** $*$

   A ball is thrown upward. Let $y(t)$ be the height as a function of time, and assume $y(0) = 0$ and $y(T) = 0$. Guess a solution for y of the form $y(t) = a_0 + a_1 t + a_2 t^2$, and explicitly calculate the action between $t = 0$ and $t = T$. Show that the action is minimized when $a_2 = -g/2$. (This gets slightly messy.)

2. $\ddot{x}$ **dependence** $**$

   Let there be $\ddot{x}$ dependence (in addition to x,$\dot{x}$,t dependence) in the Lagrangian in Theorem 5.1. There will then be the additional term $(\partial L/\partial \ddot{x}_a)\ddot{\beta}$ in eq. (5.18). It is tempting to integrate this term by parts twice, and then arrive at a modified form of eq. (5.21):

   $$\frac{\partial L}{\partial x_0} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_0}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}_0}\right) = 0. \tag{5.91}$$

   Is this a valid result? If not, where is the error in its derivation?

Section 5.3: Forces of constraint

3. **Constraint on a circle**

   A bead slides with speed v around a horizontal loop of radius R. What force does the loop apply to the bead? (Ignore gravity.)

4. **Constraint on a curve** $**$

   Let the horizontal plane be the x-y plane. A bead slides with speed v along a curve described by the function $y = f(x)$. What force does the curve apply to the bead? (Ignore gravity.)

5.10 Problems

Section 5.1: The Euler-Lagrange equations

1. **Moving plane** $**$

   A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ. The plane rests on a frictionless horizontal surface. The block is released (see Fig. 5.9). What is the horizontal acceleration of the plane?

   (This problem was posed in Chapter 2. If you haven't already done so, try solving it using $F = ma$. You will then have a much greater appreciation for the Lagrangian method.)

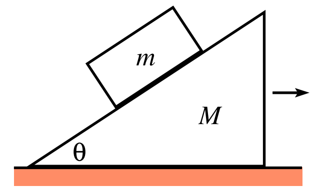


Figure 5.9

2. **Two masses, one swinging** $***$

   Two equal masses m, connected by a string, hang over two pulleys (of negligible size), as shown in Fig. 5.10. The left one moves in a vertical line, but the right one is free to swing back and forth (in the plane of the masses and pulleys). Find the equations of motion.

   Assume that the left mass starts at rest, and the right mass undergoes small oscillations with angular amplitude ϵ (with $\epsilon \ll 1$). What is the initial average acceleration (averaged over a few periods) of the left mass? In which direction does it move?

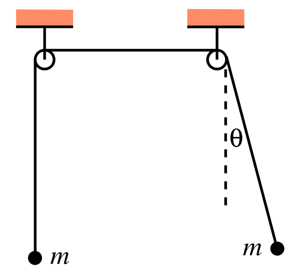


Figure 5.10

3. **Falling sticks** $**$

   Two massless sticks of length $2r$, each with a mass m fixed at its middle, are hinged at an end. One stands on top of the other, as in Fig. 5.11. The bottom end of the lower stick is hinged at the ground. They are held such that the lower stick is vertical, and the upper one is tilted at a small angle ϵ with respect to the vertical. At the instant they are released, what are the angular accelerations of the two sticks? (You may work in the approximation where ϵ is very small).

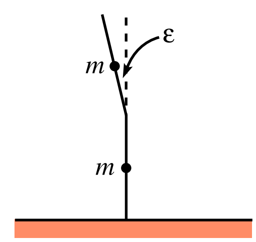


Figure 5.11

4. **Pendulum with oscillating support** $**$

   A pendulum consists of a mass m and a massless stick of length ℓ. The pendulum support oscillates horizontally with a position given by $x(t) = A\cos(\omega t)$ (see Fig. 5.12). Find the equation of motion for the angle of the pendulum.

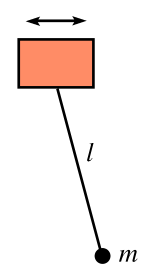


Figure 5.12

5. **Inverted pendulum** $****$

   (a) A pendulum consists of a mass m and a massless stick of length ℓ. The pendulum support oscillates vertically with a position given by $y(t) = A\cos(\omega t)$ (see Fig. 5.13). Find the equation of motion for the angle of the pendulum (measured relative to its upside-down position).

   (b) It turns out that if ω is large enough, then if the stick is initially nearly upside-down, it will, surprisingly, *not* fall over as time goes by. Instead,

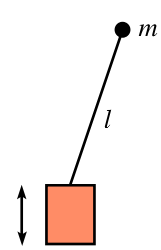


Figure 5.13

it will (sort of) oscillate like an upside-down pendulum. (This can be shown numerically.) Give a qualitative argument why the stick doesn't fall over. You don't need to be exact; just make the result believable.

Section 5.2: The principle of stationary action

6. **Minimum or saddle** $**$

In eq. (5.25), let $t_1 = 0$ and $t_2 = T$, for convenience. And let the $\xi(t)$ be an easy-to-deal-with "triangular" function, of the form

$$\xi(t) = \begin{cases} \epsilon t/T, & 0 \le t \le T/2, \\ \epsilon(1 - t/T), & T/2 \le t \le T. \end{cases} \tag{5.92}$$

Under what conditions is the ΔS in eq. (5.25) negative?

Section 5.3: Forces of constraint

7. **Mass on plane** $**$

A mass m slides down a frictionless plane which is inclined at an angle θ. Show that the normal force from the plane is the familiar $mg\cos\theta$.

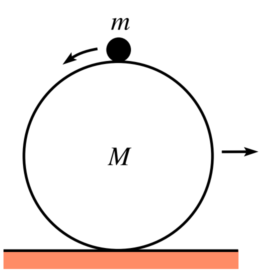


Figure 5.14

8. **Leaving the moving sphere** $***$

A particle of mass m sits on top of a frictionless sphere of mass M (see Fig. 5.14). The sphere is free to slide on the frictionless ground. The particle is given an infinitesimal kick. Let θ be the angle which the radius to the particle makes with the vertical. Find the equation of motion for θ. Also, find the force of constraint in terms of θ and $\dot\theta$.

Section 5.5: Conservation Laws

9. **Bead on stick** $*$

A stick is pivoted at the origin and swings around in a horizontal plane at constant angular speed ω. A bead of mass m slides frictionlessly along the stick. Let r be the radial position of the bead. Find the conserved quantity E given in eq. (5.52). Explain why this quantity is *not* the energy of the bead.

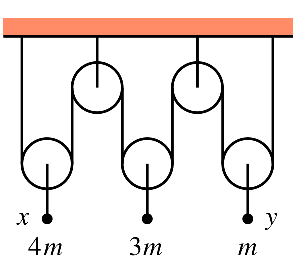


Figure 5.15

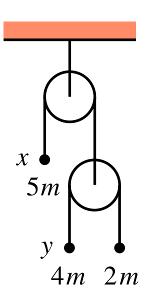
Section 5.6: Noether's Theorem

10. **Atwood's machine 1** $**$

Consider the Atwood's machine shown in Fig. 5.15. The masses are $4m$, $3m$, and m. Let x and y be the heights of the left and right masses (relative to their initial positions). Find the conserved momentum.

11. **Atwood's machine 2** $**$

Consider the Atwood's machine shown in Fig. 5.16. The masses are $5m$, $4m$, and $2m$. Let x and y be the heights of the left two masses (relative to their initial positions). Find the conserved momentum.



Section 5.7: Small oscillations

12. **Pulley pendulum** $**$

    A mass M is attached to a massless hoop (of radius R) which lies in a vertical plane. The hoop is free to rotate about its fixed center. M is tied to a string which winds part way around the hoop, then rises vertically up and over a massless pulley. A mass m hangs on the other end of the string (see Fig. 5.17). Find the equation of motion for the angle through which the hoop rotates. What is the frequency of small oscillations? (You may assume $M > m$.)

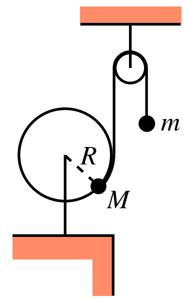


Figure 5.17

13. **Three hanging masses** $***$

    A mass M is fixed at the midpoint if a long string, at the ends of which are tied masses m. The string hangs over two frictionless pulleys (located at the same height), as shown in Fig. 5.18. The pulleys are a distance $2l$ apart and have negligible size. Assume that the mass M is constrained to move in a vertical line midway between the pulleys.

    Let θ be the angle the string from M makes with the horizontal. Find the equation of motion for θ. (This is a bit messy.) Find the frequency of small oscillations around the equilibrium point. (You may assume $M < 2m$.)

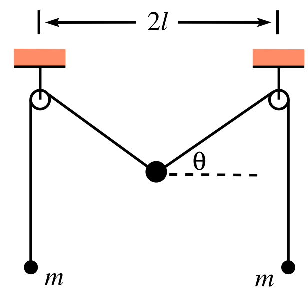


Figure 5.18

14. **Bead on rotating hoop** $**$

    A bead is free to slide along a frictionless hoop of radius R. The hoop rotates with constant angular speed ω around a vertical diameter (see Fig. 5.19). Find the equation of motion for the position of the bead. What are the equilibrium positions? What is the frequency of small oscillations about the stable equilibrium?

    There is one value of ω that is rather special. What is it, and why is it special?

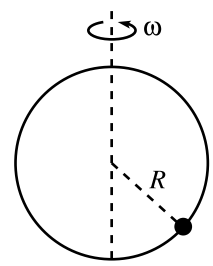


Figure 5.19

15. **Another bead on rotating hoop** $**$

    A bead is free to slide along a frictionless hoop of radius r. The plane of the hoop is horizontal, and the center of the hoop travels in a horizontal circle of radius R, with constant angular speed ω, about a given point (see Fig. 5.20). Find the equation of motion for the position of the bead. Also, find the frequency of small oscillations about the equilibrium point.

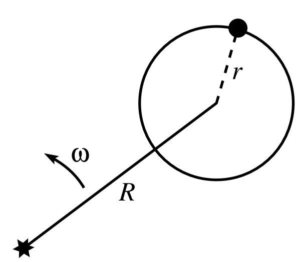


Figure 5.20

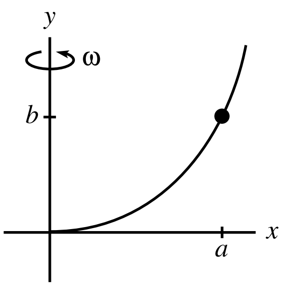


Figure 5.21

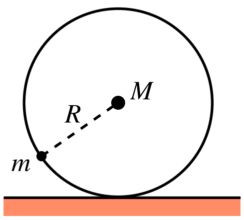


Figure 5.22

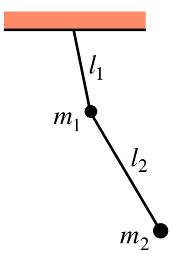


Figure 5.23

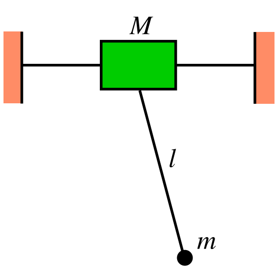


Figure 5.24

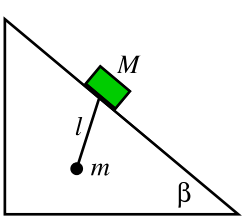


Figure 5.25

16. **Rotating curve** ∗∗∗

The curve $y = f(x) = b(x/a)^\lambda$ is rotated around the y-axis with constant frequency ω. A bead moves without friction along the curve (see Fig. 5.21). Find the frequency of small oscillations about the equilibrium point. Under what conditions do oscillations exist? (This gets a little messy.)

17. **Mass on wheel** ∗∗

A mass m is fixed to a given point at the edge of a wheel of radius R. The wheel is massless, except for a mass M located at its center (see Fig. 5.22). The wheel rolls without slipping on a horizontal table. Find the equation of motion for the angle through which the wheel rolls. For the case where the wheel undergoes small oscillations, find the frequency.

18. **Double pendulum** ∗∗∗∗

Consider a double pendulum made of two masses, m_1 and m_2, and two rods of lengths ℓ_1 and ℓ_2 (see Fig. 5.23). Find the equations of motion.

For small oscillations, find the normal modes and their frequencies for the special case $\ell_1 = \ell_2$ (and check the limits $m_1 \gg m_2$ and $m_1 \ll m_2$). Do the same for the special case $m_1 = m_2$ (and check the limits $\ell_1 \gg \ell_2$ and $\ell_1 \ll \ell_2$).

19. **Pendulum with free support** ∗∗

A pendulum of mass m and length ℓ is hung from a support of mass M which is free to move horizontally on a frictionless rail (see Fig. 5.24). Find the equations of motion. For small oscillations, find the normal modes and their frequencies.

20. **Pendulum support on inclined plane** ∗∗

A mass M slides down a frictionless plane inclined at angle β. A pendulum, with length ℓ and mass m, is attached to M (see Fig. 5.25). Find the equations of motion, and also the normal modes for small oscillations.

21. **Tilting plane** ∗∗∗

A mass M is fixed at the right-angled vertex where a massless rod of length ℓ is connected to a very long massless rod (see Fig. 5.26). A mass m is free to move frictionlessly along the long rod. The rod of length ℓ is hinged at a support, and the whole system is free to rotate, in the plane of the rods, about the support.

Let θ be the angle of rotation of the system, and let x be the distance between m and M. Find the equations of motion. Find the normal modes when θ and x are both very small.

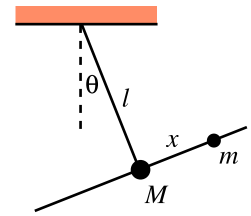


Figure 5.26

22. **Motion on a cone** ∗∗∗

A particle moves on a frictionless cone. The cone is fixed with its tip on the ground and its axis vertical. The cone has a half-angle equal to α (see Fig. 5.27). Find the equations of motion.

Let the particle move in a circle of radius r_0. What is the frequency, ω, of this circular motion? Let the particle be perturbed slightly from this motion. What is the frequency, Ω, of the oscillations about the radius r_0? Under what conditions does $\Omega = \omega$?

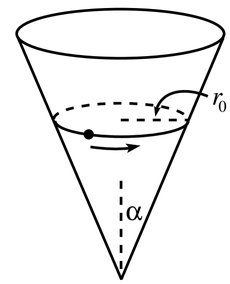


Figure 5.27

Section 5.8: Other applications

23. **Shortest distance in a plane**

In the spirit of section 5.8, show that the shortest path between two points in a plane is a straight line.

24. **Minimal surface** ∗∗

Derive the shape of the minimal surface discussed in Section 5.8, by demanding that a cross-sectional 'ring' (that is, the region between the planes $x = x_1$ and $x = x_2$) is in equilibrium; see Fig. 5.28. *Hint:* The tension must be constant throughout the surface.

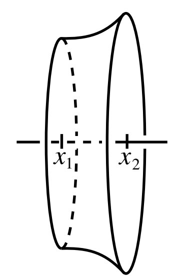


Figure 5.28

25. **The brachistochrone** ∗∗∗

A bead is released from rest and slides down a frictionless wire that connects the origin to a given point, as shown in Fig. 5.29. You wish to shape the wire so that the bead reaches the endpoint in the shortest possible time.

Let the desired curve be described by the function $y(x)$, with downward being the positive y direction, for convenience.

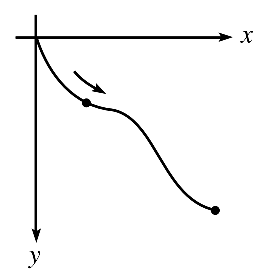


Figure 5.29

(a) Show that $y(x)$ satisfies

$$-2yy'' = 1 + y'^2, \tag{5.93}$$

and that a first integral of the motion is

$$1 + y'^2 = \frac{C}{y}. \tag{5.94}$$

where C is a constant.

(b) Show that x and y may be written as

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta). \tag{5.95}$$

You may do this by simply verifying that they satisfy eq. (5.94). But try to do it also by solving the differential equation from scratch. (Eq. (5.95) is the parametrization of a *cycloid*, which is the path taken by a point on the edge of a rolling wheel.)

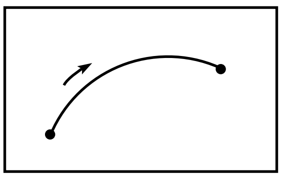


Figure 5.30

26. **Index of refraction** $**$

Assume that the speed of light in a given slab of material is proportional to the height above the base of the slab.[10] Show that light moves in circular arcs in this material; see Fig. 5.30. You may assume that light takes the path of shortest time between two points (Fermat's principle of least time).

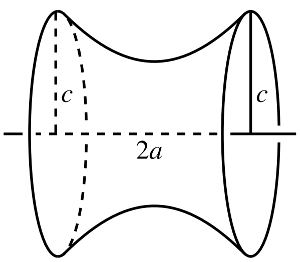


Figure 5.31

27. **Existence of minimal surface** $**$

Consider the minimal surface from Section 5.8. Consider the special case where the two rings have the same radius (that is, $c_1 = c_2 \equiv c$). Let $2a \equiv a_1 - a_2$ be the distance between the rings (see Fig. 5.31).

What is the largest value of a/c for which a minimal surface exists? (You will have to solve something numerically here.)

[10] In other words, the index of refraction of the material, n, as a function of the height, y, is given by $n(y) = y_0/y$, where y_0 is some length that is larger than the height of the slab.

5.11 Solutions

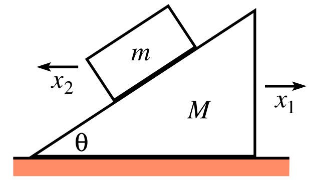


Figure 5.32

1. **Moving plane**

Let x_1 be the horizontal coordinate of the plane (with positive x_1 to the right). Let x_2 be the horizontal coordinate of the mass (with positive x_2 to the left). (See Fig. 5.32.) Then it is easy to see that the height fallen by the mass is $\Delta y = (x_1 + x_2)\tan\theta$. The Lagrangian is therefore

$$L = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\Big(\dot{x}_2^2 + (\dot{x}_1 + \dot{x}_2)^2 \tan^2\theta\Big) + mg(x_1 + x_2)\tan\theta. \qquad (5.96)$$

The equations of motion from varying x_1 and x_2 are

$$\begin{aligned} M\ddot{x}_1 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta &= mg\tan\theta, \\ m\ddot{x}_2 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta &= mg\tan\theta. \end{aligned} \qquad (5.97)$$

Note that these two equations immediately yield conservation of momentum, $M\ddot{x}_1 = m\ddot{x}_2$. We may easily solve for $\ddot{x}_1$ to obtain

$$\ddot{x}_1 = \frac{mg\tan\theta}{M(1 + \tan^2\theta) + m\tan^2\theta}\,. \qquad (5.98)$$

Remarks: For given M and m, the angle θ_0 which maximizes $\ddot{x}_1$ is found to be

$$\tan\theta_0 = \sqrt{\frac{M}{M+m}}. \qquad (5.99)$$

If $M \ll m$, then $\theta_0 \approx 0$. If $M \gg m$, then $\theta_0 \approx \pi/4$.

In the limit $M \ll m$, we have $\ddot{x}_1 \approx g/\tan\theta$. This makes sense, because m falls essentially straight down, and the plane gets squeezed out to the right.

In the limit $M \gg m$, we have $\ddot{x}_1 \approx g(m/M)\tan\theta/(1 + \tan^2\theta) = g(m/M)\sin\theta\cos\theta$. This is more transparent if we instead look at $\ddot{x}_2 = (M/m)\ddot{x}_1 \approx g\sin\theta\cos\theta$. Since the plane is essentially at rest in this limit, this value of $\ddot{x}_2$ implies that the acceleration of m *along* the plane is essentially equal to $\ddot{x}_2/\cos\theta \approx g\sin\theta$, as expected. ♣

2. **Two masses, one swinging**

Let r be the distance from the swinging mass to the pulley, and let θ be the angle of the swinging mass. Then the Lagrangian is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr + mgr\cos\theta. \qquad (5.100)$$

The equations of motion from varying r and θ are

$$\begin{aligned} 2\ddot{r} &= r\dot{\theta}^2 - g(1 - \cos\theta), \\ \frac{d}{dt}(r^2\dot{\theta}) &= -gr\sin\theta. \end{aligned} \qquad (5.101)$$

The first equation deals with the acceleration along the direction of the string. The second equation equates the torque from gravity with change in angular momentum.

If we do a (coarse) small-angle approximation and keep only terms up to first order in θ, we find at $t = 0$ (using the initial condition, $\dot{r} = 0$)

$$\begin{aligned} \ddot{r} &= 0, \\ \ddot{\theta} + \frac{g}{r}\theta &= 0. \end{aligned} \qquad (5.102)$$

These say that the left mass stays still, and the right mass behaves just like a pendulum.

If we want to find the leading term in the initial acceleration of the left mass (i.e., the leading term in $\ddot{r}$), we need to be a little less coarse in our approximation. So let's keep terms in eq. (5.101) up to second order in θ. We then have at $t = 0$ (using the initial condition, $\dot{r} = 0$)

$$\begin{aligned} 2\ddot{r} &= r\dot{\theta}^2 - \frac{1}{2}g\theta^2, \\ \ddot{\theta} + \frac{g}{r}\theta &= 0. \end{aligned} \tag{5.103}$$

The second equation says that the right mass undergoes harmonic motion. In this problem, it is given that the amplitude is ϵ. So we have

$$\theta(t) = \epsilon\cos(\omega t + \phi), \tag{5.104}$$

where $\omega = \sqrt{g/r}$. Plugging this into the first equation gives

$$2\ddot{r} = \epsilon^2 g\left(\sin^2(\omega t + \phi) - \frac{1}{2}\cos^2(\omega t + \phi)\right). \tag{5.105}$$

If we average over a few periods, both $\sin^2\alpha$ and $\cos^2\alpha$ average to 1/2, so we find

$$\ddot{r}_{\text{avg}} = \frac{\epsilon^2 g}{8}. \tag{5.106}$$

This is a small second-order effect. It is positive, so the left mass slowly begins to climb.

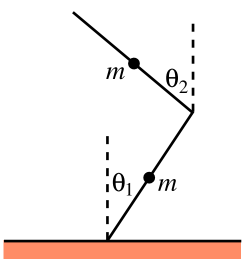


Figure 5.33

3. **Falling sticks**

Let $\theta_1(t)$ and $\theta_2(t)$ be defined as in Fig. 5.33. Then the position of the bottom mass in cartesian coordinates is $(r\sin\theta_1, r\cos\theta_1)$, and the position of the top mass is $(2r\sin\theta_1 - r\sin\theta_2, 2r\cos\theta_1 + r\cos\theta_2)$. So the potential energy of the system is

$$V(\theta_1, \theta_2) = mgr(3\cos\theta_1 + \cos\theta_2). \tag{5.107}$$

The kinetic energy is somewhat more complicated. The K.E. of the bottom mass is simply $mr^2\dot{\theta}_1^2/2$. The K.E. of the top mass is

$$\frac{1}{2}mr^2\Big((2\cos\theta_1\dot{\theta}_1 - \cos\theta_2\dot{\theta}_2)^2 + (-2\sin\theta_1\dot{\theta}_1 - \sin\theta_2\dot{\theta}_2)^2\Big). \tag{5.108}$$

Let's now simplify this, using the small-angle approximations. The terms involving $\sin\theta$ will be fourth order in the small θ's, so we may neglect them. Also, we may approximate $\cos\theta$ by 1, since we will have dropped only terms of at least fourth order. So this K.E. turns into $(1/2)mr^2(2\dot{\theta}_1 - \dot{\theta}_2)^2$. In other words, the masses move essentially horizontally. Therefore, using the small-angle approximation $\cos\theta \approx 1 - \theta^2/2$ to rewrite V, we have

$$L \approx \frac{1}{2}mr^2\Big(5\dot{\theta}_1^2 - 4\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\Big) - mgr\left(4 - \frac{3}{2}\theta_1^2 - \frac{1}{2}\theta_2^2\right). \tag{5.109}$$

The equations of motion from θ_1 and θ_2 are, respectively,

$$\begin{aligned} 5\ddot{\theta}_1 - 2\ddot{\theta}_2 &= \frac{3g}{r}\theta_1 \\ -2\ddot{\theta}_1 + \ddot{\theta}_2 &= \frac{g}{r}\theta_2. \end{aligned} \tag{5.110}$$

At the instant the sticks are released, $\theta_1 = 0$ and $\theta_2 = \epsilon$. Solving our two equations for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ gives

$$\ddot{\theta}_1 = \frac{2g}{r}\epsilon, \qquad \text{and} \qquad \ddot{\theta}_2 = \frac{5g}{r}\epsilon. \tag{5.111}$$

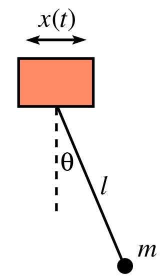


Figure 5.34

4. **Pendulum with oscillating support**

Let θ be defined as in Fig. 5.34. With $x(t) = A\cos(\omega t)$, the position of the mass m is given by

$$(X, Y)_m = (x + \ell \sin\theta, -\ell\cos\theta). \tag{5.112}$$

The square of its speed is

$$V_m^2 = \ell^2\dot{\theta}^2 + \dot{x}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta. \tag{5.113}$$

The Lagrangian is therefore

$$L = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{x}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta) + mg\ell\cos\theta. \tag{5.114}$$

The equation of motion from varying θ is

$$\ell\ddot{\theta} + \ddot{x}\cos\theta = -g\sin\theta. \tag{5.115}$$

Plugging in the explicit form of $x(t)$, we have

$$\ell\ddot{\theta} - A\omega^2\cos(\omega t)\cos\theta + g\sin\theta = 0. \tag{5.116}$$

This makes sense. Someone in the frame of the support (which has horizontal acceleration $\ddot{x} = -A\omega^2\cos(\omega t)$) may as well be living in a world where the acceleration from gravity has a component g downward and a component $A\omega^2\cos(\omega t)$ to the right. Eq. (5.123) is simply the equation for the force in the tangential direction.

A small-angle approximation gives

$$\ddot{\theta} + \omega_0^2\theta = a\omega^2\cos(\omega t), \tag{5.117}$$

where $\omega_0 \equiv \sqrt{g/\ell}$ and $a \equiv A/\ell$. This equation is simply that of a driven oscillator, which we solved in Chapter 3. The solution is

$$\theta(t) = \frac{a\omega^2}{\omega_0^2 - \omega^2}\cos(\omega t) + C\cos(\omega_0 t + \phi), \tag{5.118}$$

where C and ϕ are determined by the initial conditions.

If ω happens to equal ω_0, then the amplitude goes to infinity. But in the real world, there is a damping term which keeps the coefficient of the $\cos(\omega t)$ term finite.

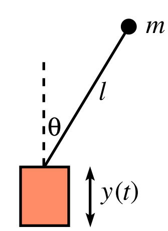


Figure 5.35

5. **Inverted pendulum**

(a) Let θ be defined as in Fig. 5.35. With $y(t) = A\cos(\omega t)$, the position of the mass m is given by

$$(X, Y)_m = (\ell\sin\theta, y + \ell\cos\theta). \tag{5.119}$$

The square of its speed is

$$V_m^2 = \ell^2\dot{\theta}^2 + \dot{y}^2 - 2\ell\dot{y}\dot{\theta}\sin\theta. \tag{5.120}$$

The Lagrangian is therefore

$$L = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{y}^2 - 2\ell\dot{y}\dot{\theta}\sin\theta) - mg(y + \ell\cos\theta). \tag{5.121}$$

The equation of motion is

$$\ell\ddot{\theta} - \ddot{y}\sin\theta = g\sin\theta. \qquad (5.122)$$

Plugging in the explicit form of $y(t)$, we have

$$\ell\ddot{\theta} + \sin\theta(A\omega^2\cos(\omega t) - g) = 0. \qquad (5.123)$$

This makes sense. Someone in the frame of the support (which has acceleration $\ddot{y} = -A\omega^2\cos(\omega t)$) may as well be living in a world where the acceleration from gravity is $(g - A\omega^2\cos(\omega t))$ downward. Eq. (5.123) is simply the equation for the force in the tangential direction.

Assuming θ is small, we may set $\sin\theta \approx \theta$, which gives

$$\ddot{\theta} + \theta\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big) = 0. \qquad (5.124)$$

where $\omega_0 \equiv \sqrt{g/\ell}$ and $a \equiv A/\ell$.

(b) Eq. (5.124) cannot be solved exactly, but we can get a general idea of how θ depends on time in two different ways.

One way is to solve the equation numerically. Figs. [numerical] show the results with parameters $\ell = 1\text{m}$, $A = 0.1\text{m}$, and $g = 10\text{m/s}^2$. In the first plot, $\omega = 10\text{s}^{-1}$; in the second plot, $\omega = 100\text{s}^{-1}$. The stick falls over in first case, but undergoes oscillatory motion in the second case. Apparently, if ω is large enough, the stick will not fall over.

Now let's explain this phenomenon in a second way: by making rough, order-of-magnitude arguments. At first glance, it seems surprising that the stick will stay up. It seems like the average of the 'effective gravity' acceleration (averaged over a few periods of the ω oscillations) in eq. (5.124) is $(-\theta g)$, since the $\cos(\omega t)$ term averages to zero (or so it appears). So one might think that there is a net downward force, making the stick fall over.

The fallacy in this reasoning is that the average of the $\theta\cos(\omega t)$ term is *not* zero, because θ undergoes tiny oscillations with frequency ω (as seen in Fig. [numerical]). The values of θ and $\cos(\omega t)$ are correlated; the θ at the t when $\cos(\omega t) = 1$ is larger than the θ at the t when $\cos(\omega t) = -1$. So there is a net positive contribution to the $\theta\cos(\omega t)$ part of the force. (And, indeed, it may be large enough to keep the pendulum up, as we show below.) This reasoning is enough to make the phenomenon believable (at least to me), but let's do a little more.

How large is this positive contribution from the $\theta\cos(\omega t)$ term? Let's make some rough approximations. We will look at the case where ω is large and $a \equiv A/\ell$ is small. (more precisely, we will assume $a \ll 1$ and $a\omega^2 \gg \omega_0^2$, for reasons seen below). Look at one of the little oscillations (with frequency ω in Fig. [pendulum]). The average position of the pendulum doesn't change much over one of these small periods, so we can look for an approximate solution to eq. (5.123) of the form

$$\theta(t) \approx C + b\cos(\omega t), \qquad (5.125)$$

where $b \ll C$. C will change over time, but on the scale of $1/\omega$ it is essentially constant.

Plugging this guess for θ into eq. (5.124), we find, at leading order (using $a \ll 1$ and $a\omega^2 \gg \omega_0^2$), $-\omega^2 b\cos(\omega t) + Ca\omega^2\cos(\omega t) = 0$. So we must have $b = aC$.

Our approximate solution for θ, on short time scales, is therefore

$$\theta \approx C\Big(1 + a\cos(\omega t)\Big). \tag{5.126}$$

From eq. (5.124), the average acceleration of θ, over a period $T = 2\pi/\omega$, is then

$$\begin{aligned}
\overline{\ddot{\theta}} &= \overline{-\theta\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big)} \\
&= \overline{-C\Big(1 + a\cos(\omega t)\Big)\Big(a\omega^2\cos(\omega t) - \omega_0^2\Big)} \\
&= -C\Big(a^2\omega^2\overline{\cos^2(\omega t)} - \omega_0^2\Big) \\
&= -C\left(\frac{a^2\omega^2}{2} - \omega_0^2\right) \\
&\equiv -C\Omega^2.
\end{aligned} \tag{5.127}$$

But from eq. (5.125) we see that the average acceleration of θ is simply $\ddot{C}$. So we have

$$\ddot{C}(t) + \Omega^2 C(t) \approx 0. \tag{5.128}$$

Therefore, C oscillates sinusoidally with frequency

$$\Omega = \sqrt{\frac{a^2\omega^2}{2} - \frac{g}{\ell}}. \tag{5.129}$$

We must have $a\omega > \sqrt{2}\omega_0$ if this frequency is to be real so that the pendulum stays up. Note that since we have assumed $a \ll 1$, we see that $a^2\omega^2 > 2\omega_0^2$ implies $a\omega^2 \gg \omega_0^2$ (for the case where the pendulum stays up), which is consistent with our assumption above.

If $a\omega \gg \sqrt{g/\ell}$, then we have $\Omega \approx a\omega/\sqrt{2}$. Such is the case if the pendulum lies in a horizontal plane where the acceleration from gravity is zero.

6. **Minimum or saddle**

For the given $\xi(t)$, the integrand in eq. (5.25) is symmetric about the midpoint, so we obtain

$$\begin{aligned}
\Delta S &= \int_0^{T/2} \left(m\Big(\frac{\epsilon}{T}\Big)^2 - k\Big(\frac{\epsilon t}{T}\Big)^2 \right) dt. \\
&= \frac{m\epsilon^2}{2T} - \frac{k\epsilon^2 T}{24}\,.
\end{aligned} \tag{5.130}$$

This is negative if $T > \sqrt{12m/k} \equiv 2\sqrt{3}/\omega$. Since the period of the oscillator is $\tau \equiv 2\pi/\omega$, we see that $T \equiv t_2 - t_1$ must be greater than $(\sqrt{3}/\pi)\tau$ in order for ΔS to be negative (provided that we are using our triangular function for ξ).

Roughly speaking, if $T \gtrsim \tau$, then the stationary point of S is a saddle point. And if $T \lesssim \tau$, then the stationary point of S is a minimum. In the latter case, the basic point is that T is small enough so that there is no way for ξ to get large, without making $\dot{\xi}$ large also.

REMARK: Our triangular function for ξ was easy to deal with, but it is undoubtedly not the function that gives the best chance of making ΔS negative. In other words, we should expect that it is possible to make T (slightly) less than $(\sqrt{3}/\pi)\tau$, and still be able to find a function ξ that makes ΔS negative. So let's try to find the smallest possible T for which ΔS can be negative.

It turns out that a function with the shape $\xi(t) \propto \sin(\pi t/T)$ gives the best chance of making ΔS negative.[11] With this form of ξ, we find $\Delta S \propto m\pi^2/T - kT$. This is negative if $T > \pi\sqrt{m/k} \equiv \pi/\omega$. In other words, if $T > \tau/2$, then the stationary value of S is a saddle point. And if $T < \tau/2$, then the stationary value of S is a minimum.

Our $T > (\sqrt{3}/\pi)\tau$ approximation using the triangular function was therefore a fairly good one. ♣

7. **Mass on plane**

First Solution: The most convenient coordinates in this problem are w and z, where w is the distance upward along the plane, and z is the distance perpendicularly away from it. The Lagrangian is then

$$\frac{1}{2}m(\dot{w}^2 + \dot{z}^2) - mg(w\sin\theta + z\cos\theta) - V(z), \tag{5.132}$$

where $V(z)$ is the (very steep) constraining potential. The two equations of motion are

$$\begin{aligned} m\ddot{w} &= -mg\sin\theta, \\ m\ddot{z} &= -mg\cos\theta - \frac{dV}{dz}\,. \end{aligned} \tag{5.133}$$

At this point we invoke the constraint $z = 0$. So $\ddot{z} = 0$, and the second equation gives us

$$F_{\rm c} \equiv -V'(0) = mg\cos\theta, \tag{5.134}$$

as desired. We also obtain the usual result of $\ddot{w} = -g\sin\theta$.

Second Solution: We can also solve this problem by using the horizontal and vertical components, x and y. We'll choose $(x, y) = (0, 0)$ to be at the top of the plane. The (very steep) constraining potential is $V(z)$, where $z \equiv x\sin\theta + y\cos\theta$ is the distance from the mass to the plane (as you can verify). The Lagrangian is then

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy - V(z) \tag{5.135}$$

Keeping mind that $z \equiv x\sin\theta + y\cos\theta$, the two equations of motion are (using the chain rule)

$$\begin{aligned} m\ddot{x} &= -\frac{dV}{dz}\frac{\partial z}{\partial x} = -V'(z)\sin\theta, \\ m\ddot{y} &= -mg - \frac{dV}{dz}\frac{\partial z}{\partial y} = -mg - V'(z)\cos\theta. \end{aligned} \tag{5.136}$$

[11] You can show this by invoking a theorem from Fourier analysis which says that any function satisfying $\xi(0) = \xi(T) = 0$ can be written as the sum $\xi(t) = \sum_1^\infty c_n \sin(n\pi t/T)$, where the c_n are numerical coefficients. When this sum is plugged into eq. (5.25), you can show that all the cross terms (terms involving two different values of n) integrate to zero. Using the fact that the average value of $\sin^2\theta$ and $\cos^2\theta$ is $1/2$, the rest of the integral is easily found to give

$$\Delta S = \frac{1}{4}\sum_1^\infty c_n^2 \left(\frac{m\pi^2 n^2}{T} - kT\right). \tag{5.131}$$

In order to obtain the smallest value of T that can make this negative, we clearly want only the $n = 1$ term to exist in the sum.

At this point we invoke the constraint condition $x = -y\cot\theta$ (that is, $z = 0$). This condition, along with the two E-L equations, allows us to solve for the three unknowns, $\ddot{x}$, $\ddot{y}$, and $V'(0)$. Using $\ddot{x} = -\ddot{y}\cot\theta$ in eqs. (5.136), we find

$$\ddot{y} = -g\sin^2\theta, \qquad \ddot{x} = g\cos\theta\sin\theta, \qquad F_c \equiv -V'(0) = mg\cos\theta. \tag{5.137}$$

The first two results here are simply the horizontal and vertical components of the acceleration along the plane.

8. **Leaving the moving sphere**

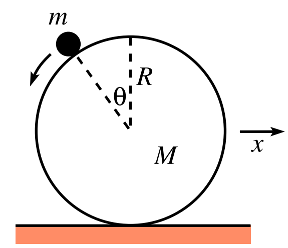


Figure 5.36

Let R be the radius of the sphere. Assume that the particle falls to the left and the sphere recoils to the right. Let θ be the angle from the top of the sphere (counterclockwise positive; see Fig. 5.36). Let x be the horizontal position of the sphere (positive to the right). Then the position of the particle (relative to the initial center of the sphere) is $(x - r\sin\theta, r\cos\theta)$, where r is constrained to be R. So the particle's velocity is $(\dot{x} - r\dot{\theta}\cos\theta, -r\dot{\theta}\sin\theta)$, and the square of its speed is $v_m^2 = \dot{x}^2 + r^2\dot{\theta}^2 - 2r\dot{x}\dot{\theta}\cos\theta$. We have ignored the negligible terms involving $\dot{r}$. To find the force of constraint, we have to consider the (very steep) potential, $V(r)$, keeping the particle on the sphere. The Lagrangian is

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + r^2\dot{\theta}^2 - 2r\dot{x}\dot{\theta}\cos\theta) - mgr\cos\theta - V(r). \tag{5.138}$$

If $\dot{x} = 0$, this reduces to the previous problem. The equations of motion from varying x, θ, and r are

$$\begin{aligned} (M+m)\ddot{x} - mr\frac{d}{dt}(\dot{\theta}\cos\theta) &= 0, \\ r\ddot{\theta} - \ddot{x}\cos\theta &= g\sin\theta, \\ mr\dot{\theta}^2 - m\dot{x}\dot{\theta}\cos\theta - mg\cos\theta - V'(r) &= 0, \end{aligned} \tag{5.139}$$

where we have ignored the $\dot{r}$ terms. The first equation (when integrated) is conservation of momentum. The second equation is $F = ma$ for the θ direction in a world where 'gravity' pulls with strength g downward and strength $\ddot{x}$ to the left (in the accelerated frame of the sphere, this is what the particle feels). The third equation gives the radial force of constraint, $F = -dV/dr$ (evaluated at $r = R$), as

$$F(\theta, \dot{\theta}, \dot{x}) = mg\cos\theta + m\dot{x}\dot{\theta}\cos\theta - mR\dot{\theta}^2. \tag{5.140}$$

Let us now eliminate x from our equations. We may use the first equation in (5.139) to eliminate $\ddot{x}$ from the second equation. The result is (with $r = R$)

$$\ddot{\theta}(M + m\sin^2\theta) + m\dot{\theta}^2\cos\theta\sin\theta - \frac{g}{R}(M+m)\sin\theta = 0. \tag{5.141}$$

We may also use the integrated form of the first equation in (5.139) to eliminate $\dot{x}$ from the expression for F. (The integrated form is $(M+m)\dot{x} - mr\dot{\theta}\cos\theta = 0$, where the constant of integration is zero, due to the initial conditions, $\dot{x} = \dot{\theta} = 0$ at $t = 0$.) The result is (with $r = R$)

$$F(\theta, \dot{\theta}) = mg\cos\theta + \frac{m^2R}{M+m}\dot{\theta}^2\cos^2\theta - mR\dot{\theta}^2. \tag{5.142}$$

If $M \gg m$, these results agree with eq. (5.31), as they should, since the sphere will remain essentially at rest.

9. **Bead on stick**

The Lagrangian for the bead is

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2. \tag{5.143}$$

Eq. (5.52) therefore gives

$$E = \frac{1}{2}m\dot{r}^2 - \frac{1}{2}mr^2\omega^2. \tag{5.144}$$

Claim 5.3 says that this quantity is conserved, because $\partial L/\partial t = 0$. But it is *not* the energy of the bead, due to the minus sign in the second term.

The main point here is that in order to keep the stick rotating at a constant angular speed, there must be an external force acting it. This force will cause work to be done on the bead, thereby changing its kinetic energy.

From the above equations, we see that $E = T - mr^2\omega^2$ is the quantity that doesn't change with time (where T is the kinetic energy).

10. **Atwood's machine 1**

First solution: If the left mass goes up by x, and the right mass goes up by y, then conservation of string says that the middle mass must go down by $x + y$. Therefore, the Lagrangian of the system is

$$\begin{aligned} L &= \frac{1}{2}(4m)\dot{x}^2 + \frac{1}{2}(3m)(-\dot{x} - \dot{y})^2 + \frac{1}{2}m\dot{y}^2 - \Big((4m)gx + (3m)g(-x-y) + mgy\Big) \\ &= \frac{7}{2}m\dot{x}^2 + 3m\dot{x}\dot{y} + 2m\dot{y}^2 - mg(x - 2y). \end{aligned} \tag{5.145}$$

This is clearly invariant under the transformation $x \to x + 2\epsilon$ and $y \to y + \epsilon$. Hence, we can use Noether's theorem, with $K_x = 2$ and $K_y = 1$. So the conserved momentum is

$$P = \frac{\partial L}{\partial \dot{x}}K_x + \frac{\partial L}{\partial \dot{y}}K_y = m(7\dot{x} + 3\dot{y})(2) + m(3\dot{x} + 4\dot{y})(1) = m(17\dot{x} + 10\dot{y}). \tag{5.146}$$

P is constant. In particular, if the system starts at rest, then $\dot{x}$ always equals $-(10/17)\dot{y}$.

Second solution: With x and y defined as in the first solution, the Euler-Lagrange equations are, from eq. (5.145),

$$\begin{aligned} 7m\ddot{x} + 3m\ddot{y} &= -mg, \\ 3m\ddot{x} + 4m\ddot{y} &= 2mg. \end{aligned} \tag{5.147}$$

Adding the second equation to twice the first gives

$$17m\ddot{x} + 10m\ddot{y} = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}\Big(17m\dot{x} + 10m\dot{y}\Big) = 0. \tag{5.148}$$

Third solution: We can also solve the problem using only $F = ma$. Since the tension, T, is the same throughout the rope, we see that the three $F = dP/dt$ equations are

$$2T - 4mg = \frac{dP_{4m}}{dt}, \qquad 2T - 3mg = \frac{dP_{3m}}{dt}, \qquad 2T - mg = \frac{dP_m}{dt}. \tag{5.149}$$

The three forces depend on only two parameters, so there will be some combination of them that adds to zero. If we set $a(2T-4mg)+b(2T-3mg)+c(2T-mg)=0$, then $a+b+c=0$ and $4a+3b+c=0$, which is satisfied by $a=2$, $b=-3$, and $c=1$. Therefore (with x and y defined as in the first solution),

$$\begin{aligned} 0 &= \frac{d}{dt}(2P_{4m}-3P_{3m}+P_m) \\ &= \frac{d}{dt}\Big(2(4m)\dot{x}-3(3m)(-\dot{x}-\dot{y})+m\dot{y}\Big) \\ &= \frac{d}{dt}(17m\dot{x}+10m\dot{y}). \end{aligned} \tag{5.150}$$

11. **Atwood's machine 2**

First solution: The average of the heights of the right two masses (relative to their initial positions) is $-x$. Therefore, the position of the right mass must be $-2x-y$.

The Lagrangian of the system is thus

$$\begin{aligned} L &= \frac{1}{2}(5m)\dot{x}^2+\frac{1}{2}(4m)\dot{y}^2+\frac{1}{2}(2m)(-2\dot{x}-\dot{y})^2-\Big((5m)gx+(4m)gy+(2m)g(-2x-y)\Big) \\ &= \frac{13}{2}m\dot{x}^2+4m\dot{x}\dot{y}+3m\dot{y}^2-mg(x+2y). \end{aligned} \tag{5.151}$$

This is clearly invariant under the transformation $x\to x+2\epsilon$ and $y\to y-\epsilon$. Hence, we can use Noether's theorem, with $K_x=2$ and $K_y=-1$. So the conserved momentum is

$$P=\frac{\partial L}{\partial \dot{x}}K_x+\frac{\partial L}{\partial \dot{y}}K_y=m(13\dot{x}+4\dot{y})(2)+m(4\dot{x}+6\dot{y})(-1)=m(22\dot{x}+2\dot{y}). \tag{5.152}$$

P is constant. In particular, if the system starts at rest, then $\dot{x}$ always equals $-(1/11)\dot{y}$.

Second solution: With x and y defined as in the first solution, the Euler-Lagrange equations are, from eq. (5.151),

$$\begin{aligned} 13m\ddot{x}+4m\ddot{y} &= -mg, \\ 4m\ddot{x}+6m\ddot{y} &= -2mg. \end{aligned} \tag{5.153}$$

Subtracting half the second equation from the first gives

$$11m\ddot{x}+m\ddot{y}=0 \qquad \Longrightarrow \qquad \frac{d}{dt}\Big(11m\dot{x}+m\dot{y}\Big)=0, \tag{5.154}$$

in agreement with (5.152).

Third solution: We can also solve the problem using only $F=ma$. Since the tension in the top rope is twice that in the bottom rope (because the net force on the massless lower pulley must be zero), we see that the three $F=dP/dt$ equations are

$$2T-5mg=\frac{dP_{5m}}{dt}, \qquad T-4mg=\frac{dP_{4m}}{dt}, \qquad T-2mg=\frac{dP_{2m}}{dt}. \tag{5.155}$$

The three forces depend on only two parameters, so there will be some combination of them that adds to zero. If we set $a(2T-5mg)+b(T-4mg)+c(T-2mg)=0$, then $2a + b + c = 0$ and $5a + 4b + 2c = 0$, which is satisfied by $a = 2$, $b = -1$, and $c = -3$. Therefore (with x and y defined as in the first solution),

$$\begin{aligned} 0 &= \frac{d}{dt}(2P_{5m} - P_{4m} - 3P_{2m}) \\ &= \frac{d}{dt}\Big(2(5m)\dot{x} - (4m)\dot{y} - 3(2m)(-2\dot{x} - \dot{y})\Big) \\ &= \frac{d}{dt}(22m\dot{x} + 2m\dot{y}). \end{aligned} \tag{5.156}$$

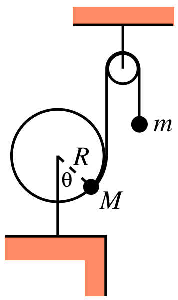


Figure 5.37

12. **Pulley pendulum**

Let the radius to M make an angle θ with the vertical (see Fig. 5.37). Then the coordinates of M are $R(\sin\theta, -\cos\theta)$. The height of the mass m, relative to its position when M is at the bottom of the hoop, is $y = -R\theta$. The Lagrangian is therefore (and yes, we've chosen a different $y = 0$ point for each mass, but such a definition only changes the potential by a constant amount, which is irrelevant)

$$L = \frac{1}{2}(M + m)R^2\dot{\theta}^2 + MgR\cos\theta + mgR\theta. \tag{5.157}$$

The equation of motion is

$$(M + m)R\ddot{\theta} = g(m - M\sin\theta). \tag{5.158}$$

This is, of course, just $F = ma$ along the direction of the string (since $Mg\sin\theta$ is the tangential component of the gravitational force on M).

Equilibrium occurs when $\dot{\theta} = \ddot{\theta} = 0$, i.e., at the θ_0 for which $\sin\theta_0 = m/M$. Letting $\theta \equiv \theta_0 + \delta$, and expanding eq. (5.158) to first order in δ gives

$$\ddot{\delta} + \frac{Mg\cos\theta_0}{(M + m)R}\delta = 0. \tag{5.159}$$

So the frequency of small oscillations is

$$\omega = \sqrt{\frac{M\cos\theta_0}{M + m}}\sqrt{\frac{g}{R}}. \tag{5.160}$$

Remarks: If $M \gg m$, then $\theta_0 \approx 0$, and $\omega \approx \sqrt{g/R}$. This makes sense, because m can be ignored, and M essentially oscillates about the bottom of the hoop, just like a pendulum of length R.

If M is only slightly greater than m, then $\theta_0 \approx \pi/2$. So $\cos\theta_0 \approx 0$, and hence $\omega \approx 0$. This makes sense; the restoring force $g(m - M\sin\theta)$ does not change much as θ changes, so it's as if we have a pendulum in a weak gravitational field.

The frequency found in eq. (5.160) can actually be figured out with no calculations at all. Look at M at the equilibrium position. The tangential forces on it cancel, so all it feels is the $Mg\cos\theta_0$ normal force from the hoop which balances the radial component of the gravitational force. Therefore, for all the mass M knows, it is sitting at the bottom of a hoop of radius R in a world where gravity has strength $g' = g\cos\theta_0$. The general formula for the frequency of a pendulum is $\omega = \sqrt{F'/M'R}$, where F' is the downward force (which is Mg' here), and M' is the total mass being accelerated (which is $M + m$ here). This gives the ω in eq. (5.160). ♣

13. **Three hanging masses**

The height of M is $-l\tan\theta$. And the length of the string to M is $l/\cos\theta$, so the height of the m's is $l/\cos\theta$ (up to an additive constant). The speed of M is therefore $l\dot\theta/\cos^2\theta$, and the speed of the m's is $l\dot\theta\sin\theta/\cos^2\theta$. The Lagrangian is therefore

$$L = \frac{Ml^2}{2\cos^4\theta}\dot\theta^2 + \frac{ml^2\sin^2\theta}{\cos^4\theta}\dot\theta^2 + Mgl\tan\theta - \frac{2mgl}{\cos\theta}. \tag{5.161}$$

The equation of motion is

$$\begin{aligned} &M\frac{d}{dt}\left(\frac{\dot\theta}{\cos^4\theta}\right) + 2m\frac{d}{dt}\left(\frac{\dot\theta\sin^2\theta}{\cos^4\theta}\right) \\ &= \frac{M}{2}\frac{d}{d\theta}\left(\frac{\dot\theta^2}{\cos^4\theta}\right) + m\frac{d}{d\theta}\left(\frac{\sin^2\theta\dot\theta^2}{\cos^4\theta}\right) + \frac{Mg}{l\cos^2\theta} - \frac{2mg\sin\theta}{l\cos^2\theta}. \end{aligned} \tag{5.162}$$

After a rather large amount of simplification, this becomes

$$\begin{aligned} &M\ddot\theta\cos\theta + 2M\dot\theta^2\sin\theta + 2m\ddot\theta\cos\theta\sin^2\theta + 2m\dot\theta^2\cos^2\theta\sin\theta + 4m\dot\theta^2\sin^3\theta \\ &= \frac{Mg}{l}\cos^3\theta - \frac{2mg}{l}\sin\theta\cos^3\theta. \end{aligned} \tag{5.163}$$

Equilibrium occurs when $\ddot\theta = \dot\theta = 0$, and therefore when $\sin\theta_0 = M/2m$. Letting $\theta = \theta_0 + \delta$, we may expand eq. (5.163) to first order in δ. The terms involving $\dot\theta^2 = \dot\delta^2$ are of second order in δ and may be dropped. We find

$$\begin{aligned} &\ddot\delta(M\cos\theta_0 + 2m\cos\theta_0\sin^2\theta_0) \\ &= \delta(-3M\cos^2\theta_0\sin\theta_0 + 6m\cos^2\theta_0\sin^2\theta_0 - 2m\cos^4\theta_0)\frac{g}{l} \end{aligned} \tag{5.164}$$

Plugging in $\sin\theta_0 = M/2m$, we find (after some simplification) that the frequency of small oscillations is

$$\omega^2 = \frac{g}{l}\left(\frac{(4m^2 - M^2)^{3/2}}{2mM(M+2m)}\right). \tag{5.165}$$

Remark: In the limit where M is just slightly less than $2m$, we have $\omega \approx 0$. This makes sense, because the equilibrium position of M is very low, and the force changes slowly with position.

In the limit $M \to 0$, we have $\omega^2 \approx (g/l)(2m/M) \to \infty$. This makes sense, because the tension in the string is huge compared to the small mass M. We can even be quantitative about this. If we tilt the picture sideways, we essentially have a pendulum of mass M and length l. And the 'effective gravity' force acting on the mass is $2mg = (2mg/M)M$. So for all the mass M knows, it is in a world where gravity has strength $g' = 2mg/M$. The frequency of such a pendulum is $\sqrt{g'/l} = \sqrt{(g/l)(2m/M)}$. ♣

14. **Bead on rotating hoop**

Let θ be the angle the radius to the bead makes with the vertical (see Fig. 5.38). Breaking the velocity up into the part along the hoop plus the part perpendicular to the hoop, we find

$$L = \frac{1}{2}m(\omega^2R^2\sin^2\theta + R^2\dot\theta^2) + mgR\cos\theta. \tag{5.166}$$

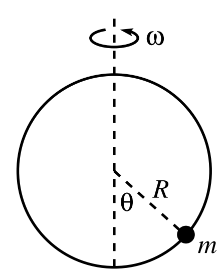


Figure 5.38

The equation of motion is

$$R\ddot{\theta} = \sin\theta(\omega^2 R\cos\theta - g). \tag{5.167}$$

This just says that the component of gravity pulling downward along the hoop accounts for the acceleration along the hoop plus the component of the centripetal acceleration along the hoop.

Equilibrium occurs when $\dot{\theta} = \ddot{\theta} = 0$. The right-hand side of eq. (5.167) equals 0 when either $\sin\theta = 0$ (i.e., $\theta = 0$ or $\theta = \pi$) or $\cos\theta = g/(R\omega^2)$. Since $\cos\theta$ must be less than 1, this second condition is possible only when $\omega^2 > g/R$. So we have two cases

- If $\omega^2 < g/R$, then $\theta = 0$ and $\theta = \pi$ are the only equilibrium points.

  The $\theta = \pi$ case is of course unstable. This can be seen mathematically by letting $\theta \equiv \pi + \delta$, where δ is small. Eq. (5.167) then becomes

  $$\ddot{\delta} - \delta(g/R + \omega^2) = 0. \tag{5.168}$$

  The coefficient of δ is negative, so this does not admit oscillatory solutions.

  The $\theta = 0$ case turns out to be stable. For small θ, eq. (5.167) becomes

  $$\ddot{\theta} + \theta(g/R - \omega^2) = 0. \tag{5.169}$$

  The coefficient of θ is positive, so we have sinusoidal solutions. The frequency of small oscillations is $\sqrt{g/R - \omega^2}$. This goes to 0 as $\omega \to \sqrt{g/R}$.

- If $\omega^2 > g/R$, then $\theta = 0$, $\theta = \pi$, and $\cos\theta_0 \equiv g/(R\omega^2)$ are all equilibrium points. But $\theta = 0$ is unstable because the coefficient of θ in eq. (5.169) is negative. (Similarly for $\theta = \pi$).

  So $\cos\theta_0 \equiv g/(R\omega^2)$ is the only stable equilibrium. To find the frequency of small oscillations, let $\theta \equiv \theta_0 + \delta$ in eq. (5.167), and expand to first order in δ. Using $\cos\theta_0 \equiv g/(R\omega^2)$, we find

  $$\ddot{\delta} + \omega^2\sin^2\theta_0\delta = 0. \tag{5.170}$$

  Therefore, the frequency of small oscillations is $\omega\sin\theta_0 = \sqrt{\omega^2 - g^2/R^2\omega^2}$.

  REMARK: This frequency goes to 0 as $\omega \to \sqrt{g/R}$. And it goes to ∞ as $\omega \to \infty$. This second limit can be looked at in the following way. For very large ω, gravity is not very important, and the bead essentially feels a centrifugal force of $mR\omega^2$ as it moves near $\theta = \pi/2$. So for all the bead knows, it is a pendulum of length R in a world where 'gravity' pulls sideways with a force $mR\omega^2 \equiv mg'$. The frequency of such a pendulum is $\sqrt{g'/R} = \sqrt{R\omega^2/R} = \omega$. ♣

The frequency $\omega = \sqrt{g/R}$ is the critical frequency above which there is a stable equilibrium at $\theta \neq 0$, i.e., above which the mass will want to move away from the bottom of the hoop.

15. **Another bead on rotating hoop**

Let the angles ωt and θ be defined as in Fig. 5.39. Then the cartesian coordinates for the bead are

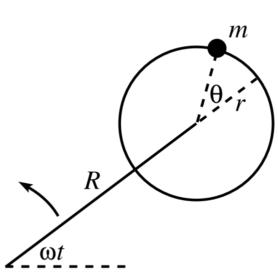


Figure 5.39

$$(x, y) = \Big(R\cos\omega t + r\cos(\omega t + \theta), R\sin\omega t + r\sin(\omega t + \theta)\Big). \tag{5.171}$$

The square of the speed is therefore

$$\begin{aligned} v^2 &= R^2\omega^2 + r^2(\omega+\dot\theta)^2 \\ &\quad +2Rr\omega(\omega+\dot\theta)\Big(\cos\omega t\cos(\omega t+\theta)+\sin\omega t\sin(\omega t+\theta)\Big) \\ &= R^2\omega^2 + r^2(\omega+\dot\theta)^2 + 2Rr\omega(\omega+\dot\theta)\cos\theta \end{aligned} \tag{5.172}$$

There is no potential energy, so the Lagrangian is simply $L = mv^2/2$. The equation of motion is then

$$r\ddot\theta + R\omega^2\sin\theta = 0. \tag{5.173}$$

Equilibrium occurs when $\dot\theta = \ddot\theta = 0$, and therefore when $\theta = 0$. (Well, $\theta = \pi$ also works, but that's an unstable equilibrium.) A small-angle approximation gives $\ddot\theta + (R/r)\omega^2\theta = 0$, so the frequency of small oscillations is $\Omega = \omega\sqrt{R/r}$.

Remarks: If $R \ll r$, then $\Omega \approx 0$. This makes sense, since the frictionless hoop is essentially not moving. If $R = r$, then $\Omega = \omega$. If $R \gg r$, then Ω is very large. In this case, we can double-check the value $\Omega = \omega\sqrt{R/r}$ in the following way. In the frame of the hoop, the bead feels a centrifugal force of $m(R+r)\omega^2$. For all the bead knows, it is in a gravitational field with strength $g' \equiv (R+r)\omega^2$. So the bead (which acts like a pendulum of length r), oscillates with frequency

$$\sqrt{\frac{g'}{r}} = \sqrt{\frac{R+r}{r}\omega^2} \approx \omega\sqrt{\frac{R}{r}}, \tag{5.174}$$

for $R \gg r$.

Note that if one tries to use this 'effective gravity' argument as a double check for smaller R values, one gets the wrong answer. For example, if $R = r$, we have would obtain an oscillation frequency of $\omega\sqrt{2R/r}$ instead of the correct value $\omega\sqrt{R/r}$. This is because in reality the centrifugal force fans out near the equilibrium point, while our 'effective' gravity' argument assumes that the field lines are parallel (and so it gives a frequency that is too large). ♣

16. **Rotating curve**

In terms of the variable x, the speed along the curve is $\dot x\sqrt{1+f'^2}$, and the speed perpendicular to the curve is ωx. So the Lagrangian is

$$\frac{1}{2}m\Big(\omega^2x^2 + \dot x^2(1+f'^2)\Big) - mgf(x). \tag{5.175}$$

The equation of motion is

$$\ddot x(1+f'^2) = \dot x^2 f'f'' + \omega^2 x - gf'. \tag{5.176}$$

Equilibrium occurs when $\dot x = \ddot x = 0$, and therefore at the x_0 for which

$$x_0 = \frac{gf'(x_0)}{\omega^2}. \tag{5.177}$$

(This simply says that the component of gravity along the curve accounts for the component of the centripetal acceleration along the curve.) Using our explicit form $f = b(x/a)^\lambda$, we find

$$x_0 = a\left(\frac{\omega^2a^2}{\lambda gb}\right)^{1/(\lambda-2)}. \tag{5.178}$$

As $\lambda \to \infty$, x_0 goes to a, as it should, since the curve essentially equals zero up to a, whereupon it rises very steeply. The reader can check numerous other limits.

Letting $x \equiv x_0 + \delta$ in eq. (5.176), and expanding to first order in δ, gives

$$\ddot{\delta}\Big(1 + f'(x_0)^2\Big) = \delta\Big(\omega^2 - gf''(x_0)\Big). \tag{5.179}$$

So the frequency of small oscillations is

$$\Omega^2 = \frac{gf''(x_0) - \omega^2}{1 + f'(x_0)^2}. \tag{5.180}$$

Using the explicit form of f, along with eq. (5.178), gives

$$\Omega^2 = \frac{(\lambda - 2)\omega^2}{1 + \frac{a^2\omega^4}{g^2}\left(\frac{a^2\omega^2}{\lambda gb}\right)^{2/(\lambda-2)}}. \tag{5.181}$$

We see that λ must be greater than 2 in order for there to be oscillatory behavior around the equilibrium point. For $\lambda < 2$, the equilibrium point is unstable, i.e., to the left the force is inward, and to the right the force is outward.

In the case $\lambda = 2$, the equilibrium condition, eq. (5.177), gives $x_0 = (2gb/a^2\omega^2)x_0$. For this to be true for some x_0, we must have $\omega^2 = 2gb/a^2$. But if this holds, then eq. (5.177) is true for all x. So in this special case, the particle feels no tangential force anywhere along the curve. (In the frame of the curve, the tangential components of the centrifugal and gravitational forces exactly cancel at all points.) If $\omega^2 \neq 2gb/a^2$, then the particle feels a force either always inward or always outward.

Remarks: For $\omega \to 0$, we have $x_0 \to 0$ and $\Omega \to 0$. And for $\omega \to \infty$, we have $x_0 \to \infty$ and $\Omega \to 0$. In both cases $\Omega \to 0$, because in both case the equilibrium position is at a place where the curve is very flat (horizontally or vertically, respectively), so the restoring force is very small.

For $\lambda \to \infty$, we have $x_0 \to a$ and $\Omega \to \infty$. The frequency is large here because the equilibrium position at a is where the curve has an abrupt corner, so the restoring force changes quickly with position. Or, you can think of it as a pendulum with a very small length (if you approximate the 'corner' by a tiny circle). ♣

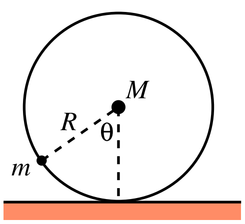


Figure 5.40

17. **Mass on wheel**

Let the angle θ be defined as in Fig. 5.40 (with the convention that θ is positive if M is to the right of m). Then the position of m in cartesian coordinates, relative to the point where m would be in contact with the ground, is

$$(x, y)_m = R(\theta - \sin\theta, 1 - \cos\theta). \tag{5.182}$$

The square of the speed of m is therefore $v_m^2 = 2R^2\dot{\theta}^2(1 - \cos\theta)$.

The position of M is $(x, y)_M = R(\theta, 1)$, so the square of its speed is $v_M^2 = R^2\dot{\theta}^2$. The Lagrangian is therefore

$$L = \frac{1}{2}MR^2\dot{\theta}^2 + mR^2\dot{\theta}^2(1 - \cos\theta) - mgR(1 - \cos\theta). \tag{5.183}$$

The equation of motion is

$$MR^2\ddot{\theta} + 2mR^2\ddot{\theta}(1 - \cos\theta) + mR^2\dot{\theta}^2\sin\theta + mgR\sin\theta = 0. \tag{5.184}$$

In the case of small oscillations, we may use $\cos\theta \approx 1 - \theta^2/2$ and $\sin\theta \approx \theta$. The second and third terms above are third order in θ and may be neglected. So we find

$$\ddot{\theta} + \frac{mg}{MR}\theta = 0. \tag{5.185}$$

The frequency of small oscillations is

$$\omega = \sqrt{\frac{m}{M}}\sqrt{\frac{g}{R}}. \tag{5.186}$$

REMARKS: If $M \gg m$, then $\omega \to 0$. This makes sense.

If $m \gg M$, then $\omega \to \infty$. This also makes sense, because the huge mg force makes the situation similar to one where the wheel is bolted to the floor, in which case the wheel vibrates with a high frequency.

Eq. (5.186) can actually be written down without doing any calculations. We'll let the reader show that for small oscillations the gravitational force on m has the effect of essentially applying a sideways force on M equal to $-mg\theta$. So the horizontal $F = Ma$ equation for M is $MR\ddot{\theta} = -mg\theta$, from which the result follows. ♣

18. **Double pendulum**

Relative to the pivot point, the cartesian coordinates of m_1 and m_2 are, respectively (see Fig. 5.41),

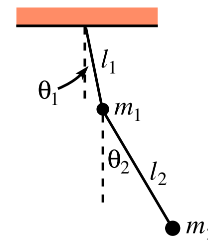


Figure 5.41

$$\begin{aligned}(x,y)_1 &= (\ell_1 \sin\theta_1, -\ell_1\cos\theta_1),\\ (x,y)_2 &= (\ell_1\sin\theta_1 + \ell_2\sin\theta_2, -\ell_1\cos\theta_1 - \ell_2\cos\theta_2).\end{aligned} \tag{5.187}$$

The squares of the speeds are therefore

$$\begin{aligned}v_1^2 &= \ell_1^2\dot{\theta}_1^2,\\ v_2^2 &= \ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2).\end{aligned} \tag{5.188}$$

The Lagrangian is then

$$\begin{aligned}L &= \frac{1}{2}m_1\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\Big(\ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\Big)\\ &\quad + m_1 g\ell_1\cos\theta_1 + m_2 g(\ell_1\cos\theta_1 + \ell_2\cos\theta_2).\end{aligned} \tag{5.189}$$

The equations of motion from varying θ_1 and θ_2 are

$$\begin{aligned}0 &= (m_1+m_2)\ell_1^2\ddot{\theta}_1 + m_2\ell_1\ell_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2\ell_1\ell_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)\\ &\quad +(m_1+m_2)g\ell_1\sin\theta_1,\\ 0 &= m_2\ell_2^2\ddot{\theta}_2 + m_2\ell_1\ell_2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2\ell_1\ell_2\dot{\theta}_1^2\sin(\theta_1-\theta_2)\\ &\quad + m_2 g\ell_2\sin\theta_2.\end{aligned} \tag{5.190}$$

This is a rather large mess, but it simplifies greatly if we consider small oscillations. Using the small-angle approximations and keeping only the leading-order terms, we have

$$\begin{aligned}0 &= (m_1+m_2)\ell_1\ddot{\theta}_1 + m_2\ell_2\ddot{\theta}_2 + (m_1+m_2)g\theta_1,\\ 0 &= \ell_2\ddot{\theta}_2 + \ell_1\ddot{\theta}_1 + g\theta_2.\end{aligned} \tag{5.191}$$

- Consider the special case $\ell_1 = \ell_2 \equiv \ell$. We may find the frequencies of the normal modes using the usual determinant method. They are

$$\omega_\pm = \sqrt{\frac{m_1 + m_2 \pm \sqrt{m_1 m_2 + m_2^2}}{m_1}}\sqrt{\frac{g}{\ell}}. \tag{5.192}$$

The normal modes are found to be, after some simplification,

$$\left(\begin{array}{c} \theta_1(t) \\ \theta_2(t) \end{array} \right)_\pm = \left(\begin{array}{c} \mp\sqrt{m_2} \\ \sqrt{m_1+m_2} \end{array} \right) \cos(\omega_\pm t + \phi_\pm). \tag{5.193}$$

Some special cases are:

Case 1: $m_1 = m_2$**.** The frequencies are

$$\omega_\pm = \sqrt{2 \pm \sqrt{2}}\sqrt{\frac{g}{\ell}}. \tag{5.194}$$

The normal modes are

$$\left(\begin{array}{c} \theta_1(t) \\ \theta_2(t) \end{array} \right)_\pm = \left(\begin{array}{c} \mp 1 \\ \sqrt{2} \end{array} \right) \cos(\omega_\pm t + \phi_\pm). \tag{5.195}$$

Case 2: $m_1 \gg m_2$**.** With $m_2/m_1 \equiv \epsilon$, the frequencies are (to leading order in ϵ)

$$\omega_\pm = (1 \pm \sqrt{\epsilon})\sqrt{\frac{g}{\ell}}. \tag{5.196}$$

The normal modes are

$$\left(\begin{array}{c} \theta_1(t) \\ \theta_2(t) \end{array} \right)_\pm = \left(\begin{array}{c} \mp\sqrt{\epsilon} \\ 1 \end{array} \right) \cos(\omega_\pm t + \phi_\pm). \tag{5.197}$$

In both modes, the upper (heavy) mass essentially stands still, and the lower (light) mass oscillates like a pendulum of length ℓ.

Case 3: $m_1 \ll m_2$**.** With $m_1/m_2 \equiv \epsilon$, the frequencies are (to leading order in ϵ)

$$\omega_+ = \sqrt{\frac{2g}{\epsilon\ell}}, \qquad \omega_- = \sqrt{\frac{g}{2\ell}}. \tag{5.198}$$

The normal modes are

$$\left(\begin{array}{c} \theta_1(t) \\ \theta_2(t) \end{array} \right)_\pm = \left(\begin{array}{c} \mp 1 \\ 1 \end{array} \right) \cos(\omega_\pm t + \phi_\pm). \tag{5.199}$$

In the first mode, the lower (heavy) mass essentially stands still, and the upper (light) mass vibrates back and forth at a high frequency (because there is a very large tension in the rods). In the second mode, the rods form a straight line, and the system is essentially a pendulum of length 2ℓ.

- Consider the special case $m_1 = m_2$. Using the determinant method, the frequencies of the normal modes are found to be

$$\omega_\pm = \sqrt{g}\sqrt{\frac{\ell_1 + \ell_2 \pm \sqrt{\ell_1^2 + \ell_2^2}}{\ell_1 \ell_2}}. \tag{5.200}$$

The normal modes are found to be, after some simplification,

$$\left(\begin{array}{c} \theta_1(t) \\ \theta_2(t) \end{array} \right)_\pm = \left(\begin{array}{c} \ell_2 \\ \ell_2 - \ell_1 \mp \sqrt{\ell_1^2 + \ell_2^2} \end{array} \right) \cos(\omega_\pm t + \phi_\pm). \tag{5.201}$$

Some special cases are:

Case 1: $\ell_1 = \ell_2$**.** This was done above.

Case 2: $\ell_1 \gg \ell_2$**.** With $\ell_2/\ell_1 \equiv \epsilon$, the frequencies are (to leading order in ϵ)

$$\omega_+ = \sqrt{\frac{2g}{\ell_2}}, \qquad \omega_- = \sqrt{\frac{g}{\ell_1}}. \tag{5.202}$$

The normal modes are

$$\begin{aligned} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_+ &= \begin{pmatrix} -\epsilon \\ 2 \end{pmatrix} \cos(\omega_+ t + \phi_+), \\ \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_- &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t + \phi_-). \end{aligned} \tag{5.203}$$

In the first mode, the masses essentially move equal distances in opposite directions, at a very high frequency (because ℓ_2 is so small). In the second mode, the string stays straight, and the masses move just like a mass of $2m$. The system is essentially a pendulum of length ℓ.

Case 3: $\ell_1 \ll \ell_2$**.** With $\ell_1/\ell_2 \equiv \epsilon$, the frequencies are (to leading order in ϵ)

$$\omega_+ = \sqrt{\frac{2g}{\ell_1}}, \qquad \omega_- = \sqrt{\frac{g}{\ell_2}}. \tag{5.204}$$

The normal modes are

$$\begin{aligned} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_+ &= \begin{pmatrix} 1 \\ -\epsilon \end{pmatrix} \cos(\omega_+ t + \phi_+), \\ \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}_- &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos(\omega_- t + \phi_-). \end{aligned} \tag{5.205}$$

In the first mode, the bottom mass essentially stays still, while the top one oscillates at a very high frequency (because ℓ_1 is so small). The factor of 2 is in the frequency because the top mass essentially lives in a world where the acceleration from gravity is $g' = 2g$ (because of the extra mg force downward from the lower mass). In the second mode, the system is essentially a pendulum of length ℓ_2. The string is slightly bent, just enough to make the tangential force on the top mass roughly 0 (because otherwise it would oscillate at a high frequency, since ℓ_1 is so small).

19. **Pendulum with free support**

Let x be the coordinate of M. Let θ be the angle of the pendulum (see Fig. 5.42). Then the position of the mass m, in cartesian coordinates, is $(x+\ell\sin\theta, -\ell\cos\theta)$. The square of the speed of m is found to be $v_m^2 = \dot{x}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta$. The Lagrangian is therefore

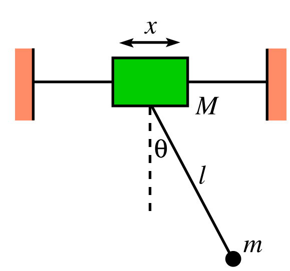


Figure 5.42

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{x}\dot{\theta}\cos\theta) + mg\ell\cos\theta. \tag{5.206}$$

The equations of motion from varying x and θ are

$$\begin{aligned} (M+m)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta &= 0, \\ \ell\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta &= 0. \end{aligned} \tag{5.207}$$

In the event that θ is very small, we may use the small angle approximations. Keeping only terms that are first-order in θ, we obtain

$$\begin{aligned}(M+m)\ddot{x}+m\ell\ddot{\theta} &= 0,\\ \ddot{x}+\ell\ddot{\theta}+g\theta &= 0.\end{aligned} \tag{5.208}$$

The first equation is momentum conservation. Integrating it twice gives

$$x=-\frac{m\ell}{M+m}\theta+At+B. \tag{5.209}$$

The second equation is $F=ma$ in the tangential direction.

Eliminating $\ddot{x}$ from eqs. (5.208) gives

$$\ddot{\theta}+\left(\frac{M+m}{M}\right)\frac{g}{\ell}\theta=0. \tag{5.210}$$

The solution to this equation is $\theta(t)=C\cos(\omega t+\phi)$, where

$$\omega=\sqrt{1+\frac{m}{M}}\sqrt{\frac{g}{\ell}}. \tag{5.211}$$

So the general solution for x and θ is

$$\theta(t)=C\cos(\omega t+\phi),\qquad x(t)=-\frac{Cm\ell}{M+m}\cos(\omega t+\phi)+At+B. \tag{5.212}$$

The constant B is irrelevant, so let's forget it.

The two normal modes are the following.

- $A=0$: In this case $x=-\theta m\ell/(M+m)$. So the masses oscillate with frequency ω, always moving in opposite directions.
- $C=0$: In this case, $\theta=0$ and $x=At$. So both masses move horizontally with the same speed.

Remarks: If $M\gg m$, then $\omega=\sqrt{g/\ell}$, as it should be, since the support essentially stays still.

If $m\gg M$, then $\omega\to\sqrt{m/M}\sqrt{g/\ell}\to\infty$. This makes sense, since the tension in the rod is so large. We can actually be quantitative about this limit. We'll let you show that for small oscillations and $m\gg M$, the gravitational force on m has the effect of essentially applying a sideways force on M equal to $mg\theta$. So the horizontal $F=Ma$ equation for M is $M\ddot{x}=mg\theta$, But $x\approx-\ell\theta$ in this limit, so we have $-M\ell\ddot{\theta}x=mg\theta$, from which the result follows. ♣

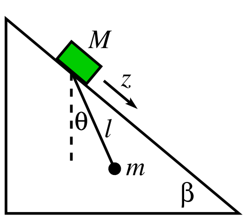


Figure 5.43

20. **Pendulum support on inclined plane**

Let z be the coordinate of M along the plane. Let θ be the angle of the pendulum (see Fig. 5.43). In cartesian coordinates, the positions of M and m are then

$$\begin{aligned}(x,y)_M &= (z\cos\beta,-z\sin\beta),\\ (x,y)_m &= (z\cos\beta+\ell\sin\theta,-z\sin\beta-\ell\cos\theta).\end{aligned} \tag{5.213}$$

The squares of the speeds are

$$\begin{aligned}v_M^2 &= \dot{z}^2,\\ v_m^2 &= \dot{z}^2+\ell^2\dot{\theta}^2+2\ell\dot{z}\dot{\theta}(\cos\beta\cos\theta-\sin\beta\sin\theta).\end{aligned} \tag{5.214}$$

The Lagrangian is therefore

$$\frac{1}{2}M\dot{z}^2+\frac{1}{2}m\Big(\dot{z}^2+\ell^2\dot{\theta}^2+2\ell\dot{z}\dot{\theta}\cos(\theta+\beta)\Big)+Mgz\sin\beta+mg(z\sin\beta+\ell\cos\theta). \quad (5.215)$$

The equations of motion from varying z and θ are

$$\begin{aligned}(M+m)\ddot{z}+m\ell\Big(\ddot{\theta}\cos(\theta+\beta)-\dot{\theta}^2\sin(\theta+\beta)\Big) &= (M+m)g\sin\beta,\\ \ell\ddot{\theta}+\ddot{z}\cos(\theta+\beta) &= -g\sin\theta. \quad (5.216)\end{aligned}$$

Let us now consider small oscillations about the equilibrium point (where $\ddot{\theta}=\dot{\theta}=0$). We first have to find where this point is. The first equation above gives $\ddot{z}=g\sin\beta$. The second equation then gives $g\sin\beta\cos(\theta+\beta)=-g\sin\theta$. By expanding this cosine, or just by inspection, we have $\theta=-\beta$ (or $\theta=\pi-\beta$, but this is an unstable equilibrium). So the equilibrium position of the pendulum is where the string is perpendicular to the plane. (This makes sense. Because the tension in the string is orthogonal to the plane, for all the pendulum bob knows, it may as well simply be sliding down a plane parallel to the given one, a distance ℓ away. Given the same initial speed, the two masses will slide down their two 'planes' at the same speed at all times.)

To find the normal modes and frequencies of small oscillations, let $\theta\equiv-\beta+\delta$, and expand eqs. (5.216) to first order in δ. Letting $\ddot{\eta}\equiv\ddot{z}-g\sin\beta$ for convenience, we have

$$\begin{aligned}(M+m)\ddot{\eta}+m\ell\ddot{\delta} &= 0,\\ \ddot{\eta}+\ell\ddot{\delta}+\delta g\cos\beta &= 0. \quad (5.217)\end{aligned}$$

Using the determinant method, the frequencies of the normal modes are found to be

$$\omega=0, \qquad \omega=\sqrt{1+\frac{m}{M}}\sqrt{\frac{g\cos\beta}{\ell}}. \quad (5.218)$$

This is the same answer as in the previous problem (with a horizontal plane), but with $g\cos\beta$ instead of g. This makes sense, because in a frame accelerating down the plane at $g\sin\beta$, the only external force on the masses is an effective gravity force $g\cos\beta$ perpendicular to the plane. For all M and m know, they live in a world where gravity has strength $g'=g\cos\beta$.

21. **Tilting plane**

Relative to the support, the position of M is $(\ell\sin\theta,-\ell\cos\theta)$. The position of m is $(\ell\sin\theta+x\cos\theta,-\ell\cos\theta+x\sin\theta)$. The squares of the velocities are therefore

$$v_M^2=\ell^2\dot{\theta}^2, \qquad v_m^2=(\ell\dot{\theta}+\dot{x})^2+x^2\dot{\theta}^2. \quad (5.219)$$

(v_m^2 can also be obtained without taking the derivative of the position; $(\ell\dot{\theta}+\dot{x})$ is the speed along the long rod, and $x\dot{\theta}$ is the speed perpendicular to it.) The Lagrangian is

$$L=\frac{1}{2}M\ell^2\dot{\theta}^2+\frac{1}{2}m\Big((\ell\dot{\theta}+\dot{x})^2+x^2\dot{\theta}^2\Big)+Mg\ell\cos\theta+mg(\ell\cos\theta-x\sin\theta). \quad (5.220)$$

The equations of motion from varying x and θ are

$$\begin{aligned}(\ell\ddot{\theta}+\ddot{x}) &= x\dot{\theta}^2-g\sin\theta,\\ M\ell^2\ddot{\theta}+m\ell(\ell\ddot{\theta}+\ddot{x})+mx^2\ddot{\theta}+2mx\dot{x}\dot{\theta} &= -Mg\ell\sin\theta-mg\ell\sin\theta\\ &\quad -mgx\cos\theta. \quad (5.221)\end{aligned}$$

Let us now consider the case where both x and θ are small (or more precisely, $\theta \ll 1$ and $x/\ell \ll 1$). Expanding eqs. (5.221) to first order in θ and x/ℓ gives

$$\begin{aligned} (\ell\ddot{\theta} + \ddot{x}) + g\theta &= 0, \\ M\ell(\ell\ddot{\theta} + g\theta) + m\ell(\ell\ddot{\theta} + \ddot{x}) + mg\ell\theta + mgx &= 0. \end{aligned} \tag{5.222}$$

We can simplify these a bit. Using the first equation to substitute $-g\theta$ for $(\ell\ddot{\theta} + \ddot{x})$ and also $-\ddot{x}$ for $(\ell\ddot{\theta} + g\theta)$ in the second equation gives

$$\begin{aligned} \ell\ddot{\theta} + \ddot{x} + g\theta &= 0, \\ -M\ell\ddot{x} + mgx &= 0. \end{aligned} \tag{5.223}$$

The normal modes can be found using the determinant method, or we can find them just by inspection. The second equation says that either $x(t) \equiv 0$ or $x(t) = A\cosh(\alpha t + \beta)$, where $\alpha = \sqrt{mg/M\ell}$. So we have two cases.

- If $x(t) = 0$, then the first equation in (5.223) says that the normal mode is

$$\begin{pmatrix} \theta \\ x \end{pmatrix} = C \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(\omega t + \phi), \tag{5.224}$$

  where $\omega \equiv \sqrt{g/\ell}$.

  This mode is fairly obvious. With proper initial conditions, m will stay right where M is. The normal force from the long rod will be exactly what is needed in order for m to undergo the same oscillatory motion as M.

- If $x(t) = A\cosh(\alpha t + \beta)$, then the first equation in (5.223) can be solved to give the normal mode

$$\begin{pmatrix} \theta \\ x \end{pmatrix} = C \begin{pmatrix} -m \\ \ell(M+m) \end{pmatrix} \cosh(\alpha t + \beta), \tag{5.225}$$

  where $\alpha = \sqrt{mg/M\ell}$.

  This mode is not so obvious. And indeed, its range of validity is rather limited. The exponential behavior will quickly make x and θ become large, and thus outside the validity of our small-variable approximations. Note that in this mode the center-of-mass remains fixed.

22. **Motion on a cone**

Let the particle be a distance r from the axis. Then its height is $r/\tan\alpha$, and the distance up along the cone is $r/\sin\alpha$. Let θ be the angle around the cone. Breaking the velocity into the components up along the cone and around the cone, we see that the square of the speed is $v^2 = \dot{r}^2/\sin^2\alpha + r^2\dot{\theta}^2$. The Lagrangian is therefore

$$L = \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\theta}^2\right) - \frac{mgr}{\tan\alpha}. \tag{5.226}$$

The equations of motion from varying θ and r are

$$\begin{aligned} \frac{d}{dt}(mr^2\dot{\theta}) &= 0 \\ \ddot{r} &= r\dot{\theta}^2\sin^2\alpha - g\cos\alpha\sin\alpha. \end{aligned} \tag{5.227}$$

(The first of these is conservation of angular momentum. The second one is more transparent if we divide through by $\sin\alpha$. With $x \equiv r/\sin\alpha$ being the distance up along the cone, it becomes $\ddot{x} = (r\dot{\theta}^2)\sin\alpha - g\cos\alpha$. This is just $F = ma$ in the x direction.)

Letting $mr^2\dot{\theta} \equiv L$, we may eliminate $\dot{\theta}$ from the second equation to obtain

$$\ddot{r} = \frac{L^2 \sin^2\alpha}{m^2 r^3} - g\cos\alpha\sin\alpha. \tag{5.228}$$

Let us now calculate the two desired frequencies.

- Frequency of circular oscillations, ω:

  For circular motion with $r = r_0$, we have $\dot{r} = \ddot{r} = 0$, so eq. (5.227) gives

$$\omega \equiv \dot{\theta} = \sqrt{\frac{g}{r_0 \tan\alpha}}. \tag{5.229}$$

- Frequency of oscillations about a circle, Ω:

  If the orbit were actually the circle $r = r_0$, then eq. (5.228) would give (with $\ddot{r} = 0$)

$$\frac{L^2\sin^2\alpha}{m^2 r_0^3} = g\cos\alpha\sin\alpha. \tag{5.230}$$

  (Of course, writing L as $mr_0^2\dot{\theta}$, this is equivalent to eq. (5.229).)

  We will now use our standard procedure of letting $r(t) = r_0 + \delta(t)$, where $\delta(t)$ is very small, and then plugging this into eq. (5.228) and expanding to first order in δ. Using

$$\frac{1}{(r_0+\delta)^3} \approx \frac{1}{r_0^3 + 3r_0^2\delta} = \frac{1}{r_0^3(1+3\delta/r_0)} \approx \frac{1}{r_0^3}\left(1 - \frac{3\delta}{r_0}\right), \tag{5.231}$$

  we have

$$\ddot{\delta} = \frac{L^2\sin^2\alpha}{m^2 r_0^3}\left(1 - \frac{3\delta}{r_0}\right) - g\cos\alpha\sin\alpha. \tag{5.232}$$

  Recalling eq. (5.230), we obtain a bit of cancellation and are left with

$$\ddot{\delta} = -\frac{3\delta L^2\sin^2\alpha}{m^2 r_0^4} \tag{5.233}$$

  Using eq. (5.230) again to eliminate L (technically, eq. (5.230) only holds for circular motion, but any error is of higher order in the following equation), we have

$$\ddot{\delta} + \delta\frac{3g}{r_0}\sin\alpha\cos\alpha = 0. \tag{5.234}$$

  So we find

$$\Omega = \sqrt{\frac{3g}{r_0}\sin\alpha\cos\alpha} = (\sqrt{3}\sin\alpha)\omega. \tag{5.235}$$

  Thus, the ratio Ω/ω is independent of r_0.

The two frequencies are equal if $\sin\alpha = 1/\sqrt{3}$, i.e., $\alpha \approx 35.3^\circ \equiv \tilde{\alpha}$. If $\alpha = \tilde{\alpha}$, then after one revolution r returns to the value it had at the beginning of the revolution. So the particle undergoes periodic motion.

If $\alpha > \tilde{\alpha}$, then $\Omega > \omega$ (i.e., the r value goes through a whole oscillation before one complete circle is traversed). If $\alpha < \tilde{\alpha}$, then $\Omega < \omega$ (i.e., more than one complete circle is needed for the r value to go through one cycle.)

REMARKS: In the limit where $\alpha \to 0$, eq. (5.235) says that $\Omega/\omega \to 0$. (In fact, eqs. (5.229) and (5.235) say that $\omega \to \infty$ and $\Omega \to 0$). So the particle spirals around many times during one complete r cycle. (This seems intuitive.)

In the limit where $\alpha \to \pi/2$ (i.e., the cone is almost a flat plane) eq. (5.235) says that $\Omega/\omega \to \sqrt{3}$. (Both Ω and ω go to 0). This result is not at all obvious (at least to me).

If $\Omega/\omega = \sqrt{3}\sin\alpha$ is a rational number, then the particle will undergo periodic motion. For example, if $\alpha = 60°$, then $\Omega/\omega = 3/2$; it takes two complete circles for r to go through three cycles. Or, if $\alpha = \arcsin(1/2\sqrt{3}) \approx 16.8°$, then $\Omega/\omega = 1/2$; it takes two complete circles for r to go through one cycle; etc.

We calculated Ω above by letting $r = r_0 + \delta$ and then expanding, but one could also use the effective potential method. Eq. (5.228) yields an effective potential

$$V_{\text{eff}} = \frac{L^2\sin^2\alpha}{2mr^2} + mgr\cos\alpha\sin\alpha. \tag{5.236}$$

The frequency of small oscillations about the minimum at r_0 is then

$$\Omega = \sqrt{\frac{V''_{\text{eff}}(r_0)}{m}}. \tag{5.237}$$

The quantity L may be eliminated in favor of r_0 by using the condition $V'_{\text{eff}}(r_0) = 0$, and we'll leave it to the reader to show that this reproduces eq. (5.235). ♣

23. **Shortest distance in a plane**

Let the two given points be (x_1, y_1) and (x_2, y_2) Let the path be described by the function $y(x)$. (Yes, we'll assume it can be written as a function. Locally, we don't have to worry about any double-valued issues.) Then the length of the path is

$$\ell = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx. \tag{5.238}$$

The 'Lagrangian' is $L = \sqrt{1 + y'^2}$, and the Euler-Lagrange equation is

$$\begin{aligned} \frac{d}{dx}\frac{\partial L}{\partial y'} &= \frac{\partial L}{\partial y} \\ \Longrightarrow \quad \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) &= 0. \end{aligned} \tag{5.239}$$

So $y'/\sqrt{1 + y'^2}$ is constant. Therefore, y' is also constant, and we have the straight line $y(x) = Ax + B$, where A and B are determined from the endpoint conditions.

24. **Minimal surface**

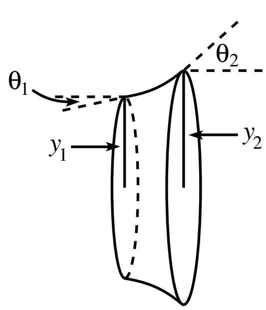


Figure 5.44

The tension throughout the surface is constant, since it is in equilibrium. The ratio of the circumferences of the circular boundaries of the ring is y_2/y_1. The condition that the horizontal forces on the ring cancel is therefore $y_1\cos\theta_1 = y_2\cos\theta_2$, where the θ's are the angles of the surface, as shown in Fig. 5.44. In other words, $y\cos\theta$ is constant throughout the surface. But $\cos\theta = 1/\sqrt{1 + y'^2}$, so we have

$$\frac{y}{\sqrt{1 + y'^2}} = C. \tag{5.240}$$

This is the same as eq. (5.75), and the solution proceeds as in Section 5.8.

25. **The Brachistochrone**

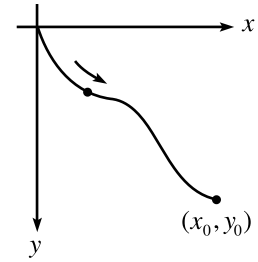


Figure 5.45

(a) In Fig. 5.45, the boundary conditions are $y(0) = 0$ and $y(x_0) = y_0$ (with downward taken to be the positive y direction). From energy conservation, the speed at position y is $v = \sqrt{2gy}$. The total time is therefore

$$T = \int_0^{x_0} \frac{ds}{v} = \int_0^{x_0} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\, dx. \tag{5.241}$$

Our 'Lagrangian' is thus

$$L \propto \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,. \tag{5.242}$$

We can apply the E-L equation now, or we can simply use Lemma 5.5, with $f(y) = 1/\sqrt{y}$. Eg. (5.87) gives

$$fy'' = f'(1+y'^2) \qquad \Longrightarrow \qquad \frac{y''}{\sqrt{y}} = -\frac{1+y'^2}{2y\sqrt{y}} \qquad \Longrightarrow \qquad -2yy'' = 1+y'^2, \tag{5.243}$$

as desired. And eq. (5.89) gives

$$1+y'^2 = Cf(y)^2 \qquad \Longrightarrow \qquad 1+y'^2 = \frac{C}{y}, \tag{5.244}$$

as desired.

(b) At this point we can either simply verify that the cycloid solution satisfies eq. (5.244), or we can separate variables and then integrate eq. (5.244). The latter method has the advantage, of course, of not requiring the solution to be given. Let's do it both ways.

Verification: Assume $x = a(\theta - \sin\theta)$, and $y = a(1-\cos\theta)$. Then

$$y' \equiv \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{1-\cos\theta}. \tag{5.245}$$

Therefore,

$$1+y'^2 = 1 + \frac{\sin^2\theta}{(1-\cos\theta)^2} = \frac{2}{1-\cos\theta} = \frac{2a}{y}, \tag{5.246}$$

which agrees with eq. (5.244), with $C \equiv 2a$.

Separation of variables: Solving for y' in eq. (5.244) and separating variables gives

$$\frac{\sqrt{y}\,dy}{\sqrt{C-y}} = \pm dx. \tag{5.247}$$

A helpful change of variables to get rid of the square root in the denominator is $y \equiv C\sin^2\phi$. Then $dy = 2C\sin\phi\cos\phi\, d\phi$, and eq. (5.247) simplifies to

$$2C\sin^2\phi d\phi = \pm dx. \tag{5.248}$$

Integrating this (using $\sin^2\phi = (1-\cos 2\phi)/2$) gives $C(2\phi - \sin 2\phi) = \pm 2x$ (plus an irrelevant constant).

Note that we may rewrite our definition of ϕ (which was $y \equiv C\sin^2\phi$) as $2y = C(1-\cos 2\phi)$. If we then define $\theta \equiv 2\phi$, we have

$$x = \pm a(\theta - \sin\theta), \qquad y = a(1-\cos\theta). \tag{5.249}$$

where $a \equiv C/2$.

26. **Index of refraction**

Let the path be described by $y(x)$. The speed at height y is $v \propto y$. The time to go from (x_0, y_0) to (x_1, y_1) is therefore

$$T = \int_{x_0}^{x_1} \frac{ds}{v} \propto \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{y}\,dx. \qquad (5.250)$$

Our 'Lagrangian' is thus

$$L \propto \frac{\sqrt{1+y'^2}}{y}\,. \qquad (5.251)$$

We can apply the E-L equation now, or we can simply use Lemma 5.5, with $f(y) = 1/y$. Eg. (5.87) gives

$$fy'' = f'(1+y'^2) \quad \Longrightarrow \quad \frac{y''}{y} = -\frac{1+y'^2}{y^2} \quad \Longrightarrow \quad -yy'' = 1+y'^2. \qquad (5.252)$$

And eq. (5.89) gives

$$1+y'^2 = Bf(y)^2 \quad \Longrightarrow \quad 1+y'^2 = \frac{B}{y^2}\,. \qquad (5.253)$$

We must now integrate this. Solving for y', and then separating variables and integrating, gives

$$\int dx = \pm \int \frac{y\,dy}{\sqrt{B-y^2}} \quad \Longrightarrow \quad x + A = \mp\sqrt{B-y^2}\,. \qquad (5.254)$$

Hence, $(x+A)^2 + y^2 = B$, which is the equation for a circle. Note that the circle is centered at a point on the bottom of the slab. (This must be the point where the perpendicular bisector of the line joining the two given points intersects the bottom of the slab.)

27. **Existence of minimal surface**

From Section 5.8, the general solution takes the form

$$y(x) = \frac{1}{b}\cosh b(x+d). \qquad (5.255)$$

If we choose the origin to be midway between the rings, we have $d = 0$. Both boundary condition are then

$$c = \frac{1}{b}\cosh ba. \qquad (5.256)$$

If a/c is larger than a certain critical value, then there is no solution for b, as we now show.

Let us define the dimensionless quantities,

$$\eta \equiv \frac{a}{c}, \qquad \text{and} \qquad z \equiv bc. \qquad (5.257)$$

Then eq. (5.256) becomes

$$z = \cosh \eta z. \qquad (5.258)$$

If we (roughly) plot the graphs of $w = z$ and $w = \cosh \eta z$ for a few values of η (see Fig. 5.46), we see that there is no solution for z if η is too big. The limiting value

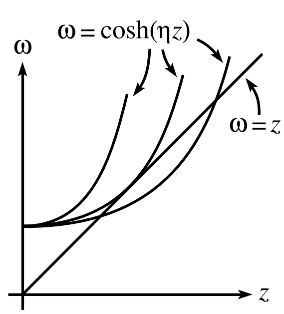


Figure 5.46

for η for which a solution exists occurs when the curves $w = z$ and $w = \cosh \eta z$ are tangent, that is, when the functions are equal and the slopes are equal. Let η_0 be this critical value for η. Let z_0 be the place where the tangency occurs. Then equality of the values and slopes gives

$$z_0 = \cosh(\eta_0 z_0), \qquad \text{and} \qquad 1 = \eta_0 \sinh(\eta_0 z_0). \tag{5.259}$$

Dividing the second of these equations by the first gives

$$1 = (\eta_0 z_0) \tanh(\eta_0 z_0). \tag{5.260}$$

This must be solved numerically. The solution is

$$\eta_0 z_0 \approx 1.200. \tag{5.261}$$

Plugging this into the second of eqs. (5.259) gives

$$\left(\frac{a}{c}\right)_{\max} \equiv \eta_0 \approx 0.663. \tag{5.262}$$

If a/c is larger than 0.663, then the Euler-Lagrange equation has no solution that is consistent with the boundary conditions. (In other words, there is no surface which is stationary with respect to small perturbations.) Above this value of a/c, a soap bubble minimizes its area by heading toward the shape of just two discs, but it will pop long before it reaches this configuration.

Remark: How does the area of the limiting minimal surface compare with the area of the two circles?

The area of the two circles is

$$A_{\rm c} = 2\pi c^2. \tag{5.263}$$

The area of the critical surface is

$$A_{\rm s} = \int_{-a}^{a} 2\pi y \sqrt{1 + y'^2}\, dx. \tag{5.264}$$

Using eq. (5.255), with $d = 0$, we find

$$\begin{aligned} A_{\rm s} &= \int_{-a}^{a} \frac{2\pi}{b} \cosh^2 bx\, dx \\ &= \int_{-a}^{a} \frac{\pi}{b}(1 + \cosh 2bx)\, dx \\ &= \frac{2a\pi}{b} + \frac{\pi \sinh 2ba}{b^2}. \end{aligned} \tag{5.265}$$

But from the definitions of η and z, we have $a = \eta_0 c$ and $b = z_0/c$ for the critical surface. Therefore,

$$A_{\rm s} = \pi c^2 \left(\frac{2\eta_0}{z_0} + \frac{\sinh 2\eta_0 z_0}{z_0^2} \right). \tag{5.266}$$

Plugging in the numerical values ($\eta_0 \approx 0.663$ and $z_0 \approx 1.810$) gives

$$A_{\rm c} \approx (6.28)\pi c^2, \qquad \text{and} \qquad A_{\rm s} \approx (7.54)\pi c^2. \tag{5.267}$$

(The ratio of these areas is approximately 1.2, which is actually $\eta_0 z_0$. We'll let you prove this.) The critical surface therefore has a larger area. This is expected, of course, because for $a/c > \eta_0$ the surface tries to run off to one with a smaller area, and there are no other stable configurations besides the cosh solution we found. ♣

Chapter 6

Central Forces

A *central force* is by definition a force that points radially and whose magnitude depends only on the distance from the source. Equivalently, we may say that a central force is one whose potential depends only on the distance from the source. That is, if the source is located at the origin, then the potential energy is of the form $V(\mathbf{r}) = V(r)$. Such a potential does indeed yield a radial force whose magnitude depends only on r, because

$$\mathbf{F}(\mathbf{r}) = -\nabla V(r) = -\frac{dV}{dr}\hat{\mathbf{r}}. \tag{6.1}$$

The relative angle to the source does not affect the force. Gravitational and electrostatic forces are central forces, with $V(r) \propto 1/r$. The spring force is also central, with $V(r) \propto (r-\ell)^2$, where ℓ is the equilibrium length.

There are two important facts concerning central forces: (1) they are ubiquitous in nature, so we had better learn how to deal with them, and (2) dealing with them is much easier than you might think, because crucial simplifications occur in the equations of motion when V is a function of r only.

6.1 Conservation of angular momentum

For a point mass, we define the angular momentum, $\mathbf{L}$, by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{6.2}$$

The vector $\mathbf{L}$ depends, of course, on where you pick the origin of your coordinate system. There are some very nice facts concerning $\mathbf{L}$, one of which is the following.[1]

Theorem 6.1 *If a particle is subject to a central force only, then its angular momentum is conserved. That is,*

$$\text{If} \quad V(\mathbf{r}) = V(r), \qquad \text{then} \quad \frac{d\mathbf{L}}{dt} = 0. \tag{6.3}$$

[1]This is a special case of the fact that torque equals the rate of change of angular momentum. We'll talk about this in Chapter 7.

Proof: We have

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}(\mathbf{r}\times\mathbf{p})\\ &= \frac{d\mathbf{r}}{dt}\times\mathbf{p}+\mathbf{r}\times\frac{d\mathbf{p}}{dt}\\ &= \mathbf{v}\times(m\mathbf{v})+\mathbf{r}\times\mathbf{F}\\ &= 0,\end{aligned} \tag{6.4}$$

because $\mathbf{F}\propto\mathbf{r}$, and the cross product of two parallel vectors is zero. ∎

We will prove this theorem again in the next section, using the Lagrangian method. Let's now prove another theorem which is probably obvious, but good to show anyway.

Theorem 6.2 *If a particle is subject to a central force only, then its motion takes place in a plane.*

Proof: At a given instant, t_0, consider the plane, P, containing the position vector $\mathbf{r}_0$ (with the source of the potential taken to be the origin) and the velocity vector $\mathbf{v}_0$. We claim that $\mathbf{r}$ lies in P at all times.[2]

P is defined as the plane orthogonal to the vector $\mathbf{n}_0\equiv\mathbf{r}_0\times\mathbf{v}_0$. But in the proof of Theorem 6.1, we showed that the vector $\mathbf{r}\times\mathbf{v}\equiv(\mathbf{r}\times\mathbf{p})/m$ does not change with time. Therefore, $\mathbf{r}\times\mathbf{v}=\mathbf{n}_0$ for all t. Since $\mathbf{r}$ is certainly orthogonal to $\mathbf{r}\times\mathbf{v}$, we see that $\mathbf{r}$ is orthogonal to $\mathbf{n}_0$ for all t. Hence, $\mathbf{r}$ must lie in P. ∎

An intuitive look at this theorem is the following. Since the position, speed, and acceleration vectors all lie in P, there is a symmetry between the two sides of P. Therefore, there is no reason why the particle would head out of P to one side rather than the other.

This theorem shows that we need only two coordinates, instead of the usual three, to describe the motion. But since we're on a roll, why stop here? We will show below that we really only need *one* variable. Not bad, three coordinates reduced to one.

6.2 The effective potential

Consider a particle of mass m subject to a central force only, described by the potential $V(r)$. Without loss of generality, let us assume that the plane in which the motion lies is the x-y plane (or the r-θ plane, in polar coordinates). In polar coordinates, the Lagrangian is (we'll use "$\mathcal{L}$" for the Lagrangian, and save "L" for the angular momentum)

$$\mathcal{L}=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)-V(r). \tag{6.5}$$

[2] P is not well-defined if $\mathbf{v}_0=\mathbf{0}$, or $\mathbf{r}_0=\mathbf{0}$, or $\mathbf{v}_0$ is parallel to $\mathbf{r}_0$. But in these cases, it is easy to show that the motion is always radial.

The equations of motion obtained from varying r and θ are

$$\begin{aligned} m\ddot{r} &= mr\dot{\theta}^2 - V'(r), \\ \frac{d}{dt}(mr^2\dot{\theta}) &= 0. \end{aligned} \tag{6.6}$$

The first equation is the force equation along the $\hat{\mathbf{r}}$ direction (complete with the centripetal acceleration). The second is conservation of angular momentum, because $mr^2\dot{\theta} = r(mr\dot{\theta}) = rp_\theta$ (where p_θ is the magnitude of the momentum in the angular direction), and because rp_θ is the magnitude of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ (which always points in the $\hat{\mathbf{z}}$ direction here). We have therefore just given a second proof of Theorem 6.1. The conservation of $\mathbf{L}$ here follows from the fact that θ is a cyclic coordinate, as we saw in Example 2 in Section 5.5.1.

Since $mr^2\dot{\theta}$ does not change in time, let us denote its constant value by

$$L \equiv mr^2\dot{\theta}. \tag{6.7}$$

L is determined by the initial conditions. (It could be specified, for example, by giving the initial values of r and $\dot{\theta}$.) Using $\dot{\theta} = L/(mr^2)$, we may rewrite the first of eqs. (6.6) as

$$m\ddot{r} = \frac{L^2}{mr^3} - V'(r). \tag{6.8}$$

Multiplying by $\dot{r}$ and integrating with respect to time yields

$$\frac{1}{2}m\dot{r}^2 + \left(\frac{L^2}{2mr^2} + V(r)\right) = E, \tag{6.9}$$

where E is a constant of integration. E is, of course, just the energy. Note that this equation could have also been obtained by simply using eq. (6.7) to eliminate $\dot{\theta}$ in the energy equation, $m(\dot{r}^2 + r^2\dot{\theta}^2)/2 + V(r) = E$.

Eq. (6.9) is rather interesting. It involves only the variable r. And it looks a lot like the equation for a particle moving in one dimension (labeled by the coordinate r) under the influence of the potential

$$\boxed{V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + V(r)}. \tag{6.10}$$

The subscript "eff" stands for "effective". $V_{\text{eff}}(r)$ is called the *effective potential*. The "effective force" is easily read off from eq. (6.8) to be

$$F_{\text{eff}}(r) = \frac{L^2}{mr^3} - V'(r) = -V'_{\text{eff}}(r). \tag{6.11}$$

This "effective" potential and "effective" force concept is a marvelous result and should be duly appreciated. It says that if we want to solve a two-dimensional problem (which could have come from a three-dimensional problem) involving a central force, we can recast the problem into a simple one-dimensional problem with a slightly modified potential. We can forget we ever had the variable θ, and we can solve this one-dimensional problem (as we'll demonstrate below) to obtain $r(t)$. Having found $r(t)$, we can use $\dot\theta(t) = L/mr^2$ to solve for $\theta(t)$ (in theory, at least).

Note that this whole procedure works only because there is a quantity involving r and θ that is independent of time. The variables r and θ are therefore *not* independent, so the problem is really one-dimensional instead of two-dimensional.

To get a general idea of how r behaves with time, we simply have to graph $V_{\text{eff}}(r)$. Consider the example where $V(r) = Ar^2$. This is the potential for a spring with equilibrium length zero. Then

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + Ar^2. \tag{6.12}$$

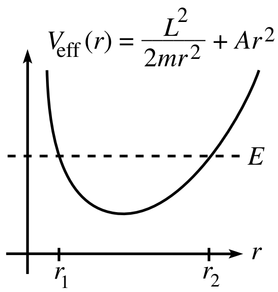


Figure 6.1

To graph $V_{\text{eff}}(r)$, we need to be given L. But the general shape looks like the curve in Fig. 6.1. The energy E (which must be given, too) is also drawn. The coordinate r will bounce back and forth between the turning points, r_1 and r_2, which satisfy $V_{\text{eff}}(r_{1,2}) = E$. (It turns out that for our r^2 potential, the motion in space is an ellipse, with semi-axis lengths r_1 and r_2. But for a general potential, the motion isn't so nice.) If E equals the minimum of $V_{\text{eff}}(r)$, then $r_1 = r_2$ and r is stuck at this one value, so the motion is a circle. Note that it is impossible for E to be less than the minimum of V_{eff}.

REMARKS: The $L^2/2mr^2$ term is sometimes called the "angular momentum barrier". It has the effect of keeping the particle from getting too close to the origin. Basically, the point is that $L \equiv mr^2\dot\theta$ is constant, so as r gets smaller, $\dot\theta$ gets bigger (in a greater manner than r is getting smaller, due to the square of the "r" in L), and eventually we end up with a tangential kinetic energy ($mr^2\dot\theta^2/2$) that is greater than what is allowed by conservation of energy.[3]

Note that it is by no means necessary to introduce the concept of the effective potential. You can simply solve the equations of motion, eqs. (6.6), as they are. But introducing V_{eff} makes it much easier to see what's going on in a central-force problem. ♣

When using potentials, effective,
Remember the one main objective:
The goal is to shun
All dimensions but one,
And then view things with 1-D perspective.

6.3 Solving the equations of motion

If we want to get quantitative, we must solve the equations of motion, eqs. (6.6). Equivalently, we must solve their integrated forms, eqs. (6.7) and (6.9), which are simply the conservation of L and E statements.

$$\begin{aligned} mr^2\dot\theta &= L, \\ \frac{1}{2}m\dot r^2 + \frac{L^2}{2mr^2} + V(r) &= E. \end{aligned} \tag{6.13}$$

[3] If $V(r)$ goes to $-\infty$ faster than $-1/r^2$, then this argument doesn't hold. This result is clear if you draw the relevant graph of $V_{\text{eff}}(r)$.

The word "solve" is a little ambiguous, since we should specify what quantities we want to solve for in terms of what other quantities. There are essentially two things we can do. We can solve for r and θ in terms of t. Or, we can solve for r in terms of θ. The former has the advantage of immediately yielding velocities (and, of course, the information of where the particle is at time t). The latter has the advantage of explicitly showing what the trajectory looks like in space (even though we don't know how quickly it is being traversed). We will deal mainly with this second case, particularly when we discuss the gravitational force and Kepler's Laws below. But let's look at both procedures now.

6.3.1 Finding $r(t)$ and $\theta(t)$

The value of $\dot{r}$ at any point is found from eq. (6.13) to be

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - \frac{L^2}{2mr^2} - V(r)}\,. \tag{6.14}$$

To get an actual $r(t)$ out of this, we must be supplied with E and L (which may be found using the initial conditions), and also the function $V(r)$. To solve this differential equation, we "simply" have to separate variables and then (in theory) integrate:

$$\int \frac{dr}{\sqrt{E - \frac{L^2}{2mr^2} - V(r)}} = \pm\int\sqrt{\frac{2}{m}}\,dt = \pm\sqrt{\frac{2}{m}}\,(t - t_0). \tag{6.15}$$

Our problem reduces to performing this (rather unpleasant) integral on the left-hand side, to obtain t as a function of r. We may then (in theory) invert the result to obtain r as a function of t. Finally, the relation $\dot{\theta} = L/mr^2$ in eq. (6.13) is a differential equation for θ, which we can (in theory) solve.

The bad news about this procedure is that for most $V(r)$'s the integral in eq. (6.15) is not calculable in closed form. There are only a few "nice" potentials $V(r)$ for which we can evaluate it (and even then, the procedure is a pain).[4] But the good news is that these "nice" potentials are precisely the ones we will be most interested in. In particular, the gravitational potential, which goes like $1/r$ and which we will spend most of our time with, leads to a calculable integral. But never mind; we're not going to apply this procedure to gravity.[5] We'll use the following strategy.

6.3.2 Finding $r(\theta)$

We may eliminate the dt from eqs. (6.13) by getting the $\dot{r}^2$ term alone on the left side of the second equation, and then dividing by the square of the first equation. The dt^2 factors cancel, and we obtain

$$\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2mE}{L^2} - \frac{1}{r^2} - \frac{2mV(r)}{L^2}\,. \tag{6.16}$$

[4] You can, of course, always evaluate the integral numerically.

[5] It's nice to know that this procedure exists, but it's rarely used to solve a problem.

At this point, we can (in theory) take a square root, separate variables, and then integrate to obtain θ as a function of r. We can then (in theory) invert to obtain r as a function of θ. To do this, of course, we need to be given a function $V(r)$. So let's now finally give ourselves a $V(r)$ and do a problem all the way through. We'll study the most important potential of all (or perhaps the second most important one), gravity.[6]

6.4 Gravity, Kepler's Laws

6.4.1 Calculation of $r(\theta)$

Our goal in this subsection will be to obtain r as a function of θ, for a gravitational potential. The gravitational potential energy between two objects, of masses M and m, is

$$V(r) = -\frac{\alpha}{r}, \qquad \text{where } \alpha = GMm. \tag{6.17}$$

In the present treatment, let us consider the mass M to be bolted down at the origin of our coordinate system. This is approximately true in the case where $M \gg m$, as in the earth-sun system. (If we want to do the problem exactly, we must use the *reduced mass*. This topic is discussed in Section 6.4.5.) Eq. (6.16) becomes

$$\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2mE}{L^2} - \frac{1}{r^2} + \frac{2m\alpha}{rL^2}. \tag{6.18}$$

As stated above, we could take a square root, separate variables, integrate to find $\theta(r)$, and then invert to find $r(\theta)$. This method, although straightforward, is terribly messy. Let's solve for $r(\theta)$ in a slick way.

With all the $1/r$ terms floating around, it might be better to solve for $1/r$ instead of r. Using $d(1/r)/d\theta = -(dr/d\theta)/r^2$, and letting $y \equiv 1/r$ for convenience, eq. (6.18) becomes

$$\left(\frac{dy}{d\theta}\right)^2 = -y^2 + \frac{2m\alpha}{L^2}y + \frac{2mE}{L^2}. \tag{6.19}$$

At this point, we could also use the separation of variables technique, but let's continue to be slick. Complete the square on the right-hand side to obtain

$$\left(\frac{dy}{d\theta}\right)^2 = -\left(y - \frac{m\alpha}{L^2}\right)^2 + \frac{2mE}{L^2} + \left(\frac{m\alpha}{L^2}\right)^2. \tag{6.20}$$

With $z \equiv y - m\alpha/L^2$ for convenience, we have

$$\begin{aligned}\left(\frac{dz}{d\theta}\right)^2 &= -z^2 + \left(\frac{m\alpha}{L^2}\right)^2\left(1 + \frac{2EL^2}{m\alpha^2}\right)\\ &\equiv -z^2 + B^2.\end{aligned} \tag{6.21}$$

[6]The two most important potentials in physics are certainly the gravitational and harmonic-oscillator ones. Interestingly, they both lead to doable integrals. And they both lead to elliptical orbits.

At this point, in the spirit of being slick, we could just look at this equation and see that $z = B\cos(\theta - \theta_0)$ is the solution. But, lest we feel guilty for not doing separation-of-variables at least once in this problem, let's do it that way, too. The integral here is nice and easy. We have

$$\begin{aligned} \int_{z_1}^{z} \frac{dz'}{\sqrt{B^2 - z'^2}} &= \int_{\theta_1}^{\theta} d\theta' \\ \Longrightarrow \qquad \cos^{-1}\left(\frac{z'}{B}\right)\Big|_{z_1}^{z} &= (\theta - \theta_1) \\ \Longrightarrow \qquad z &= B\cos\left((\theta - \theta_1) + \cos^{-1}\left(\frac{z_1}{B}\right)\right) \\ &\equiv B\cos(\theta - \theta_0). \end{aligned} \tag{6.22}$$

It is customary to pick axes so that $\theta_0 = 0$, so we'll drop the θ_0 from here on. Recalling our definition $z \equiv (1/r) - m\alpha/L^2$ and also the definition of B from eq. (6.21), we finally have

$$\frac{1}{r} = \frac{m\alpha}{L^2}(1 + \epsilon\cos\theta), \tag{6.23}$$

where

$$\epsilon \equiv \sqrt{1 + \frac{2EL^2}{m\alpha^2}} \tag{6.24}$$

is the *eccentricity* of the particle's motion. We will see shortly exactly what ϵ signifies.

This completes the derivation of $r(\theta)$ for the gravitational potential, $V(r) \propto 1/r$. It was a little messy, but not unbearably painful. And anyway, we just discovered the basic motion of virtually all of the gazillion tons of stuff in the universe. Not bad for one page of work.

Newton said as he gazed off afar,
"From here to the most distant star,
These wond'rous ellipses
And solar eclipses
All come from a 1 over r."

What are the limits on r in eq. (6.23)? The minimum value of r is obtained when the right-hand side of eq. (6.23) reaches its maximum value. This maximum value is $m\alpha(1+\epsilon)/L^2$, so we see that

$$r_{\text{min}} = \frac{L^2}{m\alpha(1+\epsilon)}. \tag{6.25}$$

What is the maximum value of r? The answer to this depends on whether ϵ is greater than or less than 1. If $\epsilon < 1$ (which corresponds to circular or elliptical orbits, as we will see below), then the minimum value of the right-hand side of eq. (6.23) is $m\alpha(1-\epsilon)/L^2$, so the maximum value of r is

$$r_{\text{max}} = \frac{L^2}{m\alpha(1-\epsilon)} \qquad (\text{if } \epsilon < 1). \tag{6.26}$$

If $\epsilon \geq 1$ (which corresponds to parabolic or hyperbolic orbits, as we will see below), then the right-hand side of eq. (6.23) can become zero (when $\cos\theta = -1/\epsilon$), so the maximum value of r is

$$r_{\text{max}} = \infty \qquad (\text{if } \epsilon \geq 1). \tag{6.27}$$

6.4.2 The orbits

Let's examine in detail the various cases for ϵ.

- **Circle** ($\epsilon = 0$)

  If $\epsilon = 0$, then eq. (6.24) says that $E = -m\alpha^2/2L^2$. The negative E simply means that the potential energy is more negative than the kinetic energy is positive. (Equivalently, the particle is trapped in the potential well.) Eqs. (6.25) and (6.26) say that $r_{\text{min}} = r_{\text{max}} = L^2/m\alpha$. Therefore, the particle moves in a circular orbit with radius $L^2/m\alpha$. (Equivalently, eq. (6.23) says that r is independent of θ.)

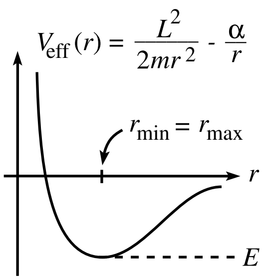
  

  Figure 6.2

  Note that it wasn't necessary to do all the work of section 6.4.1, if we just wanted to look at circular motion. For a given L, the energy $-m\alpha^2/2L^2$ is the minimum value that the E given by eq. (6.13) can take. (To achieve the minimum, we clearly want $\dot{r} = 0$. And you can show that minimizing the effective potential, $L^2/2mr^2 - \alpha/r$, yields this value for E.) If we plot $V_{\text{eff}}(r)$, we have the situation shown in Fig. 6.2. The particle is trapped at the bottom of the potential well, so it has no motion in the r direction.

- **Ellipse** ($0 < \epsilon < 1$)

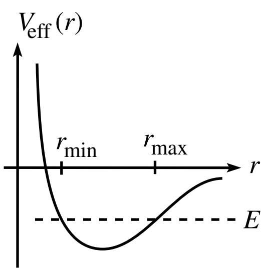
  

  Figure 6.3

  If $0 < \epsilon < 1$, then eq. (6.24) says that $-m\alpha^2/2L^2 < E < 0$. Eqs. (6.25) and (6.26) give r_{min} and r_{max}. It is not obvious that the resulting motion is an ellipse. We will demonstrate this below.

  If we plot $V_{\text{eff}}(r)$, we have the situation shown in Fig. 6.3. The particle oscillates between r_{min} and r_{max}. The energy is negative, so the particle is trapped in the potential well.

- **Parabola** ($\epsilon = 1$)

  If $\epsilon = 1$, then eq. (6.24) says that $E = 0$. This value of E means that the particle barely makes it out to infinity (its speed approaches zero as $r \to \infty$). Eq. (6.25) says $r_{\text{min}} = L^2/2m\alpha$, and eq. (6.27) says $r_{\text{max}} = \infty$. Again, it is not obvious that the resulting motion is a parabola. We will demonstrate this below.

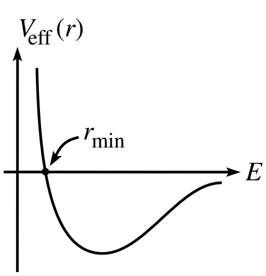
  

  Figure 6.4

  If we plot $V_{\text{eff}}(r)$, we have the situation shown in Fig. 6.4. The particle does not oscillate in the r-direction. It moves inward, turns around at $r_{\text{min}} = L^2/2m\alpha$, and heads out to infinity forever.

- **Hyperbola** ($\epsilon > 1$)

  If $\epsilon > 1$, then eq. (6.24) says that $E > 0$. This value of E means that the particle makes it out to infinity with energy to spare. (The potential goes to zero as $r \to \infty$, so the particle's speed approaches the nonzero value $\sqrt{2E/m}$ as $r \to \infty$.) Eq. (6.25) gives r_{min}, and eq. (6.27) says $r_{\text{max}} = \infty$. Again, it is not obvious that the resulting motion is a hyperbola. We will demonstrate this below.

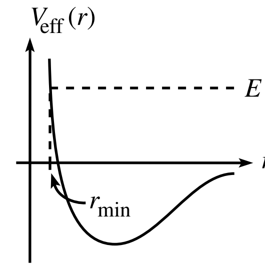


Figure 6.5

If we plot $V_{\text{eff}}(r)$, we have the situation shown in Fig. 6.5. The particle does not oscillate in the r-direction. It moves inward, turns around at r_{min}, and heads out to infinity forever.

6.4.3 Proof of conic orbits

Let's now prove that eq. (6.23) does indeed describe the conic sections stated above. We will also show that the origin (the source of the potential) is a focus of the conic section. These proofs are straightforward, although the ellipse and hyperbola cases get a bit messy.

For convenience, let

$$k \equiv \frac{L^2}{m\alpha}. \tag{6.28}$$

Multiplying eq. (6.23) through by kr, and using $\cos\theta = x/r$, gives $k = r + \epsilon x$. Solving for r and squaring yields

$$x^2 + y^2 = k^2 - 2k\epsilon x + \epsilon^2 x^2. \tag{6.29}$$

Let's look at the various cases for ϵ. We will invoke without proof various facts concerning conic sections.

- **Circle** ($\epsilon = 0$)

  In this case, eq. (6.29) becomes $x^2 + y^2 = k^2$. So we have a circle of radius $k = L^2/m\alpha$, with its center at the origin (see Fig. 6.6).

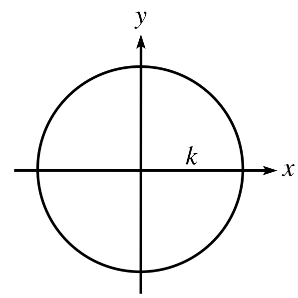


Figure 6.6

- **Ellipse** ($0 < \epsilon < 1$)

  In this case, eq. (6.29) may be written (after completing the square for the x terms, and expending a little effort)

$$\frac{\left(x + \frac{k\epsilon}{1-\epsilon^2}\right)^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{where} \;\; a = \frac{k}{1-\epsilon^2}\,, \quad \text{and} \;\; b = \frac{k}{\sqrt{1-\epsilon^2}}\,. \tag{6.30}$$

  This is the equation for an ellipse with its center located at $(-k\epsilon/(1-\epsilon^2), 0)$, and with semi-major and semi-minor axes a and b, respectively. The focal length is $c = \sqrt{a^2 - b^2} = k\epsilon/(1-\epsilon^2)$. Therefore, one focus is located at the origin (see Fig. 6.7).

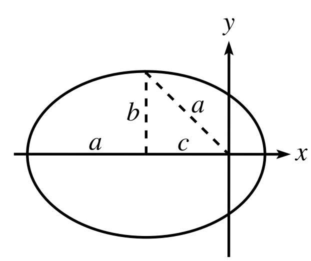


Figure 6.7

- **Parabola** ($\epsilon = 1$)

  In this case, eq. (6.29) becomes $y^2 = k^2 - 2kx$. This may be written as $y^2 = -2k(x - \frac{k}{2})$. This is the equation for a parabola with vertex at $(k/2, 0)$ and focal length $k/2$. (The focal length of a parabola written in the form $y^2 = 4ax$ is a). So we have a parabola with its focus located at the origin (see Fig. 6.8).

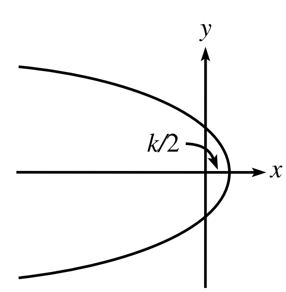


Figure 6.8

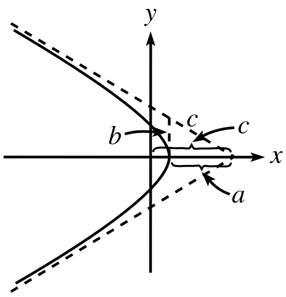


Figure 6.9

- **Hyperbola** ($\epsilon > 1$)

  In this case, eq. (6.29) may be written (after completing the square for the x terms)

$$\frac{\left(x - \frac{k\epsilon}{\epsilon^2-1}\right)^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \text{where } a = \frac{k}{\epsilon^2 - 1}, \quad \text{and } b = \frac{k}{\sqrt{\epsilon^2 - 1}}. \tag{6.31}$$

  This is the equation for a hyperbola with its center (the term used for the intersection of the asymptotes) located at $(k\epsilon/(\epsilon^2 - 1), 0)$. The focal length is $c = \sqrt{a^2 + b^2} = k\epsilon/(\epsilon^2 - 1)$. Therefore, the focus is located at the origin (see Fig. 6.9).

6.4.4 Kepler's Laws

We can now, with minimal extra work, write down Kepler's Laws. Kepler (1571–1630) lived prior Newton (1642–1727). He arrived at these laws via observational data, a rather impressive feat. It was known since the time of Copernicus (1473–1543) that the planets move around the sun, but it wasn't until Kepler and Newton that a quantitative description of the orbits was given.

- **First Law:** *The planets move in elliptical orbits with the sun at one focus.*

  We proved this in Subsection 6.4.3. (Of course, there could be objects flying past the sun in parabolic or hyperbolic orbits. But we wouldn't call these things planets, because we'd never see the same one twice.)

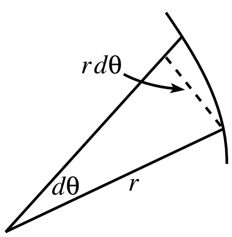


Figure 6.10

- **Second Law:** *The radius vector to a planet sweeps out area at a rate that is independent of its position in the orbit.*

  This law is nothing other than conservation of angular momentum. The area swept out by the radius vector during a short period of time is $dA = r(r\,d\theta)/2$, because $r\,d\theta$ is the base of the thin triangle in Fig. 6.10. Therefore, we have (using $L = mr^2\dot{\theta}$)

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m}, \tag{6.32}$$

  which is constant, since L is constant for a central force.

- **Third Law:** *The square of the period of an orbit, T, is proportional to the cube of the semimajor-axis length, a. More precisely,*

$$T^2 = \frac{4\pi^2 ma^3}{\alpha} \equiv \frac{4\pi^2 a^3}{GM_\odot}, \tag{6.33}$$

  where $M_\odot$ is the mass of the sun.

  Proof: Integrating eq. (6.32) over the time of a whole orbit gives

$$A = \frac{LT}{2m}. \tag{6.34}$$

But the area of an ellipse is $A = \pi ab$, where a and b are the semi-major and semi-minor axes, respectively. Squaring (6.34) and using eq. (6.30) to write $b = a\sqrt{1-\epsilon^2}$ gives

$$\pi^2 a^4 = \left(\frac{L^2}{m(1-\epsilon^2)}\right)\frac{T^2}{4m}\,. \tag{6.35}$$

We have grouped the right-hand side in this way because we may now use the $L^2 \equiv m\alpha k$ relation from eq. (6.28) to transform the term in parentheses into $\alpha k/(1-\epsilon^2) \equiv \alpha a$, where a is given in eq. (6.30). We therefore arrive at the desired result, eq. (6.33).

Our solar system is only the tip of the iceberg, of course. There's a whole universe around us, and with each generation we can see and understand a little more of it. In recent years, we've even begun to look for any friends we might have out there. Why? Because we can. There's nothing wrong with looking under the lamppost now and then. It just happens to be a very big one in this case.

As we grow up, we open an ear,
Exploring the cosmic frontier.
In this coming of age,
We turn in our cage,
All alone on a tiny blue sphere.

6.4.5 Reduced mass

We assumed in Section 6.4.1 that the sun is large enough so that it is only negligibly affected by the presence of planets. That is, it is essentially fixed at the origin. But how do we solve a problem where the masses of the two interacting bodies are comparable in size? Or equivalently, how do we solve the earth-sun problem exactly? It turns out that the only modification required is a trivial replacement of the earth's mass with the *reduced mass*, defined below. The following discussion actually holds for any central force, not just gravity.

The Lagrangian of a general central-force system consisting of the interacting masses m_1 and m_2 is

$$\mathcal{L} = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - V(|\mathbf{r}_1 - \mathbf{r}_2|). \tag{6.36}$$

We have written the potential in this form, dependent only on the distance $|\mathbf{r}_1 - \mathbf{r}_2|$, because we are assuming a central force. Let us define

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}\,, \qquad \text{and} \qquad \mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2. \tag{6.37}$$

These are simply the position of the center of mass, and the vector between the masses, respectively. Invert these equations to obtain

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}, \qquad \text{and} \qquad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}, \tag{6.38}$$

where $M \equiv m_1 + m_2$ is the total mass of the system. In terms of $\mathbf{R}$ and $\mathbf{r}$, the Lagrangian becomes

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m_1\left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}\right)^2 - V(|\mathbf{r}|) \\ &= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\dot{\mathbf{r}}^2 - V(r) \\ &= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(r), \end{aligned} \tag{6.39}$$

where the *reduced mass*, μ, is defined by

$$\frac{1}{\mu} \equiv \frac{1}{m_1} + \frac{1}{m_2}\,. \tag{6.40}$$

We now note that the Lagrangian in eq. (6.39) depends on $\dot{\mathbf{R}}$, but not on $\mathbf{R}$. Therefore, the Euler-Lagrange equations say that $\dot{\mathbf{R}}$ is constant. That is, the CM moves at constant velocity (this is just the statement that there are no external forces). The CM motion is therefore trivial, so let's ignore it. Our Lagrangian therefore essentially becomes

$$\mathcal{L} \to \frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(r). \tag{6.41}$$

But this is simply the Lagrangian for a particle of mass μ, which moves around a fixed origin, under the influence of the potential $V(r)$.

To solve the earth-sun system exactly, we therefore simply need to replace (in Section 6.4.1) the earth's mass, m, with the reduced mass, μ. The resulting value for r in eq. (6.23) is the distance between the two masses m_1 and m_2 (which are distances of $(m_2/M)r$ and $(m_1/M)r$, respectively, away from the center-of-mass, from eq. (6.38)). Note that the m's that are buried in the α, L, and ϵ in eq. (6.23) must also be changed to μ's.

For the earth-sun system, μ is essentially equal to m (it is smaller than m by only one part in $3 \cdot 10^5$), so our fixed-M approximation is a very good one.

6.5 Problems

Section 6.2: The effective potential

1. **Maximum L** $**$

   A particle moves in a potential $V(r) = -V_0 e^{-\lambda^2 r^2}$.

   (a) Given L, find the radius of the circular orbit. (An implicit equation is fine here.)

   (b) What is the largest value of L for which a circular orbit exists? What is the value of $V_{\text{eff}}(r)$ at this critical orbit?

2. **Cross section** $**$

   A particle moves in a potential $V(r) = -C/(3r^3)$.

   (a) Given L, find the maximum value of the effective potential.

   (b) Let the particle come in from infinity with speed v_0 and impact parameter b. In terms of C, m, and v_0, what is the largest value of b (call it b_{max}) for which the particle is captured by the potential? (In other words, what is the 'cross section', πb^2_{max}, for this potential?)

3. **Exponential spiral** $**$

   Given L, find the form of $V(r)$ so that the path of a particle is given by the spiral $r = Ae^{a\theta}$, where A and a are constants. (*Hint:* Obtain an expression for $\dot{r}$ that contains no θ's, and use eq. (6.9).)

4. **Power-law spiral** $**$

   Given L, find the form of $V(r)$ so that the path of a particle is given by the spiral $r = C\theta^k$, where C and k are constants. (*Hint:* Obtain an expression for $\dot{r}$ that contains no θ's, and use eq. (6.9).)

6.6 Solutions

1. **Maximum L**

(a) The effective potential is

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} - V_0 e^{-\lambda^2 r^2}. \tag{6.42}$$

A circular orbit is possible when $V'_{\text{eff}}(r) = 0$. Taking the derivative and solving for L^2 gives

$$L^2 = (2mV_0\lambda^2) r^4 e^{-\lambda^2 r^2}. \tag{6.43}$$

This implicitly determines r. As long as L isn't too large (as we will see in part (b)), there will be two solutions for r, as Fig. 6.11 indicates. The smaller solution is the one with a stable orbit.

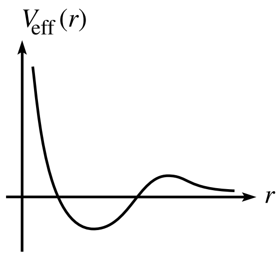


Figure 6.11

(b) The function $r^4 e^{-\lambda^2 r^2}$ on the right-hand side of eq. (6.43) clearly has a maximum value. Therefore, there is a maximum value of L for which a solution for r exists. (Above this maximum value, $V'_{\text{eff}}(r)$ is never zero, so $V'(r)$ simply decreases monotonically to zero.) The maximum of $r^4 e^{-\lambda^2 r^2}$ occurs when

$$(r^4 e^{-\lambda^2 r^2})' = e^{-\lambda^2 r^2}[4r^3 - r^4(-2\lambda^2 r)] = 0 \qquad \Longrightarrow \qquad r^2 = \frac{2}{\lambda^2} \equiv r_0^2. \tag{6.44}$$

Plugging r_0 into eq. (6.43) gives

$$L^2_{\text{max}} = \frac{8mV_0}{\lambda^2 e^2}. \tag{6.45}$$

Also, plugging r_0 into (6.42) gives

$$V_{\text{eff}}(r_0) = \frac{V_0}{e^2}, \tag{6.46}$$

which happens to be greater than zero.

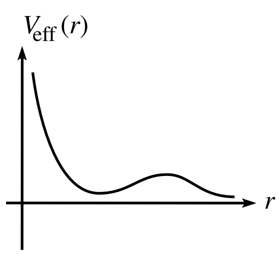


Figure 6.12

REMARK: A common error in this problem is to say that the condition for circular orbits is that $V_{\text{eff}}(r) < 0$ at the point where $V_{\text{eff}}(r)$ is minimum (because 0 is the value of V_{eff} at $r = \infty$). This gives the wrong answer ($L^2_{\text{max}} = 2mV_0/\lambda^2 e$, as you can show), because $V_{\text{eff}}(r)$ can look like the graph in Fig. 6.12. This has a local minimum with $V_{\text{eff}}(r) > 0$. (Of course, in this situation the particle would eventually tunnel out to infinity, due to quantum mechanics; but we won't worry about such things here.) ♣

2. **Cross section**

(a) The effective potential is

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{C}{3r^3}. \tag{6.47}$$

Setting the derivative equal to zero gives $r = mC/L^2$. Plugging this into $V_{\text{eff}}(r)$ gives

$$V^{\text{max}}_{\text{eff}} = \frac{L^6}{6m^3C^2}. \tag{6.48}$$

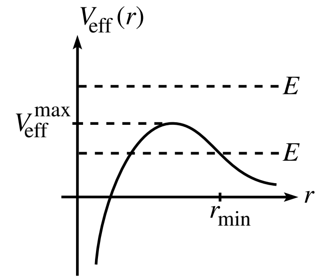


Figure 6.13

(b) If the energy of the particle, E, is less than $V_{\text{eff}}^{\text{max}}$, then the particle will reach a minimum value of r, and then head back out to infinity (see Fig. 6.13). If E is greater than $V_{\text{eff}}^{\text{max}}$, then the particle will head in to $r = 0$, never to return.
So the condition for capture is $V_{\text{eff}}^{\text{max}} < E$. Using $L = mv_0 b$ and $E = E_\infty = mv_0^2/2$, this condition becomes

$$\frac{(mv_0 b)^6}{6m^3C^2} < \frac{mv_0^2}{2}$$
$$\Longrightarrow \quad b < \left(\frac{3C^2}{m^2v_0^4}\right)^{1/6} \equiv b_{\text{max}}. \tag{6.49}$$

The cross section for capture is therefore

$$\sigma = \pi b_{\text{max}}^2 = \pi\left(\frac{3C^2}{m^2v_0^4}\right)^{1/3}. \tag{6.50}$$

It makes sense that this should increase with C and decrease with m and v_0.

3. **Exponential spiral**

The given information $r = Ae^{a\theta}$ yields (using $\dot\theta = L/mr^2$)

$$\dot r = aAe^{a\theta}\dot\theta = ar\left(\frac{L}{mr^2}\right) = \frac{aL}{mr}. \tag{6.51}$$

Plugging this into eq. (6.9) gives

$$\frac{m}{2}\left(\frac{aL}{mr}\right)^2 + \frac{L^2}{2mr^2} + V(r) = E. \tag{6.52}$$

Therefore,

$$V(r) = E - \frac{(1+a^2)L^2}{2mr^2}. \tag{6.53}$$

The total energy, E, may be chosen arbitrarily to be zero, if desired.

4. **Power-law spiral**

The given information $r = C\theta^k$ yields (using $\dot\theta = L/mr^2$)

$$\dot r = kC\theta^{k-1}\dot\theta = kC\left(\frac{r}{C}\right)^{(k-1)/k}\left(\frac{L}{mr^2}\right) = \frac{kL}{mr}\left(\frac{C}{r}\right)^{1/k}. \tag{6.54}$$

Plugging this into eq. (6.9) gives

$$\frac{m}{2}\left(\frac{kL}{mr}\right)^2\left(\frac{C}{r}\right)^{2/k} + \frac{L^2}{2mr^2} + V(r) = E. \tag{6.55}$$

Therefore,

$$V(r) = E - \frac{L^2}{2mr^2}\left(1 + k^2\left(\frac{C}{r}\right)^{2/k}\right). \tag{6.56}$$

The total energy, E, may be chosen arbitrarily to be zero, if desired.

Chapter 7

Angular Momentum, Part I (Constant $\hat{\mathrm{L}}$)

The angular momentum of a point mass, relative to a given origin, is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{7.1}$$

For a collection of particles, the total $\mathbf{L}$ is simply the sum of the $\mathbf{L}$'s of each particle.

The quantity $\mathbf{r} \times \mathbf{p}$ is a useful thing to study because it has many nice properties. One of these was presented in Theorem 6.1, which allowed us to introduce the "effective potential" in eq. 6.10. And later in this chapter we'll introduce the concept of *torque*, $\boldsymbol{\tau}$, which appears in the bread-and-butter statement, $\boldsymbol{\tau} = d\mathbf{L}/dt$ (which is analogous to Newton's $\mathbf{F} = d\mathbf{p}/dt$ law). This equation is the basic ingredient, along with $\mathbf{F} = m\mathbf{a}$, in solving any rotation problem.

There are two basic types of angular momentum problems in the world. Since the solution to any rotational problem invariably comes down to using $\boldsymbol{\tau} = d\mathbf{L}/dt$, we must determine how $\mathbf{L}$ changes in time. And since $\mathbf{L}$ is a vector, it can change because (1) its length changes, or (2) its direction changes (or through some combination of these effects). In other words, if we write $\mathbf{L} = L\hat{\mathbf{L}}$, where $\hat{\mathbf{L}}$ is the unit vector in the $\mathbf{L}$ direction, then $\mathbf{L}$ can change because L changes, or because $\hat{\mathbf{L}}$ changes, or both.

The first of these cases, that of constant $\hat{\mathbf{L}}$, is the easily understood one. If you have a spinning record (in which case $\mathbf{L} = \sum \mathbf{r} \times \mathbf{p}$ is perpendicular to the record, assuming that you've chosen the center as the origin), and if you give the record a tangential force in the proper direction, then it will speed up (in a precise way which we will soon determine). There is nothing mysterious going on here. If you push on the record, it goes faster. $\mathbf{L}$ points in the same direction as before, but it now simply has a larger magnitude. In fact, in this type of problem, you can completely forget that $\mathbf{L}$ is a vector; you can just deal with its magnitude L, and everything will be fine. This first case will be the subject of the present chapter.

In contrast, the second case, where $\mathbf{L}$ changes direction, can get rather confusing. This will be the subject of the following chapter, where we will talk about gyroscopes, tops, and other such spinning objects that have a tendency to make one's head spin also. In this case, the entire point is that **L** is actually a vector. And unlike the first case, you really have to visualize things in three dimensions to see what's going on.[1]

The angular momentum of a point mass is given by the simple expression in eq. (7.1). But in order to deal with setups in the real world, which invariably consist of many particles, we must learn how to calculate the angular momentum of an extended object. This is the task of the Section 7.1. (But we'll only deal with motion in the x-y plane in this chapter. Chapter 8 deals with general 3-D motion.)

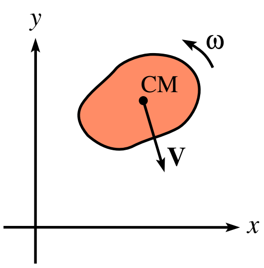


Figure 7.1

7.1 Pancake object in x-y plane

Consider a flat, rigid body undergoing arbitrary motion in the x-y plane (see Fig. 7.1). What is the angular momentum of this body, relative to the origin of the coordinate system?[2]

If we imagine the body to consist of particles of mass m_i, then the angular momentum of the entire body is the sum of the angular momenta of each m_i (which are $\mathbf{L}_i = \mathbf{r}_i \times \mathbf{p}_i$). So the total angular momentum is

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i. \tag{7.2}$$

(For a continuous distribution of mass, we'd have an integral instead of a sum.) **L** depends on the locations and momenta of the masses. The momenta in turn depend on how fast the body is translating and spinning. Our goal here is to find this dependence of **L** on the distribution and motion of its constituent masses. This result will involve the geometry of the body in a specific way, as we will show.

In this chapter, we will deal only with pancake-like objects which move in the x-y plane (or simple extensions of these). We will find **L** relative to the origin, and we will also derive an expression for the kinetic energy.

Note that since both **r** and **p** for our pancake-like objects always lie in the x-y plane, the vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ always points in the $\hat{\mathbf{z}}$ direction. This fact is what makes these pancake cases easy to deal with; **L** changes only because its length changes, not its direction. So when we eventually get to the $\boldsymbol{\tau} = d\mathbf{L}/dt$ equation, it will take on a simple form.

Let's first look at a special case, and then we will look at general motion in the x-y plane.

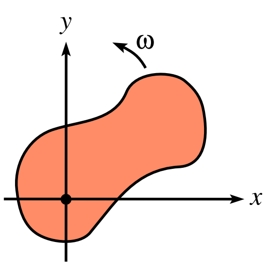


Figure 7.2

7.1.1 Rotation about the z-axis

The pancake in Fig. 7.2 rotates with angular speed ω around the z-axis, in the counterclockwise direction (as viewed from above). Consider a little piece of the

[1]The difference between these two cases is essentially the same as the difference between the two basic $\mathbf{F} = d\mathbf{p}/dt$ cases. The vector **p** can change simply because its magnitude changes, in which case we have $F = ma$. Or, **p** can change because its direction changes, in which case we end up with the centripetal acceleration statement, $F = mv^2/r$. (Or, there could be a combination of these effects). The former case seems a bit more intuitive than the latter.

[2]Remember, **L** is defined relative to a chosen origin (since it has the vector **r** in it), so it makes no sense to ask what **L** is, without specifying what origin you've chosen.

body, with mass dm and position (x, y). Let $r = \sqrt{x^2 + y^2}$. This little piece travels in a circle around the origin. Its speed[3] is $v = \omega r$. Therefore, the angular momentum of this piece (relative to the origin) is equal to $\mathbf{L} = \mathbf{r} \times \mathbf{p} = r(v\,dm)\hat{\mathbf{z}} = dm\, r^2\omega\hat{\mathbf{z}}$. The $\hat{\mathbf{z}}$ direction arises from the cross product of the (orthogonal) vectors $\mathbf{r}$ and $\mathbf{p}$. The angular momentum of the entire body is therefore

$$\begin{aligned}\mathbf{L} &= \int r^2\omega\hat{\mathbf{z}}\,dm \\ &= \int (x^2 + y^2)\omega\hat{\mathbf{z}}\,dm, \end{aligned} \tag{7.3}$$

where the integration runs over the area of the body. (If the density of the object is constant, as is usually the case, then we have $dm = \rho\,dx\,dy$.) If we define the *moment of inertia* around the z-axis to be

$$I_z \equiv \int r^2\,dm = \int (x^2 + y^2)\,dm, \tag{7.4}$$

then the z-component of $\mathbf{L}$ is

$$L_z = I_z\omega, \tag{7.5}$$

and L_x and L_y are both equal to zero. In the case where the rigid body is made up of a collection of point masses, m_i, in the x-y plane, the moment of inertia in eq. (7.4) simply takes the discretized form,

$$I_z \equiv \sum_i m_i r_i^2. \tag{7.6}$$

Given any rigid body in the x-y plane, we can calculate I_z. And given ω, we can then multiply it by I_z to find L_z. In Section 7.2.1, we will get some practice calculating many moments of inertia.

What is the kinetic energy of our object? We need to add up the energies of all the little pieces. A little piece has energy $dm\,v^2/2 = dm(r\omega)^2/2$. So the total kinetic energy is

$$T = \int \frac{1}{2} r^2\omega^2\,dm. \tag{7.7}$$

With our definition of I_z in eq. (7.4), we have

$$T = \frac{1}{2} I_z\omega^2. \tag{7.8}$$

This is easy to remember, because it looks a lot like the kinetic energy of a point mass, $(1/2)mv^2$.

7.1.2 General motion in x-y plane

How do we deal with general motion in the x-y plane? For the motion in Fig. 7.3,

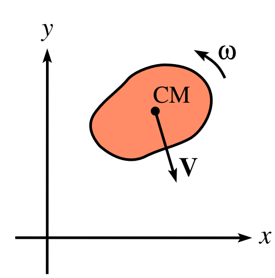


Figure 7.3

[3]The velocity is actually given by $\mathbf{v} = \vec{\omega} \times \mathbf{r}$, which reduces to $v = \omega r$ in our case. The vector $\vec{\omega}$ is the *angular velocity vector*, which is defined to point along the axis of rotation, with magnitude ω (so $\vec{\omega} = \omega\hat{\mathbf{z}}$ here). There is no great need to use the vector $\vec{\omega}$ in the constant $\hat{\mathbf{L}}$ case in this chapter, so we won't. But don't worry, in the next chapter you'll get all the practice with $\vec{\omega}$ that you could possibly hope for.

the various pieces of mass are not traveling in circles about the origin, so we cannot write $v = \omega r$, as we did above.

It turns out to be highly advantageous to write the angular momentum, $\mathbf{L}$, and the kinetic energy, T, in terms of the center-of-mass (CM) coordinates and the coordinates relative to the CM. The expressions for $\mathbf{L}$ and T take on very nice forms when written this way, as we now show.

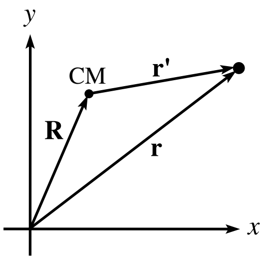


Figure 7.4

Let the coordinates of the CM be $\mathbf{R} = (X, Y)$, and let the coordinates relative to the CM be $\mathbf{r}' = (x', y')$. Then $\mathbf{r} = \mathbf{R} + \mathbf{r}'$ (see Fig. 7.4). Let the velocity of the CM be $\mathbf{V}$, and let the velocity relative to the CM be $\mathbf{v}'$. Then $\mathbf{v} = \mathbf{V} + \mathbf{v}'$. Let the body rotate with angular speed ω' around the CM (around an instantaneous axis parallel to the z-axis, so that the pancake remains in the x-y plane at all times).[4] Then $v' = \omega' r'$.

Let's look at $\mathbf{L}$ first. The angular momentum is

$$\begin{aligned}
\mathbf{L} &= \int \mathbf{r} \times \mathbf{v}\, dm \\
&= \int (\mathbf{R} + \mathbf{r}') \times (\mathbf{V} + \mathbf{v}')\, dm \\
&= M\mathbf{R} \times \mathbf{V} + \int \mathbf{r}' \times \mathbf{v}'\, dm \qquad \text{(cross terms vanish; see below)} \\
&= M\mathbf{R} \times \mathbf{V} + \left(\int r'^2 \omega' dm \right) \hat{\mathbf{z}} \\
&\equiv M\mathbf{R} \times \mathbf{V} + \left(I_z^{\mathrm{CM}} \omega' \right) \hat{\mathbf{z}}. \qquad (7.9)
\end{aligned}$$

where M is the mass of the pancake. In going from the second to the third line above, the cross terms, $\int \mathbf{r}' \times \mathbf{V}\, dm$ and $\int \mathbf{R} \times \mathbf{v}'\, dm$, vanish because of the definition of the CM (which says that $\int \mathbf{r}'\, dm = 0$, and hence $\int \mathbf{v}'\, dm = d(\int \mathbf{r}'\, dm)/dt = 0$). The quantity I_z^{CM} is the moment of inertia around an axis through the CM, parallel to the z-axis.

Eq. (7.9) is a very nice result, and it's important enough to be called a theorem. In words, it says:

Theorem 7.1 *The angular momentum (relative to the origin) of a body can be found by treating the body as a point mass located at the CM and finding the angular momentum of this point mass (relative to the origin), and by then adding on the angular momentum of the body, relative to the CM.* [5]

Note that if we have the special case where the CM travels around the origin in a circle, with angular speed Ω (so that $V = \Omega R$), then eq. (7.9) becomes $\mathbf{L} = \left(MR^2\Omega + I_z^{\mathrm{CM}} \omega' \right) \hat{\mathbf{z}}$.

[4]What we mean here is the following. Consider a coordinate system whose origin is the CM and whose axes are parallel to the fixed x- and y-axes. Then the pancake rotates with angular speed ω' in this reference frame.

[5]This theorem only works if we use the CM as the location of the imagined point mass. True, in the above analysis we could have chosen a point P other than the CM, and then written things in terms of the coordinates of P and the coordinates relative to P (which could also be described by a rotation). But then the cross terms in eq. (7.9) wouldn't vanish, and we'd end up with an unenlightening mess.

Now let's look at T. The kinetic energy is

$$\begin{aligned} T &= \int \frac{1}{2} v^2\, dm \\ &= \int \frac{1}{2} |\mathbf{V} + \mathbf{v}'|^2\, dm \\ &= \frac{1}{2} M V^2 + \int \frac{1}{2} v'^2\, dm \qquad \text{(cross term vanishes; see below)} \\ &= \frac{1}{2} M V^2 + \int \frac{1}{2} r'^2 \omega'^2 dm \\ &\equiv \frac{1}{2} M V^2 + \frac{1}{2} I_z^{\text{CM}} \omega'^2. \end{aligned} \tag{7.10}$$

In going from the second to third line above, the cross term $\int \mathbf{V} \cdot \mathbf{v}'\, dm$ vanishes by definition of the CM. Again, eq. (7.10) is a very nice result. In words, it says:

Theorem 7.2 *The kinetic energy of a body can be found by treating the body as a point mass located at the CM, and by then adding on the kinetic energy of the body due to motion relative to the CM.*

7.1.3 The parallel-axis theorem

Consider the special case where the CM rotates around the origin at the same rate as the body rotates around the CM. (This may be achieved, for example, by gluing a stick across the pancake and pivoting one end of the stick at the origin; see Fig. 7.5.) This means that we have the nice situation where all points in the pancake travel in circles around the origin. Let their angular speed be ω.

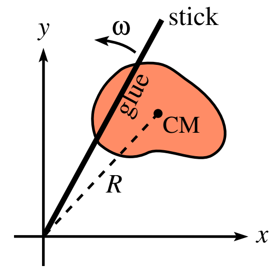


Figure 7.5

In this situation, the speed of the CM is ωR, so eq. (7.9) says that the angular momentum around the origin is

$$L_z = (MR^2 + I_z^{\text{CM}})\omega. \tag{7.11}$$

In other words, the moment of inertia around the origin is

$$\boxed{I_z = MR^2 + I_z^{\text{CM}}}. \tag{7.12}$$

This is the *parallel-axis theorem*. It says that once you've calculated the moment of inertia of an object relative to the CM (namely I_z^{CM}), then if you want to calculate I_z around an arbitrary point in the plane of the pancake, you simply have to add on MR^2, where R is the distance from the point to the CM, and M is the mass of the pancake.

The parallel-axis theorem is simply a special case of the more general result, eq. (7.9), so it is valid *only* with the CM, and not with any other point.

Likewise, in this situation, eq. (7.10) gives

$$T = \frac{1}{2}(MR^2 + I_z^{\text{CM}})\omega^2. \tag{7.13}$$

Example (A stick): Let's verify the parallel-axis theorem for a stick of mass m and length ℓ, in the case where we want to compare the moment of inertia about an end with the moment of inertia about the CM. (Both of the relevant axes will be perpendicular to the stick, and parallel to each other, of course.)

For convenience, let $\rho = m/\ell$ be the density. The moment of inertial about an end is

$$I^{\text{end}} = \int_0^\ell x^2\,dm = \int_0^\ell x^2\rho\,dx = \frac{1}{3}\rho\ell^3 = \frac{1}{3}(\rho\ell)\ell^2 = \frac{1}{3}m\ell^2. \tag{7.14}$$

The moment of inertial about the CM is

$$I^{\text{CM}} = \int_{-\ell/2}^{\ell/2} x^2\,dm = \int_{-\ell/2}^{\ell/2} x^2\rho\,dx = \frac{1}{12}\rho\ell^3 = \frac{1}{12}m\ell^2. \tag{7.15}$$

This is consistent with the parallel axis theorem, eq. (7.12), because

$$I^{\text{end}} = m\left(\frac{\ell}{2}\right)^2 + I^{\text{CM}}. \tag{7.16}$$

Remember that this only works with the CM. If you instead want to compare I^{end} with the I around a point, say, $\ell/6$ from that end, then you cannot say they differ by $m(\ell/6)^2$. But you *can* compare each of them to I^{CM} and say that they differ by $(\ell/2)^2 - (\ell/3)^2 = 5\ell^2/36$.

7.1.4 The perpendicular-axis theorem

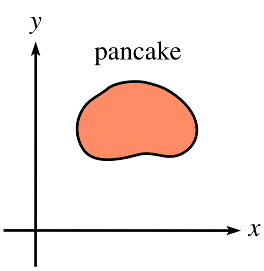


Figure 7.6

This theorem is valid *only* for pancake objects. Consider a pancake object in the x-y plane (see Fig. 7.6). Then the *perpendicular-axis theorem* says that

$$I_z = I_x + I_y, \tag{7.17}$$

where I_x and I_y are defined analogously to eq. (7.4). (That is, to find I_x, you imagine spinning the object around the x-axis at angular speed ω, and then define $I_x \equiv L_x/\omega$. Similarly for I_y.) In other words,

$$I_x \equiv \int (y^2+z^2)\,dm, \qquad I_y \equiv \int (z^2+x^2)\,dm, \qquad I_z \equiv \int (x^2+y^2)\,dm. \tag{7.18}$$

To prove the theorem, we simply use the fact that $z = 0$ for our pancake object. Hence $I_z = I_x + I_y$.

In the limited number of cases where this theorem is applicable, it may save you some trouble. A few examples are given in the next section.

7.2 Calculating moments of inertia

7.2.1 Lots of examples

Let's now compute the moments of inertia for a few objects, around specified axes. We will use ρ to denote mass density (per unit length, area, or volume, as appropriate). We will assume that this density is uniform throughout the object. For the more complicated of the objects below, it is generally a good idea to slice the object up into pieces for which I is already known. The problem then reduces to integrating over these known I's. There is usually more than one way to do this slicing. For example, a sphere may be looked at as a series of concentric shells or a collection of disks stacked on top of each other. In the examples below, you may want to play around with slicings other than the ones given.

Consider at least a few of these examples to be problems and try to work them out for yourself.

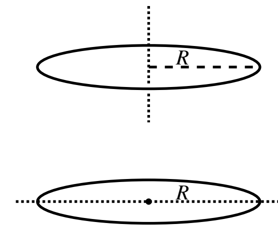


Figure 7.7

1. A ring of mass M and radius R (axis through center, perpendicular to plane; Fig. 7.7):

$$I = \int r^2\, dm = \int_0^{2\pi} R^2 \rho R\, d\theta = (2\pi R\rho)R^2 = \boxed{MR^2}, \tag{7.19}$$

as it should be (since every bit of the mass is a distance R from the axis).

2. A ring of mass M and radius R (axis through center, in plane; Fig. 7.7):

The distance from the axis is (the absolute value of) $R\sin\theta$. Therefore,

$$I = \int r^2\, dm = \int_0^{2\pi} (R\sin\theta)^2 \rho R\, d\theta = \frac{1}{2}(2\pi R\rho)R^2 = \boxed{\tfrac{1}{2}MR^2}, \tag{7.20}$$

where we have used $\sin^2\theta = (1-\cos 2\theta)/2$. You can also do this via the perpendicular axis theorem. In the notation of section 7.1.4, we have $I_x = I_y$, by symmetry. Hence, $I_z = 2I_x$. Using $I_z = MR^2$ from Example 1 gives the proper result.

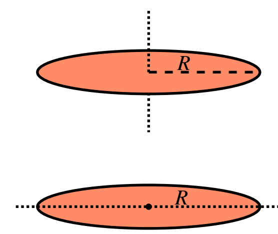


Figure 7.8

3. A disk of mass M and radius R (axis through center, perpendicular to plane; Fig. 7.8):

$$I = \int r^2\, dm = \int_0^{2\pi}\int_0^R r^2 \rho r\, dr\, d\theta = (R^4/4)2\rho\pi = \frac{1}{2}(\rho\pi R^2)R^2 = \boxed{\tfrac{1}{2}MR^2}. \tag{7.21}$$

You can save one (trivial) integration step by considering the disk to be made up of many concentric rings, and invoke Example 1. The mass of each ring is $\rho 2\pi r\, dr$. Integrating over the rings gives $I = \int_0^R (\rho 2\pi r\, dr) r^2 = \pi R^4 \rho/2 = MR^2/2$, as before. Slicing the disk up is fairly inconsequential in this example, but it will save a good deal of effort in others.

4. A disk of mass M and radius R (axis through center, in plane; Fig. 7.8):

Slice the disk up into rings, and use Example 2.

$$I = \int_0^R (1/2)(\rho 2\pi r\, dr)r^2 = (R^2/4)\rho\pi = \frac{1}{4}(\rho\pi R^2)R^2 = \boxed{\tfrac{1}{4}MR^2}. \tag{7.22}$$

Or, just use Example 3 and the perpendicular axis theorem.

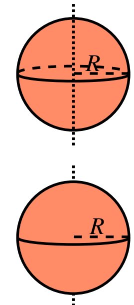


Figure 7.9

5. A spherical shell of mass M and radius R (any axis through center; Fig. 7.9):

Choose the z-axis. We may slice the sphere into a large number of horizontal ring-like strips. In spherical coordinates, the radii of the rings are given by (the absolute value of) $R\sin\theta$, where θ is the angle down from the north pole. The area of a strip is then $2\pi(R\sin\theta)Rd\theta$. Using $\int \sin^3\theta = \int \sin\theta(1-\cos^2\theta) = -\cos\theta + \cos^3\theta/3$, we have

$$\begin{aligned} I = \int (x^2+y^2)\,dm &= \int_0^\pi (R\sin\theta)^2\rho 2\pi(R\sin\theta)Rd\theta = 2\pi\rho R^4 \int_0^\pi \sin^3\theta \\ &= 2\pi\rho R^4(4/3) = \frac{2}{3}(4\pi R^2\rho)R^2 = \boxed{\tfrac{2}{3}MR^2}. \end{aligned} \tag{7.23}$$

6. A sphere of mass M and radius R (any axis through center; Fig. 7.9):

   A sphere is made up of concentric spherical shells. The volume of a shell is $4\pi r^2 dr$. Using Example 5, we have

$$I = \int_0^R (2/3)(4\pi\rho r^2 dr)r^2 = (R^5/5)(8\pi\rho/3) = \frac{2}{5}(4/3\pi R^3\rho)R^2 = \boxed{\tfrac{2}{5}MR^2}. \tag{7.24}$$

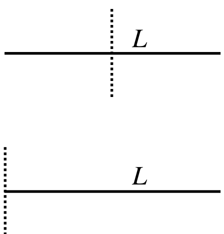


Figure 7.10

7. A thin uniform rod of mass M and length L (axis through center, perpendicular to rod; Fig. 7.10):

$$I = \int x^2 dm = \int_{-L/2}^{L/2} x^2\rho\,dx = \frac{1}{12}(\rho L)L^2 = \boxed{\tfrac{1}{12}ML^2}. \tag{7.25}$$

8. A thin uniform rod of mass M and length L (axis through end, perpendicular to rod; Fig. 7.10):

$$I = \int x^2 dm = \int_0^L x^2\rho\,dx = \frac{1}{3}(\rho L)L^2 = \boxed{\tfrac{1}{3}ML^2}. \tag{7.26}$$

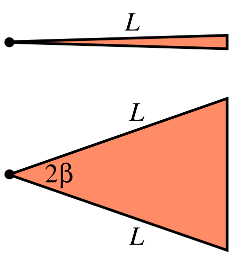


Figure 7.11

9. An infinitesimally thin triangle of mass M and length L (axis through tip, perpendicular to plane; Fig. 7.11):

   Let the base have length a (we will assume a is infinitesimally small). Then a slice at a distance x from the tip has length $a(x/L)$. If the slice has thickness dx, then it is essentially a point mass of mass $dm = \rho ax\,dx/L$. So

$$I = \int x^2 dm = \int_0^L x^2\rho ax/L\,dx = \frac{1}{2}(\rho aL/2)L^2 = \boxed{\tfrac{1}{2}ML^2}, \tag{7.27}$$

   because $aL/2$ is the area of the triangle. This of course has the same form as the disk in Example 3, because a disk is made up of many of these triangles.

10. An isosceles triangle of mass M, vertex angle 2β, and common-side length L (axis through tip, perpendicular to plane; Fig. 7.11):

    Let h be the altitude of the triangle (so $h = L\cos\beta$). Slice the triangle into thin strips parallel to the base. Let x be the distance from the vertex to a thin strip. Then the length of a strip is $\ell = 2x\tan\beta$, and its mass is $dm = \rho(2x\tan\beta\,dx)$. Using Example 7 above, along with the parallel axis theorem, we have

$$\begin{aligned} I &= \int_0^h dm\left(\frac{\ell^2}{12} + x^2\right) = \int_0^h (\rho 2x\tan\beta\,dx)\left(\frac{(2x\tan\beta)^2}{12} + x^2\right) \\ &= \int_0^h 2\rho\tan\beta\left(1 + \frac{\tan^2\beta}{3}\right)x^3\,dx = 2\rho\tan\beta\left(1+\frac{\tan^2\beta}{3}\right)\frac{h^4}{4}. \end{aligned} \tag{7.28}$$

    But the area of the whole triangle is $h^2\tan\beta$, so we have $I = (Mh^2/2)(1+\tan^2\beta/3)$. In terms of L, this is

$$I = (ML^2/2)(\cos^2\beta + \sin^2\beta/3) = \boxed{\tfrac{1}{2}ML^2\left(1 - \tfrac{2}{3}\sin^2\beta\right)}. \tag{7.29}$$

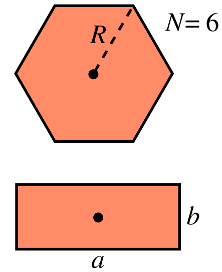


Figure 7.12

11. A regular N-gon of mass M and "radius" R (axis through center, perpendicular to plane; Fig. 7.12):

    The N-gon is made up of N isosceles triangles, so we can use Example 10, with $\beta = \pi/N$. The masses of the triangles simply add, so if M is the mass of the whole N-gon, we have

$$I = \boxed{\frac{1}{2}MR^2\left(1 - \frac{2}{3}\sin^2\frac{\pi}{N}\right)}. \qquad (7.30)$$

    Let's list the values of I for a few N. We'll use the shorthand notation (N, I/MR^2). Eq. 7.30 gives $(3, \frac{1}{4})$, $(4, \frac{1}{3})$, $(6, \frac{5}{12})$, $(\infty, \frac{1}{2})$. These values of I form a nice arithmetic progression.

12. A rectangle of mass M and sides of length a and b (axis through center, perpendicular to plane; Fig. 7.12):

    Let the z-axis be perpendicular to the plane. We know that $I_x = Mb^2/12$ and $I_y = Ma^2/12$, so the perpendicular axis theorem tells us that

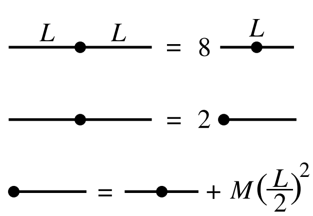
$$I_z = I_x + I_y = \boxed{\frac{1}{12}M(a^2 + b^2)}. \qquad (7.31)$$

7.2.2 A neat trick

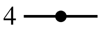
For some objects with certain symmetries, it is possible to calculate I without doing any integrals. All that is needed is a scaling argument and the parallel-axis theorem. We will illustrate this technique by finding I for a stick (Example 7, above). Other applications can be found in Problems 4 and 5.

In the present example, the basic trick is to compare I for a stick of length L with I for a stick of length $2L$. A simple scaling argument shows the latter is eight times the former. This is true because the integral $\int x^2\, dm = \int x^2 \rho\, dx$ has three powers of x in it. So a change of variables, $y = 2x$, brings in a factor of $2^3 = 8$. In other words, if we imagine expanding the smaller stick to create the larger one, then a corresponding piece will now be twice as far from the axis, and also twice as massive.

The technique is most easily illustrated with pictures. If we denote a moment of inertia of an object by a picture of the object (with a dot signifying the axis), then we have:



The first line comes from the scaling argument, the second is obvious (moments of inertia simply add; the left-hand side is two copies of the right-hand side, attached at the pivot), and the third comes from the parallel-axis theorem. Equating the right-hand sides of the first two gives

 = 4 

Plugging this expression for into the third equation gives the desired result,

 $= \frac{1}{12}ML^2$

Note that sooner or later you must use real live numbers (which enter here through the parallel axis theorem). Using only scaling arguments isn't enough, because they only provide linear equations homogeneous in the I's, and therefore give no way to pick up the proper dimensions.

Once you've mastered this trick and applied it to the fractal objects in Problem 5, you can impress your friends by saying that you can "use scaling arguments, along with the parallel-axis theorem, to calculate moments of inertia of objects with fractal dimension." (And you never know when that might come in handy.)

7.3 Torque

We will now show that (under certain conditions, stated below) the rate of change of angular momentum is equal to a certain quantity, $\boldsymbol{\tau}$, which we call the *torque*. That is, $\boldsymbol{\tau} = d\mathbf{L}/dt$. This is the rotational analog of our old friend $\mathbf{F} = d\mathbf{p}/dt$ involving linear momentum. The basic idea here is straightforward, but there are two subtle issues. One deals with internal forces within a collection of particles. The other deals with origins (the points relative to which the angular momentum is calculated) that are not fixed. To keep things straight, we'll prove the general theorem by dealing with three increasingly complicated situations.

Our derivation of $\boldsymbol{\tau} = d\mathbf{L}/dt$ here holds for completely general motion; we can take the result and use it in the following chapter, too. If you wish, you can construct a more specific proof of $\boldsymbol{\tau} = d\mathbf{L}/dt$ for the special case of a pancake object in the x-y plane. But since the general proof is no more difficult, we'll present it here in this chapter and get it over with.

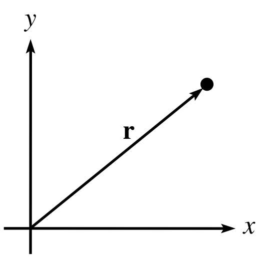


Figure 7.13

7.3.1 Point mass, fixed origin

Consider a point mass at position $\mathbf{r}$ relative to a fixed origin (see Fig. 7.13). The time derivative of the angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, is

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) \\ &= \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} \\ &= \mathbf{v} \times (m\mathbf{v}) + \mathbf{r} \times \mathbf{F} \\ &= 0 + \mathbf{r} \times \mathbf{F}, \end{aligned} \tag{7.32}$$

where $\mathbf{F}$ is the force acting on the particle. (This is the same proof as in Theorem 6.1, except that here we are considering an arbitrary force instead of a central one.) Therefore, if we define the *torque* on the particle as

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F}, \tag{7.33}$$

then we have

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}. \tag{7.34}$$

7.3.2 Extended mass, fixed origin

In an extended object, there are internal forces acting on the various pieces of the object, in addition to whatever external forces exist. For example, the external force on a given atom in a body might come from gravity, while the internal forces come from the adjacent atoms. How do we deal with these different types of forces?

In what follows, we will deal only with internal forces that are central forces, that is, where the force between two objects is directed along the line joining them. This is a valid assumption for the pushing and pulling forces between atoms in a solid. (It isn't valid, for example, when dealing with magnetic forces. But we won't be interested in such things here.) We will invoke Newton's third law, which says that the force that particle 1 applies to particle 2 is equal and opposite to the force that particle 2 applies to particle 1.

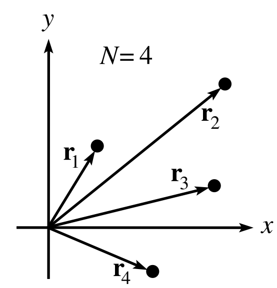


Figure 7.14

For concreteness, let us assume that we have a collection of N discrete particles labeled by the index i (see Fig. 7.14). (In the continuous case, we'd need to replace the following sums with integrals.) Then the total angular momentum of the system is

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i. \tag{7.35}$$

The force acting on each particle is $\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}} = d\mathbf{p}_i/dt$. Therefore,

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}\sum_i \mathbf{r}_i \times \mathbf{p}_i \\
&= \sum_i \frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i + \sum_i \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} \\
&= \sum_i \mathbf{v}_i \times (m\mathbf{v}_i) + \sum_i \mathbf{r}_i \times (\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}}) \\
&= 0 + \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \sum_i \boldsymbol{\tau}_i^{\text{ext}}.
\end{aligned} \tag{7.36}$$

The last line follows because $\mathbf{v}_i \times \mathbf{v}_i = 0$, and also $\sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{int}} = 0$, as you can show in Problem 15. (This is fairly obvious. It basically says that a rigid object with no external forces won't spontaneously start rotating.) Note that the right-hand side involves the *total* torque acting on the body, which may come from forces acting at many different points.

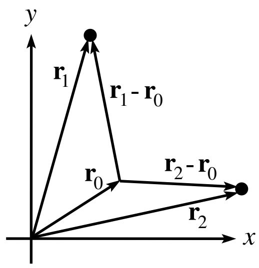


Figure 7.15

7.3.3 Extended mass, non-fixed origin

Let the position of the origin be $\mathbf{r}_0$ (see Fig. 7.15). Let the positions of the particles be $\mathbf{r}_i$ ($\mathbf{r}_0$, $\mathbf{r}_i$, and all other vectors below are measured with respect to a given fixed coordinate system). Then the total angular momentum of the system, relative to the (possibly moving) origin $\mathbf{r}_0$, is

$$\mathbf{L} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0). \tag{7.37}$$

Therefore,

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \frac{d}{dt}\left(\sum_i (\mathbf{r}_i - \mathbf{r}_0) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0)\right) \\ &= \sum_i (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0) + \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times m_i(\ddot{\mathbf{r}}_i - \ddot{\mathbf{r}}_0) \\ &= 0 + \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times (\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}} - m_i\ddot{\mathbf{r}}_0), \end{aligned} \tag{7.38}$$

because $m_i\ddot{\mathbf{r}}_i$ is the net force (namely $\mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}}$) acting on the ith particle. But a quick corollary of Problem 15 is that the term involving $\mathbf{F}_i^{\text{int}}$ vanishes (show this). And since $\sum m_i\mathbf{r}_i = M\mathbf{R}$ (where $M = \sum m_i$ is the total mass, and $\mathbf{R}$ is the position of the center-of-mass), we have

$$\frac{d\mathbf{L}}{dt} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_i^{\text{ext}} - M(\mathbf{R} - \mathbf{r}_0) \times \ddot{\mathbf{r}}_0. \tag{7.39}$$

The first term is the external torque, relative to the origin $\mathbf{r}_0$. The second term is something we wish would go away. And indeed, it usually does. It vanishes if any of the following three conditions is satisfied.

1. $\mathbf{R} = \mathbf{r}_0$. That is, the origin is the CM.
2. $\ddot{\mathbf{r}}_0 = 0$. That is, the origin is not accelerating.
3. $(\mathbf{R} - \mathbf{r}_0)$ is parallel to $\ddot{\mathbf{r}}_0$. This condition is rarely invoked.

If any of these conditions is satisfied, then we are free to write

$$\boxed{\frac{d\mathbf{L}}{dt} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_i^{\text{ext}} \equiv \sum_i \boldsymbol{\tau}_i^{\text{ext}}}. \tag{7.40}$$

That is, we can equate the total torque with the rate of change of the total angular momentum. An immediate corollary of this result is:

Corollary 7.3 *If the total torque on a system is zero, then its angular momentum is conserved. In particular, the angular momentum of an isolated system (one that is subject to no external forces) is conserved.*

In the present chapter, we are dealing only with cases where $\hat{\mathbf{L}}$ is constant. Therefore, $d\mathbf{L}/dt = d(L\hat{\mathbf{L}})/dt = (dL/dt)\hat{\mathbf{L}}$. But $L = I\omega$, so $dL/dt = I\dot{\omega} \equiv I\alpha$. Taking the magnitude of each side of eq. (7.40) therefore gives

$$\tau = I\alpha, \tag{7.41}$$

if $\hat{\mathbf{L}}$ is constant.

Invariably, we will calculate angular momentum and torque around either a fixed point or the CM. These are "safe" origins, in the sense that eq. (7.40) holds. As long as you vow to always use one of these safe origins, you can simply apply eq. (7.40) and basically ignore its derivation.

REMARK: There is one common situation where the third condition above is applicable. Consider a wheel rolling on the ground. Mark a point on the rim. At the instant this point in in contact with the ground, it is a valid choice for the origin. This is true because $(\mathbf{R}-\mathbf{r}_0)$ points vertically. And $\ddot{\mathbf{r}}_0$ also points vertically. (A point on a rolling wheel traces out a cycloid. Right before the point hits the ground, it is moving straight downward; right after it hits the ground, it is moving straight upward.) But never mind, it's still a good idea to pick your origin to be the CM or a fixed point, even if the third condition holds. ♣

For conditions that number but three,
We say, "Torque is dL by dt."
But though they're all true,
I'll stick to just two;
It's CM's and fixed points for me.

7.4 Angular impulse

In Section 4.5.1, we defined the *impulse*, $\mathcal{I}$, to be the time integral of the force applied to an object (which is the net change in linear momentum). That is,

$$\mathcal{I} \equiv \int_{t_1}^{t_2} \mathbf{F}(t)\, dt = \Delta\mathbf{p}. \tag{7.42}$$

We now define the *angular impulse*, $\mathcal{I}_\theta$, to be the time integral of the torque applied to an object (which is the net change in angular momentum). That is,

$$\mathcal{I}_\theta \equiv \int_{t_1}^{t_2} \boldsymbol{\tau}(t)\, dt = \Delta\mathbf{L}. \tag{7.43}$$

These are just definitions, devoid of any content. The place where the physics comes in is the following. Consider a situation where $\mathbf{F}(t)$ is always applied at the same position relative to the origin around which $\boldsymbol{\tau}(t)$ is calculated. Let this position be $\mathbf{R}$. Then we have $\boldsymbol{\tau}(t) = \mathbf{R} \times \mathbf{F}(t)$. Plugging this into eq. (7.43), and taking the constant $\mathbf{R}$ outside the integral, gives $\mathcal{I}_\theta = \mathbf{R} \times \mathcal{I}$. That is,

$$\Delta\mathbf{L} = \mathbf{R} \times (\Delta\mathbf{p}) \qquad \text{(for } \mathbf{F}(t) \text{ applied at one position).} \tag{7.44}$$

This is a very useful result. It deals with the net changes in $\mathbf{L}$ and $\mathbf{p}$, and not with their changes at any particular instant. Hence, even if the magnitude of $\mathbf{F}$ is changing in some arbitrary manner as time goes by, and we have no idea what $\Delta\mathbf{p}$ and $\Delta\mathbf{L}$ are, eq. (7.44) is still true. And note that Eq. (7.44) holds for general motion, so we can apply it in the next chapter, too.

In many cases, you don't have to worry about the cross product in eq. (7.44), because the lever arm, $\mathbf{R}$, is perpendicular to the change in momentum, $\Delta\mathbf{p}$. Also, in many cases the object starts at rest, so you don't have to bother with the Δ's. The following example is a classic application of this type of angular impulse situation.

Example (Striking a stick): A stick, initially at rest, is struck with a hammer. The blow is made is perpendicular to the stick, at one end. Let the stick have mass m and length ℓ. Let the blow occur quickly, so that the stick doesn't move much while the hammer is in contact. If the CM of the stick ends up moving at speed v, what are the speeds of the ends, right after the blow?

Solution: We have no idea exactly what $F(t)$ looks like, or for how long it is applied, but we do know from eq. (7.44) that $\Delta L = (\ell/2)\Delta p$, where L is calculated relative to the CM (so that the lever arm is $\ell/2$). Therefore, $(m\ell^2/12)\omega = (\ell/2)mv$. Hence, the final v and ω are related by $\omega = 6v/\ell$.

The speeds of the ends are obtained by adding (or subtracting) the rotational motion to the CM's translational motion. The rotational speeds of the ends are $\pm\omega(\ell/2) = \pm(6v/\ell)(\ell/2) = 3v$. Therefore, the end that was hit moves with speed $v + 3v = 4v$, and the other end moves with speed $v - 3v = -2v$ (that is, backwards).

What L was, he just couldn't tell.
And p? He was clueless as well.
But despite his distress,
He wrote down the right guess
For their quotient: the lever-arm's ℓ.

Impulse is also useful for "collisions" that occur over extended times (see, for example, Problem 17).

7.5 Exercises

Section 7.3: Torque

1. **Maximum frequency** $*$

   A pendulum is made of a uniform stick of length ℓ. A pivot is place somewhere along the stick, which is allowed to swing in a vertical plane. Where should the pivot be placed on the stick so that the frequency of (small) oscillations is maximum?

Section 7.4: Impulse

2. **Not hitting the pole** $*$

   A (possibly non-uniform) stick of mass m and length ℓ lies on frictionless ice. Its midpoint (which is also its CM) touches a thin pole sticking out of the ice. One end of the stick is struck with a quick blow perpendicular to the stick, so that the CM moves away from the pole. What is the minimum value of the stick's moment of inertia which allows the stick to not hit the pole?

3. **Up, down, and twisting** $**$

   A uniform stick is held horizontally and then released. At the same instant, one end is struck with a quick upwards blow. If the stick ends up horizontal when it returns to its original height, what are the possible values for the maximum height to which the stick's center rises?

4. **Repetitive bouncing** $*$

   Using the result of Problem 18, what must the relation be between v_x and ω, so that a superball will continually bounce back and forth between the same two points of contact on the ground?

5. **Bouncing under a table** $**$

   You throw a superball so that it bounces off the floor, then off the underside of a table, then off the floor again. What must the initial relation between v_x and $R\omega$ be, so that the ball returns to your hand?[6] (Use the result of Problem 18, and modifications thereof.)

6. **Bouncing between walls** $**$

   A stick of length ℓ slides on frictionless ice. It bounces between two walls, a distance L apart, in such a way that only one end touches the walls, and the stick hits the walls at an angle θ each time. What should θ be, in terms of L and ℓ? What does the situation look like in the limit $L \ll \ell$?

   What should θ be, in terms of L and ℓ, if the stick makes an additional n full revolutions between the walls? Is there a minimum value of L/ℓ for this to be possible?

[6] You are strongly encouraged to bounce a ball in such a manner and have it magically come back to your hand. It turns out that the required value of ω is rather small, so a natural throw with $\omega \approx 0$ will essentially get the job done.

7.6 Problems

Section 7.1: Pancake object in x-y plane

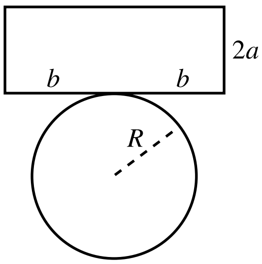


Figure 7.16

1. **Leaning rectangle** ***

   A rectangle of height $2a$ and width $2b$ rests on top of a fixed cylinder of radius R (see Fig. 7.16). The moment of inertia of the rectangle around its center is I. The rectangle is given an infinitesimal kick, and then 'rolls' on the cylinder without slipping. Find the equation of motion for the tilting angle of the rectangle. Under what conditions will it fall off the cylinder, and under what conditions will oscillate back and forth? Find the frequency of these small oscillations.

2. **Leaving the sphere** **

   A small ball with moment of inertia $\eta m r^2$ rests on top of a sphere. There is friction between the ball and sphere. The ball is given an infinitesimal kick and rolls downward without slipping. At what point does it lose contact with the sphere? (Assume that r is much less than the radius of the sphere.) How does your answer change if the size of the ball is comparable to, or larger than, the size of the sphere? (Assume that the sphere is fixed.)

   You may want to solve Problem 4.3 first, if you haven't already done so.

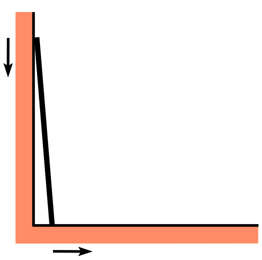


Figure 7.17

3. **Sliding ladder** ***

   A ladder of length ℓ and uniform mass density per unit length leans against a frictionless wall. The ground is also frictionless. The ladder is initially held motionless, with its bottom end an infinitesimal distance from the wall. The ladder is then released, whereupon the bottom end slides away from the wall, and the top end slides down the wall (see Fig. 7.17).

   A long time after the ladder is released, what is the horizontal component of the velocity of its center of mass?

Section 7.2: Calculating moments of inertia

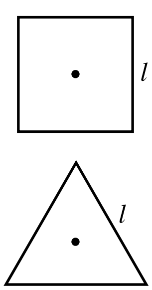


Figure 7.18

4. **Slick calculations of I** **

   In the spirit of section 7.2.2, find the moments of inertia of the following objects (see Fig. 7.18).

   (a) A uniform square of mass m and side ℓ (axis through center, perpendicular to plane).

   (b) A uniform equilateral triangle of mass m and side ℓ (axis through center, perpendicular to plane).

5. **Slick calculations of I for fractal objects** ***

   In the spirit of section 7.2.2, find the moments of inertia of the following fractal objects. (Be careful how the mass scales.)

(a) Take a stick of length ℓ, and remove the middle third. Then remove the middle third from each of the remaining two pieces. Then remove the middle third from each of the remaining four pieces, and so on, forever. Let the final object have mass m (axis through center, perpendicular to stick; see Fig. 7.19).[7]

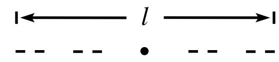


Figure 7.19

(b) Take a square of side ℓ, and remove the 'middle' square (1/9 of the area). Then remove the 'middle' square from each of the remaining eight squares, and so on, forever. Let the final object have mass m (axis through center, perpendicular to plane; see Fig. 7.20).

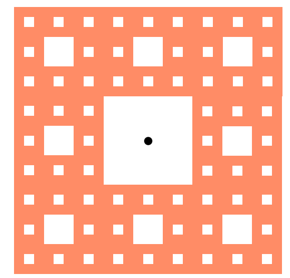


Figure 7.20

(c) Take an equilateral triangle of side ℓ, and remove the 'middle' triangle (1/4 of the area). Then remove the 'middle' triangle from each of the remaining three triangles, and so on, forever. Let the final object have mass m (axis through center, perpendicular to plane; Fig. 7.21).

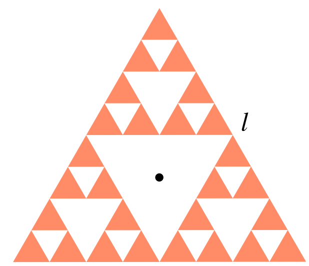


Figure 7.21

6. **Minimum I**

A moldable blob of matter of mass M is to be situated between the planes $z = 0$ and $z = 1$ (see Fig. 7.22). The goal is to have the moment of inertia around the z-axis be as small as possible. What shape should the blob take?

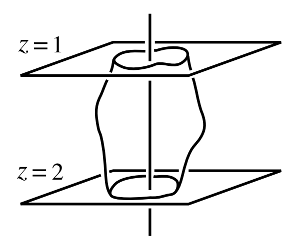


Figure 7.22

Section 7.3: Torque

7. **Removing a support**

(a) A rod of length ℓ and mass m rests on supports at its ends. The right support is quickly removed (see Fig. 7.23). What is the force on the left support immediately thereafter?

(b) A rod of length $2r$ and moment of inertia $\eta m r^2$ (where η is a numerical constant) rests on top of two supports, each of which is a distance d away from the center. The right support is quickly removed (see Fig. 7.23). What is the force on the left support immediately thereafter?

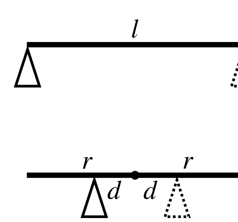


Figure 7.23

[7]This object is the Cantor set, for those who like such things. It has no length, so the density of the remaining mass is infinite. If you suddenly develop an aversion to point masses with infinite density, simply imagine the above iteration being carried out only, say, a million times.

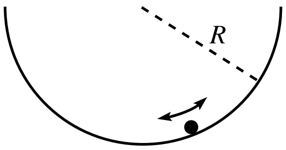


Figure 7.24

8. **Oscillating ball** $*$

A small ball (with uniform density) of radius r rolls without slipping near the bottom of a fixed cylinder of radius R (see Fig. 7.24). What is the frequency of small oscillations about the bottom? (Assume $r \ll R$.)

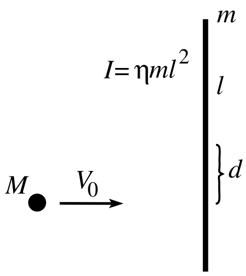


Figure 7.25

9. **Ball hitting stick** $*$

A ball of mass M hits a stick with moment of inertia $I = \eta m\ell^2$. The ball is initially traveling with velocity V_0, perpendicular to the stick. The ball strikes the stick at a distance d from its center (see Fig. 7.25). The collision is elastic.

Find the resulting translational and rotational speeds of the stick, and also the resulting speed of the ball.

10. **A ball and stick theorem** $*$

A ball of mass M hits a stick with moment of inertia I. The ball is initially traveling with velocity V_0, perpendicular to the stick. The ball strikes the stick at a distance d from its center (see Fig. 7.25). The collision is elastic.

Prove that the relative speed of the ball and the point of contact on the stick is the same before and immediately after the collision. (This theorem is analogous to the 'relative speed' theorem of two balls.)

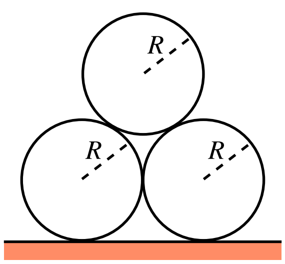


Figure 7.26

11. **A triangle of circles** $***$

Three circular objects with moments of inertia $I = \eta MR^2$ are situated in a triangle as in Fig. 7.26. Find the initial downward acceleration of the top circle, if:

(a) There is friction between the bottom two circles and the ground (so they roll without slipping), but there is no friction between any of the circles.

(b) There is no friction between the bottom two circles and the ground, but there is friction between the circles.

Which case has a larger acceleration?

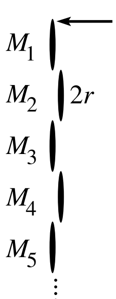


Figure 7.27

12. **Lots of sticks** $***$

This problem deals with rigid 'stick-like' objects of length $2r$, masses M_i, and moments of inertia $\eta M_i r^2$. The center-of-mass of each stick is located at the center of the stick. (All the sticks have the same r and η. Only the masses differ.) Assume $M_1 \gg M_2 \gg M_3 \gg \cdots$.

The sticks are placed on a horizontal frictionless surface, as shown in Fig. 7.27. The ends overlap a negligible distance, and the ends are a negligible distance apart.

The first (heaviest) stick is given an instantaneous blow (as shown) which causes it to translate and rotate. (The blow comes from the side of stick #1 on which stick #2 lies; the right side, as shown in the figure.) The first stick will strike the second stick, which will then strike the third stick, and so on. Assume all collisions among the sticks are elastic.

Depending on the size of η, the speed of the nth stick will either (1) approach zero, (2) approach infinity, or (3) be independent of n, as $n \to \infty$.

What is the special value of η corresponding to the third of these three scenarios? Give an example of a stick having this value of η.

(You may work in the approximation where M_1 is infinitely heavier than M_2, which is infinitely heavier than M_3, etc.)

13. **Falling stick** $*$

A massless stick of length b has one end attached to a pivot and the other end glued perpendicularly to the middle of a stick of mass m and length ℓ.

(a) If the two sticks are held in a horizontal plane (see Fig. 7.28) and then released, what is the initial acceleration of the CM?

(b) If the two sticks are held in a vertical plane (see Fig. 7.28) and then released, what is the initial acceleration of the CM?

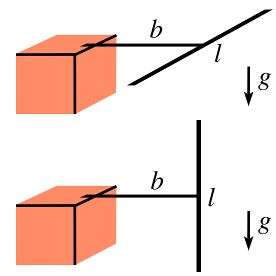


Figure 7.28

14. **Falling Chimney** $****$

A chimney initially stands upright. It is given a tiny kick, so that it topples over. At what point along its length is it most likely to break?

In doing this problem, work with the following two-dimensional simplified model of a chimney. Assume the chimney consists of boards stacked on top of each other; and each board is attached to the two adjacent ones with strings at each end (see Fig. 7.29). Assume that the boards are slightly thicker at their ends, so that they only touch each other at their endpoints. The goal is to find where the string has the maximum tension.

(In doing this problem, you may work in the approximation where the width of the chimney is very small compared to its height.)

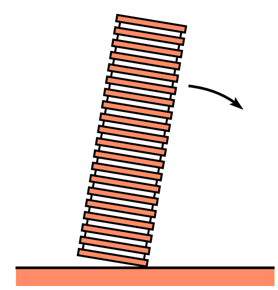


Figure 7.29

15. **Zero torque from internal forces** $**$

Given a collection of particles with positions $\mathbf{r}_i$, let the force on the ith particle, due to all the others, be $\mathbf{F}_i^{\text{int}}$. Assuming that the force between any two particles is a central force, use Newton's third law to show $\sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{int}} = 0$.

16. **Lengthening the string** $**$

A mass hangs from a string and swings around in a circle, as shown in Fig. 7.30. The length of the string is very slowly increased (or decreased). Let θ, ℓ, r, and h be defined as in the figure.

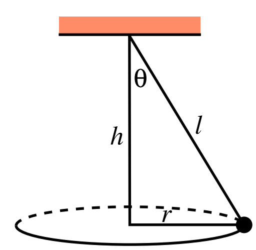


Figure 7.30

(a) Assuming θ is very small, how does r depend on ℓ?

(b) Assuming θ is very close to $\pi/2$, how does h depend on ℓ?

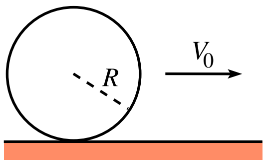


Figure 7.31

Section 7.4: Impulse

17. **Sliding to rolling** $**$

A ball initially slides, without rotating, on a horizontal surface with friction (see Fig. 7.31). The initial speed of the ball is V_0, and the moment of inertia about its center is $I = \eta m R^2$.

(a) Without knowing anything about how the friction force depends on position, find the speed of the ball when it begins to roll without slipping. Also, find the kinetic energy lost while sliding.

(b) Now consider the special case where the coefficient of sliding friction is μ, independent of position. At what time, and at what distance, does the ball begin to roll without slipping?

Verify that the work done by friction equals the loss in energy calculated in part (a) (be careful on this).

18. **The superball** $**$

A ball with radius R is thrown in the plane of the paper (the x-y plane), while also spinning around the axis perpendicular to the page. The ball bounces off the floor. Assuming that the collision is elastic, show that the v'_x and ω' after the bounce are related to the v_x and ω before the bounce by

$$\begin{pmatrix} v'_x \\ R\omega' \end{pmatrix} = \begin{pmatrix} 3/7 & 4/7 \\ 10/7 & -3/7 \end{pmatrix} \begin{pmatrix} v_x \\ R\omega \end{pmatrix}, \tag{7.45}$$

where our convention is that positive v_x is to the right, and positive ω is clockwise.[8]

19. **Many bounces** $*$

Using the result of Problem 18, describe what happens over the course of many superball bounces.

20. **Rolling over a bump** $**$

A ball with radius R (and uniform density) rolls without slipping on the ground. It encounters a step of height h and rolls up over it. Assume that the ball sticks to the corner of the step briefly (until the center of the ball is directly above the corner). And assume that the ball does not slip with respect to the corner.

Show that the minimum initial speed, V_0, required for the ball to climb over the step, is given by

$$V_0 \geq \frac{R\sqrt{14gh/5}}{7R/5 - h}. \tag{7.46}$$

[8]Assume that there is no distortion in the ball during the bounce, which means that the forces in the x- and y-directions are independent, which then means that the kinetic energies associated with the x- and y-motions are separately conserved.

7.7 Solutions

1. **Leaning rectangle**

When the rectangle has rotated through an angle θ, the position of its CM is (relative to the center of the cylinder)

$$(x, y) = R(\sin\theta, \cos\theta) + R\theta(-\cos\theta, \sin\theta) + a(\sin\theta, \cos\theta), \quad (7.47)$$

where we have added up the distances along the three shaded triangles in Fig. 7.32 (note that the contact point has moved a distance $R\theta$ along the rectangle).

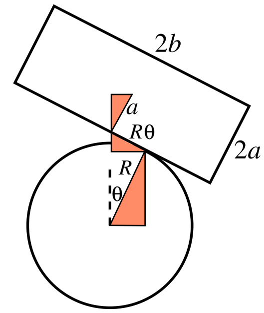


Figure 7.32

We'll use the Lagrangian method to find the equation of motion and the frequency of small oscillations. Using eq. (7.47), the square of the speed of the CM is

$$v^2 = \dot{x}^2 + \dot{y}^2 = (a^2 + R^2\theta^2)\dot{\theta}^2. \quad (7.48)$$

(There's an easy way to see this clean result. The CM instantaneously rotates around the contact point with angular speed $\dot{\theta}$, and from Fig. 7.32 the distance to the contact point is $\sqrt{a^2 + R^2\theta^2}$.)

The Lagrangian is

$$\mathcal{L} = T - V = \frac{1}{2}M(a^2 + R^2\theta^2)\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 - Mg\Big((R + a)\cos\theta + R\theta\sin\theta\Big). \quad (7.49)$$

The equation of motion is

$$(Ma^2 + MR^2\theta^2 + I)\ddot{\theta} + MR^2\theta\dot{\theta}^2 = Mga\sin\theta - MgR\theta\cos\theta. \quad (7.50)$$

Consider small oscillations. Using the small-angle approximations, and keeping terms only to first order in θ, we obtain

$$(Ma^2 + I)\ddot{\theta} + Mg(R - a)\theta = 0. \quad (7.51)$$

Therefore, oscillatory motion occurs for $a < R$ (note that this is independent of b). The frequency of small oscillations is

$$\omega = \sqrt{\frac{Mg(R - a)}{Ma^2 + I}}. \quad (7.52)$$

Some special cases: If $I = 0$ (i.e., all the mass is located at the CM), we have $\omega = \sqrt{g(R - a)/a^2}$. If the rectangle is a uniform horizontal stick, so that $a \ll R$, $a \ll b$, and $I \approx Mb^2/3$, we have $\omega \approx \sqrt{3gR/b^2}$. If the rectangle is a vertical stick (satisfying $a < R$), so that $b \ll a$ and $I \approx Ma^2/3$, we have $\omega \approx \sqrt{3g(R - a)/4a^2}$. If in addition $a \ll R$, then $\omega \approx \sqrt{3gR/4a^2}$.

Remarks:

(a) Without doing much work, there are two ways that we can determine the condition under which there is oscillatory motion. The first is to look at the height of the CM. Using small-angle approximations in eq. (7.47), the height of the CM is $y \approx (R + a) + (R - a)\theta^2/2$. Therefore, if $a < R$, the potential increases with θ, so the rectangle wants to decrease its θ and fall back down to the middle. If $a > R$, the potential decreases with θ, so the rectangle wants to keep increasing its θ, and thus falls off the cylinder.

The second way is to look at the horizontal positions of the CM and the contact point. Small-angle approximations in eq. (7.47) show that the former equals $a\theta$ and the latter equals $R\theta$. Therefore, if $a < R$ then the CM is to the left of the contact point, so the torque from gravity makes θ decrease, and the motion is stable. If $a > R$ then the torque from gravity makes θ increase, and the motion is unstable.

(b) The small-angle equation of motion, eq. (7.51), can also be derived using $\tau = dL/dt$, using the instantaneous contact point, P, on the rectangle as the origin around which we calculate τ and L. (From section (7.3.3), we know that it is legal to use this point when $\theta = 0$.)

However, point P cannot be used as the origin to use $\tau = dL/dt$ to calculate the exact equation of motion, eq. (7.50), because for $\theta \neq 0$ the third condition in section (7.3.3) does not hold.

It is possible to use the CM as the origin for $\tau = dL/dt$, but the calculation is rather messy. ♣

2. **Leaving the sphere**

In this setup, the ball still leaves the sphere when the normal force becomes zero; so eq. (4.91) is still applicable, from the solution to Problem 4.3. The only change comes in the calculation of v. The ball has rotational energy, so conservation of energy gives $mgR(1-\cos\theta) = mv^2/2 + I\omega^2/2 = mv^2/2 + \eta m r^2\omega^2/2$. Using $r\omega = v$, we have

$$v = \sqrt{\frac{2gR(1-\cos\theta)}{1+\eta}}\,. \tag{7.53}$$

Plugging this into eq. (4.91), we see that the ball leaves the sphere when

$$\cos\theta = \frac{2}{3+\eta}. \tag{7.54}$$

For, $\eta = 0$, this is $2/3$, of course. For a uniform ball with $\eta = 2/5$, we have $\cos\theta = 10/17$, so $\theta \approx 54^\circ$. For, $\eta \to \infty$, we have $\cos\theta \to 0$, so $\theta \approx 90^\circ$ (v will be very small, because most of the energy will take the form of rotational energy.)

If the size of the ball is comparable to, or bigger than, the size of the sphere, we have to take into account the fact that the CM of the ball does not move along a circle of radius R. It moves along a circle of radius $R + r$. So eq. (4.91) becomes

$$\frac{mv^2}{R+r} = mg\cos\theta. \tag{7.55}$$

Also, the conservation-of-energy equation takes the form $mg(R+r)(1-\cos\theta) = mv^2/2 + \eta m r^2\omega^2/2$. But $r\omega$ still equals v (prove this). So we have the same equations as above, except that R is replaced everywhere by $R + r$. But R didn't appear in the original answer, so the answer is unchanged.

Remark: Note that the method of the second solution to Problem 4.3 will *not* work in this problem, because there *is* a force available to make v_x decrease, namely the friction force. And indeed, v_x does decrease before the rolling ball leaves the sphere. (The v in this problem is simply $1/\sqrt{1+\eta}$ times the v in Problem 4.3, so the maximum v_x is still achieved at $\cos\theta = 2/3$, and the angle in eq. (7.54) is larger than this.) ♣

3. **Sliding ladder**

The key to this problem is the fact that the ladder will lose contact with the wall before it hits the ground. The first thing we must do is calculate exactly where this loss of contact occurs.

Let $r = \ell/2$, for convenience. It is easy to see that while the ladder is in contact with the wall, the CM of the ladder will move in a circle of radius r. (The median to the hypotenuse of a right triangle has half the length of the hypotenuse.) Let θ be the angle between the wall and the radius from the corner to the CM of the ladder; see Fig. 7.33. (This is also the angle between the ladder and the wall.)

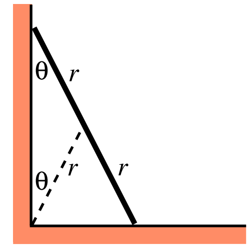


Figure 7.33

We will solve the problem by assuming that the CM always moves in a circle, and then determining the point at which the horizontal CM speed starts to decrease (i.e., the point at which the normal force from the wall becomes negative, which it of course can't do).

By conservation of energy, the kinetic energy of the ladder is equal to the loss in potential energy, which is $mgr(1-\cos\theta)$, where θ is defined above. This kinetic energy may be broken up into the CM translational energy plus the rotation energy. The CM translational energy is simply $mr^2\dot{\theta}^2/2$ (since the CM travels in a circle). The rotational energy is $I\dot{\theta}^2/2$. (The same $\dot{\theta}$ applies here as in the CM translational motion, because θ is the angle between the ladder and the vertical.) Letting $I \equiv \eta mr^2$, to be general ($\eta = 1/3$ for our ladder), we have, by conservation of energy, $(1+\eta)mr^2\dot{\theta}^2/2 = mgr(1-\cos\theta)$. Therefore, the speed of the CM, $v = r\dot{\theta}$, is

$$v = \sqrt{\frac{2gr}{1+\eta}}\sqrt{(1-\cos\theta)}\,. \tag{7.56}$$

The horizontal speed is therefore

$$v_x = \sqrt{\frac{2gr}{1+\eta}}\sqrt{(1-\cos\theta)}\cos\theta. \tag{7.57}$$

Taking the derivative of $\sqrt{(1-\cos\theta)}\cos\theta$, we see that the speed is maximum at $\cos\theta = 2/3$. (This is independent of η.)

Therefore the ladder loses contact with the wall when

$$\cos\theta = 2/3. \tag{7.58}$$

Using this value of θ in eq. (7.57) gives a horizontal speed of (letting $\eta = 1/3$)

$$v_x = \frac{\sqrt{2gr}}{3} \equiv \frac{\sqrt{g\ell}}{3}. \tag{7.59}$$

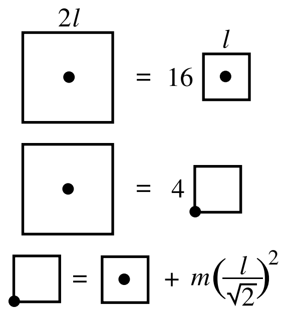
This is the horizontal speed just after the ladder loses contact with the wall, and thus is the horizontal speed from then on, because the floor exerts no horizontal force.

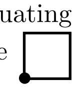
You are encouraged to compare various aspects of this problem with those in Problem 2.

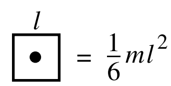
4. **Slick calculations of I**

   (a) We claim that the I for a square of side 2ℓ is 16 times the I for a square of side ℓ (where the axes pass through any two corresponding points). The factor of 16 comes in part from the fact that dm goes like the area, which is proportional to length squared. So the corresponding dm's are increased by a factor of 4. There are therefore four powers of 2 in the integral $\int r^2\,dm = \int r^2\,dx\,dy$.

   With pictures, we have:

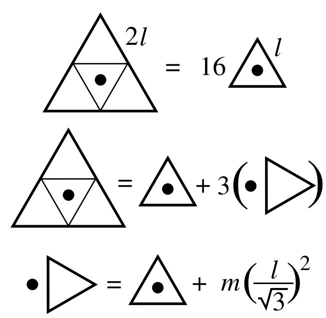


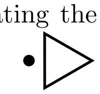
The first line comes from the scaling argument, the second is obvious (moments of inertia add), and the third comes from the parallel axis theorem. Equating the right-hand sides of the first two, and then using the third to eliminate  gives

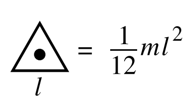


This agrees with the result of example 12 in section 7.2.1, with $a = b = \ell$.

(b) This is again a two-dimensional object, so the I for a triangle of side 2ℓ is 16 times the I for a triangle of side ℓ (where the axes pass through any two corresponding points).
Again, with pictures, we have:



The first line comes from the scaling argument, the second is obvious, and the third comes from the parallel axis theorem. Equating the right-hand sides of the first two, and then using the third to eliminate  gives

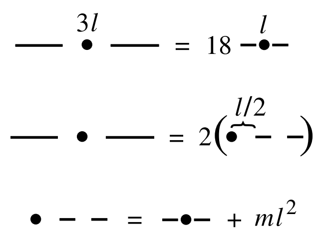


This agrees with the result of example 11 in section 7.2.1, with $N = 3$ (because the 'radius', R, used in that example equals $\ell/\sqrt{3}$).

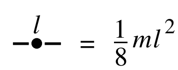
5. **Slick calculations of I for fractal objects**

(a) The scaling argument here is a little trickier than that in section 7.2.2. Our object is self-similar to an object 3 times as big, so let's increase the length by a factor of 3 and see what happens to I. In the integral $\int x^2 dm$, the x's pick up a factor of 3, so this gives a factor of 9. But what happens to the dm? Well, tripling the size of our object increases its mass by a factor of 2 (since the new object is simply made up of two of the smaller ones, plus some empty space in the middle), so the dm picks up a factor of 2. Thus the I for an object of length 3ℓ is 18 times the I for an object of length ℓ (where the axes pass through any two corresponding points).

With pictures, we have (the following symbols denote our fractal object):



The first line comes from the scaling argument, the second is obvious (moments of inertia add), and the third comes from the parallel axis theorem. Equating the right-hand sides of the first two, and then using the third to eliminate • – – gives

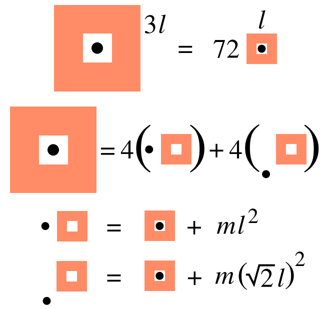


This is larger than the I for a uniform stick ($m\ell^2/12$), because the mass is generally further away from the center.

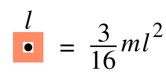
REMARK: When we increase the length of our object by a factor of 3 here, the factor of 2 in the dm is between the factor of 1 relevant to a zero-dimensional object, and the factor of 3 relevant to a one-dimensional object. So in some sense our object has a dimension between 0 and 1. It is reasonable to define the dimension, d, of an object as the number for which r^d is the increase in 'volume' when the dimensions are increased by a factor of r. In our example, we have $3^d = 2$, so $d = \log_3 2 \approx 0.63$. ♣

(b) Again, the mass scales in a strange way. Let's increase the dimensions of our object by a factor of 3 and see what happens to I. In the integral $\int x^2 dm$, the x's pick up a factor of 3, so this gives a factor of 9. But what happens to the dm? Tripling the size of our object increases its mass by a factor of 8 (since the new object is made up of eight of the smaller ones, plus an empty square in the middle), so the dm picks up a factor of 8. Thus the I for an object of side 3ℓ is 72 times the I for an object of side ℓ (where the axes pass through any two corresponding points).

Again, with pictures, we have (the following symbols denote our fractal object):



The first line comes from the scaling argument, the second is obvious, and the third and fourth come from the parallel axis theorem. Equating the right-hand sides of the first two, and then using the third and fourth to eliminate  and  gives

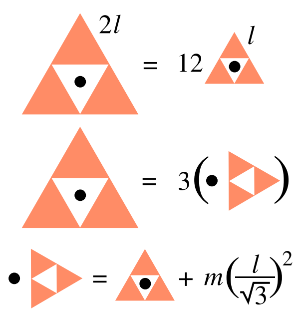


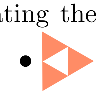
This is larger than the I for the uniform square in problem 4, because the mass is generally further away from the center.

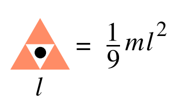
Note: Increasing the size of our object by a factor of 3 increases the 'volume' by a factor of 8. So the dimension is given by $3^d = 8$; hence $d = \log_3 8 \approx 1.89$.

(c) Again, the mass scales in a strange way. Let's increase the dimensions of our object by a factor of 2 and see what happens to I. In the integral $\int x^2 dm$, the x's pick up a factor of 2, so this gives a factor of 4. But what happens to the dm? Doubling the size of our object increases its mass by a factor of 3 (since the new object is simply made up of three of the smaller ones, plus an empty triangle in the middle), so the dm picks up a factor of 3. Thus the I for an object of side 2ℓ is 12 times the I for an object of side ℓ (where the axes pass through any two corresponding points).

Again, with pictures, we have (the following symbols denote our fractal object):



The first line comes from the scaling argument, the second is obvious, and the third comes from the parallel axis theorem. Equating the right-hand sides of the first two, and then using the third to eliminate  gives



This is larger than the I for the uniform triangle in problem 4, because the mass is generally further away from the center.

Note: Increasing the size of our object by a factor of 2 increases the 'volume' by a factor of 3. So the dimension is given by $2^d = 3$; hence $d = \log_2 3 \approx 1.58$.

6. **Minimum I**

The shape should be a cylinder with the z-axis as its symmetry axis. This is fairly obvious, and a quick proof (by contradiction) is the following.

Assume the optimal blob is not a cylinder, and consider the surface of the blob. If the blob is not a cylinder, then there exist two points on the surface, P_1 and P_2, that are located at different distances, r_1 and r_2, from the z-axis. Assume $r_1 < r_2$ (see Fig. 7.34). Then moving a small piece of the blob from P_2 to P_1 will decrease the moment of inertia, $\int r^2 \rho \, dV$. Hence, the proposed blob was not the one with smallest I.

In order to avoid this contradiction, we must have all points on the surface be equidistant from the z-axis. The only blob with this property is a cylinder.

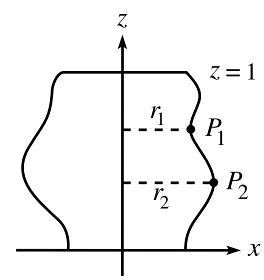


Figure 7.34

7. **Removing a support**

(a) **First Solution:** Let the desired force on the left support be F. Let the acceleration of the CM of the stick be a. Then (looking at torques around the CM, to obtain the second equation; see Fig. 7.35),

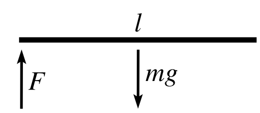


Figure 7.35

$$\begin{aligned} mg - F &= ma, \\ F\frac{\ell}{2} &= \frac{m\ell^2}{12}\alpha, \\ a &= \frac{\ell}{2}\alpha. \end{aligned} \tag{7.60}$$

Solving for F gives $F = mg/4$. (So the CM accelerates at $3g/4$, and the right end accelerates at $3g/2$.)

Second Solution: Looking at torques around the CM, we have

$$F\frac{\ell}{2} = \frac{m\ell^2}{12}\alpha. \tag{7.61}$$

Looking at torques around the fixed end, we have

$$mg\frac{\ell}{2} = \frac{m\ell^2}{3}\alpha. \tag{7.62}$$

These two equations give $F = mg/4$.

(b) **First Solution:** As in the first solution above, we have (see Fig. 7.36)

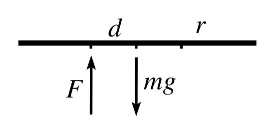


Figure 7.36

$$\begin{aligned} mg - F &= ma, \\ Fd &= (\eta m r^2)\alpha, \\ a &= d\alpha. \end{aligned} \tag{7.63}$$

Solving for F gives $F = mg(1 + d^2/\eta r^2)^{-1}$. For $d = r$ and $\eta = 1/3$, we get the answer in part (a).

Second Solution: As in the second solution above, looking at torques around the CM, we have

$$Fd = (\eta m r^2)\alpha. \tag{7.64}$$

Looking at torques around the fixed pivot, we have

$$mgd = (\eta mr^2 + md^2)\alpha. \tag{7.65}$$

These two equations give $F = mg(1 + d^2/\eta r^2)^{-1}$.

Some limits: If $d = r$, then: in the limit $\eta = 0$, $F = 0$; if $\eta = 1$, $F = mg/2$; and in the limit $\eta = \infty$, $F = mg$; these all make sense. In the limit $d = 0$, $F = mg$. And in the limit $d = \infty$, $F = 0$. (More precisely, we should be writing $d \ll \sqrt{\eta}\,r$ or $d \gg \sqrt{\eta}\,r$.)

8. **Oscillating ball**

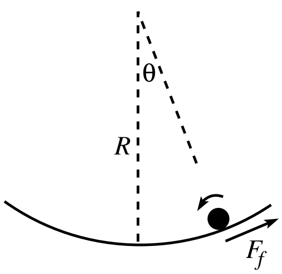


Figure 7.37

Let the angle from the bottom of the cylinder be θ (see Fig. 7.37). Let F_f be the friction force. Then $F = ma$ gives

$$F_f - mg\sin\theta = ma. \tag{7.66}$$

Looking at torque and angular momentum around the CM, we have

$$-rF_f = \frac{2}{5}mr^2\alpha. \tag{7.67}$$

Using $r\alpha = a$, this equation gives $F_f = -2ma/5$. Plugging this into eq. (7.66), and using $\sin\theta \approx \theta$, yields $mg\theta + 7ma/5 = 0$. Under the assumption $r \ll R$, we have $a \approx R\ddot{\theta}$, so we finally have

$$\ddot{\theta} + \left(\frac{5g}{7R}\right)\theta = 0. \tag{7.68}$$

This is the equation for simple harmonic motion with frequency

$$\omega = \sqrt{\frac{5g}{7R}}. \tag{7.69}$$

This answer is slightly smaller than the $\sqrt{g/R}$ answer if the ball were sliding. (The rolling ball effectively has a larger inertial mass, but the same gravitational mass.)

This problem can also be done using the contact point as the origin around which τ and L are calculated.

Remark: If we get rid if the $r \ll R$ assumption, we leave it to you to show that $r\alpha = a$ still holds, but $a = R\ddot{\theta}$ changes to $a = (R-r)\ddot{\theta}$. Therefore, the exact result for the frequency is $\omega = \sqrt{5g/7(R-r)}$. This goes to infinity as r gets close to R. ♣

9. **Ball hitting stick**

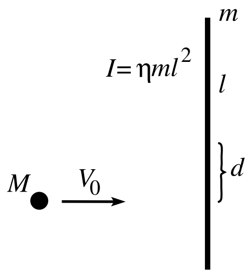


Figure 7.38

Let V be the speed of the ball after the collision. Let v be the speed of the CM of the stick after the collision. Let ω be the angular speed of the stick after the collision. Conservation of momentum, angular momentum (around the initial center of the stick), and energy give (see Fig. 7.38)

$$\begin{aligned} MV_0 &= MV + mv, \\ MV_0 d &= MVd + \eta m\ell^2\omega, \\ MV_0^2 &= MV^2 + mv^2 + \eta m\ell^2\omega^2. \end{aligned} \tag{7.70}$$

We must solve these three equations for V, v, and ω. The first two equations quickly give $vd = \eta\ell^2\omega$. Solving for V in the first equation and plugging it into the third, and then eliminating ω through $vd = \eta\ell^2\omega$ gives

$$v = V_0\frac{2}{1 + \frac{m}{M} + \frac{d^2}{\eta\ell^2}}, \qquad \text{and thus} \quad v = V_0\frac{2\frac{d}{\eta\ell^2}}{1 + \frac{m}{M} + \frac{d^2}{\eta\ell^2}}. \tag{7.71}$$

Knowing v, the first equation above gives V as

$$V = V_0 \frac{1 - \frac{m}{M} + \frac{d^2}{\eta \ell^2}}{1 + \frac{m}{M} + \frac{d^2}{\eta \ell^2}}. \tag{7.72}$$

Another solution is of course $V = V_0$, $v = 0$, and $\omega = 0$. Nowhere in eqs. (7.70) does it say that the ball actually hits the stick.

The reader is encouraged to check various limits of these answers.

10. **A ball and stick theorem**

Let V be the speed of the ball after the collision. Let v be the speed of the CM of the stick after the collision. Let ω be the angular speed of the stick after the collision. Conservation of momentum, angular momentum (around the initial center of the stick), and energy give (see Fig. 7.38)

$$\begin{aligned} MV_0 &= MV + mv, \\ MV_0 d &= MVd + I\omega, \\ MV_0^2 &= MV^2 + mv^2 + I\omega^2. \end{aligned} \tag{7.73}$$

The speed of the contact point on the stick right after the collision is $v + \omega d$. So the desired relative speed is $(v + \omega d) - V$. We can solve the three above equations for V, v, and ω and obtain our answer (i.e., use the results of problem 9), but there's a slightly more appealing method.

The first two equations quickly give $mvd = I\omega$. The last equation may be written in the form (using $I\omega^2 = (I\omega)\omega = (mvd)\omega$)

$$M(V_0 - V)(V_0 + V) = mv(v + \omega d). \tag{7.74}$$

Dividing this by the first equation, written in the from $M(V_0 - V) = mv$, gives $V_0 + V = v + \omega d$, or

$$V_0 = (v + \omega d) - V, \tag{7.75}$$

as was to be shown.

11. **A triangle of circles**

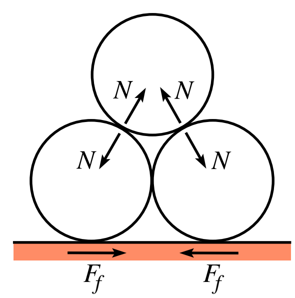


Figure 7.39

(a) Let the normal force between the circles be N. Let the friction force from the ground be F_f (see Fig. 7.39). If we consider torques around the centers of the bottom balls, then the only force we have to worry about is F_f (since N, gravity, and the normal force from the ground point through the centers).

Let a_x be the initial horizontal acceleration of the right bottom circle (so $\alpha = a_x/R$ is its angular acceleration). Let a_y be the initial vertical acceleration of the top circle (downward taken to be positive). Then

$$\begin{aligned} N\cos 60^\circ - F_f &= Ma_x, \\ Mg - 2N\sin 60^\circ &= Ma_y, \\ F_f R &= (\eta MR^2)(a_x/R). \end{aligned} \tag{7.76}$$

We have four unknowns, N, F_f, a_x, and a_y. So we need one more equation. Fortunately, a_x and a_y are related. The 'surface' of contact between the top and bottom circles lies at an angle of 30° with the horizontal. Therefore, if a bottom circle moves a distance d to the side, then the top circle moves a distance $d\tan 30°$ downward. So

$$a_x = \sqrt{3}a_y. \tag{7.77}$$

We now have four equations and four unknowns. Solving for a_y, by your method of choice, gives

$$a_y = \frac{g}{7+6\eta}. \tag{7.78}$$

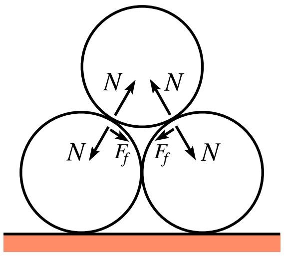


Figure 7.40

(b) Let the normal force between the circles be N. Let the friction between the circles be F_f (see Fig. 7.40). If we consider torques around the centers of the bottom balls, then the only force we have to worry about is F_f.

Let a_x be the initial horizontal acceleration of the right bottom circle. Let a_y be the initial vertical acceleration of the top circle (downward taken to be positive). From the same reasoning as in part (a), we have $a_x = \sqrt{3}a_y$. Let α be the angular acceleration of the right bottom circle (counterclockwise taken to be positive). Note that α is *not* equal to a_x/R, because the bottom circles slip. The four equations analogous to eqs. (7.76) and (7.77) are

$$\begin{aligned} N\cos 60° - F_f \sin 60° &= Ma_x, \\ Mg - 2N\sin 60° - 2F_f\cos 60° &= Ma_y, \\ F_f R &= (\eta MR^2)\alpha, \\ a_x &= \sqrt{3}a_y. \end{aligned} \tag{7.79}$$

We have five unknowns, N, F_f, a_x, a_y, and α. So we need one more equation. The tricky part is relating α to a_x. To do this, it is easiest to ignore the y motion of the top circle and imagine the bottom right circle to be rotating up and around the top circle, which is held fixed. If the bottom circle moves an infinitesimal distance d to the right, then its center moves a distance $d/\cos 30°$ up and to the right. So the angle through which the bottom circle rotates is $d/(R\cos 30°)$. Bringing back in the vertical motion of the balls does not change this result. Therefore,

$$\alpha = \frac{2}{\sqrt{3}}\frac{a_x}{R}. \tag{7.80}$$

We now have five equations and five unknowns. Solving for a_y, by your method of choice, gives

$$a_y = \frac{g}{7+8\eta}. \tag{7.81}$$

REMARK: If $\eta \neq 0$, this result is smaller than that in part (a). This is not all that intuitive. The basic reason is that the bottom circles in part (b) have to rotate a bit faster, so they take up more energy. Also, one can show that the N's are equal in (a) and (b), which is likewise not obvious. Since there's an extra force (from the friction) holding the top ball up in part (b), the acceleration is smaller. ♣

12. **Lots of sticks**

Consider the collision between two sticks. Let the speed of the end of the heavy one be V. Since this stick is essentially infinitely heavy, we may consider it to be an infinitely heavy ball, moving at speed V. (The rotational degree of freedom of the heavy stick is irrelevant, as far as the light stick is concerned.)

We will solve this problem by first finding the speed of the contact point on the light stick, and then finding the speed of the other end of the light stick.

- **Speed of contact point:**

  We can invoke the result of problem 10 to say that the point of contact on the light stick picks up a speed of $2V$. But let's prove this from scratch here in a different way: In the same spirit as the (easier) problem of the collision between two balls of greatly disparate masses, we will work things out in the rest frame of the infinitely heavy ball right before the collision. The situation then reduces to a stick of mass m, length $2r$, moment of inertia ηmr^2, and speed V, approaching a fixed wall (see Fig. 7.41). To find the behavior of the stick after the collision, we will use (1) conservation of energy, and (2) conservation of angular momentum around the contact point.

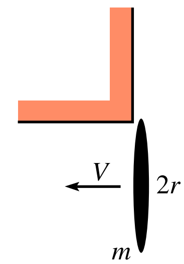


Figure 7.41

  Let u be the speed of the center of mass of the stick after the collision. Let ω be its angular velocity after the collision. Since the wall is infinitely heavy, it will acquire zero kinetic energy. So conservation of E gives

$$\frac{1}{2}mV^2 = \frac{1}{2}mu^2 + \frac{1}{2}(\eta mr^2)\omega^2. \tag{7.82}$$

  The initial angular momentum around the contact point is $L = mrV$, so conservation of L gives (breaking L after the collision up into the L of the CM plus the L relative to the CM)

$$mrV = mru + (\eta mr^2)\omega. \tag{7.83}$$

  Solving eqs. (7.82) and (7.83) for u and $r\omega$ in terms of V gives

$$u = V\frac{1-\eta}{1+\eta}, \qquad \text{and} \qquad rw = V\frac{2}{1+\eta}. \tag{7.84}$$

  (The other solution, $u = V$ and $r\omega = 0$ represents the case where the stick misses the wall.) The relative speed of the wall (i.e., the ball) and the point of contact on the light stick is

$$rw - u = V, \tag{7.85}$$

  as was to be shown.

  Going back to the lab frame (i.e., adding V onto this speed) shows that the point of contact on the light stick moves at speed $2V$.

- **Speed of other end:**

  Consider a stick struck at an end, with impulse $\mathcal{I}$. The speed of the CM is then $v_{\text{CM}} = \mathcal{I}/m$. The angular impulse is $\mathcal{I}r$, so $\mathcal{I}r = \eta mr^2\omega$, and hence $r\omega = \mathcal{I}/m\eta = v_{\text{CM}}/\eta$.

  The speed of the struck end is $v_{\text{str}} = r\omega + v_{\text{CM}}$. The speed of the other end (taking positive to be in the reverse direction) is $v_{\text{oth}} = r\omega - v_{\text{CM}}$. The ratio of these is

$$\frac{v_{\text{oth}}}{v_{\text{str}}} = \frac{v_{\text{CM}}/\eta - v_{\text{CM}}}{v_{\text{CM}}/\eta + v_{\text{CM}}} = \frac{1-\eta}{1+\eta}. \tag{7.86}$$

  In the problem at hand, we have $v_{\text{str}} = 2V$. Therefore,

$$v_{\text{oth}} = V\frac{2(1-\eta)}{1+\eta}. \tag{7.87}$$

The same analysis works in all the other collisions. Therefore, the bottom ends of the sticks move with speeds that form a geometric progression with ratio $2(1-\eta)/(1+\eta)$. If this ratio is less than 1 (i.e., $\eta > 1/3$), then the speeds go to zero, as $n \to \infty$. If it is greater than 1 (i.e., $\eta < 1/3$), then the speeds go to infinity, as $n \to \infty$. If it equals 1 (i.e., $\eta = 1/3$), then the speeds are independent of n, as $n \to \infty$. Therefore,

$$\eta = \frac{1}{3} \tag{7.88}$$

is the desired answer. A uniform stick has $\eta = 1/3$ (usually written in the form $I = m\ell^2/12$, where $\ell = 2r$).

13. **Falling stick**

(a) It is easiest to calculate τ and L relative to the pivot point. The torque is due to gravity, which effectively acts on the CM. It has magnitude mgb.

The moment of inertia of the stick around a horizontal axis through the pivot (and perpendicular to the massless stick) is simply mb^2. So when the stick starts to fall, $\tau = dL/dt$ gives $mgb = (mb^2)\alpha$. Therefore, the initial acceleration of the CM, $b\alpha$, is

$$b\alpha = g, \tag{7.89}$$

independent of ℓ and b.

This makes sense. This stick initially falls straight down, and the pivot provides no force because it doesn't know right away that the stick is moving.

(b) The only change from part (a) is the moment of inertia of the stick around a horizontal axis through the pivot (and perpendicular to the massless stick). From the parallel axis theorem, this moment is $mb^2 + m\ell^2/12$. So when the stick starts to fall, $\tau = dL/dt$ gives $mgb = (mb^2 + m\ell^2/12)\alpha$. Therefore, the initial acceleration of the CM, $b\alpha$, is

$$b\alpha = \frac{g}{1 + \frac{\ell^2}{12b^2}}. \tag{7.90}$$

As $\ell \to 0$, this goes to g, as it should. As $\ell \to \infty$, it goes to 0, as it should (a tiny movement in the CM corresponds to a very large movement in the points far out along the stick).

14. **Falling Chimney**

Let the height of the chimney be ℓ. Let the width be $2r$. The moment of inertia around the pivot point is $m\ell^2/3$ (if we ignore the width). Let the angle with the vertical be θ. Then the torque (around the pivot point) due to gravity is $\tau = mg(\ell/2)\sin\theta$. So $\tau = dL/dt$ gives $mg(\ell/2)\sin\theta = (1/3)m\ell^2\ddot{\theta}$, or

$$\ddot{\theta} = \frac{3g\sin\theta}{2\ell}. \tag{7.91}$$

Consider the chimney to consist of a chimney of height a, with another one of height $\ell - a$ placed on top of it. We will find the tensions in the strings connecting these two 'sub-chimneys'; then we will maximize one of the tensions as a function of a.

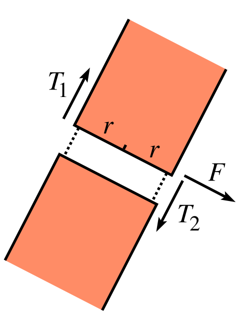


Figure 7.42

The forces on the top piece are gravity and the forces on each end of the bottom board. Let us break these latter forces up into transverse and longitudinal forces along the chimney. Let T_1 and T_2 be the two longitudinal components, and let F be the sum of the transverse components. These are shown in Fig. 7.42. We have picked the positive directions for T_1 and T_2 such that positive T_1 corresponds to a normal force, and positive T_2 corresponds to a tension in the string. This is the case we will be concerned with. (If, for example, T_1 happens to be negative, then it simply corresponds to a tension instead of a normal force.) It turns out that if $r \ll \ell$, then $T_2 \gg F$ (as we will see below), so the tension in the right string is essentially equal to T_2. We will therefore be concerned with maximizing T_2.

In writing down the force and torque equations for the top piece, we have three equations ($F_x = ma_x$, $F_y = ma_y$, and $\tau = dL/dt$ around its center-of-mass), and three unknowns (F, T_1, and T_2). Using the fact that the top piece has length $(\ell - a)$, its CM travels in a circle of radius $(\ell + a)/2$, and its mass is $m(\ell - a)/\ell$, our three equations are, respectively,

$$\begin{aligned}(T_1 - T_2)\sin\theta + F\cos\theta &= \frac{m(\ell-a)}{\ell}\left(\frac{\ell+a}{2}\ddot{\theta}\cos\theta\right),\\ (T_1 - T_2)\cos\theta - F\sin\theta - \frac{mg(\ell-a)}{\ell} &= -\frac{m(\ell-a)}{\ell}\left(\frac{\ell+a}{2}\ddot{\theta}\sin\theta\right),\\ (T_1+T_2)r - F\frac{\ell-a}{2} &= \frac{m(\ell-a)}{\ell}\left(\frac{(\ell-a)^2}{12}\ddot{\theta}\right). \qquad (7.92)\end{aligned}$$

We can solve for F by multiplying the first equation by $\cos\theta$, the second by $\sin\theta$, and subtracting. Using (7.91) to eliminate $\ddot{\theta}$ gives

$$F = \frac{mg\sin\theta}{4}(-1 + 4f - 3f^2), \qquad (7.93)$$

where $f \equiv a/\ell$ is the fraction of the way along the chimney.

We may now solve for T_2. Multiplying the second of eqs. (7.92) by r and subtracting from the third gives (to leading order in the large number ℓ/r)[9]

$$T_2 \approx F\frac{\ell-a}{4r} + \frac{m(\ell-a)}{\ell}\frac{(\ell-a)^2}{24r}\ddot{\theta}. \qquad (7.94)$$

Using eqs. (7.91) and (7.93), this may be written as

$$T_2 \approx \frac{mg\ell\sin\theta}{8r}f(1-f)^2. \qquad (7.95)$$

As stated above, this is much greater than F (since $\ell/r \gg 1$), so the tension in the right string is essentially equal to T_2. Taking the derivative with respect to f, we see that T_2 is maximum at

$$f \equiv \frac{a}{\ell} = \frac{1}{3}. \qquad (7.96)$$

So our chimney is most likely to break at a point one-third of the way up (assuming that the width is much less than the height).

15. **Zero torque from internal forces**

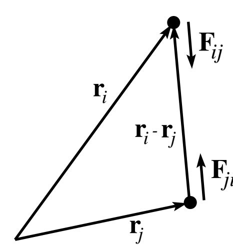


Figure 7.43

Let $\mathbf{F}_{ij}^{\text{int}}$ be the force that the ith particle feels due to the jth particle (see Fig. 7.43. Then

$$\mathbf{F}_i^{\text{int}} = \sum_j \mathbf{F}_{ij}^{\text{int}}, \qquad (7.97)$$

and Newton's third law says that

$$\mathbf{F}_{ij}^{\text{int}} = -\mathbf{F}_{ji}^{\text{int}}. \qquad (7.98)$$

[9]This result is simply the third equation with T_1 set equal to T_2. Basically, T_1 and T_2 are both very large and are essentially equal; the difference between them is of order 1.

Therefore,

$$\boldsymbol{\tau}^{\text{int}} \equiv \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{int}} = \sum_i \sum_j \mathbf{r}_i \times \mathbf{F}_{ij}^{\text{int}}. \tag{7.99}$$

But if we change the indices (which were labeled arbitrarily), we have

$$\boldsymbol{\tau}^{\text{int}} = \sum_j \sum_i \mathbf{r}_j \times \mathbf{F}_{ji}^{\text{int}} = -\sum_j \sum_i \mathbf{r}_j \times \mathbf{F}_{ij}^{\text{int}}. \tag{7.100}$$

Adding the two previous equations gives

$$2\boldsymbol{\tau}^{\text{int}} = \sum_i \sum_j (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}^{\text{int}}. \tag{7.101}$$

But our central-force assumption says that $\mathbf{F}_{ij}^{\text{int}}$ is parallel to $(\mathbf{r}_i - \mathbf{r}_j)$. Therefore, each cross-product in the sum is zero.

16. **Lengthening the pendulum**

Consider the angular momentum, $\mathbf{L}$, around the support point, P. The forces on the mass are the tension in the string and gravity. The former provides no torque around P, and the latter provides no torque in the z-direction. Therefore, L_z is constant.

Let ω_ℓ be the frequency of the circular motion, when the string has length ℓ. Then

$$mr^2\omega_\ell = L_z \tag{7.102}$$

is constant.

The frequency ω_ℓ is obtained by using $F = ma$ for the circular motion. The tension in the string is $mg/\cos\theta$, so the horizontal radial force is $mg\tan\theta$. Therefore,

$$mg\tan\theta = mr\omega_\ell^2 = m(\ell\sin\theta)\omega_\ell^2 \quad \Longrightarrow \quad \omega_\ell = \sqrt{\frac{g}{\ell\cos\theta}}. \tag{7.103}$$

plugging this into eq. (7.102) gives

$$mr^2\sqrt{\frac{g}{\ell\cos\theta}} = mr^2\sqrt{\frac{g}{h}} = L_z. \tag{7.104}$$

(a) For $\theta \approx 0$, we have $h \approx \ell$, so eq. (7.104) gives $r^2/\sqrt{\ell} \approx C$. Therefore,

$$r \propto \ell^{1/4}. \tag{7.105}$$

So r grows very slowly with ℓ.

(b) For $\theta \approx \pi/2$, we have $r \approx \ell$, so eq. (7.104) gives $\ell^2/\sqrt{h} \approx C$. Therefore,

$$h \propto \ell^4. \tag{7.106}$$

So h grows very quickly with ℓ.

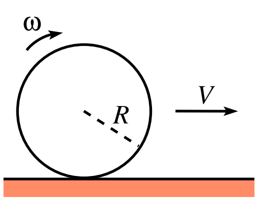


Figure 7.44

17. **Sliding to rolling**

(a) Let the ball travel to the right. Define all linear quantities to be positive to the right, and all angular quantities to be positive clockwise, as shown in Fig. 7.44. (Then, for example, the friction force F_f is negative.) The friction force slows down the translational motion and speeds up the rotational motion, according to (looking at torque around the CM)

$$\begin{aligned} F_f &= ma, \\ -F_f R &= I\alpha. \end{aligned} \tag{7.107}$$

Eliminating F_f, and using $I = \eta m R^2$, gives $a = -\eta R\alpha$. Integrating this over time, up to the time when the ball stops slipping, gives

$$\Delta V = -\eta R \Delta\omega. \tag{7.108}$$

(This is the same statement as the impulse equation, eq. (7.44).) Using $\Delta V = V_f - V_0$, and $\Delta\omega = \omega_f - \omega_0 = \omega_f$, and also $\omega_f = V_f/R$ (the non-slipping condition), we find

$$V_f = \frac{V_0}{1+\eta}, \tag{7.109}$$

independent of how F_f depends on position. (For that matter, F_f could even depend on time or speed. The relation $a = -\eta R\alpha$ would still be true at all times, and hence also eq. (7.108).)

Remark: You could also calculate τ and L relative to the instantaneous point of contact on the ground (which is a fixed point). There is zero torque relative to this point. The motion around this point is not a simple rotation, so we have to add the L of the CM plus the L relative to the CM. $\tau = dL/dt$ gives $0 = (d/dt)(mRv + \eta m R^2 \omega)$. Hence, $a = -\eta R\alpha$.

Note that it is *not* valid to calculate τ and L relative to the instantaneous point of contact *on the ball*. The ball is slowing down, so there is a horizontal component to the acceleration, and hence the third condition in section 7.3.3 does not hold. ♣

The loss in kinetic energy is given by (using eq. (7.109), and also the relation $\omega_f = V_f/R$)

$$\begin{aligned} \Delta KE &= \frac{1}{2}mV_0^2 - \left(\frac{1}{2}mV_f^2 + \frac{1}{2}I\omega_f^2\right) \\ &= \frac{1}{2}mV_0^2\left(1 - \frac{1}{(1+\eta)^2} - \frac{\eta}{(1+\eta)^2}\right) \\ &= \frac{1}{2}mV_0^2\left(\frac{\eta}{1+\eta}\right). \end{aligned} \tag{7.110}$$

For $\eta \to 0$, no energy is lost, which makes sense. And for $\eta \to \infty$, all the energy is lost, which also makes sense (this case is essentially like a sliding block which can't rotate).

(b) Let's find t. The friction force is $F_f = -\mu mg$. So $F = ma$ gives $-\mu g = a$ (so a is constant). Therefore, $\Delta V = at = -\mu g t$. But eq. (7.109) says that $\Delta V \equiv V_f - V_0 = -V_0\eta/(1+\eta)$. So we find

$$t = \frac{\eta}{\mu(1+\eta)}\frac{V_0}{g}. \tag{7.111}$$

For $\eta \to 0$, we have $t \to 0$, which makes sense. And for $\eta \to \infty$, we have $t \to V_0/(\mu g)$ which is exactly the time a sliding block would take to stop.

Now let's find d. We have $d = V_0 t + (1/2)at^2$. Using $a = -\mu g$, and plugging in t from above gives

$$d = \frac{V_0^2}{g}\frac{\eta(2+\eta)}{2\mu(1+\eta)^2}. \tag{7.112}$$

The two extreme cases for η check here.

To calculate the work done by friction, one might be tempted to take the product $F_f d$. But the result doesn't look much like the loss in kinetic energy calculated in eq. (7.110). What's wrong with this? The error is that the friction force does not act over a distance d. To find the distance over which F_f acts, we must find how far the surface of the ball moves relative to the ground.

The relative speed of the point of contact and the ground is $V_{\text{rel}} = V(t) - R\omega(t) = (V_0 + at) - R\alpha t$. Using $a = -\eta R\alpha$ and $a = -\mu g$, this becomes

$$V_{\text{rel}} = V_0 - \frac{1+\eta}{\eta}\mu g t. \tag{7.113}$$

Integrating this from $t = 0$ to the t given in eq. (7.111) yields

$$d_{\text{rel}} = \int V_{\text{rel}} dt = \frac{V_0^2 \eta}{2\mu g(1+\eta)}. \tag{7.114}$$

The work done by friction is $F_f d_{\text{rel}} = \mu m g d_{\text{rel}}$, which does indeed give the ΔKE in eq. (7.110).

18. **The superball**

The y-motion of the ball is irrelevant in this problem, because the y-velocity simply reverses direction, and the vertical impulse from the floor provides no torque around the CM of the ball.

With the positive directions for x and ω as stated in the problem, eq. (7.44) may be used to show that the horizontal impulse from the floor changes v_x and ω according to

$$I(\omega' - \omega) = -Rm(v_x' - v_x). \tag{7.115}$$

The conservation-of-energy statement is

$$\frac{1}{2}m{v_x'}^2 + \frac{1}{2}I\omega'^2 = \frac{1}{2}mv_x^2 + \frac{1}{2}I\omega^2. \tag{7.116}$$

Given v_x and ω, eqs. (7.115) and (7.116) are two equations in the two unknowns v_x' and ω'. They can be solved in a messy way using the quadratic formula, but it is much easier to use the standard trick of rewriting eq. (7.116) as

$$I(\omega'^2 - \omega^2) = -m({v_x'}^2 - v_x^2), \tag{7.117}$$

and then dividing this by eq. (7.115) to obtain

$$R(\omega' + \omega) = (v_x' + v_x). \tag{7.118}$$

Eqs. (7.118) and (7.115) are now two linear equations in the two unknowns v_x' and ω'. Using $I = (2/5)mR^2$ for a solid sphere, you can easily solve the equations to obtain the desired result, eq. (7.45).

Remark: The other solution to eqs. (7.115) and (7.116) is of course $v_x' = v_x$ and $\omega' = \omega$. This corresponds to the ball bouncing off a frictionless floor (or even just passing through the floor). Eq. (7.115) is true for any ball, but the conservation-of-energy statement in eq. (7.116) is only true for two special cases. One is the case of a frictionless floor, where there is "maximal" slipping at the point of contact. The other is the case of zero slipping, which is the case with the superball. If there is any intermediate amount of slipping, then energy is not conserved, because the friction force does work and generates heat. (Work is force times distance, and in the first special case, the force is zero; while in the second special case, the distance is zero). Therefore, in addition to being made of a very bouncy material, a superball must also have a surface that won't slip while in contact with the floor.

Note that eq. (7.118) may easily be used to show that the relative velocity of the point of contact and the ground exactly reverses direction during the bounce. ♣

19. **Many bounces**

Eq. (7.45) gives the result after one bounce, so the result after two bounces is

$$\begin{aligned} \begin{pmatrix} v_x'' \\ R\omega'' \end{pmatrix} &= \begin{pmatrix} 3/7 & 4/7 \\ 10/7 & -3/7 \end{pmatrix} \begin{pmatrix} v_x' \\ R\omega' \end{pmatrix} \\ &= \begin{pmatrix} 3/7 & 4/7 \\ 10/7 & -3/7 \end{pmatrix}^2 \begin{pmatrix} v_x \\ R\omega \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ R\omega \end{pmatrix} \\ &= \begin{pmatrix} v_x \\ R\omega \end{pmatrix}. \end{aligned} \tag{7.119}$$

The square of the matrix turns out to be the identity. Therefore, after two bounces, both v_x and ω return to their original values. The ball then repeats the motion of the previous two bounces (and so on, after every two bounces). The only difference between successive pairs of bounces is that the ball may shift horizontally. You are strongly encouraged to experimentally verify this strange periodic behavior.

20. **Rolling over a bump**

We will use the fact that the angular momentum of the ball with respect to the corner of the point (call it point P) is unchanged by the collision. This is true because any forces exerted at point P provide zero torque around P.[10] This fact will allow us to find the energy of the ball right after the collision, which we will then require to be greater than Mgh.

Breaking L into the contribution relative to the CM, plus the contribution from the ball treated like a point mass located at its CM (eq. (7.9)), gives an initial angular momentum equal to $L = (2/5)MR^2\omega_0 + MV_0(R-h)$, where ω_0 is the initial rolling angular speed. But the non-slipping condition requires that $V_0 = R\omega_0$. Hence, L may be written as

$$L = \frac{2}{5}MRV_0 + MV_0(R-h) = MV_0\left(\frac{7R}{5} - h\right). \tag{7.120}$$

Let ω' be the angular speed of the ball around point P immediately after the collision. The parallel-axis theorem says that the ball's moment of inertia around P is equal to $(2/5)MR^2 + MR^2 = (7/5)MR^2$. Conservation of L (around point P) during the collision then gives

$$MV_0\left(\frac{7R}{5} - h\right) = \frac{7}{5}MR^2\omega', \tag{7.121}$$

which gives ω'. The energy of the ball right after the collision is therefore

$$E = \frac{1}{2}\left(\frac{7}{5}MR^2\right)\omega'^2 = \frac{1}{2}\left(\frac{7}{5}MR^2\right)\left(\frac{MV_0(7R/5-h)}{(7/5)MR^2}\right)^2 = \frac{MV_0^2(7R/5-h)^2}{(14/5)R^2}. \tag{7.122}$$

[10]The torque from gravity will be relevant once the ball rises up off the ground. But during the (instantaneous) collision, L will not change.

The ball will climb up over the step if $E \geq Mgh$, which gives

$$V_0 \geq \frac{R\sqrt{14gh/5}}{7R/5 - h}\,. \tag{7.123}$$

Remarks: It is indeed possible for the ball to rise up over the step, even if $h > R$ (as long as the ball sticks to the corner, without slipping). But note that $V_0 \to \infty$ as $h \to 7R/5$. For $h \geq 7R/5$, it is impossible for the ball to make it up over the step. (The ball will actually get pushed down into the ground, instead of rising up, if $h > 7R/5$.)

For an object with a general moment of inertia $I = \eta MR^2$ (so $\eta = 2/5$ in our problem), you can easily show that the minimum initial speed is

$$V_0 \geq \frac{R\sqrt{2(1+\eta)gh}}{(1+\eta)R - h}\,. \tag{7.124}$$

This decreases as η increases. It is smallest when the "ball" is a wheel with all the mass on its rim (so that $\eta = 1$), in which case it is possible for the wheel to climb over the step even if h approaches $2R$. ♣

Chapter 8

Angular Momentum, Part II

In the previous chapter, we discussed situations where the direction of the vector **L** remains constant, and only its magnitude changes. In this chapter, we will look at the more complicated situations where the direction of **L** is allowed to change. The vector nature of **L** will prove to be vital, and we will arrive at all sorts of strange results for spinning tops and such things.

This chapter is rather long, alas. The first three sections consist of general theory, and then in Section 8.4 we start solving some actual problems.

8.1 Preliminaries concerning rotations

8.1.1 The form of general motion

Before getting started, we should make sure we're all on the same page concerning a few important things about rotations. Because rotations generally involve three dimensions, they can often be hard to visualize. A rough drawing on a piece of paper might not do the trick. For this reason, this topic is one of the more difficult ones in this book.

The next few pages consist of some definitions and helpful theorems. This first theorem describes the form of general motion. You might consider it obvious, but let's prove it anyway.

Theorem 8.1 *Consider a rigid body undergoing arbitrary motion. Pick any point P in the body. Then at any instant (see Fig. 8.1), the motion of the body may be written as the sum of the translational motion of P, plus a rotation around some axis, $\boldsymbol{\omega}$, through P (the axis $\boldsymbol{\omega}$ may change with time).*[1]

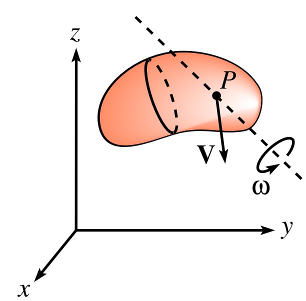


Figure 8.1

Proof: The motion of the body may be written as the sum of the translational motion of P, plus some other motion relative to P (this is true because relative coordinates are additive quantities). We must show that this latter motion is simply

[1]In other words, what we mean here is that a person at rest with respect to a frame whose origin is P, and whose axes are parallel to the fixed-frame axes, will see the body undergoing a rotation around some axis through P.

a rotation. This seems quite plausible, and it holds because the body is rigid; that is, all points keep the same relative distances. (If the body weren't rigid, then this theorem wouldn't be true.)

To be rigorous, consider a sphere fixed in the body, centered at P. The motion of the body is completely determined by the motion of the points on this sphere, so we need only examine what happens to the sphere. And because we are looking at motion relative to P, we have reduced the problem to the following: In what manner can a rigid sphere transform into itself? We claim that *any such transformation requires that two points end up where they started.*[2]

If this claim is true, then we are done, because for an infinitesimal transformation, a given point moves in only one direction (since there is no time to do any bending). So a point that ends up where it started must have always been fixed. Therefore, the diameter joining the two fixed points remains stationary (because distances are preserved), and we are left with a rotation around this axis.

This claim is quite obvious, but nevertheless tricky to prove. I can't resist making you think about it, so I've left it as a problem (Problem 1). Try to solve it on your own. ∎

We will invoke this theorem repeatedly in this chapter (often without bothering to say so). Note that it is required that P be a point in the body, since we used the fact that P keeps the same distances from other points in the body.

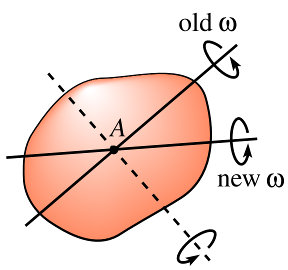


Figure 8.2

REMARK: A situation where our theorem is not so obvious is the following. Consider an object rotating around a fixed axis (see Fig. 8.2). In this case, $\boldsymbol{\omega}$ simply points along this axis. But now imagine grabbing the axis and rotating it around some other axis (the dotted line). It is not immediately obvious that the resulting motion is (instantaneously) a rotation around some new axis through A. But indeed it is. (We'll be quantitative about this in the "Rotating Sphere" example near the end of this section.) ♣

8.1.2 The angular velocity vector

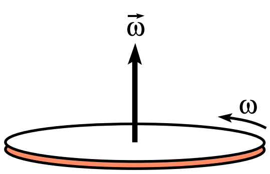


Figure 8.3

It is extremely useful to introduce the angular velocity vector, $\boldsymbol{\omega}$, which is defined to point along the axis of rotation, with a magnitude equal to the angular speed. The choice of the two possible directions is given by the right-hand rule. (Curl your right-hand fingers in the direction of the spin, and your thumb will point in the direction of $\boldsymbol{\omega}$.) For example, a spinning record has $\boldsymbol{\omega}$ perpendicular to the record, through its center (as shown in Fig. 8.3), with magnitude equal to the angular speed, ω.

REMARK: You could, of course, break the mold and use the left-hand rule, as long as you use it consistently. The direction of $\vec{\omega}$ would be opposite, but that doesn't matter, because $\vec{\omega}$ isn't really physical. Any physical result (for example, the velocity of a particle, or the force on it) will come out the same, independent of which hand you (consistently) use.

[2] This claim is actually true for *any* transformation of a rigid sphere into itself, but for the present purposes we are concerned only with infinitesimal transformations (because we are only looking at what happens at a given instant in time).

When studying vectors in school,
You'll use your right hand as a tool.
But look in a mirror,
And then you'll see clearer,
You can just use the left-handed rule. ♣

The points on the axis of rotation are the ones that (instantaneously) do not move. Of, course, the direction of $\boldsymbol{\omega}$ may change over time, so the points that were formerly on $\boldsymbol{\omega}$ may now be moving.

Remark: The fact that we can specify a rotation by specifying a vector $\boldsymbol{\omega}$ is a peculiarity to three dimensions. If we lived in one dimension, then there would be no such thing as a rotation. If we lived in two dimensions, then all rotations would take place in that plane, so we could label a rotation by simply giving its speed, ω. In three dimensions, rotations take place in $\binom{3}{2} = 3$ independent planes. And we choose to label these, for convenience, by the directions orthogonal to these planes, and by the angular speed in each plane. If we lived in four dimensions, then rotations could take place in $\binom{4}{2} = 6$ planes, so we would have to label a rotation by giving 6 planes and 6 angular speeds. Note that a vector (which has four components in four dimensions) would not do the trick here. ♣

In addition to specifying the points that are instantaneously motionless, $\boldsymbol{\omega}$ also easily produces the velocity of any point in the rotating object. Consider the case where the axis of rotation passes through the origin (which we will generally assume to be the case in this chapter, unless otherwise stated). Then we have the following theorem.

Theorem 8.2 *Given an object rotating with angular velocity $\boldsymbol{\omega}$, the velocity of any point in the object is given by (with $\mathbf{r}$ being the position of the point)*

$$\boxed{\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}}. \tag{8.1}$$

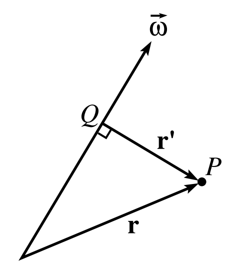


Figure 8.4

Proof: Drop a perpendicular from the point in question (call it P) to the axis $\boldsymbol{\omega}$ (call the point there Q). Let $\mathbf{r}'$ be the vector from Q to P (see Fig. 8.4). From the properties of the cross product, $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ is orthogonal to $\boldsymbol{\omega}$, $\mathbf{r}$, and also $\mathbf{r}'$ (since $\mathbf{r}'$ is a linear combination of $\boldsymbol{\omega}$ and $\mathbf{r}$). Therefore, the direction of $\mathbf{v}$ is correct (it lies in a plane perpendicular to $\boldsymbol{\omega}$, and is also perpendicular to $\mathbf{r}'$, so it describes circular motion around the axis $\boldsymbol{\omega}$; also, by the right-hand rule, it points in the proper orientation around $\boldsymbol{\omega}$). And since

$$|\mathbf{v}| = |\boldsymbol{\omega}||\mathbf{r}|\sin\theta = \omega r', \tag{8.2}$$

which is the speed of the circular motion around $\boldsymbol{\omega}$, we see that $\mathbf{v}$ has the correct magnitude. So $\mathbf{v}$ is indeed the correct velocity vector. ■

Note that if we have the special case where P lies along ω, then $\mathbf{r}$ is parallel to $\boldsymbol{\omega}$, and so the cross product gives a zero result for $\mathbf{v}$, as it should.

Eq. (8.1) is extremely useful and will be applied repeatedly in this chapter. Even if it's hard to visualize what's going on with a given rotation, all you have to do to find the speed of any given point is calculate the cross product $\boldsymbol{\omega} \times \mathbf{r}$.

Conversely, if the speed of every point in a moving body is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, then the body is undergoing a rotation with angular velocity $\boldsymbol{\omega}$ (because all points on the axis $\boldsymbol{\omega}$ are motionless, and all other points move with the proper speed for this rotation).

A very nice thing about angular velocities is that they simply add. Stated more precisely, we have the following theorem.

Theorem 8.3 *Let coordinate systems S_1, S_2, and S_3 have the same origin. Let S_1 rotate with angular velocity $\boldsymbol{\omega}_{1,2}$ with respect to S_2. Let S_2 rotate with angular velocity $\boldsymbol{\omega}_{2,3}$ with respect to S_3. Then S_1 rotates (instantaneously) with angular velocity*

$$\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3} \tag{8.3}$$

with respect to S_3.

Proof: If $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ point in the same direction, then the theorem is clear; the angular speeds just add. If, however, they don't point in the same direction, then things are a little harder to visualize. But we can prove the theorem by simply making abundant use of the definition of $\boldsymbol{\omega}$.

Pick a point P_1 at rest in S_1. Let $\mathbf{r}$ be the vector from the origin to P_1. The velocity of P_1 (relative to a very close point P_2 at rest in S_2) due to the rotation about $\boldsymbol{\omega}_{1,2}$ is $\mathbf{V}_{P_1P_2} = \boldsymbol{\omega}_{1,2} \times \mathbf{r}$. The velocity of P_2 (relative to a very close point P_3 at rest in S_3) due to the rotation about $\boldsymbol{\omega}_{2,3}$ is $\mathbf{V}_{P_2P_3} = \boldsymbol{\omega}_{2,3} \times \mathbf{r}$ (because P_2 is also located essentially at position $\mathbf{r}$). Therefore, the velocity of P_1 (relative to P_3) is $\mathbf{V}_{P_1P_2} + \mathbf{V}_{P_2P_3} = (\boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3}) \times \mathbf{r}$. This holds for any point P_1 at rest in S_1. So the frame S_1 rotates with angular velocity $(\boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3})$ with respect to S_3. ∎

Note that if $\boldsymbol{\omega}_{1,2}$ is constant in S_2, then the vector $\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3}$ will change with respect to S_3 as time goes by (because $\boldsymbol{\omega}_{1,2}$, which is fixed in S_2, is changing with respect to S_3). But at any instant, $\boldsymbol{\omega}_{1,3}$ may be obtained by simply adding the present values of $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$. Consider the following example.

Example (Rotating sphere): A sphere rotates with angular speed ω_3 around a stick that initially points in the $\hat{\mathbf{z}}$ direction. You grab the stick and rotate it around the $\hat{\mathbf{y}}$-axis with angular speed ω_2. What is the angular velocity of the sphere, with respect to the lab frame, as time goes by?

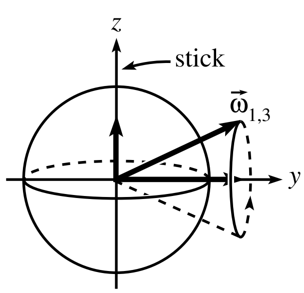


Figure 8.5

Solution: In the language of Theorem 8.3, the sphere defines the S_1 frame; the stick and the $\hat{\mathbf{y}}$-axis define the S_2 frame; and the lab frame is the S_3 frame. The instant after you grab the stick, we are given that $\boldsymbol{\omega}_{1,2} = \omega_3\hat{\mathbf{z}}$, and $\boldsymbol{\omega}_{2,3} = \omega_2\hat{\mathbf{y}}$. Therefore, the angular velocity of the sphere with respect to the lab frame is $\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3} = \omega_3\hat{\mathbf{z}} + \omega_2\hat{\mathbf{y}}$. This is shown is Fig. 8.5. As time goes by, the stick (and hence $\boldsymbol{\omega}_{1,2}$) rotates around the $\mathbf{y}$ axis, so $\boldsymbol{\omega}_{1,3} = \boldsymbol{\omega}_{1,2} + \boldsymbol{\omega}_{2,3}$ traces out a cone around the $\mathbf{y}$ axis, as shown.

REMARK: Note the different behavior of $\vec{\omega}_{1,3}$ for a slightly different statement of the problem: Let the sphere initially rotate with angular velocity $\omega_2\hat{\mathbf{y}}$. Grab the axis (which points in the $\hat{\mathbf{y}}$ direction) and rotate it with angular velocity $\omega_3\hat{\mathbf{z}}$. For this situation, $\vec{\omega}_{1,3}$ initially points in the same direction as in the above statement of the problem (it is initially equal to $\omega_3\hat{\mathbf{z}} + \omega_2\hat{\mathbf{y}}$), but as time goes by, it is the $\omega_2\hat{\mathbf{y}}$ vector that will change, so $\vec{\omega}_{1,3} = \vec{\omega}_{1,2} + \vec{\omega}_{2,3}$ traces out a cone around the **z** axis, as shown in Fig. 8.6. ♣

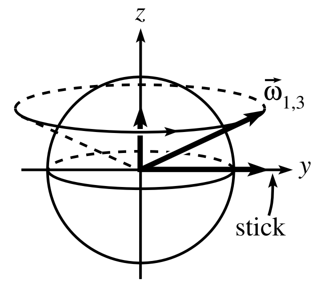


Figure 8.6

An important point concerning rotations in that they are defined with respect to a *coordinate system*. It makes no sense to ask how fast an object is rotating with respect to a certain point, or even a certain axis. Consider, for example, an object rotating with angular velocity $\boldsymbol{\omega} = \omega_3\hat{\mathbf{z}}$, with respect to the lab frame. Saying only, "The object has angular velocity $\boldsymbol{\omega} = \omega_3\hat{\mathbf{z}}$," is not sufficient, because someone standing in the frame of the object would measure $\boldsymbol{\omega} = 0$, and would therefore be very confused by your statement.

Throughout this chapter, we'll try to remember to state the coordinate system with respect to which $\boldsymbol{\omega}$ is measured. But if we forget, the default frame is the lab frame.

If you want to strain some brain cells thinking about $\boldsymbol{\omega}$ vectors, you are encouraged to solve Problem 3, and then also to look at the three given solutions.

This section was a bit abstract, so don't worry too much about it at the moment. The best strategy is probably to read on, and then come back for a second pass after digesting a few more sections. At any rate, we'll be discussing many other aspects of $\boldsymbol{\omega}$ in Section 8.7.2.

8.2 The inertia tensor

Given an object undergoing general motion, the *inertia tensor* is what relates the angular momentum, **L**, to the angular velocity, $\boldsymbol{\omega}$. This tensor[3] depends on the geometry of the object, as we will see. In finding the **L** due to general motion, we will (in the same spirit as in Section 7.1) first look at the special case of rotation around an axis through the origin. Then we will look at the most general possible motion.

8.2.1 Rotation about an axis through the origin

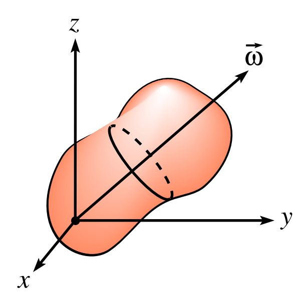


Figure 8.7

The three-dimensional object in Fig. 8.7 rotates with angular velocity $\boldsymbol{\omega}$. Consider a little piece of the body, with mass dm and position $\mathbf{r}$. The velocity of this piece is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. So the angular momentum (relative to the origin) of this piece is equal to $\mathbf{r} \times \mathbf{p} = (dm)\mathbf{r} \times \mathbf{v} = (dm)\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$. The angular momentum of the entire body is therefore

$$\mathbf{L} = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, dm, \tag{8.4}$$

where the integration runs over the volume of the body.

In the case where the rigid body is made up of a collection of point masses, m_i, the angular momentum is simply

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i). \tag{8.5}$$

[3] "Tensor" is just a fancy name for "matrix" here.

This double cross-product looks a bit intimidating, but it's actually not so bad. First, we have

$$\begin{aligned}\boldsymbol{\omega}\times\mathbf{r} &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \omega_1 & \omega_2 & \omega_3\\ x & y & z\end{vmatrix}\\ &= (\omega_2 z-\omega_3 y)\hat{\mathbf{x}}+(\omega_3 x-\omega_1 z)\hat{\mathbf{y}}+(\omega_1 y-\omega_2 x)\hat{\mathbf{z}}.\end{aligned} \tag{8.6}$$

Therefore,

$$\begin{aligned}\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r}) &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ x & y & z\\ (\omega_2 z-\omega_3 y) & (\omega_3 x-\omega_1 z) & (\omega_1 y-\omega_2 x)\end{vmatrix}\\ &= \Big(\omega_1(y^2+z^2)-\omega_2 xy-\omega_3 zx\Big)\hat{\mathbf{x}}\\ &\quad+\Big(\omega_2(z^2+x^2)-\omega_3 yz-\omega_1 xy\Big)\hat{\mathbf{y}}\\ &\quad+\Big(\omega_3(x^2+y^2)-\omega_1 zx-\omega_2 yz\Big)\hat{\mathbf{z}}.\end{aligned} \tag{8.7}$$

The angular momentum in eq. (8.4) may therefore be written in the nice, concise, matrix form,

$$\begin{aligned}\begin{pmatrix}L_1\\ L_2\\ L_3\end{pmatrix} &= \begin{pmatrix}\int(y^2+z^2) & -\int xy & -\int zx\\ -\int xy & \int(z^2+x^2) & -\int yz\\ -\int zx & -\int yz & \int(x^2+y^2)\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}\\ &\equiv \begin{pmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}\\ &\equiv \mathbf{I}\boldsymbol{\omega}\end{aligned} \tag{8.8}$$

For sake of clarity, we have not bothered to write the dm part of each integral. The matrix **I** is called the *inertia tensor*. If the word "tensor" scares you, just ignore it. **I** is simply a matrix. It acts on one vector (the angular velocity) to yield another vector (the angular momentum).

REMARKS:

1. **I** is a rather formidable-looking object. Therefore, you will undoubtedly be very pleased to hear that you will rarely have to use it. It's nice to know that it's there if you do need it, but the concept of *principal axes* in Section 8.3 provides a much better way of solving problems, which avoids the use of the inertia tensor.
2. **I** is a symmetric matrix. (This fact will be important in Section 8.3.) There are therefore only six independent entries, instead of nine.
3. In the case where the rigid body is made up of a collection of point masses, m_i, the entries in the matrix are just sums. For example, the upper left entry is $\sum m_i(y_i^2+z_i^2)$.
4. **I** depends only on the geometry of the object, and not on $\boldsymbol{\omega}$.

5. To construct an **I**, you not only need to specify the origin, you also need to specify the x,y,z axes of your coordinate system. (These basis vectors must be orthogonal, because the cross-product calculation above is valid only for an orthonormal basis.) If someone else comes along and chooses a different orthonormal basis (but the same origin), then her **I** will have different *entries*, as will her $\boldsymbol{\omega}$, as will her **L**. But her $\boldsymbol{\omega}$ and **L** will be exactly the same *vectors* as your $\boldsymbol{\omega}$ and **L**. They will only appear different because they are written in a different coordinate system. (A vector is what it is, independent of how you choose to look at it. If you each point your arm in the direction of what you calculate **L** to be, then you will both be pointing in the same direction.) ♣

All this is fine and dandy. Given any rigid body, we can calculate **I** (relative to a given origin, using a given set of axes). And given $\boldsymbol{\omega}$, we can then apply **I** to it to find **L** (relative to the origin). But what do these entries in **I** really mean? How do we interpret them? Note, for example, that the L_3 in eq. (8.8) contains terms involving ω_1 and ω_2. But ω_1 and ω_2 have to do with rotations around the x and y axes, so what in the world are they doing in L_3? Consider the following examples.

Example 1 (Point-mass in x-y plane): Consider a point-mass m traveling in a circle (centered at the origin) in the x-y plane, with frequency ω_3. Let the radius of the circle be r (see Fig. 8.8).

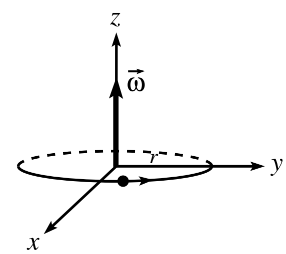


Figure 8.8

Using $\boldsymbol{\omega} = (0, 0, \omega_3)$, $x^2 + y^2 = r^2$, and $z = 0$ in eq. (8.8) (with a discrete sum of only one object, instead of the integrals), the angular momentum with respect to the origin is

$$\mathbf{L} = (0, 0, mr^2\omega_3). \tag{8.9}$$

The z-component is mrv, as it should be. And the x- and y-components are 0, as they should be. This case where $\omega_1 = \omega_2 = 0$ and $z = 0$ is simply the case we studied in the Chapter 7.

Example 2 (Point-mass in space): Consider a point-mass m traveling in a circle of radius r, with frequency ω_3. But now let the circle be centered at the point $(0, 0, z_0)$, with the plane of the circle parallel to the x-y plane (see Fig. 8.9).

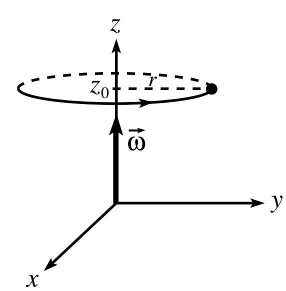


Figure 8.9

Using $\boldsymbol{\omega} = (0, 0, \omega_3)$, $x^2 + y^2 = r^2$, and $z = z_0$ in eq. (8.8), the angular momentum with respect to the origin is

$$\mathbf{L} = m\omega_3(-xz_0, -yz_0, r^2). \tag{8.10}$$

The z-component is mrv, as it should be. But, surprisingly, we have nonzero L_1 and L_2, even though our mass is simply rotating around the z-axis. What's going on?

Consider the instant when the mass is in the x-z plane. The velocity of the mass is then in the $\hat{\mathbf{y}}$ direction. Therefore, the particle most certainly has angular momentum around the x-axis, as well as the z-axis. (Someone looking at a split-second movie of the particle at this point could not tell whether the mass was rotating around the x-axis, the z-axis, or undergoing some complicated motion. But the past and future motion is irrelevant; at any instant in time, as far as the angular momentum goes, we are concerned only with what is happening at that instant.)

At this instant, the angular momentum around the x-axis is $-mz_0v$ (since z_0 is the distance from the x-axis; and the minus sign comes from the right-hand rule). Using $v = \omega_3 x$, we have $L_1 = -mxz_0\omega_3$, in agreement with eq. (8.10).

At this instant, L_2 is zero, since the velocity is parallel to the y-axis. This agrees with eq. (8.10), since $y = 0$. And you can check that eq. (8.10) is indeed correct when the mass is at a general point (x, y, z_0).

For a point mass, $\mathbf{L}$ is much more easily obtained by simply calculating $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ (you should use this to check the results of this example). But for more complicated objects, the tensor $\mathbf{I}$ must be used.

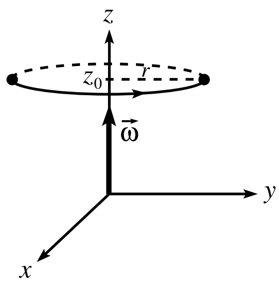


Figure 8.10

Example 3 (Two point-masses): Add another point-mass m to the previous example. Let it travel in the same circle, at the diametrically opposite point (see Fig. 8.10).

Using $\omega = (0, 0, \omega_3)$, $x^2 + y^2 = r^2$, and $z = z_0$ in eq. (8.8), you can show that the angular momentum with respect to the origin is

$$\mathbf{L} = 2m\omega_3(0, 0, r^2). \qquad (8.11)$$

The z-component is $2mrv$, as it should be. And L_1 and L_2 are zero, unlike in the previous example, because these components of the $\mathbf{L}$'s of the two particles cancel. This occurs because of the symmetry of the masses around the z-axis, which causes the I_{zx} and I_{zy} entries in the inertia tensor to vanish (because they are each the sum of two terms, with opposite x values, or opposite y values).

Let's now look at the kinetic energy of our object (which is rotating about an axis passing through the origin). To find this, we need to add up the kinetic energies of all the little pieces. A little piece has energy $(dm)\,v^2/2 = dm\,|\boldsymbol{\omega} \times \mathbf{r}|^2/2$. So, using eq. (8.6), the total kinetic energy is

$$T = \frac{1}{2}\int \Big((\omega_2 z - \omega_3 y)^2 + (\omega_3 x - \omega_1 z)^2 + (\omega_1 y - \omega_2 x)^2 \Big)\, dm. \qquad (8.12)$$

Multiplying this out, we see (after a little work) that we may write T as

$$\begin{aligned} T &= \frac{1}{2}(\omega_1, \omega_2, \omega_3) \cdot \begin{pmatrix} \int(y^2+z^2) & -\int xy & -\int zx \\ -\int xy & \int(z^2+x^2) & -\int yz \\ -\int zx & -\int yz & \int(x^2+y^2) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\ &= \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}. \end{aligned} \qquad (8.13)$$

If $\boldsymbol{\omega} = \omega_3 \hat{\mathbf{z}}$, then this reduces to the $T = I_{33}\omega_3^2/2$ result in eq. (7.8) in Chapter 7 (with a slight change in notation).

8.2.2 General motion

How do we deal with general motion in space? For the motion in Fig. 8.11, the various pieces of mass are not traveling in circles about the origin, so we cannot write $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, as we did prior to eq. (8.4).

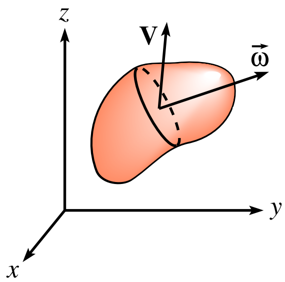


Figure 8.11

To determine $\mathbf{L}$ (relative to the origin), and also the kinetic energy T, we will invoke Theorem 8.1. In applying this theorem, we may choose any point in the body to be the point P in the theorem. However, only in the case that P is the object's CM can we extract anything useful. The theorem then says that the motion of the body is the sum of the motion of the CM plus a rotation about the CM. So, let the CM move with velocity $\mathbf{V}$, and let the body instantaneously rotate with angular velocity $\boldsymbol{\omega}'$ around the CM. (That is, with respect to the frame whose origin is the CM, and whose axes are parallel to the fixed-frame axes.)

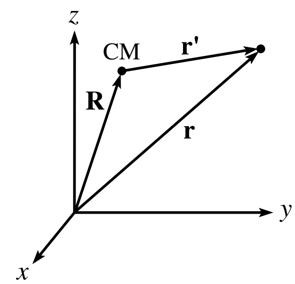


Figure 8.12

Let the CM coordinates be $\mathbf{R} = (X, Y, Z)$, and let the coordinates relative to the CM be $\mathbf{r}' = (x', y', z')$. Then $\mathbf{r} = \mathbf{R} + \mathbf{r}'$ (see Fig. 8.12). Let the velocity relative to the CM be $\mathbf{v}'$ (so $\mathbf{v}' = \boldsymbol{\omega}' \times \mathbf{r}'$). Then $\mathbf{v} = \mathbf{V} + \mathbf{v}'$.

Let's look at L first. The angular momentum is

$$
\begin{aligned}
\mathbf{L} &= \int \mathbf{r} \times \mathbf{v}\, dm \\
&= \int (\mathbf{R} + \mathbf{r}') \times \Big(\mathbf{V} + (\boldsymbol{\omega}' \times \mathbf{r}')\Big)\, dm \\
&= \int (\mathbf{R} \times \mathbf{V})\, dm + \int \mathbf{r}' \times (\boldsymbol{\omega}' \times \mathbf{r}')\, dm \\
&= M(\mathbf{R} \times \mathbf{V}) + \mathbf{L}_{\text{CM}}. \qquad (8.14)
\end{aligned}
$$

The cross terms vanish because the integrands are linear in $\mathbf{r}'$ (and so the integrals, which involve $\int \mathbf{r}'\, dm$, are zero by definition of the CM). $\mathbf{L}_{\text{CM}}$ is the angular momentum relative to the CM.[4]

As in the pancake case Section 7.1.2, we see that the angular momentum (relative to the origin) of a body can be found by treating the body as a point mass located at the CM and finding the angular momentum of this point mass (relative to the origin), and by then adding on the angular momentum of the body, relative to the CM. Note that these two parts of the angular momentum need not point in the same direction (as they did in the pancake case).

Now let's look at T. The kinetic energy is

$$
\begin{aligned}
T &= \int \frac{1}{2} v^2\, dm \\
&= \int \frac{1}{2} |\mathbf{V} + \mathbf{v}'|^2\, dm \\
&= \int \frac{1}{2} V^2\, dm + \int \frac{1}{2} v'^2\, dm \\
&= \frac{1}{2} M V^2 + \int \frac{1}{2} |\boldsymbol{\omega}' \times \mathbf{r}'|^2\, dm \\
&\equiv \frac{1}{2} M V^2 + \frac{1}{2} \boldsymbol{\omega}' \cdot \mathbf{L}_{\text{CM}}, \qquad (8.15)
\end{aligned}
$$

where the last line follows from the steps leading to eq. (8.13). The cross term $\int \mathbf{V} \cdot \mathbf{v}'\, dm = \int \mathbf{V} \cdot (\boldsymbol{\omega}' \times \mathbf{r}')\, dm$ vanishes because the integrand is linear in $\mathbf{r}'$ (and thus yields a zero integral, by definition of the CM).

[4] By this, we mean the angular momentum as measured in the coordinate system whose origin is the CM, and whose axes are parallel to the fixed-frame axes.

As in the pancake case Section 7.1.2, we see that the kinetic energy of a body can be found by treating the body as a point mass located at the CM, and by then adding on the kinetic energy of the body due to motion relative to the CM.

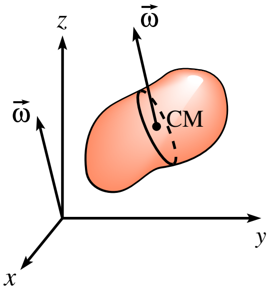


Figure 8.13

8.2.3 The parallel-axis theorem

Consider the special case where the CM rotates around the origin with the same angular velocity at which the body rotates around the CM (see Fig. 8.13). That is, $\mathbf{V} = \boldsymbol{\omega}' \times \mathbf{R}$, (This may be achieved, for example, by having a rod stick out of the body and pivoting one end of the rod at the origin.) This means that we have the nice situation where all points in the body travel in fixed circles around the axis of rotation (because $\mathbf{v} = \mathbf{V} + \mathbf{v}' = \boldsymbol{\omega}' \times \mathbf{R} + \boldsymbol{\omega}' \times \mathbf{r}' = \boldsymbol{\omega}' \times \mathbf{r}$). Dropping the prime on $\boldsymbol{\omega}$, eq. (8.14) becomes

$$\mathbf{L} = M\mathbf{R} \times (\boldsymbol{\omega} \times \mathbf{R}) + \int \mathbf{r}' \times (\boldsymbol{\omega} \times \mathbf{r}')\, dm \tag{8.16}$$

Expanding the double cross-products as in the steps leading to eq. (8.8), we may write this as

$$\begin{aligned}\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} &= M \begin{pmatrix} Y^2+Z^2 & -XY & -ZX \\ -XY & Z^2+X^2 & -YZ \\ -ZX & -YZ & X^2+Y^2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\ &\quad + \begin{pmatrix} \int (y'^2+z'^2) & -\int x'y' & -\int z'x' \\ -\int x'y' & \int (z'^2+x'^2) & -\int y'z' \\ -\int z'x' & -\int y'z' & \int (x'^2+y'^2) \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\ &\equiv (\mathbf{I}_{\mathrm{R}} + \mathbf{I}_{\mathrm{CM}})\boldsymbol{\omega}. \end{aligned} \tag{8.17}$$

This is the generalized parallel-axis theorem. It says that once you've calculated $\mathbf{I}_{\mathrm{CM}}$ for an axis through the CM, then if you want to calculate $\mathbf{I}$ around any parallel axis, you simply have to add on the $\mathbf{I}_{\mathrm{R}}$ matrix (obtained by treating the object like a point-mass at the CM). So you have to compute six numbers (there are only six, instead of nine, because the matrix is symmetric) instead of just the one MR^2 in the parallel-axis theorem in Chapter 7, given in eq. (7.12).

Likewise, if $\mathbf{V} = \boldsymbol{\omega}' \times \mathbf{R}$, then eq. (8.15) gives (dropping the prime on $\boldsymbol{\omega}$) a kinetic energy of

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot (\mathbf{I}_{\mathrm{R}} + \mathbf{I}_{\mathrm{CM}})\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}. \tag{8.18}$$

8.3 Principal axes

The cumbersome expressions in the previous section may seem a bit unsettling, but it turns out that you will rarely have to invoke them. The strategy for avoiding all the previous mess is to use the *principal axes* of a body, which we will define below.

In general, the inertia tensor $\mathbf{I}$ in eq. (8.8) has nine nonzero entries (six independent ones). In addition to depending on the origin chosen, this inertia tensor depends on the set of orthonormal basis vectors chosen for the coordinate system.

(The x,y,z variables in the integrals in $\mathbf{I}$ depend on the coordinate system with respect to which they are measured, of course.)

Given a blob of material, and given an arbitrary origin,[5] any orthonormal set of basis vectors is usable, but there is one special set that makes all our calculations very nice. These special basis vectors are called the *principal axes*. They can be defined in various equivalent ways.

- The principal axes are the orthonormal basis vectors for which $\mathbf{I}$ is diagonal, that is, for which[6]

$$\mathbf{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \tag{8.19}$$

  I_1, I_2, and I_3 are called the *principal moments*.

  For many objects, it is quite obvious what the principal axes are. For example, consider a uniform rectangle in the x-y plane, and let the CM be the origin (and let the sides be parallel to the coordinate axes). Then the principal axes are clearly the x, y, and z axes, because all the off-diagonal elements of the inertia tensor in eq. (8.8) vanish, by symmetry. For example $I_{xy} \equiv -\int xy\,dm$ equals zero, because for every point (x, y) in the rectangle, there is a corresponding point $(-x, y)$. So the contributions to $\int xy\,dm$ cancel in pairs. Also, the integrals involving z are identically zero, because $z = 0$.

- The principal axes are the orthonormal set $\hat{\boldsymbol{\omega}}_1$, $\hat{\boldsymbol{\omega}}_2$, $\hat{\boldsymbol{\omega}}_3$ with the property that

$$\mathbf{I}\hat{\boldsymbol{\omega}}_1 = I_1\hat{\boldsymbol{\omega}}_1, \qquad \mathbf{I}\hat{\boldsymbol{\omega}}_2 = I_2\hat{\boldsymbol{\omega}}_2, \qquad \mathbf{I}\hat{\boldsymbol{\omega}}_3 = I_3\hat{\boldsymbol{\omega}}_3. \tag{8.20}$$

  (That is, they are the $\boldsymbol{\omega}$'s for which $\mathbf{L}$ points in the same direction as $\boldsymbol{\omega}$.) These three statements are equivalent to eq. (8.19), because the vectors $\hat{\boldsymbol{\omega}}_1$, $\hat{\boldsymbol{\omega}}_2$, and $\hat{\boldsymbol{\omega}}_3$ are simply $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ in the frame in which they are the basis vectors.

- The principal axes are the axes around which the object can rotate with constant speed, without the need for any torque. (So in some sense, the object is "happy" to spin around a principal axis.) This is equivalent to the previous definition for the following reason. Assume the object rotates around an axis $\hat{\boldsymbol{\omega}}_1$, for which $\mathbf{L} = \mathbf{I}\hat{\boldsymbol{\omega}}_1 = I_1\hat{\boldsymbol{\omega}}_1$, as in eq. (8.20). Then, since $\hat{\boldsymbol{\omega}}_1$ is assumed to be fixed, we see that $\mathbf{L}$ is also fixed. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt = \mathbf{0}$.

The lack of need for any torque, for rotation around a principal axis $\hat{\boldsymbol{\omega}}$, means that if the object is pivoted at the origin, and if the origin is the only place where any force is applied, then the object can undergo rotation with constant angular

[5]The CM is often chosen to be the origin, but it need not be. There are principal axes associated with any origin.

[6]Technically, we should be writing I_{11} instead of I_1, etc., in this matrix, because we're talking about elements of a matrix. (The one-index object I_1 looks like a component of a vector.) But the two-index notation gets cumbersome, so we'll be sloppy and just use I_1, etc.

velocity $\boldsymbol{\omega}$. If you try to set up this scenario with a non-principal axis, it won't work.

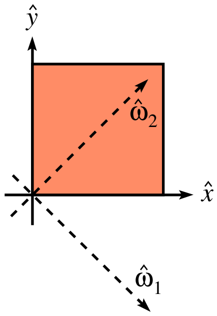


Figure 8.14

Example (Square with origin at corner): Consider the uniform square in Fig. 8.14. In Appendix G, we show that the principal axes are the dotted lines shown (and also the z-axis perpendicular to the page). But there is no need to use the techniques of the appendix to see this, because in this basis it is clear that the integral $\int x_1 x_2$ is zero, by symmetry. (And $x_3 \equiv z$ is identically zero, which makes the other off-diagonal terms in $\mathbf{I}$ also equal to zero.)

Furthermore, it is intuitively clear that the square will be happy to rotate around any one of these axes indefinitely. During such a rotation, the pivot will certainly be supplying a *force* (if the axis is $\hat{\boldsymbol{\omega}}_1$ or $\hat{\mathbf{z}}$), to provide the centripetal acceleration for the circular motion of the CM. But it will not be applying a *torque* relative to the origin (because the $\mathbf{r}$ in $\mathbf{r} \times \mathbf{F}$ is $\mathbf{0}$). This is good, because for a rotation around one of these principal axes, $d\mathbf{L}/dt = \mathbf{0}$, and there is no need for any torque.

It is fairly clear that it is impossible to make the square rotate around, say, the x-axis, assuming that its only contact with the world is through a free pivot at the origin. The square simply doesn't want to remain in that circular motion. There are various ways to demonstrate this rigorously. One is to show that $\mathbf{L}$ (relative to the origin) will not point along the x-axis, so it will therefore precess around the x-axis along with the square, tracing out the surface of a cone. This means that $\mathbf{L}$ is changing. But there is no torque available (relative to the origin) to provide for this change in $\mathbf{L}$. Hence, such a rotation cannot exist.

Note also that the integral $\int xy$ is not equal to zero (every point gives a positive contribution). So the inertia tensor is not diagonal in the x-y basis, which means that $\hat{x}$ and $\hat{y}$ are not principal axes.

At the moment, it is not at all obvious that an orthonormal set of principal axes exists for an arbitrary object. This is the task of Theorem 8.4 below. But assuming that principal axes do exist, the $\mathbf{L}$ and T in eqs. (8.8) and (8.13) take on the particularly nice forms,

$$\begin{aligned} \mathbf{L} &= (I_1\omega_1, I_2\omega_2, I_3\omega_3), \\ T &= \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2). \end{aligned} \tag{8.21}$$

in the basis of the principal axes. (The numbers ω_1, ω_2, and ω_3 are the components of a general vector $\boldsymbol{\omega}$ written in the principal-axis basis; that is, $\boldsymbol{\omega} = \omega_1\hat{\boldsymbol{\omega}}_1 + \omega_2\hat{\boldsymbol{\omega}}_2 + \omega_3\hat{\boldsymbol{\omega}}_3$.) This is a vast simplification over the general formulas in eqs. (8.8) and (8.13). We will therefore invariably work with principal axes in the remainder of this chapter.

Remark: Note that the directions of the principal axes (relative to the body) depend only on the geometry of the body. They may therefore be considered to be painted onto the object. Hence, they will generally move around in space as the body rotates. (For example, in the special case where the object is rotating happily around a principal axis, then that axis will stay fixed, and the other two principal axes will rotate around it in space.) In equations like $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$, the components ω_i and $I_i\omega_i$ are measured along the *instantaneous* principal axes $\hat{\boldsymbol{\omega}}_i$. Since these axes change with time, the components ω_i and $I_i\omega_i$ will generally change with time (except in the case where we have a nice rotation around a principal axis). ♣

Let us now prove that a set of principal axes does indeed exist, for any object, and any origin. Actually, we'll just state the theorem here. The proof involves a rather slick and useful technique, but it's slightly off the main line of thought, so we'll relegate it to Appendix F. Take a look at the proof if you wish, but if you want to simply accept the fact that the principal axes exist, that's fine.

Theorem 8.4 *Given a real symmetric* 3×3 *matrix,* $\mathbf{I}$*, there exist three orthonormal real vectors,* $\hat{\boldsymbol{\omega}}_k$*, and three real numbers,* I_k*, with the property that*

$$\mathbf{I}\hat{\boldsymbol{\omega}}_k = I_k\hat{\boldsymbol{\omega}}_k. \tag{8.22}$$

Proof: See Appendix F. ■

Since the inertia tensor in eq. (8.8) is indeed symmetric, for any body and any origin, this theorem says that we can always find three orthogonal basis-vectors for which $\mathbf{I}$ is a diagonal matrix. That is, principal axes always exist. Invariably, it is best to work in a coordinate system that has this basis. (As mentioned above, the CM is generally chosen to be the origin, but this is not necessary. There are principal axes associated with any origin.)

Problem 5 gives another way to show the existence of principal axes in the special case of a pancake object.

For an object with a fair amount of symmetry, the principal axes are usually the obvious choices and can be written down by simply looking at the object (examples are given below). If, however, you are given an unsymmetrical body, then the only way to determine the principal axes is to pick an arbitrary basis, then find $\mathbf{I}$ in this basis, then go through a diagonalization procedure. This diagonalization procedure basically consists of the steps at the beginning of the proof of Theorem 8.4 (given in Appendix F), with the addition of one more step to get the actual vectors, so we'll relegate it to Appendix G. You need not worry much about this method. Virtually every problem we encounter will involve an object with sufficient symmetry to enable you to simply write down the principal axes.

Let's now prove two very useful (and very similar) theorems, and then we'll give some examples.

Theorem 8.5 *If two principal moments are equal* ($I_1 = I_2 \equiv I$)*, then any axis (through the chosen origin) in the plane of the corresponding principal axes is a principal axis (and its moment is also* I*).*

Similarly, if all three principal moments are equal ($I_1 = I_2 = I_3 \equiv I$)*, then any axis (through the chosen origin) in space is a principal axis (and its moment is also* I*).*

Proof: (This first part was already proved at the end of the proof in Appendix F, but we'll do it again here.) Let $I_1 = I_2 \equiv I$. Then $\mathbf{I}\mathbf{u}_1 = I\mathbf{u}_1$, and $\mathbf{I}\mathbf{u}_2 = I\mathbf{u}_2$.

Hence, $\mathbf{I}(a\mathbf{u}_1 + b\mathbf{u}_2) = I(a\mathbf{u}_1 + b\mathbf{u}_2)$. Therefore, any linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ is a solution to $\mathbf{I}\mathbf{u} = I\mathbf{u}$ and is thus a principal axis, by definition.

Similarly, let $I_1 = I_2 = I_3 \equiv I$. Then $\mathbf{I}\mathbf{u}_1 = I\mathbf{u}_1$, $\mathbf{I}\mathbf{u}_2 = I\mathbf{u}_2$, and $\mathbf{I}\mathbf{u}_3 = I\mathbf{u}_3$. Hence, $\mathbf{I}(a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3) = I(a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3)$. Therefore, any linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ (that is, any vector in space) is a solution to $\mathbf{I}\mathbf{u} = I\mathbf{u}$ and is thus a principal axis, by definition.

Basically, if $I_1 = I_2 \equiv I$, then $\mathbf{I}$ is (up to a multiple) the identity matrix in the space spanned by $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. And if $I_1 = I_2 = I_3 \equiv I$, then $\mathbf{I}$ is (up to a multiple) the identity matrix in the entire space. ■

If two or three moments are equal, so that there is freedom in choosing the principal axes, then it is possible to pick a non-orthogonal group of them. We will, however, always choose ones that are orthogonal. So when we say "a set of principal axes", we mean an orthonormal set.

Theorem 8.6 *If a pancake object is symmetric under a rotation through an angle $\theta \neq 180°$ in the x-y plane (for example, a hexagon), then every axis in the x-y plane (with the origin chosen to be the center of the symmetry rotation) is a principal axis.*

Proof: Let $\hat{\boldsymbol{\omega}}_0$ be a principal axis in the plane, and let $\hat{\boldsymbol{\omega}}_\theta$ be the axis obtained by rotating $\hat{\boldsymbol{\omega}}_0$ through the angle θ. Then $\hat{\boldsymbol{\omega}}_\theta$ is also a principal axis with the same principal moment (due to the symmetry of the object). Therefore, $\mathbf{I}\hat{\boldsymbol{\omega}}_0 = I\hat{\boldsymbol{\omega}}_0$, and $\mathbf{I}\hat{\boldsymbol{\omega}}_\theta = I\hat{\boldsymbol{\omega}}_\theta$.

Now, any vector $\boldsymbol{\omega}$ in the x-y plane can be written as a linear combination of $\hat{\boldsymbol{\omega}}_0$ and $\hat{\boldsymbol{\omega}}_\theta$, provided that $\theta \neq 180°$ (this is where we use that assumption). That is, $\hat{\boldsymbol{\omega}}_0$ and $\hat{\boldsymbol{\omega}}_\theta$ span the x-y plane. Therefore, any vector $\boldsymbol{\omega}$ may be written as $\boldsymbol{\omega} = a\hat{\boldsymbol{\omega}}_0 + b\hat{\boldsymbol{\omega}}_\theta$, and so

$$\mathbf{I}\boldsymbol{\omega} = \mathbf{I}(a\hat{\boldsymbol{\omega}}_0 + b\hat{\boldsymbol{\omega}}_\theta) = aI\hat{\boldsymbol{\omega}}_0 + bI\hat{\boldsymbol{\omega}}_\theta = I\boldsymbol{\omega}. \tag{8.23}$$

Hence, $\boldsymbol{\omega}$ is also a principal axis. (Problem 6 gives another proof of this theorem.) ■

Let's now give some examples. We'll state the principal axes for the following objects (relative to the origin). Your exercise is to show that these are correct. Usually, a quick symmetry argument shows that

$$\mathbf{I} \equiv \begin{pmatrix} \int(y^2+z^2) & -\int xy & -\int zx \\ -\int xy & \int(z^2+x^2) & -\int yz \\ -\int zx & -\int yz & \int(x^2+y^2) \end{pmatrix} \tag{8.24}$$

is diagonal. In all of these examples (see Fig. 8.15), the origin for the principal axes is the origin of the given coordinate system (which is not necessarily the CM). In describing the axes, they thus all pass through the origin, in addition to having the other properties stated.

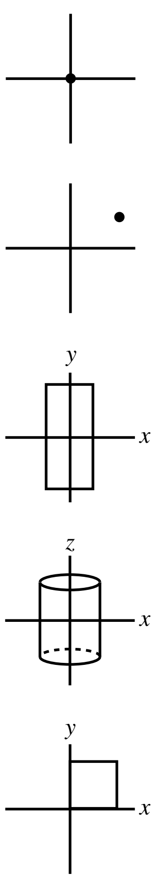


Figure 8.15

Example 1: Point mass at the origin.
principal axes: any axes.

Example 2: Point mass at the point (x_0, y_0, z_0).
principal axes: axis through point, any axes perpendicular to this.

Example 3: Rectangle centered at the origin, as shown.
principal axes: z-axis, axes parallel to sides.

Example 4: Cylinder with axis as z-axis.
principal axes: z-axis, any axes in x-y plane.

Example 5: Square with one corner at origin, as shown.
principal axes: z axis, axis through CM, axis perp to this.

8.4 Two basic types of problems

The previous three sections introduced many new, and somewhat abstract, concepts. We will now (finally) get our hands dirty and solve some actual problems. The concept of principal axes, in particular, gives us the ability to solve many kinds of problems. Two types, however, come up again and again. There are variations on these, of course, but they may be generally stated as follows.

- Strike a rigid object with an impulsive (that is, quick) blow. What is the motion of the object immediately after the blow?
- An object rotates around a fixed axis. A given torque is applied. What is the frequency of the rotation? (Or conversely, given the frequency, what is the required torque?)

Let's work through an example for each of these problems. In both cases, the solution involves a few standard steps, so we'll write them out explicitly.

8.4.1 Motion after an impulsive blow

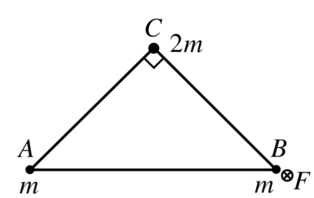


Figure 8.16

Problem: Consider the rigid object in Fig. 8.16. Three masses are connected by three massless rods, in the shape of an isosceles right triangle with hypotenuse length $4a$. The mass at the right angle is $2m$, and the other two masses are m. Label them A, B, C, as shown. Assume that the object is floating freely in space. (Alternatively, let the object hang from a long thread attached to mass C.)

Mass B is struck with a quick blow, directed into the page. Let the imparted impulse have magnitude $\int F\,dt = P$. (See Section 7.4 for a discussion of impulse and angular impulse.) What are the velocities of the three masses immediately after the blow?

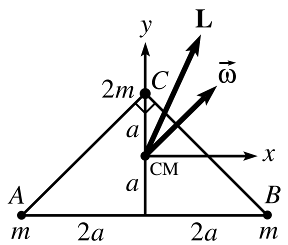


Figure 8.17

Solution: The strategy of the solution will be to find the angular momentum of the system (relative to the CM) using the angular impulse, then calculate the principal moments and find the angular velocity vector (which will give the velocities relative to the CM), and then add on the CM motion.

The altitude from the right angle to the hypotenuse has length $2a$, and the CM is easily seen to be located at its midpoint (see Fig. 8.17). Picking the CM as our origin, and letting the plane of the paper be the x-y plane, the positions of the three masses are $\mathbf{r}_A = (-2a, -a, 0)$, $\mathbf{r}_B = (2a, -a, 0)$, and $\mathbf{r}_C = (0, a, 0)$. There are now five standard steps that we must perform.

- **Find L:** The positive z-axis is directed out of the page, so the impulse vector is $\mathbf{P} \equiv \int \mathbf{F}\,dt = (0, 0, -P)$. Therefore, the angular momentum of the system (relative to the CM) is

$$\begin{aligned} \mathbf{L} = \int \boldsymbol{\tau}\,dt &= \int (\mathbf{r}_B \times \mathbf{F})\,dt = \mathbf{r}_B \times \int \mathbf{F}\,dt \\ &= (2a, -a, 0) \times (0, 0, -P) = aP(1, 2, 0), \end{aligned} \tag{8.25}$$

as shown in Fig. 8.17. We have used the fact that $\mathbf{r}_B$ is essentially constant during the blow (because the blow is assumed to happen very quickly) in taking $\mathbf{r}_B$ outside the integral in the above equation.

- **Calculate the principal moments:** The principal axes are clearly the x, y, and z axes. The moments (relative to the CM) are easily seen to be

$$\begin{aligned} I_x &= ma^2 + ma^2 + (2m)a^2 = 4ma^2, \\ I_y &= m(2a)^2 + m(2a)^2 + (2m)0^2 = 8ma^2, \\ I_z &= I_x + I_y = 12ma^2. \end{aligned} \tag{8.26}$$

We have used the perpendicular-axis theorem (eq. 7.17) to obtain I_z. But I_z will not be needed to solve the problem.

- **Find $\boldsymbol{\omega}$:** We now have two expressions for the angular momentum of the system. One expression is in terms of the given impulse, eq. (8.25). The other is in terms of the moments and the angular velocity components, eq. (8.21). Therefore,

$$\begin{aligned} (I_x\omega_x, I_y\omega_y, I_z\omega_z) &= aP(1, 2, 0) \\ \Longrightarrow \quad (4ma^2\omega_x, 8ma^2\omega_y, 12ma^2\omega_z) &= aP(1, 2, 0) \\ \Longrightarrow \quad (\omega_x, \omega_y, \omega_z) &= \frac{P}{4ma}(1, 1, 0), \end{aligned} \tag{8.27}$$

as shown in Fig. 8.17.

- **Calculate speeds relative to CM:** Right after the blow, the object rotates around the CM with the angular velocity found above. The speeds relative to the CM are therefore $\mathbf{u}_i = \boldsymbol{\omega} \times \mathbf{r}_i$. That is,

$$\begin{aligned}
\mathbf{u}_A &= \boldsymbol{\omega} \times \mathbf{r}_A = \frac{P}{4ma}(1,1,0) \times (-2a,-a,0) = (0,0,P/4m), \\
\mathbf{u}_B &= \boldsymbol{\omega} \times \mathbf{r}_B = \frac{P}{4ma}(1,1,0) \times (2a,-a,0) = (0,0,-3P/4m), \\
\mathbf{u}_C &= \boldsymbol{\omega} \times \mathbf{r}_C = \frac{P}{4ma}(1,1,0) \times (0,a,0) = (0,0,P/4m).
\end{aligned} \tag{8.28}$$

- **Add on speed of CM:** The impulse (that is, the change in linear momentum) supplied to the whole system is $\mathbf{P} = (0,0,-P)$. The total mass of the system is $M = 4m$. Therefore, the velocity of the CM is

$$V_{\text{CM}} = \frac{\mathbf{P}}{M} = (0,0,-P/4m). \tag{8.29}$$

The total velocities of the masses are therefore

$$\begin{aligned}
\mathbf{v}_A &= \mathbf{u}_A + V_{\text{CM}} = (0,0,0), \\
\mathbf{v}_B &= \mathbf{u}_B + V_{\text{CM}} = (0,0,-P/m), \\
\mathbf{v}_C &= \mathbf{u}_C + V_{\text{CM}} = (0,0,0).
\end{aligned} \tag{8.30}$$

REMARKS:

1. We see that masses A and C are instantaneously at rest immediately after the blow, and mass B acquires all of the imparted impulse. In retrospect, this is quite clear. Basically, it is possible for both A and C to remain at rest while B moves a tiny bit, so this is what happens. (If B moves into the page by a small distance ϵ, then A and C won't know that B has moved, since their distances to B will change only by a distance of order ϵ^2.) If we changed the problem and added a mass D at, say, the midpoint of the hypotenuse, then this would not be the case; it would not be possible for A, C, and D to remain at rest while B moved a tiny bit. So there would be some other motion, in addition to B's.

2. As time goes on, the system will undergo a rather complicated motion. What will happen is that the CM will move with constant velocity, and the masses will rotate around it in a messy (but understandable) manner. Since there are no torques acting on the system (after the initial blow), we know that $\mathbf{L}$ will forever remain constant. It turns out that $\boldsymbol{\omega}$ will move around $\mathbf{L}$, and the body will rotate around this changing $\boldsymbol{\omega}$. These matters are the subject of Section 8.6. (Although in that discussion, we restrict ourselves to symmetric tops; that is, ones with two equal moments.) But these issues aside, it's good to know that we can, without too much difficulty, determine what's going on immediately after the blow.

3. The body in the above problem was assumed to be floating freely in space. If we instead have an object that is pivoted at a given (fixed) point, then we simply want to use the pivot as our origin, and there is no need to perform the last step of adding on the velocity of the origin (which was the CM, above), since this velocity is now zero. Equivalently, just consider the pivot to be an infinite mass, which is therefore the location of the (motionless) CM. ♣

8.4.2 Frequency of motion due to a torque

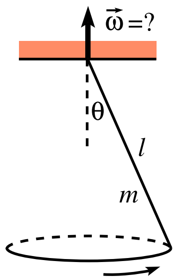


Figure 8.18

Problem: Consider a stick of length ℓ, mass m, and uniform mass density. The stick is pivoted at its top end and swings around the vertical axis. Assume conditions have been set up so that the stick always makes an angle θ with the vertical, as shown in Fig. 8.18. What is the frequency, ω, of this motion?

Solution: The strategy of the solution will be to find the principal moments and then the angular momentum of the system (in terms of ω), then find the rate of change of $\mathbf{L}$, and then calculate the torque and equate it with $d\mathbf{L}/dt$. We will choose the pivot to be the origin.[7] Again, there are five standard steps that we must perform.

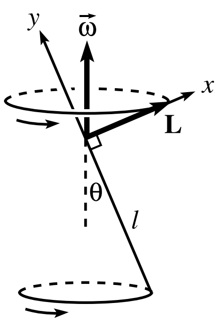


Figure 8.19

- **Calculate the principal moments:** The principal axes are clearly the axis along the stick, along with any two orthogonal axes perpendicular to the stick. So let the x- and y-axes be as shown in Fig. 8.19, and let the z-axis point out of the page. The moments (relative to the pivot) are $I_x = m\ell^2/3$, $I_y = 0$, and $I_z = m\ell^2/3$. (I_z won't be needed in this solution.)

- **Find L:** The angular velocity vector points vertically,[8] so in the basis of the principal axes, the angular velocity vector is $\boldsymbol{\omega} = (\omega \sin\theta, \omega\cos\theta, 0)$, where ω is yet to be determined. The angular momentum of the system (relative to the pivot) is therefore

$$\mathbf{L} = (I_x\omega_x, I_y\omega_y, I_z\omega_z) = (m\ell^2\omega\sin\theta/3, 0, 0). \tag{8.31}$$

- **Find $d\mathbf{L}/dt$:** The vector $\mathbf{L}$ therefore points upwards to the right, along the x-axis (at this instant; see Fig. 8.19), with magnitude $L = m\ell^2\omega\sin\theta/3$. As the stick rotates around the vertical axis, $\mathbf{L}$ traces out the surface of a cone. That is, the tip of $\mathbf{L}$ traces out a horizontal circle. The radius of this circle is the horizontal component of $\mathbf{L}$, which is $L\cos\theta$. The speed of the tip (that is, the magnitude of $d\mathbf{L}/dt$) is therefore $(L\cos\theta)\omega$, because $\mathbf{L}$ rotates around the vertical axis with the same frequency as the stick. So, $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = (L\cos\theta)\omega = \frac{1}{3}m\ell^2\omega^2\sin\theta\cos\theta, \tag{8.32}$$

and it points into the page.

Remark: In more complicated problems (where $I_y \neq 0$), $\mathbf{L}$ will point in some messy direction (not along a principal axis), and the length of the horizontal component (that is, the radius of the circle $\mathbf{L}$ traces out) won't be immediately obvious. In this case, you can either explicitly calculate the horizontal component (see the Gyroscope example in Section 8.7.5), or you can simply do things the formal (and easier) way by

[7] This is a better choice than the CM, because this way we won't have to worry about any messy forces acting at the pivot, when computing the torque.

[8] However, see the third Remark, following this solution.

finding the rate of change of $\mathbf{L}$ via the expression $d\mathbf{L}/dt = \boldsymbol{\omega} \times \mathbf{L}$ (which holds for all the same reasons that $\mathbf{v} \equiv d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r}$ holds). In the present problem, we obtain

$$d\mathbf{L}/dt = (\omega \sin\theta, \omega\cos\theta, 0) \times (m\ell^2\omega\sin\theta/3, 0, 0) = (0, 0, -m\ell^2\omega^2\sin\theta\cos\theta/3), \tag{8.33}$$

which agrees with eq. (8.32). And the direction is correct, since the negative z-axis points into the page. Note that we calculated this cross-product in the principal-axis basis. Although these axes are changing in time, they present a perfectly good set of basis vectors at any instant. ♣

- **Calculate the torque:** The torque (relative to the pivot) is due to gravity, which effectively acts on the CM of the stick. So $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ has magnitude

$$\tau = rF\sin\theta = (\ell/2)(mg)\sin\theta, \tag{8.34}$$

and it points into the page.

- **Equate $\boldsymbol{\tau}$ with $d\mathbf{L}/dt$:** The vectors $d\mathbf{L}/dt$ and $\boldsymbol{\tau}$ both point into the page (they had better point in the same direction). Equating their magnitudes gives

$$\begin{aligned} \frac{m\ell^2\omega^2\sin\theta\cos\theta}{3} &= \frac{mg\ell\sin\theta}{2} \\ \Longrightarrow \qquad \omega &= \sqrt{\frac{3g}{2\ell\cos\theta}}\,. \end{aligned} \tag{8.35}$$

REMARKS:

1. This frequency is slightly larger than the frequency obtained if we instead have a mass at the end of a massless stick of length ℓ. From Problem 12, the frequency in that case is $\sqrt{g/\ell\cos\theta}$. So, in some sense, a uniform stick of length ℓ behaves like a mass at the end of a massless stick of length $2\ell/3$, as far as these rotations are concerned.

2. As $\theta \to \pi/2$, the frequency ω goes to ∞, which makes sense. And as $\theta \to 0$, ω approaches $\sqrt{3g/2\ell}$, which isn't so obvious.

3. As explained in Problem 2, the instantaneous $\boldsymbol{\omega}$ is not uniquely defined in some situations. At the instant shown in Fig. 8.18, the stick is moving directly into the page. So let's say someone else wants to think of the stick as (instantaneously) rotating around the axis $\boldsymbol{\omega}'$ perpendicular to the stick (the x-axis, from above), instead of the vertical axis, as shown in Fig. 8.20. What is the angular speed ω'?

   Well, if ω is the angular speed of the stick around the vertical axis, then we may view the tip of the stick as instantaneously moving in a circle of radius $\ell\sin\theta$ around the vertical axis $\boldsymbol{\omega}$. So $\omega(\ell\sin\theta)$ is the speed of the tip of the stick. But we may also view the tip of the stick as instantaneously moving in a circle of radius ℓ around $\boldsymbol{\omega}'$. The speed of the tip is still $\omega(\ell\sin\theta)$, so the angular speed about this axis is given by $\omega'\ell = \omega(\ell\sin\theta)$. Hence $\omega' = \omega\sin\theta$, which is simply the x-component of $\boldsymbol{\omega}$ that we found above, right before eq. (8.31). The moment of inertia around $\boldsymbol{\omega}'$ is $m\ell^2/3$, so the angular momentum has magnitude $(m\ell^2/3)(\omega\sin\theta)$, in agreement with eq. (8.31). And the direction is along the x-axis, as it should be.

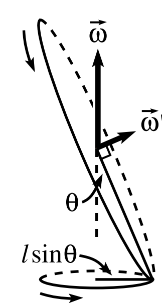


Figure 8.20

Note that although $\boldsymbol{\omega}$ is not uniquely defined at any instant, $\mathbf{L} \equiv \int (\mathbf{r} \times \mathbf{p})\, dm$ certainly is.[9] Choosing $\boldsymbol{\omega}$ to point vertically, as we did in the above solution, is in some sense the natural choice, because this $\boldsymbol{\omega}$ does not change with time. ♣

8.5 Euler's equations

Consider a rigid body instantaneously rotating around an axis $\boldsymbol{\omega}$. ($\boldsymbol{\omega}$ may change as time goes on, but all we care about for now is what it is at a given instant.) The angular momentum, $\mathbf{L}$, is given by eq. (8.8) as

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}, \tag{8.36}$$

where $\mathbf{I}$ is the inertial tensor, calculated with respect to a given set of axes (and $\boldsymbol{\omega}$ is written in the same basis, of course).

As usual, things are much nicer if we use the principal axes (relative to the chosen origin) as the basis vectors of our coordinate system. Since these axes are fixed with respect to the rotating object, they will of course rotate with respect to the fixed reference frame. In this basis, $\mathbf{L}$ takes the nice form,

$$\mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3), \tag{8.37}$$

where ω_1, ω_2, and ω_3 are the components of $\boldsymbol{\omega}$ along the principal axes. In other words, if you take the vector $\mathbf{L}$ in space and project it onto the instantaneous principal axes, then you get these components.

On one hand, writing $\mathbf{L}$ in terms of the rotating principal axes allows us to write it in the nice form of (8.37). But on the other hand, writing $\mathbf{L}$ in this way makes it nontrivial to determine how it changes in time (since the principal axes themselves are changing). The benefits outweigh the detriments, however, so we will invariably use the principal axes as our basis vectors.

The goal of this section is to find an expression for $d\mathbf{L}/dt$, and to then equate this with the torque. The result will be Euler's equations, eqs. (8.43).

Derivation of Euler's equations

If we write $\mathbf{L}$ in terms of the body frame, then we see that $\mathbf{L}$ can change (relative to the lab frame) due to two effects. $\mathbf{L}$ can change because its coordinates in the body frame may change, and $\mathbf{L}$ can also change because of the rotation of the body frame.

To be precise, let $\mathbf{L}_0$ be the vector $\mathbf{L}$ at a given instant. At this instant, imagine painting the vector $\mathbf{L}_0$ onto the body frame (so that $\mathbf{L}_0$ will then rotate with the body frame). The rate of change of $\mathbf{L}$ with respect to the lab frame may be written in the (identically true) way,

$$\frac{d\mathbf{L}}{dt} = \frac{d(\mathbf{L} - \mathbf{L}_0)}{dt} + \frac{d\mathbf{L}_0}{dt}\,. \tag{8.38}$$

[9] The non-uniqueness of $\vec{\omega}$ arises from the fact that $I_y = 0$ here. If all the moments are nonzero, then $(L_x, L_y, L_z) = (I_x\omega_x, I_y\omega_y, I_z\omega_z)$ uniquely determines $\vec{\omega}$, for a given $\mathbf{L}$.

The second term here is simply the rate of change of a body-fixed vector, which we know is $\boldsymbol{\omega} \times \mathbf{L}_0$ (which equals $\boldsymbol{\omega} \times \mathbf{L}$ at this instant). The first term is the rate of change of $\mathbf{L}$ with respect to the body frame, which we will denote by $\delta\mathbf{L}/\delta t$. So we end up with

$$\frac{d\mathbf{L}}{dt} = \frac{\delta\mathbf{L}}{\delta t} + \boldsymbol{\omega} \times \mathbf{L}. \tag{8.39}$$

This is actually a general statement, true for any vector in any rotating frame.[10] There is nothing particular to $\mathbf{L}$ that we used in the above derivation. Also, there was no need to restrict ourselves to principal axes.

In words, what we've shown is that the total change equals the change relative to the rotating frame, plus the change of the rotating frame relative to the fixed frame. Simply addition of changes.

Let us now be specific and choose our body-axes to be the principal-axes. This will put eq. (8.39) in a very usable form. Using eq. (8.37), we have

$$\frac{d\mathbf{L}}{dt} = \frac{\delta}{\delta t}(I_1\omega_1, I_2\omega_2, I_3\omega_3) + (\omega_1, \omega_2, \omega_3) \times (I_1\omega_1, I_2\omega_2, I_3\omega_3). \tag{8.40}$$

This equation equates two vectors. As is true for any vector, these (equal) vectors have an existence that is independent of what coordinate system we choose to describe them with (eq. (8.39) makes no reference to a coordinate system). But since we've chosen an explicit frame on the right-hand side of eq. (8.40), we should choose the same frame for the left-hand side; we can then equate the components on the left with the components on the right. Projecting $d\mathbf{L}/dt$ onto the instantaneous principal axes, we have

$$\left(\left(\frac{d\mathbf{L}}{dt}\right)_1, \left(\frac{d\mathbf{L}}{dt}\right)_2, \left(\frac{d\mathbf{L}}{dt}\right)_3\right) = \frac{\delta}{\delta t}(I_1\omega_1, I_2\omega_2, I_3\omega_3) + (\omega_1, \omega_2, \omega_3) \times (I_1\omega_1, I_2\omega_2, I_3\omega_3). \tag{8.41}$$

Remark: The left-hand side looks nastier than it really is. At the risk of belaboring the point, consider the following (this is a remark that has to be read very slowly): We could have written the left-hand side as $(d/dt)(L_1, L_2, L_3)$, but this might cause confusion as to whether the L_i refer to the components with respect to the rotating axes, or the components with respect to the fixed set of axes that coincide with the rotating principal axes at this instant. That is, do we project $\mathbf{L}$ onto the principal axes, and then take the derivative; or do we take the derivative and then project? The latter is what we mean in eq. (8.41). (The former is $\delta\mathbf{L}/\delta t$, by definition.) The way we've written the left-hand side of eq. (8.41), it's clear that we're taking the derivative first. We are, after all, simply projecting eq. (8.39) onto the principal axes. ♣

The time derivatives on the right-hand side of eq. (8.41) are $\delta(I_1\omega_1)/\delta t = I_1\dot{\omega}_1$ (because I_1 is constant), etc. Performing the cross product and equating the corresponding components on each side yields the three equations,

$$\left(\frac{d\mathbf{L}}{dt}\right)_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2,$$

[10] We will prove eq. (8.39) in another more mathematical way in Chapter 9.

$$\left(\frac{d\mathbf{L}}{dt}\right)_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3,$$
$$\left(\frac{d\mathbf{L}}{dt}\right)_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1. \tag{8.42}$$

If we have chosen the origin of our rotating frame to be either a fixed point or the CM (which we will always do), then the results of Section 7.3 tell us that we may equate $d\mathbf{L}/dt$ with the torque, $\boldsymbol{\tau}$. We therefore have

$$\begin{aligned}
\tau_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2,\\
\tau_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3,\\
\tau_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1.
\end{aligned} \tag{8.43}$$

These are *Euler's equations*. You need only remember one of them, because the other two can be obtained by cyclic permutation of the indices.

REMARKS:

1. We repeat that the left- and right-hand sides of eq. (8.43) are components that are measured with respect to the instantaneous principal axes. Let's say we do a problem, for example, where at all times $\tau_1 = \tau_2 = 0$, and τ_3 equals some nonzero number. This doesn't mean, of course, that $\boldsymbol{\tau}$ is a constant vector. On the contrary, $\boldsymbol{\tau}$ always points along the $\hat{\mathbf{x}}_3$ vector in the rotating frame, but this vector is changing in the fixed frame (unless $\hat{\mathbf{x}}_3$ points along $\boldsymbol{\omega}$).

   The two types of terms on the right-hand sides of eqs. (8.42) are the two types of changes that $\mathbf{L}$ can undergo. $\mathbf{L}$ can change because its components with respect to the rotating frame change, and $\mathbf{L}$ can also change because the body is rotating around $\boldsymbol{\omega}$.

2. Section 8.6.1 on the free symmetric top (viewed from the body frame) provides a good example of the use of Euler's equations. Another interesting application is the famed "tennis racket theorem" (Problem 14).

3. It should be noted that you never *have* to use Euler's equations. You can simply start from scratch and use eq. (8.39) each time you solve a problem. The point is that we've done the calculation of $d\mathbf{L}/dt$ once and for all, so you can just invoke the result in eqs. (8.43). ♣

8.6 Free symmetric top

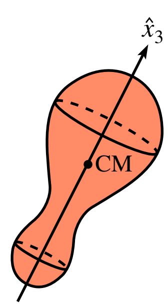


Figure 8.21

The free symmetric top is the classic example of an application of the Euler equations. Consider an object which has two of its principal moments equal (with the CM as the origin). Let the object be in outer space, far from any external forces.[11] We will choose our object to have cylindrical symmetry around some axis (see Fig. 8.21), although this is not necessary (a square cross-section, for example, would yield two equal moments). The principal axes are then the symmetry axis and any two orthogonal axes in the cross-section plane through the CM. Let the symmetry axis be chosen as the $\hat{\mathbf{x}}_3$ axis. Then our moments are $I_1 = I_2 \equiv I$, and I_3.

[11] Equivalently, the object is thrown up in the air, and we are traveling along on the CM.

8.6.1 View from body frame

Plugging $I_1 = I_2 \equiv I$ into Euler's equations, eqs. (8.43), with the τ_i equal to zero (since there are no torques, because the top is "free"), gives

$$\begin{aligned} 0 &= I\dot{\omega}_1 + (I_3 - I)\omega_3\omega_2, \\ 0 &= I\dot{\omega}_2 + (I - I_3)\omega_1\omega_3, \\ 0 &= I_3\dot{\omega}_3. \end{aligned} \tag{8.44}$$

The last equation says that ω_3 is a constant. If we then define

$$\Omega \equiv \left(\frac{I_3 - I}{I}\right)\omega_3, \tag{8.45}$$

the first two equations become

$$\dot{\omega}_1 + \Omega\omega_2 = 0, \qquad \text{and} \qquad \dot{\omega}_2 - \Omega\omega_1 = 0. \tag{8.46}$$

Taking the derivative of the first of these, and then using the second one to eliminate $\dot{\omega}_2$, gives

$$\ddot{\omega}_1 + \Omega^2\omega_1 = 0, \tag{8.47}$$

and likewise for ω_2. This is a nice simple-harmonic equation. The solutions for $\omega_1(t)$ and (by using eq. (8.46)) $\omega_2(t)$ are

$$\omega_1(t) = A\cos(\Omega t + \phi), \qquad \omega_2(t) = A\sin(\Omega t + \phi). \tag{8.48}$$

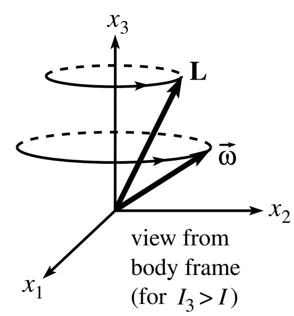


Figure 8.22

Therefore, $\omega_1(t)$ and $\omega_2(t)$ are the components of a circle in the body frame. Hence, the $\boldsymbol{\omega}$ vector traces out a cone around $\hat{\mathbf{x}}_3$ (see Fig. 8.22), with frequency Ω, as viewed by someone standing on the body. The angular momentum is

$$\mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3) = \Big(IA\cos(\Omega t + \phi), IA\sin(\Omega t + \phi), I_3\omega_3\Big), \tag{8.49}$$

so $\mathbf{L}$ also traces out a cone around $\hat{\mathbf{x}}_3$ (see Fig. 8.22), with frequency Ω, as viewed by someone standing on the body.

The frequency, Ω, in eq. (8.45) depends on the value of ω_3 and on the geometry of the object. But the amplitude, A, of the $\boldsymbol{\omega}$ cone is determined by the initial values of ω_1 and ω_2.

Note that Ω may be negative (if $I > I_3$). In this case, $\boldsymbol{\omega}$ traces out its cone in the opposite direction compared to the $\Omega > 0$ case.

Example (The earth): Let's consider the earth to be our object. Then $\omega_3 \approx 2\pi/(1\text{ day})$.[12] The bulge at the equator (caused by the spinning of the earth) makes I_3 slightly larger than I, and it turns out that $(I_3 - I)/I \approx 1/300$. Therefore, eq. (8.45) gives $\Omega \approx (1/300)\,2\pi/(1\text{ day})$. So the $\boldsymbol{\omega}$ vector should precess around its cone

[12] This isn't quite correct, since the earth rotates 366 times for every 365 days (due to the motion around the sun), but it's close enough for the purposes here.

once every 300 days, as viewed by someone on the earth. The true value is more like 400 days. The difference has to do with various things, including the non-rigidity of the earth. But at least we got an answer in the right ballpark.

How do you determine the direction of $\boldsymbol{\omega}$? Simply make an extended-time photograph exposure at night. The stars will form arcs of circles. At the center of all these circles is a point that doesn't move. This is the direction of $\boldsymbol{\omega}$.

How big is the $\boldsymbol{\omega}$ cone, for the earth? Equivalently, what is the value of A in eq. (8.48)? Observation has shown that $\boldsymbol{\omega}$ pierces the earth at a point on the order of 10 m from the north pole. Hence, $A/\omega_3 \approx (10\,\text{m})/R_E$. The half-angle of the $\boldsymbol{\omega}$ cone is therefore found to be only on the order of 10^{-4} degrees. So if you use an extended-time photograph exposure one night to see which point in the sky stands still, and then if you do the same thing 200 nights later, you probably won't be able to tell that they're really two different points.

8.6.2 View from fixed frame

Now let's see what our symmetric top looks like from a fixed frame. In terms of the principal axes, $\hat{\mathbf{x}}_1$,$\hat{\mathbf{x}}_2$,$\hat{\mathbf{x}}_3$, we have

$$\begin{aligned} \boldsymbol{\omega} &= (\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) + \omega_3\hat{\mathbf{x}}_3, \qquad \text{and} \\ \mathbf{L} &= I(\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) + I_3\omega_3\hat{\mathbf{x}}_3. \end{aligned} \tag{8.50}$$

Eliminating the $(\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2)$ term from these equations gives (in terms of the Ω defined in eq. (8.45))

$$\mathbf{L} = I(\boldsymbol{\omega} + \Omega\hat{\mathbf{x}}_3), \qquad \text{or} \qquad \boldsymbol{\omega} = \frac{L}{I}\hat{\mathbf{L}} - \Omega\hat{\mathbf{x}}_3, \tag{8.51}$$

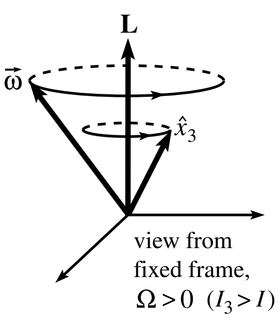


Figure 8.23

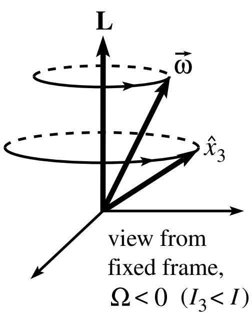


Figure 8.24

where $L = |\mathbf{L}|$, and $\hat{\mathbf{L}}$ is the unit vector in the $\mathbf{L}$ direction. The linear relationship between $\mathbf{L}$, $\boldsymbol{\omega}$, and $\hat{\mathbf{x}}_3$, implies that these three vectors lie in a plane. Since there are no torques on the system, $\mathbf{L}$ remains constant. Therefore, $\boldsymbol{\omega}$ and $\hat{\mathbf{x}}_3$ precess (as we will see below) around $\mathbf{L}$, with the three vectors always coplanar. See Fig. 8.23 for the case $I_3 > I$ (an *oblate* top, such as a coin), and Fig. 8.24 for the case $I_3 < I$ (a *prolate* top, such as a carrot).

What is the frequency of this precession, as viewed from the fixed frame? The rate of change of $\hat{\mathbf{x}}_3$ is $\boldsymbol{\omega} \times \hat{\mathbf{x}}_3$ (because $\hat{\mathbf{x}}_3$ is fixed in the body frame, so its change comes only from rotation around $\boldsymbol{\omega}$). Therefore, eq. (8.51) gives

$$\frac{d\hat{\mathbf{x}}_3}{dt} = \left(\frac{L}{I}\hat{\mathbf{L}} - \Omega\hat{\mathbf{x}}_3\right) \times \hat{\mathbf{x}}_3 = \left(\frac{L}{I}\hat{\mathbf{L}}\right) \times \hat{\mathbf{x}}_3. \tag{8.52}$$

But this is simply the expression for the rate of change of a vector rotating around the fixed vector $\tilde{\boldsymbol{\omega}} \equiv (L/I)\hat{\mathbf{L}}$. The frequency of this rotation is $|\tilde{\boldsymbol{\omega}}| = L/I$. Therefore, $\hat{\mathbf{x}}_3$ precesses around the fixed vector $\mathbf{L}$ with frequency

$$\tilde{\omega} = \frac{L}{I}, \tag{8.53}$$

in the fixed frame (and therefore $\boldsymbol{\omega}$ does also, since it is coplanar with $\hat{\mathbf{x}}_3$ and $\mathbf{L}$).

REMARKS:

1. We just said that $\boldsymbol{\omega}$ precesses around $\mathbf{L}$ with frequency L/I. What, then, is wrong with the following reasoning: "Just as the rate of change of $\hat{\mathbf{x}}_3$ equals $\boldsymbol{\omega} \times \hat{\mathbf{x}}_3$, the rate of change of $\boldsymbol{\omega}$ should equal $\boldsymbol{\omega} \times \boldsymbol{\omega}$, which is zero. Hence, $\boldsymbol{\omega}$ should remain constant." The error is that the vector $\boldsymbol{\omega}$ is not fixed in the body frame. A vector $\mathbf{A}$ must be fixed in the body frame in order for its rate of change to be given by $\boldsymbol{\omega} \times \mathbf{A}$.

2. We found in eqs. (8.49) and (8.45) that a person standing on the rotating body sees $\mathbf{L}$ (and $\boldsymbol{\omega}$) precess with frequency $\Omega \equiv \omega_3(I_3 - I)/I$ around $\hat{\mathbf{x}}_3$. But we found in eq. (8.53) that a person standing in the fixed frame sees $\hat{\mathbf{x}}_3$ (and $\boldsymbol{\omega}$) precess with frequency L/I around $\mathbf{L}$. Are these two facts compatible? Should we have obtained the same frequency from either point of view? (Answers: yes, no).

   These two frequencies are indeed consistent, as can be seen from the following reasoning. Consider the plane (call it S) containing the three vectors $\mathbf{L}$, $\boldsymbol{\omega}$, and $\hat{\mathbf{x}}_3$. We know from eq. (8.49) that S rotates with frequency $\Omega\hat{\mathbf{x}}_3$ with respect to the body. Therefore, the body rotates with frequency $-\Omega\hat{\mathbf{x}}_3$ with respect to S. And from eq. (8.53), S rotates with frequency $(L/I)\hat{\mathbf{L}}$ with respect to the fixed frame. Therefore, the total angular velocity of the body with respect to the fixed frame (using the frame S as an intermediate step) is

$$\boldsymbol{\omega}_{\text{total}} = \frac{L}{I}\hat{\mathbf{L}} - \Omega\hat{\mathbf{x}}_3. \tag{8.54}$$

   But from eq. (8.51), this is simply $\boldsymbol{\omega}$, as it should be. So the two frequencies in eqs. (8.45) and (8.53) are indeed consistent.

   For the earth, $\Omega \equiv \omega_3(I_3 - I)/I$ and L/I are much different. L/I is roughly equal to L/I_3, which is essentially equal to ω_3. Ω, on the other hand is about $(1/300)\omega_3$. Basically, an external observer sees $\boldsymbol{\omega}$ precess around its cone at roughly the rate at which the earth spins. But it's not exactly the same rate, and this difference is what causes the earth-based observer to see $\boldsymbol{\omega}$ precess with a nonzero Ω. ♣

8.7 Heavy symmetric top

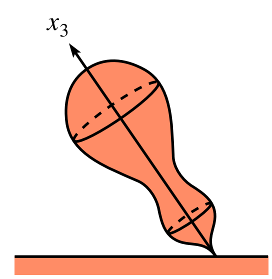


Figure 8.25

Consider now a heavy symmetrical top; that is, one that spins on a table, under the influence of gravity (see Fig. 8.25). Assume that the tip of the top is fixed on the table by a free pivot. We will solve for the motion of the top in two different ways. The first will use $\boldsymbol{\tau} = d\mathbf{L}/dt$. The second will use the Lagrangian method.

8.7.1 Euler angles

For both of these methods, it is very convenient to use the *Euler angles*, θ,ϕ,ψ, which are shown in Fig. 8.26 and are defined as follows.

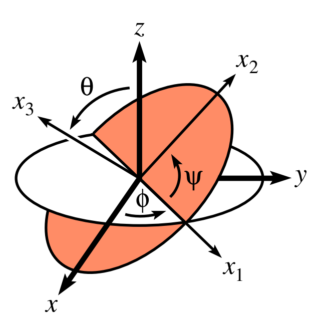


Figure 8.26

- θ**:** Let $\hat{\mathbf{x}}_3$ be the symmetry axis of the top. Define θ to be the angle that $\hat{\mathbf{x}}_3$ makes with the vertical axis $\hat{\mathbf{z}}$ of the fixed frame.

- ϕ**:** Draw the plane orthogonal to $\hat{\mathbf{x}}_3$. Let $\hat{\mathbf{x}}_1$ be the intersection of this plane with the horizontal x-y plane. Define ϕ to be the angle $\hat{\mathbf{x}}_1$ makes with the $\hat{\mathbf{x}}$ axis of the fixed frame.

- ψ**:** Let $\hat{\mathbf{x}}_2$ be orthogonal to $\hat{\mathbf{x}}_3$ and $\hat{\mathbf{x}}_1$, as shown. Let frame S be the frame whose axes are $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$. Define ψ to be the angle of rotation of the body around the $\hat{\mathbf{x}}_3$ axis in frame S. (That is, $\dot{\psi}\hat{\mathbf{x}}_3$ is the angular velocity of the body with respect to S.) Note that the angular velocity of frame S with respect to the fixed frame is $\dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{x}}_1$.

The angular velocity of the body with respect to the fixed frame is equal to the angular velocity of the body with respect to frame S, plus the angular velocity of frame S with respect to the fixed frame. In other words, it is

$$\boldsymbol{\omega} = \dot{\psi}\hat{\mathbf{x}}_3 + (\dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{x}}_1). \tag{8.55}$$

Note that the vector $\hat{\mathbf{z}}$ is not orthogonal to $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_3$. It is often more convenient to rewrite $\boldsymbol{\omega}$ entirely in terms of the orthogonal $\hat{\mathbf{x}}_1$,$\hat{\mathbf{x}}_2$,$\hat{\mathbf{x}}_3$ basis vectors. Since $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{x}}_3 + \sin\theta\hat{\mathbf{x}}_2$, eq. (8.55) gives

$$\boldsymbol{\omega} = (\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{x}}_3 + \dot{\phi}\sin\theta\hat{\mathbf{x}}_2 + \dot{\theta}\hat{\mathbf{x}}_1. \tag{8.56}$$

This form of $\boldsymbol{\omega}$ is often more useful, because $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$ are principal axes of the body. (We are assuming that we are working with a symmetrical top, with $I_1 = I_2 \equiv I$. Hence, any axes in the $\hat{\mathbf{x}}_1$-$\hat{\mathbf{x}}_2$ plane are principal axes.) Although $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are not fixed in the object, they are still good principal axes at any instant.

8.7.2 Digression on the components of $\vec{\omega}$

The previous expressions for $\boldsymbol{\omega}$ look rather formidable, but there is a very helpful diagram (see Fig. 8.27) we can draw which makes it easy to see what is going on. Let's talk a bit about this diagram before returning to the original problem of the spinning top.

In the following discussion, we will simplify things by setting $\dot{\theta} = 0$. All the interesting features of $\boldsymbol{\omega}$ remain. The $\dot{\theta}\hat{\mathbf{x}}_1$ component of $\boldsymbol{\omega}$ in eqs. (8.55) and (8.56) simply arises from the easily-visualizable rising and falling of the top. We will therefore concentrate here on the more complicated issues, namely the components of $\boldsymbol{\omega}$ in the plane of $\hat{\mathbf{x}}_3$, $\hat{\mathbf{z}}$, and $\hat{\mathbf{x}}_2$.

With $\dot{\theta} = 0$, Fig. 8.27 shows the vector $\boldsymbol{\omega}$ in the $\hat{\mathbf{x}}_3$-$\hat{\mathbf{z}}$-$\hat{\mathbf{x}}_2$ plane (the way we've drawn it, $\hat{\mathbf{x}}_1$ points into the page, in contrast to Fig. 8.26). This is an extremely useful diagram, and we will refer to it many times in the problems for this chapter. There are numerous comments to be made on it, so let's just list them out.

1. If someone asks you to "decompose" $\boldsymbol{\omega}$ into pieces along $\hat{\mathbf{z}}$ and $\hat{\mathbf{x}}_3$, what would you do? Would you draw the lines perpendicular to these axes to obtain the lengths shown (which we will label as ω_z and ω_3), or would you draw the lines parallel to these axes to obtain the lengths shown (which we will label as Ω and ω')? There is no "correct" answer to this question. The four quantities, ω_z, ω_3, Ω, ω' simply represent different things. We will interpret each of these below, along with ω_2 (the projection of $\boldsymbol{\omega}$ along $\hat{\mathbf{x}}_2$). It turns out that Ω and ω' are the frequencies that your eye can see the easiest, while ω_2 and ω_3 are what you want to use when you're doing calculations involving the angular momentum. (And as far as I can see, ω_z is not of much use.)

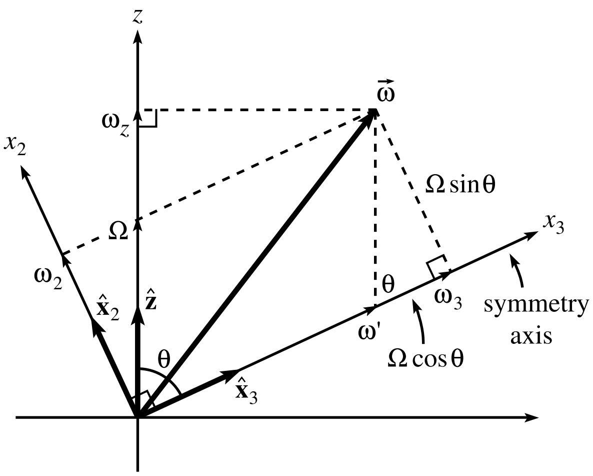


Figure 8.27

2. Note that it is true that
$$\boldsymbol{\omega} = \omega' \hat{\mathbf{x}}_3 + \Omega \hat{\mathbf{z}}, \tag{8.57}$$
but it is *not* true that $\boldsymbol{\omega} = \omega_z \hat{\mathbf{z}} + \omega_3 \hat{\mathbf{x}}_3$. Another true statement is
$$\boldsymbol{\omega} = \omega_3 \hat{\mathbf{x}}_3 + \omega_2 \hat{\mathbf{x}}_2. \tag{8.58}$$

3. In terms of the Euler angles, we see (by comparing eq. (8.57) with eq. (8.55), with $\dot{\theta} = 0$) that
$$\begin{aligned} \omega' &= \dot{\psi}, \\ \Omega &= \dot{\phi}. \end{aligned} \tag{8.59}$$
And we also have (by comparing eq. (8.58) with eq. (8.56), with $\dot{\theta} = 0$)
$$\begin{aligned} \omega_3 &= \dot{\psi} + \dot{\phi}\cos\theta = \omega' + \Omega\cos\theta, \\ \omega_2 &= \dot{\phi}\sin\theta = \Omega\sin\theta. \end{aligned} \tag{8.60}$$
These are also clear from Fig. 8.27.

   There is therefore technically no need to introduce the new ω_2, ω_3, Ω, ω' definitions in Fig. 8.27, since the Euler angles are quite sufficient. But we will be referring to this figure many times, and it is a little easier to refer to these omega's than to the various combinations of Euler angles.

4. Ω is the easiest of these frequencies to visualize. It is simply the frequency of precession of the top around the vertical $\hat{\mathbf{z}}$ axis.[13] In other words, the

[13] Although we're using the same letter, this Ω doesn't have anything to do with the Ω defined in eq. (8.45), except for the fact that they both represent a precession frequency.

symmetry axis $\hat{\mathbf{x}}_3$ traces out a cone around the $\hat{\mathbf{z}}$ axis with frequency Ω. (Note that this precession frequency is *not* ω_z.) Let's prove this.

The vector $\boldsymbol{\omega}$ is the vector which gives the speed of any point (at position $\mathbf{r}$) fixed in the top as $\boldsymbol{\omega} \times \mathbf{r}$. Therefore, since the vector $\hat{\mathbf{x}}_3$ is fixed in the top, we may write

$$\frac{d\hat{\mathbf{x}}_3}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{x}}_3 = (\omega'\hat{\mathbf{x}}_3 + \Omega\hat{\mathbf{z}}) \times \hat{\mathbf{x}}_3 = (\Omega\hat{\mathbf{z}}) \times \hat{\mathbf{x}}_3. \tag{8.61}$$

But this is precisely the expression for the rate of change of a vector rotating around the $\hat{\mathbf{z}}$ axis, with frequency Ω. (This was exactly the same type of proof as the one leading to eq. (8.52).)

Remark: In the derivation of eq. (8.61), we've basically just stripped off a certain part of $\boldsymbol{\omega}$ that points along the $\hat{\mathbf{x}}_3$ axis, because a rotation around $\hat{\mathbf{x}}_3$ contributes nothing to the motion of $\hat{\mathbf{x}}_3$. Note, however, that there is in fact an infinite number of ways to strip off a piece along $\hat{\mathbf{x}}_3$. For example, we can also break $\boldsymbol{\omega}$ up as, say, $\boldsymbol{\omega} = \omega_3\hat{\mathbf{x}}_3 + \omega_2\hat{\mathbf{x}}_2$. We then obtain $d\hat{\mathbf{x}}_3/dt = (\omega_2\hat{\mathbf{x}}_2) \times \hat{\mathbf{x}}_3$, which means that $\hat{\mathbf{x}}_3$ is instantaneously rotating around $\hat{\mathbf{x}}_2$ with frequency ω_2. Although this is true, it is not as useful as the result in eq. (8.61), because the $\hat{\mathbf{x}}_2$ axis changes with time. The point here is that the instantaneous angular velocity vector around which the symmetry axis rotates is not well-defined (Problem 2 discusses this issue).[14] But the $\hat{\mathbf{z}}$-axis is the only one of these angular velocity vectors that is fixed. When we look at the top, we therefore see it precessing around the $\hat{\mathbf{z}}$-axis. ♣

5. ω' is also easy to visualize. Imagine that you are at rest in a frame that rotates around the $\hat{\mathbf{z}}$-axis with frequency Ω. Then you will see the symmetry axis of the top remain perfectly still, and the only motion you will see is the top spinning around this axis with frequency ω'. (This is true because $\boldsymbol{\omega} = \omega'\hat{\mathbf{x}}_3 + \Omega\hat{\mathbf{z}}$, and the rotation of your frame causes you to not see the $\Omega\hat{\mathbf{z}}$ part.) If you paint a dot somewhere on the top, then the dot will trace out a fixed tilted circle, and the dot will return to, say, its maximum height at frequency ω'.

   Note that someone in the lab frame will see the dot undergo a rather complicated motion, but she must clearly observe the same frequency at which the dot returns to its highest point. Hence, ω' is something quite physical in the lab frame, also.

6. ω_3 is what you use to obtain the component of $\mathbf{L}$ along $\hat{\mathbf{x}}_3$, because $L_3 = I_3\omega_3$. It is not quite as easy to visualize as Ω and ω', but it is the frequency with which the top instantaneously rotates, as seen by someone at rest in a frame that rotates around the $\hat{\mathbf{x}}_2$ axis with frequency ω_2. (This is true because $\boldsymbol{\omega} = \omega_2\hat{\mathbf{x}}_2 + \omega_3\hat{\mathbf{x}}_3$, and the rotation of the frame causes you to not see the $\omega_2\hat{\mathbf{x}}_2$ part.) This rotation is a little harder to see, because the $\hat{\mathbf{x}}_2$ axis changes with time.

[14] The instantaneous angular velocity of the *whole body* is well defined, of course. But if you just look at the symmetry axis by itself, then there is an ambiguity (see footnote 9).

There is one physical scenario in which ω_3 is the easily observed frequency. Imagine that the top is precessing around the $\hat{\mathbf{z}}$ axis at constant θ, and imagine that the top has a frictionless rod protruding along its symmetry axis. If you grab the rod and stop the precession motion (so that the top is now spinning around its stationary symmetry axis), then this spinning will occur at frequency ω_3. This is true because when you gab the rod, you apply a torque in only the (negative) $\hat{\mathbf{x}}_2$ direction. Therefore, you don't change L_3, and hence you don't change ω_3.

7. ω_2 is similar to ω_3, of course. ω_2 is what you use to obtain the component of $\mathbf{L}$ along $\hat{\mathbf{x}}_2$, because $L_2 = I_2\omega_2$. It is the frequency with which the top instantaneously rotates, as seen by someone at rest in a frame that rotates around the $\hat{\mathbf{x}}_3$ axis with frequency ω_3. (This is true because $\boldsymbol{\omega} = \omega_2\hat{\mathbf{x}}_2 + \omega_3\hat{\mathbf{x}}_3$, and the rotation of the frame causes you to not see the $\omega_3\hat{\mathbf{x}}_3$ part.) Again, this rotation is a little harder to see, because the $\hat{\mathbf{x}}_3$ axis changes with time.

8. ω_z is not very useful (as far as I can see). The most important thing to note about it is that it is *not* the frequency of precession around the $\hat{\mathbf{z}}$-axis, even though it is the projection of $\boldsymbol{\omega}$ onto $\hat{\mathbf{z}}$. The frequency of the precession is Ω, as we found above in eq. (8.61). A true, but somewhat useless, fact about ω_z is that if someone is at rest in a frame that rotates around the $\hat{\mathbf{z}}$ axis with frequency ω_z, then she will see all points in the top instantaneously rotating around the $\hat{\mathbf{x}}$-axis with frequency ω_x, where ω_x is the projection of $\boldsymbol{\omega}$ onto the horizontal $\hat{\mathbf{x}}$ axis. (This is true because $\boldsymbol{\omega} = \omega_x\hat{\mathbf{x}} + \omega_z\hat{\mathbf{z}}$, and the rotation of the frame causes you to not see the $\omega_z\hat{\mathbf{z}}$ part.)

8.7.3 Torque method

This method of solving the heavy top will be straightforward, although a little tedious. We include it here to (1) show that this problem can be done without resorting to Lagrangians, and to (2) get some practice using $\boldsymbol{\tau} = d\mathbf{L}/dt$.

We will make use of the form of $\boldsymbol{\omega}$ given in eq. (8.56), because there it is broken up into the principal-axis components. For convenience, define $\dot\beta = \dot\psi + \dot\phi\cos\theta$, so that

$$\boldsymbol{\omega} = \dot\beta\hat{\mathbf{x}}_3 + \dot\phi\sin\theta\hat{\mathbf{x}}_2 + \dot\theta\hat{\mathbf{x}}_1. \tag{8.62}$$

Note that we've returned to the most general motion, where $\dot\theta$ is not necessarily zero.

We will choose the tip of the top as our origin, which is assumed to be fixed on the table.[15] Let the principal moments relative to this origin be $I_1 = I_2 \equiv I$, and I_3. The angular momentum of the body is then

$$\mathbf{L} = I_3\dot\beta\hat{\mathbf{x}}_3 + I\dot\phi\sin\theta\hat{\mathbf{x}}_2 + I\dot\theta\hat{\mathbf{x}}_1. \tag{8.63}$$

We must now calculate $d\mathbf{L}/dt$. What makes this nontrivial is the fact that the $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$ unit vectors change with time (they change with θ and ϕ). But let's

[15]We could use the CM as our origin, but then we would have to include the complicated forces acting at the pivot point, which is difficult.

forge ahead and take the derivative of eq. (8.63). Using the product rule (which works fine with the product of a scalar and a vector), we have

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= I_3\frac{d\dot\beta}{dt}\hat{\mathbf{x}}_3 + I\frac{d(\dot\phi\sin\theta)}{dt}\hat{\mathbf{x}}_2 + I\frac{d\dot\theta}{dt}\hat{\mathbf{x}}_1 \\ &\quad + I_3\dot\beta\frac{d\hat{\mathbf{x}}_3}{dt} + I\dot\phi\sin\theta\frac{d\hat{\mathbf{x}}_2}{dt} + I\dot\theta\frac{d\hat{\mathbf{x}}_1}{dt}\,. \end{aligned} \tag{8.64}$$

Using a little geometry, you can show

$$\begin{aligned}\frac{d\hat{\mathbf{x}}_3}{dt} &= -\dot\theta\hat{\mathbf{x}}_2 + \dot\phi\sin\theta\hat{\mathbf{x}}_1, \\ \frac{d\hat{\mathbf{x}}_2}{dt} &= \dot\theta\hat{\mathbf{x}}_3 - \dot\phi\cos\theta\hat{\mathbf{x}}_1, \\ \frac{d\hat{\mathbf{x}}_1}{dt} &= -\dot\phi\sin\theta\hat{\mathbf{x}}_3 + \dot\phi\cos\theta\hat{\mathbf{x}}_2. \end{aligned} \tag{8.65}$$

As an exercise, prove these by making use of Fig. 8.26. In the first equation, for example, show that a change in θ causes $\hat{\mathbf{x}}_3$ to move a certain distance in the $\hat{\mathbf{x}}_2$ direction; and show that a change in ϕ causes $\hat{\mathbf{x}}_3$ to move a certain distance in the $\hat{\mathbf{x}}_1$ direction. Plugging eqs. (8.65) into eq. (8.64) gives

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= I_3\ddot\beta\hat{\mathbf{x}}_3 + \Big(I\ddot\phi\sin\theta + 2I\dot\theta\dot\phi\cos\theta - I_3\dot\beta\dot\theta\Big)\hat{\mathbf{x}}_2 \\ &\quad + \Big(I\ddot\theta - I\dot\phi^2\sin\theta\cos\theta + I_3\dot\beta\dot\phi\sin\theta\Big)\hat{\mathbf{x}}_1. \end{aligned} \tag{8.66}$$

The torque on the top arises from gravity pulling down on the CM. $\boldsymbol{\tau}$ points in the $\hat{\mathbf{x}}_1$ direction and has magnitude $Mg\ell\sin\theta$, where ℓ is the distance from the pivot to CM. Equating $\boldsymbol{\tau}$ with $d\mathbf{L}/dt$ gives

$$\ddot\beta = 0, \tag{8.67}$$

for the $\hat{\mathbf{x}}_3$ component. Therefore, $\dot\beta$ is a constant, which we will call ω_3 (an obvious label, in view of eq. (8.62)). The other two components of $\boldsymbol{\tau} = d\mathbf{L}/dt$ then give

$$\begin{aligned} I\ddot\phi\sin\theta + \dot\theta(2I\dot\phi\cos\theta - I_3\omega_3) &= 0, \\ (Mg\ell + I\dot\phi^2\cos\theta - I_3\omega_3\dot\phi)\sin\theta &= I\ddot\theta. \end{aligned} \tag{8.68}$$

We will wait to fiddle with these equations until we have derived them again using the Lagrangian method.

8.7.4 Lagrangian method

Eq. (8.13) gives the kinetic energy of the top as $T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}$. Eqs. (8.62) and (8.63) give (using $\dot\psi + \dot\phi\cos\theta$ instead of the shorthand $\dot\beta$)[16]

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \frac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2 + \frac{1}{2}I(\dot\phi^2\sin^2\theta + \dot\theta^2). \tag{8.69}$$

[16] It was ok to use β in Subsection 8.7.3; we introduced it simply because it was quicker to write. But we can't use β here, because it depends on the other coordinates, and the Lagrangian method requires the use of independent coordinates. (The variational proof back in Chapter 5 assumed this independence.)

The potential energy is

$$V = Mg\ell\cos\theta, \tag{8.70}$$

where ℓ is the distance from the pivot to CM. The Lagrangian is $\mathcal{L} = T - V$ (we'll use "$\mathcal{L}$" here to avoid confusion with the angular momentum, "L"), and so the equation of motion obtained from varying ψ is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\psi} = \frac{\partial\mathcal{L}}{\partial\psi} \Longrightarrow \frac{d}{dt}(\dot\psi + \dot\phi\cos\theta) = 0. \tag{8.71}$$

Therefore, $\dot\psi + \dot\phi\cos\theta$ is a constant. Call it ω_3. The equations of motion obtained from varying ϕ and θ are then

$$\begin{aligned}\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\phi} = \frac{\partial\mathcal{L}}{\partial\phi} \quad &\Longrightarrow \quad \frac{d}{dt}\Big(I_3\omega_3\cos\theta + I\dot\phi\sin^2\theta\Big) = 0,\\ \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = \frac{\partial\mathcal{L}}{\partial\theta} \quad &\Longrightarrow \quad I\ddot\theta = (Mg\ell + I\dot\phi^2\cos\theta - I_3\omega_3\dot\phi)\sin\theta.\end{aligned} \tag{8.72}$$

These are equivalent to eqs. (8.68), as you can check. Note that there are two conserved quantities, arising from the facts that $\partial\mathcal{L}/\partial\psi$ and $\partial\mathcal{L}/\partial\phi$ equal zero. The conserved quantities are simply the angular momenta in the $\hat{\mathbf{x}}_3$ and $\hat{\mathbf{z}}$ directions, respectively. (There is no torque in the plane spanned by these vectors, since the torque points in the $\hat{\mathbf{x}}_1$ direction.)

8.7.5 Gyroscope with $\dot\theta = 0$

A special case of eqs. (8.68) occurs when $\dot\theta = 0$. In this case, the first of eqs. (8.68) says that $\dot\phi$ is a constant. The CM of the top therefore undergoes uniform circular motion in a horizontal plane. Let $\Omega \equiv \dot\phi$ be the frequency of this motion (this is the same notation as in eq. (8.59)). Then the second of eqs. (8.68) says that

$$I\Omega^2\cos\theta - I_3\omega_3\Omega + Mg\ell = 0. \tag{8.73}$$

This quadratic equation may be solved to yield two possible precessional frequencies for the top. (Yes, there are indeed two of them, provided that ω_3 is greater than a certain minimum value.)

The previous pages in this "Heavy Symmetric Top" section have been a bit abstract. So let's now pause for a moment, take a breather, and rederive eq. (8.73) from scratch. That is, we'll assume $\dot\theta = 0$ from the start of the solution, and solve things by simply finding $\mathbf{L}$ and using $\boldsymbol{\tau} = d\mathbf{L}/dt$, in the spirit of section 8.4.2.

The following Gyroscope example is the classic "top" problem. We'll warm up by solving it in an approximate way. Then we'll do it for real.

Example (Gyroscope): A symmetric top of mass M has its CM a distance ℓ from its pivot. The moments of inertia relative to the pivot are $I_1 = I_2 \equiv I$ and I_3. The top spins around its symmetry axis with frequency ω_3 (in the language of Section 8.7.2), and initial conditions have been set up so that the CM precesses in a circle around the vertical axis. The symmetry axis makes a constant angle θ with the vertical (see Fig. 8.28).

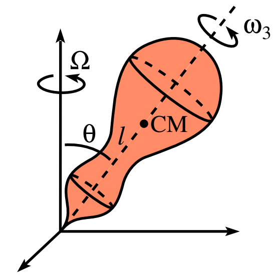


Figure 8.28

(a) Assuming that the angular momentum due to ω_3 is much larger than any other angular momentum in the problem, find an approximate expression for the frequency, Ω, of precession.

(b) Now do the problem exactly. That is, find Ω by considering all of the angular momentum.

Solution:

(a) The angular momentum (relative to the pivot) due to the spinning of the top has magnitude $L_3 = I_3\omega_3$, and it is directed along $\hat{\mathbf{x}}_3$. Let's label this angular momentum vector as $\mathbf{L}_3 \equiv L_3\hat{\mathbf{x}}_3$. As the top precesses, $\mathbf{L}_3$ traces out a cone around the vertical axis. So the tip of $\mathbf{L}_3$ moves in a circle of radius $L_3 \sin\theta$. The frequency of this circular motion is the frequency of precession, Ω. So $d\mathbf{L}_3/dt$, which is the velocity of the tip, has magnitude

$$\Omega(L_3 \sin\theta) = \Omega I_3\omega_3 \sin\theta, \tag{8.74}$$

and is directed into the page.

The torque relative to the pivot point is due to gravity acting on the CM, so it has magnitude $Mg\ell\sin\theta$. It is directed into the page. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$\Omega = \frac{Mg\ell}{I_3\omega_3}\,. \tag{8.75}$$

Note that this is independent of θ.

(b) The error in the above analysis is that we omitted the angular momentum arising from the $\hat{\mathbf{x}}_2$ (defined in Section 8.7.1) component of the angular velocity due to the precession of the top around the $\hat{\mathbf{z}}$-axis. This component has magnitude $\Omega\sin\theta$.[17] The angular momentum due to this angular velocity component has magnitude

$$L_2 = I\Omega\sin\theta, \tag{8.76}$$

and is directed along $\hat{\mathbf{x}}_2$. Let's label this as $\mathbf{L}_2 \equiv L_2\hat{\mathbf{x}}_2$. The total $\mathbf{L} = \mathbf{L}_2 + \mathbf{L}_3$ is shown in Fig. 8.29.

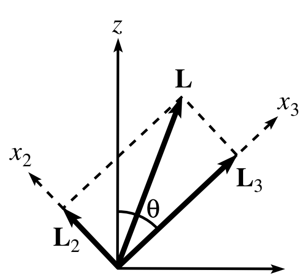


Figure 8.29

Only the horizontal component of $\mathbf{L}$ (call it $L_\perp$) changes. From the figure, $L_\perp$ is the difference in lengths of the horizontal components of $\mathbf{L}_3$ and $\mathbf{L}_2$. Therefore,

$$L_\perp = L_3\sin\theta - L_2\cos\theta = I_3\omega_3\sin\theta - I\Omega\sin\theta\cos\theta. \tag{8.77}$$

The magnitude of the rate of change of $\mathbf{L}$ is simply $\Omega L_\perp = \Omega(I_3\omega_3\sin\theta - I\Omega\sin\theta\cos\theta)$.[18] Equating this with the torque, $Mg\ell\sin\theta$, gives

$$I\Omega^2\cos\theta - I_3\omega_3\Omega + Mg\ell = 0, \tag{8.78}$$

in agreement with eq. (8.73), as we wanted to show. The quadratic formula quickly gives the two solutions for Ω, which may be written as

$$\Omega_\pm = \frac{I_3\omega_3}{2I\cos\theta}\left(1 \pm \sqrt{1 - \frac{4MIg\ell\cos\theta}{I_3^2\omega_3^2}}\right). \tag{8.79}$$

[17] The angular velocity due to the precession is $\Omega\hat{\mathbf{z}}$. We may break this up into components along the orthogonal directions $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_3$. The $\Omega\cos\theta$ component along $\hat{\mathbf{x}}_3$ was absorbed into the definition of ω_3 (see Fig. 8.27).

[18] This result can also be obtained in a more formal way. Since $\mathbf{L}$ precesses with angular velocity $\Omega\hat{\mathbf{z}}$, the rate of change of $\mathbf{L}$ is $d\mathbf{L}/dt = \Omega\hat{\mathbf{z}} \times \mathbf{L}$. This cross product is easily computed in the x_2-x_3 basis, and gives the same result.

REMARK: Note that if $\theta = \pi/2$, then eq. (8.78) is actually a linear equation, so there is only one solution for Ω, which is the one in eq. (8.75). $\mathbf{L}_2$ points vertically, so it doesn't change. Only $\mathbf{L}_3$ contributes to $d\mathbf{L}/dt$. For this reason, a gyroscope is much easier to deal with when its symmetry axis is horizontal. ♣

The two solutions in eq. (8.79) are known as the *fast-precession* and *slow-precession* frequencies. For large ω_3, you can show that the slow-precession frequency is

$$\Omega_- \approx \frac{Mg\ell}{I_3\omega_3}\,, \tag{8.80}$$

in agreement with the solution found in eq. (8.75).[19] This task, along with many other interesting features of this problem (including the interpretation of the fast-precession frequency, Ω_+), is the subject of Problem 16, which you are strongly encouraged to do.

8.7.6 Nutation

Let us now solve eqs. (8.68) in a somewhat more general case, where θ is allowed to vary slightly. That is, we will consider a slight perturbation to the circular motion associated with eq. (8.73). We will assume ω_3 is large here, and we will assume that the original circular motion corresponds to the slow precession, so that $\dot\phi$ is small. Under these assumptions, we will find that the top will bounce around slightly as it travels (roughly) in a circle. This bouncing is known as *nutation*.

Since $\dot\theta$ and $\dot\phi$ are small, we can (to a good approximation) ignore the quadratic terms in eqs. (8.68) and obtain

$$\begin{aligned} I\ddot\phi\sin\theta - \dot\theta I_3\omega_3 &= 0, \\ (Mg\ell - I_3\omega_3\dot\phi)\sin\theta &= I\ddot\theta. \end{aligned} \tag{8.81}$$

We must somehow solve these equations for $\theta(t)$ and $\phi(t)$. Taking the derivative of the first equation and dropping the quadratic term gives $\ddot\theta = (I\sin\theta/I_3\omega_3)\, d^2\dot\phi/dt^2$. Substituting this into the second equation gives

$$\frac{d^2\dot\phi}{dt^2} + \omega_n^2(\dot\phi - \Omega_s) = 0, \tag{8.82}$$

where

$$\omega_n \equiv \frac{I_3\omega_3}{I} \qquad \text{and} \qquad \Omega_s = \frac{Mg\ell}{I_3\omega_3} \tag{8.83}$$

are, respectively, the frequency of nutation (as we shall soon see), and the slow-precession frequency given in eq. (8.75). Shifting variables to $y \equiv \dot\phi - \Omega_s$ in eq. (8.82) gives us a nice harmonic-oscillator equation. Solving this and then shifting back to $\dot\phi$ yields

$$\dot\phi(t) = \Omega_s + A\cos(\omega_n t + \gamma), \tag{8.84}$$

[19]This is fairly clear. If ω_3 is large enough compared to Ω, then we can ignore the first term in eq. (8.78). That is, we can ignore the effects of $\mathbf{L}_2$, which is exactly what we did in the approximate solution in part (a).

where A and γ are determined from initial conditions. Integrating this gives

$$\phi(t) = \Omega_s t + \left(\frac{A}{\omega_n}\right)\sin(\omega_n t + \gamma), \tag{8.85}$$

plus an irrelevant constant.

Now let's solve for $\theta(t)$. Plugging $\phi(t)$ into the first of eqs. (8.81) gives

$$\dot\theta(t) = -\left(\frac{I\sin\theta}{I_3\omega_3}\right)A\omega_n\sin(\omega_n t + \gamma) = -A\sin\theta\sin(\omega_n t + \gamma). \tag{8.86}$$

Since $\theta(t)$ doesn't change much, we may set $\sin\theta \approx \sin\theta_0$, where θ_0 is, say, the initial value of $\theta(t)$. (Any errors here are second-order effects in small quantities.) Integration then gives

$$\theta(t) = B + \left(\frac{A}{\omega_n}\sin\theta_0\right)\cos(\omega_n t + \gamma), \tag{8.87}$$

where B is a constant of integration.

Eqs. (8.85) and (8.87) show that both ϕ (neglecting the uniform $\Omega_s t$ part) and θ oscillate with frequency ω_n, and with amplitudes inversely proportional to ω_n. Note that eq. (8.83) says ω_n grows with ω_3.

Example (Sideways kick): Assume that uniform circular precession is initially taking place with $\theta = \theta_0$ and $\dot\phi = \Omega_s$. You then give the top a quick kick along the direction of motion, so that $\dot\phi$ is now equal to $\Omega_s + \Delta\Omega$ ($\Delta\Omega$ may be positive or negative). Find $\phi(t)$ and $\theta(t)$.

Solution: This is simply an exercise in initial conditions. We are given the initial values for $\dot\phi$, $\dot\theta$, and θ. So we will need to solve for the unknowns A, B and γ in eqs. (8.84), (8.86), and (8.87). $\dot\theta$ is initially zero, so eq. (8.86) gives $\gamma = 0$. And $\dot\phi$ is initially $\Omega_s + \Delta\Omega$, so eq. (8.84) gives $A = \Delta\Omega$. Finally, θ is initially θ_0, so eq. (8.87) gives $B = \theta_0 - (\Delta\Omega/\omega_n)\sin\theta_0$. Putting it all together, we have

$$\begin{aligned}\phi(t) &= \Omega_s t + \left(\frac{\Delta\Omega}{\omega_n}\right)\sin\omega_n t,\\ \theta(t) &= \left(\theta_0 - \frac{\Delta\Omega}{\omega_n}\sin\theta_0\right) + \left(\frac{\Delta\Omega}{\omega_n}\sin\theta_0\right)\cos\omega_n t.\end{aligned} \tag{8.88}$$

And for future reference (in the problems for this chapter), we'll also list the derivatives,

$$\begin{aligned}\dot\phi(t) &= \Omega_s + \Delta\Omega\cos\omega_n t,\\ \dot\theta(t) &= -\Delta\Omega\sin\theta_0\sin\omega_n t.\end{aligned} \tag{8.89}$$

REMARKS:

(a) With the initial conditions we have chosen, eq. (8.88) shows that θ always stays on one side of θ_0. If $\Delta\Omega > 0$, then $\theta(t) \le \theta_0$ (that is, the top is always at a higher position, since θ is measured from the vertical). If $\Delta\Omega < 0$, then $\theta(t) \ge \theta_0$ (that is, the top is always at a lower position).

(b) The $\sin\theta_0$ coefficient of the $\cos\omega_n t$ term in eq. (8.88) implies that the amplitude of the θ oscillation is $\sin\theta_0$ times the amplitude of the ϕ oscillation. This is precisely the factor needed to make the CM travel in a circle around its average precessing position (because a change in θ causes a displacement of $\ell\, d\theta$, whereas a change in ϕ causes a displacement of $\ell \sin\theta_0\, d\phi$).

(c) Fig. [nutate] shows plots of $\theta(t)$ vs. $\phi(t)$ for various values of $\Delta\Omega$. ♣

8.8 Exercises

Section 8.2: The inertia tensor

1. **Inertia tensor** $*$

   Calculate the $\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$ double cross-product in eq. (8.7) by using the vector identity,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}. \tag{8.90}$$

Section 8.3: Principal axes

2. **Tennis racket theorem** $**$

   Problem 14 gives the statement of the "tennis racket theorem," and the solution there involves Euler's equations.

   Demonstrate the theorem here by using conservation of L^2 and conservation of rotational kinetic energy in the following way. Produce an equation which says that if ω_2 and ω_3 (or ω_1 and ω_2) start small, then they must remain small. Also, produce the analogous equation which says that if ω_1 and ω_3 start small, then they need *not* remain small.[20]

Section 8.4: Two basic types of problems

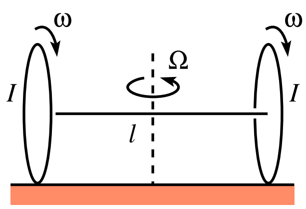


Figure 8.30

3. **Rotating axle** $**$

   Two wheels (with moment of inertia I) are connected by a rod of length ℓ, as shown in Fig. 8.30. The system rests on a frictionless surface, and the wheels rotate with frequency ω around the axis of the rod. Additionally, the whole system rotates with frequency Ω around the vertical axis through the center of the rod. What is largest value of Ω for which both wheels stay on the ground?

Section 8.7: Heavy symmetric top

4. **Relation between Ω and ω'** $**$

   Initial conditions have been set up so that a symmetric top undergoes precession, with its symmetry axis always making an angle θ with the vertical. The top has mass M, and the principal moments are I_3 and $I \equiv I_1 = I_2$. The CM is a distance ℓ from the pivot. In the language of Fig. 8.27, show that ω' must be related to Ω by

$$\omega' = \frac{Mg\ell}{I_3\Omega} + \Omega\cos\theta\left(\frac{I - I_3}{I_3}\right). \tag{8.91}$$

   Note: You could just plug $\omega_3 = \omega' + \Omega\cos\theta$ (from eq. (8.60)) into eq. (8.73), and then solve for ω'. But do this problem from scratch, using $\boldsymbol{\tau} = d\mathbf{L}/dt$.

[20] It's another matter to show that they actually *won't* remain small. But don't bother with that here.

5. **Sliding lollipop** ∗∗∗

Consider a lollipop made of a solid sphere of mass m and radius r, which is radially pierced by massless stick. The free end of the stick is pivoted on the ground, which is frictionless (see Fig. 8.31). The sphere slides along the ground (keeping the same contact point), with its center moving in a circle of radius R, with frequency Ω.

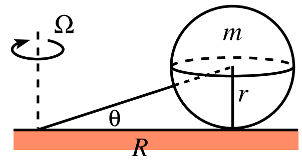


Figure 8.31

Show that the normal force between the ground and the sphere is $F_N = mg + mr\Omega^2$ (which is independent of R). Solve this by:

(a) Using a simple $\mathbf{F} = m\mathbf{a}$ argument.[21]

(b) Using a (more complicated) $\boldsymbol{\tau} = d\mathbf{L}/dt$ argument.

6. **Rolling wheel and axle** ∗∗∗

A massless stick has one end attached to a wheel (which is a uniform disc of mass m and radius r) and the other end pivoted on the ground (see Fig. 8.32). The wheel rolls on the ground without slipping, with the axle inclined at an angle θ. The point of contact with the ground traces out a circle with frequency Ω.

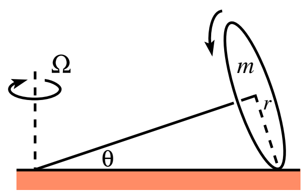


Figure 8.32

(a) Show that $\boldsymbol{\omega}$ points horizontally to the right (at the instant shown), with magnitude $\omega = \Omega/\tan\theta$.

(b) Show that the normal force between the ground and the wheel is

$$N = mg\cos^2\theta + mr\Omega^2\left(\frac{3}{2}\cos^3\theta + \frac{1}{4}\cos\theta\sin^2\theta\right). \qquad (8.92)$$

[21] This method happens to work here, due to the unusually nice nature of the sphere's motion. For more general motion (for example, in Problem 21, where the sphere is spinning), you must use $\vec{\tau} = d\mathbf{L}/dt$.

8.9 Problems

Section 8.1: Preliminaries concerning rotations

1. **Fixed points on a sphere** $**$

   Consider a transformation of a rigid sphere into itself. Show that two points on the sphere end up where they started.

2. **Many different $\vec{\omega}$'s** $*$

   Consider a particle at the point $(a, 0, 0)$, with velocity $(0, v, 0)$. This particle may be considered to be rotating around many different $\boldsymbol{\omega}$ vectors passing through the origin. (There is no one 'correct' $\boldsymbol{\omega}$.) Find all the possible $\boldsymbol{\omega}$'s (that is, find their directions and magnitudes).

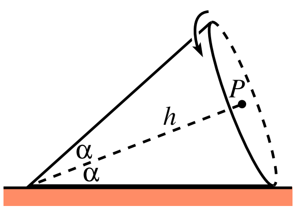


Figure 8.33

3. **Rolling cone** $**$

   A cone rolls without slipping on a table. The half-angle at the vertex is α, and the axis of the cone has length h (see Fig. 8.33). Let the speed of point P (the middle of the base) be v. What is the angular velocity of the cone with respect to the lab frame (at the instant shown)?

   (There are many ways to do this problem, so you may want to take a look at the three given solutions, even if you solve it.)

Section 8.2: The inertia tensor

4. **Parallel axis theorem**

   Let (X, Y, Z) be the position of an object's CM, and let (x', y', z') be the position relative to the CM. Prove the parallel axis theorem by setting $x = X + x'$, $y = Y + y'$, and $z = Z + z'$ in eq. (8.8).

Section 8.3: Principal axes

5. **Existence of principal axes for a pancake** $*$

   Given a pancake object in the x-y plane, show that there exist principal axes by considering what happens to the integral $\int xy$ as the coordinate axes are rotated about the origin.

6. **Symmetries and principal axes for a pancake** $**$

   A rotation of the axes in the x-y plane through an angle θ transforms the coordinates according to

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{8.93}$$

   Use this to show that if a pancake object in the x-y plane has a symmetry under a rotation through $\theta \neq \pi$, then all axes (through the origin) in the plane are principal axes.

7. **A nice cylinder** $*$

What must the ratio of height to diameter of a cylinder be so that every axis is a principal axis (with the CM as the origin)?

8. **Rotating square** $*$

Here's an exercise in geometry. Theorem 8.5 says that if the moments of inertia of two principal axes are equal, then any axis in the plane of these axes is a principal axis. This means that the object will rotate happily about any axis in this plane (no torque is needed). Demonstrate this explicitly for four masses m in the shape of a square (which obviously has two moments equal), with the CM as the origin (see Fig. 8.34). Assume that the masses are connected with strings to the axis, as shown. Your task is to show that the tensions in the strings are such that there is no torque about the center of the square.

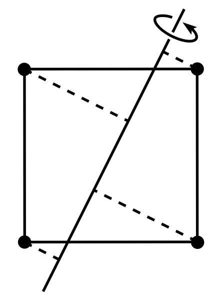


Figure 8.34

Section 8.4: Two basic types of problems

9. **Rotating rectangle** $*$

A flat rectangle with sides of length a and b sits it space (not rotating). You strike the corners at the ends of one diagonal, with equal and opposite forces (see Fig. 8.35). Show that the resulting initial $\boldsymbol{\omega}$ points along the other diagonal.

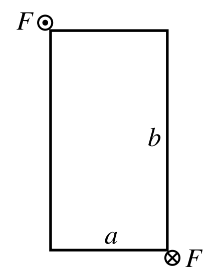


Figure 8.35

10. **Rotating stick** $**$

A stick of mass m and length ℓ spins with frequency ω around an axis, as shown in Fig. 8.36. The stick makes an angle θ with the axis and is pivoted at its center. The stick is kept in this motion by two strings which are perpendicular to the axis. What is the tension in the strings?

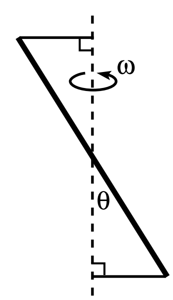


Figure 8.36

11. **Another rotating stick** $**$

A stick of mass m and length $2r$ is arranged to have its CM motionless and its top end slide in a circle on a frictionless rail (see Fig. 8.37). The stick makes an angle θ with the vertical. What is the frequency of this motion?

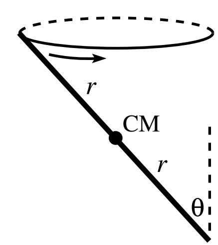


Figure 8.37

12. **Spherical pendulum** $**$

Consider a pendulum made of a massless rod of length ℓ and a point mass m. Assume conditions have been set up so that the mass moves in a horizontal circle, with the rod always making an angle θ with respect to the vertical. Find the frequency, Ω, of this circular motion in three different ways.

(a) Use $\mathbf{F} = m\mathbf{a}$. (The net force accounts for the centripetal acceleration.)[22]

(b) Use $\boldsymbol{\tau} = d\mathbf{L}/dt$ with the pendulum pivot as the origin.

(c) Use $\boldsymbol{\tau} = d\mathbf{L}/dt$ with the mass as the origin.

[22]This method works only if you have a point mass. With an extended object, you have to use one of the following methods involving torque.

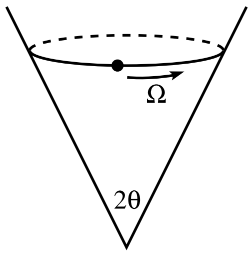


Figure 8.38

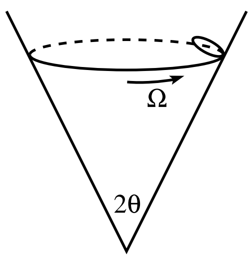


Figure 8.39

13. **Rolling on a cone** ***

(a) A fixed cone stands on its tip, with its axis in the vertical direction. The half-angle at the vertex is θ. A particle of negligible size slides on the inside surface of the cone (see Fig. 8.38). This surface is frictionless.

Assume conditions have been set up so that the particle moves in a circle at height h above the tip. What is the frequency, Ω, of this circular motion?

(b) Assume now that the surface has friction, and a small ring of radius r rolls without slipping on the surface. Assume conditions have been set up so that (1) the point of contact between the ring and the cone moves in a circle at height h above the tip, and (2) the plane of the ring is at all times perpendicular to the line joining the point of contact and the tip of the cone (see Fig. 8.39).

What is the frequency, Ω, of this circular motion? How does it compare to the answer in part (a)?

(Note: You may work in the approximation where r is much less than the radius of the circular motion, $h\tan\theta$.)

Section 8.5: Euler's equations

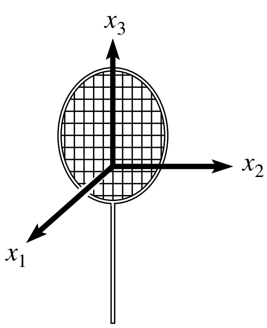


Figure 8.40

14. **Tennis racket theorem** ***

If you try to spin a tennis racket (or a book, etc.) around any of its three principal axes, you will notice that different things happen with the different axes. Assuming that the principal moments (relative to the CM) are labeled according to $I_1 > I_2 > I_3$ (see Fig. 8.40), you will find that the racket will spin nicely around the $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_3$ axes, but it will wobble in a rather messy manner if you try to spin it around the $\hat{\mathbf{x}}_2$ axis.

(a) Verify this claim experimentally with a book (preferably lightweight, and wrapped with a rubber band), or a tennis racket (if you happen to study with one on hand).

(b) Verify this claim mathematically. The main point here is that you clearly can't start the motion off with $\boldsymbol{\omega}$ pointing *exactly* along a principal axis. Therefore, what you want to show is that the motion around the $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_3$ axes is *stable* (that is, small errors in the initial conditions remain small); whereas the motion around the $\hat{\mathbf{x}}_2$ axis is *unstable* (that is, small errors in the initial conditions get larger and larger, until the motion eventually doesn't resemble rotation around the $\hat{\mathbf{x}}_2$ axis).[23]

Your task is to use Euler's equations to prove these statements about stability.

[23]If you try for a long enough time, you will eventually be able to get the initial $\vec{\omega}$ pointing close enough to $\hat{\mathbf{x}}_2$ so that the book remains rotating (almost) around $\hat{\mathbf{x}}_2$ for the entire time of its flight. There is, however, probably a better use for your time, as well as the book . . .

(Exercise (2) gives another derivation of this result.)

Section 8.6: Free symmetric top

15. **Free-top angles** $*$

    In section 8.6.2, we showed that for a free symmetric top, the angular momentum $\mathbf{L}$, the angular velocity $\boldsymbol{\omega}$, and the symmetry axis $\hat{\mathbf{x}}_3$, all lie in a plane. Let α be the angle between $\hat{\mathbf{x}}_3$ and $\mathbf{L}$, and let β be the angle between $\hat{\mathbf{x}}_3$ and $\boldsymbol{\omega}$ (see Fig. 8.41). Find the relationship between α and β in terms of the principal moments, I and I_3.

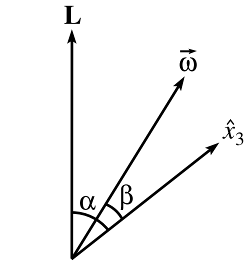


Figure 8.41

Section 8.7: Heavy symmetric top

16. **Gyroscope** $**$ This problem deals with the gyroscope example in section 8.7.5, and uses the result for Ω in eq. (8.79).

    (a) What is the minimum ω_3 for which circular precession is possible?

    (b) Let ω_3 be very large, and find approximate expressions for $\Omega_\pm$.
    The phrase "very large" is rather meaningless. What mathematical statement should replace it?

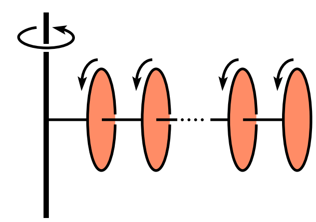


Figure 8.42

17. **Many gyroscopes** $***$

    N identical plates and massless sticks are arranged as shown in Fig. 8.42. Each plate is glued to the stick on its left. And each plate is attached by a free pivot to the stick on its right. (And the leftmost stick is attached by a free pivot to a pole.)

    You wish to set up a circular precession with the sticks always forming a straight horizontal line. What should the relative angular speeds of the plates be so that this is possible?

Figure 8.43

18. **Heavy top on slippery table** $*$

    Solve the problem of a heavy symmetric top spinning on a frictionless table (see Fig. 8.43). You may do this by simply stating what modifications are needed in the derivation in section 8.7.

19. **Fixed highest point** $**$

    Consider a top made of a uniform disc of radius R, connected to the origin by a massless stick (which is perpendicular to the disc) of length ℓ. Label the highest point on the top as P (see Fig. 8.44). You wish to set up uniform circular precession, with the stick making a constant angle θ with the vertical, and with P always being the highest point on the top.

    What relation between R and ℓ must be satisfied for such motion to be possible? What is the frequency of precession, Ω?

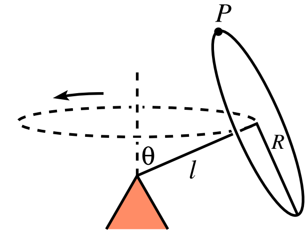


Figure 8.44

20. **Basketball on rim** ∗∗∗

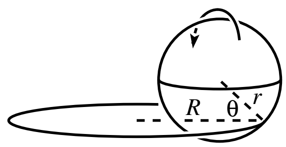


Figure 8.45

A basketball rolls without slipping around a basketball rim in such a way that the contact points trace out a great circle on the ball, and the CM moves around in a horizontal circle with frequency Ω. The radii of the ball and rim are r and R, respectively, and the ball's radius to the contact point makes an angle θ with the horizontal (see Fig. 8.45). Assume that the ball's moment of inertia around its center is $I = (3/5)mr^2$. Find Ω.

21. **Rolling lollipop** ∗∗∗

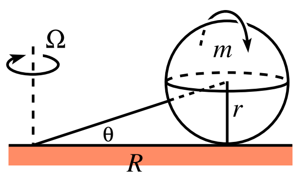


Figure 8.46

Consider a lollipop made of a solid sphere of mass m and radius r, which is radially pierced by massless stick. The free end of the stick is pivoted on the ground (see Fig. 8.46). The sphere rolls on the ground without slipping, with its center moving in a circle of radius R, with frequency Ω.

(a) Find the angular velocity vector, $\boldsymbol{\omega}$.

(b) What is the normal force between the ground and the sphere?

22. **Rolling coin** ∗∗∗∗

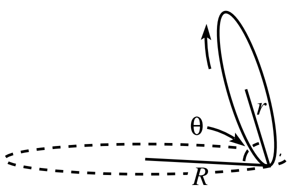


Figure 8.47

Initial conditions have been set up so that a coin of radius r rolls around in a circle, as shown in Fig. 8.47. The contact point on the ground traces out a circle of radius R, and the coin always makes an angle θ with the horizontal. The coin rolls without slipping. (Assume that the friction with the ground is as large as needed.)

What is the frequency of the circular motion of the contact point on the ground? Show that such motion exists only if $R > (5/6)r\cos\theta$.

23. **Wobbling coin** ∗∗∗∗

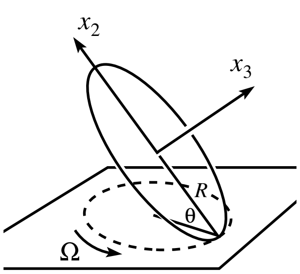


Figure 8.48

If you spin a coin around a vertical diameter on a table, it will slowly lose energy and begin a wobbling motion. The angle between the coin and the table will decrease, and eventually the coin will come to rest.

Assume that this process is slow, and consider the situation when the coin makes an angle θ with the table (see Fig. 8.48). You may assume that the CM is essentially motionless. Let R be the radius of the coin, and let Ω be the frequency at which the point of contact on the table traces out its circle. Assume that the coin rolls without slipping.

(a) Show that the angular velocity vector of the coin is $\boldsymbol{\omega} = \Omega\sin\theta\hat{\mathbf{x}}_2$, where $\hat{\mathbf{x}}_2$ points upward along the coin, directly away from the contact point (as in the notation of Fig. 8.27)

(b) Show that $\Omega = 2\sqrt{g/R}/\sqrt{\sin\theta}$.

(c) Show that Abe (or Tom, Franklin, George, John, Dwight, Sue, or Sacagawea) appears to rotate with frequency $2(1-\cos\theta)\sqrt{g/R}/\sqrt{\sin\theta}$, when viewed from above.

24. **Nutation cusps** $**$

    (a) Using the notation and initial conditions of the example in Section 8.7.6, prove that kinks occur in nutation if and only if $\Delta\Omega = \pm\Omega_s$. (A kink is where the plot of $\theta(t)$ vs. $\phi(t)$ has a discontinuity in its slope.)

    (b) Prove that these kinks are in fact cusps. (A cusp is a kink where the parametric plot reverses direction in the ϕ-θ plane).

25. **Nutation circles** $**$

    (a) Using the notation and initial conditions of the example in section 8.7.6, and assuming $\omega_3 \gg \Delta\Omega \gg \Omega_s$, find (approximately) the direction of the angular momentum right after the sideways kick takes place.

    (b) Use eqs. (8.88) to then show that the CM travels (roughly) in a circle around $\mathbf{L}$. And show that this 'circular' motion is just what you would expect from the reasoning in section 8.6.2 (in particular, eq. (8.53)), concerning the free top.

Additional problems

26. **Rolling straight?** $**$

    The velocity of the CM of the coin in Problem 22 changes direction as time goes by. Consider now a uniform sphere rolling on the ground without slipping. Is it possible for the velocity of its CM to change direction? Justify your answer rigorously.

27. **Ball on paper** $***$

    A ball rolls across the (horizontal) floor without slipping. It rolls onto a piece of paper, which you slide around in an arbitrary manner. (Any horizontal motion of the paper is allowed, even jerky motions which cause the ball to slip with respect to it.) The ball eventually returns to the floor and finally resumes rolling without slipping.

    Show that the final velocity (i.e., both speed and direction) of the ball is the same as the initial velocity.

    (This one is a gem. Don't peek at the answer too soon!)

28. **Ball on turntable** $****$

    A ball (with uniform mass density) rolls without slipping on a turntable. Show that the ball moves in a circle (as viewed from the inertial lab frame), with a frequency equal to 2/7 times the frequency of the turntable.

8.10 Solutions

1. **Fixed points on a sphere**

First solution: For the purposes of Theorem 8.1, we only need to show that two points remain fixed for an *infinitesimal* transformation. But since it's possible to prove this result for a general transformation, we'll consider the general case here.

Consider the point A that ends up furthest away from where it started. (If there is more than one such point, pick any one of them.) Label the ending point B. Draw the great circle, C_{AB}, through A and B. Draw the great circle, C_A, which is perpendicular to C_{AB} at A; and draw the great circle, C_B, which is perpendicular to C_{AB} at B.

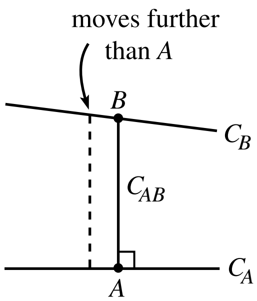


Figure 8.49

We claim that the transformation must take C_A to C_B. This is true for the following reason. The image of C_A is certainly a great circle through B, and this great circle must be perpendicular to C_{AB}; otherwise, there would exist another point which ended up further away from its starting point than A did (see Fig. 8.49).

Now consider the two points, P_1 and P_2, where C_A and C_B intersect. (Any two great circles must intersect.) Let's look at P_1. The distances P_1A and P_1B are equal. Therefore, the point P_1 is not moved by the transformation. (If it did move, it would have to move to a different point along C_B, and hence its final distance from B would be different from its initial distance from A. This is impossible, since distances are preserved on the rigid sphere.) Likewise for P_2.

Note that for a non-infinitesimal transformation, every point on the sphere may move at some time during the transformation. What we just showed is that two of the points end up back where they started.

Second solution: In the spirit of the above solution, we can give simpler solution, but which is valid only in the case of infinitesimal transformations.

Pick any point, A, that moves during the transformation. Draw the great circle that passes through A and is perpendicular to A's motion. All points on this great circle must move (if they move at all) perpendicularly to the great circle, because otherwise their distances to A would change. But they cannot all move in the same direction, because then the center of the sphere would move (but it is assumed to remain fixed). Therefore, at least one point on the great circle moves in the opposite direction from how A moves. Therefore (by continuity), some point (and hence also its diametrically opposite point) on the great circle remains fixed.

2. **Many different $\vec{\omega}$'s**

We want to find all the vectors, $\boldsymbol{\omega}$, such that $\boldsymbol{\omega} \times a\hat{\mathbf{x}} = v\hat{\mathbf{y}}$. Since $\boldsymbol{\omega}$ is orthogonal to this cross product, $\boldsymbol{\omega}$ must lie in the x-z plane. Let $\boldsymbol{\omega}$ make an angle θ with the x-axis. Is this a possible direction for $\boldsymbol{\omega}$, and if so, what is its magnitude? The answers are 'yes', and '$v/(a\sin\theta)$'. Indeed, if $\boldsymbol{\omega}$ has magnitude $v/(a\sin\theta)$, then

$$\boldsymbol{\omega} \times a\hat{\mathbf{x}} = |\boldsymbol{\omega}||a\hat{\mathbf{x}}|\sin\theta\hat{\mathbf{y}} = v\hat{\mathbf{y}}. \tag{8.94}$$

(Alternatively, that $\boldsymbol{\omega}$ may be written as

$$\boldsymbol{\omega} = \frac{v}{a\sin\theta}(\cos\theta, 0, \sin\theta) = \left(\frac{v}{a\tan\theta}, 0, \frac{v}{a}\right), \tag{8.95}$$

and only the z-component here is relevant in the cross product with $a\hat{\mathbf{x}}$.)

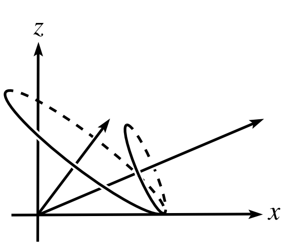


Figure 8.50

It is clear that the magnitude of $\boldsymbol{\omega}$ is $v/(a\sin\theta)$, because the particle is traveling in a circle of radius $a\sin\theta$ around $\boldsymbol{\omega}$, at speed v.

A few possible $\boldsymbol{\omega}$'s are drawn in Fig. 8.50. Technically, it is possible to have $\pi <$

$\theta < 2\pi$, but then the $v/(a \sin\theta)$ coefficient in eq. (8.95) is negative, so $\boldsymbol{\omega}$ really points upward in the x-z plane. (It is clear that $\boldsymbol{\omega}$ must point upward if the particle's velocity is to be in the positive y-direction.) Note that ω_z is independent of θ, so all the possible $\boldsymbol{\omega}$'s look like those in Fig. 8.51.

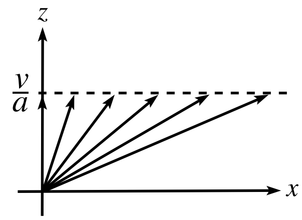


Figure 8.51

For $\theta = \pi/2$, we have $\omega = v/a$, which makes sense. If θ is very small, then ω is very large. This makes sense, because the particle is traveling around in a very small circle at the speed v.

REMARK: The point of this problem is that the particle may be in the process of having its position vector trace out a cone around one of many possible axes (or perhaps undergoing some other not-so-nice motion). If we are handed only the given information of position and velocity, then there is no possible way to determine which of these motions is happening. Indeed, it is irrelevant (and meaningless). All that we ever need to know in a rotational problem is the position and velocity at a given instant; the past and future motion is unimportant. ♣

3. **Rolling cone**

At the risk of overdoing it, we'll give three solutions. The second and third solutions are the type that tend to make your head hurt, so you may want to reread them after studying the discussion on the angular velocity vector in Section 8.7.2.

First solution: Without doing any calculations, we know that $\boldsymbol{\omega}$ points along the line of contact of the cone with the table (because these are the points on the cone that are instantaneously at rest). And we know that as time goes by, $\boldsymbol{\omega}$ rotates around in the horizontal plane with angular speed $v/(h \cos\alpha)$ (because P travels at speed v in a circle of radius $h \cos\alpha$ around the z-axis).

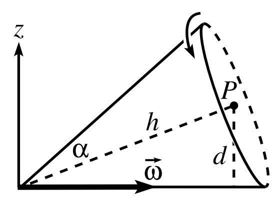


Figure 8.52

The magnitude of $\boldsymbol{\omega}$ may be found as follows. At a given instant, P may be considered to be rotating in a circle of radius $d = h \sin\alpha$ around $\boldsymbol{\omega}$. (see Fig. 8.52). Since the speed of P is v, the angular speed of this rotation is v/d. Therefore,

$$\omega = \frac{v}{h \sin\alpha} \,. \tag{8.96}$$

Second solution: We can use Theorem 8.3 with the following frames. S_1 is fixed in the cone; S_2 is the frame whose z-axis is fixed in the direction shown in Fig. 8.53, and whose y-axis is the axis of the cone; and S_3 is the lab frame. (Note that after the cone moves a little, we will need to use a new S_2 frame. But at each stage, the z-axis of S_2 is fixed.) In the language of Theorem 8.3, $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ point in the directions shown. We must find their magnitudes and add these vectors to find the angular velocity of S_1 with respect to S_3.

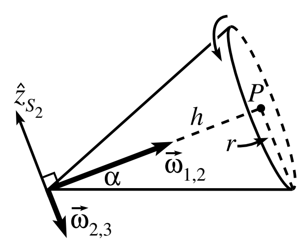


Figure 8.53

It is easy to see that

$$|\boldsymbol{\omega}_{1,2}| = v/r, \tag{8.97}$$

where $r = h \tan\alpha$ is the radius of the base of the cone. (This is true because someone fixed in S_2 will see the endpoint of this radius moving at speed v, since it is stationary with respect to the table. Hence the cone must be spinning with frequency v/r in S_2.)

Also,

$$|\boldsymbol{\omega}_{2,3}| = v/h, \tag{8.98}$$

since point P moves with speed v in a circle of radius h around $\boldsymbol{\omega}_{2,3}$. The addition of $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ is shown in Fig. 8.54. The result has magnitude $v/(h \sin\alpha)$, and it points horizontally (because $|\boldsymbol{\omega}_{2,3}|/|\boldsymbol{\omega}_{1,2}| = \tan\alpha$).

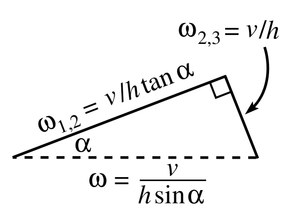


Figure 8.54

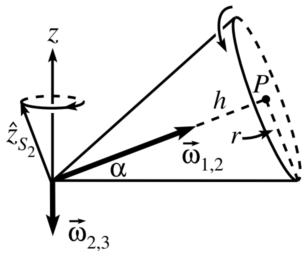


Figure 8.55

Third solution: We can use Theorem 8.3 with the following frames. S_1 is fixed in the cone; and S_3 is the lab frame (as in the second solution). But now let S_2 be the frame which initially has its z-axis pointing in the direction shown in Fig. 8.55, and let $\boldsymbol{\omega}_{2,3}$ point along the vertical axis in the lab frame (so the z-axis of S_2 precesses around the vertical axis of the lab frame; note that we can keep using this same S_2 frame as time goes by, unlike the S_2 frame in the second solution). $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ point in the directions shown. As above, we must find their magnitudes and add these vectors to find the angular velocity of S_1 with respect to S_3.

The easy one is

$$|\boldsymbol{\omega}_{2,3}| = v/(h\cos\alpha), \tag{8.99}$$

since point P moves with speed v in a circle of radius $h\cos\alpha$ around $\boldsymbol{\omega}_{2,3}$.

It is a little trickier, however, to find $|\boldsymbol{\omega}_{1,2}|$. Define Q to be the point on the table that touches the base of the cone. Q moves in a circle of radius $h/\cos\alpha$ with angular speed $|\boldsymbol{\omega}_{2,3}| = v/(h\cos\alpha)$ around the vertical. Therefore, Q moves with speed $v/\cos^2\alpha$ around the vertical. Since there is no slipping, Q also moves with this speed around the base of the cone. Hence Q (which is a fixed point in S_2) moves with angular speed $v/(r\cos^2\alpha)$ with respect to S_1. Thus,

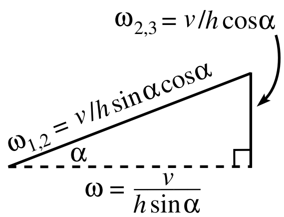


Figure 8.56

$$|\boldsymbol{\omega}_{1,2}| = v/(r\cos^2\alpha) = v/(h\sin\alpha\cos\alpha). \tag{8.100}$$

The addition of $\boldsymbol{\omega}_{1,2}$ and $\boldsymbol{\omega}_{2,3}$ is shown in Fig. 8.56. The result has magnitude $v/(h\sin\alpha)$, and it points horizontally (because $|\boldsymbol{\omega}_{2,3}|/|\boldsymbol{\omega}_{1,2}| = \sin\alpha$).

4. **Parallel axis theorem**

Consider the $\mathbf{I}_{11} = \int(y^2+z^2)$ entry. In terms of the new variables, this is

$$\begin{aligned}\mathbf{I}_{11} &= \int\Big[(Y+y')^2+(Z+z')^2\Big] \\ &= \int(Y^2+Z^2)+\int(y'^2+z'^2) \\ &= M(Y^2+Z^2)+\int(y'^2+z'^2).\end{aligned} \tag{8.101}$$

The cross terms vanish because, e.g., $\int Yy' = Y\int y' = 0$, by definition of the CM. Similarly, consider an off-diagonal term, say, $\mathbf{I}_{12}$. We have

$$\begin{aligned}\mathbf{I}_{12} &= -\int(X+x')(Y+y') \\ &= -\int XY - \int x'y' \\ &= -M(XY)-\int x'y'.\end{aligned} \tag{8.102}$$

The cross terms likewise vanish. So we may rewrite eq. (8.8) as

$$\begin{aligned}\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} &= M\begin{pmatrix} (Y^2+Z^2) & -XY & -ZX \\ -XY & (Z^2+X^2) & -YZ \\ -ZX & -YZ & (X^2+Y^2) \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\ &+ \begin{pmatrix} \int(y'^2+z'^2) & -\int x'y' & -\int z'x' \\ -\int x'y' & \int(z'^2+x'^2) & -\int y'z' \\ -\int z'x' & -\int y'z' & \int(x'^2+y'^2) \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix},\end{aligned} \tag{8.103}$$

in agreement with eq. (8.17).

5. **Existence of principal axes for a pancake**

For our pancake object, the inertia tensor $\mathbf{I}$ takes the form in eq. (8.8), with $z = 0$. Therefore, if we can find a set of axes for which $\int xy = 0$, then $\mathbf{I}$ is diagonal, and we have found our principal axes. We can prove, using a continuity argument, that such a set of axes exists.

Pick a set of axes, and write down the quantity $\int xy \equiv I_0$. If $I_0 = 0$, then we are done. If $I_0 \neq 0$, then rotate these axes by an angle $\pi/2$, so that the new $\hat{\mathbf{x}}$ is the old $\hat{\mathbf{y}}$, and the new $\hat{\mathbf{y}}$ is the old $-\hat{\mathbf{x}}$ (see Fig. 8.57). Write down the new $\int xy \equiv I_{\pi/2}$. Since the new and old coordinates are related by $x_{\text{new}} = y_{\text{old}}$ and $y_{\text{new}} = -x_{\text{old}}$, we have $I_{\pi/2} = -I_0$. Therefore, since $\int xy$ switched sign during our rotation of the axes, it must have been equal to zero for some intermediate angle.

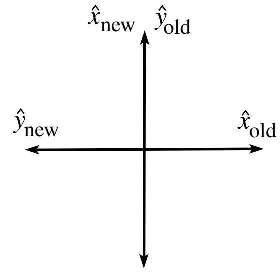


Figure 8.57

6. **Symmetries and principal axes for a pancake**

First Solution: In view of the form of the inertia tensor given in eq. (8.8), we want to show that if a pancake object has a symmetry under a rotation through $\theta \neq \pi$, then $\int xy = 0$ for any set of axes (through the origin).

Take an arbitrary set of axes and rotate them through an angle $\theta \neq \pi$. The new coordinates are $x' = (x\cos\theta + y\sin\theta)$ and $y' = (-x\sin\theta + y\cos\theta)$, so the new matrix entries, in terms of the old ones, are

$$\begin{aligned} I'_{xx} \equiv \int x'^2 &= I_{xx}\cos^2\theta + 2I_{xy}\sin\theta\cos\theta + I_{yy}\sin^2\theta, \\ I'_{yy} \equiv \int y'^2 &= I_{xx}\sin^2\theta - 2I_{xy}\sin\theta\cos\theta + I_{yy}\cos^2\theta, \\ I'_{xy} \equiv \int x'y' &= -I_{xx}\sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta) + I_{yy}\sin\theta\cos\theta. \end{aligned} \quad (8.104)$$

If the object looks exactly like it did before the rotation, then $I'_{xx} = I_{xx}$, $I'_{yy} = I_{yy}$, and $I'_{xy} = I_{xy}$. The first two of these are actually equivalent statements, so we'll just use the first and third. These give

$$\begin{aligned} 0 &= -I_{xx}\sin^2\theta + 2I_{xy}\sin\theta\cos\theta + I_{yy}\sin^2\theta, \\ 0 &= -I_{xx}\sin\theta\cos\theta - 2I_{xy}\sin^2\theta + I_{yy}\sin\theta\cos\theta. \end{aligned} \quad (8.105)$$

Multiplying the first of these by $\cos\theta$ and the second by $\sin\theta$, and subtracting, gives

$$2I_{xy}\sin\theta = 0. \quad (8.106)$$

Under the assumption $\theta \neq \pi$ (and $\theta \neq 0$, of course), we must therefore have $I_{xy} = 0$. Our initial axes were arbitrary; hence, any set of axes (through the origin) in the plane is a set of principal axes.

REMARK: If you don't trust this result, then you may want to show explicitly that the moments around two orthogonal axes are equal for, say, an equilateral triangle centered at the origin. (Then by Theorem 8.5, all axes in the plane are principal axes.) ♣

Second Solution: If an object is invariant under a rotation through an angle θ, then it is clear that θ must be of the form $\theta = 2\pi/N$, for some integer N.[24]

Consider a regular N-gon with 'radius' R, with point-masses m located at the vertices. Any object that is invariant under a rotation through $\theta = 2\pi/N$ can be considered

[24] If N is divisible by 4, then a quick application of Theorem 8.5 shows that any axis in the plane is a principal axis. But if N is not divisible by 4, then the result is not so obvious.

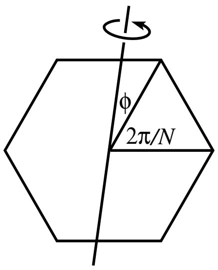


Figure 8.58

to be built up out of many regular point-mass N-gons of various sizes. Therefore, if we can show that any axis in the plane of a regular point-mass N-gon is a principal axis, then we're done. We may do this as follows.

In Fig. 8.58, let ϕ be the angle between the axis and the nearest point-mass to its right. Label the N masses clockwise from 0 to $N-1$, starting with this one. Then the angle between the axis and point-mass k is $\phi + 2\pi k/N$. And the distance from point-mass k to the axis is $r_k = |R\sin(\phi + 2\pi k/N)|$.

The moment of inertia around the axis is $I_\phi = \sum_{k=0}^{N-1} mr_k^2$. In view of Theorem 8.5, if we can show that $I_\phi = I_{\phi'}$, for some $\phi \neq \phi'$ (and $\phi \neq \phi' + \pi$), then we have shown that every axis is a principal axis. We will do this by showing that I_ϕ is independent of ϕ. We'll use a nice math trick, involving writing a trig function as the real part of a complex exponential. If $N \neq 2$, then

$$\begin{aligned} I_\phi &= mR^2 \sum_{k=0}^{N-1} \sin^2\left(\phi + \frac{2\pi k}{N}\right) \\ &= \frac{mR^2}{2} \sum_{k=0}^{N-1} \left(1 - \cos\left(2\phi + \frac{4\pi k}{N}\right)\right) \\ &= \frac{NmR^2}{2} - \frac{mR^2}{2} \sum_{k=0}^{N-1} \cos\left(2\phi + \frac{4\pi k}{N}\right) \\ &= \frac{NmR^2}{2} - \frac{mR^2}{2} \sum_{k=0}^{N-1} \mathrm{Re}\left(e^{i(2\phi + 4\pi k/N)}\right) \\ &= \frac{NmR^2}{2} - \frac{mR^2}{2} \mathrm{Re}\left(e^{2i\phi}\left(1 + e^{4\pi i/N} + e^{8\pi i/N} + \cdots + e^{4(N-1)\pi i/N}\right)\right) \\ &= \frac{NmR^2}{2} - \frac{mR^2}{2} \mathrm{Re}\left(e^{2i\phi}\left(\frac{e^{4N\pi i/N} - 1}{e^{4\pi i/N} - 1}\right)\right) \\ &= \frac{NmR^2}{2}. \end{aligned} \tag{8.107}$$

(We summed the geometric series to get the sixth line.) This is independent of ϕ, so the moments around all axes in the plane are equal. Therefore, every axis in the plane is a principal axis.

Remarks: Given that the moments around all the axes in the plane are equal, they must be equal to $NmR^2/2$, because the perpendicular axis theorem says that they all must be one-half of the moment around the axis perpendicular to the plane (which is NmR^2).

If $N = 2$, then I_ϕ does depend on ϕ (find where the above calculation fails). This is why we needed the $\theta \neq \pi$ condition. ♣

7. **A nice cylinder**

Three axes that are certainly principal axes are two orthogonal diameters and the symmetry axis. The moments around the former two are equal (call them I). Therefore, by Theorem 8.5, if the moment around the symmetry axis is also equal to I, then every axis is a principal axis.

Let the mass of the cylinder be M. Let its radius be R and its height be h. Then the moment around the symmetry axis is $MR^2/2$.

Let D_{CM} be a diameter through the CM. The moment around D_{CM} may be calculated as follows. Slice the cylinder into horizontal disks of thickness dy. Let ρ be the mass per unit height (so $\rho = M/h$). The mass of each disk is then $\rho\,dy$. The moment of the disk around a diameter through the disk is $(\rho\,dy)R^2/4$. So by the parallel axis theorem, the moment of a disk at height y (where $-h/2 \le y \le h/2$) around D_{CM} is $(\rho\,dy)R^2/4 + (\rho\,dy)y^2$. Therefore, the moment of the entire cylinder around D_{CM} is

$$I = \int_{-h/2}^{h/2} \left(\frac{\rho R^2}{4} + \rho y^2 \right) dy = \frac{\rho R^2 h}{4} + \frac{\rho h^3}{12} = \frac{MR^2}{4} + \frac{Mh^2}{12}. \tag{8.108}$$

We want this to equal $MR^2/2$. Therefore,

$$h = \sqrt{3}R. \tag{8.109}$$

Note: if the origin was instead taken to be the center of one of the circular faces, then the answer would be $h = \sqrt{3}R/2$.

8. **Rotating square**

Label two of the masses A and B, as shown in Fig. 8.59. Let ℓ_A be the distance along the axis from the CM to A's string, and let r_A be the length of A's string. Likewise for B.

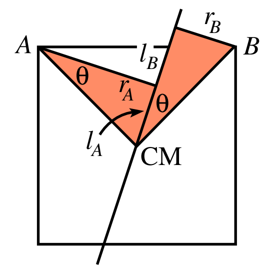


Figure 8.59

The force, F_A, in A's string must account for the centripetal acceleration of A. Hence, $F_A = mr_A\omega^2$. The torque around the CM due to F_A is therefore

$$\tau_A = mr_A\ell_A\omega^2. \tag{8.110}$$

Likewise, the torque around the CM due to B's string is $\tau_B = mr_B\ell_B\omega^2$, in the opposite direction.

But the two shaded triangles in Fig. 8.59 are congruent (they have the same hypotenuse and the same angle θ). Therefore, $\ell_A = r_B$ and $\ell_B = r_A$. Hence, $\tau_A = \tau_B$, and the torques cancel. The torques from the other two masses likewise cancel.

(A uniform square is made up of many sets of these squares of point masses, so we've also shown that no torque is needed for a uniform square.)

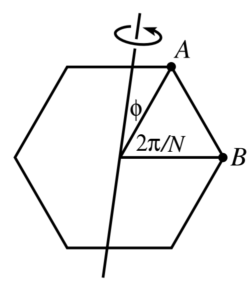


Figure 8.60

REMARK: For a general N-gon of point masses, problem 6 shows that any axis in the plane is a principal axis. We should be able to use the above torque argument to prove this. This can be done as follows (time for a nice math trick). Using eq. (8.110), we see that the torque from mass A in Fig. 8.60 is $\tau_A = m\omega^2R^2\sin\theta\cos\theta$. Likewise, the torque from mass B is $\tau_B = m\omega^2R^2\sin(\theta + 2\pi/N)\cos(\theta + 2\pi/N)$, and so on. The total torque around the CM is therefore

$$\begin{aligned}
\tau &= mR^2\omega^2 \sum_{k=0}^{N-1} \sin\left(\theta + \frac{2\pi k}{N}\right)\cos\left(\theta + \frac{2\pi k}{N}\right) \\
&= \frac{mR^2\omega^2}{2} \sum_{k=0}^{N-1} \sin\left(2\theta + \frac{4\pi k}{N}\right) \\
&= \frac{mR^2\omega^2}{2} \sum_{k=0}^{N-1} \mathrm{Im}\left(e^{i(2\theta + 4\pi k/N)}\right) \\
&= \frac{mR^2\omega^2}{2} \mathrm{Im}\Big(e^{2i\theta}\left(1 + e^{4\pi i/N} + e^{8\pi i/N} + \cdots + e^{4(N-2)\pi i/N} + e^{4(N-1)\pi i/N}\right)\Big) \\
&= \frac{mR^2\omega^2}{2} \mathrm{Im}\left(e^{2i\theta}\left(\frac{e^{4N\pi i/N} - 1}{e^{4\pi i/N} - 1}\right)\right) \\
&= 0.
\end{aligned} \tag{8.111}$$

This was essentially another proof of problem 6. To prove that the torque was 0 (which is one of the definitions of principal axes), we showed here that $\sum r_i \ell_i = 0$. In terms of the chosen axes, this is equivalent to showing that $\sum xy = 0$, i.e., showing that the off-diagonal terms in the inertia tensor vanish (which is simply another definition of the principal axes). ♣

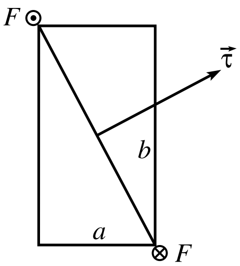


Figure 8.61

9. **Rotating rectangle**

Without loss of generality, let the force on the upper left corner be out of the page, and that on the lower right corner be into the page. Then the torque, $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, points upward to the right, as shown in Fig. 8.61, with $\boldsymbol{\tau} \propto (b, a)$. The angular momentum equals $\int \boldsymbol{\tau}\, dt$. Therefore, immediately after the strike, $\mathbf{L}$ is proportional to (b, a).

The principal moments are $I_x = mb^2/12$ and $I_y = ma^2/12$. The angular momentum may be written as $\mathbf{L} = (I_x\omega_x, I_y\omega_y)$. Since we know $\mathbf{L} \propto (b, a)$, we have

$$(\omega_x, \omega_y) \propto \left(\frac{b}{I_x}, \frac{a}{I_y}\right) \propto \left(\frac{b}{b^2}, \frac{a}{a^2}\right) \propto (a, b), \tag{8.112}$$

which is the direction of the other diagonal.

This answer checks in the limit $a = b$, and also in the limit where either a or b goes to zero.

10. **Rotating stick**

Break $\boldsymbol{\omega}$ up into its components along the principal axes of the stick (which are parallel and perpendicular to the stick). The moment of inertia around the stick is zero. Therefore, to compute $\mathbf{L}$, we only need to know the component of $\boldsymbol{\omega}$ perpendicular to the stick. This component is $\omega \sin\theta$. The associated moment of inertia is $m\ell^2/12$. Therefore, the angular momentum at any time has magnitude

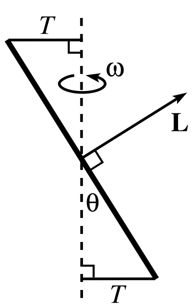


Figure 8.62

$$L = \frac{1}{12} m\ell^2 \omega \sin\theta, \tag{8.113}$$

and it points as shown in Fig. 8.62. The tip of the vector $\mathbf{L}$ traces out a circle in a horizontal plane, with frequency ω. The radius of this circle is the horizontal component of $\mathbf{L}$, namely $L_\perp \equiv L\cos\theta$. The rate of change of $\mathbf{L}$ therefore has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = \omega L_\perp = \omega L \cos\theta = \omega\left(\frac{1}{12} m\ell^2\omega\sin\theta\right)\cos\theta, \tag{8.114}$$

and at the instant shown, it is directed into the paper.

Let the tension in the strings be T. Then the torque from the strings is $\boldsymbol{\tau} = 2T(\ell/2)\cos\theta$, directed into the paper at this instant. So $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$T\ell\cos\theta = \omega\left(\frac{1}{12} m\ell^2\omega\sin\theta\right)\cos\theta, \tag{8.115}$$

and so

$$T = \frac{1}{12} m\omega^2 \ell \sin\theta. \tag{8.116}$$

For $\theta \to 0$, this goes to 0, which makes sense. For $\theta \to \pi/2$, it goes to the finite value $m\ell\omega^2/12$, which isn't entirely obvious.

11. **Another rotating stick**

We will deal with torque and angular momentum around the CM. The torque around the CM (due to the vertical force from the rail; there is no horizontal force from the rail, since the CM does not move) has magnitude $mgr\sin\theta$.

The angular momentum around the CM may be found as follows. The principal axes of the stick in the plane of the paper are $\hat{\boldsymbol{\omega}}_1$ along the stick and $\hat{\boldsymbol{\omega}}_2$ perpendicular to the stick (and $\hat{\boldsymbol{\omega}}_3$ is into the page), with $I_1 = 0$ and $I_2 = mr^2/3$. The components of $\boldsymbol{\omega}$ are $\omega_1 = \omega\cos\theta$ and $\omega_2 = \omega\sin\theta$ (and $\omega_3 = 0$). Therefore, $\mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3) = (0, (mr^2/3)\omega\sin\theta, 0)$. So $\mathbf{L}$ points upward to the right with magnitude $(mr^2/3)\omega\sin\theta$, as shown in Fig. 8.63.

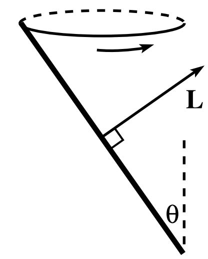


Figure 8.63

The change in $\mathbf{L}$ comes from the horizontal component, which has length $L\cos\theta$, traveling in a circle. Hence, $|d\mathbf{L}/dt| = \omega L\cos\theta$. $\boldsymbol{\tau} = d\mathbf{L}/dt$ then gives

$$mgr\sin\theta = \omega\left(\frac{mr^2\omega\sin\theta}{3}\right)\cos\theta, \tag{8.117}$$

and so

$$\omega = \sqrt{\frac{3g}{r\cos\theta}}. \tag{8.118}$$

For $\theta \to \pi/2$, this goes to infinity, as it should. For $\theta \to 0$, it goes to the constant $\sqrt{3g/r}$, which isn't so obvious.

Note that if we instead have a massless stick with equal masses on the ends, then our answer would be $\omega = \sqrt{g/(r\cos\theta)}$. For $\theta \to 0$, this goes to the nice result $\sqrt{g/r}$.

12. **Spherical pendulum**

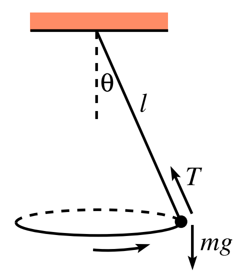


Figure 8.64

(a) The forces on the mass are gravity, mg, and the tension, T, from the rod (see Fig. 8.64). The fact that there is no vertical acceleration means that $T\cos\theta = mg$. The unbalanced horizontal force from the tension is therefore $T\sin\theta = mg\tan\theta$. This force accounts for the centripetal acceleration, $m(\ell\sin\theta)\Omega^2$. Hence,

$$\Omega = \sqrt{\frac{g}{\ell\cos\theta}}. \tag{8.119}$$

For $\theta \approx 0$, this is the same as that for a simple pendulum. For $\theta \approx \pi/2$, it goes to infinity, which makes sense. Note that θ must be less than $\pi/2$ for circular motion to be possible. (This restriction does not hold for a gyroscope with extended mass.)

(b) The only force that applies a torque relative to the pivot is the gravitational force. The torque is $\tau = mg\ell\sin\theta$, directed into the page (see Fig. 8.65).

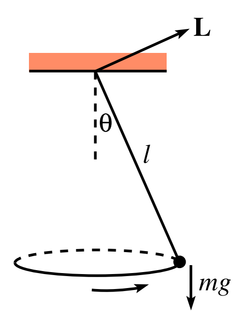


Figure 8.65

At this instant in time, the mass has a speed $(\ell\sin\theta)\Omega$, directed into the page. Therefore, $\mathbf{L} = \mathbf{r}\times\mathbf{v}$ has magnitude $m\ell^2\sin\theta\Omega$, and is directed upward to the right, as shown.

The tip of $\mathbf{L}$ traces out a circle of radius $L\cos\theta$, at angular frequency Ω. Therefore, $d\mathbf{L}/dt$ has magnitude $L\cos\theta\Omega$, directed into the page.

Hence, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives $mg\ell\sin\theta = m(\ell^2\sin\theta\Omega)\cos\theta\Omega$. This yields eq. (8.119).

(c) The only force that applies a torque relative to the mass is that from the pivot. There are two components to this force (see Fig. 8.66).

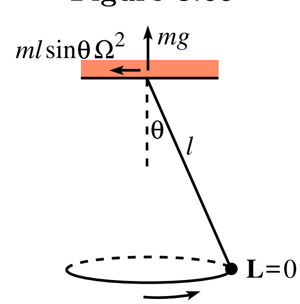


Figure 8.66

There is the vertical piece, which is mg. Relative to the mass, this provides a torque of $mg(\ell\sin\theta)$, which is directed into the page.

There is also the horizontal piece, which accounts for the centripetal acceleration. This equals $m(\ell \sin\theta)\Omega^2$. Relative to the mass, this provides a torque of $m\ell \sin\theta\Omega^2(\ell\cos\theta)$, which is directed out of the page.

Relative to the mass, there is no angular momentum. Therefore, $d\mathbf{L}/dt = 0$. Hence, there must be no torque; so the above two torques cancel. This implies $mg(\ell\sin\theta) = m\ell\sin\theta\Omega^2(\ell\cos\theta)$, which yields eq. (8.119).

REMARK: In problems that are more complicated than this one, it is often easier to work with a fixed pivot as the origin (if there is one) instead of the CM, because then you don't have to worry about messy pivot forces contributing to the torque. ♣

13. **Rolling on a cone**

(a) The forces on the particle are gravity (mg) and the normal force (N) from the cone. In our situation, there is no net force in the vertical direction, so

$$N\sin\theta = mg. \tag{8.120}$$

Therefore, the inward horizontal force, $N\cos\theta$, equals $mg/\tan\theta$. This force must account for the centripetal acceleration of the particle moving in a circle of radius $h\tan\theta$. Hence, $mg/\tan\theta = m(h\tan\theta)\Omega^2$, and so

$$\Omega = \frac{1}{\tan\theta}\sqrt{\frac{g}{h}}. \tag{8.121}$$

(b) The forces on the ring are gravity (mg), the normal force (N) from the cone, and a friction force (F) pointing up along the cone. In our situation, there is no net force in the vertical direction, so

$$N\sin\theta + F\cos\theta = mg. \tag{8.122}$$

The fact that the inward horizontal force accounts for the centripetal acceleration yields

$$N\cos\theta - F\sin\theta = m(h\tan\theta)\Omega^2. \tag{8.123}$$

We must now consider the torque, $\boldsymbol{\tau}$, on the ring (relative to the CM). The torque is due solely to F (because gravity provides no torque, and N points though the center of the ring, by assumption (2) in the problem). So

$$\tau = rF, \tag{8.124}$$

and $\boldsymbol{\tau}$ points in the direction along the circular motion. Since $\boldsymbol{\tau} = d\mathbf{L}/dt$, we must now find $d\mathbf{L}/dt$.

We are assuming $r \ll h\tan\theta$. Hence, the frequency of the spinning of the ring (call it ω) is much greater than the frequency of precession, Ω. We will therefore neglect the latter in the computation of $\mathbf{L}$. So we have $L = mr^2\omega$, and $\mathbf{L}$ points upward along the cone. The horizontal component of $\mathbf{L}$ is $L_\perp \equiv L\sin\theta$. This traces out a circle at frequency Ω. Therefore, $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = \Omega L_\perp = \Omega L\sin\theta = \Omega(mr^2\omega)\sin\theta, \tag{8.125}$$

and it points in the direction along the circular motion.

We know that ω and Ω are related by $r\omega = (h\tan\theta)\Omega$ (the rolling-without-slipping condition)[25]. Using this in eq. (8.125) gives

$$\left|\frac{d\mathbf{L}}{dt}\right| = \Omega^2 mrh\tan\theta\sin\theta, \qquad (8.126)$$

Equating this with the torque from eq. (8.124) gives

$$F = m\Omega^2 h\tan\theta\sin\theta. \qquad (8.127)$$

Eqs. (8.122), (8.123), and (8.127) are three equations with three unknowns, N, F, and Ω. We can eliminate N by multiplying eq. (8.122) by $\cos\theta$, and eq. (8.123) by $\sin\theta$, and taking the difference, to obtain

$$F = mg\cos\theta - m\Omega^2(h\tan\theta)\sin\theta. \qquad (8.128)$$

Equating this expression for F with that in eq. (8.127) gives

$$\Omega = \frac{1}{\tan\theta}\sqrt{\frac{g}{2h}}. \qquad (8.129)$$

This frequency is $1/\sqrt{2}$ times the frequency found in part (a).

REMARK: If you consider an object with moment of inertia ηmr^2 (our ring has $\eta = 1$), then you can show by the above reasoning that the "2" in eq. (8.129) is simply replaced by $(1+\eta)$. ♣

14. **Tennis racket theorem**

(a) Presumably this worked out as it was supposed to.

(b) • *Rotation around* $\hat{\mathbf{x}}_1$: If the racket is rotated (nearly) around the $\hat{\mathbf{x}}_1$ axis, then the initial ω_2 and ω_3 are much smaller than ω_1. To emphasize this, relabel $\omega_2 \to \epsilon_2$ and $\omega_3 \to \epsilon_3$. Then eqs. (8.43) become (with $\boldsymbol{\tau} = \mathbf{0}$, because only gravity acts on the racket)

$$\begin{aligned} 0 &= \dot{\omega}_1 - A\epsilon_2\epsilon_3, \\ 0 &= \dot{\epsilon}_2 + B\omega_1\epsilon_3, \\ 0 &= \dot{\epsilon}_3 - C\omega_1\epsilon_2, \end{aligned} \qquad (8.130)$$

where we have defined (for convenience)

$$A \equiv \frac{I_2 - I_3}{I_1}, \qquad B \equiv \frac{I_1 - I_3}{I_2}, \qquad C \equiv \frac{I_1 - I_2}{I_3}. \qquad (8.131)$$

Note that A, B, and C are all positive (this fact will be very important). Our goal here is to show that if the ϵ's start out small, then they remain small. Assuming that they are small (which is true initially), the first equation says that $\dot{\omega}_1 \approx 0$ (to first order in the ϵ's). So we may assume that ω_1 is essentially constant (when the ϵ's are small). Taking the derivative of the second equation then gives $0 = \ddot{\epsilon}_2 + B\omega_1\dot{\epsilon}_3$. Plugging the value of $\dot{\epsilon}_3$ from the third equation into this yields

$$\ddot{\epsilon}_2 = -(BC\omega_1^2)\epsilon_2. \qquad (8.132)$$

[25] Actually, this isn't quite true, for the same reason that the earth spins around 366 instead of 365 times in a year. But it's valid enough, in the limit of small r.

Due to the negative coefficient on the right-hand side, this equation describes simple harmonic motion. Therefore, ϵ_2 oscillates sinusoidally around 0. Hence, if it starts small, it stays small. By the same reasoning, ϵ_3 remains small.

We see that $\boldsymbol{\omega} \approx (\omega_1, 0, 0)$ at all times, which implies that $\mathbf{L} \approx (I_1\omega_1, 0, 0)$ at all times. That is, $\mathbf{L}$ always points (nearly) along the $\hat{\mathbf{x}}_1$ direction (which is fixed in the racket frame). But the direction of $\mathbf{L}$ is fixed in the lab frame (since there is no torque); therefore the direction of $\hat{\mathbf{x}}_1$ must also be (nearly) fixed in the lab frame. In other words, the racket doesn't wobble.

- *Rotation around* $\hat{\mathbf{x}}_3$: The calculation goes through exactly as above, except with "1" and "3" interchanged. We find that if ϵ_1 and ϵ_2 start small, they remain small. And $\boldsymbol{\omega} \approx (0, 0, \omega_3)$ at all times.
- *Rotation around* $\hat{\mathbf{x}}_2$: If the racket is rotated (nearly) around the $\hat{\mathbf{x}}_2$ axis, then the initial ω_1 and ω_3 are much smaller than ω_2. As above, let's emphasize this by relabeling $\omega_1 \to \epsilon_1$ and $\omega_3 \to \epsilon_3$. Then as above, eqs. (8.43) become

$$\begin{aligned} 0 &= \dot{\epsilon}_1 - A\omega_2\epsilon_3, \\ 0 &= \dot{\omega}_2 + B\epsilon_1\epsilon_3, \\ 0 &= \dot{\epsilon}_3 - C\omega_2\epsilon_1, \end{aligned} \tag{8.133}$$

Our goal here is to show that if the ϵ's start out small, then they do *not* remain small. Assuming that they are small (which is true initially), the second equation says that $\dot{\omega}_2 \approx 0$ (to first order in the ϵ's). So we may assume that ω_2 is essentially constant (when the ϵ's are small). Taking the derivative of the first equation then gives $0 = I_1\ddot{\epsilon}_1 - A\omega_2\dot{\epsilon}_3$. Plugging the value of $\dot{\epsilon}_3$ from the third equation into this yields

$$\ddot{\epsilon}_1 = (AC\omega_2^2)\epsilon_1. \tag{8.134}$$

Due to the positive coefficient on the right-hand side, this equation describes an exponentially growing motion, instead of an oscillatory one. Therefore, ϵ_1 grows quickly from its initial small value. Hence, if it starts small, it becomes large. By the same reasoning, ϵ_3 becomes large. (Of course, once the ϵ's become large, then our assumption of $\dot{\omega}_2 \approx 0$ isn't valid anymore. But once the ϵ's become large, we've shown what we wanted to.)

We see that $\boldsymbol{\omega}$ does *not* remain equal to $(0, \omega_2, 0)$ at all times, which implies that $\mathbf{L}$ does *not* remain equal to $(0, I_2\omega_2, 0)$ at all times. That is, $\mathbf{L}$ does not always points (nearly) along the $\hat{\mathbf{x}}_2$ direction (which is fixed in the racket frame). But the direction of $\mathbf{L}$ is fixed in the lab frame (since there is no torque); therefore the direction of $\hat{\mathbf{x}}_2$ must change in the lab frame. In other words, the racket wobbles.

15. **Free-top angles**

In terms of the principal axes, $\hat{\mathbf{x}}_1$,$\hat{\mathbf{x}}_2$,$\hat{\mathbf{x}}_3$, we have

$$\begin{aligned} \boldsymbol{\omega} &= (\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) + \omega_3\hat{\mathbf{x}}_3, \qquad \text{and} \\ \mathbf{L} &= I(\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) + I_3\omega_3\hat{\mathbf{x}}_3. \end{aligned} \tag{8.135}$$

Let $(\omega_1\hat{\mathbf{x}}_1 + \omega_2\hat{\mathbf{x}}_2) \equiv \omega_\perp\hat{\boldsymbol{\omega}}_\perp$ be the component of $\boldsymbol{\omega}$ orthogonal to $\boldsymbol{\omega}_3$. Then, by definition, we have

$$\tan\beta = \frac{\omega_\perp}{\omega_3}, \qquad \text{and} \qquad \tan\alpha = \frac{I\omega_\perp}{I_3\omega_3}. \tag{8.136}$$

Therefore,

$$\frac{\tan\alpha}{\tan\beta} = \frac{I}{I_3}. \tag{8.137}$$

If $I > I_3$, then $\alpha > \beta$, and we have the situation shown in Fig. 8.67. A top with this property is called a 'prolate top'. An example is a football or a pencil.

If $I < I_3$, then $\alpha < \beta$, and we have the situation shown in Fig. 8.68 . A top with this property is called an 'oblate top'. An example is a coin or a Frisbee.

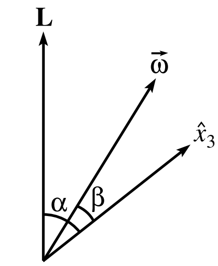


Figure 8.67

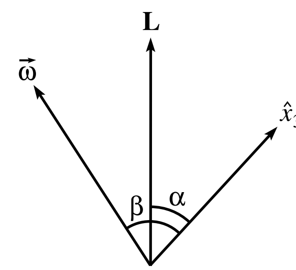


Figure 8.68

16. **Gyroscope**

(a) In order for there to exist real solutions for Ω in eq. (8.79), the discriminant must be non-negative, i.e.,

$$\omega_3 \geq \sqrt{\frac{4MIg\ell\cos\theta}{I_3^2}} \equiv \tilde{\omega}_3. \tag{8.138}$$

If $\theta \geq \pi/2$, then $\cos\theta \leq 0$, so the discriminant is automatically positive. But if $\theta < \pi/2$, then $\tilde{\omega}_3$ is the lower limit on ω_3 for there to be circular precession. Note that at this critical value, eq. (8.79) gives

$$\Omega_+ = \Omega_- = \frac{I_3\tilde{\omega}_3}{2I\cos\theta} = \sqrt{\frac{Mg\ell}{I\cos\theta}} \equiv \Omega_0. \tag{8.139}$$

(b) One limiting case of eq. (8.79) is easy to address, namely that of large ω_3. Of course, "large ω_3" is a meaningless description. What we really want is for the fraction in the square root in eq. (8.79) to be very small. That is, $\epsilon \equiv (4MIg\ell\cos\theta)/(I_3^2\omega_3^2) \ll 1$. In this case, we may use $\sqrt{1-\epsilon} \approx 1 - \epsilon/2 + \cdots$ to write

$$\Omega_\pm \approx \frac{I_3\omega_3}{2I\cos\theta}\left(1 \pm \left(1 - \frac{2MIg\ell\cos\theta}{I_3^2\omega_3^2}\right)\right). \tag{8.140}$$

Therefore, the two solutions for Ω are

$$\Omega_+ \approx \frac{I_3\omega_3}{I\cos\theta}, \qquad \text{and} \qquad \Omega_- \approx \frac{Mg\ell}{I_3\omega_3}. \tag{8.141}$$

These are known as the frequencies of 'fast' and 'slow' precession, respectively. Ω_- is the solution found in part (a). It was obtained here under the assumption $\epsilon \ll 1$, which is equivalent to

$$\omega_3 \gg \sqrt{\frac{MIg\ell\cos\theta}{I_3^2}} \qquad (\text{i.e., } \omega_3 \gg \tilde{\omega}_3). \tag{8.142}$$

This, therefore, is the condition for the result in part (a) to be a good approximation. Note that if I is of the same order as I_3 (so that they are both of the order $M\ell^2$), and if $\cos\theta$ is of order 1, then this condition can be written as $\omega \gg \sqrt{g/\ell}$, which is the frequency of a pendulum of length ℓ.

REMARKS:

i. The Ω_+ solution is a fairly surprising result. Two strange features of Ω_+ are that it grows with ω_3, and that it is independent of g. To see what is going on with this precession, note that Ω_+ is the value of Ω that makes the $L_\perp$ in eq. (8.77) essentially equal to zero. So **L** points nearly along the vertical axis. The rate of change of **L** is the product of a very small radius (of the circle the tip traces out) and a very large Ω (if we've picked ω_3 to be large). The product of these equals the 'medium sized' torque $Mg\ell \sin\theta$.

In the limit of large ω_3, this fast precession should look basically like the motion of a free top (since $\mathbf{L}$ is essentially staying constant), discussed in section 8.6.2. On short time scales, the effect of the gravitational torque is negligible compared to $\mathbf{L}$ (which is very large), so the motion should look similar to the case where there is no gravity. Indeed, Ω_+ is independent of g. We'll leave it to you to show that $\Omega_+ \approx L/I$, which is the precession frequency of a free top (eq. (8.53)), as viewed from a fixed frame.

ii. We can plot the $\Omega_\pm$ of eq. (8.79) as functions of ω_3. With the definitions of $\tilde{\omega}_3$ and Ω_0 in eqs. (8.138) and (8.139), we can rewrite eq. (8.79) as

$$\Omega_\pm = \frac{\omega_3 \Omega_0}{\tilde{\omega}_3}\left(1 \pm \sqrt{1 - \frac{\tilde{\omega}_3^2}{\omega_3^2}}\right). \tag{8.143}$$

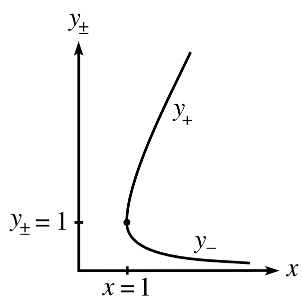


Figure 8.69

It is easier to work with dimensionless quantities, so let's rewrite this as

$$y_\pm = x \pm \sqrt{x^2 - 1}, \qquad \text{with } y_\pm \equiv \frac{\Omega_\pm}{\Omega_0}, \quad x \equiv \frac{\omega_3}{\tilde{\omega}_3}. \tag{8.144}$$

A rough plot of $y_\pm$ vs. x is shown in Fig. 8.69. ♣

17. **Many gyroscopes**

Consider the system to be made up of N rigid bodies, each consisting of a plate and the massless stick glued to it on its left (see Fig. 8.70). Label these sub-systems as S_i, with S_1 being the one closest to the pole.

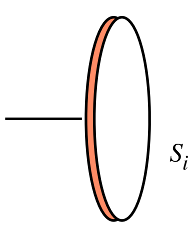


Figure 8.70

Let each plate have mass m and moment of inertia I, and let each stick have length ℓ. Let the angular speeds be ω_i. The relevant angular momentum of S_i is then $L_i = I\omega_i$, and it points horizontally.[26] Let the desired precession frequency be Ω. Then the magnitude of $d\mathbf{L}_i/dt$ is $L_i\Omega = (I\omega_i)\Omega$ (and it points perpendicular to $\mathbf{L}_i$.)

Consider the torque $\boldsymbol{\tau}_i$ on S_i, around its CM. Let's first look at S_1. The wall provides an upward force of Nmg (this force is what keeps all the gyroscopes up), so it provides a torque of $Nmg\ell$ around the CM of S_1. The downward force from the stick to the right provides no torque around the CM (since it acts at the CM). So $\boldsymbol{\tau}_1 = d\mathbf{L}_1/dt$ gives $Nmg\ell = (I\omega_1)\Omega$, and thus

$$\omega_1 = \frac{Nmg\ell}{I\Omega}. \tag{8.145}$$

Now look at S_2. S_1 provides an upward force of $(N-1)mg$ (this force is what keeps S_2 through S_N up), so it provides a torque of $(N-1)mg\ell$ around the CM of S_2. The downward force from the stick to the right provides no torque around the CM of S_2. So $\boldsymbol{\tau}_2 = d\mathbf{L}_2/dt$ gives $(N-1)mg\ell = (I\omega_2)\Omega$, and thus

$$\omega_2 = \frac{(N-1)mg\ell}{I\Omega}. \tag{8.146}$$

The same reasoning applies to the other S_i. For S_i, $\boldsymbol{\tau}_i = d\mathbf{L}_i/dt$ gives $(N-(i-1))mg\ell = (I\omega_i)\Omega$, and thus

$$\omega_i = \frac{(N+1-i)mg\ell}{I\Omega}. \tag{8.147}$$

[26] We are ignoring the angular momentum arising from the precession. This part of $\mathbf{L}$ points vertically (because the gyroscopes all point horizontally) and therefore does not change. Hence, it does not enter into $\vec{\tau} = d\mathbf{L}/dt$.

The ω_i are therefore in the ratio

$$\omega_1 : \omega_2 : \cdots : \omega_{N-1} : \omega_N = N : (N-1) : \cdots : 2 : 1. \tag{8.148}$$

Note that we needed to apply $\boldsymbol{\tau} = d\mathbf{L}/dt$ many times, using each CM as an origin. Using only the pivot point on the wall as the origin would have given only one piece of information, whereas we needed N pieces.

REMARKS:

(a) As a double-check, we can verify that these ω's make $\vec{\tau} = d\mathbf{L}/dt$ true, where $\vec{\tau}$ and $\mathbf{L}$ are the total torque and angular momentum about the CM of the entire system. (Using the pivot point as the origin would give the same equation here.) The CM of the whole system is $(N+1)\ell/2$ from the wall. So the torque from wall's upward force is

$$\tau = Nmg\frac{(N+1)\ell}{2}. \tag{8.149}$$

The total angular momentum is (using eq. (8.147))

$$\begin{aligned} L &= I(\omega_1 + \omega_2 + \cdots + \omega_N) \\ &= \frac{mg\ell}{\Omega}\Big(N + (N-1) + (N-2) + \cdots + 2 + 1\Big) \\ &= \frac{mg\ell}{\Omega}\frac{N(N+1)}{2}. \end{aligned} \tag{8.150}$$

So indeed, $\tau = L\Omega = |d\mathbf{L}/dt|$.

(b) One can also pose this problem for the case where all the ω_i are equal (call them ω), and the goal is the find the lengths of the sticks that will allow the desired motion. The same reasoning applies, and eq. (8.147) takes the modified form

$$\omega = \frac{(N+1-i)mg\ell_i}{I\Omega}, \tag{8.151}$$

where ℓ_i is the length of the ith stick. Therefore, the ℓ_i are in the ratio

$$\ell_1 : \ell_2 : \cdots : \ell_{N-1} : \ell_N = \frac{1}{N} : \frac{1}{N-1} : \cdots : \frac{1}{2} : 1. \tag{8.152}$$

Again, one can verify that these ℓ's make $\vec{\tau} = d\mathbf{L}/dt$ true, where $\vec{\tau}$ and $\mathbf{L}$ are the total torque and angular momentum about the CM of the entire system. We'll let the reader show that the CM is simply a distance ℓ_N from the wall. So the torque from wall's upward force is (using eq. (8.151))

$$\tau = Nmg\ell_N = Nmg(\omega I\Omega/mg) = NI\omega\Omega. \tag{8.153}$$

The total angular momentum is simply $L = NI\omega$. So indeed, $\tau = L\Omega = |d\mathbf{L}/dt|$. ♣

18. **Heavy top on slippery table**

In section 8.7, we looked at $\boldsymbol{\tau}$ and $\mathbf{L}$ relative to the pivot point. Such quantities are of no use here, because we can't use $\boldsymbol{\tau} = d\mathbf{L}/dt$ relative to the pivot point (since it is accelerating). We will therefore look at $\boldsymbol{\tau}$ and $\mathbf{L}$ relative to the CM, which is always a legal origin around which we can apply $\boldsymbol{\tau} = d\mathbf{L}/dt$.

The only force the floor applies is the normal force Mg. So the torque relative to the CM has the same magnitude, $Mg\ell\sin\theta$, and the same direction as in section 8.7. If we choose the CM as the origin of our coordinate system, then all the Euler angles are the same as before. The only change in the whole analysis is the change in the $I_1 = I_2 \equiv I$ moment of inertia. We are now measuring them with respect to the CM, instead of the pivot point. By the parallel axis theorem, they are now equal to

$$I' = I - M\ell^2. \tag{8.154}$$

So, changing I to $I - M\ell^2$ is the only modification needed.

19. **Fixed highest point**

The crucial thing to note is that every point in the top moves in a fixed circle around the $\hat{\mathbf{z}}$-axis; therefore, $\boldsymbol{\omega}$ points vertically. Hence, if we let Ω be the frequency of precession (in the language of Fig. 8.27), we have $\boldsymbol{\omega} = \Omega\hat{\mathbf{z}}$.

(Another way to see that $\boldsymbol{\omega}$ points vertically is to view things in the frame that rotates with angular velocity $\Omega\hat{\mathbf{z}}$. In this frame, the top has no motion whatsoever. It is not even spinning, since the point P is always the highest point. In the language of Fig. 8.27, we therefore have $\omega' = 0$. Hence, $\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} + \omega'\hat{\mathbf{x}}_3 = \Omega\hat{\mathbf{z}}$.)

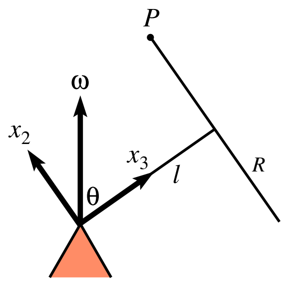


Figure 8.71

The principal moments (with the pivot as the origin; see Fig. 8.71) are $I_3 = MR^2/2$, and $I \equiv I_1 = I_2 = M\ell^2 + MR^2/4$ (from the parallel axis theorem). The components of $\boldsymbol{\omega}$ along the principal axes are $\omega_3 = \Omega\cos\theta$, and $\omega_2 = \Omega\sin\theta$. Therefore (keeping things in terms of the general moments, I_3 and I),

$$\mathbf{L} = I_3\Omega\cos\theta\hat{\mathbf{x}}_3 + I\Omega\sin\theta\hat{\mathbf{x}}_2. \tag{8.155}$$

The horizontal component of $\mathbf{L}$ is thus $L_\perp = (I_3\Omega\cos\theta)\sin\theta - (I\Omega\sin\theta)\cos\theta$, and so the rate of change of $\mathbf{L}$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = L_\perp\Omega = \Omega^2\sin\theta\cos\theta(I_3 - I), \tag{8.156}$$

and is directed into the page (or out of the page, if this quantity is negative). This must equal the torque, which has magnitude $|\boldsymbol{\tau}| = Mg\ell\sin\theta$, and is directed into the page. Therefore,

$$\Omega = \sqrt{\frac{Mg\ell}{(I_3 - I)\cos\theta}}\,. \tag{8.157}$$

We see that for a general symmetric top, such precessional motion (where the same 'side' always points up) is possible only if

$$I_3 > I. \tag{8.158}$$

Note that this condition is independent of θ. For the problem at hand, I_3 and I are given above, and we find

$$\Omega = \sqrt{\frac{4g\ell}{(R^2 - 4\ell^2)\cos\theta}}\,, \tag{8.159}$$

along with the requirement $R > 2\ell$.

REMARKS:

(a) It is intuitively clear that Ω should become very large as $\theta \to \pi/2$ (although it is by no means intuitively clear that such motion should exist at all for angles near $\pi/2$).

(b) Ω approaches a non-zero constant as $\theta \to 0$, which isn't entirely obvious.

(c) If both R and ℓ are scaled up by the same factor, then Ω decreases (this is clear from dimensional analysis).

(d) The condition $I_3 > I$ is easily seen in the following way. If $I_3 = I$, then $\mathbf{L} \propto \vec{\omega}$, and thus $\mathbf{L}$ points vertically along $\vec{\omega}$. If $I_3 > I$, then $\mathbf{L}$ points somewhere to the right of the $\hat{\mathbf{z}}$-axis (at the instant shown in Fig. 8.71). This means that the tip of $\mathbf{L}$ is moving into the page, along with the top. This is what we need, since $\vec{\tau}$ points into the page. If, however, $I_3 < I$, then $\mathbf{L}$ points somewhere to the left of the $\hat{\mathbf{z}}$-axis, so $d\mathbf{L}/dt$ points out of the page, and hence it cannot be equal to $\vec{\tau}$. ♣

20. **Basketball on rim** ***

Let Ω be the desired frequency around the $\hat{\mathbf{z}}$-axis, and look at things in the frame rotating with angular velocity $\Omega\hat{\mathbf{z}}$. In this frame, the center of the ball is at rest. Therefore, if the contact points are to form a great circle, the ball must be spinning around the (negative) $\hat{\mathbf{x}}_3$ axis shown in Fig. 8.72. Let the frequency of this spinning be ω' (in the language of Fig. 8.27). Then the nonslipping condition says that $\omega' r = \Omega R$, so $\omega' = \Omega R/r$. Therefore, the total angular velocity vector of the ball in the lab frame is

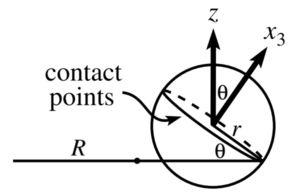


Figure 8.72

$$\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} - \omega'\hat{\mathbf{x}}_3 = \Omega\hat{\mathbf{z}} - (R/r)\Omega\hat{\mathbf{x}}_3. \tag{8.160}$$

Choose the center of the ball as the origin around which $\boldsymbol{\tau}$ and $\mathbf{L}$ are calculated. Then every axis in the ball is a principal axis, with moment of inertia $I = (3/5)mr^2$. The angular momentum is therefore

$$\mathbf{L} = I\boldsymbol{\omega} = I\Omega\hat{\mathbf{z}} - I(R/r)\Omega\hat{\mathbf{x}}_3. \tag{8.161}$$

Only the $\hat{\mathbf{x}}_3$ piece has a horizontal component which will contribute to $d\mathbf{L}/dt$. This component has length $L_\perp = I(R/r)\Omega\sin\theta$. Therefore, $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = \Omega L_\perp = \frac{3}{5}\Omega^2 mrR\sin\theta, \tag{8.162}$$

and points out of the page.

The torque (relative to the center of the ball) comes from the force at the contact point. There are two components of this force. The vertical component is mg, and the horizontal component is $m(R-r\cos\theta)\Omega^2$ (pointing to the left), because the center of the ball moves in a circle of radius $(R - r\cos\theta)$. The torque is then easily seen to have magnitude

$$|\boldsymbol{\tau}| = mg(r\cos\theta) - m(R - r\cos\theta)\Omega^2(r\sin\theta), \tag{8.163}$$

with outward from the page taken to be positive. $\boldsymbol{\tau} = d\mathbf{L}/dt$ therefore gives

$$\Omega^2 = \frac{g\cos\theta}{\frac{8}{5}R\sin\theta - r\sin\theta\cos\theta}\,. \tag{8.164}$$

Remarks:

(a) $\Omega \to \infty$ as $\theta \to 0$, which makes sense.

(b) Also, $\Omega \to \infty$ when $R = (5/8)r\cos\theta$. This case, however, is not physical, since $R > r\cos\theta$ is required in order for the other side of the rim to outside the basketball.

(c) You can also work out the problem in the case where the contact points trace out a circle other than a great circle (say, one that makes an angle β with respect to the great circle). The expression for the torque in eq. (8.163) remains unchanged, but the value of ω' and the angle of the $\hat{\mathbf{x}}_3$ axis both change. Eq. (8.162) is therefore modified. The resulting Ω, however, isn't very illuminating. ♣

21. **Rolling lollipop**

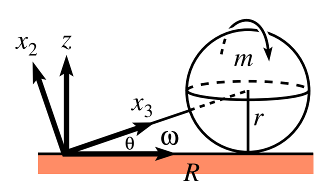


Figure 8.73

(a) We claim that $\boldsymbol{\omega}$ points horizontally to the right (at the instant shown in Fig. 8.73), with magnitude $(R/r)\Omega$. This may be seen in (at least) two ways.

The first method is to note that we essentially have the same scenario as in the "Rolling cone" setup of Problem 3. The sphere's contact point with the ground is at rest (the non-slipping condition), therefore $\boldsymbol{\omega}$ must pass through this point. It is then easy to see that it must point along the horizontal axis. The center of the sphere moves with speed ΩR. And since the center may be considered to be moving with frequency ω in a circle of radius r around the horizontal axis, we see that $\omega = (R/r)\Omega$

The second method is to write $\boldsymbol{\omega}$ as $\boldsymbol{\omega} = -\Omega\hat{\mathbf{z}} + \omega'\hat{\mathbf{x}}_3$ (in the language of Fig. 8.27). We know that ω' is the frequency of the spinning as viewed by someone rotating around the (negative) $\hat{\mathbf{z}}$ axis with frequency Ω. Since the contact points form a circle of radius R on the ground, and since they also form a circle of radius $r\cos\theta$ on the sphere (where $\cos\theta$ is the angle between the stick and the ground), the non-slipping condition implies that $\Omega R = \omega'(r\cos\theta)$. Hence, $\omega' = \Omega R/(r\cos\theta)$. Therefore,

$$\boldsymbol{\omega} = -\Omega\hat{\mathbf{z}} + \omega'\hat{\mathbf{x}}_3 = -\Omega\hat{\mathbf{z}} + \left(\frac{\Omega R}{r\cos\theta}\right)(\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}}) = (R/r)\Omega\hat{\mathbf{x}}, \qquad (8.165)$$

where we have used $\tan\theta = r/R$.

(b) Choose the pivot as the origin. The principal axes are then $\hat{\mathbf{x}}_3$ along the stick, along with any two directions orthogonal to the stick. Choose $\hat{\mathbf{x}}_2$ to be in the plane of the paper. Then the components of $\boldsymbol{\omega}$ along the principal axes are

$$\omega_3 = (R/r)\Omega\cos\theta, \qquad \text{and} \qquad \omega_2 = -(R/r)\Omega\sin\theta. \qquad (8.166)$$

The principal moments are

$$I_3 = (2/5)mr^2, \qquad \text{and} \qquad I_2 = (2/5)mr^2 + m(r^2+R^2), \qquad (8.167)$$

where we have used the parallel-axis theorem. The angular momentum is $\mathbf{L} = I_3\omega_3\hat{\mathbf{x}}_3 + I_2\omega_2\hat{\mathbf{x}}_2$, so its horizontal component of this has length $L_\perp = I_3\omega_3\cos\theta - I_2\omega_2\sin\theta$. Therefore, $d\mathbf{L}/dt$ has magnitude

$$\begin{aligned}
\left|\frac{d\mathbf{L}}{dt}\right| &= \Omega L_\perp \\
&= \Omega(I_3\omega_3\cos\theta - I_2\omega_2\sin\theta) \\
&= \Omega\left(\left(\frac{2}{5}mr^2\right)\left(\frac{R}{r}\Omega\cos\theta\right)\cos\theta - \left(\frac{2}{5}mr^2 + m(r^2+R^2)\right)\left(-\frac{R}{r}\Omega\sin\theta\right)\sin\theta\right) \\
&= \Omega^2 m\frac{R}{r}\left(\frac{2}{5}r^2 + (r^2+R^2)\sin^2\theta\right) \\
&= \frac{7}{5}mrR\Omega^2, \qquad (8.168)
\end{aligned}$$

and it points out of the page.

The torque (relative to the pivot) is due to the gravitational force acting at the CM, along with the normal force, N, acting at the contact point. (Any horizontal friction at the contact point will yield zero torque relative to the pivot.) Therefore, $\boldsymbol{\tau}$ points out of the page with magnitude $|\boldsymbol{\tau}| = (N - mg)R$. Equating this with the $|d\mathbf{L}/dt|$ from eq. (8.168) gives

$$N = mg + \frac{7}{5}mr\Omega^2. \qquad (8.169)$$

This has the interesting property of being independent of R (and hence θ).

REMARK: The pivot must provide a downward force of $N - mg = (7/5)mr\Omega^2$, to make the net vertical force on the lollipop equal to zero. (This result is slightly larger than the $mr\Omega^2$ result for the "sliding" situation in Exercise 5.)

The sum of the horizontal forces at the pivot and the contact point must equal the required centripetal force of $mR\Omega^2$. (But it is impossible to say how this force is divided up, without being given more information.) ♣

22. **Rolling coin**

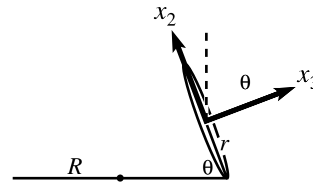


Figure 8.74

Choose the CM as the origin. The principal axes are then $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_3$ (as shown in Fig. 8.74), along with $\hat{\mathbf{x}}_1$ pointing into the paper. Let Ω be the desired frequency. Look at things in the frame rotating around the $\hat{\mathbf{z}}$-axis with frequency Ω. In this frame, the CM remains fixed, and the coin rotates with frequency ω' (in the language of Fig. 8.27) around the negative $\hat{\mathbf{x}}_3$-axis. The non-slipping condition says that $\omega' r = \Omega R$, so $\omega' = \Omega R/r$. Therefore, the total angular velocity vector of the coin in the lab frame is

$$\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} - \omega'\hat{\mathbf{x}}_3 = \Omega\hat{\mathbf{z}} - \frac{R}{r}\Omega\hat{\mathbf{x}}_3. \tag{8.170}$$

But $\hat{\mathbf{z}} = \sin\theta\hat{\mathbf{x}}_2 + \cos\theta\hat{\mathbf{x}}_3$, so we may write $\boldsymbol{\omega}$ in terms of the principal axes as

$$\boldsymbol{\omega} = \Omega\sin\theta\hat{\mathbf{x}}_2 - \Omega\left(\frac{R}{r} - \cos\theta\right)\hat{\mathbf{x}}_3. \tag{8.171}$$

The principal moments are

$$I_3 = (1/2)mr^2, \qquad \text{and} \qquad I_2 = (1/4)mr^2. \tag{8.172}$$

The angular momentum is $\mathbf{L} = I_2\omega_2\hat{\mathbf{x}}_2 + I_3\omega_3\hat{\mathbf{x}}_3$, so its horizontal component has length $L_\perp = I_2\omega_2\cos\theta - I_3\omega_3\sin\theta$, with leftward taken to be positive. Therefore, $d\mathbf{L}/dt$ has magnitude

$$\begin{aligned}
\left|\frac{d\mathbf{L}}{dt}\right| &= \Omega L_\perp \\
&= \Omega(I_2\omega_2\cos\theta - I_3\omega_3\sin\theta) \\
&= \Omega\left(\left(\frac{1}{4}mr^2\right)\Big(\Omega\sin\theta\Big)\cos\theta - \left(\frac{1}{2}mr^2\right)\Big(-\Omega(R/r - \cos\theta)\Big)\sin\theta\right) \\
&= \frac{mr\Omega^2\sin\theta}{4}(2R - r\cos\theta),
\end{aligned} \tag{8.173}$$

with positive numbers corresponding to $d\mathbf{L}/dt$ pointing out of the page (at the instant shown).

The torque (relative to the CM) comes from the force at the contact point. There are two components of this force. The vertical component is mg, and the horizontal component is $m(R - r\cos\theta)\Omega^2$ (pointing to the left), because the center of the CM moves in a circle of radius $(R - r\cos\theta)$. The torque is then easily seen to have magnitude

$$|\boldsymbol{\tau}| = mg(r\cos\theta) - m(R - r\cos\theta)\Omega^2(r\sin\theta), \tag{8.174}$$

with outward from the page taken to be positive. Equating this $|\boldsymbol{\tau}|$ with the $|d\mathbf{L}/dt|$ from eq. (8.173) gives

$$\Omega^2 = \frac{g}{\frac{3}{2}R\tan\theta - \frac{5}{4}r\sin\theta}\,. \tag{8.175}$$

The right-hand-side must be positive if a solution for Ω is to exist. Therefore, we must have

$$R > \frac{5}{6} r\cos\theta \tag{8.176}$$

in order for the described motion to be possible.

REMARKS:

(a) For $\theta \to \pi/2$, eq. (8.175) gives $\Omega \to 0$, as it should. And for $\theta \to 0$, we obtain $\Omega \to \infty$, which also makes sense.

(b) Note that for $(5/6)r\cos\theta < R < r\cos\theta$, the CM of the coin lies to the *left* of the $\hat{\mathbf{z}}$-axis (at the instant shown). The centripetal force, $m(R - r\cos\theta)\Omega^2$, is therefore negative (which means that it is directed radially outward, to the right), but the motion is still possible.

(c) We may consider a more general coin, whose density depends on only the distance from the center, and which has $I_3 = \eta m r^2$. (For example, a uniform coin has $\eta = 1/2$, and a coin with all its mass on the edge has $\eta = 1$.) By the perpendicular axis theorem, $I_1 = I_2 = (1/2)\eta m r^2$, and you can show that the above methods yield

$$\Omega^2 = \frac{g}{(1+\eta)R\tan\theta - (1+\eta/2)r\sin\theta}\,. \tag{8.177}$$

The condition for such motion to exist is then

$$R > \left(\frac{1+\eta/2}{1+\eta}\right) r\cos\theta.\ \clubsuit \tag{8.178}$$

23. **Wobbling coin**

(a) Look at the situation in the frame rotating with angular velocity $\Omega\hat{\mathbf{z}}$. In this frame, the place of contact remains fixed, and the coin rotates with frequency ω' (in the language of Fig. 8.27) around the negative $\hat{\mathbf{x}}_3$ axis. The radius of the circle of contact points on the table in $R\cos\theta$. Therefore, the non-slipping condition says that $\omega' R = \Omega(R\cos\theta)$, so $\omega = \Omega\cos\theta$. Hence, the total angular velocity vector of the coin in the lab frame is

$$\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} - \omega'\hat{\mathbf{x}}_3 = \Omega(\sin\theta\hat{\mathbf{x}}_2 + \cos\theta\hat{\mathbf{x}}_3) - (\Omega\cos\theta)\hat{\mathbf{x}}_3 = \Omega\sin\theta\hat{\mathbf{x}}_2. \tag{8.179}$$

In retrospect, it is clear that $\boldsymbol{\omega}$ must point in the $\hat{\mathbf{x}}_2$ direction. Both the CM and the instantaneous contact point on the coin are at rest, so $\boldsymbol{\omega}$ must lie along the line containing these two points (that is, along the $\hat{\mathbf{x}}_2$-axis).

(b) Choose the CM as the origin. The principal moment around the $\hat{\mathbf{x}}_2$-axis is $I = mR^2/4$. The angular momentum is $\mathbf{L} = I\omega_2\hat{\mathbf{x}}_2$, so its horizontal component has length $L_\perp = L\cos\theta = (I\omega_2)\cos\theta$. Therefore, $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = \Omega L_\perp = \Omega\left(\frac{mR^2}{4}\right)(\Omega\sin\theta)\cos\theta, \tag{8.180}$$

and it points out of the page.

The torque (relative to the CM) is due to the normal force at the contact point (there is no sideways friction force at the contact point, since the CM is motionless) so it has magnitude

$$|\boldsymbol{\tau}| = mgR\cos\theta, \tag{8.181}$$

and it also points out of the page. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives

$$\Omega = \frac{2}{\sqrt{\sin\theta}}\sqrt{\frac{g}{R}}. \qquad (8.182)$$

REMARKS:

i. $\Omega \to \infty$ as $\theta \to 0$. This is quite evident if you do the experiment; the contact point travels very quickly around the circle.

ii. $\Omega \to 2\sqrt{g/R}$ as $\theta \to \pi/2$. This isn't intuitive (to me, at least). In this case, $\mathbf{L}$ is nearly vertical, and it traces out a tiny cone, due to a tiny torque.

iii. In this limit $\theta \to \pi/2$, Ω is also the frequency at which the plane of the coin spins around the vertical axis. Therefore, if you spin a coin very fast about a vertical diameter, it will initially undergo a pure spinning motion with only one contact point; then it will gradually lose energy due to friction, until the spinning frequency slows down to $2\sqrt{g/R}$, at which time it will begin to wobble. (We're assuming, of course, that the coin is very thin, so that it can't balance on its edge.)
In the case where the coin is a quarter (with $R \approx .012\,\text{m}$), this critical frequency of $2\sqrt{g/R}$ turns out to be $\Omega \approx 57\,\text{rad/s}$, which corresponds to about 9 Hertz.
At this critical frequency, the coin's kinetic energy is $T = I\Omega^2/2 = (mR^2/4)(4g/R)/2 = mgR/2$, which happens to be half of the its potential energy. You can show that this fact is independent of the value of I.

iv. The result in eq. (8.182) is a special case of the result in eq. (8.175) of Problem (22). The CM of the coin in Problem (22) will be motionless if $R = r\cos\theta$. Plugging this into eq. (8.175) gives $\Omega^2 = 4g/(r\sin\theta)$, which agrees with eq. (8.182), since r was the coin's radius in Problem 22. ♣

(c) Consider one revolution of the point of contact around the $\hat{\mathbf{z}}$-axis. Since the radius of the circle on the table is $R\cos\theta$, the contact point moves a distance $2\pi R\cos\theta$ around the coin during this time. Hence, the new point of contact on the coin is a distance $2\pi R - 2\pi R\cos\theta$ away from the original point of contact. The coin therefore appears to have rotated by a fraction $(1-\cos\theta)$ of a full turn during this time. The frequency with which you see it turn is therefore

$$(1-\cos\theta)\Omega = \frac{2(1-\cos\theta)}{\sqrt{\sin\theta}}\sqrt{\frac{g}{R}}. \qquad (8.183)$$

REMARKS:

i. If $\theta \approx \pi/2$, then this frequency of Abe's rotation is essentially equal to Ω. This makes sense, because the top of Abe's head will be, say, always near the top of the coin, and this point will trace out a small circle around the $\hat{\mathbf{z}}$-axis, with nearly the same frequency as the contact point.

ii. As $\theta \to 0$, Abe appears to rotate with frequency $\theta^{3/2}\sqrt{g/R}$ (using $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$). Therefore, although the contact point moves infinitely quickly in this limit, we nevertheless see Abe rotating infinitely slowly.

iii. All of the results for frequencies in this problem have to look like some multiple of $\sqrt{g/R}$, by dimensional analysis. But whether the multiplication factor is zero, infinite, or something inbetween, is not at all obvious.

iv. A incorrect answer for the frequency of Abe's turning (when viewed from above) is to say that it equals the vertical component of $\vec{\omega}$, which is $\omega_z = \omega\sin\theta = (\Omega\sin\theta)\sin\theta = 2(\sin\theta)^{3/2}\sqrt{g/R}$. This does not equal the result in eq. (8.183). (It agrees at $\theta = \pi/2$, but is off by a factor of 2 for $\theta \to 0$.) This answer is incorrect because there is simply no reason why the vertical component of $\vec{\omega}$

should equal the frequency of revolution of, say, Abe's nose, around the vertical axis. For example, at moments when $\vec{\omega}$ passes through the nose, then the nose isn't moving at all, so it certainly cannot be described as moving around the vertical axis with frequency ω_z.
The result in eq. (8.183) is a sort of average measure of the frequency of rotation. Even though any given point in the coin is not undergoing uniform circular motion, your eye will see the whole coin as (approximately) rotating uniformly. ♣

24. **Nutation cusps**

(a) Since both $\dot\phi$ and $\dot\theta$ are continuous functions of time, it is clear that we must have $\dot\phi = \dot\theta = 0$ at a kink. (Otherwise, either $d\theta/d\phi = \dot\theta/\dot\phi$ or $d\phi/d\theta = \dot\phi/\dot\theta$ would be well-defined at the kink.) Let the kink occur at $t = t_0$. Eq. (8.89) then says that $\sin(\omega_n t_0) = 0$. Therefore, $\cos(\omega_n t_0) = \pm 1$, and using eq. (8.89) again, we find

$$\Delta\Omega = \mp\Omega_s, \tag{8.184}$$

as was to be shown.

Remark: Note that if $\cos(\omega_n t_0) = 1$, then $\Delta\Omega = -\Omega_s$, so eq. (8.88) says that the kink occurs at the smallest value of θ, i.e., at the highest point of the top's motion. And if $\cos(\omega_n t_0) = -1$, then $\Delta\Omega = \Omega_s$, so eq. (8.88) again says that the kink occurs at the highest point of the top's motion. (See Fig. [nutate].) ♣

(b) To show that these kinks are cusps, we will show that the slope of the θ vs. ϕ plot is infinite on either side of the kink. That is, we will show $d\theta/d\phi = \dot\theta/\dot\phi = \pm\infty$. For simplicity, we will look at the case where $\cos(\omega_n t_0) = 1$ and $\Delta\Omega = -\Omega_s$ (the $\cos(\omega_n t_0) = -1$ case proceeds the same). With $\Delta\Omega = -\Omega_s$, eqs. (8.89) give

$$\frac{\dot\theta}{\dot\phi} = \frac{\sin\theta_0 \sin\omega_n t}{1 - \cos\omega_n t}. \tag{8.185}$$

Letting $t = t_0 + \epsilon$, we have (using $\sin(\omega_n t_0) = 0$ and $\cos(\omega_n t_0) = 1$, and expanding to lowest order in ϵ)

$$\frac{\dot\theta}{\dot\phi} = \frac{\sin\theta_0 \omega_n \epsilon}{\omega_n^2 \epsilon^2/2} = \frac{2\sin\theta_0}{\omega_n \epsilon}. \tag{8.186}$$

For infinitesimal ϵ, this switches from $-\infty$ to $+\infty$ as ϵ passes through zero.

25. **Nutation circles**

(a) Note that a change in angular speed of $\Delta\Omega$ around the fixed $\hat{\mathbf{z}}$ axis corresponds to a change in angular speed of $\sin\theta_0\Delta\Omega$ around the $\hat{\mathbf{x}}_2$ axis. The kick therefore produces an angular momentum (relative to the pivot) component of $I\sin\theta_0\Delta\Omega$ in the $\hat{\mathbf{x}}_2$ direction.

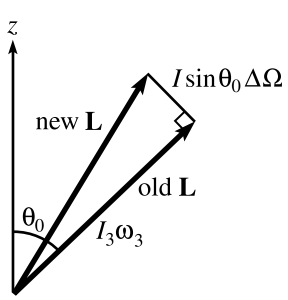


Figure 8.75

The original $\mathbf{L}$ pointed along the $\hat{\mathbf{x}}_3$ direction, with magnitude equal to $I_3\omega_3$ (these two statements hold to a good approximation if $\omega_3 \gg \Omega_s$). By definition, $\mathbf{L}$ made an angle θ_0 with the vertical $\hat{\mathbf{z}}$-axis. Therefore, from Fig. 8.75, the new $\mathbf{L}$ makes an angle

$$\theta_0' = \theta_0 - \frac{I\sin\theta_0\Delta\Omega}{I_3\omega_3} \equiv \theta_0 - \frac{\sin\theta_0\Delta\Omega}{\omega_n} \tag{8.187}$$

with the $\hat{\mathbf{z}}$-axis. So we see that the kick makes $\mathbf{L}$ quickly change its θ value (by a small amount, since we are assuming $\omega_n \sim \omega_3 \gg \Delta\Omega$). And the ϕ value doesn't immediately change.

(b) The torque (relative to the pivot) has magnitude $Mg\ell\sin\theta$ and is directed horizontally. Since θ doesn't change appreciably, the magnitude of the torque is essentially constant, so $\mathbf{L}$ traces out a circle at a constant rate. This rate is simply Ω_s (none of the relevant quantities in $\boldsymbol{\tau} = d\mathbf{L}/dt$ changed much from the original circular-precession case, so the precession frequency remains basically the same). So the new $\mathbf{L}$ has its (ϕ, θ) coordinates given by

$$(\phi(t), \theta(t))_{\mathbf{L}} = \left(\Omega_s t,\ \theta_0 - \frac{\sin\theta_0 \Delta\Omega}{\omega_n}\right). \tag{8.188}$$

Looking at eqs. (8.88), we see that the coordinates of the CM relative to $\mathbf{L}$ are

$$(\phi(t), \theta(t))_{\mathrm{CM}-\mathbf{L}} = \left(\left(\frac{\Delta\Omega}{\omega_n}\right)\sin\omega_n t, \left(\frac{\Delta\Omega}{\omega_n}\sin\theta_0\right)\cos\omega_n t\right). \tag{8.189}$$

The $\sin\theta_0$ factor in $\theta(t)$ is exactly what is needed for the CM to trace out a circle around $\mathbf{L}$ (since a change in ϕ corresponds to a CM spatial change of $\ell\Delta\phi\sin\theta_0$, and a change in θ corresponds to a CM spatial change of $\ell\Delta\theta$.)

This circular motion is exactly what we expect from the results in section 8.6.2, for the following reason. For ω_n very large and Ω_s very small, $\mathbf{L}$ essentially sits still, and the CM traces out a circle around it at frequency ω_n. Since $\mathbf{L}$ is essentially constant, the top should therefore behave very much like the free top, as viewed from a fixed frame. (The effects of the gravitational torque are negligible compared to the magnitude of $\mathbf{L}$, so we can basically ignore gravity [as long as the time scale isn't too large; eventually the effects of gravity are noticeable, and we see $\mathbf{L}$ move].)

But eq. (8.53) in section 8.6.2 says that the frequency of the precession of $\hat{\mathbf{x}}_3$ around $\mathbf{L}$ is L/I. The frequency of the precession of $\hat{\mathbf{x}}_3$ around $\mathbf{L}$ was obtained above as ω_n. This had better be equal to L/I. And indeed, L is essentially equal to $I_3\omega_3$, so $L/I = I_3\omega_3/I \equiv \omega_n$.

Therefore, for short enough time scales (short enough so that $\mathbf{L}$ doesn't move much), a nutating top (with $\Delta\Omega \gg \Omega_s$) looks very much like a free top.

REMARK: We need the $\Delta\Omega \gg \Omega_s$ condition so that the nutation motion looks like circles (that is, it looks like the top plot in Fig. [nutate], rather than the others). This can be seen by the following reasoning. The time for one period of the nutation motion is $2\pi/\omega_n$. From eq. (8.88), $\phi(t)$ increases by $\Delta\phi = 2\pi\Omega_s/\omega_n$ in this time. And also from eq. (8.88), the width, w, of the 'circle' along the ϕ axis is roughly $w = 2\Delta\Omega/\omega_n$. The motion looks like basically like a circle if $w \gg \Delta\phi$, that is, if $\Delta\Omega \gg \Omega_s$. ♣

26. **Rolling straight?**

Intuitively, it is fairly clear that the sphere cannot change direction, but it is a little tricky to prove. Qualitatively, we can reason as follows. Assume there is a nonzero friction force at the contact point. (The normal force is irrelevant here, since it never provides a torque about the CM. This is what is special about a sphere.) Then the ball will accelerate in the direction of this force. However, it is easy to show with the right-hand rule that this force will produce a torque which will cause angular momentum to change in a way that corresponds to the ball accelerating in the direction *opposite* to the friction force. There is thus a contradiction, unless the friction force equals zero.

Let's now be rigorous. Let the angular velocity of the ball be $\boldsymbol{\omega}$. The non-slipping condition says that the ball's velocity equals

$$\mathbf{v} = \boldsymbol{\omega} \times (a\hat{\mathbf{z}}). \tag{8.190}$$

where a is the radius of the sphere. The ball's angular momentum is

$$\mathbf{L} = I\boldsymbol{\omega}. \tag{8.191}$$

The friction force from the ground at the contact point is responsible for changing both the momentum and the angular momentum of the ball. $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}\,, \tag{8.192}$$

and $\boldsymbol{\tau} = d\mathbf{L}/dt$ (relative to the center of the ball) gives

$$(-a\hat{\mathbf{z}}) \times \mathbf{F} = \frac{d\mathbf{L}}{dt}\,, \tag{8.193}$$

since the force is applied at position $-a\hat{\mathbf{z}}$ relative to the ball's center.

We will show that the preceding four equations imply $\dot{\boldsymbol{\omega}} = \mathbf{0}$ (which implies $\mathbf{v}$ is constant), in the following way. Plug the $\mathbf{v}$ from eq. (8.190) into eq. (8.192), and then plug the resulting $\mathbf{F}$ into eq. (8.193). Also, plug the $\mathbf{L}$ from eq. (8.191) into eq. (8.193). The result is

$$(-a\hat{\mathbf{z}}) \times \big(m\dot{\boldsymbol{\omega}} \times (a\hat{\mathbf{z}})\big) = I\dot{\boldsymbol{\omega}}. \tag{8.194}$$

Since the vector $\dot{\boldsymbol{\omega}}$ lies in the horizontal plane, it is easy to see work out that $\hat{\mathbf{z}} \times (\dot{\boldsymbol{\omega}} \times \hat{\mathbf{z}}) = \dot{\boldsymbol{\omega}}$. Therefore, we have

$$-ma^2\dot{\boldsymbol{\omega}} = (2/5)ma^2\dot{\boldsymbol{\omega}}, \tag{8.195}$$

and hence $\dot{\boldsymbol{\omega}} = \mathbf{0}$.

27. **Ball on paper**

It turns out that the solution to this problem is virtually the same as the solution to the previous problem.

Let the angular velocity of the ball be $\boldsymbol{\omega}$. The ball's angular momentum is

$$\mathbf{L} = I\boldsymbol{\omega}. \tag{8.196}$$

The friction force from the ground at the contact point is responsible for changing both the momentum and the angular momentum of the ball. $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}\,, \tag{8.197}$$

and $\boldsymbol{\tau} = d\mathbf{L}/dt$ (relative to the center of the ball) gives

$$(-a\hat{\mathbf{z}}) \times \mathbf{F} = \frac{d\mathbf{L}}{dt}\,, \tag{8.198}$$

since the force is always applied at position $-a\hat{\mathbf{z}}$ relative to the ball's center (where a is the radius of the ball).

Plugging the $\mathbf{L}$ from eq. (8.196) and the $\mathbf{F}$ from eq. (8.197) into eq. (8.198) gives

$$-a\hat{\mathbf{z}} \times m\dot{\mathbf{v}} = I\dot{\boldsymbol{\omega}}. \tag{8.199}$$

Integrating from the initial to final times yields

$$-a\hat{\mathbf{z}} \times m\Delta\mathbf{v} = I\Delta\boldsymbol{\omega}. \tag{8.200}$$

(This is simply the relation between impulse and angular impulse in eq. (7.31). It is true no matter what slipping or jerky motions take place.)

The initial and final non-slipping conditions say that

$$\mathbf{v} = \boldsymbol{\omega} \times (a\hat{\mathbf{z}}). \tag{8.201}$$

at these times. Therefore,

$$\Delta\mathbf{v} = \Delta\boldsymbol{\omega} \times (a\hat{\mathbf{z}}). \tag{8.202}$$

Plugging this into eq. (8.200) gives

$$-a\hat{\mathbf{z}} \times m(\Delta\boldsymbol{\omega} \times a\hat{\mathbf{z}}) = I\Delta\boldsymbol{\omega}. \tag{8.203}$$

Since the vector $\dot{\boldsymbol{\omega}}$ lies in the horizontal plane, it is easy to see work out that $\hat{\mathbf{z}} \times (\Delta\boldsymbol{\omega} \times \hat{\mathbf{z}}) = \Delta\boldsymbol{\omega}$, and so

$$-ma^2\Delta\boldsymbol{\omega} = (2/5)ma^2\Delta\boldsymbol{\omega}. \tag{8.204}$$

Therefore $\Delta\boldsymbol{\omega} = \mathbf{0}$, and hence $\Delta\mathbf{v} = \mathbf{0}$, as was to be shown.

REMARKS:

(a) Note that it is perfectly allowed to move the paper in a jerky motion, so that the ball slips around on it. We assumed nothing about the nature of the friction force in the above reasoning. We used the non-slipping condition only at the initial and final times. The intermediate motion is arbitrary.

(b) As a special case, if you start a ball at rest on a piece of paper, then no matter how you choose to (horizontally) slide the paper out from underneath the ball, the ball will be at rest in the end.

(c) You are encouraged to experimentally verify that all these crazy claims are true. Make sure that the paper doesn't wrinkle (this would allow a force to be applied at a point other than the contact point). And balls that don't squish are much better, of course (for the same reason). ♣

28. **Ball on turntable**

The angular velocity of the turntable is $\Omega\hat{\mathbf{z}}$. Let the angular velocity of the ball be $\boldsymbol{\omega}$. If the ball is at position $\mathbf{r}$ (with respect to the lab frame), then its velocity (with respect to the lab frame) may be broken up into the velocity of the turntable (at position $\mathbf{r}$) plus the ball's velocity relative to the turntable. The non-slipping condition says that this latter velocity is given by $\boldsymbol{\omega} \times (a\hat{\mathbf{z}})$. (We'll use "$a$" to denote the radius of the sphere.) The ball's velocity with respect to the lab frame is thus

$$\mathbf{v} = (\Omega\hat{\mathbf{z}}) \times \mathbf{r} + \boldsymbol{\omega} \times (a\hat{\mathbf{z}}). \tag{8.205}$$

The angular momentum of the ball is

$$\mathbf{L} = I\boldsymbol{\omega}. \tag{8.206}$$

The friction force from the ground is responsible for changing both the momentum and the angular momentum of the ball. $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt}\,, \tag{8.207}$$

and $\boldsymbol{\tau} = d\mathbf{L}/dt$ (relative to the center of the ball) gives

$$(-a\hat{\mathbf{z}}) \times \mathbf{F} = \frac{d\mathbf{L}}{dt}\,, \tag{8.208}$$

since the force is applied at position $-a\hat{\mathbf{z}}$ relative to the ball's center.

We will now use the previous four equations to demonstrate that the ball undergoes circular motion. Our goal will be to produce an equation of the form $d\mathbf{v}/dt = \Omega'\hat{\mathbf{z}}\times\mathbf{v}$, since this describes circular motion, with frequency Ω' (to be determined).

Plugging the expressions for $\mathbf{L}$ and $\mathbf{F}$ from eqs. (8.206) and (8.207) into eq. (8.208) gives

$$\begin{aligned}(-a\hat{\mathbf{z}})\times\left(m\frac{d\mathbf{v}}{dt}\right) &= I\frac{d\boldsymbol{\omega}}{dt} \\ \Longrightarrow \qquad \frac{d\boldsymbol{\omega}}{dt} &= -\left(\frac{am}{I}\right)\hat{\mathbf{z}}\times\frac{d\mathbf{v}}{dt}\,. \end{aligned} \tag{8.209}$$

Taking the derivative of eq. (8.205) then gives

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \Omega\hat{\mathbf{z}}\times\frac{d\mathbf{r}}{dt}+\frac{d\boldsymbol{\omega}}{dt}\times(a\hat{\mathbf{z}}) \\ &= \Omega\hat{\mathbf{z}}\times\mathbf{v}-\left(\left(\frac{am}{I}\right)\hat{\mathbf{z}}\times\frac{d\mathbf{v}}{dt}\right)\times(a\hat{\mathbf{z}}). \end{aligned} \tag{8.210}$$

Since the vector $d\mathbf{v}/dt$ lies in the horizontal plane, it is easy to work out the cross-product in the right term (or just use the identity $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{B}\cdot\mathbf{C})\mathbf{A}$) to obtain

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \Omega\hat{\mathbf{z}}\times\mathbf{v}-\left(\frac{a^2m}{I}\right)\frac{d\mathbf{v}}{dt} \\ \Longrightarrow \qquad \frac{d\mathbf{v}}{dt} &= \left(\frac{\Omega}{1+(a^2m/I)}\right)\hat{\mathbf{z}}\times\mathbf{v}. \end{aligned} \tag{8.211}$$

For a uniform sphere, $I = (2/5)ma^2$, so we obtain

$$\frac{d\mathbf{v}}{dt} = \left(\frac{2}{7}\Omega\right)\hat{\mathbf{z}}\times\mathbf{v}. \tag{8.212}$$

The ball therefore undergoes circular motion, with a frequency equal to 2/7 times the frequency of the turntable. This result for the frequency does not depend on initial conditions.

REMARKS:

(a) Integrating eq. (8.212) from the initial time to some later time gives

$$\mathbf{v}-\mathbf{v}_0 = \left(\frac{2}{7}\Omega\right)\hat{\mathbf{z}}\times(\mathbf{r}-\mathbf{r}_0). \tag{8.213}$$

This may be written in the more suggestive form,

$$\mathbf{v} = \left(\frac{2}{7}\Omega\right)\hat{\mathbf{z}}\times\left(\mathbf{r}-\left(\mathbf{r}_0+\frac{7}{2\Omega}(\hat{\mathbf{z}}\times\mathbf{v}_0)\right)\right). \tag{8.214}$$

This equation describes circular motion, with the center located at the point

$$\mathbf{r}_c = \mathbf{r}_0+(7/2\Omega)(\hat{\mathbf{z}}\times\mathbf{v}_0), \tag{8.215}$$

and with radius $(7/2\Omega)|\hat{\mathbf{z}}\times\mathbf{v}_0| = 7v_0/2\Omega$. (Eq. (8.214) does indeed describe circular motion, because it says that $\mathbf{v}$ is always perpendicular to $\mathbf{r}-\mathbf{r}_c$. Hence, the distance to the point $\mathbf{r}_c$ doesn't change.)

(b) There are a few special cases to consider:

- If $v_0 = 0$ (that is, if the spinning motion of the ball exactly cancels the rotational motion of the turntable), then the ball will always remain in the same place (of course).
- If the ball is initially not spinning, and just moving along with the turntable, then $v_0 = \Omega r_0$, so the radius of the circle is $(7/2)r_0$.
- If we want the center of the circle be the center of the turntable, then eq. (8.215) say that we need $(7/2\Omega)\hat{\mathbf{z}} \times \mathbf{v}_0 = -\mathbf{r}_0$. This implies that $\mathbf{v}_0$ has magnitude $v_0 = (2/7)\Omega r_0$ and points tangentially in the same direction as the turntable moves. (That is, the ball moves at 2/7 times the velocity of the turntable beneath it.)

(c) The fact that the frequency $(2/7)\Omega$ is a rational multiple of Ω means that the ball will eventually return to the same point on the turntable. In the lab frame, the ball will trace out two circles in the time it takes the turntable to undergo seven revolutions. And from the point of view of someone on the turntable, the ball will 'spiral' around five times before returning to the original position.

(d) If we look at a ball with moment of inertia $I = \eta m a^2$ (so a uniform sphere has $\eta = 2/5$), then it is easy to show that the "2/7" in the above result gets replaced by "$\eta/(1+\eta)$". If a ball has most of its mass concentrated at its center (so that $\eta \to 0$), then the frequency of the circular motion goes to 0, and the radius goes to ∞. ♣

Chapter 9

Accelerated Frames of Reference

Newton's laws hold only in inertial frames of reference. However, there are many non-inertial (that is, accelerated) frames of reference that we might reasonably want to study (such as elevators, merry-go-rounds, as so on). Is there any possible way to modify Newton's laws so that they hold in non-inertial frames, or do we have to give up entirely on $\mathbf{F} = m\mathbf{a}$?

It turns out that we can indeed hold onto our good friend $\mathbf{F} = m\mathbf{a}$, provided that we introduce some new "fictitious" forces. These are forces that a person in the accelerated frame thinks exist. If he applies $\mathbf{F} = m\mathbf{a}$, while including these new forces, he will get the correct answer for the acceleration, $\mathbf{a}$, as measured with respect to his frame.

To be quantitative about all this, we'll have to spend some time determining how the coordinates (and their derivatives) in an accelerated frame relate to those in an inertial frame. But before diving into that, let's look at a simple example which demonstrates the basic idea of fictitious forces.

Example (The train): Imagine that you are standing on a train which is accelerating to the right with acceleration a. If you wish to remain in the same spot on the train, your feet must apply a friction force, $F_\mathrm{f} = ma$, pointing to the right. Someone standing in the inertial frame of the ground will simply interpret the situation as, "The friction force, $F_\mathrm{f} = ma$, causes your acceleration, a."

How do you interpret the situation, in the frame of the train? (Imagine that there are no windows, so that all you see is the inside of the train.) As we will show below in eq. (9.11), you will feel a fictitious *translation* force, $F_\mathrm{trans} = -ma$, pointing to the left. You will therefore interpret the situation as, "In my frame (the frame of the train), the friction force $F_\mathrm{f} = ma$ pointing to my right exactly cancels the mysterious $F_\mathrm{trans} = -ma$ force pointing to my left, resulting in zero acceleration (in my frame)."

Of course, if the floor of the train is frictionless and your feet aren't able to provide a force, then you will say that the net force on you is $F_\mathrm{trans} = -ma$, pointing to the left. You will therefore accelerate with acceleration a to the left, with respect to your frame (the train). In other words, you will remain motionless with respect to the inertial frame of the ground, which is all quite obvious to someone standing on the ground.

In the case where your feet are able to supply a nonzero force, but not a large enough one to balance out the whole $F_{\rm trans} = -ma$ force, you will end up being jerked toward the back of the train a bit (until your feet or hands are able to balance out $F_{\rm trans}$), which is what usually happens on a subway train (at least here in Boston, where hands are often necessary).

Let's now derive the fictitious forces in their full generality. This endeavor will require a bit of math, since we have to relate the coordinates in an accelerated frame with those in an inertial frame.

9.1 Relating the coordinates

Consider an inertial coordinate system with axes $\hat{\mathbf{x}}_{\rm I}$, $\hat{\mathbf{y}}_{\rm I}$, and $\hat{\mathbf{z}}_{\rm I}$, and let there be another (possibly accelerating) coordinate system with axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. These axes will be allowed to change in an arbitrary manner with respect to the inertial frame. That is, the origin may undergo acceleration, and the axes may rotate. (This is the most general possible motion, as we saw in Section 8.1.) These axes may be considered to be functions of the inertial axes. Let $O_{\rm I}$ and O be the origins of the two coordinate systems.

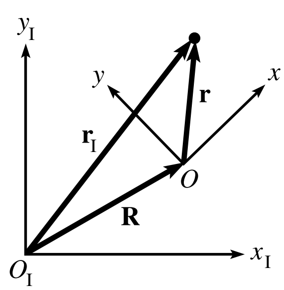


Figure 9.1

Let the vector from $O_{\rm I}$ to O be $\mathbf{R}$. Let the vector from $O_{\rm I}$ to a given point be $\mathbf{r}_{\rm I}$. And let the vector from O to the given point be $\mathbf{r}$. (See Fig. 9.1 for the 2-D case.) Then

$$\mathbf{r}_{\rm I} = \mathbf{R} + \mathbf{r}. \tag{9.1}$$

These vectors have an existence independent of any specific coordinate system, but let us write them in terms of some definite coordinates. We may write

$$\begin{aligned} \mathbf{R} &= (X\hat{\mathbf{x}}_{\rm I} + Y\hat{\mathbf{y}}_{\rm I} + Z\hat{\mathbf{z}}_{\rm I}), \\ \mathbf{r}_{\rm I} &= (x_{\rm I}\hat{\mathbf{x}}_{\rm I} + y_{\rm I}\hat{\mathbf{y}}_{\rm I} + z_{\rm I}\hat{\mathbf{z}}_{\rm I}), \\ \mathbf{r} &= (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}). \end{aligned} \tag{9.2}$$

For reasons that will become clear, we have chosen to write $\mathbf{R}$ and $\mathbf{r}_{\rm I}$ in terms of the inertial-frame coordinates, and $\mathbf{r}$ in terms of the accelerated-frame coordinates. If desired, eq. (9.1) may be written as

$$x_{\rm I}\hat{\mathbf{x}}_{\rm I} + y_{\rm I}\hat{\mathbf{y}}_{\rm I} + z_{\rm I}\hat{\mathbf{z}}_{\rm I} = (X\hat{\mathbf{x}}_{\rm I} + Y\hat{\mathbf{y}}_{\rm I} + Z\hat{\mathbf{z}}_{\rm I}) + (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})\,. \tag{9.3}$$

Our goal is to take the second time derivative of eq. (9.1), and to interpret the result in an $\mathbf{F} = m\mathbf{a}$ form. The second derivative of $\mathbf{r}_{\rm I}$ is simply the acceleration of the particle with respect to the inertial system (and so Newton's second law tells us that $\mathbf{F} = m\ddot{\mathbf{r}}_{\rm I}$). The second derivative of $\mathbf{R}$ is the acceleration of the origin of the moving system. The second derivative of $\mathbf{r}$ is the tricky part. Changes in $\mathbf{r}$ can come about in two ways. First, the coordinates (x, y, z) of $\mathbf{r}$ (which are measured relative to the moving axes) may change. And second, the axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ themselves may change; so even if (x, y, z) remain fixed, the position of the particle may change. Let us be quantitative about this.

Calculation of $d^2\mathbf{r}/dt^2$

We should clarify our goal here. We would like to obtain $d^2\mathbf{r}/dt^2$ in terms of the coordinates in the *moving* frame, because we want to be able to work entirely in terms of the coordinates of the accelerated frame, so that a person in this frame can write down an $\mathbf{F} = m\mathbf{a}$ equation in terms of only his coordinates, and not have to consider the underlying inertial frame at all. (In terms of the inertial frame, $d^2\mathbf{r}/dt^2$ is simply $d^2(\mathbf{r}_\text{I} - \mathbf{R})/dt^2$, but this is not very enlightening by itself.)

The following exercise in taking derivatives works for a general vector $\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$ in the moving frame. So we'll work with a general $\mathbf{A}$ and then set $\mathbf{A} = \mathbf{r}$ when we're done.

To take d/dt of $\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$, we use the product rule to obtain

$$\frac{d\mathbf{A}}{dt} = \left(\frac{dA_x}{dt}\hat{\mathbf{x}} + \frac{dA_y}{dt}\hat{\mathbf{y}} + \frac{dA_z}{dt}\hat{\mathbf{z}}\right) + \left(A_x\frac{d\hat{\mathbf{x}}}{dt} + A_y\frac{d\hat{\mathbf{y}}}{dt} + A_z\frac{d\hat{\mathbf{z}}}{dt}\right). \qquad (9.4)$$

Yes, the product rule works with vectors too. We're doing nothing more than expanding $(A_x + dA_x)(\hat{\mathbf{x}} + d\hat{\mathbf{x}}) - A_x\hat{\mathbf{x}}$, etc., to first order.

The first of the two terms in eq. (9.4) is simply the rate of change of $\mathbf{A}$, as measured with respect to the moving frame. We will denote this quantity by $\delta\mathbf{A}/\delta t$.

The second term arises because the coordinate axes are moving. In what manner are they moving? We have already extracted the motion of the origin of the moving system (by introducing the vector $\mathbf{R}$), so the only thing left is a rotation about some axis $\boldsymbol{\omega}$ through the origin (see Theorem 8.1). This axis may be changing in time, but at any instant a unique axis of rotation describes the system. (The fact that the axis may change will be relevant in finding the second derivative of $\mathbf{r}$, but not in finding the first derivative.)

We saw in Theorem 8.2 that a vector $\mathbf{B}$, of fixed length, rotating with angular velocity $\boldsymbol{\omega} \equiv \omega\hat{\boldsymbol{\omega}}$ changes at a rate $d\mathbf{B}/dt = \boldsymbol{\omega}\times\mathbf{B}$. In particular, $d\hat{\mathbf{x}}/dt = \boldsymbol{\omega}\times\hat{\mathbf{x}}$, etc. So in eq. (9.4), the $A_x(d\hat{\mathbf{x}}/dt)$ term, for example, equals $A_x(\boldsymbol{\omega}\times\hat{\mathbf{x}}) = \boldsymbol{\omega}\times(A_x\hat{\mathbf{x}})$. Adding on the y and z terms gives $\boldsymbol{\omega}\times(A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) = \boldsymbol{\omega}\times\mathbf{A}$. Therefore, eq. (9.4) yields

$$\boxed{\frac{d\mathbf{A}}{dt} = \frac{\delta\mathbf{A}}{\delta t} + \boldsymbol{\omega}\times\mathbf{A}}\,. \qquad (9.5)$$

This agrees with the result obtained in Section 8.5, eq. (8.39). We've basically given the same proof here, but with a little more mathematical rigor.

We still have to take one more time derivative. The time derivative of eq. (9.5) yields

$$\frac{d^2\mathbf{A}}{dt^2} = \frac{d}{dt}\left(\frac{\delta\mathbf{A}}{\delta t}\right) + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{A} + \boldsymbol{\omega}\times\frac{d\mathbf{A}}{dt}\,. \qquad (9.6)$$

Applying eq. (9.5) to the first term (with $\delta\mathbf{A}/\delta t$ instead of $\mathbf{A}$), and plugging eq. (9.5) into the third term, gives

$$\begin{aligned}\frac{d^2\mathbf{A}}{dt^2} &= \left(\frac{\delta^2\mathbf{A}}{\delta t^2} + \boldsymbol{\omega}\times\frac{\delta\mathbf{A}}{\delta t}\right) + \left(\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{A}\right) + \left(\boldsymbol{\omega}\times\frac{\delta\mathbf{A}}{\delta t} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A})\right)\\ &= \frac{\delta^2\mathbf{A}}{\delta t^2} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{A}) + 2\boldsymbol{\omega}\times\frac{\delta\mathbf{A}}{\delta t} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{A}.\end{aligned} \qquad (9.7)$$

At this point, we will now set $\mathbf{A} = \mathbf{r}$, so we have

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{\delta^2\mathbf{r}}{\delta t^2} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathbf{v} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}, \tag{9.8}$$

where $\mathbf{v} \equiv \delta\mathbf{r}/\delta t$ is the velocity of the particle, as measured with respect to the moving frame.

9.2 The fictitious forces

From eq. (9.1) we have

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2\mathbf{r}_\text{I}}{dt^2} - \frac{d^2\mathbf{R}}{dt^2}\,. \tag{9.9}$$

Let us equate this expression for $d^2\mathbf{r}/dt^2$ with the one in eq. (9.8), and then multiply through by the mass m of the particle. Recognizing that the $m(d^2\mathbf{r}_\text{I}/dt^2)$ term is simply the force $\mathbf{F}$ acting on the particle ($\mathbf{F}$ may be gravity, a normal force, friction, tension, etc.), we may write the result as

$$\begin{aligned} m\frac{\delta^2\mathbf{r}}{\delta t^2} &= \mathbf{F} - m\frac{d^2\mathbf{R}}{dt^2} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v} - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} \\ &\equiv \mathbf{F} + \mathbf{F}_\text{translation} + \mathbf{F}_\text{centrifugal} + \mathbf{F}_\text{Coriolis} + \mathbf{F}_\text{azimuthal}, \end{aligned} \tag{9.10}$$

where the *fictitious forces* are defined as

$$\begin{aligned} \mathbf{F}_\text{trans} &\equiv -m\frac{d^2\mathbf{R}}{dt^2}\,, \\ \mathbf{F}_\text{cent} &\equiv -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \\ \mathbf{F}_\text{Cor} &\equiv -2m\boldsymbol{\omega} \times \mathbf{v}, \\ \mathbf{F}_\text{az} &\equiv -m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}. \end{aligned} \tag{9.11}$$

We have taken the liberty of calling these quantities "forces", because the left-hand side of eq. (9.10) is simply m times the acceleration, as measured by someone in the moving frame. This person should therefore be able to interpret the right-hand side as some effective force. In other words, if a person wishes to calculate $m(\delta^2\mathbf{r}/\delta t^2)$, he simply needs to take the true force $\mathbf{F}$, and then add on all the other terms on the right-hand side, which he will then quite reasonably interpret as forces (in his frame).

These extra terms of course are not actual forces. The constituents of $\mathbf{F}$ are the only real forces in the problem. All we are saying is that if our friend in the moving frame assumes the extra terms are real forces, and then adds them to $\mathbf{F}$, then he will get the correct answer for $m(\delta^2\mathbf{r}/\delta t^2)$, the mass times acceleration in his frame.

For example, consider a box (far away from other objects, in outer space) which accelerates at a rate of $g = 10$ m/s^2 in some direction. A person in the box will feel a fictitious force of $\mathbf{F}_\text{trans} = mg$ down into the floor. For all he knows, he is in a box on the surface of the earth. If he performs various experiments under this assumption, the results will always be what he expects. The surprising fact that no local experiment can distinguish between the fictitious force in the accelerated box and the real gravitational force on the earth is what led Einstein to his Equivalence Principle (discussed in Chapter 13). These fictitious forces are more meaningful than you might think.

> As Einstein explored elevators,
> And studied the spinning ice-skaters,
> He eyed as suspicious,
> The forces fictitious,
> Of gravity's great imitators.

Let's now look at each of the above "forces" in detail. The translational and centrifugal forces are easy to understand. The Coriolis force is a little more difficult. And the azimuthal force can be easy or difficult, depending on how exactly $\boldsymbol{\omega}$ is changing (we'll mainly deal with the easy case).

9.2.1 Translation force: $-md^2\mathbf{R}/dt^2$

This is the most intuitive of the fictitious forces. We've already discussed this force in the train example in the introduction to this chapter. If $\mathbf{R}$ is the position of the train, then $\mathbf{F}_{\text{trans}} \equiv -m\, d^2\mathbf{R}/dt^2$ is the fictitious force you feel in the accelerated frame.

9.2.2 Centrifugal force: $-m\vec{\omega} \times (\vec{\omega} \times \mathbf{r})$

This force goes hand-in-hand with the $mv^2/r = mr\omega^2$ centripetal acceleration as viewed by someone in an inertial frame.

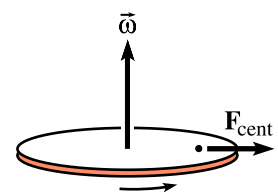


Figure 9.2

Example 1 (Standing on a carousel): Consider a person standing motionless on a carousel. Let the carousel rotate in the x-y plane with angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ (see Fig. 9.2). What is the centrifugal force felt by a person standing at a distance r from the center?

Solution: $\boldsymbol{\omega} \times \mathbf{r}$ has magnitude ωr and points in the tangential direction, in the direction of motion. (It's just the velocity as viewed by someone on the ground, after all.) So $m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ points radially inward and has magnitude $mr\omega^2$. Therefore, the centrifugal force, $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$, points radially outward, with magnitude $mr\omega^2$.

Remark: If the person is not moving with respect to the carousel, and if $\vec{\omega}$ is constant, then the centrifugal force is the only non-zero fictitious force in eq. (9.10). Since the person is not accelerating in her rotating frame, the net force (as measured in her frame) must be zero. The forces in her frame are (1) gravity pulling downward, (2) a normal force pushing upward (which cancels the gravity), (3) a friction force pushing inward at her feet, and (4) the centrifugal force pulling outward. So we conclude that the last two of these must cancel. That is, the friction force equals $mr\omega^2$.

Of course, someone standing on the ground will observe only the first three of these forces, so the net force will not be zero. And indeed, there is a centripetal acceleration of $v^2/r = r\omega^2$, which is arises because of the friction force. To sum up: in the inertial frame, the friction force is there to provide an acceleration. In the rotating frame, the friction force is there to balance out this mysterious new centrifugal force, in order to yield zero acceleration. ♣

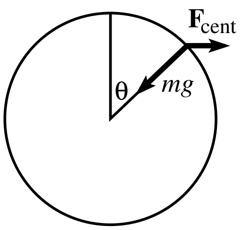


Figure 9.3

Example 2 (Effective gravity force, $m\mathbf{g}_{\text{eff}}$): Consider a person standing motionless on the earth, at a polar angle θ. (See Fig. 9.3. The way we've defined it, θ equals $\pi/2$ minus the latitude angle.) She will feel a force due to gravity, directed toward the center of the earth. But in her rotating frame, she will also feel a centrifugal force, directed away from the rotation axis. The sum of these two forces (that is, what she thinks is gravity) will not point radially (unless she is at the equator or at a pole). Let us denote the sum of these forces as $m\mathbf{g}_{\text{eff}}$.

To calculate $m\mathbf{g}_{\text{eff}}$, we must calculate $-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$. The $\boldsymbol{\omega}\times\mathbf{r}$ part has magnitude $R\omega\sin\theta$ (where R is the radius of the earth), and is directed tangentially along the latitude circle of radius $R\sin\theta$. So $-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ points outward from the z-axis, with magnitude $mR\omega^2\sin\theta$ (which is just what we expect for something traveling in a circle of radius $R\sin\theta$). Therefore, the effective gravitational force,

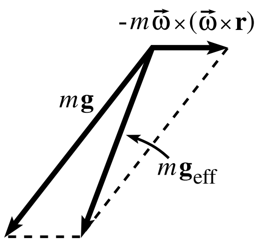


Figure 9.4

$$m\mathbf{g}_{\text{eff}} \equiv m\Big(\mathbf{g}-\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\Big), \tag{9.12}$$

points slightly in the southerly direction (for someone in the northern hemisphere), as shown in Fig. 9.4. The magnitude of the correction term, $mR\omega^2\sin\theta$, is small compared to g. Using $\omega\approx 7.3\cdot 10^{-5}\,\text{s}^{-1}$ (that is, one revolution per day, which is 2π radians per 86,400 seconds) and $R\approx 6.4\cdot 10^6\,\text{m}$, we find $R\omega^2\approx .03\,\text{m/s}^2$. Therefore, the correction to g is about 0.3% at the equator.

Remark: In the construction of buildings, and in similar matters, it is of course $\mathbf{g}_{\text{eff}}$, and not $\mathbf{g}$, that determines the "upward" direction in which the building should point. The exact direction to the center of the earth is irrelevant. A plumb bob hanging from the top of a skyscraper touches exactly at the base. Both the bob and the building point in a direction slightly different from the radial, but no one cares. ♣

9.2.3 Coriolis force: $-2m\vec{\omega}\times\mathbf{v}$

While the centrifugal force is very intuitive concept (everyone has gone around a corner in a car), the same thing cannot be said about the Coriolis force. This force requires a non-zero velocity $\mathbf{v}$ relative to the accelerated frame (and people do not normally move appreciably with respect to their car while rounding a corner). To get a feel for this force, let's look at two special cases.

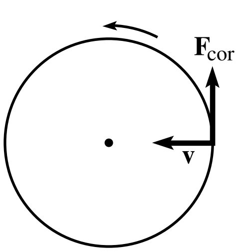


Figure 9.5

Case 1 (Moving radially on a carousel): Consider someone walking radially inward on a carousel, with speed v at radius r (see Fig. 9.5). The force $-2m\boldsymbol{\omega}\times\mathbf{v}$ points tangentially, in the direction of the motion of the carousel (that is, to the person's right, in our scenario), with magnitude

$$F_{\text{Cor}} = 2m\omega v. \tag{9.13}$$

Let's assume that the person counters this force with a tangential friction force of $2m\omega v$ (pointing to his left) at his feet, so that he continues to walk on the same radial line.[1]

Why is this Coriolis force (and hence the tangential friction force) there? It exists so that the resultant friction force changes the angular momentum of the person (measured with respect to the lab frame) in the proper way. To see this, take d/dt of $L = mr^2\omega$ (where ω is the person's angular speed with respect to the lab frame, which is also the carousel's angular speed). Using $dr/dt = -v$, we have

$$\frac{dL}{dt} = -2mr\omega v + mr^2(d\omega/dt). \tag{9.14}$$

But $d\omega/dt = 0$, since the person is staying on one radial line (and we're assuming that the carousel is arranged to keep a constant ω). Eq. (9.14) then gives $dL/dt = -2mr\omega v$. So the L of the person changes at a rate of $-(2m\omega v)r$. This is simply the radius times the tangential friction force applied by the carousel, that is, the torque applied to the person.

REMARK: What if the person does not apply a tangential friction force at his feet? Then the Coriolis force of $2m\omega v$ produces a tangential acceleration of $2\omega v$ in his frame (and hence the lab frame, too). In this case, this acceleration exists essentially to keep the angular momentum (measured with respect to the lab frame) of the person constant. (It *is* constant in this scenario, since there are no tangential forces.) To see that this tangential acceleration is consistent with conservation of angular momentum, set $dL/dt = 0$ in eq. (9.14) to obtain $2\omega v = r(d\omega/dt)$. The right-hand side of this is by definition the tangential acceleration. Therefore, saying that L is conserved is the same as saying that $2\omega v$ is the tangential acceleration (for this situation where the inward radial speed is v). ♣

Case 2 (Moving tangentially on a carousel): Consider someone walking tangentially on a carousel (in the direction of the carousel's motion), with speed v at radius r (see Fig. 9.6). The force $-2m\boldsymbol{\omega} \times \mathbf{v}$ points radially outward with magnitude $2m\omega v$. Assume the person applies the friction forces necessary to continue moving at radius r.

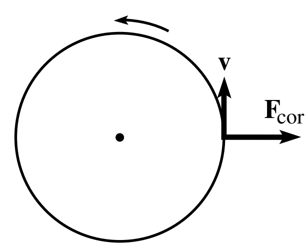


Figure 9.6

There is a simple way to see why this force of $2m\omega v$ exists. Let $V \equiv \omega r$ (that is, V is the speed of a point on the carousel at radius r, as viewed by an outside observer). If the person moves tangentially (in the same direction as the spinning) with speed v, then his speed as viewed by an outside observer is $V + v$. So the outside observer sees a centripetal acceleration of $(V + v)^2/r$. If the person is moving at constant r, then the outside observer knows that this acceleration must be accounted for by the inward-pointing friction force at the person's feet, so

$$F_{\text{friction}} = \frac{m(V+v)^2}{r} = \frac{mV^2}{r} + \frac{2mVv}{r} + \frac{mv^2}{r}. \tag{9.15}$$

This friction force is of course the same in any frame. How, then, does our person on the carousel interpret the three pieces of the inward-pointing friction force in eq. (9.15)? The first simply balances the outward centrifugal force due to the rotation of the frame, which he always feels. The third is simply the inward force his feet must apply if he is to walk in a circle of radius r at speed v, which is exactly what he thinks

[1]There is also the centrifugal force, which is countered by a radial friction force at the person's feet. This effect won't be important here.

he is doing. The middle term is the additional inward friction force he must apply, which balances the outward Coriolis force of $2m\omega v$ (using $V \equiv \omega r$).

Said in an equivalent way, the person on the carousel will write down an $F = ma$ equation of the form (taking radially inward to be positive),

$$\begin{aligned} m\frac{v^2}{r} &= m\frac{(V+v)^2}{r} - m\frac{V^2}{r} - 2m\frac{Vv}{r}\,, \qquad \text{or} \\ ma &= F_{\text{friction}} - F_{\text{cent}} - F_{\text{Cor}}. \end{aligned} \tag{9.16}$$

So the net force he feels is indeed equal to his ma (where a is measured with respect to his rotating frame).

For cases in between the two special ones above, things aren't so clear, but that's the way it goes. Note that no matter what direction you move on a carousel, the Coriolis force always points in the same direction relative to your motion. Whether it's to the right or to the left depends on the direction of the rotation. But given $\boldsymbol{\omega}$, you're stuck with the same relative direction of force.

On a merry-go-round in the night,
Coriolis was shaken with fright.
Despite how he walked,
'Twas like he was stalked
By some fiend always pushing him right.

Example 1 (Dropped ball): A ball is dropped from a height h, at a polar angle θ (the angle down from the north pole). How far to the east is the ball deflected, by the time it hits the ground?

Solution: Note that the ball is indeed deflected to the east, independent of which hemisphere it is in. The Coriolis force, $-2m\omega \times \mathbf{v}$, is directed eastward and has a magnitude $2m\omega v \sin\theta$, where $v = gt$ is the speed at time t (t runs from 0 to the usual $\sqrt{2h/g}$).[2] So the eastward acceleration at time t equals $2\omega g t \sin\theta$. Integrating this to get the eastward speed (with an initial eastward speed of 0) gives $v_{\text{east}} = \omega g t^2 \sin\theta$. Integrating once more to get the eastward deflection (with an initial eastward deflection of 0) gives $d_{\text{east}} = \omega g t^3 \sin\theta/3$. Plugging in $t = \sqrt{2h/g}$ gives

$$d_{\text{east}} = h\left(\frac{2\sqrt{2}}{3}\right)\left(\omega\sqrt{\frac{h}{g}}\right)\sin\theta. \tag{9.17}$$

This is valid up to second-order effects in the small quantity $\omega \approx 7.3 \cdot 10^{-5}\,\text{s}^{-1}$ (or, more precisely, in the small dimensionless quantity $\omega\sqrt{h/g}$).

---

[2]Technically, $v = gt$ isn't quite correct. Due to the Coriolis force, the ball will pick up a small sideways component in its velocity (this is the point of the problem). We may, however, ignore this component in the calculation of the Coriolis force; the error is a small second-order effect.

Example 2 (Foucault's pendulum): This is the classic example of a consequence of the Coriolis force. It unequivocally shows that the earth rotates. The basic point is that due to the rotation of the earth, the plane of a swinging pendulum rotates slowly, with a calculable frequency.

In the special case where the pendulum is at one of the poles, this rotation is easy to understand. Consider the north pole. An external observer, hovering above the north pole and watching the earth rotate, sees the pendulum's plane stay fixed (with respect to the distant stars) while the earth rotates counterclockwise beneath it.[3] Therefore, to an observer on the earth, the pendulum's plane rotates clockwise (viewed from above). The frequency of this rotation is of course just the frequency of the earth's rotation. The earth-based observer sees the pendulum's plane make one revolution each day.

What if the pendulum is not at one of the poles? What is the frequency of the precession? Let the pendulum be located at the polar angle θ on the earth. We will work in the approximation where the velocity of the pendulum bob is horizontal. This is essentially true if the pendulum's string is very long; the correction due to the rising and falling of the bob is negligible. The Coriolis force, $-2m\boldsymbol{\omega} \times \mathbf{v}$, points in some complicated direction, but fortunately we are concerned only with the component that lies in the horizontal plane. The vertical component serves only to modify the apparent force of gravity and is therefore negligible. (Although the frequency of the pendulum depends on g, the resulting modification is very small.)

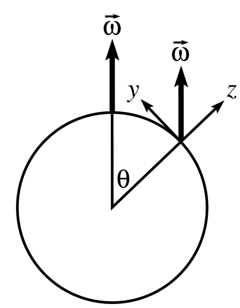


Figure 9.7

With this in mind, let's break $\boldsymbol{\omega}$ into vertical and horizontal components in a coordinate system located at the pendulum. From Fig. 9.7, we see that

$$\boldsymbol{\omega} = \omega \cos\theta \hat{\mathbf{z}} + \omega \sin\theta \hat{\mathbf{y}}. \tag{9.18}$$

We'll ignore the y-component, since it produces a Coriolis force in the $\hat{\mathbf{z}}$ direction (because $\mathbf{v}$ lies in the horizontal x-y plane). So for our purposes, $\boldsymbol{\omega}$ is essentially equal to $\omega \cos\theta \hat{\mathbf{z}}$.

From this point on, the problem of finding the frequency of precession can be done in numerous ways. We'll present two solutions.

First solution (The slick way): The horizontal component of the Coriolis force has magnitude

$$F_{\text{Cor}} = |-2m(\omega \cos\theta \hat{\mathbf{z}}) \times \mathbf{v}| = 2m(\omega \cos\theta)v, \tag{9.19}$$

and is perpendicular to $\mathbf{v}(t)$. Therefore, as far as the pendulum is concerned, it is located at the north pole of a planet called Terra Costhetica which has rotational frequency $\omega \cos\theta$. But as we saw above, the precessional frequency of a Foucault pendulum located at the north pole of such a planet is simply

$$\omega_F = \omega \cos\theta, \tag{9.20}$$

in the clockwise direction. So that's our answer. (As mentioned above, the situation isn't *exactly* like that on the new planet; there will be a vertical component of the Coriolis force for the pendulum on the earth, but this effect is negligible.)

Second solution (In the pendulum's frame): Let's work in the frame of the vertical plane that the Foucault pendulum sweeps through. The goal is to find the

[3]Assume that the pivot of the pendulum is a frictionless bearing, so that it can't provide any torque to twist the pendulum's plane.

rate of precession of this frame. With respect to a frame fixed on the earth (with axes $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$), we know that this plane rotates with frequency $\boldsymbol{\omega}_F = -\omega\hat{\mathbf{z}}$ if we're at the north pole ($\theta = 0$), and frequency $\boldsymbol{\omega}_F = 0$ if we're at the equator ($\theta = \pi/2$). So if there's any justice in the world, the general answer has got to be $\boldsymbol{\omega}_F = -\omega\cos\theta\hat{\mathbf{z}}$, and that's what we'll now show.

Working in the frame of the plane of the pendulum is useful, because we may take advantage of the fact that *the pendulum feels no sideways forces in this frame* (because otherwise it would move outside of the plane).

The frame fixed on the earth rotates with frequency $\boldsymbol{\omega} = \omega\cos\theta\hat{\mathbf{z}} + \omega\sin\theta\hat{\mathbf{y}}$, with respect to the inertial frame. Let the pendulum rotate with frequency $\boldsymbol{\omega}_F = \omega_F\hat{\mathbf{z}}$ with respect to the earth frame. Then the angular velocity of the pendulum's frame with respect to the inertial frame is

$$\boldsymbol{\omega} + \boldsymbol{\omega}_F = (\omega\cos\theta + \omega_F)\hat{\mathbf{z}} + \omega\sin\theta\hat{\mathbf{y}}. \tag{9.21}$$

To find the horizontal component of the Coriolis force in this rotating frame, we only care about the $\hat{\mathbf{z}}$ part of this frequency. The horizontal force therefore has magnitude $2m(\omega\cos\theta + \omega_F)v$. But in the frame of the pendulum, there is no horizontal force, so this must be zero. Therefore,

$$\omega_F = -\omega\cos\theta. \tag{9.22}$$

This agrees with eq. (9.20), where we just wrote down the magnitude of ω_F.

9.2.4 Azimuthal force: $-m(d\boldsymbol{\omega}/dt) \times \mathbf{r}$

In this section, we will restrict ourselves to the simple and intuitive case where $\boldsymbol{\omega}$ changes only in magnitude (that is, not in direction).[4] In this case, the azimuthal force may be written as

$$\mathbf{F}_{\text{az}} = -m\dot{\omega}\hat{\boldsymbol{\omega}} \times \mathbf{r}. \tag{9.23}$$

This force is easily understood by considering a person standing motionless on a rotating carousel. If the carousel speeds up, then a force must be applied to the person, if he is to remain fixed on the carousel. Therefore, he feels a friction force at his feet. But from his point of view, he is not moving, so there must be some other mysterious force which balances this friction. This is the azimuthal force. Quantitatively, when $\hat{\boldsymbol{\omega}}$ is orthogonal to $\mathbf{r}$, we have $|\hat{\boldsymbol{\omega}} \times \mathbf{r}| = r$, so the azimuthal force in eq. (9.23) has magnitude $mr\dot{\omega}$. But $r\dot{\omega} = a$. So the azimuthal force (and hence also the friction force) has magnitude ma, as expected.

What we have here is exactly the same effect as we had with the translation force on the accelerating train; if the floor speeds up beneath you, then you must apply a friction force if you don't want to be thrown backwards (with respect to the floor).

We can also view things in terms of rotational quantities, as opposed to the linear ones above. If the carousel speeds up, then a torque must be applied to the person, if he is to remain fixed on the carousel (because his angular momentum in the fixed frame increases). Therefore, he feels a friction force at his feet.

[4] The more complicated case where $\boldsymbol{\omega}$ changes direction is left for Problem 8.

Quantitatively, it's easy to see that this friction force (which exists to cancel the azimuthal force, in the rotating frame) exactly accounts for the change in the angular momentum of the person (in the fixed frame). Since $L = mr^2\omega$, we have $dL/dt = mr^2\dot{\omega}$ (assuming r is fixed). And since $dL/dt = \tau = rF$, we see that the required friction force is $F = mr\dot{\omega}$. This force must equal the azimuthal force, if the person is to remain motionless in the rotating frame. And indeed, when $\hat{\boldsymbol{\omega}}$ is orthogonal to $\mathbf{r}$, the azimuthal force in eq. (9.23) equals $mr\dot{\omega}$ (in the direction opposite to the carousel's motion).

Example (Spinning ice skater): We have all seen ice skaters increase their angular speed by bringing their arms in close to their body. This is easily understood in terms of angular momentum (since a smaller moment of inertia requires a larger ω, to keep L constant). But let us analyze the situation here in terms of fictitious forces. We will idealize things by giving the skater massive hands at the end of massless arms attached to a massless body.[5] Let the hands have total mass m, and let them be drawn in radially.

Let's look at things in the skater's frame (which has an increasing ω), defined as the vertical frame containing the hands. The crucial thing to realize is that the skater remains in the skater's frame (a fine tautology, indeed). Therefore, the skater can feel no net tangential force in her frame (because otherwise she would accelerate with respect to it). The hands are being drawn in by a muscular force that works against the centrifugal force, but there is no net tangential force on the hands in the skater's frame.

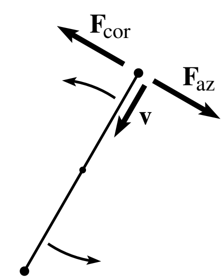


Figure 9.8

What are the tangential forces in the skater's frame? (See Fig. 9.8.) Let the hands be drawn in at speed v. Then there is a Coriolis force (in the same direction as the spinning) with magnitude $2m\omega v$. There is also an azimuthal force with magnitude $mr\dot{\omega}$ (in the direction opposite the spinning, as you can check). Since the net tangential force is zero in the skater's frame, we must have

$$2m\omega v = mr\dot{\omega}. \tag{9.24}$$

Does this relation make sense? Well, the total angular momentum of the hands is constant. Therefore, $d(mr^2\omega)/dt = 0$. Taking this derivative and using $dr/dt \equiv -v$ (we defined v to be positive) gives eq. (9.24).

A word of advice about using fictitious forces: Decide which frame you are going to work in (the lab frame or the accelerated frame), and then stick with it. The common mistake is to work a little in one frame and a little on the other, without realizing it. For example, you might introduce a centrifugal force on someone sitting at rest on a carousel, but then also give him a centripetal acceleration. This is incorrect. In the lab frame, there is a centripetal acceleration (caused by the friction force) and no centrifugal force. In the rotating frame, there is a centrifugal force (which cancels the friction force) and no centripetal acceleration (since the person is sitting at rest on the carousel).

[5] This reminds me of a joke about a spherical cow . . .

9.3 Exercises

Section 9.2: The fictitious forces

1. **Gravity, to ω^2** $**$

   To second order in ω, what is the downward acceleration of a mass dropped at the equator? (Careful, there's a second-order Coriolis effect, in addition to the centrifugal term.)

2. **Southern deflection** $**$

   A ball is dropped from a height h (small compared to the radius of the earth), at a polar angle θ. How far to the *south* (in the northern hemisphere) is it deflected away from the $\mathbf{g}_{\text{eff}}$ direction, by the time it hits the ground? (This is a second order Coriolis effect.)

3. **Oscillations across equator** $*$

   A bead lies on a frictionless wire which lies in the north-south direction across the equator. The wire takes the form of an arc of a circle; all points are the same distance from the center of the earth. The bead is released from rest a short distance from the equator. Because $\mathbf{g}_{\text{eff}}$ does not point directly toward the earth's center, the bead will head toward the equator and then undergo oscillatory motion. What is the frequency of these oscillations?

4. **Spinning bucket** $**$

   A bucket containing water is spun at frequency ω. If the water is at rest with respect to the bucket, find the shape of the water's surface.

5. **Coin on turntable** $***$

   A coin stands upright at an arbitrary point on a rotating turntable, and rotates (without slipping) with the required speed to make its center remain motionless in the lab frame. In the frame of the turntable, the coin will roll around in a circle with the same frequency as that of the turntable. In the frame of the turntable, show that

   (a) $\mathbf{F} = d\mathbf{p}/dt$, and

   (b) $\boldsymbol{\tau} = d\mathbf{L}/dt$.

6. **Precession viewed from rotating frame** $***$

   Consider a top made of a wheel with all its mass on the rim. A massless rod (perpendicular to the plane of the wheel) connects the CM to the pivot. Initial conditions have been set up so that the top undergoes precession, with the rod always horizontal.

   In the language of the figure in Section 8.7.2 in Chapter 8, we may write the angular velocity of the top as $\boldsymbol{\omega} = \Omega\hat{\mathbf{z}} + \omega'\hat{\mathbf{x}}_3$ (where $\hat{\mathbf{x}}_3$ is horizontal here). Consider things in the frame rotating around the $\hat{\mathbf{z}}$-axis with angular speed Ω.

In this frame, the top spins with angular speed ω' around its *fixed* symmetry axis. Therefore, in this frame $\boldsymbol{\tau} = 0$, because there is no change in $\mathbf{L}$.

Verify explicitly that $\boldsymbol{\tau} = 0$ (calculated with respect to the pivot) in this rotating frame (you will need to find the relation between ω' and Ω). In other words, show that the torque due to gravity is exactly canceled by the torque do to the Coriolis force. (You can easily show that the centrifugal force provides no net torque.)

9.4 Problems

Section 9.2: The fictitious forces

1. **$\mathbf{g}_{\text{eff}}$ vs. g** $*$

   For what θ is the angle between $m\mathbf{g}_{\text{eff}}$ and $\mathbf{g}$ maximum?

2. **Longjumping in $\mathbf{g}_{\text{eff}}$** $*$

   If a longjumper can jump 8 meters at the north pole, how far can he jump at the equator?

   (Ignore effects of wind resistance, temperature, and runways made of ice. And assume that the jump is made in the north-south direction at the equator, so that there is no Coriolis force.)

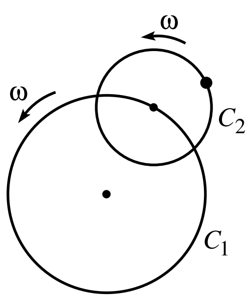


Figure 9.9

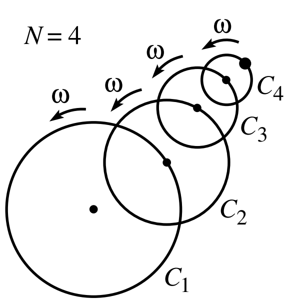


Figure 9.10

3. **Lots of circles** $**$

   (a) Two circles in a plane, C_1 and C_2, each rotate with frequency ω (relative to an inertia frame). (See Fig. 9.9.) The center of C_1 is fixed in an inertial frame. The center of C_2 is fixed on C_1. A mass is fixed on C_2. The position of the mass relative to the center of C_1 is $\mathbf{R}(t)$. Find the fictitious force felt by the mass.

   (b) N circles in a plane, C_i, each rotate with frequency ω (relative to an inertia frame). (See Fig. 9.10.) The center of C_1 is fixed in an inertial frame. The center of C_i is fixed on C_{i-1} (for $i = 2, \ldots, N$). A mass is fixed on C_N. The position of the mass relative to the center of C_1 is $\mathbf{R}(t)$. Find the fictitious force felt by the mass.

4. **Mass on turntable** $*$

   A mass rests motionless with respect to the lab frame, while a frictionless turntable rotates beneath it. The frequency of the turntable is ω, and the mass is located at radius r. In the frame of the turntable, what are the forces on the mass?

5. **Released mass** $*$

   A mass is bolted down to a frictionless turntable. The frequency of rotation is ω, and the mass is located at a radius a. The mass is released. Viewed from an inertial frame, it travels in a straight line. In the rotating frame, what path does the mass take?

6. **Coriolis circles** $*$

   A puck slides with speed v on frictionless ice. The surface is "level", in the sense that it is orthogonal to $\mathbf{g}_{\text{eff}}$ at all points. Show that the puck moves in a circle (as seen in the earth's rotating frame). What is the radius of the circle? What is the frequency of the motion? (You may assume that the radius of the circle is small compared to the radius of the earth.)

7. **Shape of the earth** ∗∗∗

The earth bulges slightly at the equator, due to the centrifugal force in the earth's rotating frame. Show that the height of a point on the earth (relative to a spherical earth), is given by

$$h = R\left(\frac{R\omega^2}{6g}\right)(3\sin^2\theta - 2), \tag{9.25}$$

where θ is the polar angle (the angle down from the north pole), and R is the radius of the earth.

8. **Changing $\boldsymbol{\omega}$'s direction** ∗∗∗

Consider the special case where a reference frame's $\boldsymbol{\omega}$ changes only in direction (and not in magnitude). In particular, consider a cone rolling on a table, which is the classic example of such a situation.

The instantaneous $\boldsymbol{\omega}$ for a rolling cone is its line of contact with the table. This line precesses around the origin. Let the frequency of this precession be $\boldsymbol{\Omega}$. The origin of our rotating cone will be the tip of the cone. This point remains fixed in the inertial frame.

In order to isolate the azimuthal force, consider the special case of a point P on the cone which lies on the instantaneous $\boldsymbol{\omega}$ and which is motionless with respect to the cone (see Fig. 9.11). From eq. (9.11), we then see that the centrifugal, Coriolis, and translation forces are zero. The only remaining fictitious force is the azimuthal force, and it arises from the fact that P is accelerating up away from the table.

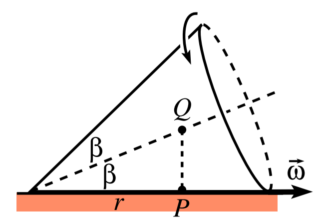


Figure 9.11

(a) Find the acceleration of P.

(b) Calculate the azimuthal force on a mass located at P, and show that the result is consistent with part (a).

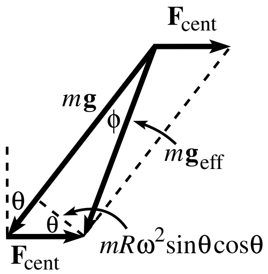


Figure 9.12

9.5 Solutions

1. **$\mathbf{g}_{\text{eff}}$ vs. g**
The forces $m\mathbf{g}$ and $\mathbf{F}_{\text{cent}}$ are shown in Fig. 9.12. The magnitude of $\mathbf{F}_{\text{cent}}$ is $mR\omega^2\sin\theta$, so the component of $\mathbf{F}_{\text{cent}}$ orthogonal to $m\mathbf{g}$ is $mR\omega^2\sin\theta\cos\theta = mR\omega^2(\sin 2\theta)/2$. For small $\mathbf{F}_{\text{cent}}$, maximizing the angle between $\mathbf{g}_{\text{eff}}$ and $\mathbf{g}$ is equivalent to maximizing this orthogonal component. Therefore, we obtain the maximum angle when

$$\theta = \frac{\pi}{4}. \tag{9.26}$$

The maximum angle achieved is $\phi \approx \sin\phi \approx \Big(mR\omega^2(\sin\frac{\pi}{2})/2\Big)/mg = R\omega^2/2g \approx 0.0017$. This is about $0.1°$. The line along $\mathbf{g}_{\text{eff}}$ misses the center of the earth by about 10 km.

2. **Longjumping in $\mathbf{g}_{\text{eff}}$**
Let the jumper take off with speed v, at an inclination θ. Then $d = v_x t = vt\cos\theta$, and $g_{\text{eff}}(t/2) = v_y = v\sin\theta$, where t is the time in the air and d is the distance traveled. Eliminating t gives $d = (v^2/g_{\text{eff}})\sin 2\theta$. (This is maximum when $\theta = \pi/4$, as we well know.) So we see that $d \propto 1/\sqrt{g_{\text{eff}}}$. Taking $g_{\text{eff}} \approx 10$ m/s^2 at the north pole, and $g_{\text{eff}} \approx (10 - 0.03)$ m/s^2 at the equator, we find that the jump at the equator is approximately 1.0015 times as long as the one on the north pole. So the longjumper gains about one centimeter.

Remark: For a longjumper, the optimal angle of takeoff is undoubtedly not equal to $\pi/4$. To change his direction abruptly from horizontal to such an inclination would entail a significant loss in speed. The best angle is some hard-to-determine angle less than $\pi/4$. But this won't change our general $d \propto 1/\sqrt{g_{\text{eff}}}$ result, so our answer still holds. ♣

3. **Lots of circles**

(a) The fictitious force, $\mathbf{F}_{\text{f}}$, on the mass has an $\mathbf{F}_{\text{cent}}$ part and an $\mathbf{F}_{\text{trans}}$ part, since the center of C_2 is moving. So the fictitious force is

$$\mathbf{F}_{\text{f}} = m\omega^2\mathbf{r}_2 + \mathbf{F}_{\text{trans}}, \tag{9.27}$$

where $\mathbf{r}_2$ is the position of the mass in the frame of C_2.
But $\mathbf{F}_{\text{trans}}$ is simply the centrifugal force felt by a point on C_1. Therefore,

$$\mathbf{F}_{\text{trans}} = m\omega^2\mathbf{r}_1, \tag{9.28}$$

where $\mathbf{r}_1$ is the position of the center of C_2, in the frame of C_1. Substituting this into eq. (9.27) gives

$$\begin{aligned}\mathbf{F}_{\text{f}} &= m\omega^2\mathbf{r}_2 + m\omega^2\mathbf{r}_1 \\ &= m\omega^2\mathbf{R}(t).\end{aligned} \tag{9.29}$$

(b) The fictitious force, $\mathbf{F}_{\text{f}}$, on the mass has an $\mathbf{F}_{\text{cent}}$ part and an $\mathbf{F}_{\text{trans}}$ part, since the center of the Nth circle is moving. So the fictitious force is

$$\mathbf{F}_{\text{f}} = m\omega^2\mathbf{r}_N + \mathbf{F}_{\text{trans},N}. \tag{9.30}$$

But $\mathbf{F}_{\text{trans},N}$ is simply the centrifugal force felt by a point on the $(N-1)$st circle, plus the translation force coming from the movement of the center of the $(N-1)$st circle. Therefore,

$$\mathbf{F}_{\text{trans},N} = m\omega^2\mathbf{r}_{N-1} + \mathbf{F}_{\text{trans},N-1}. \tag{9.31}$$

Substituting this into eq. (9.30) and successively rewriting the $\mathbf{F}^{\text{eff}}_{\text{trans}}$ terms in a similar manner, gives

$$\begin{aligned} \mathbf{F}_{\text{f}} &= m\omega^2\mathbf{r}_N + m\omega^2\mathbf{r}_{N-1} + \cdots + m\omega^2\mathbf{r}_1 \\ &= m\omega^2\mathbf{R}(t). \end{aligned} \tag{9.32}$$

The whole point here is that $\mathbf{F}_{\text{cent}}$ is linear in $\mathbf{r}$.

4. **Mass on turntable**

In the lab frame, the force on the mass is zero, of course, because it is sitting still. But in the rotating frame, the mass thinks it is traveling in a circle of radius r, with frequency ω. So it knows that in its frame there must be a force of $m\omega^2 r$ inward to account for the centripetal acceleration. And indeed, it feels a centrifugal force of $m\omega^2 r$ outward, and a Coriolis force of $2m\omega v = 2m\omega^2 r$ inward, which sum to the desired force (see Fig. 9.13).

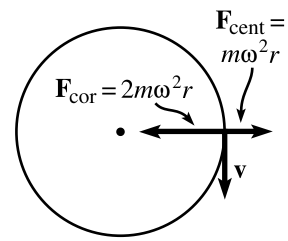


Figure 9.13

Remark: This inward force in this problem is a little different from that for someone swinging around in a circle in an inertial frame. If a skater maintains a circular path by holding onto a rope whose other end is fixed, she has to use her muscles to maintain the position of her torso with respect to her arm, and her head with respect to her torso, etc. But if a person takes the place of the mass in this problem, she needs to exert no effort to keep her body from being pulled outward (as is obvious, when looked at from the inertial frame), because each atom in her body is moving at (essentially) the same speed and therefore feels the same Coriolis force. So she doesn't really *feel* this force, in the same sense that one doesn't feel gravity when in free-fall with no wind. ♣

5. **Released mass**

Let the x'- and y'-axes of the rotating frame coincide with the x- and y-axes of the inertial frame at the moment the mass is released (at $t = 0$). Then after a time t, the situation looks like that in Fig. 9.14. The speed of the mass is $v = a\omega$, so it has traveled a distance $a\omega t$. The angle that its position vector makes with the inertial x-axis is therefore $\tan^{-1}\omega t$ (with counterclockwise taken to be positive). Hence, the angle that its position vector makes with the rotating x-axis is $\theta(t) = -(wt - \tan^{-1}\omega t)$. And the radius is of course $r(t) = a\sqrt{1+\omega^2 t^2}$. So for large t, $r(t) \approx a\omega t$ and $\theta(t) \approx -wt + \pi/2$, which make sense.

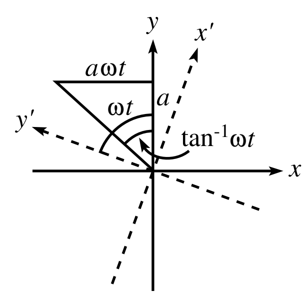


Figure 9.14

6. **Coriolis circles** *

By construction, the normal force from the ice will exactly cancel all effects of the gravitational and centrifugal forces. We therefore need only concern ourselves with the Coriolis force. This force equals $\mathbf{F}_{\text{cor}} = -2m\boldsymbol{\omega} \times \mathbf{v}$. Let the angle from the north pole be θ (we assume the circle is small enough so that θ is essentially constant throughout the motion). Then the component of the Coriolis force that points horizontally along the surface (the vertical component will simply modify the required normal force) has magnitude $f = 2m\omega v\cos\theta$, and it is perpendicular to the direction of motion. Such a force produces circular motion, with a radius given by

$$2m\omega v\cos\theta = \frac{mv^2}{r} \qquad \Longrightarrow \qquad r = \frac{v}{2\omega\cos\theta}\,. \tag{9.33}$$

The frequency of the circular motion is

$$\omega' = \frac{v}{r} = 2\omega\cos\theta. \tag{9.34}$$

REMARKS: To get a rough idea of the size of the circle, you can show (using $\omega \approx 7.3\cdot 10^{-5}\,\text{s}^{-1}$) that $r \approx 10\,\text{km}$ when $v = 1\,\text{m/s}$ and $\theta = 45°$. Even the tiniest bit of friction will clearly make this effect essentially impossible to see.

For the $\theta \approx \pi/2$ (that is, near the equator), the component of the Coriolis force along the surface is negligible, so r becomes large, and ω' goes to 0.

For the $\theta \approx 0$ (that is, near the north pole), the Coriolis force essentially points along the surface. The above equations give $r \approx v/(2\omega)$, and $\omega' \approx 2\omega$. For the special case where the center of the circle is the north pole, this $\omega' \approx 2\omega$ result might seem incorrect, because you might want to say that this setup is achieved by having the puck remain motionless in the inertial fame, while the earth rotates beneath it (thus making $\omega' = \omega$). The error in this reasoning is that the "level" earth is not spherical, due to the non-radial direction of $\mathbf{g}_{\text{eff}}$. If the puck is motionless in the inertial frame, then it will be drawn toward the north pole, due to the component of gravity in that direction. In order to not fall toward the pole, The puck needs to travel with frequency ω (relative to the inertial frame) in the direction opposite to the earth's rotation. The puck therefore moves at frequency 2ω relative to the frame of the earth. ♣

7. **Shape of the earth**

In the reference frame of the earth, the forces on an atom at the surface are: earth's gravity, the centrifugal force, and the normal force from the ground below it. These three forces must sum to zero. Therefore, the sum of the gravity plus centrifugal forces must be normal to the surface. Said differently, the gravity-plus-centrifugal force must have no component along the surface. Said in yet another way, the potential energy function derived from the gravity-plus-centrifugal force must be constant along the surface. (Otherwise, a piece of the earth would want to move along the surface, which would mean we didn't have the correct surface to begin with.)

If x is the distance from the earth's axis, then the centrifugal force is $F_c = m\omega^2 x$, directed outward. The potential energy function for this force is $V_c = -m\omega^2 x^2/2$, up to an arbitrary additive constant. The potential energy for the earth's gravitation force is simply mgh. (We've arbitrarily chosen the original spherical surface have zero potential; any other choice would add on an irrelevant constant. Also, we've assumed that the slight distortion of the earth won't make the mgh result invalid. This is true to lowest order in h/R, which you can demonstrate if you wish.)

The equal-potential condition is therefore

$$mgh - \frac{m\omega^2 x^2}{2} = C, \tag{9.35}$$

where C is a constant to be determined. Using $x = r\sin\theta$, we obtain

$$h = \frac{\omega^2 r^2 \sin^2\theta}{2g} + B, \tag{9.36}$$

where $B \equiv C/(mg)$ is another constant. We may replace the r here with the radius of the earth, R, with negligible error.

Depending what the constant B is, this equation describes a whole family of surfaces. We may determine the correct value of B by demanding that the volume of the earth be the same as it would be in its spherical shape if the centrifugal force were turned off. This is equivalent to demanding that the integral of h over the surface of the earth is zero. The integral of $(a\sin^2\theta + b)$ over the surface of the earth is (the integral is easy if we write $\sin^2\theta$ as $1-\cos^2\theta$)

$$\begin{aligned}\int_0^\pi \big(a(1-\cos^2\theta)+b\big)\,2\pi R^2\sin\theta\,d\theta &= \int_0^\pi \big(-a\cos^2\theta+(a+b)\big)\,2\pi R^2\sin\theta\,d\theta \\ &= 2\pi R^2\left(\frac{a\cos^3\theta}{3}-(a+b)\cos\theta\right)\Big|_0^\pi \\ &= 2\pi R^2\left(-\frac{2a}{3}+2(a+b)\right). \qquad (9.37)\end{aligned}$$

Hence, we need $b=-(2/3)a$ for this integral to be zero. Plugging this result into eq. (9.36) gives

$$h = R\left(\frac{R\omega^2}{6g}\right)(3\sin^2\theta-2), \qquad (9.38)$$

as desired.

8. **Changing ω's direction**

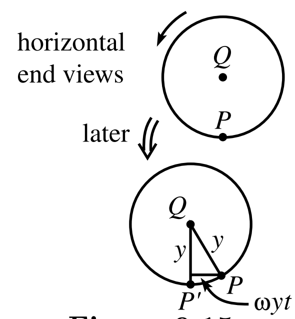


Figure 9.15

(a) Let Q be the point which lies on the axis of the cone and which is directly above P (see Fig. 9.15). If P is a distance r from the origin, and the half-angle of the cone is β, then Q is a distance $y=r\tan\beta$ above P.

Consider the situation an infinitesimal time, t, later. Let P' be the point now directly below Q (see Fig. 9.15). Since the angular speed of the cone is ω, Q moves horizontally at a speed $v_Q=\omega y=\omega r\tan\beta$. So in the infinitesimal time t, Q moves to the side a distance ωyt.

This is also (essentially) the horizontal distance between P and P'. Therefore, a little geometry tells us that P is now at a distance

$$h(t) = y-\sqrt{y^2-(\omega yt)^2}\approx\frac{(\omega t)^2 y}{2} \qquad (9.39)$$

above the table. Since P started on the table with zero speed, this means that P is undergoing an acceleration of $\omega^2 y$ in the vertical direction. A mass located at P must therefore feel a force $F_P=m\omega^2 y$ (friction, normal, or other) in the upward vertical direction, if it is to remain motionless with respect to the cone.

(b) The precession frequency Ω (that is, how fast $\boldsymbol{\omega}$ swings around the origin) is equal to the speed of Q, divided by r (because Q is always directly above $\boldsymbol{\omega}$, so it moves in a circle of radius r around the z-axis). Therefore, $\mathbf{\Omega}$ has magnitude $v_Q/r=\omega y/r$, and it points in the vertical direction. Hence, $d\boldsymbol{\omega}/dt=\mathbf{\Omega}\times\boldsymbol{\omega}$ has magnitude $\omega^2 y/r$, and it points in the horizontal direction. Therefore, $\mathbf{F}_{\mathrm{az}}=-m(d\boldsymbol{\omega}/dt)\times\mathbf{r}$ has magnitude $m\omega^2 y$, and it points in the downward vertical direction.

A person of mass m at point P therefore interprets the situation as, "I am not accelerating with respect to the cone. The net force on me is therefore zero. And indeed, the normal force F_P upward from the cone is exactly balanced by this mysterious F_{az} force downward."

Chapter 10

Relativity (Kinematics)

We now come to Einstein's theory of relativity. This is where we find out that everything we've done so far in this book has been wrong. Well, perhaps "incomplete" would be a better word. The point is that Newtonian physics is a limiting case of the more correct Relativistic theory. Newtonian physics works just fine when the speeds you are dealing with are much less than the speed of light (which is about $3 \cdot 10^8$m/s). It would be silly, to put it mildly, to use relativity to solve a problem involving length of a baseball trajectory. But in problems involving large speeds, you must use the relativistic theory. This will be the topic of the next four chapters.[1]

The theory of Relativity is certainly one of the most exciting and talked-about topics in physics. It is well-known for its "paradoxes", which are quite conducive to discussion. There is, however, nothing at all paradoxical about it. The theory is logically and experimentally sound, and the whole subject is actually quite straightforward, provided that you proceed calmly and keep a firm hold of your wits.

The theory rests upon certain postulates. The one that most people find counterintuitive is that the speed of light has the same value in any inertial (that is, non-accelerating) reference frame. This speed is much greater than the speed of everyday objects, so most of the consequences of this new theory are not noticeable. If we lived in a world similar to ours except for the fact that the speed of light was 100 mph, then the consequences of relativity would be ubiquitous, and we wouldn't think twice about time dilations, length contractions, and so on.

I have included a large number of puzzles and "paradoxes" in the text and in the problems. When attacking one of these, be sure to follow it through to completion, and do not say, "I could finish it if I wanted to, but all I'd have to do would be such-and-such, so I won't bother," because the essence of the paradox may very well be contained in the such-and-such, and you will have missed out on all the fun. Most of the paradoxes arise because different frames of reference *seem* to give

[1]At any rate, you shouldn't feel too bad about having spent so much time learning about a theory that is simply the limiting case of another one, because relativity is also the limiting case of yet another theory (quantum field theory). And likewise, you shouldn't feel too bad about spending so much time on relativity, because quantum field theory is also the limiting case of yet another theory (string theory). And likewise . . . well, you get the idea. It really *is* turtles all the way down.

different answers. So, in explaining the paradox, you not only have to give the correct reasoning; you also have to say what's wrong with incorrect reasoning.

There are two main topics in relativity. One is Special Relativity (which does not deal with gravity), and the other is General Relativity (which does). We will deal mostly with the former, but Chapter 13 contains some GR.

Special Relativity may be divided into two topics, *kinematics* and *dynamics*. Kinematics deals with lengths, times, speeds, etc. It is basically concerned with only the space and time coordinates, and not with masses, forces, energy, momentum, etc. Dynamics, on the other hand, does deal with these quantities.

This chapter will cover kinematics. Chapter 11 will cover dynamics. Most of the fun paradoxes fall into the kinematics part, so the present chapter will be the longer of the two. In Chapter 12, we will introduce the concept of 4-vectors, which ties a lot of the material in Chapters 10 and 11 together.

10.1 The postulates

Various approaches can be taken in deriving the consequences of the Special Relativity theory. Different approaches use different postulates. Some start with the invariance of the speed of light in any inertial frame. Others start with the invariance of the spacetime interval (discussed in Section 10.4 below). Others start with the invariance of the inner product of 4-momentum vectors (discussed in Chapter 12). Postulates in one approach are theorems in another. There is no "good" or "bad" route to take; they are all equally valid. However, some approaches are simpler and more intuitive (if there is such a thing as intuition in relativity) than others. I will choose to start with the speed-of-light postulate.

- *The speed of light has the same value in any inertial frame.*

I do not claim that this statement is obvious, or even believable. But I do claim that it's easy to understand what the statement says (even if you think it's too silly to be true). It says the following. Consider a train moving along the ground at constant velocity (that is, it is not accelerating — this is what an inertial frame is). Someone on the train shines a light from one point on the train to another. Let the speed of the light with respect to the train be c ($\approx 3 \cdot 10^8$m/s). Then the above postulate says that a person on the ground also sees the light move at speed c.

This is a rather bizarre statement. It does not hold for everyday objects. If a baseball is thrown on the train, then the speed of the baseball is not the same in different frames, of course. The observer on the ground must add the velocity of the train and the velocity of the ball (with respect to the train) to get the velocity of the ball with respect to the ground.[2]

The truth of our this postulate cannot be demonstrated from first principles. No statement with any physical content in physics (that is, one that isn't purely mathematical, such as, "two apples plus two apples gives four apples,") can be

[2] Actually, this isn't quite true, as Section 10.3.3 on velocity addition shows. But it's true enough for the point we are making here.

proven. In the end, we must rely on experiment. And all the consequences of the above postulate have been verified countless times during this century. Indeed, the consequences are being verified continually each day in high-energy particle accelerators, where elementary particles reach speeds very close to c.

The most well-known of the early experiments was the one performed by Michelson and Morley, who tried to measure the effect of the earth's motion on the speed of light. If light moves at speed c with respect to only one special frame, the frame of the 'ether' (analogous to the way sound travels through air), then the speed of light on the earth should be faster or slower than c, depending on which way the earth is moving through the ether. In particular, if the speed of light is measured at one time, and then measured again six months later, then the results should be different, due to the earth's motion around the sun. Michelson and Morley were not able to measure any such differences in the speed of light. Nor has anyone else been able to do so.

> The findings of Michelson–Morley
> Allow us to say very surely.
> If this ether is real,
> Then it has no appeal,
> And shows itself off rather poorly.

The collection of all the data from various experiments over many years allows us to conclude with reasonable certainty that our starting assumption is correct (or is at least the limiting case of a more accurate theory).

There is one more postulate in the Special Relativity theory. It is much more believable than the one above, so you may take it for granted and forget to consider it. But like any postulate, of course, it is crucial.

- *All inertial frames are 'equivalent'.*

This basically says that one inertial frame is no better than any another. There is no preferred reference frame. That is, it makes no sense to say that something is moving; it only makes sense to say that one thing is moving with respect to another. It also says that if the laws of physics hold in one inertial frame (and presumably they hold in the one in which I now sit),[3] then they hold in all others. It also says that if we have two frames S and S', then S should see things in S' in exactly the same way that S' sees things in S (because we could simply switch the labels of S and S'). It also says that empty space is homogeneous, that is, that all points look the same (because we could pick any point to be, say, the origin of a coordinate system). It also says that empty space is isotropic, that is, that all directions look the same (because we could pick any axis to be, say, the x-axis of a coordinate system).

[3]Technically, the earth is spinning while revolving around the sun; and there are also little vibrations in the floor beneath my chair, etc., so I'm not *really* in an inertial frame. But it's close enough for me.

Unlike the first claim, this one is entirely reasonable. We've gotten used to having no special places in the universe. We gave up having the earth as the center, so let's not give any other point a chance, either.

> Copernicus gave his reply
> To those who had pledged to deny.
> "All your addictions
> To ancient convictions
> Won't bring back your place in the sky."

Everything we've said here about our second postulate refers to empty space. If we have a chunk of mass, then of course there is a difference between the position of the mass and a point a meter away. To incorporate mass into the theory, we would have to delve into the General Relativity theory. But we won't have anything to say about that in this chapter. We will deal only with empty space, containing perhaps a few observant souls sailing along in rockets or floating aimlessly on little spheres. Though it may sound boring at first, it will turn out to be more exciting than you'd think.

10.2 The fundamental effects

The most striking effects of the above postulates are: (1) the loss of simultaneity, (2) length contraction, and (3) time dilation. In this section we will discuss these three effects using some time-honored concrete examples (which are always nice to fall back on). In the following section we will derive the Lorentz transformations using these three results.

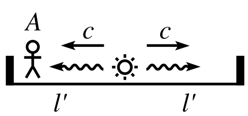


Figure 10.1

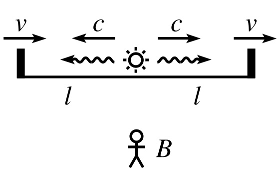


Figure 10.2

10.2.1 Loss of Simultaneity

Consider the following setup. In A's reference frame, a light source is placed midway between two receivers, ℓ' meters from each (see Fig. 10.1). The light source emits a flash. From A's point of view, the two receivers receive the light at the same time (ℓ'/c seconds after the flash). Another observer, B, is traveling by to the left at speed v. From her point of view, does the light hit the receivers at the same time? We will show that it does not.

In B's frame, the situation looks like that in Fig. 10.2. The receivers (and the whole train) are moving to the right at speed v, and the light is traveling in both directions at speed c (with respect to B, and *not* with respect to the light source, in B's frame; this is the whole point). So the relative speed of the light and the left receiver is $c + v$, and the relative speed of the light and the right receiver is $c - v$ (as viewed by B).[4]

[4] Yes, it is legal to simply add and subtract these speeds to obtain the relative speeds *as viewed by B*. This means that it is perfectly legal for the result to any number up to $2c$. Both the v and c here are measured with respect to the *same* person, namely B, so our intuition works just fine. We don't need to use the "velocity-addition formula", which we will derive in Section 10.3.3, and which

Let ℓ be the distance from the source to the receivers, as measured by B. (We will see in the next subsection that ℓ is not equal to ℓ', but this won't be important here.) Then the light hits the left receiver at t_l and the right receiver at t_r, where

$$t_l = \frac{\ell}{c+v}, \qquad t_r = \frac{\ell}{c-v}. \tag{10.1}$$

These are not equal[5] if $v \neq 0$.

The moral of this exercise is that it makes no sense whatsoever to say that one event happens at the same time as another (unless they take place at the same location). Simultaneity depends on the frame in which the observations are made.

Of the many effects, miscellaneous,
The loss of events, simultaneous,
Allows A to claim
There's no pause in B's frame,

REMARKS:

1. The invariance of the speed of light was used in saying that the two relative speeds above were $c + v$ and $c - v$. If we were talking about baseballs instead of light beams, then the relative speeds wouldn't look like this. If v_b is the speed at which the baseballs are thrown in A's (the train's) frame, then B sees the balls move at speeds $v_b - v$ to the left and $v_b + v$ to the right.[6] (These are of course not equal to v_b, as they would be for light.) The relative speeds between the balls and the left and right receivers are therefore $(v_b - v) + v = v_b$ and $(v_b + v) - v = v_b$. These are of course equal, and B sees the balls hit the receivers at the same time, as we know very well from everyday experience.

2. Yes, it is legal in eq. (10.1) to obtain the times simply by dividing ℓ by the relative speed, $c + v$ or $c - v$. But if you want a more formal method, then consider the following. The position of the right photon is given by ct, and the position of the right receiver (which had a head start of ℓ) is given by $\ell + vt$. Equating these two positions gives the desired result. Likewise for the left photon.

3. There is always a difference between the time an event happens and the time someone *sees* the event happen, because light takes time to travel from the event to the observer. What we calculated above was the time at which the events really happen. We could, of course, calculate the times at which B *sees* the events occur, but such times are rarely important, and in general we will not be concerned with them. They can simply be calculated by adding on a (distance)/c time difference, relevant to the path of the photons. Of course, if B did the above experiment to find t_r and t_l, she would do it by writing down the times at which she saw the events occur, and then subtracting off the relevant (distance)/c time differences to find when the events really happened.

is relevant in a different setup. I include this footnote just in case you've seen the velocity-addition formula and think it is relevant here. But if it didn't occur to you, then never mind.

[5]The one exception is when $\ell = 0$, in which case the two events happen at the same place and same time in all frames.

[6]The velocity-addition formula in Section 10.3.3 shows that these formulas aren't actually correct. But they're close enough for our purposes here.

To sum up, the $t_r \neq t_l$ result in eq. (10.1) is due to the fact that the events truly occur at different times. *It has nothing to do with the time it takes light to travel to your eye.* In this chapter, we will sometimes use language of the sort, "What time does Bob see event Q happen?" But we don't really mean, "When do Bob's eyes register that Q happened?" We mean, "What time does Bob *know* that event Q happened?" If we ever want to use "see" in the former sense, we will explicitly say so (as in Section 10.6 on the Doppler effect). ♣

Where this last line is not so extraneous.

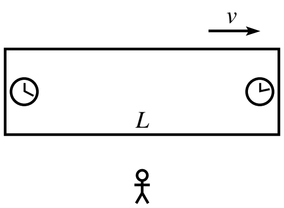


Figure 10.3

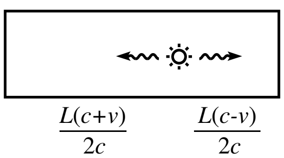


Figure 10.4

Example (The head start): Two clocks are positioned at the ends of a train of length L (as measured in its own frame). They are synchronized in the train frame. If the train travels past you at speed v, it turns out that you will observe the rear clock showing a higher reading than the front clock (see Fig. 10.3). By how much?

Solution: As above, let's put a light source on the train, but now let's position it so that the light hits the clocks at the ends of the train at the same time *in your frame.* As above, the relative speeds of the photons and the clocks are $c+v$ and $c-v$ (as viewed in your frame). We therefore need to divide the train into lengths in this ratio. The desired lengths are easily seen to be $L(c+v)/2c$ and $L(c-v)/2c$.

Consider now the situation in the frame of the train (see Fig. 10.4). The light must travel an extra distance of $L(c+v)/2c - L(c-v)/2c = Lv/c$ to reach the rear clock. The extra time is therefore Lv/c^2. Hence, the rear clock reads Lv/c^2 more when it is hit by the backward photon, compared to what the front clock reads when it is hit by the forward photon.

Now, let the instant you look at the clocks be the instant the photons hit them (that's why we chose the hittings to be simultaneous in your frame). Then you observe the rear clock reading more than the front clock by an amount,

$$\text{(difference in readings)} = \frac{Lv}{c^2}. \tag{10.2}$$

Remarks: The fact that the rear clock is *ahead* of the front clock in your frame means that the light hits the rear clock *after* it hits the front clock in the train frame.

Note that our result does *not* say that you see the rear clock ticking at a faster rate than the front clock. They run at the same rate. (Both have the same time-dilation factor relative to you; see Section 10.2.2.) The back clock is simply a fixed time ahead of the front clock, as seen by you. ♣

10.2.2 Time dilation

We present here the classic example of a light beam traveling vertically on a train.

Let there be a light source on the floor of the train and a mirror on the ceiling, which is at a height h above the floor. Let observer A be on the train, and observer B be on the ground. The speed of the train with respect to the ground[7] is v. A

[7] Technically, the words "with respect to . . ." should *always* be included when talking about speeds, since there is no absolute reference frame, and hence no absolute speed. But in the future, when it is clear what we mean (as in the case of a train moving on the ground), we will reserve the right to be sloppy, and occasionally drop the "with respect to . . ."

flash of light is emitted. The light travels up to the mirror and then back down to the source.

In A's frame, the train is at rest. The path of the light is shown in Fig. 10.5. It takes the light a time h/c to reach the ceiling and then a time h/c to return to the source. The roundtrip time is therefore

$$t_A = \frac{2h}{c}. \tag{10.3}$$

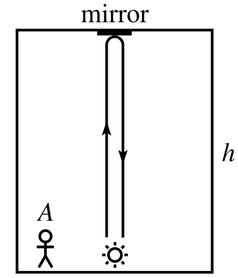


Figure 10.5

In B's frame, the train is moving at speed v. The path of the light is shown in Fig. 10.6. The crucial fact to remember is that the speed of light in B's frame is still c. This means that the light travels along its diagonally upward path at speed c. (The vertical component of its speed is *not* c, as would be the case if light behaved like a baseball.) Since the horizontal component of the light's velocity is v, the vertical component must be $\sqrt{c^2 - v^2}$, as shown in Fig. 10.7. (Yes, the Pythagorean theorem works fine here.) The time it takes to reach the mirror is therefore $h/\sqrt{c^2 - v^2}$,[8] so the roundtrip time is

$$t_B = \frac{2h}{\sqrt{c^2 - v^2}}. \tag{10.4}$$

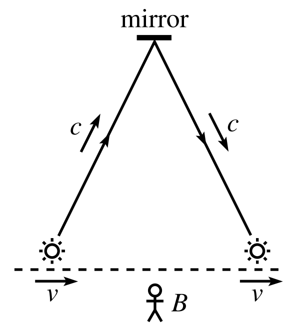


Figure 10.6

(Equivalently, the light travels a distance of $2hc/\sqrt{c^2 - v^2}$ on its crooked path, at speed c.) Dividing eq. (10.4) by eq. (10.3) gives

$$t_B = \gamma t_A, \tag{10.5}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{10.6}$$

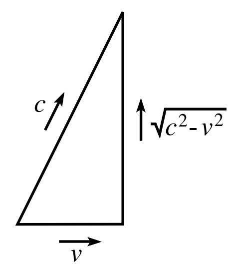


Figure 10.7

This γ factor is ubiquitous in special relativity. Note that it is always greater than or equal to 1. This means that the roundtrip time is longer in B's frame than is A's frame.

What are the implications of this? Let us for concreteness set $v/c = 3/5$, and so $\gamma = 5/4$. Then we may say the following. If A is standing next to the light source, and if B is standing on the ground, and if A claps his hands at $t_A = 4$ second intervals, then B will observe claps at $t_B = 5$ second intervals (after having subtracted off the time for the light to travel, of course). We may say this because both A and B must agree on the number of ups-and-downs the light beam took between claps. (And if we have a train that does not contain one of our special clocks, that's no matter. We *could* have built one if we wanted to, so the same results concerning the claps must hold.)

Therefore, B will observe A moving strangely slowly. B will observe A's heartbeat beating slowly; his blinks will be a bit lethargic; and his sips of coffee will be slow enough to suggest that he needs another cup.

[8]We've assumed that the height of the train in B's frame is still h. We will see below that there is length contraction along the direction of motion. But there is none in the direction perpendicular to the motion. (See Problem 1.)

The effects of dilation of time
Are magical, strange, and sublime.
In your frame, this verse,
Which you'll see is not terse,
Can be read in the same amount of time it takes someone
else in another frame to read a similar sort of rhyme.

Note that we may make these conclusions only if A is at rest with respect to the train. If A is moving with respect to the train, then eq. (10.5) does not hold (since we can *not* say that both A and B must agree on the number of ups-and-downs the light beam took between claps, because of the problem of simultaneity; it cannot be said which flash of light happened at the same time as a clap).

Remarks:

1. The time dilation derived in eq. (10.5) is a bit strange, no doubt, but there seems to be nothing downright incorrect about it until we look at the situation from A's point of view. A sees B flying by at a speed v in the other direction. The ground is no more fundamental than a train, so the same reasoning applies. The time dilation factor, γ, doesn't depend on the sign of v, so A sees the same time dilation factor that B sees: A sees B's clock running slow. But how can this be? Are we claiming that A's clock is slower than B's, and also that B's clock is slower than A's? Well . . . yes and no.

   Remember that the above time-dilation reasoning applies only to a situation where something is motionless in the appropriate frame. In the second situation (where A sees B flying by), the statement $t_A = \gamma t_B$ holds only when the events happen at the same place in B's frame. But for two such events, they are not in the same place in A's frame, so the $t_B = \gamma t_A$ result of eq. (10.5) does *not* hold. The conditions of being motionless in each frame never both hold (unless $v = 0$, in which case $\gamma = 1$ and $t_A = t_B$). So, the answer to the question at the end of the previous paragraph is "yes" if you ask the questions in the appropriate frames, and "no" if you think the answer should be frame independent.

2. Concerning the fact that A sees B's clock run slow, *and* B sees A's clock run slow, consider the following statement. "This is a contradiction. It is essentially the same as saying, 'I have two apples on a table. The left one is bigger than the right one, and the right one is bigger than the left one.' " How would you reply to this?

   Well, it is not a contradiction. Twins A and B are using *different coordinates* to measure time. The times measured in each of their frames are quite different things. They are not comparing apples and apples; they are comparing apples and oranges.

   A more correct analogy would be the following. An apple and an orange sit on a table. The apple says to the orange, "You are a much uglier apple than I am," and the orange says to the apple, "You are a much uglier orange than I am."

3. One might view the statement, "A sees B's clock running slowly, and also B sees A's clock running slowly," as somewhat unsettling. But in fact, it would be a complete disaster for the theory if A and B viewed each other in different ways. A critical postulate in the theory is that A sees B in exactly the same way as B sees A. ♣

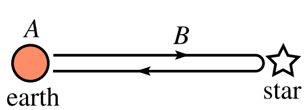


Figure 10.8

Example (Twin Paradox): Twin A stays on the earth, while twin B flies quickly to a distant star and back (see Fig. 10.8). Show that B is younger when they meet up again.

Solution: From A's point of view, B's clock is running slow by a factor γ, on both the outward and return parts of the trip. Therefore, B is younger when they meet up again.

That's all there is to it. But although the above reasoning is correct, it leaves one main point unaddressed. The "Paradox" part of this problem's title comes from the following reasoning. You may say that in B's frame, A's clock is running slow by a factor γ, and so A is younger when they meet up again.

It is definitely true that when the two twins are standing next to each other, we can't have both B younger than A, and A younger than B. So, what is wrong with the reasoning in the previous paragraph? The error lies in the fact that there is no "one frame" that B is in. The inertial frame for the outward trip is different from the inertial frame for the return trip. The derivation of our time-dilation result applies only to one inertial frame. Said in a different way, B accelerates when he turns around, and our time-dilation result holds only from the point of view of an *inertial* observer.[9]

The above paragraph shows what is wrong with the "A is younger" reasoning, but it doesn't show how to modify it quantitatively to obtain the correct answer. There are many different ways of doing this, and I'll let you do some of these in the problems.

Example (Muon decay): Elementary particles called *muons* (which are identical to electrons, except that they are about 200 times as massive) are created in the upper atmosphere when cosmic rays collide with air molecules. The muons have an average lifetime of about $2 \cdot 10^{-6}$ seconds[10] (then they decay into electrons, neutrinos, and the like), and move at nearly the speed of light.

Assume for simplicity that a certain muon is created at a height of 50 km, moves straight downward, has a speed $v = .99998\,c$, decays in exactly $T = 2 \cdot 10^{-6}$ seconds, and doesn't collide with anything on the way down.[11] Will the muon reach the earth before it decays?

Solution: The naive thing to say is that the distance traveled by the muon is $d = vT = 600\,\text{m}$, and that this is less than 50 km, so the muon does not reach the earth. This reasoning is incorrect, because of the time-dilation effect. The muon lives longer in the earth frame, by a factor of γ (which is $\gamma = 1/\sqrt{1-v^2/c^2} \approx 160$ here). The correct distance traveled in the earth frame is therefore $v(\gamma T) \approx 100\,\text{km}$. Hence, the muon travels the 50 km, with room to spare.

The real-life fact that we actually do detect muons reaching the surface of the earth in the predicted abundances (while the naive $d = vT$ reasoning would predict that we shouldn't see any) is one of the many experimental tests that support the relativity theory.

---

[9] For the entire outward and return parts of the trip, B *does* observe A's clock running slow, but enough strangeness occurs during the turning-around period to have A end up older. Note, however, that a discussion of acceleration is not required to quantitatively understand the paradox, as Problem 21 shows.

[10] This is the "proper" lifetime. That is, the lifetime as measured in the frame of the muon.

[11] In the real world, the muons are created at various heights, move in different directions, have different speeds, decay in lifetimes that vary according to a standard half-life formula, and may very well bump into air molecules. So technically we've got everything wrong here. But that's no matter. This example will work just fine for the present purpose.

10.2.3 Length contraction

Consider the following scenario. Person A stands on a train which he measures to have length ℓ', and person B stands on the ground. A light source is located at the back of the train, and a mirror is located at the front. The train moves at speed v with respect to the ground. The source emits a flash of light which heads to the mirror, bounces off, then heads back to the source. By looking at how long this process takes in the two reference frames, we can determine the length of the train, as viewed by B.[12]

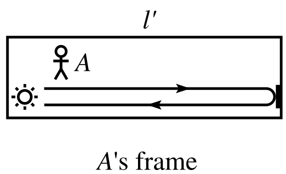


Figure 10.9

In A's frame (see Fig. 10.9), the round-trip time for the light is simply

$$t_A = \frac{2\ell'}{c}. \tag{10.7}$$

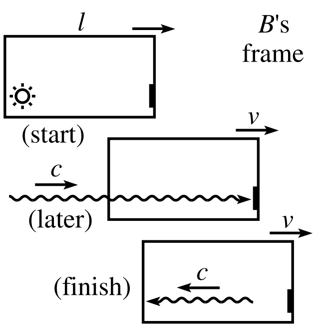


Figure 10.10

Things are a little more complicated in B's frame (see Fig. 10.10). Let the length of the train, as viewed by B, be ℓ. (For all we know at this point, ℓ may equal ℓ', but we'll soon find that it doesn't.) The relative speed of the light and the mirror during the first part of the trip is $c - v$. The relative speed during the second part is $c + v$. So the total round-trip time is

$$t_B = \frac{\ell}{c-v} + \frac{\ell}{c+v} = \frac{2\ell c}{c^2 - v^2} \equiv \frac{2\ell}{c}\gamma^2. \tag{10.8}$$

But we know from eq. (10.5) that

$$t_B = \gamma t_A. \tag{10.9}$$

Substituting the results for t_A and t_B from eqs. (10.7) and (10.8) into this equation, we find

$$\ell = \frac{\ell'}{\gamma}. \tag{10.10}$$

Since $\gamma \geq 1$, we see that B measures the train to be shorter than what A measures.

Note that we could not have used this experiment to find the length contraction if we had not already found the time dilation in eq. (10.5).

The term *proper length* is used to describe the length of an object in its rest frame. (So ℓ' is the proper length of the above train.)

Relativistic limericks have the attraction
Of being shrunk by a Lorentz contraction.
But for readers, unwary,
The results may be scary,
When a fraction . . .

[12]The second remark below gives another (quicker) derivation of length contraction.

REMARKS:

1. As with the time dilation, this is all a bit strange, but there seems to be nothing actually paradoxical about it until we look at the situation from A's point of view. To make a nice symmetrical situation, let's say B is standing on an identical train (which is motionless with respect to the ground). A sees B flying by at speed v in the other direction. The ground is no more fundamental than a train, so the same reasoning applies, and A sees the same length contraction factor that B sees; A measures B's train to be short. But how can this be?

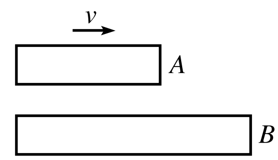


Figure 10.11

Are we claiming that A's train is shorter than B's, and also that B's train is shorter than A's? Does the situation look like that in Fig. 10.11, or does it look like that in Fig. 10.12? Well . . . it depends.

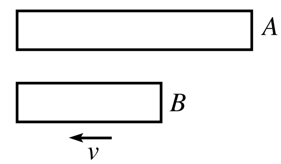


Figure 10.12

The word "is" in the above paragraph is a very bad word to use, and is generally the cause of all the confusion. There is no such thing as "is-ness" It makes no sense to say what the length of the train really *is*. It only makes sense to say what the length is in a given frame. The situation doesn't really *look like* one thing in particular. The look depends on the frame in which the looking is being done.

Let's be a little more specific. How do you measure a length? You write down the coordinates of the ends of something *measured simultaneously*, and you then take the difference. The point is that simultaneous events in one frame are not simultaneous events in another.

Stated more precisely, what we are really claiming is: Let B write down simultaneous coordinates of the ends of A's train, and also simultaneous coordinates of the ends of his own train. Then the difference between the former is smaller than the difference between the latter. Also, let A write down simultaneous coordinates of the ends of B's train, and also simultaneous coordinates of the ends of his own train. Then the difference between the former is smaller than the difference between the latter. There is no contradiction here, because the times at which A and B are writing down the coordinates don't have much to do with each other. Again, we are comparing apples and oranges.

2. There is an easy argument to show that time dilation implies length contraction, and vice versa. Let B stand on the ground, next to a stick of length ℓ. Let A fly past the stick at speed v. In B's frame, it takes A a time of ℓ/v to traverse the length of the stick. Therefore (assuming that we have demonstrated the time-dilation result), a watch on A's wrist will advance by a time of $\ell/\gamma v$ while he traverses the length of the stick.

How does A view the situation? He sees the ground and the stick fly by with speed v. The time between the two ends passing him is $\ell/\gamma v$. To get the length of the stick in his frame, he simply multiplies the speed times the time. That is, he measures the length to be $(\ell/\gamma v)v = \ell/\gamma$, which is the desired contraction.

The same argument also shows that length contraction implies time dilation. ♣

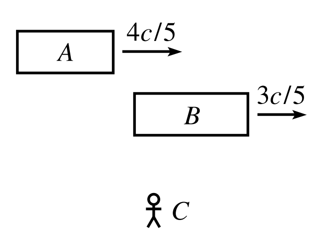


Figure 10.13

Example: Two trains, A and B, each have proper length L and move in the same direction. A's speed is $4c/5$, and B's speed is $3c/5$. A starts behind B (see Fig. 10.13). How long, as viewed by person C on the ground, does it take for A to overtake B? (By this we mean the time between the front of A passing the back of B, and the back of A passing the front of B.)

Solution: Relative to C on the ground, the γ factors associated with A and B are $5/3$ and $5/4$, respectively. Therefore, their lengths in the ground frame are $3L/5$ and $4L/5$. While overtaking B, A must travel further than B, by an excess distance equal to the sum of the lengths of the trains, which is $7L/5$. The relative speed of the two trains (as viewed by C on the ground) is the difference of the speeds, which is $c/5$. The total time is therefore

$$t_C = \frac{7L/5}{c/5} = \frac{7L}{c}\,. \tag{10.11}$$

Example (Muon decay, again): Consider the "Muon decay" example from section 10.2.2. From the muon's point of view, it lives for a time of $T = 2 \cdot 10^{-6}$ seconds, and the earth is speeding toward it at $v = .99998c$. How, then, does the earth (which will travel only $d = vT = 600\,\text{m}$ before the muon decays) reach the muon?

Solution: The point is that in the muon's frame, the distance to the earth is contracted by a factor $\gamma \approx 160$. Therefore, the earth starts only $50\,\text{km}/160 \approx 300\,\text{m}$ away. Since the earth can travel a distance of 600 m during the muon's lifetime, the earth collides with the muon, with time to spare.

As stated in the second remark above, time dilation and length contraction are intimately related; we can't have one without the other. In the earth's frame, the muon's arrival on the earth is explained by time dilation. In the muon's frame, it is explained by length contraction.

Observe that for muons, created,
The dilation of time is related
To Einstein's insistence
Of shrunken-down distance
In the frame where decays aren't belated.

10.3 The Lorentz transformations

10.3.1 The derivation

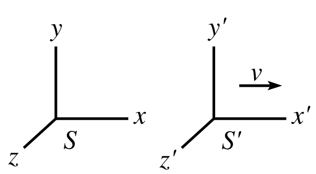


Figure 10.14

Consider a coordinate system, S', moving relative to another system, S (see Fig. 10.14). Let the relative speed be v. Let the corresponding axes of S and S' point in the same direction, and let the origin of S' move along the x-axis of S (in the positive direction). Nothing exciting happens here with the y and z directions, so we'll ignore them.[13]

Our goal in this section is to look at two events in spacetime and relate the Δx and Δt of the coordinates in one frame to the $\Delta x'$ and $\Delta t'$ of the coordinates in another. To avoid writing the Δ's over and over, it is customary to pick the first event to be the origin in both frames. Thus, $\Delta x = x$, etc., and our goal is to relate

[13]See Problem 1.

x and t to x' and t'. We therefore want to find the constants, A,B,C,D, in the following relations,

$$\begin{aligned} x &= Ax' + Bt', \\ t &= Ct' + Dx'. \end{aligned} \qquad (10.12)$$

The four constants here will end up depending on v (which is constant, given the two inertial frames). We will not explicitly write this dependence, for simplicity.

Remarks:

1. We have assumed that x and t are linear functions of x' and t'. And we have also assumed that A, B, C, and D are constants (that is, independent of x,t,x',t').

   The first of these assumptions is fairly clear. Any finite transformation can be built up out of a series of many infinitesimal ones. Hence, any terms such as, for example, t'^2 (which is really $(\Delta t')^2$) are negligible compared to the linear terms.

   The second assumption may be justified in various ways. One is that all inertial frames should agree on what "non-accelerating" motion is. That is, if $\Delta x' = a\Delta t'$, then we should also have $\Delta x = b\Delta t$, for some constant b. This is true only if the above coefficients are constants. Another justification comes from the first of our two relativity postulates, which says that all points in (empty) space should be indistinguishable. With this in mind, let's say we had a transformation of the form $\Delta x = A\Delta x' + B\Delta t' + Ex'\Delta x'$. The x' in the last term implies that the absolute location in spacetime (and not just the relative position) is important. This last term, therefore, cannot exist.

2. If these relations were the usual Galilean transformations (which are the ones that hold for everyday relative speeds, v) then we would have $x = x'+vt$ and $t = t'$. We will find, under the assumptions of Special Relativity, that the Galilean transformations do *not* hold. ♣

The constants A, B, C, D in eqs. (10.12) are four unknowns, and we can solve for them by using four facts we previous found in Section 10.2. These four facts are:

	fact	condition	result	eq. in text
1	Time dilation	$x' = 0$	$t = \gamma t'$	(10.5)
2	Length contraction	$t' = 0$	$x' = x/\gamma$	(10.10)
3	Relative v of frames	$x = 0$	$x' = -vt'$	
4	"Head-start" effect	$t = 0$	$t' = -vx'/c^2$	(10.2)

Again, these should all be Δx's and Δt's, etc., since we are always concerned with the difference between the coordinates of two events in spacetime. But we won't write the Δ's here, lest things get too messy.

You should pause for a moment and verify that the "results" in the above table are in fact the proper mathematical expressions for the four effects, given the stated "conditions".[14]

[14] Of course, there are other ways to state the effects. For example, time dilation may be given as $t' = \gamma t$ when $x = 0$. But the expressions in the table are the ones that will allow us to (very) quickly solve for our unknowns.

We may now (very easily) solve for the unknowns A, B, C, D.

(1) gives $C = \gamma$.

(2) gives $A = \gamma$.

(3) gives $B/A = v \Longrightarrow B = \gamma v$.

(4) gives $D/C = v/c^2 \Longrightarrow D = \gamma v/c^2$.

The Lorentz transformations are therefore given by

$$\begin{aligned} x &= \gamma(x' + vt'), \\ t &= \gamma(t' + vx'/c^2), \\ y &= y', \\ z &= z', \end{aligned} \tag{10.13}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{10.14}$$

We have tacked on the trivial transformations for y and z, but we won't bother writing these in the future.

Solving for x' and t' in terms of x and t, we see that inverse Lorentz transformations are given by

$$\begin{aligned} x' &= \gamma(x - vt), \\ t' &= \gamma(t - vx/c^2). \end{aligned} \tag{10.15}$$

Of course, which ones are the "inverse" transformations depends simply on your point of view. But it's intuitively clear that the only difference between the two sets of equations is the sign of v, because S is simply moving backwards with respect to S.

The reason why the derivation of eqs. (10.13) was so quick is, of course, that we had already done most of the work in Section 10.2, when deriving the fundamental effects. If you wanted to derive the Lorentz transformations from scratch, that is, by starting with the two postulates in Section 10.1, then the derivation would be much longer. In Appendix I we give such a derivation, where it is clear what information comes from each of the postulates. The procedure there is somewhat cumbersome, but it's worth taking a look at, because we will invoke the results in a very cool way in Section 10.8.

REMARKS:

1. We emphasize again that x,t,x',t' refer to separations between two events. Technically, they should all have a "Δ" in front of them. But we get tired of writing the Δ's.
2. In the limit $v \ll c$, eqs. (10.13) reduce to $x = x' + vt$ and $t = t'$, that is, simply the Galilean transformations. This of course must be the case, since we know from everyday experience (where $v \ll c$) that the Galilean transformations work just fine.

3. Eqs. (10.13) exhibit a nice symmetry between x and ct. With $\beta \equiv v/c$, they become

$$\begin{aligned} x &= \gamma\big(x' + \beta(ct')\big), \\ ct &= \gamma\big((ct') + \beta x'\big). \end{aligned} \tag{10.16}$$

Equivalently, in units where $c = 1$ (for example, where one unit of distance equals $3 \cdot 10^8$ meters, or where one unit of time equals $1/(3 \cdot 10^8)$ seconds), eqs. (10.13) take the symmetric form

$$\begin{aligned} x &= \gamma(x' + vt'), \\ t &= \gamma(t' + vx'). \end{aligned} \tag{10.17}$$

4. In matrix form, eqs. (10.16) are

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}. \tag{10.18}$$

This looks similar to a rotation matrix. More about this in Section 10.7, and in Problem 26.

5. The plus or minus sign on the right-hand side of the L.T.'s in eqs. (10.13) and (10.15) corresponds to which way the coordinate system on the left-hand side sees the coordinate system on the right-hand side moving. But if you get confused about the sign, simply write down $x_A = \gamma(x_B \pm vt_B)$, and then imagine sitting in system A and looking at a fixed point in B, which satisfies (putting the Δ's back in to avoid any mixup) $\Delta x_B = 0 \Longrightarrow \Delta x_A = \pm\gamma v \Delta t_B$. If the point moves to the right (that is, if it increases as time increases), then pick the "+". If it moves to the left, then pick the "−".

6. One very important thing we must check is that two successive Lorentz transformations (from S_1 to S_2 and then from S_2 to S_3) again yield a Lorentz transformation (from S_1 to S_3). This must be true because we showed that any two frames must be related by eq. (10.13). If we composed two L.T.'s and found that the transformation from S_1 to S_3 was not of the form of eqs. (10.13) (for some new v), then the whole theory would be inconsistent, and we would have to drop one of our postulates. We'll let you show that the composition of an L.T. (with speed v_1) with an L.T. (with speed v_2) does indeed yield an L.T. (with speed $(v_1 + v_2)/(1 + v_1 v_2/c^2)$). This is the task of Problem 26 (which is stated in terms of *rapidity*, introduced in Section 10.7), but you should try it in the present language, too. ♣

Example: A train with proper length L moves with speed $5c/13$ with respect to the ground. A ball is thrown from the back of the train to the front. The speed of the ball with respect to the train is $c/3$. As viewed by someone on the ground, how much time does the ball spend in the air, and how far does it travel?

Solution: The γ factor associated with the speed $5c/13$ is $\gamma = 13/12$. The two events we are concerned with are "ball leaving back of train" and "ball arriving at front of train". The spacetime separation between these events is easy to calculate on the train. We have $\Delta x_T = L$, and $\Delta t_T = L/(c/3) = 3L/c$. The Lorentz transformations for the coordinates on the ground are

$$\begin{aligned} x_G &= \gamma(x_T + vt_T), \\ t_G &= \gamma(t_T + vx_T/c^2). \end{aligned} \tag{10.19}$$

Therefore,

$$\begin{aligned} x_G &= \frac{13}{12}\left(L + \left(\frac{5c}{13}\right)\left(\frac{3L}{c}\right)\right) = \frac{7L}{3}\,, \\ t_G &= \frac{13}{12}\left(\frac{3L}{c} + \frac{\frac{5c}{13}L}{c^2}\right) = \frac{11L}{3c}\,. \end{aligned} \tag{10.20}$$

In a given problem, one of the frames generally allows for an easy calculation of Δx and Δt, so you simply have to mechanically plug these quantities into the L.T.'s to obtain $\Delta x'$ and $\Delta t'$ in the other frame, where they may not be as obvious.

Relativity is a subject where there are usually many ways to do a problem. If you are trying to find some Δx's and Δt's, then you can use the L.T.'s, or perhaps the invariance of the spacetime interval (introduced in Section 10.4), or maybe a velocity-addition approach (introduced in Section 10.3.3), or even the sending-of-light-signals strategy used in Section 10.2. Depending on the specific problem and what your personal preferences are, certain approaches will be more enjoyable than others. But no matter which method you choose, you should take advantage of the myriad of possibilities by picking a second method to double-check your answer. Personally, I find the L.T.'s to be the perfect option for this, because the other methods are generally more fun when solving a problem the first time, while the L.T.'s are usually quick and easy to apply (perfect for a double-check).[15]

The excitement will build in your voice,
As you rise from your seat and rejoice,
"A Lorentz transformation
Provides confirmation
Of my alternate method of choice!"

10.3.2 The fundamental effects

Let's now see how the Lorentz transformations imply the three fundamental effects (namely, loss of simultaneity, time dilation, and length contraction) discussed in Section 10.2. Of course, we just used these effects to *derive* the Lorentz transformation, so we know everything will work out. We'll just be going in circles. But since these fundamental effects are, well, fundamental, let's belabor the point and discuss them one more time.

Loss of Simultaneity

Let two events occur simultaneously in frame S'. Then the separation between them, as measured by S', is $(x', t') = (x', 0)$. (As usual, we are not bothering to write the Δ's in front of the coordinates.) Using the second of eqs. (10.13), we see that the time between the events, as measured by S, is $t = \gamma v x'/c^2$. This is not

[15] I would, however, be very wary of solving a problem using only the L.T.'s, with no other check, because it's very easy to mess up a sign in the transformations. And since there's nothing to do except mechanically plug in numbers, there's not much opportunity for an intuitive check, either.

equal to 0 (unless $x' = 0$). Hence, the events do not occur simultaneously in the S frame.

Time dilation

Consider two events that occur in the same place in S'. Then the separation between them is $(x', t') = (0, t')$. Using the second of eqs. (10.13), we see that the time between the events, as measured by S, is

$$t = \gamma t' \qquad (\text{if } x' = 0). \tag{10.21}$$

The factor γ is greater than or equal to 1, so $t \geq t'$. The passing of one second on S''s clock takes more than one second on S's clock. S sees S' drinking his coffee very slowly.

The same strategy works if we interchange S and S'. Consider two events that occur in the same place in S. The separation between them is $(x, t) = (0, t)$. Using the second of eqs. (10.15), we see that the time between the events, as measured by S', is

$$t' = \gamma t \qquad (\text{if } x = 0). \tag{10.22}$$

Hence, $t' \geq t$. (Another way to derive this is to use the first of eqs. (10.13) to write $x' = -vt'$, and substitute this into the second equation.)

Remark: The above equations, $t = \gamma t'$ and $t' = \gamma t$, appear to contradict each other. The apparent contradiction arises from the sloppy notation. The former equation follows from the assumption $x' = 0$. The latter equation follows from the assumption $x = 0$. They have nothing to do with each other. It would perhaps be better to write the equations as

$$\begin{aligned} t_{x'=0} &= \gamma t', \\ t'_{x=0} &= \gamma t, \end{aligned} \tag{10.23}$$

but this is somewhat cumbersome. ♣

Length contraction

This proceeds just like the time dilation above, except now we want to set certain time intervals equal to zero, instead of certain space intervals. We want to do this because to measure a length, you simply measure the distance between two points whose positions are measured *simultaneously*. That's what a length is.

Consider a stick at rest in S', where it has length ℓ'. We want to find the length in S. Measurements of the coordinates of the ends of the stick in S yield a separation of $(x, t) = (x, 0)$. Using the first of eqs. (10.15), we have

$$x' = \gamma x \qquad (\text{if } t = 0). \tag{10.24}$$

But x is (by definition) the length in S, and x' is the length in S' (because the stick is not moving in S').[16] Therefore, $\ell = \ell'/\gamma$. And since $\gamma \geq 1$, we have $\ell \leq \ell'$. S sees the stick shorter than S' sees it.

Now interchange S and S'. Consider a stick at rest in S, where it has length ℓ. We want to find the length in S'. Measurements of the coordinates of the ends of the stick in S' yield a separation of $(x', t') = (x', 0)$. Using the first of eqs. (10.13), we have

$$x = \gamma x' \qquad (\text{if } t' = 0). \tag{10.25}$$

But x' is (by definition) the length in S', and x is the length in S (because the stick is not moving in S). Therefore, $\ell' = \ell/\gamma$. So $\ell' \leq \ell$.

Remark: As with the time dilation, the above equations, $\ell = \ell'/\gamma$ and $\ell' = \ell/\gamma$, appear to contradict each other. And as before, the apparent contradiction arises from the sloppy notation. The former equation follows from the assumptions that $t = 0$ and that the stick is at rest in S'. The latter equation follows from the assumptions that $t' = 0$ and that the stick is at rest in S. We should really write the equations as

$$\begin{aligned} x_{t=0} &= x'/\gamma, \\ x'_{t'=0} &= x/\gamma, \end{aligned} \tag{10.26}$$

and then identify x' in the first equation with ℓ' only after invoking the further assumption that the stick is at rest in S'; likewise for x in the second equation. But this is a pain. ♣

10.3.3 Velocity addition

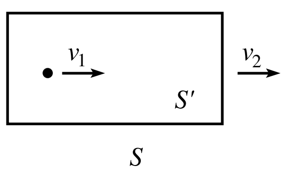


Figure 10.15

Longitudinal velocity addition

Consider the following situation. An object moves at speed v_1 with respect to frame S'. And frame S' moves at speed v_2 with respect to frame S (in the same direction as the motion of the object; see Fig. 10.15). What is the speed, u, of the object with respect to frame S?

The Lorentz transformations may be used to easily answer this question. The relative speed of the frames is v_2. Consider two events along the object's path (for example, say it emits two flashes of light). We are given that $\Delta x'/\Delta t' = v_1$. Our goal is to find $u \equiv \Delta x/\Delta t$.

The Lorentz transformations from S' to S, eqs. (10.13), are

$$\Delta x = \frac{\Delta x' + v\Delta t'}{\sqrt{1 - v^2/c^2}}, \qquad \text{and} \qquad \Delta t = \frac{\Delta t' + v\Delta x'/c^2}{\sqrt{1 - v^2/c^2}}, \tag{10.27}$$

where $v = v_2$ here. Therefore,

$$u \equiv \frac{\Delta x}{\Delta t} \quad = \quad \frac{\Delta x' + v_2 \Delta t'}{\Delta t' + v_2 \Delta x'/c^2}$$

[16]The measurements of the ends made by S will *not* be simultaneous in the S' frame. In the S' frame, the separation between the events is (x', t'), where both x' and t' are nonzero. This does not satisfy our definition of a length measurement in S' (because $t' \neq 0$), but the stick is not moving in S', so S' can measure the ends whenever he feels like it, and he will always get the same difference. So x' is indeed the length in the S' frame.

$$= \frac{\Delta x'/\Delta t' + v_2}{1 + v_2(\Delta x'/\Delta t')/c^2}$$
$$= \frac{v_1 + v_2}{1 + v_1 v_2/c^2}\,. \qquad (10.28)$$

This is the *velocity-addition formula* (for adding velocities in the same direction). Let's look at some of its properties. It is symmetric with respect to v_1 and v_2, as it should be (because we could switch the roles of the object and frame S.) For $v_1 v_2 \ll c^2$, it reduces to $u \approx v_1 + v_2$, which we know holds perfectly fine for everyday speeds. If $v_1 = c$ or $v_2 = c$, then we find $u = c$, as should be the case, since anything that moves with speed c in one frame moves with speed c in another. The maximum (or minimum) of u in the region $-c < v_1, v_2 < c$ is c (or $-c$), which can be seen by noting that $\partial u/\partial v_1 = 0$ and $\partial u/\partial v_2 = 0$ only when $v_2 = \pm c$ and $v_1 = \pm c$.

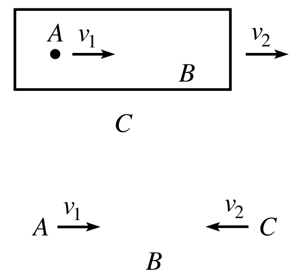


Figure 10.16

REMARK: The velocity-addition formula applies to both scenarios in Fig. 10.16, if we want to find the speed of A with respect to C. (The second scenario is simply the first scenario, as observed in B's frame.) That is, the velocity-addition formula applies when we ask, "If A moves at v_1 with respect to B, and B moves at v_2 with respect to C (which means, of course, that C moves with speed v_2 with respect to B), how fast does A move with respect to C?" The formula does *not* apply if we ask the more mundane question, "What is the relative speed of A and C, as viewed by B?" The answer here is of course just $v_1 + v_2$.

The point is that if the two speeds are given with respect to B, and if you are asking for a relative speed as measured B, then you simply add the speeds.[17] But if you are asking for a relative speed as measured by A or C, then you have to use the velocity-addition formula. ♣

Example: Consider again the scenario in the first example in Section 10.2.3.

(a) How long, as viewed by A and as viewed by B, does it take for A to overtake B?

(b) Let event E_1 be "the front of A passing the back of B", and let event E_2 be "the back of A passing the front of B". Person D walks at constant speed from the back of train B to its front (see Fig. 10.17), such that he coincides with both events E_1 and E_2. How long does the "overtaking" process take, as viewed by D?

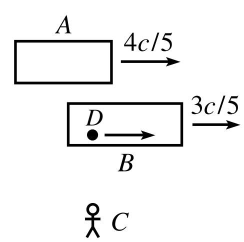


Figure 10.17

Solution:

(a) First consider B's point of view. From the velocity-addition formula, B sees A moving with speed

$$u = \frac{\frac{4c}{5} - \frac{3c}{5}}{1 - \frac{4}{5}\cdot\frac{3}{5}} = \frac{5c}{13}. \qquad (10.29)$$

[17] Note that the resulting speed can certainly be greater than c. If I see a ball heading toward me at $.9c$ from the right, and another one heading toward me at $.9c$ from the left, then the relative speed of the balls in my frame is $1.8c$. In the frame of one of the balls, however, the relative speed is $(1.8/1.81)c \approx (.9945)c$, from eq. (10.28).

The γ factor associated with this speed is $\gamma = 13/12$. Therefore, B sees A's train contracted to a length $12L/13$. During the overtaking, A must travel a distance equal to the sum of the lengths of the trains, which is $L + 12L/13 = 25L/13$. Since A moves at speed $5c/13$, the total time is

$$t_B = \frac{25L/13}{5c/13} = \frac{5L}{c}\,. \tag{10.30}$$

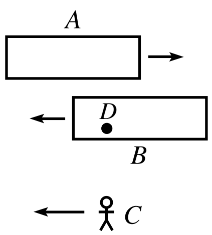


Figure 10.18

The exact same reasoning holds from A's point of view, so we have $t_A = t_B = 5L/c$.

(b) Look at things from D's point of view. D is at rest, and the two trains move with equal and opposite speeds, v, as shown in Fig. 10.18. The relativistic addition of v with itself must equal the $5c/13$ speed found in part (a). Therefore,

$$\frac{2v}{1+v^2/c^2} = \frac{5c}{13} \qquad \Longrightarrow \qquad v = \frac{c}{5}\,. \tag{10.31}$$

The γ factor associated with this speed is $\gamma = 5/2\sqrt{6}$. Therefore, D sees both trains contracted to a length $2\sqrt{6}L/5$. During the overtaking, each train must travel a distance equal to its length (since both events, E_1 and E_2, take place right at D). The total time is therefore

$$t_D = \frac{2\sqrt{6}L/5}{c/5} = \frac{2\sqrt{6}L}{c}\,. \tag{10.32}$$

Remarks: There are a few double-checks we can perform. The speed of D with respect to the ground my be obtained by either relativistically adding $3c/5$ and $c/5$, or subtracting $c/5$ from $4c/5$). Fortunately, these both give the same answer, namely $5c/7$. The γ factor between the ground and D is therefore $7/2\sqrt{6}$. We may now use time dilation to say that someone on the ground sees the overtaking take a time of $(7/2\sqrt{6})t_D$. Using eq. (10.32), this gives $7L/c$, in agreement with the result of the first example in Section 10.2.3.

Likewise, the gamma factor between D and either train is $5/2\sqrt{6}$. So the time of the overtaking as viewed by either A or B is $(5/2\sqrt{6})t_D = 5L/c$, in agreement with the result of part (a).

Note that we can *not* use simple time dilation between the ground and A or B, because the two events don't happen at the same place in the frame of the trains. D's frame does have this property, since both events take place at D. ♣

Transverse velocity addition

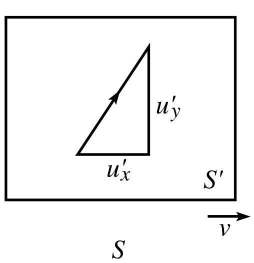


Figure 10.19

Consider the following more general situation. An object moves with velocity (u'_x, u'_y) with respect to frame S'. And frame S' moves at speed v with respect to frame S, in the x-direction (see Fig. 10.19). What is the velocity, (u_x, u_y), of the object with respect to frame S?

The existence of motion in the y-direction doesn't affect the preceding derivation of the speed in the x-direction, so eq. (10.28) is still valid. In the present notation, it becomes

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}\,. \tag{10.33}$$

To find u_y, we may again make easy use of the Lorentz transformations. Consider two events along the object's path. We are given that $\Delta x'/\Delta t' = u'_x$, and $\Delta y'/\Delta t' = u'_y$. Our goal is to find $u_y \equiv \Delta y/\Delta t$.

The relevant Lorentz transformations from S' to S, eqs. (10.13), are

$$\Delta y = \Delta y', \qquad \text{and} \qquad \Delta t = \gamma(\Delta t' + v\Delta x'/c^2). \tag{10.34}$$

Therefore,

$$\begin{aligned} u_y \equiv \frac{\Delta y}{\Delta t} &= \frac{\Delta y'}{\gamma(\Delta t' + v\Delta x'/c^2)} \\ &= \frac{\Delta y'/\Delta t'}{\gamma(1 + v(\Delta x'/\Delta t')/c^2)} \\ &= \frac{u'_y}{\gamma(1 + u'_x v/c^2)}. \end{aligned} \tag{10.35}$$

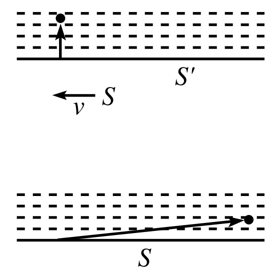


Figure 10.20

REMARK: In the special case where $u'_x = 0$, we have $u_y = u'_y/\gamma$. When u'_y is small and v is large, this result can be seen to be a special case of time dilation, in the following way. Consider a series of equally spaced lines parallel to the x-axis (see Fig. 10.20). Imagine that the object's clock ticks once every time it crosses a line. Since u'_y is small, the object's frame is essentially the frame S', so the object is essentially moving at speed v with respect to S. Therefore, S sees the clock run slow by a factor γ. This means that S sees the object cross the lines at a slower rate, by a factor γ. Since distances in the y-direction are the same in the two frames, we conclude that $u_y = u'_y/\gamma$. (We will see this gamma factor again, when we deal with momentum in the next chapter.)

To sum up: if you run in the x-direction, then an object's y-speed slows down (or speeds up, if u'_x and v have the opposite sign) from your point of view. Strange indeed, but no stranger than other effects we've seen.

Problem 17 deals with the case where $u'_x = 0$, but where u'_y is not necessarily small. ♣

10.4 The spacetime interval

Consider the quantity,

$$s^2 \equiv c^2t^2 - x^2. \tag{10.36}$$

(Again, we should be writing $\Delta s^2 \equiv c^2\Delta t^2 - \Delta x^2$.) Using eq. (10.13), we can write s^2 in terms of the S' coordinates, x' and t'. The result is

$$\begin{aligned} c^2t^2 - x^2 &= \frac{c^2(t' + vx'/c^2)^2}{1 - v^2/c^2} - \frac{(x' + vt')^2}{1 - v^2/c^2} \\ &= \frac{t'^2(c^2 - v^2) - x'^2(1 - v^2/c^2)}{1 - v^2/c^2} \\ &= c^2t'^2 - x'^2 \\ &\equiv s'^2. \end{aligned} \tag{10.37}$$

We see that the Lorentz transformations imply that the quantity $c^2t^2 - x^2$ does not depend on the frame. This result is more than we bargained for, for the following reason. The "invariance of the speed of light" postulate says that if $c^2t'^2 - x'^2 = 0$ then $c^2t^2 - x^2 = 0$. But eq. (10.37) says that if $c^2t'^2 - x'^2 = b$ then $c^2t^2 - x^2 = b$, for *any* b. This is, as you might guess, very useful. The fact that s^2 is invariant under Lorentz transformations of x and t is exactly analogous to the fact that r^2 is invariant under rotations in the x-y plane. The invariance of s^2 is just a special case of the more general results involving inner products and 4-vectors, which we'll discuss in Chapter 12.

What is the physical significance of $s^2 = c^2t^2 - x^2$? There are three cases to consider.

Case 1: $s^2 > 0$ (timelike separation)

In this case, we say that the two events are *timelike* separated. We have $c^2t^2 > x^2$, and so $|x/t| < c$. Consider a frame S' moving at speed v with respect to S. The Lorentz transformation for x is

$$x' = \gamma(x - vt). \tag{10.38}$$

Since $|x/t| < c$, there exists a v which is less than c (namely $v = x/t$) that makes $x' = 0$. In other words, if two events are timelike separated, it is possible to find a frame S' in which the two events happen at the same place. (This is obvious. The inequality $|x/t| < c$ means than it is possible for a particle to travel from one event to the other.) The invariance of s^2 then gives $s^2 = c^2t'^2 - x'^2 = c^2t'^2$. So we see that s/c is simply the time between the events in the frame where the events occur at the same place. This time is called the *proper time*.

Case 2: $s^2 < 0$ (spacelike separation)

In this case, we say that the two events are *spacelike* separated. We have $c^2t^2 < x^2$, and so $|t/x| < 1/c$. Consider a frame S' moving at speed v with respect to S. The Lorentz transformation for t' is

$$t' = \gamma(t - vx/c^2). \tag{10.39}$$

Since $|t/x| < 1/c$, there exists a v which is less than c (namely $v = c^2t/x$) that makes $t' = 0$. In other words, if two events are spacelike separated, it is possible to find a frame S' in which the two events happen at the same time. (This statement is not as obvious as the corresponding one in the timelike case above. But if you draw a Minkowski diagram, described in the next section, it is quite evident.) The invariance of s^2 then gives $s^2 = c^2t'^2 - x'^2 = -x'^2$. So we see that $|s|$ is simply the distance between the events in the frame where the events occur at the same time. This distance is called the *proper distance*.

Case 3: $s^2 = 0$ (lightlike separation)

In this case, we say that the two events are *lightlike* separated. We have $c^2t^2 = x^2$, and so $|x/t| = c$. It is not possible to find a frame S' in which the two events happen at the same place or the same time. In any frame, a photon emitted at one of the events will arrive at the other.

Example (Time dilation): An easy illustration of the usefulness of the invariance of s^2 is a derivation of time dilation. Let frame S' move past frame S, at speed v. Consider two events at the origin of S', separated by time t'. Picking the first event to be the origin in both frames, the separation between the events is

$$\begin{aligned} \text{in } S': &\quad (x', t') = (0, t'), \\ \text{in } S: &\quad (x, t) = (vt, t). \end{aligned} \tag{10.40}$$

The invariance of s^2 implies $c^2t'^2 - 0 = c^2t^2 - v^2t^2$. Therefore,

$$t = \frac{t'}{\sqrt{1 - v^2/c^2}}\,. \tag{10.41}$$

Here it is clear that this result rests on the assumption that $x' = 0$.

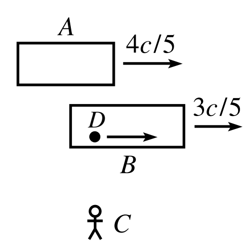


Figure 10.21

Example: Consider again the scenario of the examples in Sections 10.2.3 and 10.3.3. Verify that the s^2 between the events E_1 and E_2 is the same in all of the frames, A, B, C, and D (see Fig. 10.21).

Solution: The only quantity that we'll need that we haven't already found in the two previous examples is the distance between E_1 and E_2 in C's frame (the ground frame). In this frame, train A travels at a rate $4c/5$ for a time $t_C = 7L/c$, covering a distance of $28L/5$. But event E_2 occurs at the back of the train, which is a distance $3L/5$ behind the front end (this is the contracted length in the ground frame). Therefore, the distance between events E_1 and E_2 in the ground frame is $28L/5 - 3L/5 = 5L$. (You can do the same line of reasoning using train B, in which the $5L$ takes the form $(3c/5)(7L/c) + 4L/5$.)

Putting all the previous results together, we have the following separations between events:

	A	B	C	D
Δt	$5L/c$	$5L/c$	$7L/c$	$2\sqrt{6}L/c$
Δx	$-L$	L	$5L$	0

From the table, we see that $s^2 \equiv c^2\Delta t^2 - \Delta x^2 = 24L^2$ for all four frames, as desired.

We could have, of course, worked backwards and used the $s^2 = 24L^2$ result from frame B, C, or D, to deduce that $\Delta x = 5L$ in frame A.

In Problem 11, you are asked to perform the mundane task of checking that the values in the above table satisfy the Lorentz transformations between the various pairs of frames.

10.5 Minkowski diagrams

Minkowski diagrams (or "space-time" diagrams) are extremely useful in seeing how coordinates transform between different reference frames. If you want to produce exact numbers in a problem, you may have to use the Lorentz transformations. But as far as getting the overall intuitive picture goes (if there is indeed any such thing as intuition in relativity), there is no better tool than a Minkowski diagram. Here's how you make one.

Let frame S' move at speed v with respect to frame S (along the x-axis, as usual; and ignore the y and z components). Draw the x and ct axes of frame S.[18] What do the x' and ct' axes of S' look like, superimposed on this graph? That is, at what angles are the axes inclined, and what is the size of one unit on these axes? (There is no reason why one unit on the x' and ct' axes should have the same length on the paper as one unit on the x and ct axes.) We can figure this out using the Lorentz transformations, eqs. (10.13). We'll first look at the ct' axis, and then at the x' axis.

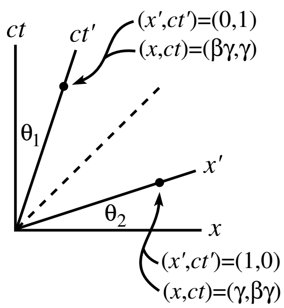


Figure 10.22

ct'-axis angle and unit size

Look at the point $(x', ct') = (0, 1)$, which lies on the ct' axis, one ct' unit from the origin (see Fig. 10.22). Eqs. (10.13) tell us that this point is the point $(x, ct) = (\gamma v/c, \gamma)$. The angle between the ct' and ct axes is therefore given by $\tan\theta_1 = x/ct = v/c$. With $\beta \equiv v/c$, we have

$$\tan\theta_1 = \beta. \tag{10.42}$$

Alternatively, the ct' axis is simply the "worldline" of the origin of S'. (A worldline is simply the path of an object as it travels through spacetime.) The origin moves at speed v with respect to S. Therefore, points on the ct' axis satisfy $x/t = v$, or $x/ct = v/c$.

On the paper, the point $(x', ct') = (0, 1)$, which we just found to be the point $(x, ct) = (\gamma v/c, \gamma)$, is a distance $\gamma\sqrt{1 + v^2/c^2}$ from the origin. Using the definitions of β and γ, we see that

$$\frac{\text{one } ct' \text{ unit}}{\text{one } ct \text{ unit}} = \sqrt{\frac{1+\beta^2}{1-\beta^2}}\,, \tag{10.43}$$

as measured on a grid where the x and ct axes are orthogonal.

Note that this ratio approaches infinity as $\beta \to 1$. And it of course equals 1 if $\beta = 0$.

x'-axis angle and unit size

The same basic argument holds here. Look at the point $(x', ct') = (1, 0)$, which lies on the x' axis, one x' unit from the origin (see Fig. 10.22). Eqs. (10.13) tell us that

[18]We choose to plot ct instead of t on the vertical axis, so that the trajectory of a light beam lies at a nice 45° angle. Alternatively, we could choose units where $c = 1$.

this point is the point $(x, ct) = (\gamma, \gamma v/c)$. The angle between the x' and x axes is therefore given by $\tan\theta_2 = ct/x = v/c$. So, as in the ct'-axis case,

$$\tan\theta_2 = \beta. \tag{10.44}$$

On the paper, the point $(x', ct') = (1, 0)$, which we just found to be the point $(x, ct) = (\gamma, \gamma v/c)$, is a distance $\gamma\sqrt{1 + v^2/c^2}$ from the origin. So, as in the ct'-axis case,

$$\frac{\text{one } x' \text{ unit}}{\text{one } x \text{ unit}} = \sqrt{\frac{1+\beta^2}{1-\beta^2}}\,, \tag{10.45}$$

as measured on a grid where the x and ct axes are orthogonal. Both the x' and ct' axes are therefore stretched by the same factor, and tilted by the same angle, relative to the x and ct axes.

REMARKS: If $v/c \equiv \beta = 0$, then $\theta_1 = \theta_2 = 0$, so the ct' and x' axes coincide with the ct and x axes, as they should. If β is very close to 1, then the x' and ct' axes are both very close to the 45° light-ray line. Note that since $\theta_1 = \theta_2$, the light-ray line bisects the x' and ct' axes; therefore, the scales on these axes must be the same (since a light ray must satisfy $x' = ct'$), as we verified above. ♣

We now know what the x' and ct' axes look like. Therefore, given any two points in a Minkowski diagram (that is, given any two events in spacetime), we can simply read off (if our graph is accurate enough) the Δx, Δct, $\Delta x'$, and $\Delta ct'$ quantities that our two observers would measure. These quantities must of course be related by the Lorentz transformation. But the advantage of a Minkowski diagram is that you can actually see geometrically what's going on.

Note that there are very useful interpretations of the ct' and x' axes. If you stand at the origin of S', then the ct' axis is the "here" axis, and the x' axis is the "now" axis. That is, all events on the ct' axis take place at your position (the ct' axis is your worldline, after all), and all events on the x' axis take place simultaneously (they all have $t' = 0$).

Example (Length contraction): For both parts of this problem, use a Minkowski diagram where the axes in frame S are orthogonal.

(a) The relative speed of S' and S is v (along the x direction). A 1-meter stick (as measured by S') lies along the x' axis and is at rest in S'. S measures its length. What is the result?

(b) Do the same problem, except with S and S' interchanged.

Solution:

(a) Without loss of generality, pick the left end of the stick to be at the origin in S'. Then the worldlines of the two ends are shown in Fig. 10.23. The distance AC is 1 meter in S''s frame (because A and C are the endpoints of the stick at simultaneous times in the S' frame; this is how a length is measured). But one unit on the x' axis has length $\sqrt{1+\beta^2}/\sqrt{1-\beta^2}$. So this is the length on the paper of the segment AC.

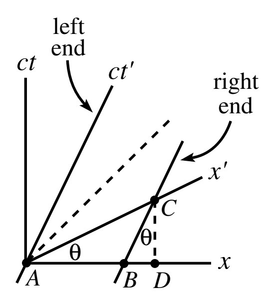


Figure 10.23

How does S measure the length of the stick? He simply writes down the x coordinates of the ends at simultaneous times (as measured by him, of course), and takes the difference. Let the time he makes the measurements be $t = 0$. Then he measures the ends to be at the points A and B.[19]

Now it's time to do some geometry. We have to find the length of segment AB in Fig. 10.23, given that segment AC has length $\sqrt{1+\beta^2}/\sqrt{1-\beta^2}$. We know that the primed axes are tilted at an angle θ, where $\tan\theta = \beta$. Therefore, $CD = (AC)\sin\theta$. And since $\angle BCD = \theta$, we have $BD = (CD)\tan\theta = (AC)\sin\theta\tan\theta$. Therefore (using $\tan\theta = \beta$),

$$\begin{aligned} AB &= AD - BD \\ &= (AC)\cos\theta - (AC)\sin\theta\tan\theta \\ &= (AC)\cos\theta(1-\tan^2\theta) \\ &= \sqrt{\frac{1+\beta^2}{1-\beta^2}}\frac{1}{\sqrt{1+\beta^2}}(1-\beta^2) \\ &= \sqrt{1-\beta^2}\,. \end{aligned} \tag{10.46}$$

Therefore, S sees the meter stick shortened by a factor $\sqrt{1-\beta^2}$, which is the standard length-contraction result.

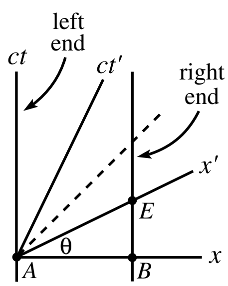


Figure 10.24

(b) The stick is now at rest in S, and we want to find the length that S' measures. Pick the left end of the stick to be at the origin in S. Then the worldlines of the two ends are shown in Fig. 10.24. The distance AB is 1 meter in S's frame.

In measuring the length of the stick, S' writes down the x' coordinates of the ends at simultaneous times (as measured by him), and takes the difference. Let the time he makes the measurements be $t' = 0$. Then he measures the ends to be at the points A and E.

Now we do the geometry, which is very easy in this case. The length of AE is simply $1/\cos\theta = \sqrt{1+\beta^2}$. Since one unit along the x' axis has length $\sqrt{1+\beta^2}/\sqrt{1-\beta^2}$, we see that AE is $\sqrt{1-\beta^2}$ of one unit, in S''s frame. Therefore, S' sees the meter stick shortened by a factor $\sqrt{1-\beta^2}$, which again is the standard length-contraction result.

10.6 The Doppler effect

10.6.1 Longitudinal Doppler effect

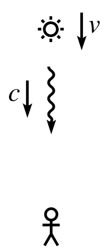


Figure 10.25

Consider a source that emits flashes at frequency f' (in its own frame), while moving directly toward you at speed v (see Fig. 10.25). With what frequency do the flashes hit your eye?

In these Doppler-effect problems, you must be careful to distinguish between the time at which an event *occurs* in your frame, and the time at which you see the

[19]If S measures the ends in a dramatic fashion by, say, blowing them up, then S' will see the right end blow up first (the event at B, which certainly has a negative t' coordinate, since it lies below the x' axis), and then a little while later he will see the left end blow up (the event at A, which has $t' = 0$). So S measures the ends at different times in S''s frame. This is part of the reason why S' should not be at all surprised that S's measurement is smaller than 1m.

event occur. This is one of the few situations where we will be concerned with the latter.

There are two effects contributing to the longitudinal Doppler effect. The first is relativistic time dilation; the light flashes occur at a smaller frequency in your frame. The second is the everyday Doppler effect (as with sound), arising from the motion of the source; successive flashes have a smaller distance (or larger, if v is negative) to travel to reach your eye. This effect increases the frequency at which the flashes hit your eye (or decreases it, if v is negative).

Let's now be quantitative and find the observed frequency. The time between emissions in the source's frame is $\Delta t' = 1/f'$. Therefore, the time between emissions in your frame is $\Delta t = \gamma \Delta t'$, by the usual time dilation. So the photons of one flash have traveled a distance (in your frame) of $c\Delta t = c\gamma\Delta t'$ by the time the next flash occurs. During this time between emissions, the source has traveled a distance $v\Delta t = v\gamma\Delta t'$ toward you in your frame. Hence, at the instant the next flash occurs, the photons of this next flash are a distance (in your frame) of $c\Delta t - v\Delta t = (c-v)\gamma\Delta t'$ behind the photons of the previous flash. This result holds for all adjacent flashes. The time, ΔT, between the arrivals of the flashes at your eye is $1/c$ times this distance. Therefore,

$$\Delta T = \frac{1}{c}(c-v)\gamma\Delta t' = \frac{1-\beta}{\sqrt{1-\beta^2}}\,\Delta t' = \sqrt{\frac{1-\beta}{1+\beta}}\,\Delta t', \qquad (10.47)$$

where $\beta = v/c$. Hence, the frequency you see is

$$f = \frac{1}{\Delta T} = \sqrt{\frac{1+\beta}{1-\beta}}\,f'. \qquad (10.48)$$

If $\beta > 0$ (that is, the source is moving toward you), then $f > f'$; the everyday Doppler effect wins out over the time-dilation effect. If $\beta < 0$ (that is, the source is moving away from you), then $f < f'$; both effects serve to decrease the frequency.

10.6.2 Transverse Doppler effect

Consider a source that emits flashes at frequency f' (in its own frame), while moving across your field of vision at speed v. There are two reasonable questions we may ask about the frequency you observe:

- **Case 1:**

  At the instant the source is at its closest approach to you, with what frequency do the flashes hit your eye?

- **Case 2:**

  When you see the source at its closest approach to you, with what frequency do the flashes hit your eye?

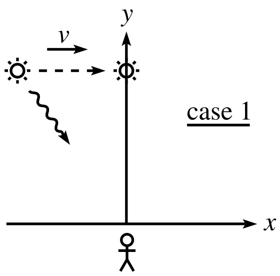


Figure 10.26

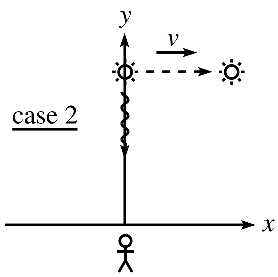


Figure 10.27

The difference between these two scenarios is shown in Fig. 10.26 and Fig. 10.27, where the source's motion is taken to be parallel to the x-axis.

In the first case, the photons you see must have been emitted at an earlier time, because the source will have moved during the non-zero time it takes the light to reach you (and in this scenario, we are dealing with photons that hit your eye *when* the source crosses the y-axis). You will therefore see the photon come in at an angle with respect to the y-axis.

In the second case, you will see the photons come in along the y-axis (by the definition of this scenario). At the instant you observe such a photon, the source will be at a position past the y-axis.

Let's now find the observed frequency in these two scenarios.

Case 1

Let your frame be S, and let the source's frame be S'. Consider the situation from S''s point of view. S' sees you moving across his field of vision at speed v. The relevant photons hit your eye when you cross the y'-axis of S''s frame. Because of the time dilation, your clock ticks slowly (in S''s frame) by a factor γ. That is, $\Delta t' = \gamma \Delta t$.

Now, S' sees you get hit by a flash every $\Delta t' = 1/f'$ seconds in his frame. (This is true because when you are very close to the y'-axis, all points on your path are essentially equidistant from the source. So we don't have to worry about any longitudinal effects.) This means that you get hit by a flash every $\Delta T = \Delta t'/\gamma = 1/(f'\gamma)$ seconds in your frame. Therefore, the frequency in your frame is

$$f = \frac{1}{\Delta T} = \gamma f' = \frac{f'}{\sqrt{1-\beta^2}}. \tag{10.49}$$

Hence, f is greater than f'; you see the flashes at a higher frequency than S' emits them.

Case 2

Again, let your frame be S, and let the source's frame be S'. Consider the situation from your point of view. Because of the time dilation, S''s clock ticks slowly (in your frame) by a factor of γ. That is, $\Delta t = \gamma \Delta t'$. When you see the source cross the y-axis, you therefore observe a frequency of

$$f = \frac{1}{\Delta T} = \frac{1}{\gamma \Delta t'} = \frac{f'}{\gamma} = f'\sqrt{1-\beta^2}. \tag{10.50}$$

(We have used the fact the relevant photons are emitted from points that are essentially equidistant from you. So they all travel the same distance, and we don't have to worry about any longitudinal effects.)

Hence, f is smaller than f'; you see the flashes at a lower frequency than S' emits them.

REMARKS:

1. When people talk about the "transverse Doppler effect", they sometimes mean Case 1, and they sometimes mean Case 2. The title "transverse Doppler" is ambiguous, so you should remember to state exactly which scenario you are talking about.

2. The two scenarios may alternatively be described, respectively (as you can convince yourself), in the following ways (see Fig. 10.28).

   - **Case 1:**
     A receiver moves with speed v in a circle around a source. What frequency does the receiver register?
   - **Case 2:**
     A source moves with speed v in a circle around a receiver. What frequency does the receiver register?

   These setups involve accelerating objects. We must therefore invoke the fact (which is verified very well experimentally) that if an inertial observer observes the clock of a moving object, then only the instantaneous speed of the object is important in computing the time dilation; the acceleration is irrelevant.[20]

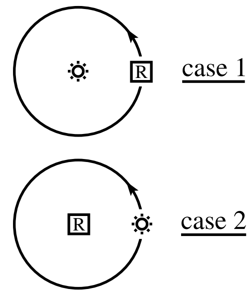


Figure 10.28

3. Beware of the following incorrect reasoning for Case 1, leading to an incorrect version of eq. (10.49). "S sees things in S' slowed down by a factor γ (that is, $\Delta t = \gamma \Delta t'$), by the usual time dilation effect. Hence, S sees the light flashing at a slower pace. Therefore, $f = f'/\gamma$." This reasoning puts the γ in the wrong place. Where is the error? The error lies in confusing the time at which an event *occurs* in S's frame, with the time at which S *sees* (with his eyes) the event occur. The flashes certainly *occur* at a lower frequency in S, but due to the motion of S' relative to S, it turns out that the pulses meet S's eye at a faster rate (because the source is moving slightly towards S while it is emitting the relevant photons). We'll let you work out the details of the situation from S's point of view.[21]

   Alternatively, the error can be stated as follows. The time dilation result $\Delta t = \gamma \Delta t'$ rests on the assumption that the $\Delta x'$ between the two events is 0. This applies fine to two emissions of light from the source. However, the two events in question are the absorption of two light pulses by your eye (which is moving in S'), so $\Delta t = \gamma \Delta t'$ is not applicable. Instead, $\Delta t' = \gamma \Delta t$ is the relevant result, valid when $\Delta x = 0$.

4. Other cases that are "inbetween" the longitudinal and transverse cases may also be considered. But they can get a little messy. ♣

10.7 Rapidity

Definition

Let us define the *rapidity*, ϕ, by

$$\tanh \phi \equiv \beta \equiv \frac{v}{c}\,. \tag{10.51}$$

[20]Of course, the acceleration is very important if things are considered from the accelerating object's point of view. But we'll wait until Chapter 13 on General Relativity to talk about this.

[21]This is a fun exercise, but it should convince you that it is much easier to look at things in the frame in which there are no longitudinal effects, as we did in our solutions above.

This quantity ϕ is very useful in relativity because many of our expressions take on a particularly nice form when written in terms of it.

Consider, for example, the velocity-addition formula. Let $\beta_1 = \tanh\phi_1$ and $\beta_2 = \tanh\phi_2$. Then if we add β_1 and β_2 using the velocity-addition formula, eq. (10.28), we obtain

$$\frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} = \frac{\tanh\phi_1 + \tanh\phi_2}{1 + \tanh\phi_1 \tanh\phi_2} = \tanh(\phi_1 + \phi_2), \tag{10.52}$$

where we have used the addition formula for $\tanh\phi$ (easily proven by writing things in terms of the exponentials $e^{\pm\phi}$). Therefore, while the velocities add in the strange manner of eq. (10.28), the rapidities add by standard addition.

The Lorentz transformations also take a nice form when written in terms of the rapidity. Our friendly γ factor can be written as

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\tanh^2\phi}} = \cosh\phi. \tag{10.53}$$

Also,

$$\gamma\beta \equiv \frac{\beta}{\sqrt{1-\beta^2}} = \frac{\tanh\phi}{\sqrt{1-\tanh^2\phi}} = \sinh\phi. \tag{10.54}$$

Therefore, the Lorentz transformations in matrix form, eqs. (10.18), become

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}. \tag{10.55}$$

This looks similar to a rotation in a plane, which is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}, \tag{10.56}$$

except that we now have hyperbolic trig functions instead of trig functions. The fact that the interval $s^2 \equiv c^2t^2 - x^2$ does not depend on the frame is clear from eq. (10.55); the cross terms in the squares cancel, and $\cosh^2\phi - \sinh^2\phi = 1$. (Compare with the invariance of $r^2 \equiv x^2 + y^2$ for rotations in a plane.)

The quantities associated with a Minkowski diagram also take a nice form when written in terms of the rapidity. The angle between the S and S' axes satisfies

$$\tan\theta = \beta = \tanh\phi. \tag{10.57}$$

And the size of one unit on the x' or ct' axes is, from eq. (10.43),

$$\sqrt{\frac{1+\beta^2}{1-\beta^2}} = \sqrt{\frac{1+\tanh^2\phi}{1-\tanh^2\phi}} = \sqrt{\cosh^2\phi + \sinh^2\phi} = \sqrt{\cosh 2\phi}. \tag{10.58}$$

For large ϕ, this is approximately equal to $e^\phi/\sqrt{2}$.

Physical meaning

The fact that the rapidity, ϕ, makes many of our formulas look nice and pretty is reason enough to consider it. But in addition, it turns out to have a very meaningful physical interpretation.

Consider the following situation. A spaceship is initially at rest in the lab frame. At a given instant, it starts to accelerate. Let a be the *proper acceleration*, which is defined as follows. Let t be the time coordinate in the spaceship's frame.[22] If the proper acceleration is a, then at time $t+dt$, the spaceship is moving at a speed $a\,dt$ relative to the frame it was in at time t. Equivalently, the astronaut feels a force of ma applied to his body by the spaceship; if he is standing on a scale, then the scale shows a reading of $F = ma$.

What is the relative speed of the spaceship and lab frame at (the spaceship's) time t?

We can answer this question by considering two nearby times and using the velocity-addition formula, eq. (10.28). From the definition of a, eq. (10.28) gives (with $v_1 \equiv v(t)$ and $v_2 \equiv a\,dt$)

$$v(t+dt) = \frac{v(t) + a\,dt}{1 + v(t)a\,dt/c^2}\,. \tag{10.59}$$

Expanding this to first order in dt yields[23]

$$\frac{dv}{dt} = a\left(1 - \frac{v^2}{c^2}\right). \tag{10.60}$$

Separating variables and integrating gives, using $1/(1-b^2) = 1/2(1-b) + 1/2(1+b)$,

$$\int_0^v \left(\frac{1}{1-v/c} + \frac{1}{1+v/c}\right) dv = 2a\int_0^t dt. \tag{10.61}$$

This yields $\ln\left((1+v/c)/(1-v/c)\right) = 2at/c$. Solving for v, we find[24]

$$v(t) = c\left(\frac{e^{2at/c}-1}{e^{2at/c}+1}\right) = c\tanh(at/c). \tag{10.62}$$

Note that for small a or small t (more precisely, if $at/c \ll 1$), we obtain $v(t) \approx at$, as we should. And for $at/c \gg 1$, we obtain $v(t) \approx c$, as we should.

If a happens to be a function of time, $a(t)$, then it is easy to see that the above derivation yields

$$v(t) = c\tanh\left(\frac{1}{c}\int_0^t a(t)\,dt\right). \tag{10.63}$$

[22] This frame is changing, of course, as time goes on (since the spaceship is accelerating). The time t is simply the spaceship's proper time. Normally, we would denote this by t', but we don't want to have to keep writing the primes over and over in the following calculation.

[23] Equivalently, just take the derivative of $(v+w)/(1+vw/c^2)$ with respect to w, and then set $w=0$.

[24] You can also use the result of Problem 14 to find $v(t)$. See the remark in the solution to that problem (after trying to solve it, of course!).

We therefore see that the rapidity, ϕ, as defined in eq. (10.51), is given by

$$\phi(t) \equiv \frac{1}{c}\int_0^t a(t)\,dt. \tag{10.64}$$

Note that whereas v has c as a limiting value, ϕ can become arbitrarily large. The ϕ associated with a given v is simply $1/mc$ times the time integral of the force (felt by the astronaut) needed to bring the astronaut up to speed v. By applying a force for an arbitrarily long time, we can make ϕ arbitrarily large.

The integral $\int a(t)\,dt$ may be described as the naive, incorrect speed. That is, it is the speed the astronaut might *think* he has, if he has his eyes closed and knows nothing about the theory of relativity. (And indeed, his thinking would be essentially correct for small speeds.) This quantity $\int a(t)\,dt$ seems like a reasonably physical thing, so if there is any justice in the world, $\int a(t)\,dt = \int F(t)\,dt/m$ should have *some* meaning. And indeed, although it doesn't equal v, all you have to do to get v is take a tanh and throw in some factors of c.

The fact that rapidities add via simple addition when using the velocity-addition formula, as we saw in eq. (10.52), is evident from eq. (10.63). There is really nothing more going on here than the fact that

$$\int_{t_0}^{t_2} a(t)\,dt = \int_{t_0}^{t_1} a(t)\,dt + \int_{t_1}^{t_2} a(t)\,dt. \tag{10.65}$$

To be explicit, let a force be applied from t_0 to t_1 that brings a mass up to speed $\beta_1 = \tanh\phi_1 = \tanh(\int_{t_0}^{t_1} a\,dt)$, and then let an additional force be applied from t_1 to t_2 that adds on an additional speed of $\beta_2 = \tanh\phi_2 = \tanh(\int_{t_1}^{t_2} a\,dt)$ (relative to the speed at t_1). Then the resulting speed may be looked at in two ways: (1) it is the result of relativistically adding the speeds $\beta_1 = \tanh\phi_1$ and $\beta_2 = \tanh\phi_2$, and (2) it is the result of applying the force from t_0 to t_2 (you get the same final speed, of course, whether or not you bother to record the speed along the way at t_1), which is $\beta = \tanh(\int_{t_0}^{t_2} adt) = \tanh(\phi_1 + \phi_2)$, where the last equality comes from the obvious statement, eq. (10.65). Therefore, the relativistic addition of $\tanh\phi_1$ and $\tanh\phi_2$ gives $\tanh(\phi_1 + \phi_2)$, as was to be shown.

10.8 Relativity without c

In Section 10.1, we introduced the two basic postulates of Special Relativity, namely the speed-of-light postulate and the relativity postulate. In Appendix I we show that together these imply that the coordinates in two frames must be related by the Lorentz transformations, eqs. (10.13).

It is interesting to see what happens if we relax these postulates. It is hard to imagine a "reasonable" universe where the relativity postulate does not hold, but it is easy to imagine a universe where the speed of light depends on the frame of reference. (Light could behave as sound does, for example.) So let's drop the speed-of-light postulate and see what we can say about the coordinate transformations between frames, using only the relativity postulate.

In Appendix I, the form of the transformations, just prior to invoking the speed-of-light postulate, was given in eq. (14.76) as

$$\begin{aligned} x &= A_v(x' + vt'), \\ t &= A_v\left(t' + \frac{1}{v}\Big(1 - \frac{1}{A_v^2}\Big)x'\right). \end{aligned} \tag{10.66}$$

We'll put a subscript on A in this section, to remind you of the v dependence. Can we say anything about A_v, without invoking the speed-of-light postulate? Indeed we can.

Define V_v by

$$\frac{1}{V_v^2} \equiv \frac{1}{v^2}\left(1 - \frac{1}{A_v^2}\right), \qquad \text{so that} \qquad A_v = \frac{1}{\sqrt{1 - v^2/V_v^2}}. \tag{10.67}$$

Eqs. (10.66) then become

$$\begin{aligned} x &= \frac{1}{\sqrt{1 - v^2/V_v^2}}(x' + vt'), \\ t &= \frac{1}{\sqrt{1 - v^2/V_v^2}}\left(\frac{v}{V_v^2}x' + t'\right). \end{aligned} \tag{10.68}$$

All we've done so far is make a change of variables. But we now make the following claim.

Claim 10.1 *V_v^2 is independent of v.*

Proof: As stated in the last remark in Section 10.3.1, we know that two successive applications of the transformations in eq. (10.68) must again yield a transformation of the same form.

Consider a transformation characterized by velocity v_1, and another one characterized by velocity v_2. For simplicity, define

$$\begin{aligned} V_1 &\equiv V_{v_1}, & V_2 &\equiv V_{v_2}, \\ \gamma_1 &\equiv \frac{1}{\sqrt{1 - v_1^2/V_1^2}}, & \gamma_2 &\equiv \frac{1}{\sqrt{1 - v_2^2/V_2^2}}. \end{aligned} \tag{10.69}$$

To calculate the composite transformation, it is easiest to use matrix notation. The composite transformation for the vector (x, t) is given by the matrix

$$\begin{pmatrix} \gamma_2 & \gamma_2 v_2 \\ \gamma_2 \frac{v_2}{V_2^2} & \gamma_2 \end{pmatrix}\begin{pmatrix} \gamma_1 & \gamma_1 v_1 \\ \gamma_1 \frac{v_1}{V_1^2} & \gamma_1 \end{pmatrix} = \gamma_1\gamma_2\begin{pmatrix} 1 + \frac{v_1 v_2}{V_1^2} & v_2 + v_2 \\ \frac{v_1}{V_1^2} + \frac{v_2}{V_2^2} & 1 + \frac{v_1 v_2}{V_2^2} \end{pmatrix}. \tag{10.70}$$

The composite transformation must still be of the form of eq. (10.68). But this implies that the upper-left and lower-right entries of the composite matrix must be equal. Therefore, $V_1^2 = V_2^2$. Since this holds for arbitrary v_1 and v_2, we see that V_v^2 must be a constant, independent of v. ■

Denote the constant value of V_v^2 by V^2. Then the coordinate transformations in eq. (10.68) become

$$\begin{aligned} x &= \frac{1}{\sqrt{1-v^2/V^2}}(x'+vt'), \\ t &= \frac{1}{\sqrt{1-v^2/V^2}}\left(t'+\frac{v}{V^2}x'\right). \end{aligned} \tag{10.71}$$

We have obtained this result using only the relativity postulate. These transformations have the same form as the Lorentz transformations, eqs. (10.13). The only extra information in eqs. (10.13) is that V is equal to the speed of light, c. It is remarkable that we were able to prove so much by using only the relativity postulate.

We can say a few more things. There are four possibilities for the value of V^2. Two of these, however, are not physical.

$V^2 = \infty$:

This gives the Galilean transformations.

$0 < V^2 < \infty$:

This gives transformations of the Lorentz type. V is the limiting speed of an object.

$V^2 = 0$:

This case is not physical, because any nonzero value of v will make the γ factor imaginary (and infinite).

$V^2 < 0$:

It turns out that this case is also not physical. You might be concerned that the square of V is less than zero, but this is fine because V appears in the transformations (10.71) only through its square (there's no need for V to actually be the speed of anything). The trouble is that the nature of eqs. (10.71) implies the possibility of time reversal. This opens the door for causality violation and all the other problems associated with time reversal. We therefore reject this case.

To be a little more explicit, define $b^2 \equiv -V^2$, where b is a positive number. Then eqs. (10.71) may be written in the form,

$$\begin{aligned} x &= x'\cos\theta + (bt')\sin\theta, \\ bt &= -x'\sin\theta + (bt')\cos\theta, \end{aligned} \tag{10.72}$$

where $\tan\theta = v/b$. This transformation is simply a rotation in the plane, through an angle of $-\theta$. We have the usual trig functions here, instead of the hyperbolic trig functions in the Lorentz transformations in eq. (10.55).

Eqs. (10.72) satisfy the requirement that the composition of two transformations is again a transformation of the same form. (Rotation by θ_1 and then by θ_2 yields a rotation by $\theta_1 + \theta_2$.) However, if the resulting rotation is through an angle, θ, that is greater than 90°, then we have a problem. The tangent of such an angle is negative. Hence, $\tan\theta = v/b$ implies that v is negative.

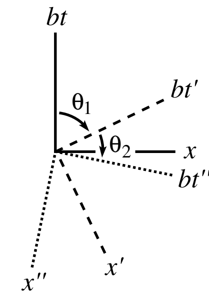


Figure 10.29

The situation is shown in Fig. 10.29. Frame S'' moves at speed $v_2 > 0$ with respect to frame S', which moves at speed $v_1 > 0$ with respect to frame S. But from the figure, we see that the origin of S'' moves at a *negative* speed with respect to S. Adding two positive velocities and obtaining a negative one is clearly absurd. Viewed in another way, we see that someone standing at the origin of S'' (that is, someone whose worldline is the t''-axis) will travel backwards in time in S. That is, he will die before he is born. This is not good.

Note that all of the finite $0 < V^2 < \infty$ possibilities are essentially the same. Any difference in the numerical definition of V can be absorbed into the definitions of the unit sizes for x and t. Given that V is finite, it has to be *something*, so it doesn't make sense to put much importance on its numerical value.

There is therefore only one decision to be made when constructing the spacetime structure of an (empty) universe. You just have to say whether V is finite or infinite (that is, whether the universe is Lorentzian or Galilean). Equivalently, all you have to say is whether or not there is an upper limit for the speed of any object. If there is, then you can simply postulate the existence of something that moves with this limiting speed. In other words, to create your universe, you simply have to say, "Let there be light."

10.9 Exercises

Section 10.2: The fundamental effects

1. **Effectively speed** c $*$

   A rocket flies between two planets that are one light-year apart. What should the rocket's speed be so that the time elapsed on the captain's watch is one year?

2. **A passing train**

   A train of length $15\,c$s moves at speed $3c/5$.[25] How much time does it take to pass a person standing on the ground? Solve this by working in the frame of the person, and then again by working in the frame of the train.

3. **The twin paradox** $*$

   Person A stays on the earth, while person B flies at speed v to a distant star, which is at rest relative to the earth, a distance L away. The star's clock is synchronized with A's, and B's clock is set to agree with A's when the trip begins.

   State what is wrong with the following twin-paradox reasoning, and give a quantitative correction to it:

   The earth-star distance in B's frame is L/γ, so B's clock advances by $L/\gamma v$ during the trip. But B sees the star's clock running slow by a factor γ, so the star's clock only advances by $(L/\gamma v)/\gamma$. The star's clock therefore reads $L/\gamma^2 v$ on B's arrival. In other words, people on the star (and hence the earth, since they're in the same frame) age more slowly than B.

4. **Coinciding runner** $**$

   A train of length L moves at speed $4c/5$ eastward, and a train of length $3L$ moves at speed $3c/5$ westward. How fast must someone run along the ground if he is to coincide with both the fronts-passing-each-other and backs-passing-each-other events?

5. **Another Train in Tunnel** $***$

   Consider the scenario of Problem 6, with the only change being that the train now has length $r\ell$, where r is some numerical factor.

   What is the largest value of r, in terms of v, for which it is possible for the bomb to not explode? (Verify that you obtain the same answer working in the frame of the train and working in the frame of the tunnel.)

[25] $1\,c$s equals $(3 \cdot 10^8\ \text{m/s})(1\,\text{s}) = 3 \cdot 10^8$ m; that is, one "light-second".

Section 10.3: The Lorentz transformations

6. **Pythagorean triples** $*$

   Let (a, b, h) be a pythagorean triplet. (We'll use h to denote the hypotenuse, instead of c, for obvious reasons.) Consider the relativistic addition or subtraction of the two speeds, $\beta_1 = a/h$ and $\beta_2 = b/h$. Show that the numerator and denominator of the result are members of another pythagorean triplet, and find the third member. What is the associated γ factor?

7. **Running on a train** $*$

   A train of length L moves at speed v_1 with respect to the ground. A passenger runs from the back of the train to the front, at speed v_2 with respect to the train. How much time does this take, as viewed by someone on the ground? Solve this problem in two different ways:

   (a) Find the relative speed of the passenger and the train (as viewed by someone on the ground), and then find the time it takes for the passenger to erase the initial "head start" that the front of the train had.

   (b) Find the time it takes on the passenger's clock, and then use time dilation to get the time elapsed on a ground clock.

8. **Velocity addition** $**$

   Derive the velocity-addition formula by using the following setup: A train of length L moves at speed a with respect to the ground, and a ball is thrown at speed b with respect to the train, from the back to the front. Let the speed of the ball with respect to the ground be V.

   Calculate the time of the ball's journey, as measured by an observer on the ground, in the following two different ways, and then set them equal to solve for V in terms of a and b.

   (a) First way: Find the relative speed of the ball and the train (as viewed by someone on the ground), and then find the time it takes for the ball to erase the initial "head start" that the the front of the train had.

   (b) Second way: Find the time it takes on the ball's clock, and then use time dilation to get the time elapsed on a ground clock.

9. **Bullets on a train** $**$

   A train moves at speed v. Bullets are successively fired at speed u (relative to the train) from the back of a train to the front. A new bullet is fired at the instant (as measured in the train frame) the previous bullet hits the front. In the frame of the ground, what fraction of the way along the train is a given bullet, at the instant (as measured in the ground frame) the next bullet is fired? What is the maximum number of balls an observer on the ground can see in flight at any given instant?

10. **Some γ's**

    Show that the relativistic addition (or subtraction) of the velocities u and v has a γ factor given by $\gamma = \gamma_u \gamma_v (1 \pm uv)$.

11. **Angled photon** $*$

    A photon moves at an angle θ with respect to the x'-axis in frame S'. Frame S' moves at speed v with respect to frame S (along the x' axis). Calculate the components of the photon's velocity in S, and verify that the speed is c.

Section 10.4: The spacetime interval

12. **Head start**

    Derive the Lv/c^2 "head-start" result (given in eq. 10.2) by making use of the invariant spacetime interval.

13. **Passing trains** $***$

    Train A of length L moves eastward at speed v, and train B of length $2L$ moves westward also at speed v. How much time does it take for the trains to pass each other (defined as the time between the front of B coinciding with the front of A, and the back of B coinciding with the back of A):

    (a) As viewed by A?

    (b) As viewed by B?

    (c) As viewed by the ground?

    (d) Verify that the invariant interval is indeed the same in all three frames.

Section 10.5: Minkowski diagrams

14. **Simultaneous claps** $***$

    With respect to the ground, A moves to the right at speed $c/\sqrt{3}$, and B moves to the left, also at speed $c/\sqrt{3}$. At the instant they are a distance d apart (as measured in the ground frame), A claps his hands. B then claps his hands simultaneously (as measured by B) with A's clap. A then claps his hands simultaneously (as measured by A) with B's clap. B then claps his hands simultaneously (as measured by B) with A's second clap, and so on. As measured in the ground frame, how far apart are A and B when A makes his nth clap? What is the answer if $c/\sqrt{3}$ is replaced by a general speed v?

15. **Train in tunnel** $**$

    Repeat Exercise 5, but now solve it by using a Minkowski diagram. (Do this from the point of view of the train, and also of the tunnel.)

10.10 Problems

Section 10.2: The fundamental effects

1. **No transverse length contraction** $*$

   Two meter sticks, A and B, move past each other as shown in Fig. 10.30. Stick A has paint brushes at its ends. Use this setup to show that in the frame of one stick, the other stick still looks 1 m long.

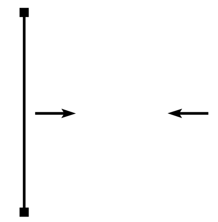


Figure 10.30

2. **Explaining time dilation** $**$

   A spaceship gathers speed and achieves a speed v with respect to two stars (which are at rest with respect to each other, a distance L apart, with synchronized clocks). After it has achieved speed v, the spaceship flies past star A and synchronizes its clock with the A's (they both set their clocks to zero). It then flies past star B and compares its clock to B's.

   We know that when the spaceship reaches B, B's clock will simply read L/v. And the spaceship's clock will read $L/\gamma v$, since it runs slow by a factor of γ, compared to the stars' clocks.

   How would someone on the spaceship quantitatively explain to you why B's clock reads L/v (which is *more* than its own $L/\gamma v$), considering that the spaceship sees B's clock running *slow*?

3. **Explaining Length contraction** $**$

   Two bombs lie on a train platform a distance L apart. As a train passes by at speed v, the bombs explode simultaneously (in the platform frame) and leave marks on the train. Due to the length contraction of the train, we know that the marks on the train will be a distance γL apart when viewed in the train's frame (since this distance is what is length-contracted down to the given distance L in the platform frame).

   How would someone on the train quantitatively explain to you why the marks are γL apart, considering that the bombs are only a distance L/γ apart in the train frame?

4. **A passing stick** $**$

   A stick of length L moves past you at speed v. There is a time interval between the front end coinciding with you and the back end coinciding with you. What is this time interval in

   (a) your frame? (Calculate this by working in your frame.)

   (b) your frame? (Work in the stick's frame.)

   (c) the stick's frame? (Work in your frame. This is the tricky one.)

   (d) the stick's frame? (Work in the stick's frame.)

5. **Rotated square** $*$

   A square with side L flies by you at speed v, in a direction parallel to two of its sides. You stand in the plane of the square. When you see the square at its nearest point to you, show that it *looks* to you like it is simply rotated, instead of contracted. (Assume that L is small compared to the distance between you and the square.)

6. **Train in Tunnel** $**$

   A train and a tunnel both have proper lengths ℓ. The train speeds toward the tunnel, with speed v. A bomb is located at the front of the train. The bomb is designed to explode when the front of the train passes the far end of the tunnel. A deactivation sensor is located at the back of the train. When the back of the train passes the near end of the tunnel, this sensor tells the bomb to disarm itself. Does the bomb explode?

7. **Seeing behind the stick** $**$

   A ruler is positioned perpendicular to a wall. A stick of length ℓ flies by at speed v. It travels in front of the ruler, so that it obscures part of the ruler from your view. When the stick hits the wall it stops.

   In your reference frame, the stick is shorter than ℓ. Therefore, right before it hits the wall, you will be able to see a mark on the ruler which is less than ℓ units from the wall (see Fig. 10.31).

   But in the stick's frame, the marks on the ruler are closer together. Therefore, when the wall hits the stick, the closest mark on the ruler to the wall that you can see is greater than ℓ units (see Fig. 10.31).

   Which view is correct (and what is wrong with the incorrect one)?

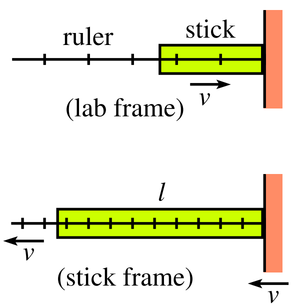


Figure 10.31

8. **Cookie cutter** $**$

   Cookie dough (chocolate chip, of course) lies on a conveyor belt which moves along at speed v. A circular stamp stamps out cookies as the dough rushes by beneath it. When the conveyor belt is brought to a stop, what is the shape of the cookies? (Are they squashed or stretched in the direction of the belt, or are they circular?)

9. **The twin paradox** $**$

   Consider the usual twin paradox: Person A stays on the earth, while person B flies quickly to a distant star and back. B is younger than A when they meet up again. The paradox is that one might argue that although A will see B's clock moving slowly, B will also see A's clock moving slowly, so A should be younger than B.

   There are many resolutions to this 'paradox'. Perform the following one: Let B's path to the distant star be lined with a wire that periodically zaps B as he flies along (see Fig. 10.32). Let this be accomplished by having every point in the wire emit a 'zap' simultaneously in A's frame. Let t_A be the time between

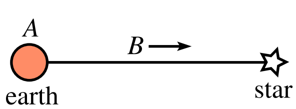


Figure 10.32

zaps in A's frame. Find the time between zaps in B's frame, and then use the fact that both A and B agree on the total number of times B gets zapped.

Section 10.3: The Lorentz transformations

10. **Throwing on a train**

    A train with proper length ℓ' moves at speed $(4/5)c$ with respect to the ground. A ball is thrown with speed $(1/2)c$ (with respect to the train) from the back of the train to the front. How much time does the ball spend in the air, as viewed by someone on the ground?

11. **A bunch of L.T.'s** $*$

    Verify that the values in the table in the example of section 10.5 satisfy the Lorentz transformations between the six pairs of frames, namely AB, AC, AD, BC, BD, CD (see Fig. 10.33). (On second thought, just do it for a couple pairs; this can get tedious.)

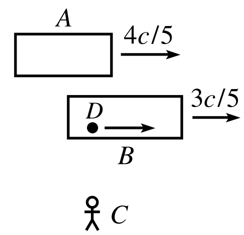


Figure 10.33

12. **A new frame**

    In one reference frame, Event 1 happens at $x = 0$, $ct = 0$, and Event 2 happens at $x = 2$, $ct = 1$. Find a frame where the two events are simultaneous.

13. **Velocity Addition from scratch** $***$

    A ball moves at speed v_1 with respect to a train. The train moves at speed v_2 with respect to the ground. What is the speed of the ball with respect to the ground?

    Solve this problem (i.e., derive the velocity addition formula) in the following way. (Do not use any time dilation, length contraction, etc. Use only the fact that the speed of light is the same in any inertial frame.) Let the ball be thrown from the back of the train. At this instant, a photon is released next to it (see Fig. 10.34). The photon heads to the front of the train, bounces off a mirror, heads back, and eventually runs into the ball. In both frames, find the fraction of the way along the train the meeting occurs, and then equate these fractions.

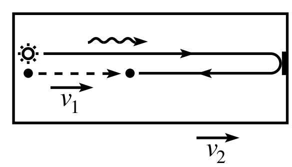


Figure 10.34

14. **Many velocity additions** $**$

    An object moves at speed $v_1/c \equiv \beta_1$ with respect to S_1, which moves at speed β_2 with respect to S_2, which moves at speed β_3 with respect to S_3, and so on, until finally S_{N-1} moves at speed β_N with respect to S_N (see Fig. 10.35). Show that the speed, $\beta_{(N)}$, of the object with respect to S_N can be written as

$$\beta_{(N)} = \frac{P_N^+ - P_N^-}{P_N^+ + P_N^-}, \qquad (10.73)$$

where

$$P_N^+ \equiv \prod_{i=1}^{N}(1+\beta_i), \qquad \text{and} \qquad P_N^- \equiv \prod_{i=1}^{N}(1-\beta_i). \qquad (10.74)$$

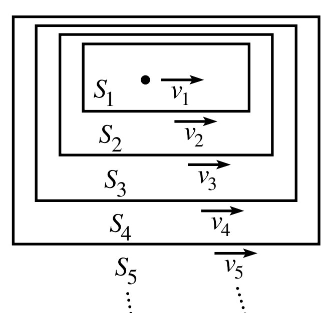


Figure 10.35

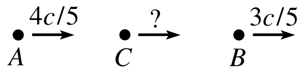


Figure 10.36

15. **The "midpoint"**

A and B travel at $4c/5$ and $3c/5$, respectively, as shown in Fig. 10.36. Let C move at the speed such that he sees A and B approaching him at the same rate, u. What is u?

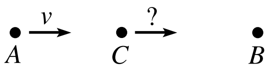


Figure 10.37

16. **Another "midpoint"** $*$

A moves at speed v, and B is at rest, as shown in Fig. 10.37. How fast must C travel, so that she sees A and B approaching her at the same rate?

In the lab frame (B's frame), what is the ratio between the distances CB and AC? (The answer to this is very nice and clean. Can you think of a simple intuitive explanation for the result?)

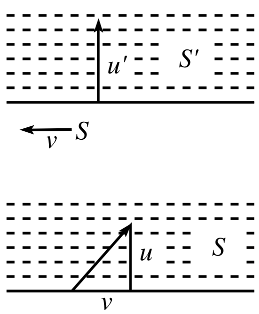


Figure 10.38

17. **Transverse velocity addition** $**$

Derive the transverse velocity addition formula (eq. 10.35), for the special case $u'_x = 0$, in the following way.

In frame S', a particle moves with speed u' in the y'-direction. Frame S moves to the left with speed v, so that the situation in S looks like that in Fig. 10.38, with the y-speed now u. Consider a series of equally spaced dotted lines, as shown. The ratio of times between passes of the dotted lines in frames S and S' is $T_S/T_{S'} = (1/u)/(1/u') = u'/u$.

Assume that a clock on the particle shows a time T between successive passes of the dotted lines. Derive another expression for the ratio $T_S/T_{S'}$, by using time dilation arguments, and then equate the two ratios to solve for u in terms of u' and v.

18. **Equal transverse speeds** $*$

An object moves with velocity (u_x, u_y). You move with speed v in the x-direction. What must v be so that you also see the object moving with speed u_y in your y-direction?

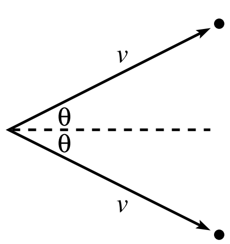


Figure 10.39

19. **Relative speed** $*$

In the lab frame, two particles move with speed v along the paths shown in Fig. 10.39. The angle between the trajectories is 2θ. What is the speed of one particle, as viewed by the other?

(This problem is posed again in chapter 12, where it can be solved in a very simple way, using 4-vectors.)

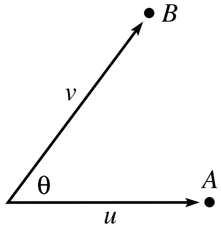


Figure 10.40

20. **Another relative speed** $**$

In the lab frame, two particles, A and B, move with speeds u and v along the paths shown in Fig. 10.40. The angle between the trajectories is θ. What is the speed of one particle, as viewed by the other?

(This problem is posed again in chapter 12, where it can be solved in a very simple way, using 4-vectors.)

21. **Modified twin paradox** ∗∗∗

Consider the following variation of the twin paradox, described in the Minkowski diagram in Fig. 10.41. A, B, and C each have a clock. In A's reference frame, B moves to the right with speed v. When B passes A, they both set their clocks to zero. Also, in A's reference frame, C moves to the left with speed v. When B and C pass each other, C sets his clock to read the same as B's. Finally, when C passes A, they compare the readings on their clocks. At this event, let A's clock read T_A, and let C's clock read T_C.

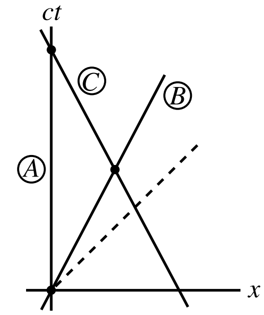


Figure 10.41

(a) Working in A's frame, show that $T_C = T_A/\gamma$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

(b) Working in B's frame, show again that $T_C = T_A/\gamma$.

(c) Working in C's frame, show again that $T_C = T_A/\gamma$.

(This form of the twin paradox loses a bit of the punch of the usual statement of the problem, since we don't compare the ages of the same two people before and after a journey. But on the other hand, this version of the problem does not involve accelerations.)

Section 10.5: Minkowski diagrams

22. **Minkowski diagram units** ∗

Consider the Minkowski diagram in Fig. 10.42. In frame S, the hyperbola $c^2t^2 - x^2 = 1$ is drawn. Also drawn are the axes of frame S', which moves past S with speed v. Use the invariance of the interval $s^2 = c^2t^2 - x^2$ to derive the ratio of the unit sizes on the ct' and ct axes (and check the result with eq. (10.43)).

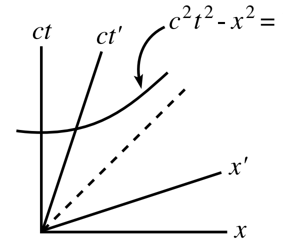


Figure 10.42

23. **Velocity Addition via Minkowski** ∗

An object moves at speed v_1 with respect to frame S'. Frame S' moves at speed v_2 with respect to frame S. (in the same direction as the motion of the object). What is the speed, u, of the object with respect to frame S?

Solve this problem (i.e., derive the velocity addition formula) by drawing a Minkowski diagram with frames S and S', drawing the worldline of the object, and doing a little geometry.

24. **Acceleration and redshift** ∗∗∗

Use a Minkowski diagram to do the following problem:

Two people stand a distance d apart. They simultaneously start accelerating in the same direction (along the line between them), each with proper acceleration a. At the instant they start to move, how fast does each person see the other person's clock tick?

Figure 10.43

25. **Break or not break?** ∗∗∗

Two spaceships float in space and are at rest relative to each other. They are connected by a string (see Fig. 10.43). The string is strong, but it cannot withstand an arbitrary amount of stretching. At a given instant, the spaceships simultaneously start accelerating (along the direction of the line between them) with the same acceleration. (Assume they bought identical engines from the same store, and they put them on the same setting.)

Will the string eventually break?

Section 10.7: Rapidity

26. **Successive Lorentz transformations**

The Lorentz transformation in eq. (10.55) may be written in matrix form as

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}. \tag{10.75}$$

Show that by applying an L.T. with $v_1 = \tanh\phi_1$, and then another L.T. with $v_2 = \tanh\phi_2$, you do indeed obtain the L.T. with $v = \tanh(\phi_1 + \phi_2)$.

27. **Accelerator's time** $*$

A spaceship is initially at rest in the lab frame. At a given instant, it starts to accelerate. Let this happen when the lab clock reads $t = 0$ and the spaceship clock reads $t' = 0$. The proper acceleration is a. (That is, at time $t' + dt'$, the spaceship is moving at a speed $a\,dt'$ relative to the frame it was in at time t'.) Later on, a person in the lab measures t and t'. What is the relation between them?

10.11 Solutions

1. **No transverse length contraction**

   Assume that the paint brushes point toward stick B, so that if B is long enough, or if A short enough, then the brushes will leave marks on B.

   We will invoke the postulate that the frames of the sticks are equivalent. That is, if A sees B shorter than (or longer than) (or equal to) itself, then B also sees A shorter than (or longer than) (or equal to) itself. The contraction factor must be the same in going each way between the frames.

   Let's say A sees B short; then B won't reach out to the ends of A, so there will be no marks on B. But in this case, B must also see A short; so there *will* be marks on B (see Fig. 10.44). This is a contradiction.

   Likewise, if we assume that A sees B long, we also reach a contradiction. Hence, they each must see the other stick as 1m long.

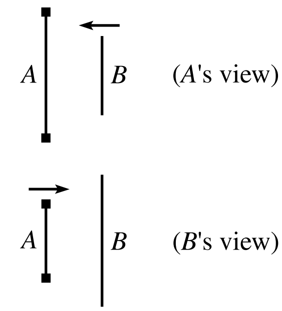


Figure 10.44

2. **Explaining time dilation**

   The key to the explanation is the "head start" that B's clock has over A's clock (in the spaceship frame). From eq. (10.2), we know that in the spaceship's frame, B's clock reads Lv/c^2 more than A's. (The two stars may be considered to be at the ends of the train in the example in Section 10.2.1.)

   So, what a person in the spaceship says is this: "My clock advances by $L/\gamma v$ during the whole trip. I see B's clock running slow by a factor γ; therefore I see B's clock advance by only $(L/\gamma v)/\gamma = L/\gamma^2 v$. However, B's clock started not at zero but at Lv/c^2. Therefore, the final reading on B's clock when I get there is

$$\frac{Lv}{c^2} + \frac{L}{\gamma^2 v} = \frac{L}{v}\left(\frac{v^2}{c^2} + \left(1 - \frac{v^2}{c^2}\right)\right) = \frac{L}{v}\,, \qquad (10.76)$$

   as it should be."

3. **Explaining Length contraction**

   The resolution to the "paradox" is that the explosions do not occur simultaneously in the train frame. As the platform rushes past the train, the rear bomb explodes before the front bomb explodes. The front bomb therefore gets to travel further before it explodes and leaves its mark (thus making the distance between the marks larger than one might naively expect). Let's be quantitative about this.

   If both bombs contain clocks that read a time t when they explode (they are synchronized in the ground frame), then in the frame of the train, the front bomb's clock reads only $t - Lv/c^2$ (the "head start" result from eq. (10.2)) when the rear bomb explodes when showing a time t. The front bomb's clock must therefore advance by a time of Lv/c^2 before it explodes. Since the train sees the platform's clocks running slow by a factor γ, we conclude that in the frame of the train, the front bomb explodes a time of $Lv\gamma/c^2$ after the rear bomb explodes. During this time of $Lv\gamma/c^2$, the platform moves a distance $(Lv\gamma/c^2)v$ relative to the train.

   So, what a person on the train says is this: "Due to length contraction, the distance between the bombs is L/γ. The front bomb is therefore a distance L/γ ahead of the rear bomb when the latter explodes. The front bomb then travels a further distance of $L\gamma v^2/c^2$ by the time it explodes, at which point it is a distance of

$$\frac{L}{\gamma} + \frac{L\gamma v^2}{c^2} = L\gamma\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) = L\gamma\left(\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right) = L\gamma \qquad (10.77)$$

   ahead of the rear bomb's mark, as we wanted to show."

4. **A passing stick**

   (a) The stick has length L/γ in your frame. It moves with speed v. Therefore, the time taken in your frame to cover the distance of L/γ is $L/\gamma v$.

   (b) The stick sees you fly by at speed v. The stick has length L in its own frame, so the time elapsed in the stick frame is simply L/v. During this time, the stick will see the watch on your wrist run slow, by a factor γ. Therefore, a time of $L/\gamma v$ elapses on your watch, in agreement with part (a).

   REMARK: Logically, the two solutions (a) and (b) differ in that one uses length contraction and the other uses time dilation. Mathematically, they differ simply in the order in which the divisions by γ and v occur. ♣

   (c) You see the rear clock on the train showing a time of Lv/c^2 more than the front clock. In addition to this head start, more time will of course elapse on the rear clock by the time it reaches you. The time in your frame is $L/\gamma v$ (since the train has length L/γ in your frame). But the train's clocks run slow, so a time of only $L/\gamma^2 v$ will elapse on the rear clock by the time it reaches you. The total extra time the rear clock shows is

$$\frac{Lv}{c^2} + \frac{L}{\gamma^2 v} = \frac{L}{v}\left(\frac{v^2}{c^2} + \left(1 - \frac{v^2}{c^2}\right)\right) = \frac{L}{v}, \tag{10.78}$$

   in agreement with the quick calculation below in part (d).

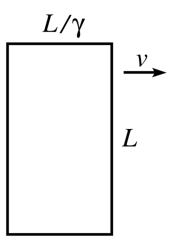


Figure 10.45

   (d) The stick sees you fly by at speed v. The stick has length L in its own frame, so the time elapsed in the stick frame is simply L/v.

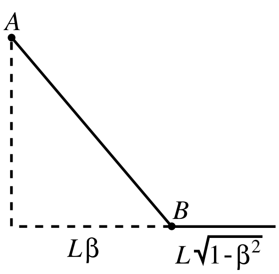


Figure 10.46

5. **Rotated square**

   Fig. 10.45 shows a top view of the position of the square at the instant (in your frame) when it is closest to you. Its length is contracted along the direction of motion, so it takes the shape of a rectangle with sides L and L/γ (with $\gamma = 1/\sqrt{1-\beta^2}$). That's what the shape *is* in your frame (where *is*-ness is defined by where all the points of an object are at simultaneous times). But what does the square *look* like to you. That is, what is the nature of the photons hitting your eye at a given instant?[26]

   Photons from the far side of the square have to travel an extra distance L to get to your eye, compared to ones from the near side. So they need an extra time L/c of flight. During this time L/c, the square moves a distance $Lv/c = L\beta$ sideways. Therefore, referring to Fig. 10.46, a photon emitted at point A reaches your eye at the same time as a photon emitted from point B.

   This means that the trailing side of length L takes up a distance $L\beta$ across your field of vision, while the near side takes up a distance $L\sqrt{1-\beta^2}$ across your field of vision. But this is exactly what a rotated square of side L looks like, as shown in Fig. 10.47 (where the angle of rotation satisfies $\sin\theta = \beta$).

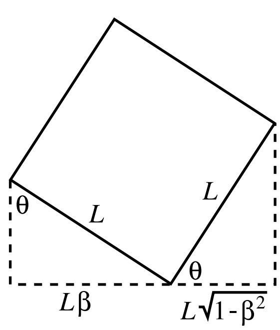


Figure 10.47

6. **Train in Tunnel**

   Yes, the bomb explodes. This is obvious in the frame of the train (see Fig. 10.48). In this frame, the train has length ℓ, and the tunnel has length $\ell\sqrt{1-v^2/c^2} \equiv \ell/\gamma$,

(train frame)

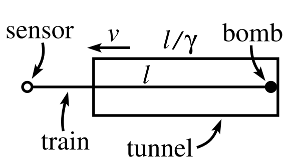


Figure 10.48

[26] In relativity problems, we virtually always subtract off the time it takes light to travel from the object to your eye (i.e., we find out what *is*). As with the Doppler effect in the previous problem, this problem is one of the few exceptions where we actually want to determine what your eye registers.

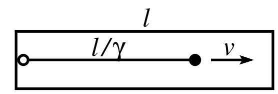


Figure 10.49

which is less than ℓ. So the far end of the tunnel passes the front of the train before the near end passes the back, and the bomb explodes.

We may, however, look at this from the tunnel's point of view (see Fig. 10.49). Here the tunnel has length ℓ, and the train has length ℓ/γ, which is less than ℓ. Therefore, the deactivation device gets triggered *before* the front of the train passes the far end of the tunnel, so one might think that the bomb does *not* explode. We appear to have a paradox.

The resolution of this paradox is that the deactivation device cannot instantaneously tell the bomb to deactivate itself. It takes a finite time for the signal to travel the length of the train from the sensor to the bomb. This transmission time makes it impossible for the deactivation signal to get to the bomb before the bomb gets to the far end of the tunnel, no matter how fast the train is moving. Let's prove this.

Clearly, the signal has the best chance of winning this 'race' if it has a speed c. So let us assume this is the case. Then it is clear that the signal gets to the bomb before the bomb gets to the far end of the tunnel if and only if a light pulse emitted from the near end of the tunnel (at the instant the back of the train goes by) reaches the far end of the tunnel before the front of the train does.

The former takes a time ℓ/c. The latter takes a time $\ell(1-1/\gamma)/v$ (since the front of the train is already a distance ℓ/γ through the tunnel). So if the bomb is to not explode, we must have $\ell/c < \ell(1-1/\gamma)/v$, or $\beta < 1-\sqrt{1-\beta^2}$. So $\sqrt{1-\beta^2} < 1-\beta$, or $\sqrt{1+\beta} < \sqrt{1-\beta}$. This can never be true. The signal always arrives too late, and the bomb always explodes.

7. **Seeing behind the stick**

The first reasoning is correct; you will be able to see a mark on the ruler which is less than ℓ units from the wall. The whole point of this problem (as with many others) is that signals do not travel instantaneously; the back of the stick does not know that the front of the stick has hit the wall until a finite time has passed. Let's be quantitative. What is the closest mark to the wall you can see?

Consider your reference frame. The stick has length ℓ/γ. Therefore, when the stick hits the wall, you can see mark a distance ℓ/γ from the wall. You will, however, be able to see a mark even closer to the wall, because the back end of the stick will keep moving forward, since it doesn't yet know that the front end has hit the wall. The signal takes time to travel.

Let's assume that the signal traveling along the stick moves with speed c. (One can work with a general speed u. But a speed c is simpler, and it has all the important features.) Where will the signal reach the back end? Starting from the time the stick hits the wall, the signal travels backward from the wall at speed c, and the back end travels forward at speed v (from a point ℓ/γ away from the wall). So they meet at a distance $(\ell/\gamma)c/(c+v)$ from the wall. So the closest point to the wall you can see is the

$$\frac{\ell}{\gamma}\frac{1}{1+\beta} = \ell\sqrt{\frac{1-\beta}{1+\beta}} \tag{10.79}$$

mark on the ruler.

Now consider the stick's reference frame. The wall is moving toward it at speed v. After the wall hits the end, the signal moves to the left with speed c, and the wall keeps moving to the left at speed v. Where is the wall when the signal reaches the left end? The wall travels v/c as fast as the signal, so it has traveled a distance $\ell v/c$ in this time. So it is $\ell(1-v/c)$ away from the left end of the stick. In the stick's frame, this corresponds to a distance $\gamma\ell(1 - v/c)$ on the ruler. So the left end of the stick is at the

$$\ell\gamma(1-\beta) = \ell\sqrt{\frac{1-\beta}{1+\beta}} \tag{10.80}$$

mark on the ruler. This agrees with eq. (10.79).

8. **Cookie cutter**

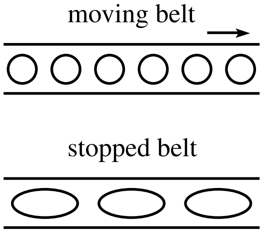


Figure 10.50

Let the diameter of the cookie cutter be d. In the frame of the cookie cutter, the dough is contracted, so the diameter d corresponds to a distance larger than d (namely γd) in the dough's frame. So when the conveyor belt stops, the cookies are stretched out by a factor γ in the direction of the belt (see Fig. 10.50).[27]

However, one might give the following reasoning. In the frame of the dough, the cookie cutter appears to be contracted in the direction of motion; it appears to have length d/γ. So in the frame of the dough, the cookies are squashed by a factor γ in the direction of the belt. So when the conveyor belt stops, the cookies are squashed by a factor γ.

Which reasoning is correct? The first is; the cookies are indeed stretched out. The fallacy in the second reasoning is that the various parts of the cookie cutter do *not* strike the dough simultaneously (as observed by the dough). These events are simultaneous in the cookie cutter's frame, but not in the dough's frame. What the dough sees is this: The cutter moves to the right. The left side of the cutter stamps the dough, then nearby parts of the cutter stamp it, and so on, until finally the right side of the cutter stamps the dough. But by this time the front of the cutter has moved further to the right. So the cookie turns out to be longer than d.

REMARK: The above argument makes it believable that there is no paradox here, but let's work things out quantitatively in the frame of the dough, just to be sure. There are various ways to do this. We'll let you work it out using the Lorentz transformations. We'll do it here using the standard procedure of sending out light signals to coordinate the timing of events.

In the frame of the cutter, let a light source be placed at the center of the circular stamp. It sends out a flash. When the flash reaches a point on the stamp, it strikes the dough. With this setup, all points on the cutter do their stamping simultaneously in the cutter's frame.

What does the dough see? For simplicity, we'll just look at the left and right edges of the cutter. The cutter moves to the right, and the flash is emitted. The relative speed between the flash and the left edge of the cutter is $c + v$. The relative speed between the flash and the right edge of the cutter is $c - v$. So the difference in time (in the dough's frame) between the left and right stamps is

$$\Delta t_{\text{dough}} = \frac{d'/2}{c-v} - \frac{d'/2}{c+v} = d'\gamma^2\frac{v}{c^2}, \tag{10.81}$$

where $d' = d/\gamma$ is the length of the cutter in the dough's frame. The distance the right edge moved in this time is $v\Delta t_{\text{dough}} = d'\gamma^2 v^2/c^2 = d'\gamma^2\beta^2$. The right edge had a head start of d' over the left edge, so the total length of the cookie is $d' + d'\gamma^2\beta^2 = \gamma^2 d' = \gamma d$, as desired. ♣

9. **The twin paradox**

The main point is that because the zaps occur simultaneously in A's frame, they do *not* occur simultaneously in B's frame. The zaps further ahead of B occur earlier (as

[27] The shape is an ellipse, since that's what a stretched-out circle is. The eccentricity of an ellipse is the focal distance divided by the semi-major axis length. We'll let you show that this equals $\beta \equiv v/c$ here.

the reader can show), so there is less time between zaps in B's frame than one might think (since B is constantly moving toward the zaps ahead, which happen earlier).

Consider two successive zapping events. The Δx_B between them is zero, so the Lorentz transformation $\Delta t_A = \gamma(\Delta t_B - v\Delta x_B/c^2)$ gives

$$\Delta t_B = \Delta t_A/\gamma. \tag{10.82}$$

So if t_A is the time between zaps in A's frame, then $t_B = t_A/\gamma$ is the time between zaps in B's frame. (This is the usual time dilation result.)

Let N be the total number of zaps B gets. Then the total time in A's frame is $T_A = Nt_A$, while the total time in B's frame is $T_B = Nt_B = N(t_A/\gamma)$. Therefore,

$$T_B = \frac{T_A}{\gamma}. \tag{10.83}$$

So B is younger.

This can all be seen quite clearly if we draw a Minkowski diagram. Fig. 10.51 shows our situation where the zaps occur simultaneously in A's frame. We know from eq. (10.43) of section 6 that the unit size on B's ct axis on the paper is $\sqrt{(1+\beta^2)/(1-\beta^2)}$ times the unit size of A's ct axis. Since the two pieces of B's ct axis are $\sqrt{1+\beta^2}$ times as long as the corresponding piece of A's ct axis, we see that only $\sqrt{1-\beta^2}$ as many time units fit on B's worldline as fit on A's worldline. So B is younger.

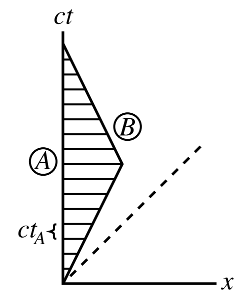


Figure 10.51

REMARK: Eq. (10.83) is the result we wanted to show. But the obvious question is: Why doesn't it work the other way around? That is, if we let A be zapped by zaps along a wire that occur simultaneously in B's frame, why don't we conclude that A is younger? The answer is that there is no *one* B frame; B has a different frame going out and coming in.

As usual, the best way to see what is going on is to draw a Minkowski diagram. Fig. 10.52 shows the situation where the zaps occur simultaneously in B's frame. The lines of simultaneity (as viewed by B) are tilted one way on the trip outward, and the other way on the trip back. The result is that A gets zapped frequently for a while, then no zaps occur for a while, then he gets zapped frequently again. The overall result, as we will now show, is that more time elapses in A's frame than in B's frame.

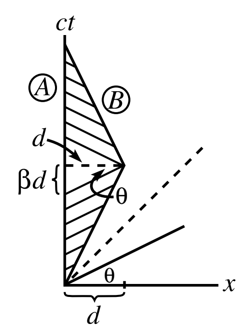


Figure 10.52

Let the distant star be a distance d from the earth, in A's frame. Let the zaps occur at intervals Δt_B in B's frame. Then they occur at intervals $\Delta t_B/\gamma$ in A'a frame. (The same factor applies to both parts of the journey.) Let there be N total zaps during the journey. Then the total times registered by B and A are, respectively, $T_B = N\Delta t_B$, and $T_A = N(\Delta t_B/\gamma) + t$, where t is the time where no zaps occur in A's frame (in the middle of the journey). The distance between the earth and star in B's frame is $vT_B/2$. But we know that this distance also equals d/γ. Therefore, $N\Delta t_B = T_B = 2d/(\gamma v)$. So we have

$$\begin{aligned} T_B &= \frac{2d}{\gamma v}, \\ T_A &= \frac{2d}{\gamma^2 v} + t. \end{aligned} \tag{10.84}$$

We must now calculate t. Since the slopes of B's lines of simultaneity in the figure are $\pm\beta$, we see that $ct = 2d\tan\theta = 2d\beta$. Therefore,

$$\begin{aligned} T_B &= \frac{2d}{\gamma v}, \\ T_A &= \frac{2d}{\gamma^2 v} + \frac{2d\beta}{c} = \frac{2d}{v}. \end{aligned} \tag{10.85}$$

Hence, $T_B = T_A/\gamma$.

(We know of course, without doing any calculations, that $T_A = 2d/v$. But it is reassuring to add up the times when A is getting zapped and when he is not getting zapped, to show that we still get the same answer.) ♣

10. **Throwing on a train**

First solution:

Let the train be the S' frame, and the ground be the S frame. Then the Lorentz transformations are

$$\begin{aligned} x &= \gamma(x' + vt'), \\ t &= \gamma(t' + vx'/c^2), \end{aligned} \tag{10.86}$$

where $v = (4/5)c$ and $\gamma = 1/\sqrt{1-(4/5)^2} = 5/3$. Let the throwing of the ball be the origin in both frames. In the S' frame, the ball hits the front of the train at $t' = \ell'/(c/2)$. And $x' = \ell'$, of course. Plugging $(x', t') = (\ell', 2\ell'/c)$ into the second of eqs. (10.86) gives

$$t = \frac{5}{3}\left(\frac{2\ell'}{c} + \frac{4\ell'}{5c}\right) = \frac{14}{3}\frac{\ell'}{c}. \tag{10.87}$$

Second Solution:

Working in the frame of the ground, we will find both the position of the ball and the position of the front of the train as functions of t. We will then find the value of t that makes these positions equal.

From the velocity addition formula, eq. (10.28), the speed of the ball, as viewed from the ground, is (with $v = (4/5)c$ and $v_b = (1/2)c$)

$$u = \frac{v + v_b}{1 + vv_b/c^2} = \frac{13c}{14}. \tag{10.88}$$

So the position of the ball at time t is $x = ut = 13ct/14$.

An observer on the ground sees a train with length $\ell = \ell'/\gamma = 3\ell'/5$. So the front starts with position $3\ell'/5$ and moves forward at speed $v = (4/5)c$. So the position of the front of the train at time t is $x = 3\ell'/5 + 4ct/5$.

Equating the position of the ball with the position of the front of the train gives $13ct/14 = 3\ell'/5 + 4ct/5$. Therefore, $t = (14/3)(\ell'/c)$, as above.

11. **A bunch of L.T.'s**

The relative speeds of frames for the pairs AB, AC, AD, BC, BD, CD, and their associated γ factors are (using the results from the examples in sections 10.3.3, 10.4.3, and 10.5)

	AB	AC	AD	BC	BD	CD
v	$5c/13$	$4c/5$	$c/5$	$3c/5$	$c/5$	$5c/7$
γ	$13/12$	$5/3$	$5/2\sqrt{6}$	$5/4$	$5/2\sqrt{6}$	$7/2\sqrt{6}$

The separations of the two events in the various frames are (from the example in section 10.5)

	A	B	C	D
Δx	$-L$	L	$5L$	0
Δt	$5L/c$	$5L/c$	$7L/c$	$2\sqrt{6}L/c$

The Lorentz transformations are

$$\begin{aligned} x &= \gamma(x' + vt'), \\ t &= \gamma(t' + vx'/c^2). \end{aligned} \tag{10.89}$$

For each of the six pairs, we'll transform from the faster frame to the slower one. In other words, the coords of the faster frame will be on the right-hand side of the L.T.'s. (The sign on the right-hand side of the L.T.'s will therefore always be a "+".) In the AB case, we will write, for example, "Frames B and A," in that order, to signify that the B coordinates are on the left-hand side, and the A coordinates are on the right-hand side. We'll simply list the L.T.'s for the six cases, and you can check that they all work out.

Frames B and A:

$$\begin{aligned} L &= \frac{13}{12}\left(-L + \left(\frac{5c}{13}\right)\left(\frac{5L}{c}\right)\right), \\ \frac{5L}{c} &= \frac{13}{12}\left(\frac{5L}{c} + \frac{\frac{5c}{13}(-L)}{c^2}\right). \end{aligned} \tag{10.90}$$

Frames C and A:

$$\begin{aligned} 5L &= \frac{5}{3}\left(-L + \left(\frac{4c}{5}\right)\left(\frac{5L}{c}\right)\right), \\ \frac{7L}{c} &= \frac{5}{3}\left(\frac{5L}{c} + \frac{\frac{4c}{5}(-L)}{c^2}\right). \end{aligned} \tag{10.91}$$

Frames D and A:

$$\begin{aligned} 0 &= \frac{5}{2\sqrt{6}}\left(-L + \left(\frac{c}{5}\right)\left(\frac{5L}{c}\right)\right), \\ \frac{2\sqrt{6}L}{c} &= \frac{5}{2\sqrt{6}}\left(\frac{5L}{c} + \frac{\frac{c}{5}(-L)}{c^2}\right). \end{aligned} \tag{10.92}$$

Frames C and B:

$$\begin{aligned} 5L &= \frac{5}{4}\left(L + \left(\frac{3c}{5}\right)\left(\frac{5L}{c}\right)\right), \\ \frac{7L}{c} &= \frac{5}{4}\left(\frac{5L}{c} + \frac{\frac{3c}{5}L}{c^2}\right). \end{aligned} \tag{10.93}$$

Frames B and D:

$$\begin{aligned} L &= \frac{5}{2\sqrt{6}}\left(0 + \left(\frac{c}{5}\right)\left(\frac{2\sqrt{6}L}{c}\right)\right), \\ \frac{5L}{c} &= \frac{5}{2\sqrt{6}}\left(\frac{2\sqrt{6}L}{c} + \frac{\frac{c}{5}(0)}{c^2}\right). \end{aligned} \tag{10.94}$$

Frames C and D:

$$\begin{aligned} 5L &= \frac{7}{2\sqrt{6}}\left(0 + \left(\frac{5c}{7}\right)\left(\frac{2\sqrt{6}L}{c}\right)\right), \\ \frac{7L}{c} &= \frac{7}{2\sqrt{6}}\left(\frac{2\sqrt{6}L}{c} + \frac{\frac{5c}{7}(0)}{c^2}\right). \end{aligned} \tag{10.95}$$

12. **A new frame**

First Solution: Let the original frame be S, and let the desired frame be S'. Let S' move at speed v with respect to S. Our goal is to find v.

The Lorentz transformations are

$$\Delta x' = \frac{\Delta x - v\Delta t}{\sqrt{1 - v^2/c^2}}, \qquad \Delta t' = \frac{\Delta t - v\Delta x/c^2}{\sqrt{1 - v^2/c^2}}. \tag{10.96}$$

We want to make $\Delta t'$ equal to zero, so the second of these equations yields $\Delta t - v\Delta x/c^2 = 0$, or $v = c^2\Delta t/\Delta x$. We are given $\Delta x = 2$, and $\Delta t = 1/c$, so the desired v is

$$v = c^2\Delta t/\Delta x = c/2. \tag{10.97}$$

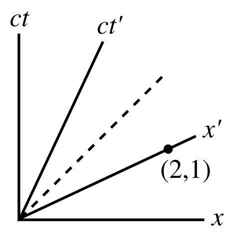


Figure 10.53

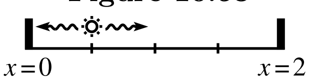


Figure 10.54

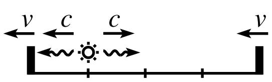


Figure 10.55

Second Solution: Consider the Minkowski diagram in Fig. 10.53. Event 1 is at the origin, and Event 2 is at the point $(2, 1)$, in the frame S.

Consider a frame S' whose x' axis passes through the point $(2, 1)$. Since all points on the x' axis are simultaneous in the S' frame (they all have $t' = 0$), we see that S' is the desired frame. From section 6, the slope of the x' axis is equal to $\beta \equiv v/c$. Since the slope is $1/2$, we have $v = c/2$.

(Looking at our Minkowski diagram, it is clear that if $v > c/2$, then Event 2 occurs before Event 1 in the new frame. And if $v < c/2$, then Event 2 occurs after Event 1 in the new frame.)

Third Solution: Consider the setup in Fig. 10.54, which explicitly constructs two such given events. Receivers are located at $x = 0$ and $x = 2$. A light source is located at $x = 1/2$. This source emits a flash of light, and when the light hits a receiver we will say an event has occurred. So the left event happens at $x = 0$, $ct = 1/2$; and the right event happens at $x = 2$, $ct = 3/2$. (We may shift our clocks by $-1/(2c)$ seconds in order to make the events happen at $ct = 0$ and $ct = 1$, but this shift will be irrelevant since all we are concerned with is differences in time.)

Now consider an observer flying by to the right at speed v. She sees the apparatus flying by to the left at speed v (see Fig. 10.55). Our goal is to find the v for which she sees the photons hit the receivers at the same time.

Consider the photons moving to the left. She sees them moving at speed c, but the left-hand receiver is retreating at speed v. So the relative speed of the photons and the left-hand receiver is $c - v$. By similar reasoning, the relative speed of the photons and the right-hand receiver is $c + v$.

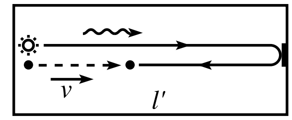
The light source is three times as far from the right-hand receiver as it is from the left-hand receiver. Therefore, if the light is to reach the two receivers at the same time, we must have $c + v = 3(c - v)$. This gives $v = c/2$.

13. **Velocity addition from scratch**

Let the train's frame be S'. Let the ground's frame be S. For concreteness, let's say that when the light hits the ball, a small explosion takes place which leaves a mark on the floor.

The basic ingredient in this solution is the fact that the mark on the floor occurs at the same fraction of the way along the train, independent of the frame. (Overall distances may change, depending on the frame, but this fraction must remain the same.) Another key point it that the photon and ball are released simultaneously in every frame, because they are simultaneous in S', and they are released at the same location.

We'll compute the desired fraction in the train's frame S', and then in the ground's frame S.



(frame S')

Figure 10.56

- Frame S': Let the train have length ℓ'.

  Let's first calculate the time of the explosion (see Fig. 10.56). Light takes a time ℓ'/c to reach the mirror. At this time, the ball has traveled a distance $v_1\ell'/c$. So the separation between the light and the ball at this time is $\ell'(1 - v_1/c)$. From this point on, the relative speed of the light and ball is $c + v_1$. So it takes them an additional time $\ell'(1-v_1/c)/(c+v_1)$ to meet. The total time before the explosion is therefore $\ell'(1 - v_1/c)/(c + v_1) + \ell'/c = 2\ell'/(c + v_1)$.

  The distance the object has traveled is therefore $2v_1\ell'/(c + v_1)$. So the desired fraction, F', is

$$F' = \frac{2v_1}{c + v_1}. \tag{10.98}$$

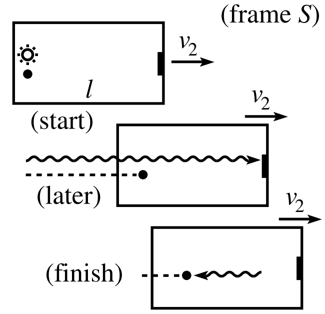


Figure 10.57

- Frame S: Let the speed of the ball in S be v. Let the train have length ℓ.

  Again, let's first calculate the time of the explosion (see Fig. 10.57). Light takes a time $\ell/(c - v_2)$ to reach the mirror (since the mirror is receding at a speed v_2). At this time, the ball has traveled a distance $v\ell/(c - v_2)$, and the light has traveled a distance $c\ell/(c - v_2)$. So the separation between the light and the ball at this time is $\ell(c - v)/(c - v_2)$. From this point on, the relative speed of the light and ball is $c + v$. So it takes them an additional time $\ell(c-v)/[(c-v_2)(c+v)]$ to meet. The total time before the explosion is therefore $\ell(c - v)/[(c - v_2)(c + v)] + \ell/(c - v_2) = 2c\ell/[(c - v_2)(c + v)]$.

  The distance the ball has traveled is therefore $2vc\ell/[(c - v_2)(c + v)]$. But the distance the back of the train has traveled in this time is $2v_2c\ell/[(c - v_2)(c+v)]$. So the distance between the ball and the back of the train is $2(v - v_2)c\ell/[(c - v_2)(c + v)]$. The desired fraction, F, is therefore

$$F = \frac{2(v - v_2)c}{(c - v_2)(c + v)}. \tag{10.99}$$

For convenience, let us define $\beta \equiv v/c$, $\beta_1 \equiv v_1/c$, and $\beta_2 \equiv v_2/c$. Then, equating the expressions for F' and F above gives

$$\frac{\beta_1}{1 + \beta_1} = \frac{\beta - \beta_2}{(1 - \beta_2)(1 + \beta)}. \tag{10.100}$$

Solving for β in terms of β_1 and β_2 gives[28]

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}. \tag{10.101}$$

14. **Many velocity additions**

Let's check the formula when N equals 1 or 2. For $N = 1$, the formula gives

$$\beta_{(1)} = \frac{P_1^+ - P_1^-}{P_1^+ + P_1^-} = \frac{(1 + \beta_1) - (1 - \beta_1)}{(1 + \beta_1) + (1 - \beta_1)} = \beta_1, \tag{10.102}$$

[28]N. David Mermin does this problem in Am. J. Phys., **51**, 1130 (1983), and then takes things one step further in Am. J. Phys., **52**, 119 (1984).

as it should. For $N = 2$, the formula gives

$$\beta_{(2)} = \frac{P_2^+ - P_2^-}{P_2^+ + P_2^-} = \frac{(1+\beta_1)(1+\beta_2) - (1-\beta_1)(1-\beta_2)}{(1+\beta_1)(1+\beta_2) + (1-\beta_1)(1-\beta_2)} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}, \tag{10.103}$$

which agrees with the velocity addition formula.

Let's now prove the formula for general N. We will use induction. That is, we will assume that the result holds for N and then show that it holds for $N+1$. To find the speed, $\beta_{(N+1)}$, of the object with respect to S_{N+1}, we can find the speed of the object with respect to S_N (which is $\beta_{(N)}$), and then combine this (using the velocity addition formula) with the speed of S_N with respect to S_{N+1} (which is β_{N+1}). In other words,

$$\beta_{(N+1)} = \frac{\beta_{N+1} + \beta_{(N)}}{1 + \beta_{N+1}\beta_{(N)}}. \tag{10.104}$$

Under the assumption that our formula holds for N, this becomes

$$\begin{aligned}
\beta_{(N+1)} &= \frac{\beta_{N+1} + \frac{P_N^+ - P_N^-}{P_N^+ + P_N^-}}{1 + \beta_{N+1}\frac{P_N^+ - P_N^-}{P_N^+ + P_N^-}} \\
&= \frac{\beta_{N+1}(P_N^+ + P_N^-) + (P_N^+ - P_N^-)}{(P_N^+ + P_N^-) + \beta_{N+1}(P_N^+ - P_N^-)} \\
&= \frac{P_N^+(1+\beta_{N+1}) - P_N^-(1-\beta_{N+1})}{P_N^+(1+\beta_{N+1}) + P_N^-(1-\beta_{N+1})} \\
&\equiv \frac{P_{N+1}^+ - P_{N+1}^-}{P_{N+1}^+ + P_{N+1}^-},
\end{aligned} \tag{10.105}$$

as we wanted to show. So if the result holds for N, then it holds for $N+1$. We know that the result holds for $N = 1$. Therefore, it holds for all N.

The expression for $\beta_{(N)}$ has some expected properties. It is symmetric in the β_i. And if at least one of the β_i equals 1, then $P_N^- = 0$; so $\beta_{(N)} = 1$, as it should. And if at least one of the β_i equals -1, then $P_N^+ = 0$; so $\beta_{(N)} = -1$, as it should.

Remark: We can use the result of this problem to derive the $v(t)$ given in eq. (10.62). First, note that if all the β_i in this problem are equal to a β which is much less than 1, then

$$\beta_{(N)} = \frac{(1+\beta)^N - (1-\beta)^N}{(1+\beta)^N + (1-\beta)^N} \approx \frac{e^{\beta N} - e^{-\beta N}}{e^{\beta N} + e^{-\beta N}} = \tanh(\beta N). \tag{10.106}$$

Let β equal $a\,dt/c$; this is the relative speed of two frames at nearby times, in the scenario leading up to eq. (10.62). If we let $N = t/dt$ be the number of frames, then we have produced the same scenario (if we take the limit $dt \to 0$). Therefore, $v(t) = c\beta_{(N)}$. With these N and β, eq. (10.106) gives $\beta_{(N)} = \tanh(at/c)$, as desired. ♣

15. **The "midpoint"**

First Solution: The relativistic subtraction of u from $4c/5$ must equal the relativistic addition of u to $3c/5$ (since both results are the speed of C). Hence (dropping the c's),

$$\frac{\frac{4}{5} - u}{1 - \frac{4}{5}u} = \frac{\frac{3}{5} + u}{1 + \frac{3}{5}u}. \tag{10.107}$$

Simplification gives $0 = 5u^2 - 26u + 5 = (5u - 1)(u - 5)$. The $u = 5$ root represents a speed larger than c, so our answer is

$$u = \frac{1}{5}c. \tag{10.108}$$

(Plugging this into eq. (10.107) shows that C's speed is $5c/7$ in the lab frame.)

Second Solution: The relative speed of A and B is (dropping the c's)

$$\frac{\frac{4}{5} - \frac{3}{5}}{1 - \frac{4}{5} \cdot \frac{3}{5}} = \frac{5}{13}\,. \tag{10.109}$$

From C's point of view, this $5/13$ is the result of relativistically adding u with another u. Therefore,

$$\frac{5}{13} = \frac{2u}{1+u^2} \qquad \Longrightarrow \qquad 5u^2 - 26u + 5 = 0, \tag{10.110}$$

as in the first solution.

16. **Another "midpoint"**

Let u be the speed at which C sees A and B approaching her. From C's point of view, v is the result of relativistically adding u with another u. Therefore (dropping the c's),

$$v = \frac{2u}{1+u^2} \qquad \Longrightarrow \qquad u = \frac{1 - \sqrt{1-v^2}}{v}. \tag{10.111}$$

(The quadratic equation for u has another solution with a "+" in front of the square root, but this solution is unphysical, because it goes to infinity as v goes to zero.) Note that our solution for u has the proper limit of $u \to v/2$ as $v \to 0$ (obtained by Taylor-expanding the square root).

The ratio of the distances CB and AC in the lab frame is the same as the ratio of the differences in velocity,

$$\begin{aligned} \frac{CB}{AC} = \frac{V_C - V_B}{V_A - V_C} &= \frac{\frac{1-\sqrt{1-v^2}}{v} - 0}{v - \frac{1-\sqrt{1-v^2}}{v}} \\ &= \frac{1 - \sqrt{1-v^2}}{\sqrt{1-v^2} - (1-v^2)} \\ &= \frac{1}{\sqrt{1-v^2}} \equiv \gamma. \end{aligned} \tag{10.112}$$

We see that C is γ times as far from B as she is from A. (For nonrelativistic speeds, we have $\gamma \approx 1$, and C is of course midway between A and B.)

An intuitive reason for the simple factor of γ is the following. Imagine that A and B are carrying identical jousting sticks as they run toward C. Consider what the situation looks like when the tips of the sticks reach C. In the lab frame (in which B is at rest), B's stick is uncontracted, but A's stick is contracted by a factor γ. Therefore, A is closer to C than B is, by a factor γ.

17. **Transverse velocity addition**

In frame S', the speed of the particle is u', so the time dilation factor is (dropping the c's) $\gamma' = 1/\sqrt{1-u'^2}$. The time between successive passes of the dotted lines is therefore $T_S = \gamma' T$.

In frame S, the speed of the particle is $\sqrt{v^2+u^2}$ (yes, the Pythagorean theorem still holds for the speeds, since both speeds are measured with respect to the same frame), so the time dilation factor is $\gamma = 1/\sqrt{1-v^2-u^2}$. The time between successive passes of the dotted lines is therefore $T_S = \gamma T$.

Equating our two expressions for $T_S/T_{S'}$ gives

$$\frac{u'}{u} = \frac{T_S}{T_{S'}} = \frac{\sqrt{1-u'^2}}{\sqrt{1-v^2-u^2}} \tag{10.113}$$

Solving for u gives

$$u = u'\sqrt{1-v^2} \equiv \frac{u'}{\gamma_v}, \tag{10.114}$$

as desired.

18. **Equal transverse speeds**

From your point of view, the lab frame is moving with speed v in the negative x-direction. The transverse velocity addition formula (eq. 10.35) therefore gives the y-speed in your frame as $u_y/\gamma(1-u_x v)$. (We'll drop the c's.) Demanding that this equal u_y gives

$$\gamma(1-u_x v) = 1 \qquad \Longrightarrow \qquad \sqrt{1-v^2} = (1-u_x v) \qquad \Longrightarrow \qquad v = \frac{2u_x}{1+u_x^2}\,. \tag{10.115}$$

(Another root is $v = 0$, of course.)

Remark: This answer makes sense. The fact that v is simply the relativistic addition of u_x with itself means that both your frame and the original lab frame move at speed u_x (but in opposite directions) relative to the frame in which the object has no speed in the x-direction. By symmetry, therefore, the y-speed of the object must be the same in your frame and the lab frame. ♣

19. **Relative speed**

Consider the frame, S', traveling along with the point P midway between the particles. S' moves at speed $v\cos\theta$, so the γ factor between this frame and the lab frame is

$$\gamma = \frac{1}{\sqrt{1-v^2\cos^2\theta}}\,. \tag{10.116}$$

In S', each particle moves along the vertical axis away from P with speed

$$u' = \gamma v \sin\theta. \tag{10.117}$$

The γ factor here comes from the time dilation between the lab frame and S'. (Clocks run slow in S', and transverse distances don't change, so the speed in S' is greater than the $v\sin\theta$ speed in the lab, by a factor γ.) Alternatively, just use the transverse velocity addition formula, eq. (10.35), to write $v\sin\theta = u'/\gamma$ (this nice clean result comes from the fact that there is no x'-speed in frame S').

Therefore, the speed of one particle as viewed by the other is, via the velocity addition formula,

$$V = \frac{2u'}{1+u'^2} = \frac{\frac{2v\sin\theta}{\sqrt{1-v^2\cos^2\theta}}}{1+\frac{v^2\sin^2\theta}{1-v^2\cos^2\theta}} = \frac{2v\sin\theta\sqrt{1-v^2\cos^2\theta}}{1-v^2\cos 2\theta}\,. \tag{10.118}$$

If desired, this can be rewritten as

$$V = \sqrt{1 - \frac{(1-v^2)^2}{(1-v^2\cos 2\theta)^2}}\,. \tag{10.119}$$

REMARK: If $2\theta = 180°$, then $V = 2v/(1+v^2)$, as it should. And if $\theta = 0°$, then $V = 0$, as it should. If θ is very small, then the result reduces to $V \approx 2v\sin\theta/\sqrt{1-v^2}$, which is simply the relative speed in the lab frame, multiplied by the time dilation factor between the frames. ♣

20. **Modified twin paradox**

(a) In A's reference frame, B's clock runs slow by a factor $1/\gamma = \sqrt{1-v^2/c^2}$. So if A's clock reads t when B meets C, then B's clock will read t/γ when he meets C. So the time he gives to C is t/γ.

In A's reference frame, the time between this event and the event where C meets A is again t (since B and C travel at the same speed). But A sees C's clock run slow by a factor $1/\gamma$, so A sees C's clock increase by t/γ.

Therefore, when A and C meet, A's clock reads $2t$, and C's clock reads $2t/\gamma$, i.e., $T_C = T_A/\gamma$.

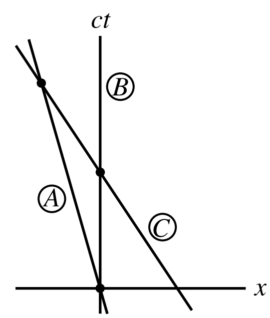


Figure 10.58

(b) Now let's do everything in B's frame. The worldlines of A, B, and C in this frame are shown in Fig. 10.58. One might think that looking at the problem in B's frame yields a paradox. Apparently, from part (a), we must have C's clock reading less time than A's, when A and C meet. However, from B's point of view, he sees A's clock run slowly, so when he transfers his time to C, at that moment he sees C's clock read *more* than A's. The resolution of this 'paradox' is that during the remainder of the travels of A and C, C's clock runs so much slower than A's (since the relative speed of C and B is greater than the relative speed of A and B) that C's clock in the end shows less time than A's. Let's be precise about this.

First of all, we have to calculate the relative speed of C and B (as viewed by B). It is *not* $2v$. We must use the velocity addition formula, because the only speeds we were given in the problem were ones with respect to A. For convenience, we will use the notation where $V_{A,B}$ denotes the velocity of A with respect to B, etc. The addition formula then gives

$$V_{C,B} = \frac{V_{C,A} + V_{A,B}}{1 + V_{C,A}V_{A,B}/c^2} = \frac{-v-v}{1+v^2/c^2} \equiv \frac{-2v}{1+\beta^2}. \tag{10.120}$$

Let B's clock read t_B when he meets C. Then at this time, B sees A's clock read t_B/γ, and he sees C's clock read t_B.

We must now determine how much time the remainder of the problem takes (as viewed by B). B sees C flying by to the left at speed $2v/(1+\beta^2)$. He also sees A flying by to the left at speed v, but A had a 'head-start' ahead of C of a distance vt_B. So if t is the time between the meeting of B and C and the meeting of A and C (as viewed from B), then

$$\frac{2v}{1+\beta^2}t = vt + vt_B. \tag{10.121}$$

This gives

$$t = t_B\left(\frac{1+\beta^2}{1-\beta^2}\right). \tag{10.122}$$

During this time, B sees A's and C's clocks increase by t divided by the relevant time dilation factor. The dilation factor for A is the usual $\gamma = 1/\sqrt{1 - v^2/c^2}$. The dilation factor for C is $1/\sqrt{1 - V_{C,B}^2/c^2}$. Using eq. (10.120), this turns out nicely to be $(1 + \beta^2)/(1 - \beta^2)$.

So the total time shown on A's clock when A and C meet is

$$\begin{aligned} T_A &= \frac{t_B}{\gamma} + t\sqrt{1 - \beta^2} \\ &= t_B\sqrt{1 - \beta^2} + t_B\left(\frac{1 + \beta^2}{1 - \beta^2}\right)\sqrt{1 - \beta^2} \\ &= \frac{2t_B}{\sqrt{1 - \beta^2}}. \end{aligned} \tag{10.123}$$

The total time shown on C's clock when A and C meet is

$$\begin{aligned} T_C &= t_B + t\sqrt{1 - V_{C,B}^2/c^2} \\ &= t_B + t_B\left(\frac{1 + \beta^2}{1 - \beta^2}\right)\left(\frac{1 - \beta^2}{1 + \beta^2}\right) \\ &= 2t_B. \end{aligned} \tag{10.124}$$

Therefore, $T_C = T_A\sqrt{1 - \beta^2} \equiv T_A/\gamma$.

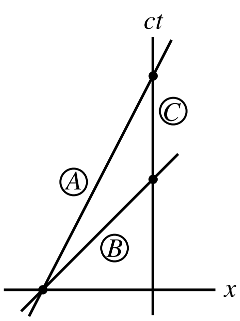


Figure 10.59

(c) Now let's work in C's frame. The worldlines of A, B, and C in this frame are shown Fig. 10.59.

As in part (b), the relative speed of B and C is $V_{B,C} = 2v/(1 + \beta^2)$, and the time dilation factor between B and C is $(1 + \beta^2)/(1 - \beta^2)$.

Let B and C meet when B's clock reads t_B. So t_B is the time that B hands off to C. (We will find all relevant times in terms of this t_B.) From C's point of view, B has traveled for a time $t_B^{\text{travel}} = t_B(1 + \beta^2)/(1 - \beta^2)$. In this time, B has traveled a distance (from when he met A) of

$$d = t_B\left(\frac{1 + \beta^2}{1 - \beta^2}\right)V_{B,C} = t_B\left(\frac{1 + \beta^2}{1 - \beta^2}\right)\frac{2v}{1 + \beta^2} = \frac{2vt_B}{1 - \beta^2}, \tag{10.125}$$

in C's frame. A must of course travel this same distance (from when he met B) to meet up with A.

Let's find T_A. Since A must travel a distance d to meet C, it takes A a time of $t_A^{\text{travel}} = d/v = 2t_B/(1 - \beta^2)$ to meet C (as viewed by C). Therefore, A's clock will read this time divided by the dilation factor γ. So A's clock reads

$$T_A = \frac{2t_B}{\sqrt{1 - \beta^2}}, \tag{10.126}$$

when A meets C.

Now let's find T_C. To find this, we must take t_B and add to it the extra time it takes A to reach C compared to the time it takes B to reach C. From above, this extra time is $t_A^{\text{travel}} - t_B^{\text{travel}} = 2t_B/(1 - \beta^2) - t_B(1 + \beta^2)/(1 - \beta^2) = t_B$. Therefore, C's clock reads

$$T_C = 2t_B, \tag{10.127}$$

when A meets C. Therefore, $T_C = T_A\sqrt{1 - \beta^2} \equiv T_A/\gamma$.

21. **Another relative speed**

Let the velocity **u** point in the x-direction, as shown in Fig. 10.60. Let S' be the lab frame, and let S be A's frame. (so frame S' moves at speed $-u$ with respect to S). The x- and y-speeds of B in frame S' are $v\cos\theta$ and $v\sin\theta$. Therefore the longitudinal and transverse velocity addition formulas, eqs. (10.28) and (10.35), give the components of B's speed in S as (dropping the c's)

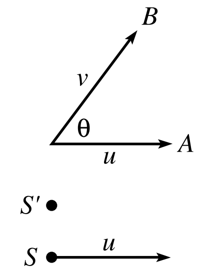


Figure 10.60

$$\begin{aligned} V_x &= \frac{v\cos\theta - u}{1 - uv\cos\theta}, \\ V_y &= \frac{v\sin\theta}{\gamma_u(1 - uv\cos\theta)} = \frac{\sqrt{1-u^2}\; v\sin\theta}{1 - uv\cos\theta}. \end{aligned} \qquad (10.128)$$

The total speed of B in frame S (that is, from the point of view of A) is therefore

$$\begin{aligned} V &= \sqrt{V_x^2 + V_y^2} \\ &= \sqrt{\Big(\frac{v\cos\theta - u}{1 - uv\cos\theta}\Big)^2 + \Big(\frac{\sqrt{1-u^2}\; v\sin\theta}{1 - uv\cos\theta}\Big)^2} \\ &= \frac{\sqrt{u^2 + v^2 - 2uv\cos\theta - u^2v^2\sin^2\theta}}{1 - uv\cos\theta}. \end{aligned} \qquad (10.129)$$

If desired, this can be rewritten as

$$V = \sqrt{1 - \frac{(1-u^2)(1-v^2)}{(1 - uv\cos\theta)^2}}. \qquad (10.130)$$

(The reason why this can be written in such an organized form will become clear in Chapter 12.)

Remark: If $u = v$, then this reduces to the result of the previous problem (if we replace θ by 2θ). If $2\theta = 180°$, then $V = (u+v)/(1+uv)$, as it should. And if $\theta = 0°$, then $V = (v-u)/(1-uv)$, as it should. ♣

22. **Minkowski diagram units**

All points on the ct'-axis have the property that $x' = 0$. All points on the hyperbola have the property that $c^2t'^2 - x'^2 = 1$, due to the invariance of s^2. So the ct' value at the intersection point, A, equals 1. Therefore, we simply have to determine the distance from A to the origin (see Fig. 10.61).

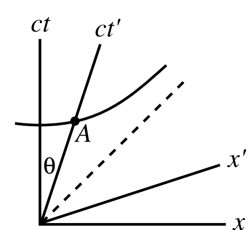


Figure 10.61

We'll do this by finding the (x, ct) coordinates of A. We know that $\tan\theta = \beta \equiv v/c$. Therefore, $x = \beta(ct)$ (i.e., $x = vt$). Plugging this into the given information, $c^2t^2 - x^2 = 1$, we find $ct = 1/\sqrt{1-\beta^2}$. So the distance from A to the origin is $\sqrt{c^2t^2 + x^2} = ct\sqrt{1+\beta^2} = \sqrt{(1+\beta^2)/(1-\beta^2)}$. The ratio of the unit sizes on the ct' and ct axes is therefore

$$\sqrt{\frac{1+\beta^2}{1-\beta^2}}, \qquad (10.131)$$

which agrees with eq. (10.43).

(Exactly the same analysis holds for the x-axis unit size ratio, of course.)

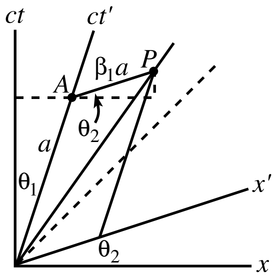


Figure 10.62

23. **Velocity Addition via Minkowski**

Pick a point P on the object's worldline. Let the coordinates of P in frame S be (x, ct). Our goal is to find the speed $u = x/t$. Throughout this problem, it will be easier to work with the quantities $\beta \equiv v/c$; so our goal is to find $\beta_u \equiv x/(ct)$.

The coordinates of P in S', namely (x', ct'), are shown in Fig. 10.62. For convenience, let ct' have length a on the paper. Then from the given information, we have $x' = v_1 t' \equiv \beta_1(ct') = \beta_1 a$. In terms of a, we can easily determine the coordinates (x, ct) of P. The coordinates of point A (shown in the figure) are simply

$$(x, ct)_A = (a\sin\theta_2, a\cos\theta_2). \tag{10.132}$$

The coordinates of P, relative to A, are

$$(x, ct)_{P-A} = (\beta_1 a\cos\theta_2, \beta_1 a\sin\theta_2). \tag{10.133}$$

So the coordinates of point P are

$$(x, ct)_P = (a\sin\theta_2 + \beta_1 a\cos\theta_2, a\cos\theta_2 + \beta_1 a\sin\theta_2). \tag{10.134}$$

The ratio of x to ct at the point P is therefore

$$\begin{aligned}\beta_u \equiv \frac{x}{ct} &= \frac{\sin\theta_2 + \beta_1\cos\theta_2}{\cos\theta_2 + \beta_1\sin\theta_2} \\ &= \frac{\tan\theta_2 + \beta_1}{1 + \beta_1\tan\theta_2} \\ &= \frac{\beta_2 + \beta_1}{1 + \beta_1\beta_2},\end{aligned} \tag{10.135}$$

where we have used $\tan\theta_2 = v_2/c \equiv \beta_2$, because S' moves at speed v_2 with respect to S. If we change from the β's back to the v's, the result is $u = (v_2 + v_1)/(1 + v_1 v_2/c^2)$.

24. **Acceleration and redshift**

There are various ways to do this problem (for example, by sending photons between the people, or by invoking the gravitational equivalence principle in GR, etc.). We'll do it here by using a Minkowski diagram, to demonstrate that it can be solved perfectly fine using only basic Special Relativity.

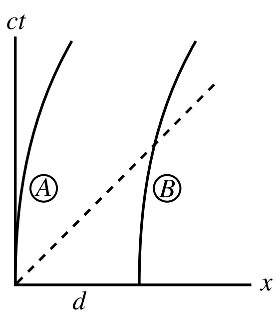


Figure 10.63

Draw the world lines of the two people, A and B, as seen by an observer, C, in the frame where they were both initially at rest. We have the situation shown in Fig. 10.63.

Consider an infinitesimal time Δt, as measured by C. At this time (in C's frame), A and B are both moving at speed $a\Delta t$. The axes of the A frame are shown in Fig. 10.64. Both A and B have moved a distance $a(\Delta t)^2/2$, which can be neglected since Δt is small.[29] Also, the special-relativity time-dilation factor between any of the A,B,C frames can be neglected. (Any relative speeds are no greater than $v = a\Delta t$, so the time-dilation factors differ from 1 by at most order $(\Delta t)^2$.) Let A make a little explosion, E_1, at this time. Then Δt (which was defined to be the time as measured by C) is also the time of the explosion, as measured by A (up to an error of order $(\Delta t)^2$).

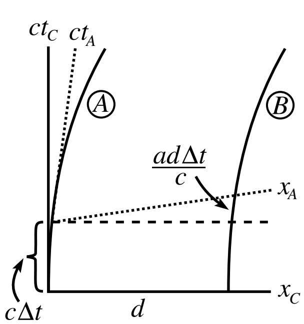


Figure 10.64

Let's figure out where A's x-axis (i.e., the 'now' axis in A's frame) meets B's worldline. The slope of A's x-axis in the figure is $v/c = a\Delta t/c$. So the axis starts at a height

[29] It will turn out that the leading-order terms in the result below are of order Δt. Any $(\Delta t)^2$ terms can therefore be ignored.

$c\Delta t$, and then climbs up by the amount $ad\Delta t/c$, over the distance d. Therefore, the axis meets B's worldline at a height $c\Delta t + ad\Delta t/c$, as viewed by C; i.e, at the time $\Delta t + ad\Delta t/c^2$, as viewed by C. But C's time is the same as B's time (up to order $(\Delta t)^2$). So B's clock reads $\Delta t(1 + ad/c^2)$. Let's say that B makes a little explosion, E_2, at this time.

A sees both E_1 and E_2 occur at the same time (they both lie along a line of constant time in A's frame). In other words, A sees B's clock read $\Delta t(1+ad/c^2)$ when he sees his own clock read Δt. Therefore, A sees B's clock sped up by a factor

$$\frac{\Delta t_B}{\Delta t_A} = 1 + \frac{ad}{c^2}. \tag{10.136}$$

We can perform the same procedure to see how B views A's clocks. Drawing B's x-axis at time Δt, we easily find that B sees A's clock slowed down by a factor

$$\frac{\Delta t_A}{\Delta t_B} = 1 - \frac{ad}{c^2}. \tag{10.137}$$

REMARK: In the usual special-relativity situation where two observers fly past each other with relative speed v, they *both* see the other person's time slowed down by the same factor. This had better be the case, since the situation is symmetric between the observers. But in this problem, A sees B's clock sped up, and B sees A's clock slowed down. This difference is possible because the situation is *not* symmetric between A and B. The acceleration vector determines a direction in space, and one person (namely B) is further along this direction than the other person (A). ♣

25. **Break or not break?**

There are two possible reasonings.

(1) To an observer in the original rest frame, the spaceships stay the same distance, d, apart. Therefore, in the frame of the spaceships, the distance between them, d', must be greater than d. This is the case because d equals d'/γ, by the usual length contraction. After a long enough time, γ will differ appreciably from 1, and the string will be stretched by a large factor. Therefore, it *will* break.

(2) Let A be the back spaceship, and let B be the front spaceship. From the point of view of A (B's point of view would work just as well), it looks like B is doing exactly what A is doing. It looks like B undergoes the same acceleration as A, so B should stay the same distance ahead of A. Therefore, the string should *not* break.

The second reasoning is incorrect. The first reasoning is (mostly) correct. The trouble with the second reasoning is that the two spaceships are in different frames. A in fact sees B's clock sped up, and B sees A's clock slowed down (from the previous problem). A sees B's engine working faster, and B therefore pulls away from A. So the string eventually breaks.

The first reasoning is mostly correct. The only trouble with it is that there is no one "frame of the spaceships". Their frames differ. It is not clear exactly what is meant by the 'length of the string', because it is not clear what frame the measurement should take place in.

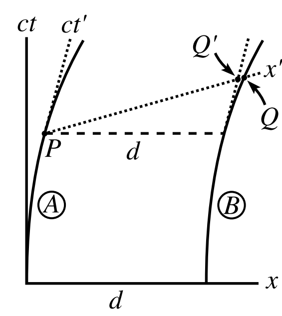


Figure 10.65

Everything becomes more clear once we draw a Minkowski diagram. Fig. 10.65 shows the x' and ct' axes of A's frame. The x'-axis is tilted up, so it meets B's worldline further to the right than one might think. The distance PQ along the x'-axis is the distance that A measures the string to be. Although it is not obvious that this distance in A's frame is larger than d (because the unit size on the x' axis is larger than that in C's frame), we can easily demonstrate this. In A's frame, the distance PQ is greater than the distance PQ'. But PQ' is simply the length of something in A's frame which has length d in C's frame. So PQ' is γd in A's frame. Since $PQ > \gamma d > d$ in A's frame, the string breaks.

REMARKS:

(a) If you want there to eventually be a well-defined "frame of the spaceships", you can modify the problem by stating that after a while, the spaceships both stop accelerating simultaneously (as measured by C). Equivalently, both A and B turn off their engines after equal proper times.

What A sees is the following. B pulls away from A. Then B turns off his engine. The gap continues to widen. But A continues firing his engine until be reaches B's speed. Then they sail onward, in a common frame, keeping a constant separation (which is greater than the original separation.)

(b) The main issue in this problem is that it depends exactly how you choose to accelerate an extended object. If you accelerate a stick by pushing on the back end (or by pulling on the front end), the length will remain essentially the same in its own frame, and it will become shorter in the original frame. But if you arrange for each end (or perhaps a number of points on the stick) to speed up in such a way that they always move at the same speed with respect to the original frame, then the stick will get torn apart. ♣

26. **Successive Lorentz transformations**

It is not necessary, of course, to use matrices in this problem; but things look nicer if you do. The desired composite L.T. is obtained by multiplying the matrices for the individual L.T.'s. So we have

$$\begin{aligned} L &= \begin{pmatrix} \cosh\phi_2 & \sinh\phi_2 \\ \sinh\phi_2 & \cosh\phi_2 \end{pmatrix} \begin{pmatrix} \cosh\phi_1 & \sinh\phi_1 \\ \sinh\phi_1 & \cosh\phi_1 \end{pmatrix} \\ &= \begin{pmatrix} \cosh\phi_1\cosh\phi_2 + \sinh\phi_1\sinh\phi_2 & \sinh\phi_1\cosh\phi_2 + \cosh\phi_1\sinh\phi_2 \\ \cosh\phi_1\sinh\phi_2 + \sinh\phi_1\cosh\phi_2 & \sinh\phi_1\sinh\phi_2 + \cosh\phi_1\cosh\phi_2 \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\phi_1+\phi_2) & \sinh(\phi_1+\phi_2) \\ \sinh(\phi_1+\phi_2) & \cosh(\phi_1+\phi_2) \end{pmatrix}. \end{aligned} \tag{10.138}$$

This is the L.T. with $v = \tanh(\phi_1 + \phi_2)$, as desired. This proof is just like the one for successive rotations in the plane (except for a few minus signs).

27. **Accelerator's time**

Eq. (10.62) gives the speed as a function of the spaceship's time,

$$\beta(t') \equiv \frac{v(t')}{c} = \tanh(at'/c). \tag{10.139}$$

The person in the lab sees the spaceship's clock slowed down by a factor $1/\gamma = \sqrt{1-\beta^2}$, i.e., $dt = dt'/\sqrt{1-\beta^2}$. So we have

$$\begin{aligned} t = \int_0^t dt &= \int_0^{t'} \frac{dt'}{\sqrt{1-\beta(t')^2}} \\ &= \int_0^{t'} \cosh(at'/c)\, dt' \\ &= \frac{c}{a}\sinh(at'/c). \end{aligned} \tag{10.140}$$

Note that for small a or t' (more precisely, if $at'/c \ll 1$), we obtain $t \approx t'$, as we should. For very large times, we essentially have

$$t \approx \frac{c}{2a} e^{at'/c}, \qquad \text{or} \qquad t' = \frac{c}{a} \ln(2at/c). \tag{10.141}$$

The lab frame will see the astronaut read all of "Moby Dick", but it will take an exponentially long time.

Chapter 11

Relativity (Dynamics)

In the previous chapter, we dealt only with abstract particles flying through space and time. We didn't concern ourselves with the nature of the particles, how they got to be moving the way they were moving, or what would happen if various particles interacted. In this chapter we will deal with these issues. That is, we will discuss masses, forces, energy, momentum, etc.

The two main results of this chapter are that the momentum and energy of a particle are given by

$$\mathbf{p} = \gamma m \mathbf{v}, \qquad \text{and} \qquad E = \gamma m c^2, \tag{11.1}$$

where $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$, and m is the mass of the particle.[1] When $v \ll c$, the expression for $\mathbf{p}$ reduces to $\mathbf{p} = m\mathbf{v}$, as it should for a non-relativistic particle. When $v = 0$, the expression for E reduces to the well-known $E = mc^2$.

11.1 Energy and momentum

Let's give some justification for eqs. (11.1). The reasoning in this section should convince you of their truth. An alternative, and perhaps more convincing, motivation comes from the 4-vector formalism in Chapter 12. In the end, however, the justification for eqs. (11.1) is obtained through experiments. Every day, experiments in high-energy accelerators are verifying the truth of these expressions. (More precisely, they are verifying that these energy and momenta are *conserved* in any type of collision.) We therefore conclude, with reasonable certainty, that eqs. (11.1) are the correct expressions for energy and momentum.

But actual experiments aside, let's consider a few thought-experiments that motivate the above expressions.

[1]Some people use the word "mass" in different ways in relativity. They talk about "rest mass" and "relativistic mass". These terms, however, are misleading. (See Section 11.8 for a discussion of this.) There is only one thing that can reasonably be called "mass" in relativity. It is the same thing that we call "mass" in Newtonian physics (and what some people would call "rest mass", although the qualifier "rest" is redundant).

11.1.1 Momentum

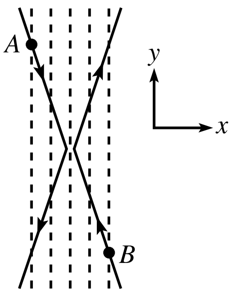


Figure 11.1

Consider the following system. In the lab frame, identical particles A and B move as shown in Fig. 11.1. They move with equal and opposite small speeds in the x-direction, and with equal and opposite large speeds in the y-direction. Their paths are arranged so that they glance off each other and reverse their motion in the x-direction.

For clarity, imagine a series of equally spaced vertical lines for reference. Assume that both A and B have identical clocks that tick every time they cross one of the lines.

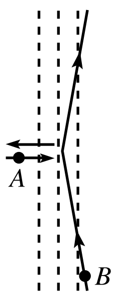


Figure 11.2

Now go to the reference frame which moves in the y-direction, with the same v_y as A. In this frame, the situation looks like Fig. 11.2. The bounce simply changes the sign of the x-velocities of the particles. Therefore, the x-momenta of the two particles must be the same.[2]

However, the x-speeds of the two particles are *not* the same in this frame. A is essentially at rest in this frame, and B is moving with a very large speed, v. Therefore, B's clock is running slower than A's, by a factor essentially equal to $1/\gamma \equiv \sqrt{1 - v^2/c^2}$. And since B's clock ticks once for every vertical line it crosses (this fact is independent of the frame), B must therefore be moving slower in the x-direction, by a factor of $1/\gamma$.

Therefore, the Newtonian expression, $p_x = mv_x$, cannot be the correct one for momentum, because B's momentum would be smaller than A's (by a factor of $1/\gamma$), due to their different v_x's. But the γ factor in

$$p_x = \gamma m v_x \equiv \frac{m v_x}{\sqrt{1 - v^2/c^2}} \tag{11.2}$$

precisely takes care of this problem, since $\gamma \approx 1$ for A, and $\gamma = 1/\sqrt{1 - v^2/c^2}$ for B (which precisely cancels the effect of B's smaller v_x).

To obtain the three-dimensional form for $\mathbf{p}$, we simply note that the vector $\mathbf{p}$ must point in the same direction as the vector $\mathbf{v}$ points.[3] Therefore, eq. (11.2) implies that the momentum vector must be

$$\mathbf{p} = \gamma m \mathbf{v} \equiv \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}\,, \tag{11.3}$$

in agreement with eq. (11.1).

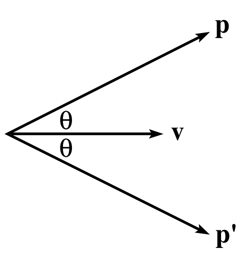


Figure 11.3

Remark: What we've shown here is that the only possible vector of the form $f(v)m\mathbf{v}$ (where f is some function) that has any chance at being conserved is $\gamma m\mathbf{v}$ (or some constant multiple of this). We haven't proven that it actually *is* conserved; this is the duty of experiments. But we've shown that it would be a waste of time to consider, say, the vector $\gamma^5 m\mathbf{v}$. ♣

[2] This is true because if, say, A's p_x were larger than B's p_x, then the total p_x would point to the right before the bounce, and to the left after the bounce. Since momentum is something we want to be conserved, this cannot be the case.

[3] This is true because any other direction for $\mathbf{p}$ would violate rotation invariance. If someone claims that $\mathbf{p}$ points in the direction shown in Fig. 11.3, then he would be hard-pressed to explain why it doesn't instead point along the direction $\mathbf{p}'$ shown. In short, the direction of $\mathbf{v}$ is the only preferred direction in space.

11.1.2 Energy

Having found that momentum is given by $\mathbf{p} = \gamma m\mathbf{v}$, we will now convince you that energy is given by

$$E = \gamma mc^2. \tag{11.4}$$

More precisely, we will show that γmc^2 is *conserved* in an interaction. There are various ways to do this. The best way, perhaps, is to use the 4-vector formalism in Chapter 12. But we'll consider one nice scenario here that should do the job.

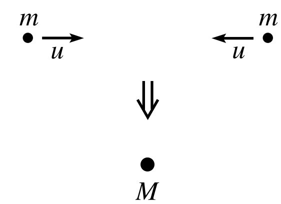


Figure 11.4

Consider the following system. Two identical particles of mass m head toward each other, both with speed u, as shown in Fig. 11.4. They stick together and form a particle with mass M. (At the moment we cannot assume anything about M. We will find that it does *not* equal the naive value of $2m$.) M is at rest, due to the symmetry of the situation.

This is a fairly uninteresting setup (momentum conservation gives $0 = 0$), but now consider the less trivial view from a frame moving to the left at speed u. This situation is shown in Fig. 11.5. Here, the right mass is at rest, M moves to the right at speed u, and the left mass moves to the right at speed $v = 2u/(1+u^2)$ (from the velocity addition formula).[4] Note that the γ-factor associated with this speed v is given by

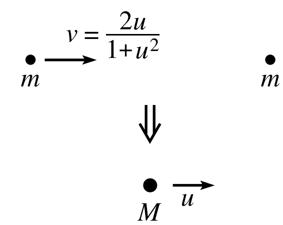


Figure 11.5

$$\gamma_v \equiv \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\left(\frac{2u}{1+u^2}\right)^2}} = \frac{1+u^2}{1-u^2}\,. \tag{11.5}$$

Therefore, conservation of momentum in this collision gives

$$\begin{aligned} \gamma_v mv + 0 &= \gamma_u Mu \\ \Longrightarrow \qquad m\left(\frac{1+u^2}{1-u^2}\right)\left(\frac{2u}{1+u^2}\right) &= \frac{Mu}{\sqrt{1-u^2}} \\ \Longrightarrow \qquad M = \frac{2m}{\sqrt{1-u^2}}\,. \end{aligned} \tag{11.6}$$

Conservation of momentum thus tells us that M is *not* equal to $2m$. (But if u is very small, then M is approximately equal to $2m$, as we know from everyday experience.)

Let's now see if our candidate for energy, $E = \gamma mc^2$, is conserved in this collision. There is no freedom left in any of the parameters, so γmc^2 is either conserved or it isn't. In the original frame where M is at rest, E is conserved if

$$\gamma_0 Mc^2 = 2(\gamma_u mc^2) \qquad \Longleftrightarrow \qquad M = \frac{2m}{\sqrt{1-u^2}}\,. \tag{11.7}$$

But this is exactly what eq. (11.6) says. So E is indeed conserved in this frame.

Let's also check that E is conserved in the frame where the right mass is at rest. It is conserved if

$$\gamma_v mc^2 + \gamma_0 mc^2 = \gamma_u Mc^2, \qquad \text{or}$$

[4] We're going to set $c = 1$ for a little while here, because this calculation would get too messy if we kept in the c's. See the subsection at the end of this section for more comments on this.

$$\begin{aligned} \left(\frac{1+u^2}{1-u^2}\right)m + m &= \frac{M}{\sqrt{1-u^2}}\,, \qquad \text{or} \\ \frac{2m}{1-u^2} &= \left(\frac{2m}{\sqrt{1-u^2}}\right)\frac{1}{\sqrt{1-u^2}}\,, \end{aligned} \tag{11.8}$$

which is indeed true. So E is conserved in this frame also.

Hopefully at this point you're convinced that γmc^2 is a believable expression for the energy of a particle. But just as in the case of momentum, we haven't proven that γmc^2 actually *is* conserved; this is the duty of experiments. But we've shown that it would be a waste of time to consider, say, the quantity $\gamma^4 mc^2$.

One thing that we certainly need to check is that if E and p are conserved in one reference frame, then they are conserved in any other. (A conservation law shouldn't depend on what frame you're in.) We'll demonstrate this in Section 11.2.

REMARKS:

1. To be precise, we should say that technically we're not trying to justify eqs. (11.1) here. These two equations by themselves are devoid of any meaning. All they do is define the letters $\mathbf{p}$ and E. Our goal is to make a meaningful physical statement, not just a definition.

   The meaningful physical statement we want make is that the quantities $\gamma m\mathbf{v}$ and γmc^2 are *conserved* in an interaction among particles (and this is what we tried to justify above). This fact then makes these quantities worthy of special attention, because conserved quantities are very helpful in understanding what is happening in a given physical situation. And anything worthy of special attention certainly deserves a label, so we may then attach the names "momentum" and "energy" to $\gamma m\mathbf{v}$ and γmc^2. Any other names would work just as well, of course, but we choose these because in the limit of small speeds, $\gamma m\mathbf{v}$ and γmc^2 reduce (as we will soon show) to some other nicely conserved quantities, which someone already tagged with "momentum" and "energy" long ago.

2. As mentioned above, the fact of the matter is that we can't *prove* that $\gamma m\mathbf{v}$ and γmc^2 are conserved. In Newtonian physics, conservation of $\mathbf{p} \equiv m\mathbf{v}$ is basically postulated via Newton's third law, and we're not going to be able to do any better than that here. All we can hope to do as physicists is provide some motivation for considering $\gamma m\mathbf{v}$ and γmc^2, show that it is consistent for $\gamma m\mathbf{v}$ and γmc^2 to be conserved during an interaction, and gather a large amount of experimental evidence, all of which is consistent with $\gamma m\mathbf{v}$ and γmc^2 being conserved. That's how physics works. You can't prove anything. So you learn to settle for the things you can't disprove.

   Consider, when seeking gestalts,
   The theories that physics exalts.
   It's not that they're known
   To be written in stone.
   It's just that we can't say they're false.

   As far as the experimental evidence goes, suffice it to say that high-energy accelerators are continually verifying everything we think is true about relativistic dynamics. If the theory is not correct, then we know that it must be the limiting theory of a more complete one (just as Newtonian physics is a limiting theory of relativity). But all this experimental induction has to count for something . . .

"To three, five, and seven, assign
A name," the prof said, "We'll define."
But he botched the instruction
With woeful induction
And told us the next prime was nine.

3. Note that conservation of energy in relativistic mechanics is actually a much simpler concept than it is in nonrelativistic mechanics, because $E = \gamma m$ is conserved, period. We don't have to worry about the generation of heat, which ruins conservation of the nonrelativistic $E = mv^2/2$. The heat is simply built into γm. In the example above, the two m's collide and generate heat in the resulting mass M. The heat shows up as an increase in mass, which makes M larger than $2m$. The energy that arises from this increase in mass accounts for the initial kinetic energy of the two m's.

4. Problem 1 gives an alternate derivation of the energy and momentum expressions in eq. (11.1). This derivation uses additional facts, namely that the energy and momentum of a photon are given by $E = h\nu$ and $p = h\nu/c$. ♣

Any multiple of γmc^2 is also conserved, of course. Why did we pick γmc^2 to label "E" instead of, say, $5\gamma mc^3$? Consider the approximate form γmc^2 takes in the Newtonian limit (that is, in the limit $v \ll c$). We have, using the Taylor series expansion for $(1-x)^{-1/2}$,

$$\begin{aligned} E \equiv \gamma mc^2 &= \frac{mc^2}{\sqrt{1 - v^2/c^2}} \\ &= mc^2\left(1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \cdots\right) \\ &= mc^2 + \frac{1}{2}mv^2 + \cdots . \end{aligned} \tag{11.9}$$

The dots represent higher-order terms in v^2/c^2, which may be neglected if $v \ll c$. Since the quantity mc^2 has a fixed value, we see that conservation of $E \equiv \gamma mc^2$ reduces to the familiar conservation of kinetic energy, $mv^2/2$, in the limit of slow speeds.

Likewise, we picked $\mathbf{p} \equiv \gamma m\mathbf{v}$, instead of, say, $6\gamma mc^4\mathbf{v}$, because the former obviously reduces to the familiar momentum, $m\mathbf{v}$, in the limit of slow speeds.

Whether abstract, profound, or just mystic,
Or boring, or somewhat simplistic,
A theory must lead
To results that we need
In limits, nonrelativistic.

Whenever we use the term "energy", we will mean the total energy, γmc^2. If we use the term "kinetic energy", we will mean a particle's excess energy over its energy when it is motionless, namely $\gamma mc^2 - mc^2$. Note that kinetic energy is *not* necessarily conserved in a collision, because mass is not necessarily conserved, as we saw in eq. (11.6) in the above scenario, where $M = 2m/\sqrt{1-u^2}$.

Note the following extremely important relation.

$$\begin{aligned} E^2 - |\mathbf{p}|^2c^2 &= \gamma^2m^2c^4 - \gamma^2m^2|\mathbf{v}|^2c^2 \\ &= \gamma^2m^2c^4\left(1 - \frac{v^2}{c^2}\right) \\ &= m^2c^4. \end{aligned} \tag{11.10}$$

This is the primary ingredient in solving relativistic collision problems, as we will soon see. It replaces the KE $= p^2/2m$ relation between kinetic energy and momentum in Newtonian physics. It can be derived in more profound ways, as we will see in Chapter 12. Let's put in in a box, since it's so important.

$$\boxed{E^2 = p^2c^2 + m^2c^4}. \tag{11.11}$$

In the case where $m = 0$ (as with photons), eq. (11.11) says that $E = pc$. This is the key equation for massless objects. The two equations, $\mathbf{p} = \gamma m\mathbf{v}$ and $E = \gamma mc^2$, don't tell us much, because $m = 0$ and $\gamma = \infty$. But $E^2 - |\mathbf{p}|^2c^2 = m^2c^4$ still holds, and we conclude that $E = pc$.

Note that a massless particle must have $\gamma = \infty$ (that is, it must travel at speed c), in order for $E = \gamma mc^2$ to be nonzero (which it must be, if we are to observe the particle).

Setting $c = 1$

For the remainder of our treatment of relativity, we will invariably work in units where $c = 1$. (For example, instead of 1 m being the unit of distance, we will make $3 \cdot 10^8$ m equal to one unit.) In these units, our various expressions become

$$\mathbf{p} = \gamma m\mathbf{v}, \qquad E = \gamma m, \qquad E^2 = p^2 + m^2. \tag{11.12}$$

Said in another way, you can simply ignore all the c's in your calculations (which will generally save you a lot of strife), and then put them back into your final answer to make the units correct. For example, let's say the goal of a certain problem is to find the time of some event. If your answer comes out to be ℓ, where ℓ is a given length, then you know that the correct answer (in terms of the usual mks units) has to be ℓ/c, because this has units of time. In order for this procedure to work, there must be only one way to put the c's back in at the end. It is clear that this is the case.

Another nice relation is

$$\frac{\mathbf{p}}{E} = \mathbf{v}, \tag{11.13}$$

or $\mathbf{p}/E = \mathbf{v}/c^2$, with the c's.

11.2 Transformations of E and $\vec{p}$

Consider a one-dimensional situation, where all motion is along the x-axis. Consider a particle that has energy E' and momentum p' in frame S'. Let frame S' move

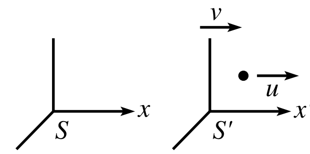


Figure 11.6

with speed v with respect to frame S (in the positive x-direction; see Fig. 11.6). What are E and p in S?

Let u be the particle's speed in S'. From the velocity addition formula, the particle's speed in S is (dropping the factors of c)

$$V = \frac{u+v}{1+uv}. \tag{11.14}$$

The γ-factor associated with this speed is

$$\gamma_V = \frac{1}{\sqrt{1-\left(\frac{u+v}{1+uv}\right)^2}} = \frac{1+uv}{\sqrt{(1-u^2)(1-v^2)}} \equiv \gamma_u\gamma_v(1+uv). \tag{11.15}$$

The energy and momentum in S' are

$$E' = \gamma_u m, \qquad \text{and} \qquad p' = \gamma_u m u, \tag{11.16}$$

while the energy and momentum in S are (using eq. (11.15))

$$\begin{aligned} E &= \gamma_V m = \gamma_u\gamma_v(1+uv)m, \\ p &= \gamma_V m V = \gamma_u\gamma_v(1+uv)m\left(\frac{u+v}{1+uv}\right) = \gamma_u\gamma_v(u+v)m. \end{aligned} \tag{11.17}$$

Using the E' and p' from eq. (11.16), we may rewrite E and p as (with $\gamma \equiv \gamma_v$)

$$\begin{aligned} E &= \gamma(E' + vp'), \\ p &= \gamma(p' + vE'). \end{aligned} \tag{11.18}$$

These are transformations of E and p between frames. If you want to put the factors of c back in, then the vE' term becomes vE'/c^2. These equations are easy to remember, because they look *exactly* like the Lorentz transformations of the coordinates t and x in eq. (10.17). This is no coincidence, as we will see in Chapter 12.

Note that since the transformations are linear, they also hold if E and p represent the total energy and momentum of a collection of particles. This is a very important and useful result. It is important enough so that we will write it out explicitly,

$$\begin{aligned} \sum E &= \gamma\left(\sum E' + v\sum p'\right), \\ \sum p &= \gamma\left(\sum p' + v\sum E'\right). \end{aligned} \tag{11.19}$$

Indeed, any linear function of the energies and momenta would be valid, in place of the sums.

Remark: As a double check, if $u = 0$ (so that $p' = 0$ and $E' = m$), then $E = \gamma m$ and $p = \gamma m v$, as they should. Also, if $u = -v$ (so that $p' = -\gamma m v$ and $E' = \gamma m$), then $E = m$ and $p = 0$, as they should. ♣

You can use eq. (11.18) to easily show that

$$E^2 - p^2 = E'^2 - p'^2, \tag{11.20}$$

just like the $t^2 - x^2 = t'^2 - x'^2$ result in eq. (10.37). For one particle, we already knew this was true, because both sides are equal to m^2 (from eq. (11.10)). For many particles, the invariant $E_{\text{total}}^2 - p_{\text{total}}^2$ is equal to the square of the total energy in the CM frame (which reduces to m^2 for one particle), because $p_{\text{total}} = 0$ in the CM frame, by definition.

REMARKS:

1. In the previous section, we said that we needed to show that if E and p are conserved in one reference frame, then they are conserved in any other frame (because a conservation law shouldn't depend on what frame you're in). Eq. (11.18) makes this fact trivially clear, because the E and p in one frame are linear functions of the E' and p' in another frame. If the total $\Delta E'$ and $\Delta p'$ in S' are zero, then eq. (11.18) says that the total ΔE and Δp in S must also be zero. (We have used the fact that ΔE is a linear combination of the E's, etc., so that eq. (11.18) applies to this linear combination.)

2. Eq. (11.18) also makes it clear that if you accept the fact that $p = \gamma mv$ is conserved in all frames, then you must also accept the fact that $E = \gamma m$ is conserved in all frames (and vice versa). This is true because the second of eqs. (11.18) says that if Δp and $\Delta p'$ are both zero, then $\Delta E'$ must also be zero. E and p have no choice but to go hand in hand. ♣

Eq. (11.18) applies to the x-component of the momentum. How do the transverse components, p_y and p_z, transform? Just as with the y and z coordinates in the Lorentz transformations, p_y and p_z do not change between frames. The analysis in Chapter 12 makes this obvious, so for now we'll simply state that

$$\begin{aligned} p_y &= p'_y, \\ p_z &= p'_z, \end{aligned} \tag{11.21}$$

if the relative velocity between the frames is in the x-direction.

11.3 Collisions and decays

The strategy for studying relativistic collisions is the same as that for studying nonrelativistic ones. You simply have to write down all the conservation of energy and momentum equations, and then solve for whatever variables you want to solve for. In doing so, one equation you will use over and over is $E^2 - p^2 = m^2$.

In writing down the conservation of energy and momentum equations, it proves extremely useful to put E and $\mathbf{p}$ together into one four-component vector,

$$P \equiv (E, \mathbf{p}) \equiv (E, p_x, p_y, p_z). \tag{11.22}$$

This is called the *energy-momentum 4-vector*, or the *4-momentum*, for short.[5] Our notation in this chapter will be to use an uppercase P to denote a 4-momentum, and a lowercase p or $\mathbf{p}$ to denote a spatial momentum. The components of a 4-momentum are generally indexed from 0 to 3, so that $P_0 \equiv E$, and $(P_1, P_2, P_3) \equiv \mathbf{p}$. For one particle, we have

$$P = (\gamma m, \gamma m v_x, \gamma m v_y, \gamma m v_z). \quad (11.23)$$

The 4-momentum for a collection particles simply consists of the total E and total $\mathbf{p}$ of all the particles.

There are deep reasons for forming this four-component vector,[6] but for now we will view it as simply a matter on convenience. If nothing else, it helps with bookkeeping. Conservation of energy and momentum in a collision reduce to the concise statement,

$$P_{\text{before}} = P_{\text{after}}, \quad (11.24)$$

where these are the total 4-momenta of all the particles.

If we define the inner product between two 4-momenta, $A \equiv (A_0, A_1, A_2, A_3)$ and $B \equiv (B_0, B_1, B_2, B_3)$, to be

$$A \cdot B \equiv A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3, \quad (11.25)$$

then the relation $E^2 - p^2 = m^2$ (which is true for one particle) may be concisely written as

$$P^2 \equiv P \cdot P = m^2. \quad (11.26)$$

In other words, the square of a particle's 4-momentum equals its mass squared. This relation is frame-independent, as we saw in eq. (11.20).

This inner product is different from the one we're used to in three-dimensional space. It has one positive sign and three negative signs, in contrast with the usual three positive signs. But we are free to define it however we wish, and we clearly picked a good definition, because our inner product is invariant under a Lorentz-transformation of the coordinates (just as the usual 3-D inner product is invariant under a rotation of the coordinates).

Example (Relativistic billiards): A particle with energy E and mass m approaches an identical particle at rest. They collide (elastically) in such a way that they both scatter at an angle θ relative to the incident direction (see Fig. 11.7). What is θ in terms of E and m?

What is θ in the relativistic and non-relativistic limits?

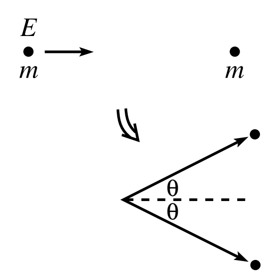


Figure 11.7

Solution: The first thing you should always do is write down the 4-momenta. The 4-momenta before the collision are

$$P_1 = (E, p, 0, 0), \qquad P_2 = (m, 0, 0, 0), \quad (11.27)$$

[5] If we were keeping in the factors of c, then the first term would be E/c (although some people instead multiply the $\mathbf{p}$ by c; either convention is fine).

[6] This will become clear in Chapter 12. You may read Sections 12.1 through 12.4 now if you wish, but it is not necessary to do so to understand the remainder of this chapter.

where $p = \sqrt{E^2 - m^2}$. The 4-momenta after the collision are

$$P_1' = (E', p'\cos\theta, p'\sin\theta, 0), \qquad P_2' = (E', p'\cos\theta, -p'\sin\theta, 0), \tag{11.28}$$

where $p' = \sqrt{E'^2 - m^2}$. Conservation of energy gives $E' = (E+m)/2$, and conservation of momentum gives $p'\cos\theta = p/2$. Therefore, the 4-momenta after the collision are

$$P_{1,2}' = \left(\frac{E+m}{2}\,,\, \frac{p}{2}\,,\, \pm\frac{p}{2}\tan\theta,\, 0\right). \tag{11.29}$$

From eq. (11.26), the squares of these must be m^2. Therefore,

$$\begin{aligned} m^2 &= \left(\frac{E+m}{2}\right)^2 - \left(\frac{p}{2}\right)^2 (1 + \tan^2\theta) \\ \Longrightarrow \qquad 4m^2 &= (E+m)^2 - \frac{(E^2 - m^2)}{\cos^2\theta} \\ \Longrightarrow \qquad \cos^2\theta &= \frac{E^2 - m^2}{E^2 + 2Em - 3m^2} = \frac{E+m}{E+3m}\,. \end{aligned} \tag{11.30}$$

The relativistic limit is $E \gg m$, which yields $\cos\theta \approx 1$. Therefore, both particles scatter almost directly forward.

The nonrelativistic limit is $E \approx m$ (note: it's *not* $E \approx 0$), which yields $\cos\theta \approx 1/\sqrt{2}$. Therefore, $\theta \approx 45^\circ$, and the particles scatter with a 90° angle between them. This agrees with the result in Section 4.7.2 (a result which pool players are very familiar with).

Decays are basically the same as collisions. All you have to do is conserve energy and momentum, as the following example shows.

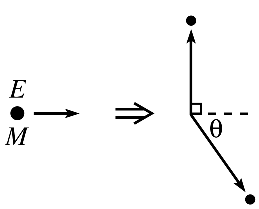


Figure 11.8

Example (Decay at an angle): A particle with mass M and energy E decays into two identical particles. In the lab frame, they are emitted at angles 90° and θ, as shown in Fig. 11.8.

What are the energies of the created particles? (We'll give two solutions. The second one shows how 4-momenta can be used in a very clever and time-saving way.)

First Solution: The 4-momentum before the decay is

$$P = (E, p, 0, 0), \tag{11.31}$$

where $p = \sqrt{E^2 - M^2}$. Let the created particles have mass m. The 4-momenta after the collision are

$$P_1 = (E_1, 0, p_1, 0), \qquad P_2 = (E_2, p_2\cos\theta, -p_2\sin\theta, 0). \tag{11.32}$$

Conservation of p_x immediately gives $p_2\cos\theta = p$. Conservation of p_y says that the final p_y's are opposites. Therefore, the 4-momenta after the collision are

$$P_1 = (E_1, 0, p\tan\theta, 0), \qquad P_2 = (E_2, p, -p\tan\theta, 0). \tag{11.33}$$

Conservation of energy gives $E = E_1 + E_2$. Writing this in terms of the momenta and masses gives

$$E = \sqrt{p^2\tan^2\theta + m^2} + \sqrt{p^2(1 + \tan^2\theta) + m^2}. \tag{11.34}$$

Putting the first radical on the left side, squaring, and solving for that radical (which is E_1) gives

$$E_1 = \frac{E^2 - p^2}{2E} = \frac{M^2}{2E}\,. \qquad (11.35)$$

Similarly, we find E_2 to be

$$E_2 = \frac{E^2 + p^2}{2E} = \frac{2E^2 - M^2}{2E}\,. \qquad (11.36)$$

These add up to E, as they should.

Second Solution: With the 4-momenta defined as in eqs. (11.31) and (11.32), conservation of energy and momentum takes the form $P = P_1 + P_2$. Therefore,

$$\begin{aligned} P - P_1 &= P_2, \\ \Longrightarrow \quad (P - P_1)\cdot(P - P_1) &= P_2 \cdot P_2, \\ \Longrightarrow \quad P^2 - 2P\cdot P_1 + P_1^2 &= P_2^2, \\ \Longrightarrow \quad M^2 - 2EE_1 + m^2 &= m^2, \\ \Longrightarrow \quad E_1 &= \frac{M^2}{2E}\,. \end{aligned} \qquad (11.37)$$

And then $E_2 = E - E_1 = (2E^2 - M^2)/2E$.

This solution should convince you that 4-momenta can save you a lot of work. What happened here was that the expression for P_2 was fairly messy, but we arranged things so that it only appeared in the form P_2^2, which is simply m^2. 4-momenta provide a remarkably organized method for sweeping unwanted garbage under the rug.

11.4 Particle-physics units

A branch of physics that uses relativity as its main ingredient is Elementary-Particle Physics, which is the study of the building blocks of matter (electrons, quarks, pions, etc.). It is unfortunately the case that most of the elementary particles we want to study don't exist naturally in the world. We therefore have to create them in laboratories by colliding other particles together at very high energies. The high speeds involved require the use of relativistic dynamics. Newtonian physics is essentially useless.

What is a typical size of an energy, γmc^2, of an elementary particle? The rest-energy of a proton (which isn't really elementary; it's made up of quarks, but never mind) is

$$E = m_p c^2 = (1.67 \cdot 10^{-27}\text{kg})(3 \cdot 10^8\text{m/s})^2 = 1.5 \cdot 10^{-10}\ \text{Joules}. \qquad (11.38)$$

This is very small, of course. So a Joule is probably not the best unit to work with. We would get very tired of writing the negative exponents over and over.

We could perhaps work with "microjoules" or "nanojoules", but particle-physicists like to work instead with the "eV", the *electron-volt*. This is the amount of energy gained by an electron when it passes through a potential of one volt. The electron charge is $e = 1.6022\cdot 10^{-19}$ C, and a volt is defined as 1 V = 1 J/C. So the conversion from eV to Joules is[7]

$$1 \text{ eV} = (1.6022 \cdot 10^{-19}\text{C})(1 \text{ J/C}) = 1.6022 \cdot 10^{-19} \text{ J}. \tag{11.39}$$

Therefore, in terms of eV, the rest-energy of a proton is $938 \cdot 10^6$ eV. We now seem to have the opposite problem of having a large exponent hanging around. This is easily remedied by the prefix "M", which stands for "mega", or "million". So we finally have

$$E_{\text{proton}} = 938 \text{ MeV}. \tag{11.40}$$

You can work out for yourself that the electron has a rest-energy of $E_{\text{e}} = 0.511$ MeV. The rest-energies of various particles are listed in the following table. The ones preceded by a "≈" are the averages of differently charged particles (which differ by a couple MeV). These (and the many other) elementary particles have specific properties (spin, charge, etc.), but for the present purposes they need only be thought of as point objects having a definite mass.

particle	rest-energy (MeV)
electron (e)	0.511
muon (μ)	105.7
tau (τ)	1784
proton (p)	938.3
neutron (n)	939.6
lambda (Λ)	1115.6
sigma (Σ)	≈ 1193
delta (Δ)	≈ 1232
pion (π)	≈ 137
kaon (K)	≈ 496

We now come to a slight abuse of language. When particle-physicists talk about masses, they say things like, "The mass of a proton is 938 MeV." This, of course, makes no sense, because the units are wrong; a mass can't equal an energy. But what they mean is that if you take this energy and divide it by c^2, then you get the mass. It would truly be a pain to keep saying, "The mass is such-and-such an energy, divided by c^2." For a quick conversion back to kilograms, you can show that

$$1 \text{ MeV}/c^2 = 1.783 \cdot 10^{-30} \text{ kg}. \tag{11.41}$$

[7]This is getting a little picky, but "eV" should actually be written as "eV". At any rate, you should remember that "eV" stands for two things that are multiplied together (in contrast with the "kg" symbol for "kilogram"), one of which is the electron charge, which is usually denoted by e.

11.5 Force

11.5.1 Force in one dimension

"Force" is a fairly intuitive concept. It's how hard you push or pull on something. We were told long ago that $\mathbf{F}$ equals $m\mathbf{a}$, and this makes sense. If you push an object in a certain direction, then it accelerates in that direction. But, alas, we've now outgrown the $\mathbf{F} = m\mathbf{a}$ definition. It's time to look at things a different way.

The force on an object is hereby *defined* to be the rate of change of momentum,

$$F = \frac{dp}{dt}\,. \tag{11.42}$$

(We'll just deal with one-dimensional motion for now.) It turns out that F does *not* equal ma. In the nonrelativistic case, $p = mv$, so we do have $F = ma$. But in the relativistic case, $p = \gamma mv$, and we need to remember that γ changes with time.

To see what form F takes in terms of the acceleration, a, note that

$$\frac{d\gamma}{dt} \equiv \frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2}}\right) = \frac{v\dot{v}}{(1-v^2)^{3/2}} \equiv \gamma^3 va. \tag{11.43}$$

Therefore,

$$\begin{aligned} F = \frac{d(\gamma mv)}{dt} &= m(\dot{\gamma}v + \gamma\dot{v}) \\ &= ma(\gamma^3 v^2 + \gamma) \\ &= \gamma^3 ma, \end{aligned} \tag{11.44}$$

assuming that m is constant. This doesn't look as nice as $F = ma$, but that's the way it goes. For reasons we will see in the next chapter, it doesn't make any sense to claim that $F = ma$ is a physical law. But $F = dp/dt$ works just fine.

> They *said*, "F is ma, bar none."
> What they *meant* sounded not as much fun.
> It's dp by dt,
> Which just happens to be
> Good ol' "ma" when γ is 1.

Consider now the quantity dE/dx, where E is the energy, $E = \gamma m$. We have

$$\begin{aligned} \frac{dE}{dx} = \frac{d(\gamma m)}{dx} &= m\frac{d(1/\sqrt{1-v^2})}{dx} \\ &= \gamma^3 mv\frac{dv}{dx}\,. \end{aligned} \tag{11.45}$$

But $v(dv/dx) = dv/dt \equiv a$. Therefore, combining eq. (11.45) with eq. (11.44), we find

$$F = \frac{dE}{dx}\,. \tag{11.46}$$

Note that eqs. (11.42) and (11.46) take exactly the same form as in the nonrelativistic case. The only new thing in the relativistic case is that the expressions for p and E are modified.

REMARKS:

1. Eq. (11.42) is devoid of any physical content, because all it does is define F. If F were instead defined through eq. (11.46), then eq. (11.46) would be devoid of any content. The whole point of this section, and the only thing of any substance, is that (with the definitions $p = \gamma m v$ and $E = \gamma m$)
$$\frac{dp}{dt} = \frac{dE}{dx}\,. \tag{11.47}$$
This is the physically meaningful statement. If we then want to label both sides of the equation with the letter F, so be it. But "force" is simply a name.

2. The result in eq. (11.46) suggests another way to arrive at the $E = \gamma m$ relation. The reasoning is exactly the same as in the nonrelativistic derivation of energy conservation in Section 4.1. Define F, as we have done, through eq. (11.42). Then integrate eq. (11.44) from x_1 to x_2 to obtain
$$\begin{aligned}\int_{x_1}^{x_2} F\,dx &= \int_{x_1}^{x_2} (\gamma^3 m a)dx \\ &= \int_{x_1}^{x_2} \left(\gamma^3 m v \frac{dv}{dx}\right) dx \\ &= \gamma m\Big|_{v_1}^{v_2}. \end{aligned}\tag{11.48}$$
If we then define the "potential energy" as
$$V(x) \equiv -\int_{x_0}^{x} F(x)dx, \tag{11.49}$$
where x_0 is an arbitrary reference point, we obtain
$$V(x_1) + \gamma m\Big|_{v_1} = V(x_2) + \gamma m\Big|_{v_2}. \tag{11.50}$$
Hence, the quantity $V + \gamma m$ is independent of x. It is therefore worthy of a name, and we use the name "energy", due to the similarity with the Newtonian result.[8]

   The work-energy theorem (that is, $\int F\,dx = \Delta E$) holds in relativistic dynamics, just as it does in the nonrelativistic case. The only difference is that E is γm instead of $mv^2/2$. ♣

11.5.2 Force in two dimensions

In two dimensions, the concept of force becomes a little strange. In particular, as we will see, the acceleration of an object need not point in the same direction as the force.

[8] Actually, this derivation only suggests that E is given by γm up to an additive constant. For all we know, E might take the form $E = \gamma m - m$, which would make the energy of a motionless particle equal to zero. An argument along the lines of Section 11.1.2 is required to show that the additive constant is zero.

We start with the definition

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}\,. \tag{11.51}$$

This is a vector equation. Without loss of generality, let us deal with only two spatial dimensions. Consider a particle moving in the x-direction, and let us apply a force $\mathbf{F} = (F_x, F_y)$.

The momentum is

$$\mathbf{p} = \frac{m(v_x, v_y)}{\sqrt{1 - v_x^2 - v_y^2}}\,. \tag{11.52}$$

Taking the derivative of this, and using the fact that v_y is initially zero, we obtain

$$\begin{aligned}
\mathbf{F} = \frac{d\mathbf{p}}{dt}\bigg|_{v_y=0} &= m\left(\frac{\dot{v}_x}{\sqrt{1-v^2}} + \frac{v_x(v_x\dot{v}_x + v_y\dot{v}_y)}{(\sqrt{1-v^2})^3}\,,\ \frac{\dot{v}_y}{\sqrt{1-v^2}} + \frac{v_y(v_x\dot{v}_x + v_y\dot{v}_y)}{(\sqrt{1-v^2})^3}\right)\bigg|_{v_y=0} \\
&= m\left(\frac{\dot{v}_x}{(\sqrt{1-v^2})^3}\,,\ \frac{\dot{v}_y}{\sqrt{1-v^2}}\right) \\
&\equiv m(\gamma^3 a_x, \gamma a_y).
\end{aligned} \tag{11.53}$$

Note that this is *not* proportional to (a_x, a_y). The first component agrees with eq. (11.44), but the second component has only one factor of γ. The difference comes from the fact that γ has a first-order change if v_x changes, but not if v_y changes (assuming v_y is initially zero).

The particle therefore responds differently to forces in the x- and y-directions. It's easier to accelerate something in the transverse direction.

11.5.3 Transformation of forces

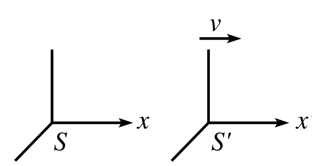


Figure 11.9

Let a force act on a particle. How are the components of the force in the particle's frame[9] (S') related to the components of the force in another frame (S)? Let the relative motion be along the x- and x'-axes, as in Fig. 11.9.

In frame S, eq. (11.53) says

$$(F_x, F_y) = m\left(\gamma^3 \frac{d^2x}{dt^2}\,, \gamma\frac{d^2y}{dt^2}\right). \tag{11.54}$$

And in frame S', $\gamma = 1$, so we just have the usual expression

$$(F'_x, F'_y) = m\left(\frac{d^2x'}{dt'^2}\,, \frac{d^2y'}{dt'^2}\right). \tag{11.55}$$

where $t' \equiv \tau$ is the proper time as measured by the particle.

[9] To be more precise, S' is an instantaneous inertial frame of the particle. Once the force is applied, the particle's rest frame will no longer be S'. But for a very small elapsed time, the frame will still essentially be S'.

Consider two events located at the particle, (for example, say it emits two flashes of light). Then you can show that the Lorentz transformations give $\Delta t' = \Delta t/\gamma$, and $\Delta x' = \gamma(\Delta x - v\Delta t)$, and $\Delta y' = \Delta y$. Therefore, eq. (11.55) may be written as

$$\begin{aligned}(F'_x, F'_y) &= m\left(\gamma^2 \frac{d^2x'}{dt^2}\,,\,\gamma^2 \frac{d^2y'}{dt^2}\right) \\ &= m\left(\gamma^3 \frac{d^2x}{dt^2}\,,\,\gamma^2 \frac{d^2y}{dt^2}\right). \end{aligned} \tag{11.56}$$

Comparing eqs. (11.54) and (11.56), we find

$$F_x = F'_x, \qquad \text{and} \qquad F_y = \frac{F'_y}{\gamma}\,. \tag{11.57}$$

We see that the longitudinal force is the same in the two frames, but the transverse force is larger by a factor of γ in the particle's frame.

Remarks:

1. What if someone comes along and relabels the primed and unprimed frames in eq. (11.57), and concludes that the transverse force is *smaller* in the particle's frame? He certainly can't be correct (given that eq. (11.57) is true), but where is the error?

   The error lies in the fact that we (correctly) used $\Delta t' = \Delta t/\gamma$, because this is the relevant expression concerning two events along the particle's worldline. (We are interested in two such events, because we want to see how the particle moves.) The inverted expression, $\Delta t = \Delta t'/\gamma$, deals with two events located at the same position in S, and therefore has nothing to do with the situation at hand. (Similar reasoning holds for the relation between Δx and $\Delta x'$). There is indeed one frame here that is special among all the possible frames, namely the particle's instantaneous inertial frame.

2. If you want to compare forces in two frames, neither of which is the particle's rest frame, then just use eq. (11.57) twice and relate each of the forces to the rest-frame forces. You can easily show that for another frame S'', the result is $F''_x = F_x$, and $\gamma'' F''_y = \gamma F_y$ (where the γ's are measured relative to the rest fame S'). ♣

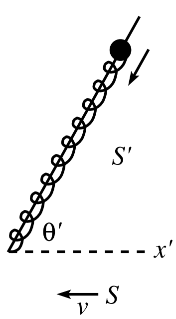


Figure 11.10

Example (Bead on a rod): A spring with a tension has one end attached to the end of a rod, and the other end attached to a bead which is constrained to move along the rod. The rod makes an angle θ' with respect to the x'-axis, and is fixed at rest in the S' frame (see Fig. 11.10). The bead is released and is pulled along the rod.

When the bead is released, what does the situation look like in the frame (S) of someone moving to the left at speed v? In answering this, draw the directions of (1) the rod, (2) the acceleration of the bead, and (3) the force on the bead.

In frame S, does the wire exert a force of constraint?

Solution: In frame S:

(1) The horizontal span of the rod is decreased by a factor γ, due to length contraction.

(2) The acceleration has to point along the rod, because the bead is being pulled by the spring in that direction.

(3) The y-component of the force on the bead is decreased by a factor γ, by eq. (11.57).

The situation is therefore as shown in Fig. 11.11.

As a double-check that **a** does indeed point along the rod, we can use eq. (11.53) to write $a_y/a_x = \gamma^2 F_y/F_x$. Then eq. (11.57) gives $a_y/a_x = \gamma F'_y/F'_x = \gamma \tan\theta' = \tan\theta$, which is the direction of the wire.

The wire does *not* exert a force of constraint; the bead need not touch the wire in S', so it need not touch it in S. Basically, there is no need to have an extra force to combine with **F** to make the result point along **a**. **F** simply doesn't have to be collinear with **a**.

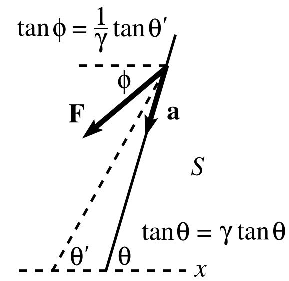


Figure 11.11

11.6 Rocket motion

Up to this point, we've dealt with situations where the masses of our particles are constant, or where they change abruptly (as in a decay, where the sum of the masses of the products is less than the mass of the initial particle). But in many problems, the mass of an object changes continuously. A rocket is the classic example of this type of situation. Hence, we will use the term "rocket motion" to describe the general class of problems where the mass changes continuously.

The relativistic rocket itself encompases all the important ideas, so let's do that example here. (Many more examples are left for the problems.) We'll present three solutions, the last of which is rather slick. In the end, the solutions are all basically the same, but it should be helpful to see various ways of looking at the problem.

Example (Relativistic rocket): Assume that a rocket propels itself by continually converting mass into photons and firing them out the back. Let m be the instantaneous mass of the rocket, and let v be the instantaneous speed with respect to the ground. Show that

$$\frac{dm}{m} + \frac{dv}{1-v^2} = 0. \qquad (11.58)$$

If the initial mass is M, and the initial v is zero, integrate eq.(11.58) to obtain

$$m = M\sqrt{\frac{1-v}{1+v}}\,. \qquad (11.59)$$

First solution: The strategy of this solution will be to use conservation of momentum in the ground frame.

Consider the effect of a small mass being converted into photons. The mass of the rocket goes from m to $m + dm$ (dm will be negative here). So in the frame of the rocket, photons with total energy $E_r = -dm$ (which is positive) are fired out the back. In the frame of the rocket, these photons have momentum $p_r = dm$ (which is negative).

Let the rocket move with speed v with respect to the ground. Then the momentum of the photons in the ground frame, p_g, may be found via the Lorentz transformation,

$$p_g = \gamma(p_r + vE_r) = \gamma\big(dm + v(-dm)\big) = \gamma(1-v)\,dm, \tag{11.60}$$

which is still negative, of course.

REMARK: A common error is to say that the converted mass $(-dm)$ takes the form of photons of energy $(-dm)$ in the ground frame. This is incorrect, because although the photons have energy $(-dm)$ in the rocket frame, they are redshifted (due to the Doppler effect) in the ground frame. From eq. (10.48), we see that the frequency (and hence the energy) of the photons decreases by a factor of $\sqrt{(1-v)/(1+v)}$ when going from the rocket frame to the ground frame. This factor equals the $\gamma(1-v)$ factor in eq. (11.60). ♣

We may now use conservation of momentum in the ground frame to say that

$$(m\gamma v)_{\text{old}} = \gamma(1-v)\,dm + (m\gamma v)_{\text{new}} \qquad \Longrightarrow \qquad \gamma(1-v)\,dm + d(m\gamma v) = 0. \tag{11.61}$$

The $d(m\gamma v)$ term may be expanded to give

$$\begin{aligned} d(m\gamma v) &= (dm)\gamma v + m(d\gamma)v + m\gamma(dv) \\ &= \gamma v\,dm + m(\gamma^3 v\,dv)v + m\gamma\,dv \\ &= \gamma v\,dm + m\gamma(\gamma^2v^2+1)\,dv \\ &= \gamma v\,dm + m\gamma^3\,dv. \end{aligned} \tag{11.62}$$

Therefore, eq. (11.61) gives

$$\begin{aligned} 0 &= \gamma(1-v)\,dm + \gamma v\,dm + m\gamma^3 dv \\ &= \gamma\,dm + m\gamma^3\,dv. \end{aligned} \tag{11.63}$$

Thus,

$$\frac{dm}{m} + \frac{dv}{1-v^2} = 0, \tag{11.64}$$

in agreement with eq. (11.58). We must now integrate this. With the given initial values, we have

$$\int_M^m \frac{dm}{m} + \int_0^v \frac{dv}{1-v^2} = 0. \tag{11.65}$$

We can simply look up the dv integral in a table, but let's do it from scratch.[10] Writing $1/(1-v^2)$ as the sum of two fractions gives

$$\begin{aligned} \int_0^v \frac{dv}{1-v^2} &= \frac{1}{2}\int_0^v \left(\frac{1}{1+v} + \frac{1}{1-v}\right) dv \\ &= \frac{1}{2}\Big(\ln(1+v) - \ln(1-v)\Big)\Big|_0^v \\ &= \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right). \end{aligned} \tag{11.66}$$

[10] Tables often list the integral of $1/(1-v^2)$ as $\tanh^{-1}(v)$, which you can show is equivalent to the result in eq. (11.66).

Eq. (11.65) therefore gives

$$\begin{aligned} \ln\left(\frac{m}{M}\right) &= \frac{1}{2}\ln\left(\frac{1-v}{1+v}\right) \\ \Longrightarrow \qquad m &= M\sqrt{\frac{1-v}{1+v}}\,, \end{aligned} \tag{11.67}$$

in agreement with eq. (11.59). Note that this result is independent of the rate at which the mass is converted into photons. It is also independent of the frequency of the emmitted photons; only the total mass expelled matters.

Remark: From eq. (11.60), or from the previous remark, we see that ratio of the energy of the photons in the ground frame to that in the rocket frame is $\sqrt{(1-v)/(1+v)}$. This factor is the same as the factor in eq. (11.67). In other words, the photons' energy decreases in exactly the same manner as the mass of the rocket (assuming that the photons are ejected with the same frequency in the rocket frame throughout the process). Therefore, in the ground frame, the ratio of the photons' energy to the mass of the rocket is constant for all time. (There must be nice intuitive explanation for this, but it eludes me.) ♣

Second solution: The strategy of this solution will be to use $F = dp/dt$ in the ground frame.

Let τ denote the time in the rocket frame. Then in the rocket frame, $dm/d\tau$ is the rate at which the mass of the rocket decreases and is converted into photons (dm is negative). The photons therefore acquire momentum at the rate $dp/d\tau = dm/d\tau$ in the rocket frame. Since force is the rate of change of momentum, we see that a force of $dm/d\tau$ pushes the photons backwards, and an equal and opposite force of $F = -dm/d\tau$ pushes the rocket forwards in the rocket frame.

Now go to the ground frame. We know that the longitudinal force is the same in both frames (from eq. (11.57)), so $F = -dm/d\tau$ is also the force on the rocket in the ground frame. And since $t = \gamma\tau$, where t is the time on the ground (the photon emissions occur at the same place in the rocket frame, so we have indeed put the time-dilation factor of γ in the right place), we have

$$F = -\gamma\frac{dm}{dt}\,. \tag{11.68}$$

Remark: We may also calculate the force on the rocket by working entirely in the ground frame. Consider a mass $(-dm)$ that is converted into photons. Initially, this mass is traveling along with the rocket, so it has momentum $(-dm)\gamma v$. After it is converted into photons, it has momentum $\gamma(1-v)\,dm$ (from the first solution above). The change in momentum is therefore $\gamma(1-v)\,dm - (-dm)\gamma v = \gamma\,dm$. Since force is the rate of change of momentum, a force of $\gamma\,dm/dt$ pushes the photons backwards, and an equal and opposite force of $F = -\gamma\,dm/dt$ therefore pushes the rocket forwards. ♣

Now things get a little tricky. It is tempting to write down $F = dp/dt = d(m\gamma v)/dt = (dm/dt)\gamma v + m\,d(\gamma v)/dt$. This, however, is not correct, because the dm/dt term is not relevant here. When the force is applied to the rocket at an instant when the rocket has mass m, the only thing the force cares about is that the mass of the rocket is m.

It does not care that m is changing.[11] Therefore, the correct expression we want is

$$F = m\frac{d(\gamma v)}{dt}\,. \tag{11.69}$$

As in the first solution above, or in eq. (11.44), we have $d(\gamma v)/dt = \gamma^3\, dv/dt$. Using the F from eq. (11.68), we arrive at

$$-\gamma\frac{dm}{dt} = m\gamma^3\frac{dv}{dt}\,, \tag{11.70}$$

which is equivalent to eq. (11.63). The solution procedes as before.

Third solution: The strategy of this solution will be to use conservation of energy and momentum in the ground frame, in a slick way.

Consider a clump of photons fired out the back. The energy and momentum of these photons are equal in magnitude and opposite in sign (with the convention that the photons are fired in the negative direction). By conservation of energy and momentum, the same statement must be true about the changes in energy and momentum of the rocket. That is,

$$d(\gamma m) = -d(\gamma m v) \qquad \Longrightarrow \qquad d(\gamma m + \gamma m v) = 0. \tag{11.71}$$

Therefore, $\gamma m(1+v)$ is a constant. We are given that $m = M$ when $v = 0$. Hence, the constant must be M. Therefore,

$$\gamma m(1+v) = M \qquad \Longrightarrow \qquad m = M\sqrt{\frac{1-v}{1+v}}\,. \tag{11.72}$$

Now, *that's* a quick solution, if there ever was one!

11.7 Relativistic strings

Consider a "massless" string[12] with constant (that is, independent of length) tension T. We consider such an object for two reasons. First, such things (or reasonable approximations thereof) actually do occur in nature. For example, the gluon force which holds quarks together is approximately constant over distance. And second, these strings open the door to a whole new supply of problems you can do, like the following one.

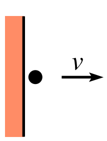


Figure 11.12

Example (Mass connected to a wall): A mass m is connected to a wall by a relativistic string with constant tension T. The mass starts next to the wall and has initial speed v away from it (see Fig. 11.12).

How far will it get from the wall? How much time will it take to reach this point?

[11] Said in a different way, the momentum associated with the missing mass still exists. It's simply not part of the rocket anymore. This issue is expanded on in Appendix E.

[12] By "massless", we mean that the string has no mass in its unstretched (that is, zero-length) state. Once it is stretched, it will have energy, and hence mass.

Solution: Let ℓ be the maximum distance from the wall. The initial energy of the mass is $E = m/\sqrt{1-v^2} \equiv \gamma m$. The final energy at $x = \ell$ is simply m, because the mass is at rest there. Integrating $F = dE/dx$ (and using the fact that the force always equals $-T$), we have

$$F\Delta x = \Delta E \quad \Longrightarrow \quad (-T)\ell = m - \gamma m \quad \Longrightarrow \quad \ell = \frac{m(\gamma - 1)}{T}\,. \qquad (11.73)$$

Let t be the time it takes to reach this point. The initial momentum of the mass is $p = \gamma m v$. Integrating $F = dp/dt$ (and using the fact that the force always equals $-T$), we have

$$F\Delta t = \Delta p \quad \Longrightarrow \quad (-T)t = 0 - \gamma m v \quad \Longrightarrow \quad t = \frac{\gamma m v}{T}\,. \qquad (11.74)$$

Note that you *cannot* use $F = ma$ to do this problem. F does not equal ma. It equals dp/dt (and also dE/dx).

Relativistic strings may seem a bit strange, but there is nothing more to solving a one-dimensional problem than the two equations,

$$F = \frac{dp}{dt}\,, \qquad \text{and} \qquad F = \frac{dE}{dx}\,. \qquad (11.75)$$

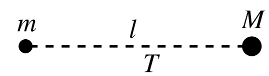


Figure 11.13

Example (Where the masses meet): A relativistic string of length ℓ and tension T connects a mass m and a mass M (see Fig. 11.13). The masses are released from rest. Where do they meet?

Solution: Let the masses meet at a distance x from the initial position of m. At this point, $F = dE/dx$ says that the energy of m is $m + Tx$, and the energy of M is $M + T(\ell - x)$. Using $p = \sqrt{E^2 - m^2}$ we see that the magnitudes of the momenta at this point are

$$p_m = \sqrt{(m + Tx)^2 - m^2} \qquad \text{and} \qquad p_M = \sqrt{(M + T(\ell - x))^2 - M^2}\,. \qquad (11.76)$$

These must be equal, because $F = dp/dt$, and because the same force (in magnitude, but opposite in direction) acts on the two masses for the same time. Equating the above p's gives

$$x = \frac{\ell\big(T(\ell/2) + M\big)}{M + m + T\ell}\,. \qquad (11.77)$$

This is reassuring, since the answer is simply the location of the initial center-of-mass, with the string being treated (quite correctly) like a stick of length ℓ and mass $T\ell$ (divided by c^2).

Remark: We can check a few limits. In the limit of large T or ℓ (more precisely, in the limit $T\ell \gg Mc^2$ and $T\ell \gg mc^2$), we have $x = \ell/2$. This makes sense, because in this case the masses are negligible and thus both move essentially at speed c, and hence meet in the middle. In the limit of small T or ℓ (more precisely, in the limit $T\ell \ll Mc^2$ and $T\ell \ll mc^2$), we have $x = M/(M + m)$, which is simply the Newtonian result for an everyday-strength spring. ♣

11.8 Mass

Some treatments of relativity refer to the mass of a motionless particle as the "rest-mass", m, and the mass of moving particle as the "relativistic mass", $m_{\text{rel}} = \gamma m$. This terminology is misleading and should be avoided. There is no such thing as "relativistic mass". There is only one "mass" associated with an object. This mass is what the above treatments would call the "rest mass".[13] And since there is only one type of mass, there is no need to use the qualifier, "rest", or the subscript "0". We therefore simply use the notation, "m". In this section, we will explain why "relativistic mass" is not a good concept to use.[14]

Why might someone want to call $m_{\text{rel}} \equiv \gamma m$ the mass of a moving particle? The basic reason is that the momentum takes the nice Newtonian form of $\mathbf{p} = m_{\text{rel}}\mathbf{v}$. The tacit assumption here is that the goal is to assign a mass to the particle such that all the Newtonian expressions continue to hold, with the only change being a modified mass. That is, we want our particle to act just like a particle of mass γm would, according to our everyday intuition.[15]

If we insist on hanging onto our Newtonian rules, let's see what they imply. If we want our particle to act as a mass γm does, then we must have $\mathbf{F} = (\gamma m)\mathbf{a}$. However, we saw in section 11.5.2 that although this equation is true for transverse forces, it is *not* true for longitudinal forces. The γm would have to be replaced by $\gamma^3 m$ if the force is longitudinal. As far as acceleration goes, a mass reacts differently to different forces, depending on their direction. We therefore see that it is impossible to assign a unique mass to a moving particle, such that it behaves in a Newtonian way under all circumstances. Not only is this goal of thinking of things in a Newtonian way ill-advised, it is doomed to failure.

"Force is my a times my 'mass',"
Said the driver, when starting to pass.
But from what we've just learned,
He was right when he turned,
But wrong when he stepped on the gas.

The above argument closes the case on this subject, but there are a few other arguments that show why it is not good to think of γm as a mass.

The word "mass" is used to describe what is on the right-hand side of the equation $E^2 - |\mathbf{p}|^2 = m^2$. The m^2 here is an *invariant*, that is, it is something that

[13]For example, the mass of an electron is $9.11 \cdot 10^{-31}$ kg, and the mass of a liter of water is 1 kg, independent of the speed.

[14]Of course, you can *define* the quantity γm to be any name you want. You can call it "relativistic mass", or you can call it "pumpkin pie". The point is that the connotations associated with these definitions will mislead you into thinking certain things are true when they are not. The quantity γm does *not* behave as you might want a mass to behave (as we will show). And it also doesn't make for a good dessert.

[15]This goal should send up a red flag. It is similar to trying to think about quantum mechanics in terms of classical mechanics. It simply cannot be done. All analogies will eventually break down and lead to incorrect conclusions. It is quite silly to try to think about a (more) correct theory (relativity or quantum mechanics) in terms of an incorrect theory (classical mechanics), simply because our intuition (which is limited and incorrect) is based on the latter.

is independent of the frame of reference. E and the components of $\mathbf{p}$, on the other hand, are components of a 4-vector. They depend on the frame. If "mass" is to be used in this definite way to describe an invariant, then it doesn't make sense to also use it to describe the quantity γm, which is frame-dependent. And besides, there is certainly no need to give γm another name. It already goes by the name "E", up to factors of c.

It is often claimed that γm is the "mass" that appears in the expression for gravitational force. If this were true, then it might be reasonable to use "mass" as a label for the quantity γm. But, in fact, it is not true. The gravitational force depends in a somewhat complicated way on the motion of the particle. For example, the force depends on whether the particle is moving longitudinally or transversely to the source. We cannot demonstrate this fact here, but suffice it to say that if one insists on using the naive force law, $F = Gm_1m_2/r^2$, then it is impossible to label the particle with a unique mass.

11.9 Exercises

Section 11.4: Particle-physics units

1. **Pion-muon race** ∗

   A pion and a muon each have energy 10 GeV. They have a 100m race. By how much distance does the muon win?

Section 11.5: Force

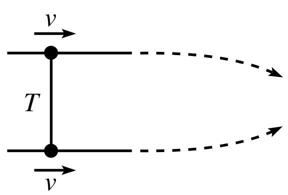


Figure 11.14

2. **Momentum paradox** ∗∗∗

   Two equal masses are connected by a massless string with tension T. The masses are constrained to move with speed v along parallel lines, as shown in Fig. 11.14. The constraints are then removed, and the masses are eventually drawn together. They collide and make one blob which continues to move to the right. Is the following reasoning correct?

   "The forces on the masses point in the y-direction. Therefore, there is no change in momentum in the x-direction. But the mass of the resulting blob is greater than the sum of the initial masses (since they collided with some relative speed). Therefore, the speed of the resulting blob must be less than v (to keep p_x constant), so the whole apparatus slows down in the x-direction."

   State what is invalid about whichever of the four sentences in this reasoning is/are invalid, if any.

Section 11.7: Relativistic strings

3. **Two masses** ∗

   A mass m is placed right in front of an identical one. They are connected by a relativistic string with tension T. The front one suddenly acquires a speed of $3c/5$. How far from the starting point will the masses collide with each other?

11.10 Problems

Section 11.1: Energy and momentum

1. **Deriving E and p** $**$

   Accepting the facts that the energy and momentum of a photon are $E = h\nu$ and $p = h\nu/c$, derive the relativistic formulas for the energy and momentum of a massive particle, $E = \gamma mc^2$ and $p = \gamma mv$. (*Hint:* Consider a mass m that decays into two photons. Look at this decay in both the rest frame of the mass, and a frame in which the mass moves at speed v. You'll need to use the Doppler effect.)

Section 11.3: Collisions and decays

2. **Colliding photons**

   Two photons each have energy E. They collide at an angle θ and create a particle of mass M. What is M?

3. **Increase in mass**

   A large mass M, moving at speed v, collides and sticks to a small mass m, initially at rest. What is the mass of the resulting object? (Work in the approximation where $M \gg m$.)

4. **Compton scattering** $**$

   A photon collides elastically with a (charged) particle of mass m. If the photon scatters at an angle θ (see Fig. 11.15), show that the resulting wavelength, λ', is given in terms of the original wavelength, λ, by

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta). \qquad (11.78)$$

   (Note: The energy of a photon is $E = h\nu = hc/\lambda$.)

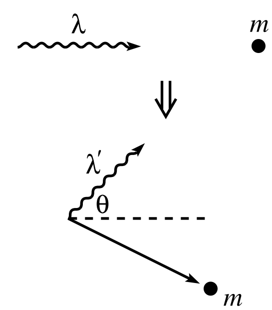


Figure 11.15

5. **Bouncing backwards** $**$

   (a) A ball of mass m and energy E collides (elastically) head-on with a stationary ball of mass M. Show that the final energy of mass m is

$$E' = \frac{2m^2M + E(M^2 + m^2)}{M^2 + m^2 + 2EM}\,. \qquad (11.79)$$

   (*Hint:* This problem is a little messy, but you can save yourself a lot of trouble by noting that $E' = E$ must be a root of an equation you get for E'.)

   (b) Fix m and M, and make E arbitrarily large. Find (approximately) the final speed of mass m.

6. **Two-body decay** $*$

   A mass M_A decays into masses M_B and M_C. What are the energies of M_B and M_C? What are their momenta?

7. **Threshold energy** $*$

   A particle of mass m and energy E collides with an identical stationary particle. What is the threshold energy for a final state containing N particles of mass m? ('Threshold energy' is the minimum energy for which the process occurs.)

Section 11.5: Force

8. **Relativistic harmonic oscillator** $**$

   A particle of mass m moves along the x-axis under a force $F = -m\omega^2 x$. The amplitude is b. Show that the period, T_0, is given by

$$T_0 = \frac{4}{c}\int_0^b \frac{\gamma}{\sqrt{\gamma^2 - 1}}dx, \tag{11.80}$$

   where

$$\gamma = 1 + \frac{\omega^2}{2c^2}(b^2 - x^2). \tag{11.81}$$

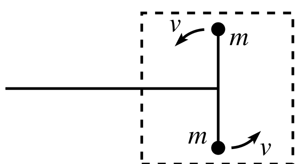


Figure 11.16

9. **System of masses** $**$

   Consider a dumbbell made of two equal masses, m. It spins around, with its center pivoted at the end of a stick (see Fig. 11.16). If the speed of the masses is v, then the energy of the system is $2\gamma m$. Treated as a whole, the system is at rest. Therefore, the mass of the system must be $2\gamma m$. (Imagine enclosing it in a box, so that you can't see what is going on inside.)

   Convince yourself that the system does indeed behave like a mass of $M = 2\gamma m$, by pushing on the stick (when the dumbbell is in the 'transverse' position shown in the figure) and showing that $F \equiv dp/dt = Ma$.

Section 11.6: Rocket motion

10. **Relativistic rocket** $**$

    Consider the relativistic rocket from section 11.6. Let mass be converted to photons at a rate σ in the rest frame of the rocket. Find the time, t, in the ground frame as a function of v.[16] (Alas, it is not possible to invert this, to get v as a function of t.)

11. **Relativistic dustpan I** $*$

    A dustpan of mass M is given an initial relativistic speed. It gathers up dust with mass density λ per unit length on the floor (as measured in the lab frame). At the instant the speed is v, find the rate (as measured in the lab frame) at which the mass of the dustpan-plus-dust-inside is increasing.

[16] This involves a slightly tricky integral. Pick your favorite method – pencil, book, or computer.

12. **Relativistic dustpan II** ∗∗

Consider the setup in Problem 11. If the initial speed of the dustpan is V, what is its speed as a function of distance and as a function of time in the lab frame?

13. **Relativistic dustpan III** ∗∗

Consider the setup in Problem 11. Calculate, in both the dustpan frame and lab frame, the force on the dustpan-plus-dust-inside as a function of v, and show that the results are equal.

14. **Relativistic cart I** ∗∗∗∗

A (very long) cart moves at a constant relativistic speed v. Sand is dropped into the cart at a rate of $dm/dt = \sigma$ in the ground frame.

Assume that you stand on the ground next to where the sand falls in, and you push on the cart to keep it moving at speed v. What is the force between your feet and the ground?

Calculate this force in both the ground frame (your frame) and the cart frame, and show that the results are equal.

15. **Relativistic cart II** ∗∗∗∗

A (very long) cart moves at a constant relativistic speed v. Sand is dropped into the cart at a rate of $dm/dt = \sigma$ in the ground frame.

Assume that you grab the front of the cart, and you pull on it to keep it moving at speed v (while running with it). What force does your hand apply to the cart? (Assume that the cart is made of the most rigid material possible.)

Calculate the force in both the ground frame and the cart frame (your frame), and show that the results are equal.

Section 11.6: Relativistic strings

16. **Different frames** ∗∗

(a) Two masses, m, are connected by a string with length ℓ and constant tension T. The masses are released simultaneously. They collide and stick together. What is the mass, M, of the resulting blob?

(b) Consider this scenario from the point of view of a frame moving to the left with speed v (see Fig. 11.17). The energy of the resulting blob must be γMc^2, from part (a). Show that you obtain this same result by computing the work done on the two masses.

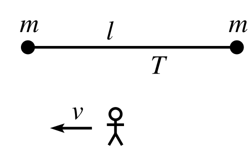


Figure 11.17

17. **Relativistic leaky bucket** ∗∗∗

(a) A massless string with constant tension T (i.e., independent of length) has one end attached to a wall and the other end attached to a mass M. The initial length of the string is ℓ (see Fig. 11.18).

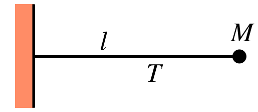


Figure 11.18

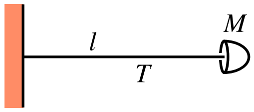


Figure 11.19

The mass is released. Halfway to the wall, the back half of the mass breaks away from the front half (with zero initial relative speed). What is the total time it takes the front half to reach the wall?

(b) The mass M in part (a) is replaced by a massless bucket containing an initial mass M of sand (see Fig. 11.19). On the way to the wall, the bucket leaks sand at a rate $dm/dx = M/\ell$, where m denotes the mass at later positions (so the rate is constant with respect to distance, not time).

i. What is the energy of the bucket, as a function of distance to the wall? What is its maximum value? What is the maximum value of the kinetic energy?

ii. What is the momentum of the bucket, as a function of distance to the wall? Where is it maximum?

18. **Relativistic bucket** ***

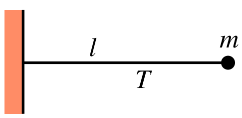


Figure 11.20

(a) A massless string with constant tension T (i.e., independent of length) has one end attached to a wall and the other end attached to a mass m The initial length of the string is ℓ (see Fig. 11.20). The mass is released. How long does it take to reach the wall?

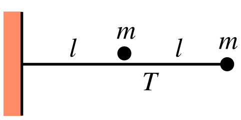


Figure 11.21

(b) Let the string now have length 2ℓ, with a mass m on the end. Let another mass m be positioned next to the ℓ mark on the string (but not touching the string). See Fig. 11.21. The right mass is released. It heads toward the wall (while the other mass is still motionless), and then sticks to the other mass to make one large blob, which then heads toward the wall.[17] How long does it take to reach the wall?[18]

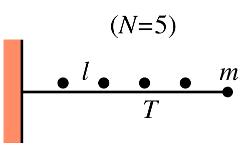


Figure 11.22

(c) Let there now be N masses and a string of length $N\ell$, as shown in Fig. 11.22. How long does it take to reach the wall?

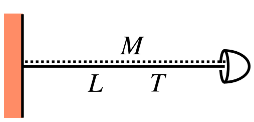


Figure 11.23

(d) Consider now a massless bucket at the end of the string (of length L) which gathers up a continuous stream of sand (of total mass M), as it gets pulled to the wall (see Fig. 11.23).

How long does it take to reach the wall?

What is the mass of the blob right before it hits the wall?

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[17] The left mass could actually be attached to the string, and we would still have the same situation. (It wouldn't move during the first part of the process, because there would be equal tensions on both sides of it).

[18] You can do this in various ways, but one method that generalizes nicely for the next part is to show that $\Delta(p^2) = (E_2^2 - E_1^2) + (E_4^2 - E_3^2)$, where the energies of the moving object (i.e., the initial m or the resulting blob) are: E_1 right at the start, E_2 just before the collision, E_3 just after the collision, E_4 right before the wall. Note that this method does not require knowledge of the mass of the blob (which is *not* $2m$).

11.11 Solutions

1. **Deriving E and p**

We'll derive the energy formula, $E = \gamma mc^2$, first. Let the given mass decay into two photons, and let E_0 be the energy of the mass in its rest frame. Then each of the resulting photons has energy $E_0/2$.

Now look at the decay in a frame where the mass moves at speed v. From eq. (10.48), the frequencies of the photons are Doppler-shifted by the factors $\sqrt{(1+v)/(1-v)}$ and $\sqrt{(1-v)/(1+v)}$. Since the photons have $E = h\nu$, their energies are shifted by the same factors. Conservation of energy therefore says that the mass (which is moving at speed v) has energy

$$E = \frac{E_0}{2}\sqrt{\frac{1+v}{1-v}} + \frac{E_0}{2}\sqrt{\frac{1-v}{1+v}} = \gamma E_0. \tag{11.82}$$

We therefore see that a moving mass has an energy which is γ times its rest energy.

We will now use the correspondence principle to find E_0 in terms of m. We just found that the difference between the energies of a moving mass and a stationary one is $\gamma E_0 - E_0$. This must reduce to the familiar kinetic energy, $mv^2/2$, in the limit $v \ll c$. In other words,

$$\begin{aligned} \frac{mv^2}{2} &\approx \frac{E_0}{\sqrt{1-v^2/c^2}} - E_0 \\ &\approx E_0\left(1 + \frac{v^2}{2c^2}\right) - E_0 \\ &= (E_0/c^2)\frac{v^2}{2}. \end{aligned} \tag{11.83}$$

Therefore $E_0 = mc^2$, and so $E = \gamma mc^2$.

The momentum formula, $p = \gamma mv$, is derived in a similar way. Let the magnitude of the photons' (equal and opposite) momenta in the particle's rest frame be $p_0/2$.[19] Using the Doppler-shifted frequencies as above, we see that the total momentum of the photons in the frame where the mass moves at speed v is

$$p = \frac{p_0}{2}\sqrt{\frac{1+v}{1-v}} - \frac{p_0}{2}\sqrt{\frac{1-v}{1+v}} = \gamma p_0 v. \tag{11.84}$$

Putting the c's back in, this equals $\gamma p_0 v/c$. By conservation of momentum, this is the momentum of the mass m moving at speed v.

We now use the correspondence principle again. If $p = \gamma(p_0/c)v$ is to reduce to the familiar $p = mv$ result in the limit $v \ll c$, then we must have $p_0 = mc$. Therefore, $p = \gamma mv$.

2. **Colliding photons**

The 4-momenta of the photons are (see Fig. 11.24)

$$P_{\gamma_1} = (E, E, 0, 0), \qquad \text{and} \qquad P_{\gamma_2} = (E, E\cos\theta, E\sin\theta, 0). \tag{11.85}$$

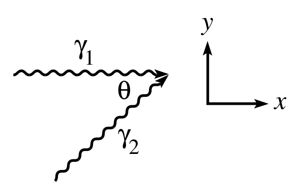

Figure 11.24

[19] With the given information that a photon has $E = h\nu$ and $p = h\nu/c$, we may use the preceding $E_0 = mc^2$ result to quickly conclude that $p_0 = mc$. But let's pretend that we haven't found E_0 yet. (This will give us an excuse the use the correspondence principle again.)

Energy and momentum are conserved, so the 4-momentum of the final particle is $P_M = (2E, E + E\cos\theta, E\sin\theta, 0)$. Hence,

$$P_M \cdot P_M = (2E)^2 - (E + E\cos\theta)^2 - (E\sin\theta)^2 = M^2. \tag{11.86}$$

Therefore, the desired mass is

$$M = E\sqrt{2(1-\cos\theta)}. \tag{11.87}$$

If $\theta = 180^\circ$ then $M = 2E$, as it should (none of the final energy is kinetic). And if $\theta = 0^\circ$ then $M = 0$, as it should (all of the final energy is kinetic).

3. **Increase in mass**

In the lab frame, the energy of the resulting object is $\gamma M + m$, and its momentum is still γMV. The mass of the object is therefore

$$M' = \sqrt{(\gamma M + m)^2 - (\gamma MV)^2} = \sqrt{M^2 + 2\gamma Mm + m^2}\,. \tag{11.88}$$

The m^2 term is negligible compared to the other two, so we may approximate M' as

$$M' \approx M\sqrt{1 + \frac{2\gamma m}{M}} \approx M\left(1 + \frac{\gamma m}{M}\right) = M + \gamma m, \tag{11.89}$$

where we have used the Taylor series, $\sqrt{1+\epsilon} \approx 1 + \epsilon/2$.

Therefore, the increase in mass is γ times the mass of the stationary object. (The increase must clearly be greater than the nonrelativistic answer of "m", because heat is generated during the collision, and this heat shows up as mass in the final object.)

Remark: This result is quite obvious if we work in the frame where M is initially at rest. In this frame, mass m comes flying in with energy γm, and essentially all of this energy shows up as mass in the final object (that is, essentially none of it shows up as overall kinetic energy of the object).

This is a general result. Stationary large objects pick up negligible kinetic energy when hit by small objects. This is true because the speed of the large object is porportional to m/M, by momentum conservation (there's a factor of γ if things are relativistic), so the kinetic energy goes like $Mv^2 \propto M(m/M)^2 \approx 0$. In other words, the smallness of v wins out over the largeness of M. When a snowball hits a tree, all of the initial energy goes into heat to melt the snowball; none of it goes into changing the speed of the earth.

Of course, you can alternatively just work things out in M's frame just as we did for the lab frame. ♣

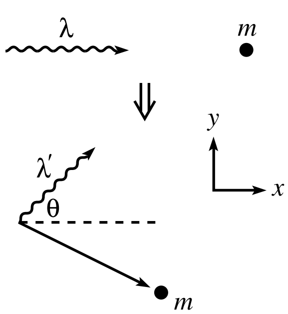

Figure 11.25

4. **Compton scattering**

The 4-momenta before the collision are (see Fig. 11.25)

$$P_\gamma = \left(\frac{hc}{\lambda}, \frac{hc}{\lambda}, 0, 0\right), \qquad P_m = (mc^2, 0, 0, 0). \tag{11.90}$$

The 4-momenta after the collision are

$$P'_\gamma = \left(\frac{hc}{\lambda'}, \frac{hc}{\lambda'}\cos\theta, \frac{hc}{\lambda'}\sin\theta, 0\right), \qquad P'_m = (\text{we won't need this}). \tag{11.91}$$

If we wanted to, we could write P'_m in terms of its momentum and scattering angle. But the nice thing about the following method is that we don't need to introduce these quantities which we're not interested in.

Conservation of energy and momentum give $P_\gamma + P_m = P'_\gamma + P'_m$. Therefore,

$$\begin{aligned} (P_\gamma + P_m - P'_\gamma)^2 &= P'^2_m \\ \Longrightarrow P^2_\gamma + P^2_m + P'^2_\gamma + 2P_m(P_\gamma - P'_\gamma) - 2P_\gamma P'_\gamma &= P'^2_m \\ \Longrightarrow 0 + m^2c^4 + 0 + 2mc^2\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right) - 2\frac{hc}{\lambda}\frac{hc}{\lambda'}(1 - \cos\theta) &= m^2c^4. \end{aligned} \tag{11.92}$$

Multiplying through by $\lambda\lambda'/(hmc^3)$ gives the desired result,

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta). \tag{11.93}$$

The nice thing about this solution is that all the unknown garbage in P'_m disappeared when we squared it.

If $\theta \approx 0$ (that is, not much scattering), then $\lambda' \approx \lambda$, as it should.

If $\theta = \pi$ (that is, backward scattering) and additionally $\lambda \ll h/mc$ (i.e., $mc^2 \ll hc/\lambda = E_\gamma$), then

$$E'_\gamma = \frac{hc}{\lambda'} \approx \frac{hc}{\frac{2h}{mc}} = \frac{1}{2}mc^2. \tag{11.94}$$

Therefore, the photon bounces back with an essentially fixed E'_γ, independent of the initial E_γ (as long as E_γ is large enough). This isn't entirely intuitive.

5. **Bouncing backwards**

(a) The 4-momenta before the collision are

$$P_m = (E, p, 0, 0), \qquad P_M = (M, 0, 0, 0), \tag{11.95}$$

where $p = \sqrt{E^2 - m^2}$. The 4-momenta after the collision are

$$P'_m = (E', p', 0, 0), \qquad P'_M = (\text{we won't need this}), \tag{11.96}$$

where $p' = \sqrt{E'^2 - m^2}$. If we wanted to, we could write P'_M in terms of its momentum. But we don't need to introduce it. Conservation of energy and momentum give $P_m + P_M = P'_m + P'_M$. Therefore,

$$\begin{aligned} & (P_m + P_M - P'_m)^2 = P'^2_M \\ \Longrightarrow\quad & P^2_m + P^2_M + P'^2_m + 2P_M(P_m - P'_m) - 2P_m P'_m = P'^2_M \\ \Longrightarrow\quad & m^2 + M^2 + m^2 + 2M(E - E') - 2(EE' - pp') = M^2 \qquad (11.97) \\ \Longrightarrow\quad & \Big((m^2 - EE') + M(E - E')\Big)^2 = p^2p'^2 = \left(\sqrt{E^2 - m^2}\sqrt{E'^2 - m^2}\right)^2 \\ \Longrightarrow\quad & m^2(E^2 - 2EE' + E'^2) + 2(m^2 - EE')M(E - E') + M^2(E - E')^2 = 0. \end{aligned}$$

As claimed, $E' = E$ is a root of this equation (because $E' = E$ and $p' = p$ certainly satisfy conservation of energy and momentum with the initial conditions, by definition). Dividing through by $(E - E')$ gives $m^2(E - E') + 2M(m^2 - EE') + M^2(E - E') = 0$. Solving for E' gives the desired result,

$$E' = \frac{2m^2M + E(M^2 + m^2)}{M^2 + m^2 + 2EM}. \tag{11.98}$$

We should double-check a few limits:

i. $E \approx m$ (barely moving): then $E' \approx m$ (because m is still barely moving).

ii. $M \gg E$ (brick wall): then $E' \approx E$ (because the heavy mass M picks up no energy).

iii. $m \gg M$: then $E' \approx E$ (because it's essentially like M is not there). (Actually, this only holds if E isn't too big; more precisely, we need $EM \ll m^2$.)

iv. $m = M$: then $E' = m$ (because m stops and M picks up all the energy that m had).

v. $E \gg M \gg m$: then $E' \approx M/2$ (not obvious, but similar to an analogous limit in Compton scattering). m moves essentially with speed c, which is consistent with the result of part (b) below.

(b) In the limit $E \gg M, m$, we have

$$\frac{M^2 + m^2}{2M} = E' = \frac{m}{\sqrt{1 - v'^2}} \qquad \Longrightarrow \qquad v' = \frac{m^2 - M^2}{m^2 + M^2}. \tag{11.99}$$

If we also have $m \gg M$, then $v' = 1$. If we instead also have $M \gg m$, then $v' = -1$. If $m = M$, then $v' = 0$ (which is true no matter what E is).

Note that our reasoning only determined v' up to a sign. You can get the correct sign by solving for p' above. Or, you can just use the fact that if $m \gg M$, then v' certainly has to be positive.

6. **Two-body decay**

B and C have equal and opposite momenta. Therefore,

$$E_B^2 - M_B^2 = p^2 = E_C^2 - M_C^2. \tag{11.100}$$

Also, conservation of energy gives

$$E_B + E_C = M_A. \tag{11.101}$$

Solving the two previous equations for E_B and E_C gives (using the shorthand $a \equiv M_A$, etc.)

$$E_B = \frac{a^2 + b^2 - c^2}{2a}, \qquad \text{and} \qquad E_C = \frac{a^2 + c^2 - b^2}{2a}. \tag{11.102}$$

Eq. (11.100) then gives the momentum of the particles as

$$p = \frac{\sqrt{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2}}{2a}. \tag{11.103}$$

Remark: It turns out that the quantity under the radical may be factored into

$$(a + b + c)(a + b - c)(a - b + c)(a - b - c). \tag{11.104}$$

This makes it clear that if $a = b + c$, then $p = 0$ (because there is no leftover energy for the particles to be able to move). ♣

7. **Threshold energy**

The initial 4-momenta are

$$(E, p, 0, 0), \qquad \text{and} \qquad (m, 0, 0, 0), \tag{11.105}$$

where $p = \sqrt{E^2 - m^2}$. Therefore, the final 4-momentum is $(E + m, p, 0, 0)$. The quantity $(E + m)^2 - p^2$ is an invariant, and it equals the square of the energy in the CM frame. At threshold, there is no relative motion between the final N particles (because there is no leftover energy for such motion). So the energy in the CM frame is simply Nm. We therefore have

$$(E+m)^2 - (E^2 - m^2) = (Nm)^2 \qquad \Longrightarrow \qquad E = \left(\frac{N^2}{2} - 1\right) m. \tag{11.106}$$

Note that $E \propto N^2$, for large N.

8. **Relativistic harmonic oscillator**

$F = dp/dt$ gives $-m\omega^2 x = d(m\gamma v)/dt$. Using eq. (11.44), we have

$$-\omega^2 x = \gamma^3 \frac{dv}{dt}. \tag{11.107}$$

We have to somehow solve this differential equation. A helpful thing to do is to multiply both sides by v to obtain $-\omega^2 x\dot{x} = \gamma^3 v\dot{v}$. From the derivation leading to eq. (11.44), the right-hand side is simply $d\gamma/dt$. Integration then gives $-\omega^2 x^2/2 + C = \gamma$, where C is a constant of integration. We know that $\gamma = 1$ when $x = b$, so we find

$$\gamma = 1 + \frac{\omega^2}{2c^2}(b^2 - x^2), \tag{11.108}$$

where we have put the c's back in to make the units right.

The period is given by

$$T_0 = 4\int_0^b \frac{dx}{v}. \tag{11.109}$$

But $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$, and so $v = c\sqrt{\gamma^2 - 1}/\gamma$. Therefore,

$$T_0 = \frac{4}{c}\int_0^b \frac{\gamma}{\sqrt{\gamma^2 - 1}} dx. \tag{11.110}$$

In the limit $\omega b \ll c$ (so that $\gamma \approx 1$, from eq. (11.108), i.e., the speed is always small), we must recover the Newtonian limit. Indeed, to lowest nontrivial order, $\gamma^2 \approx 1 + (\omega^2/c^2)(b^2 - x^2)$, and so

$$T_0 \approx \frac{4}{c}\int_0^b \frac{dx}{(\omega/c)\sqrt{b^2 - x^2}}. \tag{11.111}$$

This is the correct result, because conservation of energy gives $v^2 = \omega^2(b^2 - x^2)$ for a nonrelativistic spring.

9. **System of masses**

Let the speed of the stick go from 0 to ϵ, where $\epsilon \ll v$. Then the final speeds of the two masses are obtained by relativistically adding and subtracting ϵ from v. Repeating the derivation leading to eq. (11.17), we see that the final momenta of the two masses have magnitudes $\gamma_v\gamma_\epsilon(v \pm \epsilon)m$. But since ϵ is small, we may set $\gamma_\epsilon \approx 1$, to first order.

Therefore, the forward-moving mass now has momentum $\gamma_v(v+\epsilon)m$, and the backward-moving mass now has momentum $-\gamma_v(v - \epsilon)m$. The net increase in momentum is thus (with $\gamma_v \equiv \gamma$) $\Delta p = 2\gamma m\epsilon$. Hence,

$$F \equiv \frac{\Delta p}{\Delta t} = 2\gamma m \frac{\epsilon}{\Delta t} \equiv 2\gamma m a = Ma. \tag{11.112}$$

10. **Relativistic rocket**

The relation between m and v obtained in eq. (11.59) is independent of the rate at which the mass is converted to photons. We now assume a certain rate, in order to obtain a relation between v and t.

In the frame of the rocket, we have $dm/d\tau = -\sigma$. From the usual time dilation, we then have $dm/dt = -\sigma/\gamma$, since the ground frame sees the rocket's clocks run slow (that is, $t = \gamma\tau$).

From eq. (11.59), you can show

$$dm = \frac{-M\,dv}{(1+v)\sqrt{1-v^2}}\,. \tag{11.113}$$

Using $dm = -(\sigma/\gamma)dt$, this becomes

$$\int_0^v \frac{dv}{(1+v)(1-v^2)} = \int_0^t \frac{\sigma}{M}\,dt\,. \tag{11.114}$$

We can simply look up the dv integral in a table, but let's do it from scratch. Using a few partial-fraction tricks, we have

$$\begin{aligned}\int \frac{dv}{(1+v)(1-v^2)} &= \int \frac{dv}{(1+v)(1-v)(1+v)} \\ &= \frac{1}{2}\int\left(\frac{1}{1+v}+\frac{1}{1-v}\right)\frac{dv}{1+v} \\ &= \frac{1}{2}\int\frac{dv}{(1+v)^2} + \frac{1}{4}\int\left(\frac{1}{1+v}+\frac{1}{1-v}\right)dv \\ &= \frac{-1}{2(1+v)} + \frac{1}{4}\ln\left(\frac{1+v}{1-v}\right)\,. \end{aligned}\tag{11.115}$$

Equation (11.114) therefore gives

$$\frac{\sigma t}{M} = \frac{1}{2} - \frac{1}{2(1+v)} + \frac{1}{4}\ln\left(\frac{1+v}{1-v}\right)\,. \tag{11.116}$$

REMARKS: If $v \ll 1$ (or more precisely, if $v \ll c$), then we may Taylor-expand the quantities in eq. (11.116) to obtain $\sigma t/M \approx v$. This may be written as $\sigma \approx M(v/t) \equiv Ma$. But σ is simply the force acting on the rocket (or rather σc, to make the units correct), because this is the change in momentum of the photons. We therefore obtain the expected nonrelativistic $F = ma$ equation.

If v is very close to the speed of light (that is, $v = 1-\epsilon$, where ϵ is very small), then we can make approximations in eq. (11.116) to obtain $\epsilon \approx 2e^{1-4\sigma t/M}$. We see that the difference between v and 1 decreases exponentially with t. ♣

11. **Relativistic dustpan I**

(This is essentially the same problem as Problem 3.)

Let M be the mass of the dustpan-plus-dust-inside (henceforth denoted by "A") when its speed is v. After a small time dt in the lab frame, A has moved a distance $v\,dt$, so A has basically "collided" with an infinitesimal mass $\lambda v\,dt$. Its energy therefore increases to $\gamma M + \lambda v\,dt$. Its momentum is still $\gamma M v$, so its mass is now

$$M' = \sqrt{(\gamma M + \lambda v\,dt)^2 - (\gamma M v)^2} \approx \sqrt{M^2 + 2\gamma M \lambda v\,dt}\,, \tag{11.117}$$

where we have dropped the second-order dt^2 terms. Using the Taylor series $\sqrt{1+\epsilon} \approx 1+\epsilon/2$, w may approximate M' as

$$M' \approx M\sqrt{1+\frac{2\gamma\lambda v\,dt}{M}} \approx M\left(1+\frac{\gamma\lambda v\,dt}{M}\right) = M + \gamma\lambda v\,dt, \qquad (11.118)$$

Therefore, the rate of increase in A's mass is $\gamma\lambda v$. (This increase must certainly be greater than the nonrelativistic answer of "λv", because heat is generated during the collision, and this heat shows up as mass in the final object.)

REMARK: This result is quite obvious if we work in the frame where A is at rest. In this frame, a mass $\lambda v\,dt$ comes flying in with energy $\gamma\lambda v\,dt$, and essentially all of this energy shows up as mass (heat) in the final object. (That is, essentially none of it shows up as overall kinetic energy of the object, which is a general result when a small object hits a stationary large object.)

Note that the rate at which the mass increases, as measured in A's frame, is $\gamma^2\lambda v$, due to time dilation. (The dust-entering-dustpan events happen at the same location in the dustpan frame, so we have indeed put the extra γ factor in the correct place.) Alternatively, you can view it in terms of length contraction. A sees the dust contracted, so its density is increased to $\gamma\lambda$. ♣

12. **Relativistic dustpan II**

The initial momentum is $\gamma_V MV \equiv P$. Since there are no outside forces acting on the system, the momentum of the dustpan-plus-dust-inside (henceforth denoted by "A") always equals P. That is, $\gamma mv = P$, where m and v are the mass and speed of A at later times.

The energy of A, namely γm, increases due to the acquisition of new dust. Therefore, $d(\gamma m) = \lambda\,dx$, or

$$d\left(\frac{P}{v}\right) = \lambda\,dx. \qquad (11.119)$$

Integrating this and using the initial conditions gives $P/v - P/V = \lambda x$. Therefore,

$$v = \frac{V}{1+\frac{V\lambda x}{P}}. \qquad (11.120)$$

Note that for large x, this approaches $P/(\lambda x)$. This makes sense, because the mass of the dustpan-plus-dust is essentially equal to λx, and it is moving at a slow, non-relativistic speed.

To find v as a function of time, write the dx in eq. (11.119) as $v\,dt$ to obtain $(-P/v^2)\,dv = \lambda v\,dt$. Hence,

$$-\int_V^v \frac{P\,dv}{v^3} = \int_0^t \lambda\,dt \quad\Longrightarrow\quad \frac{P}{v^2} - \frac{P}{V^2} = 2\lambda t \quad\Longrightarrow\quad v = \frac{V}{\sqrt{1+\frac{2\lambda V^2 t}{P}}}. \qquad (11.121)$$

13. **Relativistic dustpan III**

Let A denote the dustpan-plus-dust-inside.

In A's frame, the density of the dust is $\gamma\lambda$, due to length contraction. Therefore, in a time $d\tau$ (τ is the time in the dustpan frame), a mass of $\gamma\lambda v\,d\tau$ crashes into A and loses its (negative) momentum of $(\gamma\lambda v\,d\tau)(\gamma v) = -\gamma^2 v^2\lambda\,d\tau$. The force on this mass is therefore $\gamma^2 v^2\lambda$. The force on A is equal and opposite to this, or

$$F = -\gamma^2 v^2\lambda. \qquad (11.122)$$

Now consider the lab frame. In time dt (t is the time in the lab frame), a mass of $\lambda v\,dt$ on the floor gets picked up by the dustpan. What is the change in momentum of this mass? It is tempting to say that it is $(\lambda v\,dt)(\gamma v)$, but this would lead to a force of $-\gamma v^2\lambda$ on the dustpan, which doesn't agree with the result from the dustpan frame (which is cause for concern, since longitudinal forces are supposed to be the same in different frames).

The key point to realize is that the mass of A increases at a rate $\gamma\lambda v$, and not λv. (This was the task of Problem 11.) We therefore see that the change in momentum of the additional mass in the dustpan is $(\gamma\lambda v\,dt)(\gamma v) = \gamma^2 v^2\lambda\,dt$. A therefore loses this much momentum,[20] and so the force on it is $F = -\gamma^2 v^2\lambda$, which agrees with the result from the dustpan frame.

14. **Relativistic cart I**

Ground frame (your frame): Using reasoning similar to that in Problem 3 or Problem 11, we see that the mass of the cart-plus-sand-inside system (denoted by "A") increases at a rate $\gamma\sigma$. Therefore, its momentum increases at a rate

$$\frac{dP}{dt} = \gamma(\gamma\sigma)v = \gamma^2\sigma v. \tag{11.123}$$

This is the force you exert on the cart, so it is also the force the ground exerts on your feet.

Cart frame: The sand-entering-cart events happen at the same location in the ground frame, so time dilation says that the sand enters the cart at a slower rate in the cart frame, that is, at the rate σ/γ. This sand flies in at speed v, and then eventually comes at rest on the cart, so its momentum decreases at a rate $\gamma(\sigma/\gamma)v = \sigma v$.

If this were the only change in momentum in the problem, then we would be in trouble, because the force on your feet would be σv in the cart frame, whereas we found above that it is $\gamma^2\sigma v$ in the ground frame. This would contradict the fact that longitudinal forces are the same in different frames. What is the resolution of this apparent paradox?

The resolution is that while you are pushing on the cart, *your mass is decreasing*. You are moving with speed v in the cart frame, and mass is continually being transferred from you (who are moving) to the cart (which is at rest). This is the missing change in momentum we need. Let's be quantitative about this.

Go back to the ground frame for a moment. We found above that the mass of A increases at rate $\gamma\sigma$ in the ground frame. Therefore, the energy of A increases at a rate $\gamma(\gamma\sigma)$ in the ground frame. The sand provides σ of this energy, so you must provide the remaining $(\gamma^2-1)\sigma$ part. Therefore, since you are losing energy at this rate, you must also be losing mass at this rate, in the ground frame.

Now go back to the cart frame. Due to time dilation, you lose mass at a rate of only $(\gamma^2-1)\sigma/\gamma$. This mass goes from moving at speed v (that is, along with you), to speed zero (that is, at rest on the cart). Therefore, the rate of change in momentum of this mass is $\gamma\big((\gamma^2-1)\sigma/\gamma\big)v = (\gamma^2-1)\sigma v$.

Adding this result to the σv result due to the sand, we see that the total rate of change of momentum is $\gamma^2\sigma v$. This, then, is the force the ground applies to your feet, in agreement with the calculation in the ground frame.

[20] A is defined here to not include the additional small bit of mass the dustpan acquires during the dt time interval. If it did, then the momentum wouldn't change, and the total force would be zero, of course.

15. **Relativistic cart II**

Ground frame: Using reasoning similar to that in Problem 3 or Problem 11, we see that the mass of the cart-plus-sand-inside system (denoted by "A") increases at a rate $\gamma\sigma$. Therefore, its momentum increases at a rate $\gamma(\gamma\sigma)v = \gamma^2\sigma v$.

However, this is *not* the force that your hand exerts on the cart. The reason is that the sand enters the cart at a location far from your hand, so your hand cannot immediately be aware of the additional need for momentum. (No matter how rigid the cart is, it can't transmit information any faster than c.) In a sense, there is a sort of Doppler effect going on, and your hand only needs to be responsible for a certain fraction of the momentum increase. Let's be quantitative about this.

Consider two grains of sand that enter the cart a time t apart. What is the difference between the two times that your hand becomes aware that the grains have entered the cart? Assuming maximal rigidity (that is, assuming that signals propagate along the cart at speed c), then the relative speed (as measured by someone on the ground) of the signals and your hand is $c-v$ (because your hand is moving at speed v, and the signals can travel no faster than c in the ground frame). The distance between the two signals is ct. Therefore, they arrive at your hand separated by a time of $ct/(c-v)$. In other words, the rate at which you feel sand entering the cart is $(c-v)/c$ times the given σ rate. This is the factor by which we must multiply the naive $\gamma^2\sigma v$ result for the force we found above. The force you must apply is therefore

$$F = \left(1 - \frac{v}{c}\right)\gamma^2\sigma v = \frac{\sigma v}{1+v}\,. \tag{11.124}$$

Cart frame (your frame): The sand-entering-cart events happen at the same location in the ground frame, so time dilation says that the sand enters the cart at a slower rate in the cart frame, that is, σ/γ. This sand flies in at speed v, and then eventually comes to rest on the cart, so its momentum decreases at a rate $\gamma(\sigma/\gamma)v = \sigma v$.

But again, this is *not* the force that your hand exerts on the cart. As before, the sand enters the cart at a location far from your hand, so your hand cannot immediately be aware of the additional need for momentum. Let's be quantitative about this.

Consider two grains of sand that enter the cart a time t apart. What is the difference between the two times that your hand becomes aware that the grains have entered the cart? Assuming maximal rigidity (that is, assuming that signals propagate along the cart at speed c), then the relative speed (as measured by someone on the cart) of the signals and your hand is c (because you are at rest). The distance between the two signals is $ct + vt$ (because the sand source is moving away from you at speed v). Therefore, the signals arrive at your hand separated by a time of $(c+v)t/c$. In other words, the rate at which you feel sand entering the cart is $c/(c+v)$ times the σ/γ rate found above. This is the factor by which we must multiply the naive σv result for the force we found above. The force you must apply is therefore

$$F = \left(\frac{1}{1+v/c}\right)\sigma v = \frac{\sigma v}{1+v}\,, \tag{11.125}$$

in agreement with eq. (11.124).

In a nutshell, the two naive results, $\gamma^2\sigma v$ and σv, differ by two factors of γ. The ratio of the two 'Doppler-effect' factors (which arose from the impossibility of absolute rigidity) precisely remedies this discrepancy.

16. **Different frames**

(a) The energy of the resulting blob is $2m + T\ell$. It is at rest, so

$$M = 2m + T\ell. \tag{11.126}$$

(b) Let this new frame be frame S. Let the original frame be S'. The crucial point to realize is that in frame S the left mass begins to accelerate before the right mass does. (Events that are simultaneous in S' are not simultaneous in S.)

Note that a longitudinal force does not change between frames, so the masses still feel a tension T in frame S.

Consider the two events when the two masses start to move. Let the left mass and right mass start moving at positions x_l and x_r in S. The Lorentz transformation $\Delta x = \gamma(\Delta x' + v\Delta t')$ tells us that $x_r - x_l = \gamma\ell$ (since $\Delta x' = \ell$ and $\Delta t' = 0$ for these events).

Let the masses collide at position x_c in S. Then the gain in energy of the left mass is $T(x_c - x_l)$, and the gain in energy of the right mass is $(-T)(x_c - x_r)$ (so this is negative if $x_c > x_r$). So the gain in the sum of the energies is

$$\Delta E = T(x_c - x_l) + (-T)(x_c - x_r) = T(x_r - x_l) = \gamma T\ell. \tag{11.127}$$

The initial sum of energies was $2\gamma m$, so the final energy is

$$E = 2\gamma m + \gamma T\ell = \gamma M, \tag{11.128}$$

as desired.

17. **Relativistic leaky bucket**

(a) We'll calculate the times for the two parts of the process to occur.

The energy of the mass right before it breaks is $E_b = M + T(\ell/2)$. So the momentum is $p_b = \sqrt{E_b^2 - M^2} = \sqrt{MT\ell + T^2\ell^2/4}$. Using $F = dp/dt$, the time for the first part of the process is (since T is constant)

$$t_1 = \frac{1}{T}\sqrt{MT\ell + T^2\ell^2/4}. \tag{11.129}$$

The momentum of the front half of the mass immediately after it breaks is $p_a = p_b/2 = (1/2)\sqrt{MT\ell + T^2\ell^2/4}$. The energy at the wall is $E_w = M/2 + 3T\ell/4$, so the momentum at the wall is $p_w = \sqrt{E_w^2 - (M/2)^2} = (1/2)\sqrt{3MT\ell + 9T^2\ell^2/4}$. The change in momentum during the second part of the process is therefore $\Delta p = p_w - p_a = (1/2)\sqrt{3MT\ell + 9T^2\ell^2/4} - (1/2)\sqrt{MT\ell + T^2\ell^2/4}$. The time for the second part is then

$$t_2 = \frac{1}{2T}\left(\sqrt{3MT\ell + 9T^2\ell^2/4} - \sqrt{MT\ell + T^2\ell^2/4}\right). \tag{11.130}$$

The total time is $t_1 + t_2$, which simply changes the minus sign in the above expression to a plus sign.

(b) i. Let the wall be at $x = 0$, and let the initial position be at $x = \ell$. Consider a small interval during which the bucket moves from x to $x + dx$ (where dx is negative). The bucket's energy changes by $T(-dx)$ due to the string, and also changes by a fraction dx/x, due to the leaking. Therefore, $dE = T(-dx) + E dx/x$, or

$$\frac{dE}{dx} = -T + \frac{E}{x}. \qquad (11.131)$$

In solving this differential equation, it is convenient to introduce the variable $y = E/x$. Then $E' = xy' + y$. So eq. (11.131) becomes $xy' = -T$, or

$$dy = \frac{-T dx}{x}. \qquad (11.132)$$

Integration gives $y = -T \ln x + C$, which we may write as $y = -T \ln(x/\ell) + B$ (since it's much nicer to have dimensionless arguments in a log). Therefore, $E = xy$ is given by

$$E = Bx - Tx \ln(x/\ell). \qquad (11.133)$$

The reasoning up to this point is valid for both the total energy and the kinetic energy. Let's now look at each of these cases.

- **Total energy:**
  Eq. (11.133) gives

  $$E = M(x/\ell) - Tx \ln(x/\ell), \qquad (11.134)$$

  where the constant of integration, B, has been chosen so that $E = M$ when $x = \ell$.
  To find the maximum of E, it is more convenient to work with the fraction $z \equiv x/\ell$, in terms of which $E = Mz - T\ell z \ln z$. Setting dE/dz equal to zero gives

  $$\ln z = \frac{M}{T\ell} - 1 \qquad \Longrightarrow \qquad E_{\text{max}} = \frac{T\ell}{e} e^{M/T\ell}. \qquad (11.135)$$

  The fraction z must satisfy $0 < z < 1$, so we must have $-\infty < \ln z < 0$. Therefore, a solution for z exists only for $M < T\ell$. If $M > T\ell$, then the energy decreases all the way to the wall.
  If M is just slightly less then $T\ell$, then E quickly achieves a maximum of slightly more than M, then decreases for the rest of the way to the wall.
  If $M \ll T\ell$, then E achieves its maximum value of $T\ell/e$ at $x/\ell \equiv z \approx 1/e$.
- **Kinetic energy:**
  Eq. (11.133) gives

  $$KE = -Tx \ln(x/\ell), \qquad (11.136)$$

  where the constant of integration, B, has been chosen so that $KE = 0$ when $x = \ell$. (Equivalently, $E - KE$ must equal the mass $M(x/\ell)$.)
  In terms of the fraction $z \equiv x/\ell$, we have $KE = -T\ell z \ln z$. Setting $d(KE)/dz$ equal to zero gives

  $$z = \frac{1}{e} \qquad \Longrightarrow \qquad KE_{\text{max}} = \frac{T\ell}{e}, \qquad (11.137)$$

  independent of M. Since this result is independent of M, it must hold in the nonrelativistic limit. And indeed, the analogous 'leaky-bucket' problem in chapter 4 gave the same result.

ii. Eq. (11.134) gives, with $z \equiv x/\ell$,

$$p = \sqrt{E^2 - (Mz)^2} = \sqrt{(Mz - T\ell z \ln z)^2 - (Mz)^2}$$
$$= \sqrt{-2MT\ell z^2 \ln z + T^2\ell^2 z^2 \ln^2 z}\,. \quad (11.138)$$

Setting the derivative equal to zero gives $T\ell \ln^2 z + (T\ell - 2M)\ln z - M = 0$. Therefore, the maximum p occurs at

$$\ln z = \frac{2M - T\ell - \sqrt{T^2\ell^2 + 4M^2}}{2T\ell}. \quad (11.139)$$

(The other root is ignored, since it gives $\ln z > 0$.)

If $M \ll T\ell$, then this implies $z \approx 1/e$. (The bucket becomes relativistic, so we have $E \approx pc$. Therefore, both E and p should achieve their maxima at the same place. This agrees with the result for E above.)
If $M \gg T\ell$, then this implies $z \approx 1/\sqrt{e}$. (In this case the bucket is nonrelativistic, and the result here agrees with the analogous 'leaky-bucket' problem in chapter 4).

18. **Relativistic bucket**

(a) The mass's energy just before it hits the wall is $E = m + T\ell$. Therefore, the momentum just before the wall is $p = \sqrt{E^2 - m^2} = \sqrt{2mT\ell + T^2\ell^2}$. $F = \Delta p/\Delta t$ then gives (using the fact that the tension is constant)

$$\Delta t = \frac{\Delta p}{F} = \frac{\sqrt{2mT\ell + T^2\ell^2}}{T}. \quad (11.140)$$

If $m \ll T\ell$, then $\Delta t \approx \ell$ (or ℓ/c in normal units), which is correct, since the mass must travel at speed c.

If $m \gg T\ell$, then $\Delta t \approx \sqrt{2m\ell/T}$. This the nonrelativistic limit. The answer agrees with the familiar $\ell = at^2/2$, where $a = T/m$ is the acceleration.

(b) **Straightforward method:** The energy of the blob right before it hits the wall is $E_w = 2m + 2T\ell$. If we can find the mass, M, of the blob, then we can use $p = \sqrt{E^2 - M^2}$ to get the momentum, and then $\Delta t = \Delta p/F$ to get the time.

The momentum right before the collision is $p_b = \sqrt{2mT\ell + T^2\ell^2}$, and this is also the momentum of the blob right after the collision, p_a.

The energy of the blob right after the collision is $E_a = 2m + T\ell$. So the mass of the blob after the collision is $M = \sqrt{E_a^2 - p_a^2} = \sqrt{4m^2 + 2mT\ell}$.

Therefore, the momentum at the wall is $p_w = \sqrt{E_w^2 - M^2} = \sqrt{6mT\ell + 4T^2\ell^2}$, and hence

$$\Delta t = \frac{\Delta p}{F} = \frac{\sqrt{6mT\ell + 4T^2\ell^2}}{T}. \quad (11.141)$$

Note that if $m = 0$ then $t = 2\ell$, as it should.

Better method: In the notation in the footnote in the problem, the change in p^2 from the start to just before the collision is $\Delta(p^2) = E_2^2 - E_1^2$. This is true because

$$E_1^2 - m^2 = p_1^2, \qquad \text{and} \qquad E_2^2 - m^2 = p_2^2, \quad (11.142)$$

and since m is the same throughout the first half of the process, we have $\Delta(E^2) = \Delta(p^2)$.

Likewise, the change in p^2 during the second half of the process is $\Delta(p^2) = E_4^2 - E_3^2$, because

$$E_3^2 - M^2 = p_3^2, \qquad \text{and} \qquad E_4^2 - M^2 = p_4^2, \tag{11.143}$$

and since M is the same throughout the second half of the process,[21] we have $\Delta(E^2) = \Delta(P^2)$.

The total change in p^2 is the sum of the above two changes, so the final p^2 is

$$\begin{aligned} p^2 &= (E_2^2 - E_1^2) + (E_4^2 - E_3^2) \\ &= \Big((m + T\ell)^2 - m^2\Big) + \Big((2m + 2T\ell)^2 - (2m + T\ell)^2\Big) \\ &= 6mT\ell + 4T^2\ell^2, \end{aligned} \tag{11.144}$$

as in the first solution above. (The first solution basically performs the same calculation, but in a more obscure manner.)

(c) The reasoning in part (b) tells us that the final p^2 equals the sum of the $\Delta(E^2)$ over the N parts of the process. So we have (using an indexing notation analogous to that in part (b))

$$\begin{aligned} p^2 &= \sum_{k=1}^{N} \Big(E_{2k}^2 - E_{2k-1}^2\Big) \\ &= \sum \Big((km + kT\ell)^2 - \big(km + (k-1)T\ell\big)^2\Big) \\ &= \sum \Big(2kmT\ell + \big(k^2 - (k-1)^2\big)T^2\ell^2\Big) \\ &= N(N+1)mT\ell + N^2T^2\ell^2. \end{aligned} \tag{11.145}$$

Therefore,

$$\Delta t = \frac{\Delta p}{F} = \frac{\sqrt{N(N+1)mT\ell + N^2T^2\ell^2}}{T}. \tag{11.146}$$

This checks with the results from parts (a) and (b).

(d) We want to take the limit $N \to \infty$, $\ell \to 0$, $m \to 0$, while requiring that $N\ell = L$ and $Nm = M$. Written in terms of M and L, the result in part (c) is

$$\Delta t = \frac{\sqrt{(1 + 1/N)MTL + T^2L^2}}{T} \quad \longrightarrow \quad \frac{\sqrt{MTL + T^2L^2}}{T}, \tag{11.147}$$

as $N \to \infty$. Note that this takes the same time as one particle of mass $m = M/2$, from part (a).

The mass, $\mathcal{M}$, of the final blob at the wall is

$$\mathcal{M} = \sqrt{E_w^2 - p_w^2} = \sqrt{(M + TL)^2 - (MTL + T^2L^2)} = \sqrt{M^2 + MTL}\,. \tag{11.148}$$

If $TL \ll M$, then $\mathcal{M} \approx M$, which makes sense. If $M \ll TL$, then $\mathcal{M} \approx \sqrt{MTL}$; so $\mathcal{M}$ is the geometric mean between the given mass and the energy stored in the string, which isn't entirely obvious.

[21] M happens to be $\sqrt{4m^2 + 2mT\ell}$, but the nice thing about this solution is that we don't need to know this. All we need to know is that it is constant.

Chapter 12

4-vectors

We now come to a most useful concept in relativity, namely that of *4-vectors*. Although it is possible to derive everything in special relativity without the use of 4-vectors (which is the route, give or take, that we took in the previous two chapters), they are *extremely* helpful in making calculations and concepts much simpler and far more transparent.

I have chosen to postpone their introduction in order to make it clear that everything in relativity can be derived without them. In encountering relativity for the first time, it's nice to know that no "advanced" techniques are required. But now that you've seen everything once, let's go back and derive some things in an easier way.

This situation, where 4-vectors are helpful but not necessary, is more pronounced in general relativity, where the concept of *tensors* (the generalization of 4-vectors) is, for all practical purposes, completely necessary for an understanding of the subject. We won't have time to go very deeply into GR in Chapter 13, so you'll have to just accept this fact. But suffice it to say that an eventual understanding of GR requires a firm understanding of special-relativity 4-vectors.

12.1 Definition of 4-vectors

Definition 12.1 *The 4-tuplet $A = (A_0, A_1, A_2, A_3)$ is a "4-vector" if the A_i transform under a Lorentz transformation in the same way that (cdt, dx, dy, dz) do. In other words, they must transform like (assuming the LT is along the x-direction; see Fig. 12.1):*

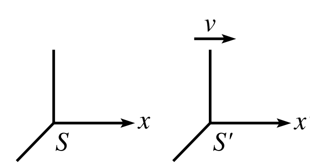

Figure 12.1

$$\begin{aligned} A_0 &= \gamma(A'_0 + (v/c)A'_1), \\ A_1 &= \gamma(A'_1 + (v/c)A'_0), \\ A_2 &= A'_2, \\ A_3 &= A'_3. \end{aligned} \tag{12.1}$$

Remarks:

1. Similar equations must hold, of course, for Lorentz transformations in the y- and z-directions.

2. Additionally, the last three components have to be a vector in 3-space (that is, they have to transform like a usual vector under rotations in 3-space).
3. We'll use a capital roman letter to denote a 4-vector. A bold-face letter will denote, as usual, a vector in 3-space.
4. The A_i may be functions of the dx_i, the x_i, their derivatives, and also v and any invariants (that is, frame-independent quantities) like the mass m.
5. Lest we get tired of writing the c's over and over, we will henceforth work in units where $c = 1$.
6. The first component of a 4-vector is sometimes referred to as the "time component". The other three are the "space components".
7. 4-vectors are the obvious generalization of vectors in regular space. A vector in 3-dimensions, after all, is something that transforms under a rotation just like (dx, dy, dz) does. We have simply generalized a 3-D rotation to a 4-D Lorentz transformation. ♣

12.2 Examples

So far, we have only one 4-vector at our disposal, namely (dt, dx, dy, dz). What are some others? Well, $(7dt, 7dx, 7dy, 7dz)$ certainly works, as does any other constant multiple of (dt, dx, dy, dz). Indeed, $m(dt, dx, dy, dz)$ is also a 4-vector, since m is an invariant (that is, independent of frame).

How about $A = (dt, 2dx, dy, dz)$? No, this isn't a 4-vector, because on one hand it has to transform like

$$\begin{aligned} dt \equiv A_0 &= \gamma(A'_0 + vA'_1) \equiv \gamma(dt' + v(2dx')), \\ 2dx \equiv A_1 &= \gamma(A'_1 + vA'_0) \equiv \gamma((2dx') + vdt'), \\ dy \equiv A_2 &= A'_2 \equiv dy', \\ dz \equiv A_3 &= A'_3 \equiv dz', \end{aligned} \tag{12.2}$$

from the definition of a 4-vector. But on the other hand, it has to transform like

$$\begin{aligned} dt &= \gamma(dt' + vdx'), \\ 2dx &= 2\gamma(dx' + vdt'), \\ dy &= dy', \\ dz &= dz', \end{aligned} \tag{12.3}$$

because this is how the dx_i transform. The two preceding sets of equations are inconsistent, so $A = (dt, 2dx, dy, dz)$ is not a 4-vector. Note that if we had instead considered the 4-tuplet $A = (dt, dx, 2dy, dz)$, then the two preceding equations would have been consistent. But if we had then looked at how A transforms under an LT in the y-direction, we would have found that it is not a 4-vector.

The moral of this story is that the above definition of a 4-vector is nontrivial because there are two possible ways a 4-tuplet can transform. We can transform it according to the 4-vector definition, as in eq. (12.2). Or, we can simply transform each A_i separately (knowing how the dx_i transform), as in eq. (12.3). Only for certain special 4-tuplets do these two methods give the same result. By definition, we call these 4-vectors.

Let us now construct some less trivial examples of 4-vectors. In constructing these, we will make abundant use of the fact that the proper-time interval, $d\tau \equiv \sqrt{dt^2 - d\mathbf{r}^2}$, is an invariant.

- **Velocity 4-vector:** We can divide (dt, dx, dy, dz) by $d\tau$, where $d\tau$ is the proper time between two events (the same two events that yielded the dt, etc.). The result is indeed a 4-vector, because $d\tau$ is independent of the frame in which it is measured. Using $d\tau = dt/\gamma$, we see that

$$V \equiv \frac{1}{d\tau}(dt, dx, dy, dz) = \gamma\left(1, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (\gamma, \gamma\mathbf{v}) \tag{12.4}$$

  is a 4-vector. This is known as the *velocity 4-vector*. (With the c's, we have $V = (\gamma c, \gamma\mathbf{v})$.) In the rest frame of the object, V reduces to $V = (1, 0, 0, 0)$.

- **Energy-momentum 4-vector:** If we multiply the velocity 4-vector by the invariant m, we obtain another 4-vector,

$$P \equiv mV = (\gamma m, \gamma m\mathbf{v}) = (E, \mathbf{p}), \tag{12.5}$$

  which is known as the *energy-momentum 4-vector* (or the *4-momentum* for short), for obvious reasons. (With the c's, we have $P = (\gamma mc, \gamma m\mathbf{v}) = (E/c, \mathbf{p})$.) In the rest frame of the object, P reduces to $P = (m, 0, 0, 0)$.

- **Acceleration 4-vector:** We can also take the derivative of the velocity 4-vector with respect to τ. The result is indeed a 4-vector, because taking the derivative is essentially taking the difference between two 4-vectors (which results in a 4-vector because eq. (12.1) is linear), and then dividing by the invariant $d\tau$ (which again results in a 4-vector). We obtain

$$A \equiv \frac{dV}{d\tau} = \frac{d}{d\tau}(\gamma, \gamma\mathbf{v}) = \gamma\left(\frac{d\gamma}{dt}, \frac{d(\gamma\mathbf{v})}{dt}\right). \tag{12.6}$$

  Using $d\gamma/dt = v\dot{v}/(1 - v^2)^{3/2} = \gamma^3 v\dot{v}$, we have

$$A = (\gamma^4 v\dot{v}\,,\, \gamma^4 v\dot{v}\mathbf{v} + \gamma^2\mathbf{a}), \tag{12.7}$$

  where $\mathbf{a} \equiv d\mathbf{v}/dt$. A is known as the *acceleration 4-vector*. In the rest frame of the object (or, rather, the instantaneous inertial frame), A reduces to $A = (0, \mathbf{a})$.

  As we always do, we will pick the relative velocity, $\mathbf{v}$, to point in the x-direction. Hence, $\mathbf{v} = (v_x, 0, 0)$, $v = v_x$, and $\dot{v} = \dot{v}_x \equiv a_x$. We then have

$$\begin{aligned} A &= (\gamma^4 v_x a_x\,,\, \gamma^4 v_x^2 a_x + \gamma^2 a_x\,,\, \gamma^2 a_y\,,\, \gamma^2 a_z) \\ &= (\gamma^4 v_x a_x\,,\, \gamma^4 a_x\,,\, \gamma^2 a_y\,,\, \gamma^2 a_z). \end{aligned} \tag{12.8}$$

  We can keep taking derivatives with respect to τ to create other 4-vectors, but they aren't very relevant to the real world.

- **Force 4-vector:** We define the *force 4-vector* as

$$F \equiv \frac{dP}{d\tau} = \gamma\left(\frac{dE}{dt}, \frac{d\mathbf{p}}{dt}\right) = \gamma\left(\frac{dE}{dt}, \mathbf{f}\right), \tag{12.9}$$

where $\mathbf{f}$ is the usual 3-force. (We'll use $\mathbf{f}$ instead of $\mathbf{F}$ in this chapter, to avoid confusion with the 4-force, F.)

In the case where m is constant,[1] F can be written as $F = d(mV)/d\tau = m\,dV/d\tau = mA$. We therefore still have a nice "F equals mA" law of physics, but it's now a 4-vector equation instead of the old 3-vector one. In terms of the acceleration 4-vector, we may write (if m is constant)

$$F = mA = (\gamma^4 mv\dot{v}\,,\, \gamma^4 mv\dot{v}\mathbf{v} + \gamma^2 m\mathbf{a}). \tag{12.10}$$

In the rest frame of the object (or, rather, the instantaneous inertial frame), F reduces to $F = (0, \mathbf{f})$, and mA reduces to $mA = (0, m\mathbf{a})$. So $F = mA$ becomes $\mathbf{f} = m\mathbf{a}$.

12.3 Properties of 4-vectors

The appealing thing about 4-vectors is that they have many useful properties. Let's look at these.

- **Linear combinations:** If A and B are 4-vectors, then $C \equiv aA + bB$ is also a 4-vector (as we noted above when deriving the acceleration 4-vector). This is true because the transformations in eq. (12.1) are linear; so the transformation of, say, the time component is

$$\begin{aligned} C_0 \equiv (aA + bB)_0 = aA_0 + bB_0 &= a(A'_0 + vA'_1) + b(B'_0 + vB'_1) \\ &= (aA'_0 + bB'_0) + v(aA'_1 + bB'_1) \\ &\equiv C'_0 + vC'_1, \end{aligned} \tag{12.11}$$

which is the proper transformation for the time component of a 4-vector. Likewise for the other components. This property holds, of course, just as it does for linear combinations of vectors in 3-space.

- **Inner product invariance:** Consider two arbitrary 4-vectors, A and B. Define their inner product to be

$$A \cdot B \equiv A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3 \equiv A_0B_0 - \mathbf{A} \cdot \mathbf{B}. \tag{12.12}$$

Then $A \cdot B$ is invariant (that is, independent of the frame in which it is calculated). This is easily shown by direct calculation, using the transformations

[1] m would not be constant if the object were being heated, or if extra mass were being added to it. We won't concern ourselves with such cases here.

in eq. (12.1).

$$\begin{aligned}
A \cdot B &\equiv A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3 \\
&= \Big(\gamma(A'_0 + vA'_1)\Big)\Big(\gamma(B'_0 + vB'_1)\Big) - \Big(\gamma(A'_1 + vA'_0)\Big)\Big(\gamma(B'_1 + vB'_0)\Big) \\
&\qquad\qquad -A'_2 B'_2 - A'_3 B'_3 \\
&= \gamma^2\Big(A'_0 B'_0 + v(A'_0 B'_1 + A'_1 B'_0) + v^2 A'_1 B'_1\Big) \\
&\quad -\gamma^2\Big(A'_1 B'_1 + v(A'_1 B'_0 + A'_0 B'_1) + v^2 A'_0 B'_0\Big) \\
&\qquad\qquad -A'_2 B'_2 - A'_3 B'_3 \\
&= A'_0 B'_0(\gamma^2 - \gamma^2 v^2) - A'_1 B'_1(\gamma^2 - \gamma^2 v^2) - A'_2 B'_2 - A'_3 B'_3 \\
&= A'_0 B'_0 - A'_1 B'_1 - A'_2 B'_2 - A'_3 B'_3 \\
&\equiv A' \cdot B'. \qquad (12.13)
\end{aligned}$$

The importance of this result cannot be overstated. This invariance is analogous, of course, to the invariance of the inner product $\mathbf{A} \cdot \mathbf{B}$ for rotations in 3-space. The above inner product is also clearly invariant under rotations in 3-space, since it involves the combination $\mathbf{A} \cdot \mathbf{B}$.

The minus signs in the inner product may seem a little strange. But the goal was to find a combination of two arbitrary vectors that is invariant under a Lorentz transformation (because such combinations are very useful in seeing what's going on in a problem). The nature of the LT's demands that there be opposite signs in the inner product, so that's the way it is.

- **Norm:** As a corollary to the invariance of the inner product, we can look at the inner product of a 4-vector with itself (which is by definition the square of the norm). We see that

$$|A|^2 \equiv A \cdot A \equiv A_0 A_0 - A_1 A_1 - A_2 A_2 - A_3 A_3 = A_0^2 - |\mathbf{A}|^2 \qquad (12.14)$$

is invariant. This is analogous, of course, to the invariance of the norm $\sqrt{\mathbf{A} \cdot \mathbf{A}}$ for rotations in 3-space.

- **A theorem:** Here's a nice little theorem:

If a certain one of the components of a 4-vector is 0 in every frame, then all four components are 0 in every frame.

Proof: If one of the space components (say, A_1) is 0 in every frame, then the other space components must also be 0 in every frame (otherwise a rotation would make $A_1 \neq 0$). Also, the time component A_0 must be 0 in every frame (otherwise a Lorentz transformation in the x-direction would make $A_1 \neq 0$).

If the time component, A_0, is 0 in every frame, then the space components must also be 0 in every frame (otherwise a Lorentz transformation in the appropriate direction would make $A_0 \neq 0$). ■

12.4 Energy, momentum

12.4.1 Norm

Many useful things arise from the simple fact that the P in eq. (12.5) is a 4-vector. The invariance of the norm implies that $P \cdot P = E^2 - |\mathbf{p}|^2$ is invariant. If we are dealing with only one particle, we may find the value of P^2 by conveniently picking the rest-frame of the particle (so that $\mathbf{v} = \mathbf{0}$), to obtain

$$E^2 - p^2 = m^2, \tag{12.15}$$

or $E^2 - p^2c^2 = m^2c^4$, with the c's. We already knew this, of course, from just writing out $E^2 - p^2 = \gamma^2 m^2 - \gamma^2 m^2 v^2 = m^2$.

For a collection of particles, knowledge of the norm is very useful. If a process involves many particles, then we can say that for *any* subset of the particles,

$$\left(\sum E\right)^2 - \left(\sum \mathbf{p}\right)^2 \qquad \text{is invariant}, \tag{12.16}$$

because this is simply the norm of the sum of the energy-momentum 4-vectors of the chosen particles, and the sum is again a 4-vector, due to the linearity of eqs. (12.1).

What is the value of this invariant? The most concise description (which is basically a tautology) is that it is the square of the energy in the CM frame (that is, in the frame where $\sum \mathbf{p} = \mathbf{0}$). For one particle, this reduces to m^2.

Note that the sums are taken before squaring in eq. (12.16). Squaring before adding would simply give the sum of the squares of the masses.

12.4.2 Transformation of E,p

We already know how the energy and momentum transform (see Section 11.2), but we'll derive the transformation again in a very quick and easy manner.

We know that (E, p_x, p_y, p_z) is a 4-vector. So it must transform according to eq. (12.1). Therefore (for an LT in the x-direction),

$$\begin{aligned} E &= \gamma(E' + v p'_x), \\ p_x &= \gamma(p'_x + vE'), \\ p_y &= p'_y, \\ p_z &= p'_z. \end{aligned} \tag{12.17}$$

That's all there is to it.

Remark: The fact that E and $\mathbf{p}$ are part of the same 4-vector provides an easy way to see that if one of them is conserved, then the other is also. Consider an interaction among a set of particles. Look at the 4-vector $\Delta P \equiv P_{\text{after}} - P_{\text{before}}$. If E is conserved in every frame, then the time component of ΔP is 0 in every frame. But then the theorem in the previous section says that all four components of ΔP are 0 in every frame. Hence, $\mathbf{p}$ is conserved. Likewise for the case where one of the p_i is known to be conserved. ♣

12.5 Force and acceleration

Throughout this section, we will deal with objects with constant mass (which we will call "particles"). The treatment can be generalized to cases where the mass changes (for example, the object is being heated, or extra mass is being dumped on it), but we won't concern ourselves with these.

12.5.1 Transformation of forces

Let us first look at the force 4-vector in the instantaneous inertial frame of a given particle (frame S'). Eq. (12.9) gives

$$F' = \gamma\left(\frac{dE'}{dt}, \mathbf{f}'\right) = (0, \mathbf{f}'). \tag{12.18}$$

The first component is zero because $dE'/dt = d(m/\sqrt{1-v'^2})/dt$ carries a factor of v', which is zero in this frame. (Equivalently, you can just use eq.(12.10), with a speed of zero.)

We may now write down two expressions for the 4-force, F, in another frame, S. First, since F is a 4-vector, it transforms according to eq. (12.1). So we have (using eq. (12.18))

$$\begin{aligned} F_0 &= \gamma(F_0' + vF_1') = \gamma v f_x', \\ F_1 &= \gamma(F_1' + vF_0') = \gamma f_x', \\ F_2 &= F_2' = f_y', \\ F_3 &= F_3' = f_z'. \end{aligned} \tag{12.19}$$

And second, from the definition in eq. (12.9), we have

$$\begin{aligned} F_0 &= \gamma dE/dt, \\ F_1 &= \gamma f_x, \\ F_2 &= \gamma f_y, \\ F_3 &= \gamma f_z. \end{aligned} \tag{12.20}$$

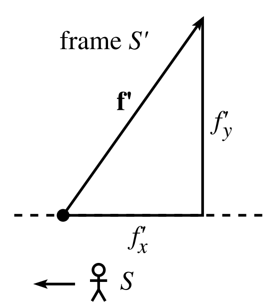

Figure 12.2

Eqs. (12.19) and (12.20) give

$$\begin{aligned} dE/dt &= v f_x', \\ f_x &= f_x', \\ f_y &= f_y'/\gamma, \\ f_z &= f_z'/\gamma. \end{aligned} \tag{12.21}$$

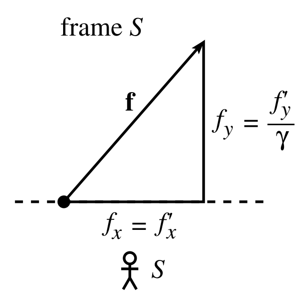

Figure 12.3

We therefore recover the results of Section 11.5.3. The longitudinal force is the same in both frames, but the transverse forces are larger by a factor of γ in the particle's frame. Hence, f_y/f_x decreases by a factor of γ when going from the particle's frame to the lab frame (see Fig. 12.2 and Fig. 12.3).

As a bonus, the F_0 component tells us (after multiplying through by dt) that $dE = f_x dx$, which is the work-energy result. (In other words, we just proved again the result $dE/dx = dp/dt$ from Section 11.5.1.)

As noted in Section 11.5.3, we can't switch the S and S' frames and write $f'_y = f_y/\gamma$. When talking about the forces on a particle, there is indeed one preferred frame of reference, namely that of the particle. All frames are not equivalent here. When forming all of our 4-vectors in Section 12.2, we explicitly used the $d\tau$, dt, dx, etc., from two events, and it was understood that these two events were located at the particle.

12.5.2 Transformation of accelerations

The procedure here is similar to the above treatment of the force.

Let us first look at the acceleration 4-vector in the instantaneous inertial frame of a given particle (frame S'). Eq. (12.7) or eq. (12.8) gives

$$A' = (0, \mathbf{a}'), \tag{12.22}$$

since $v' = 0$ in S'.

We may now write down two expressions for the 4-acceleration, A, in another frame, S. First, since A is a 4-vector, it transforms according to eq. (12.1). So we have (using eq. (12.22))

$$\begin{aligned} A_0 &= \gamma(A'_0 + vA'_1) = \gamma v a'_x, \\ A_1 &= \gamma(A'_1 + vA'_0) = \gamma a'_x, \\ A_2 &= A'_2 = a'_y, \\ A_3 &= A'_3 = a'_z. \end{aligned} \tag{12.23}$$

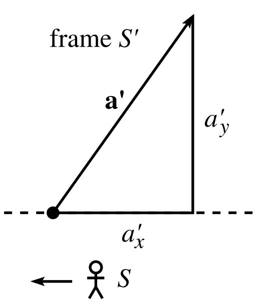

Figure 12.4

And second, from the definition in eq. (12.8), we have

$$\begin{aligned} A_0 &= \gamma^4 v a_x, \\ A_1 &= \gamma^4 a_x, \\ A_2 &= \gamma^2 a_y, \\ A_3 &= \gamma^2 a_z. \end{aligned} \tag{12.24}$$

Eqs. (12.23) and (12.24) give

$$\begin{aligned} a_x &= a'_x/\gamma^3, \\ a_x &= a'_x/\gamma^3, \\ a_y &= a'_y/\gamma^2, \\ a_z &= a'_z/\gamma^2. \end{aligned} \tag{12.25}$$

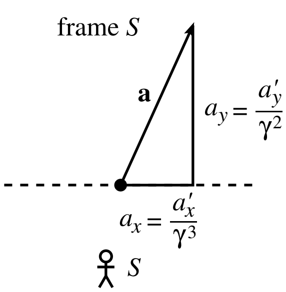

Figure 12.5

We see that a_y/a_x increases by a factor of γ when going from the particle's frame to the lab frame (see Fig. 12.4 and Fig. 12.5). This is the opposite of the effect on f_y/f_x.[2] This makes it clear that an $\mathbf{f} = m\mathbf{a}$ law wouldn't make any sense. If it's true in one frame, it might not be true in another.

Note also that the increase in a_y/a_x in going to the lab frame is consistent with length contraction, as the Bead-on-a-rod example in Section 11.5.3 showed.

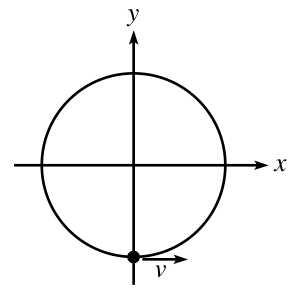

Figure 12.6

Example (Acceleration for circular motion): A particle moves with constant speed v along the circle $x^2 + y^2 = r^2$, $z = 0$, in the lab frame. At the instant the particle crosses the negative y-axis (see Fig. 12.6), find the three-acceleration and four-acceleration in both the lab frame and the instantaneous rest frame of the particle (with axes chosen parallel to the lab's axes).

Solution: Let the lab frame be S, and let the particle's instantaneous inertial frame be S' when it crosses the negative y-axis. Then S and S' are related by a boost in the x-direction.

The 3-acceleration in S is simply

$$\mathbf{a} = (0, v^2/r, 0). \tag{12.26}$$

Eq. (12.7) or (12.8) then gives the 4-acceleration in S as

$$A = (0, 0, \gamma^2 v^2/r, 0). \tag{12.27}$$

To find the vectors in S', we use the fact that the transformation between S' and S involves a boost in the x-direction. Therefore, the A_2 component is unchanged. So the 4-acceleration in S' is the same,

$$A' = A = (0, 0, \gamma^2 v^2/r, 0). \tag{12.28}$$

In the particle's frame, $\mathbf{a}'$ is simply the space part of A (using eq. (12.7) or (12.8), with $v = 0$ and $\gamma = 1$), so the 3-acceleration in S' is

$$\mathbf{a}' = (0, \gamma^2 v^2/r, 0). \tag{12.29}$$

REMARK: We can also arrive at the two factors of γ in $\mathbf{a}'$ by using a simple time-dilation argument. We have

$$a'_y = \frac{d^2y'}{d\tau^2} = \frac{d^2y'}{d(t/\gamma)^2} = \gamma^2\frac{d^2y}{dt^2} = \gamma^2\frac{v^2}{r}, \tag{12.30}$$

where we have used the fact that transverse lengths are the same in the two frames. ♣

12.6 The form of physical laws

One of the postulates of special relativity is that all inertial frames are equivalent. Therefore, if a physical law holds in one frame, then it must hold in all frames (otherwise it would be possible to differentiate between frames).

[2]In a nutshell, the difference is due to the fact that γ changes with time. When talking about acceleration, there are γ's that we have to differentiate. This isn't the case with forces, because the γ is absorbed into the definition of $\mathbf{p} = \gamma m\mathbf{v}$.

As noted in the previous section, the statement "$\mathbf{f} = m\mathbf{a}$" cannot be a physical law. The two sides transform differently when going from one frame to another, so the statement cannot be true in all frames.

If a statement has any chance of being true in all frames, it must involve only 4-vectors. Consider a 4-vector equation (say, "$A = B$") which is true in frame S. Then if we apply to this equation a Lorentz transformation from S to another frame S' (call it $\mathcal{M}$), we have

$$\begin{aligned} A &= B, \\ \Longrightarrow \quad \mathcal{M}A &= \mathcal{M}B, \\ \Longrightarrow \quad A' &= B'. \end{aligned} \tag{12.31}$$

The law is therefore true in frame S', also.

Of course, there are many 4-vector equations that are simply not true (for example, $F = P$). Only a small set of such equations (for example, $F = dP/d\tau$) correspond to the real world.

Physical laws may also take the form of scalar equations, such as $P \cdot P = m^2$. A scalar is by definition a quantity that is frame-independent (as we have shown the inner product to be). So if this statement is true in one frame, then it is true in any other. (Physical laws may also be higher-rank "tensor" equations, such as arise in electromagnetism and general relativity. We won't discuss such things here, but suffice it to say that tensors may be thought of as things built up from 4-vectors. Scalars and 4-vectors are special cases of tensors.)

This is exactly analogous, of course, to the situation in 3-D space. In Newtonian mechanics, $\mathbf{f} = m\mathbf{a}$ is a possible law, because both sides are 3-vectors. But $\mathbf{f} = m(2a_x, a_y, a_z)$ is not a possible law, because the right-hand side is not a 3-vector; it depends on which axis you label as the x-axis. Another example is the statement that a given stick has a length of 2 meters. This is fine, but if you say that the stick has an x-component of 1.7 meters, then this cannot be true in all frames.

> God said to his cosmos directors,
> "I've added some stringent selectors.
> One is the clause
> That your physical laws
> Shall be written in terms of 4-vectors."

12.7 Problems

1. **Velocity addition**

   In A's frame, B moves to the right with speed v, and C moves to the left with speed u. What is the speed of B with respect to C? (In other words, use 4-vectors to derive the velocity addition formula.)

2. **Relative speed** $*$

   In the lab frame, two particles move with speed v along the paths shown in Fig. 12.7. The angle between the trajectories is 2θ. What is the speed of one particle, as viewed by the other?

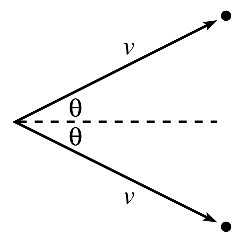

Figure 12.7

3. **Another relative speed** $*$

   In the lab frame, two particles, A and B, move with speeds u and v along the paths shown in Fig. 12.8. The angle between the trajectories is θ. What is the speed of one particle, as viewed by the other?

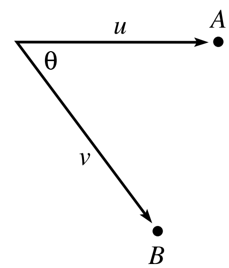

Figure 12.8

4. **Acceleration for linear motion** $*$

   A spaceship starts at rest with respect to S and accelerates with constant proper acceleration a. In section 10.8, we showed that the speed of the spaceship w.r.t. S is given by $v(\tau) = \tanh(a\tau)$, where τ is the spaceship's proper time (and $c = 1$).

   Let V be the spaceship's 4-velocity, and let A be the spaceship's 4-acceleration.

   In terms of the proper time τ (it's easier to do the problem in terms of τ than in terms of the t of frame S),

   (a) Find V and A in frame S (by explicitly using $v(\tau) = \tanh(a\tau)$).

   (b) Write down V and A in the spaceship's frame, S'.

   (c) Verify that V and A transform like 4-vectors between the two frames.

12.8 Solutions

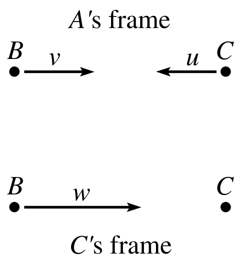

Figure 12.9

1. **Velocity addition**

   Let the desired speed of B with respect to C be w (see Fig. 12.9).

   In A's frame, the 4-velocity of B is $(\gamma_v, \gamma_v v)$, and the 4-velocity of C is $(\gamma_u, -\gamma_u u)$ (suppressing the y and z components).

   In C's frame the 4-velocity of B is $(\gamma_w, \gamma_w w)$, and the 4-velocity of C is $(1, 0)$.

   The invariance of the inner product implies

$$\begin{aligned} (\gamma_v, \gamma_v v) \cdot (\gamma_u, -\gamma_u u) &= (\gamma_w, \gamma_w w) \cdot (1, 0) \\ \Longrightarrow \quad \gamma_u \gamma_v (1 + uv) &= \gamma_w \\ \Longrightarrow \quad \frac{1 + uv}{\sqrt{1 - u^2}\sqrt{1 - v^2}} &= \frac{1}{\sqrt{1 - w^2}} . \end{aligned} \tag{12.32}$$

   Squaring, and solving for w gives

$$w = \frac{u + v}{1 + uv}. \tag{12.33}$$

2. **Relative speed**

   In the lab frame, the 4-velocities of the particles are (suppressing the z component)

$$(\gamma_v, \gamma_v v \cos\theta, -\gamma_v v \sin\theta) \qquad \text{and} \qquad (\gamma_v, \gamma_v v \cos\theta, \gamma_v v \sin\theta). \tag{12.34}$$

   Let w be the desired speed of one particle as viewed by the other. Then in the frame of one particle, the 4-velocities are (suppressing two spatial components)

$$(\gamma_w, \gamma_w w) \qquad \text{and} \qquad (1, 0). \tag{12.35}$$

   (We have rotated the axes so that the relative motion is along the x-axis in this frame.) Since the 4-vector inner product is invariant under Lorentz transformations and rotations, we have (using $\cos 2\theta = \cos^2\theta - \sin^2\theta$)

$$\begin{aligned} (\gamma_v, \gamma_v v \cos\theta, -\gamma_v v \sin\theta) \cdot (\gamma_v, \gamma_v v \cos\theta, \gamma_v v \sin\theta) &= (\gamma_w, \gamma_w w) \cdot (1, 0) \\ \Longrightarrow \quad \gamma_v^2 (1 - v^2 \cos 2\theta) &= \gamma_w . \end{aligned} \tag{12.36}$$

   Using the definitions of the γ's, squaring, and solving for w gives

$$w = \frac{\sqrt{2v^2(1 - \cos 2\theta) - v^4 \sin^2 2\theta}}{1 - v^2 \cos 2\theta} . \tag{12.37}$$

   If desired, this can be rewritten (using some double-angle formulas) in the form

$$w = \frac{2v \sin\theta \sqrt{1 - v^2 \cos^2\theta}}{1 - v^2 \cos 2\theta} . \tag{12.38}$$

   Remark: If $2\theta = 180°$, then $w = 2v/(1 + v^2)$, as it should. And if $\theta = 0°$, then $w = 0$, as it should. If θ is very small, then the result reduces to $w \approx 2v \sin\theta/\sqrt{1 - v^2}$, which is simply the relative speed in the lab frame, multiplied by the time dilation factor between the frames. (The particles' clocks run slow, and transverse distances don't change, so the motion is faster in a particle's frame.) ♣

3. **Another relative speed**

   For you to do.

4. **Acceleration for linear motion**

   (a) Using $v(\tau) = \tanh(a\tau)$, we have $\gamma = 1/\sqrt{1-v^2} = \cosh(a\tau)$. Therefore (suppressing the two transverse components of V, which are 0),

$$V = (\gamma, \gamma v) = (\cosh(a\tau), \sinh(a\tau)), \tag{12.39}$$

   and so

$$A = \frac{dV}{d\tau} = a(\sinh(a\tau), \cosh(a\tau)). \tag{12.40}$$

   (b) The spaceship is at rest in its instantaneous inertial frame, so

$$V' = (1, 0). \tag{12.41}$$

   In the rest frame, we also have

$$A' = (0, a). \tag{12.42}$$

   Equivalently, these are obtained by setting $\tau = 0$ in the results above (because the spaceship hasn't started moving at $\tau = 0$, as is always the case in the instantaneous rest frame).

   (c) The Lorentz transformation matrix from S' to S is

$$\mathcal{M} = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix} = \begin{pmatrix} \cosh(a\tau) & \sinh(a\tau) \\ \sinh(a\tau) & \cosh(a\tau) \end{pmatrix}. \tag{12.43}$$

   We must check that

$$\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \mathcal{M} \begin{pmatrix} V'_0 \\ V'_1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} A_0 \\ A_1 \end{pmatrix} = \mathcal{M} \begin{pmatrix} A'_0 \\ A'_1 \end{pmatrix}. \tag{12.44}$$

   These are easily seen to be true.

Chapter 13

General Relativity

This will be somewhat of a strange chapter, because we won't have enough time to get to the heart of General Relativity (GR). The only thing we will be able to do is give you a flavor for the subject, and give you a few examples of some GR results.

One crucial idea in GR is the Equivalence Principle. This basically says that gravity is equivalent to acceleration. We will have much to say about this issue in the sections below.

Another crucial concept in GR is that of coordinate independence. The laws of physics should not depend on what coordinate system you choose. This seemingly innocuous statement has surprisingly far-reaching consequences. However, discussion of this topic is one of the many things we won't have time for. We would need a whole class on GR to do it justice. Fortunately, it is possible to get a sense of the nature of GR without having to master such things. This is the route will we take in this chapter.

13.1 The Equivalence Principle

13.1.1 Statement of the principle

Einstein's Equivalence Principle says that it is impossible to locally distinguish between gravity and acceleration. This may be stated more precisely in (at least) three ways.

- Let person A be enclosed in a small box (far from any massive objects) undergoing uniform acceleration (say, g). Let person B stand at rest on the earth (see Fig. 13.1). The Equivalence Principle says that there are no local physical experiments these two people can perform that will tell them which of the two settings they are in. The physics of each setting is the same.

- Let person A be enclosed in a small box that is in free-fall near a planet. Let person B float freely in space, far away from any massive objects (see Fig. 13.2). The Equivalence Principle says that there are no local physical experiments these two people can perform that will tell them which of the two settings they are in. The physics of each setting is the same.

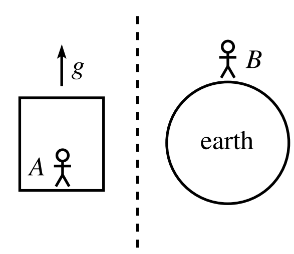

Figure 13.1

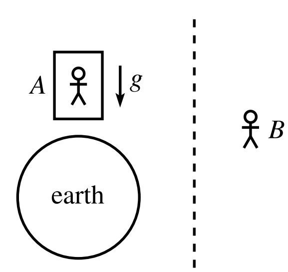

Figure 13.2

- "Gravitational" mass is equal to (or proportional to) "inertial" mass. Gravitational mass is the m_g that appears in the formula $F = GMm_g/r^2 \equiv m_g g$. Inertial mass is the m_i that appears in the formula $F = m_i a$. There is no *a priori* reason why these two m's should be the same (or proportional). An object that is dropped on the earth will have acceleration $a = (m_g/m_i)g$. For all we know, the ratio m_g/m_i for plutonium is different from what it is for copper. But experiments with various materials have detected no difference in the ratios. The Equivalence Principle states that the ratios are equal for any type of mass.

  This definition of the Equivalence Principle is equivalent to, say, the second one above for the following reason. Two different masses near B will stay right where they are. But two different masses near A will diverge from each other if their accelerations are not equal.

These statements are all quite believable. Consider the first one, for example. When standing on the earth, you have to keep your legs firm to avoid falling down. When standing in the accelerating box, you have to keep your legs firm to maintain the same position relative to the floor (that is, to avoid falling down). You certainly can't naively tell the difference between the two scenarios. The Equivalence Principle says that it's not just that you're too inept to figure out a way to differentiate between them, but instead that there is no possible local experiment you can perform to tell the difference, no matter how clever you are.

REMARK: Note the inclusion of the words "small box" and "local" above. On the surface of the earth, the lines of the gravitational force are not parallel; they diverge from the center. The gravitational force also varies with height. Therefore, an experiment performed over a non-negligible distance (for example, dropping two balls next to each other, and watching them converge; or dropping two balls on top of each other and watching them diverge) will have different results from the same experiment in the accelerating box. The equivalence principle says that if your laboratory is small enough, or if the gravitational field is sufficiently uniform, then the two scenarios look essentially the same. ♣

13.1.2 Time dilation

The equivalence principle has a striking consequence concerning the behavior of clocks in a gravitational field. It implies that higher clocks run faster than lower clocks. If you put a watch on a 100 ft tower, and then stand on the ground, you will see the watch on the tower tick faster than an identical watch on you wrist. When you take the former watch down and compare it with the latter, it will show more time elapsed.[1] Likewise, someone standing on top of the tower will see a clock on the ground run slow. Let's be quantitative about this. Consider the following two scenarios.

[1]This will be true only if you keep the watch on the tower for a long enough time; the movement of the watch will cause it to run slow due to the usual special-relativistic time dilation. But the (speeding-up) effect due to the height can be made arbitrarily large compared to the (slowing-down) effect due to the motion, by simply keeping the watch on the tower for an arbitrarily long time.

- A light source at the top of a tower of height h emits flashes at time intervals t_s. A receiver on the ground receives the flashes at time intervals t_r (see Fig. 13.3). What is t_r in terms of t_s?
- A rocket with length h accelerates with acceleration g. A light source at the front end emits flashes at time intervals t_s. A receiver at the back end receives the flashes at time intervals t_r (see Fig. 13.4). What is t_r in terms of t_s?

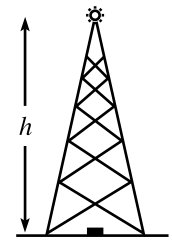

Figure 13.3

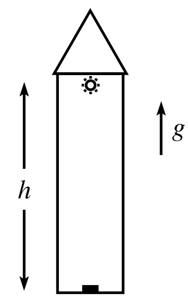

Figure 13.4

The equivalence principle tells us that these two scenarios look exactly the same, as far as the sources and receivers are concerned. Hence, the relation between t_r and t_s is the same in each. Therefore, to find out what is going on in the first scenario, we will study the second scenario (because we can figure out how this one behaves).

Consider an instantaneous inertial frame, S, of the rocket. In this frame, the rocket is momentarily at rest (at, say, $t = 0$), and it then accelerates out of the frame with acceleration g. The following discussion will be made with respect to the frame S.

Consider a series of quick light pulses emitted from the source, starting at $t = 0$. The distance the rocket has traveled out of S is $gt^2/2$, so if we assume that t_s is very small, then we may say that many light pulses are emitted before the rocket moves appreciably. Likewise, the speed of the source, namely gt, is also very small. We may therefore ignore the motion of the rocket, as far as the light source is concerned.

However, the light takes a finite time to reach the receiver, and by then the receiver will be moving. We therefore *cannot* ignore the motion of the rocket when dealing with the receiver. The time it takes the light to reach the receiver is h/c.[2] The receiver has a speed of $v = g(h/c)$ at this time; therefore, by the usual classical Doppler effect, the time between the received pulses is[3]

$$t_r = \frac{t_s}{1 + (v/c)}. \tag{13.1}$$

Therefore, the frequencies, $f_r = 1/t_r$ and $f_s = 1/t_s$, are related by

$$f_r = \left(1 + \frac{v}{c}\right) f_s = \left(1 + \frac{gh}{c^2}\right) f_s. \tag{13.2}$$

Returning to the clock-on-tower scenario, we see (using the equivalence principle) that an observer on the ground will see the clock on the tower running fast, by a factor $1 + gh/c^2$. This means that the upper clock really *is* running fast, compared to the lower clock. A twin from Denver will be younger than his twin from Boston, when they meet up at a family reunion (all other things being equal, of course).

[2] The receiver moves a tiny bit during this time, so the "h" here should really be replaced by a slightly smaller distance, but this yields a negligible second-order effect in the small quantity gh/c^2, as you can show. To sum up, the displacement of the source, the speed of the source, and the displacement of the receiver are all unimportant. But the speed of the receiver is quite relevant.

[3] Quick proof of the Doppler effect: When the receiver and a particular pulse meet, the next pulse is a distance ct_s behind. The receiver and this next pulse then travel toward each other with relative speed $c + v$ (as measured by someone in S). The time difference between receptions is therefore $t_r = ct_s/(c + v)$.

Greetings! Dear brother from Boulder,
I hear that you've gotten much older.
And please tell me why
My lower left thigh
Hasn't aged quite as much as my shoulder.

REMARK: You might object to the above derivation, because t_r is the time as measured by someone in the inertial frame, S. And since the receiver is now moving with respect to S, we should multiply the above f_r by the usual time dilation factor $1/\sqrt{1-(v/c)^2}$ (because the receiver's clocks are running slow relative to S, so the frequency measured by the receiver is greater than that measured in S). However, this is a second-order effect in the small quantity $v/c = gh/c^2$. We already dropped other effects of the same order, so we have no right to keep this one. Of course, if the leading effect in our final answer was second-order in v/c, then we would know that our answer was garbage. But the leading effect happens to be first order, so we can afford to be careless with the second-order effects. ♣

After a finite time has passed, the frame S will no longer be of any use to us. But we can always pick an instantaneous rest frame of the rocket, so we can repeat the above analysis at any later time. Hence, the result in eq. (13.2) holds at all times.

This time-dilation effect was first measured by Pound and Rebka in 1960, here at Harvard (in the northern end of Jefferson Lab). They sent gamma rays up a 20 m tower and measured the redshift (that is, the decrease in frequency) at the top. This was a notable feat indeed, considering that they were able to measure a frequency shift of gh/c^2 (which is only a few parts in 10^{15}) to within 1% accuracy.

13.2 Uniformly accelerated frame

Before reading this section, you should think carefully about the "Break or not break" problem (Problem 25) in Chapter 10. Don't look at the solution too soon, because chances are you will change your answer after a few more minutes of thought. This is a classic problem, so don't waste it by peeking!

Technically, this uniformly accelerating frame we will construct has nothing to do with GR. We will not need to leave the realm of special relativity for the analysis in this section. The reason we choose to study this special-relativistic setup in detail is that it shows many similarities to genuine GR situations (such as black holes).

13.2.1 Uniformly accelerated point particle

In order to understand a uniformly accelerated frame, we need to understand a uniformly accelerated point particle. In Section 10.7, we briefly discussed the motion of a uniformly accelerated particle (that is, one that feels a constant force in its instantaneous rest frame). Let us now take a closer look at such a particle.

Let the particle's frame be S', and let it start from rest in the inertial frame S. Let its mass be m. We know that the longitudinal force is the same in the two frames (from Section 11.5.3). Hence, it is constant in frame S, also. Call it f. For convenience, let $g \equiv f/m$ (so g is the proper acceleration felt by the particle). Then in frame S we have (using the fact that f is constant),

$$f = \frac{dp}{dt} = \frac{d(m\gamma v)}{dt} \qquad \Longrightarrow \qquad \gamma v = gt \qquad \Longrightarrow \qquad v = \frac{gt}{\sqrt{1+(gt)^2}}\,, \tag{13.3}$$

where we have set $c = 1$. As a double-check, this has the correct behavior for $t \to 0$ and $t \to \infty$. (If you want to keep the c's in, then $(gt)^2$ becomes $(gt/c)^2$, to make the units correct.)

Having found the speed in frame S at time t, the position in frame S at time t is given by

$$x = \int_0^t v\,dt = \int_0^t \frac{gt\,dt}{\sqrt{1+(gt)^2}} = \frac{1}{g}\left(\sqrt{1+(gt)^2} - 1\right). \tag{13.4}$$

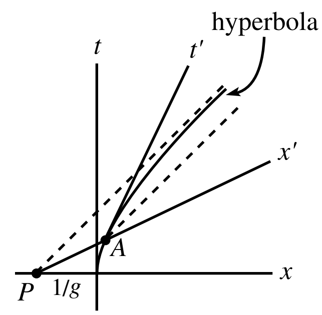

Figure 13.5

For convenience, let P be the point (see Fig. 13.5)

$$(x_P, t_P) = (-1/g, 0). \tag{13.5}$$

Then eq. (13.4) yields

$$(x - x_P)^2 - t^2 = \frac{1}{g^2}\,. \tag{13.6}$$

This is the equation for a hyperbola with a focus at point P. For a large acceleration g, the point P is very close to the particle's starting point. For a small acceleration, it is far away.

Everything has been fairly normal up to this point, but now the fun begins. Consider a point A on the particle's worldline at time t. From eq. (13.4), A has coordinates

$$(x_A, t_A) = \left(\frac{1}{g}\left(\sqrt{1+(gt)^2} - 1\right),\, t\right). \tag{13.7}$$

The slope of the line PA is therefore

$$\frac{t_A - t_P}{x_A - x_P} = \frac{gt}{\sqrt{1+(gt)^2}}\,. \tag{13.8}$$

Looking at eq. (13.3) we see that this slope is simply the speed of the particle at point A. And we know very well that the speed v is the slope of the particle's instantaneous x'-axis (see eq. (10.44)). Therefore, the line PA and the particle's x'-axis are the same line! This holds for any arbitrary time, t. So we may say that *at any point along the particle's worldline, the line PA is the instantaneous x'-axis of the particle.* Or, said another way, *no matter where the particle is, the event at P is simultaneous with an event located at the particle, as measured in the instantaneous frame of the particle.* In other words, the particle always says that P happens "now".

Remark: This is actually quite believable, considering that a flash of light from the event at P will never quite reach the particle. Indeed, all events in the 45° cone to the left of P are causally disconnected from the particle. They do not exist, as far as the particle is concerned. In some sense, the particle sees time run backwards in this region. But this is a meaningless statement, because the regions are causally disconnected.

The point P is very much like the event horizon of a black hole. Time seems to stand still at P. And if we went more deeply into GR, we would find that time seems to stand still at the edge of a black hole, too (as viewed by someone further away). ♣

Here is another strange fact. What is the distance from P to A, as measured in an instantaneous rest frame of the particle? The γ factor between frames S and S' is, using eq. (13.3), $\gamma = \sqrt{1+(gt)^2}$. The distance between P and A in frame S is $x_A - x_P = \sqrt{1+(gt)^2}\,/g$. So the distance between P and A in frame S' is (using the Lorentz transformation $\Delta x = \gamma(\Delta x' + v\Delta t')$, with $\Delta t' = 0$)

$$x'_A - x'_P = \frac{1}{\gamma}(x_A - x_P) = \frac{1}{g}\,. \tag{13.9}$$

This is independent of t! Therefore, not only do we find that P is always simultaneous with the particle, in the particle's frame; we also find that P *always remains the same distance (namely* $1/g$*) away from the particle, as measured in the particle's instantaneous rest-frame.* This is rather strange. The particle accelerates away from point P, but it does not get further away from it (in its own frame).

13.2.2 Uniformly accelerated frame

Let's now put a collection of uniformly accelerated particles together to make a uniformly accelerated frame. The goal will be to create a frame where the distances between particles (as measured in any particle's instantaneous rest frame) remain constant.

Why is this our goal? We know from the "Break or not break" problem in Chapter 10 that if all the particles accelerate with the same proper acceleration, g, then the distances (as measured in a particle's instantaneous rest frame) grow larger. While this is a perfectly possible frame to construct, it is not desirable here for the following reason. Einstein's Equivalence Principle states that an accelerated frame is equivalent to a frame sitting on, say, the earth. We may therefore study the effects of gravity by studying an accelerated frame. But if we want this frame to look anything like the surface of the earth, we certainly can't have distances that change.

We would therefore like to construct a *static* frame (that is, one where distances do not change). This allows us to say that if we enclose the frame by windowless walls, then for all a person inside knows, he is standing motionless in a static gravitational field (which has a certain definite form, as we shall see).

Let's figure out how to construct the frame. We'll discuss only the acceleration of two particles here. Others may be added in an obvious manner. In the end, the desired frame as a whole is constructed by accelerating each atom in the floor of the frame with a specific proper acceleration.

From Section 13.2.1, we already have a particle A which is being "pivoted" around the focal point, P. We claim that every other particle in the frame should also "pivot" around P.

Consider another particle, B. Let a and b be the initial distances from P to particles A and B. If both particles are to pivot around P, then their proper accelerations must be, from eq. (13.5),

$$g_A = \frac{1}{a}, \qquad \text{and} \qquad g_B = \frac{1}{b}. \tag{13.10}$$

Therefore, in order to have all points in the frame pivot around P, we simply have to make their proper accelerations inversely proportional to their initial distance form P.

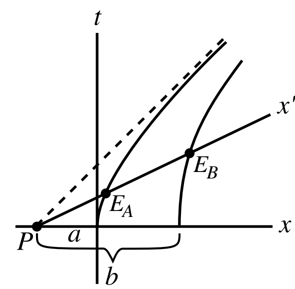

Figure 13.6

Why do we want every particle to pivot around P? Consider two events, E_A and E_B, such that P, E_A, and E_B are collinear in Fig. 13.6. Due to construction, the line PE_AE_B is the x'-axis for both particle A and particle B, at the positions shown. Since A is always a distance a from P, and since B is always a distance b from P, and since A and B measure their distances along the x'-axis of the same frame (at the events shown in the figure), we see that both A and B measure the distance between them as $b - a$. This is independent of t, so A and B measure a constant distance between them. We have therefore constructed our desired static frame.

If a person walks around in this frame, he will think he lives in a static world where the acceleration due to gravity takes the form $g(z) \propto 1/z$, where z is the distance to a certain magical point which is located at the end of the known "universe".

What if he releases himself from the frame, so that he forever sails through space at constant speed? He thinks he is falling, and you should convince yourself that he passes by the "magical point" in a finite proper time. But his friends who are still in the frame see him take an infinitely long time to get to the "magical point" P. This is similar to the situation with a black hole. An outside observer will see it take an infinitely long time for a falling person to reach the "boundary" of a black hole, even though it will take a finite proper time for the person.

Our analysis shows that A and B feel a different proper acceleration, because $a \neq b$. It is impossible to mimic a constant gravitational field (over a finite distance) by using an accelerated frame. (Well, for that matter, it's impossible to *create* an actual constant gravitational field without using an infinite amount of matter; an infinite sheet would do the trick.) There is no way to construct a static frame where all points feel the same proper acceleration.

13.3 Maximal-proper-time principle

The maximal-proper-time principle says: Given two events in spacetime, a particle under the influence of only gravity takes the path in spacetime that maximizes the proper time. For example, if you throw a ball from given coordinates $(\mathbf{x}_1, t_1)$, and it lands at given coordinates $(\mathbf{x}_2, t_2)$, then the claim is that the ball takes the path which maximizes its proper time.

This is clear for a ball in outer space, far from any massive objects. The ball travels at constant speed from one point to another, and we know that this constant-speed motion is the one with the maximal proper time. (This ball (A) would see any other ball (B) have a time dilation, if there were a relative speed between them. Hence, B would show a shorter elapsed time. This argument does not work the other way around, because B is not in an inertial frame and therefore cannot use the time-dilation results from special relativity.)

Consistency with Newtonian physics

The maximal-proper-time principle sounds like a plausible idea, but we already know from Chapter 5 that the path an object takes is the one that minimizes the classical action, $\int(T-V)$. We must therefore demonstrate that the maximal-proper-time principle reduces to the "least-action" principle, in the limit of small velocities. If this were not the case, then we'd have to throw out our theory of gravitation.

Consider a ball thrown vertically on the earth. Assume that the initial and final coordinates are fixed to be (y_1, t_1) and (y_2, t_2). Our plan will be to assume that the maximal-proper-time principle holds, and to then show that this leads to the least-action principle.

Before being quantitative, let's get a qualitative handle on what's going on with the ball. There are two competing effects, as far as maximizing the proper time goes. On one hand, the ball wants to climb very high, because its clock will run faster there (due to the GR time dilation). But on the other hand, if it climbs very high, then it must move very fast to get there (because the total time, $t_2 - t_1$, is fixed), and this will make its clock run slow (due to the SR time dilation). So there is a tradeoff. Let's now look quantitatively at the implications of this tradeoff.

The goal is to maximize

$$\tau = \int_{t_1}^{t_2} d\tau. \tag{13.11}$$

Due to the motion of the ball, we have the usual time dilation, $d\tau = \sqrt{1 - v^2/c^2}\, dt$. But due to the height of the ball, we have the gravitational time dilation, $d\tau = (1 + gy/c^2)dt$. Combining these effects gives[4]

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}\left(1 + \frac{gy}{c^2}\right) dt. \tag{13.12}$$

Using the Taylor expansion for $\sqrt{1-\epsilon}$, and dropping terms of order $1/c^4$ and smaller, we see that we want to maximize

$$\begin{aligned}
\int_{t_1}^{t_2} d\tau &\approx \int_{t_1}^{t_2} \left(1 - \frac{v^2}{2c^2}\right)\left(1 + \frac{gy}{c^2}\right) dt \\
&\approx \int_{t_1}^{t_2} \left(1 - \frac{v^2}{2c^2} + \frac{gy}{c^2}\right) dt. \qquad (13.13)
\end{aligned}$$

[4]This result is technically not correct; the two effects are intertwined in a somewhat more complicated way. But it is valid up to order v^2/c^2, which is all we are concerned with, since we are assuming $v \ll c$.

The "1" term gives a constant, so maximizing this is the same as minimizing

$$mc^2 \int_{t_1}^{t_2} \left(\frac{v^2}{2c^2} - \frac{gy}{c^2} \right) dt = \int_{t_1}^{t_2} \left(\frac{mv^2}{2} - mgy \right) dt, \tag{13.14}$$

which is the classical action, as desired.

In retrospect, it is not surprising that this all works out. The factor of 1/2 in the kinetic energy comes about in exactly the same way as in the derivation in equation 11.9, where we showed that the relativistic form of energy reduces to the familiar Newtonian expression.

13.4 Twin paradox revisited

Let's take another look at the standard twin-paradox, this time from the perspective of General Relativity. We should emphasize that GR is by no means necessary for an understanding of the original formulation of the paradox (the first scenario below). We were able to solve it in Section 10.2.2, after all. The present discussion is given simply to show that the answer to an alternative formulation (the second scenario below) is consistent with what we've learned about GR. Consider the two following twin-paradox setups.

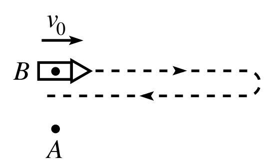

Figure 13.7

- Twin A floats freely in outer space. Twin B flies past A in a spaceship, with speed v_0 (see Fig. 13.7). At the instant they are next to each other, they both set their clocks to zero. At this same instant, B turns on the reverse thrusters of his spaceship and decelerates with proper deceleration g. B eventually reaches a farthest point from A and then accelerates back toward A, finally passing him with speed v_0 again. When they are next to each other, they compare the readings on their clocks. Which twin is younger?

- Twin B stands on the earth. Twin A is thrown upward with speed v_0 (let's say he is fired from a cannon in a hole in the ground). See Fig. 13.8. At the instant they are next to each other, they both set their clocks to zero. A rises up and then falls back down, finally passing B with speed v_0 again. When they are next to each other, they compare the readings on their clocks. Which twin is younger?

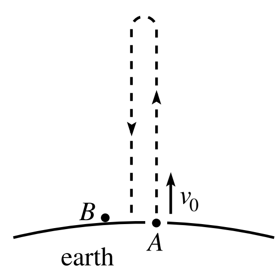

Figure 13.8

The first scenario is easily solved using special relativity. Since A is in an inertial frame, he may apply the results of special relativity. In particular, A sees B's clock run slow, due to the usual time dilation. Therefore, B ends up younger at the end. (B cannot use the reverse reasoning, because he is not in an inertial frame.)

What about the second scenario? The key point to realize is that the Equivalence Principle says that these two scenarios are *exactly the same*, as far as the twins are concerned. Twin B has no way of telling whether he is in a spaceship accelerating at g or on the surface of the earth. And A has no way of telling whether he is floating freely in outer space or in free-fall in a gravitational field.[5] We therefore conclude that B must be younger in the second scenario, too.

At first glance, this seems incorrect, because B is sitting motionless, and A is the one who is moving. It seems that B should see A's clock running slow, due to the usual time dilation, and hence A should be younger. This reasoning is incorrect because it fails to take into account the gravitational time dilation. The fact of the matter is that A is higher in the gravitational field, and therefore his clock runs faster. This effect does indeed win out over the special-relativistic time dilation, and A ends up older. (You can explicitly show this in Problem 7.)

Note that the reasoning in this section is another way to conclude that the Equivalence Principle implies that higher clocks must run faster, in one way or another. (But it takes some more work to show that the factor is actually $1+gh/c^2$.)

Also note that the fact that A is older is consistent with the maximal-proper-time principle. In both scenarios, A is under the influence of only gravity (zero gravity in the first scenario), whereas B feels a normal force from either the spaceship's floor or the ground.

[5] This fact is made possible by the equivalence of inertial and gravitational mass. Were it not for this, different parts of A's body would accelerate at different rates in the gravitational field in the second scenario. This would certainly clue him in to the fact that he was not floating freely in space.

13.5 Exercises

Section 13.1: The Equivalence Principle

1. **Driving on a hill**

   You drive up and down a hill, of height h, at constant speed. What should your speed be so that you age the same as someone standing at the base of the hill? (Assume that the hill is in the shape of an isosceles triangle with altitude h.)

Section 13.2: Uniformly accelerated frame

2. **Various quantities** $*$

   A particle starts at rest and accelerates with proper acceleration g. Let τ be the time showing on the particle's clock. Show that the time t in the original frame, the speed of the particle, and the associated γ factor are given by (with $c = 1$)

$$gt = \sinh(g\tau), \qquad v = \tanh(g\tau), \qquad \gamma = \cosh(g\tau). \tag{13.15}$$

3. **Gravity and speed combined** $**$

   Use a Minkowski diagram to do this problem (in the spirit of Problem 10.24).

   A rocket accelerates with proper acceleration g. As measured in the instantaneous inertial frame of the rocket, a planet is a distance x away and moving at speed v.

   As measured in the *accelerating* frame of the rocket, show that the planet's clock runs at a rate (with c=1)

$$dt_p = dt_r(1 + gx)\sqrt{1 - v^2}\,, \tag{13.16}$$

   and that the planet's speed is

$$V = (1 + gx)v. \tag{13.17}$$

   Note that these two results may be quickly combined to eliminate v and give the result that a clock moving at speed V at height h in a gravitation field is seen by someone on the ground to run at a rate (putting the c's back in)

$$\sqrt{\left(1 + \frac{gh}{c^2}\right)^2 - \frac{V^2}{c^2}}\ . \tag{13.18}$$

4. **Accelerating stick** $**$

   Consider a uniformly accelerated frame consisting of a stick, the ends of which have worldlines given by the curves in Fig. 13.6 (so the stick has proper length $b-a$). An observer in the original inertial frame will see the stick being length-contracted by different factors along its length (because different points move at different speeds, at a given time in the original frame). Show, by doing an appropriate integral, that this observer will conclude that the stick always has proper length $b - a$.

Section 13.4: Twin paradox revisited

5. **Symmetric twin non-paradox** $**$

Two twins travel in opposite directions at speed v with respect to the earth ($v \ll c$). They synchronize their clocks when they pass each other. They each travel to a star located a distance ℓ from the earth, and then decelerate and accelerate back up to speed v in the opposite direction (uniformly, and in a short time compared to the total journey time).

From the frame of the earth, it is obvious that both twins age the same amount by the time they pass each other again. Reproduce this result by working in the frame of one of the twins.

13.6 Problems

Section 13.1: The Equivalence Principle

1. **Airplane's speed**

   A plane flies at constant height h. What should its speed be so that an observer on the ground sees the plane's clock tick at the same rate as a ground clock? (Assume $v \ll c$.)

2. **Clock on tower** $**$

   A clock starts on the ground and then moves up a tower at constant speed v. It sits on the top for a time T and then descends at constant speed v. If the tower has height h, how long should the clock sit at the top so that it comes back showing the same time as a clock that remained on the ground? (Assume $v \ll c$.)

3. **Circular motion** $**$

   Person B moves at speed v ($v \ll c$) in a circle of radius r around person A. By what fraction does B's clock run slower than A's? Calculate this in three ways. Work in:

   (a) A's frame.

   (b) The frame whose origin is B and whose axes remain parallel to an inertial set of axes.

   (c) The rotating frame which is centered at A and rotates around A with the same frequency as B.

4. **More circular motion** $**$

   A and B move at speed v ($v \ll c$) in a circle of radius r, at diametrically opposite points. They both see their clocks ticking at the same rate. Show this in three ways. Work in:

   (a) The lab frame (the inertial frame whose origin is the center of the circle).

   (b) The frame whose origin is B and whose axes remain parallel to an inertial set of axes.

   (c) The rotating frame that is centered at the origin and rotates with the same frequency as A and B.

Section 13.2: Uniformly accelerated frame

5. **Getting way ahead** $****$

   A rocket with proper length L accelerates from rest, with proper acceleration g (where $gL \ll c^2$). Clocks are located at the front and back of the rocket. If we look at this setup in the frame of the rocket, then the general-relativistic time dilation says that the times on the two clocks are related by $t_f = (1+gL/c^2)t_b$.

Therefore, if we look at things in the ground frame, then the times showing on the two clocks are related by

$$t_f = t_b\left(1 + \frac{gL}{c^2}\right) - \frac{Lv}{c^2}\,, \tag{13.19}$$

where the last term is the standard special-relativistic "head start" result. Derive this relation by working entirely in the frame of the ground.[6]

6. **Accelerator's point of view** ∗∗∗

   A rocket starts at rest relative to a planet, a distance L away. It accelerates toward the planet with proper acceleration g. Let τ and t be the readings on the rocket's and planet's clocks, respectively.

   (a) Show that when the astronaut's clock reads τ, he observes the rocket-planet distance, x, (as measured in his instantaneous inertial frame) to be given by

   $$1 + gx = \frac{1 + gL}{\cosh(g\tau)}\,, \tag{13.20}$$

   (b) Show that when the astronaut's clock reads τ, he observes the time, t, on the planet's clock to be given by

   $$gt = (1 + gL)\tanh(g\tau). \tag{13.21}$$

   The results from Exercise 3 may be useful in this problem.

Section 13.4: Twin paradox revisited

7. **Twin paradox times** ∗∗∗

   (a) In the first scenario in section 13.4, calculate the ratio of B's age to A's age, in terms of v_0 and g. Assume $v_0 \ll c$, and drop high-order terms.

   (b) In the second scenario in section 13.4, calculate the ratio of B's age to A's age, in terms of v_0 and g. Assume $v_0 \ll c$, and drop high-order terms. (Do this from scratch using the time dilations. Then check that your answer agrees with part (a), as the equivalence principle demands.)

8. **Twin paradox** ∗

   A spaceship travels at speed v ($v \ll c$) to a distant star. Upon reaching the star, it decelerates and then accelerates back up to speed v in the opposite direction (uniformly, and in a short time compared to the total journey time). By what fraction does the traveler age less than his twin on earth? Work in:

[6] You may find the relation surprising, because it implies that the front clock will eventually be an arbitrarily large time ahead of the back clock, in the ground frame. (The subtractive Lv/c^2 term will be at most L/c and will therefore eventually become negligible compared to the additive, and unbounded, $(gL/c^2)t_b$ term.) But both clocks are doing basically the same thing relative to the ground frame, so how can they eventually differ by so much? Your job is to find out.

(a) The earth frame.

(b) The spaceship frame.

9. **Twin paradox again** $**$

   (a) Answer the previous problem, except now let the spaceship turn around by moving in a small semicircle while maintaining speed v.

   (b) Answer the previous problem, except now let the spaceship turn around by moving in an arbitrary manner. The only constraints are that the turn-around is done quickly (compared to the total journey time), and is contained in a small region of space (compared to the earth-star distance).

13.7 Solutions

1. **Airplane's speed**

An observer on the ground sees the plane's clock run slow by a factor $\sqrt{1-v^2/c^2}$ due to the special-relativistic effect. But he also sees it run fast by a factor $1+gh/c^2$ due to the general-relativistic effect. We therefore want the product of these two factors to equal 1. Using the standard approximation for slow speeds in the first factor, we find

$$\left(1-\frac{v^2}{2c^2}\right)\left(1+\frac{gh}{c^2}\right)=1 \quad\Longrightarrow\quad 1-\frac{v^2}{2c^2}+\frac{gh}{c^2}-(\text{small})=1, \tag{13.22}$$

where we have neglected the very small term proportional to $1/c^4$. The 1's cancel, and we can solve for v to obtain $v=\sqrt{2gh}$.

Interestingly, $\sqrt{2gh}$ is the answer to a standard question from Newtonian physics, namely, how fast must you throw a ball straight up if you want it to reach a height h?

2. **Clock on tower**

The special-relativistic time-dilation factor is $\sqrt{1-v^2/c^2}\approx 1-v^2/2c^2$. The clock therefore loses a fraction $v^2/2c^2$ of the time elapsed during its motion up and down the tower. The journey upwards takes a time h/v, and likewise for the downwards trip, so the time loss due to the special-relativistic effect is

$$\left(\frac{v^2}{2c^2}\right)\left(\frac{2h}{v}\right)=\frac{vh}{c^2}\,. \tag{13.23}$$

Our goal is to balance this time loss with the time gain due to the general-relativistic effect. If the clock sits on the tower for a time T, then the time gain is

$$\left(\frac{gh}{c^2}\right)T. \tag{13.24}$$

But we must not forget also the increase in time due to the height gained while the clock is in motion. During its motion, the clock's average height is $h/2$. The total time in motion is $2h/v$, so the general-relativistic time increase while the clock is moving is

$$\left(\frac{g(h/2)}{c^2}\right)\left(\frac{2h}{v}\right)=\frac{gh^2}{c^2v}\,. \tag{13.25}$$

Setting the total change in the clock's time equal to zero gives

$$-\frac{vh}{c^2}+\frac{gh}{c^2}T+\frac{gh^2}{c^2v}=0 \quad\Longrightarrow\quad -v+gT+\frac{gh}{v}=0. \tag{13.26}$$

Therefore,

$$T=\frac{v}{g}-\frac{h}{v}\,. \tag{13.27}$$

Remarks: Note that we must have $v>\sqrt{gh}$ in order for a positive solution for T to exist. (If $v=\sqrt{gh}$ then $T=0$, which is essentially the same situation as in Exercise 1.) Note also that if v is very large compared to $\sqrt{gh}$ (but still small compared to c, so that our $1/\gamma\approx 1-v^2/2c^2$ approximation is valid), then $T\approx v/g$ (which is independent of h). ♣

3. **Circular motion**

   (a) In A's frame, there is only a Special-Relativistic effect. A sees B move at speed v, so B's clock runs slow by a factor of $\sqrt{1 - v^2/c^2}$. And since $v \ll c$, we may use the Taylor series to approximate this as $1 - v^2/2c^2$.

   (b) In this frame, there are both SR and GR effects. A moves at speed v with respect to B in this frame, so there is the SR effect that A's clock runs slow by a factor $\sqrt{1 - v^2/c^2} \approx 1 - v^2/2c^2$.

   But B is undergoing an acceleration of $a = v^2/r$ toward A, so there is also the GR effect that A's clock runs fast by a factor $1 + ar/c^2 = 1 + v^2/c^2$.

   Multiplying these two effects together, we find (to lowest order) that A's clock runs fast by a factor $1 + v^2/2c^2$. This agrees with the answer to part (a), to lowest order in v^2/c^2.

   (c) In this frame, there is no relative motion between A and B. Hence, there is only the GR effect. The gravitational field (i.e., the centripetal acceleration) at a distance x from the center is $g_x = x\omega^2 = x(v/r)^2$. If we imagine lining up a series of clocks along a radius, with separation dx, then each clock loses a fraction $g_x\, dx/c^2 = x\omega^2\, dx/c^2$ relative to the clock just inside it. Integrating these fractions from $x = 0$ to $x = r$ shows that B's clock loses a fraction $r^2\omega^2/2c^2 = v^2/2c^2$, compared to A's clock. This agrees with the results in parts (a) and (b).

4. **More circular motion**

   (a) In the lab frame, the situation is symmetric with respect to A and B. Therefore, if A and B are decelerated in a symmetric manner and brought together, then their clocks must read the same time.

   Assume (in the interest of obtaining a contradiction), that A sees B's clock run slow. Then after an arbitrarily long time, A will see B's clock an arbitrary amount behind his. Now bring A and B to a stop. There is no possible way that the stopping motion can make B's clock gain an arbitrary amount of time, as seen by A. This is true because everything takes place in a finite region of space, so there is an upper bound on the GR effect (since it behaves like gh/c^2, and h is bounded). Therefore, A will end up seeing B's clock reading less. This contradicts the result of the previous paragraph.

   REMARK: Note how this problem differs from the problem where A and B move with equal speeds directly away from each other, and then reverse directions and head back to meet up again.

   For this new problem, the symmetry reasoning in the first paragraph above still holds; they will indeed have the same clock readings when they meet up again. But the reasoning in the second paragraph does not hold (it better not, since each person does *not* see the other person's clock running at the same rate). The error is that in the present scenario the experiment is not contained in a small region of space, so the turning-around effects of order gh/c^2 become arbitrarily large as the time of travel becomes arbitraril large, since h grows with time. (See Problem 8.) ♣

   (b) In this frame, there are both SR and GR effects. A moves at speed $2v$ with respect to B in this frame (we don't need to use the relativistic velocity addition formula, since $v \ll c$), so there is the SR effect that A's clock runs slow by a factor $\sqrt{1 - (2v)^2/c^2} \approx 1 - 2v^2/c^2$.

But B is undergoing an acceleration of $a = v^2/r$ toward A, so there is also the GR effect that A's clock runs fast by a factor $1 + a(2r)/c^2 = 1 + 2v^2/c^2$ (since they are separated by a distance $2r$).

Multiplying these two effects together, we find (to lowest order) that the two clocks run at the same rate.

(c) In this frame, there is no relative motion between A and B. Hence, there is only the GR effect. But A and B are both at the same gravitational potential. Hence, they both see the clocks running at the same rate.

5. **Getting way ahead**

The explanation of why the two clocks show different times in the ground frame is the following. The rocket will become increasingly length contracted in the ground frame, and so the front end won't be traveling as fast as the back end. Therefore, the time-dilation factor for to the front clock won't be as large as that for the back clock. So the front clock will lose less time and hence end up ahead the front clock. Of course, it's not at all obvious that everything will work out quantitatively, and that the front clock will eventually end up an arbitrarily large time ahead of the back clock, but let's now show that it does.

Let the back of the rocket be located at position x. Then the front is located at $x + L\sqrt{1-v^2}$. (We will set $c = 1$ throughout this solution.) Taking the time derivatives, we see that the speeds of the back and front are (with $dx/dt = v$)[7]

$$v_b = v, \qquad v_f = v(1 - L\gamma\dot{v}). \tag{13.28}$$

For v_b, we will simply invoke the result in eq. (13.3),

$$v_b = v = \frac{gt}{\sqrt{1+(gt)^2}}\,, \tag{13.29}$$

where t is the time in the ground frame. Note that the γ factor associated with this velocity is

$$\gamma = \sqrt{1+(gt)^2}\,. \tag{13.30}$$

For $v_f = v(1 - L\gamma\dot{v})$, we first need to calculate $\dot{v}$. From eq. (13.29), we find $\dot{v} = g/(1+g^2t^2)^{3/2}$, and so

$$v_f = \frac{gt}{\sqrt{1+(gt)^2}}\left(1 - \frac{gL}{1+g^2t^2}\right). \tag{13.31}$$

The γ factor (or rather $1/\gamma$, which is what we'll be concerned with) associated with this velocity is

$$\begin{aligned}\frac{1}{\gamma_f} = \sqrt{1-v_f^2} &\approx \sqrt{1 - \frac{g^2t^2}{1+g^2t^2}\left(1 - \frac{2gL}{1+g^2t^2}\right)} \\ &= \frac{1}{\sqrt{1+g^2t^2}}\sqrt{1 + \frac{2g^3t^2L}{1+g^2t^2}} \\ &\approx \frac{1}{\sqrt{1+g^2t^2}}\left(1 + \frac{g^3t^2L}{1+g^2t^2}\right).\end{aligned} \tag{13.32}$$

[7] Since these speeds are unequal, there is of course an ambiguity concerning which speed we should use in the length-contraction factor, $\sqrt{1-v^2}$. Equivalently, the rocket actually doesn't have one inertial frame that describes the whole thing. But you can show that any differences arising from this ambiguity are of higher order in gL/c^2 than we need to be concerned with.

In the first line, we ignored the higher-order $(gL)^2$ term, because it is really $(gL/c^2)^2$, and we are assuming that gL/c^2 is small. In obtaining the third line, we used the Taylor-series approximation $\sqrt{1-\epsilon} \approx 1 - \epsilon/2$.

Let's now calculate the time showing on each of the clocks, at time t in the ground frame. The time on the back clock changes according to $dt_b = dt/\gamma_b$, so eq. (13.30) gives

$$t_b = \int_0^t \frac{dt}{\sqrt{1+g^2t^2}} . \tag{13.33}$$

The integral[8] of $1/\sqrt{1+x^2}$ equals $\sinh^{-1} x$. You can show this leads to

$$gt_b = \sinh^{-1}(gt). \tag{13.34}$$

The time on the front clock changes according to $dt_f = dt/\gamma_f$, so eq. (13.32) gives

$$t_f = \int_0^t \frac{dt}{\sqrt{1+g^2t^2}} + \int_0^t \frac{g^3t^2L\,dt}{(1+g^2t^2)^{3/2}} . \tag{13.35}$$

The integral[9] of $x^2/(1+x^2)^{3/2}$ equals $\sinh^{-1} x - x/\sqrt{1+x^2}$. You can show this leads to

$$gt_f = \sinh^{-1}(gt) + (gL)\left(\sinh^{-1}(gt) - \frac{gt}{\sqrt{1+g^2t^2}}\right). \tag{13.36}$$

Using eqs. (13.34) and (13.29), we may rewrite this as

$$gt_f = gt_b(1+gL) - gLv. \tag{13.37}$$

Dividing by g and putting the c's back in to make the units correct gives

$$t_f = t_b\left(1 + \frac{gL}{c^2}\right) - \frac{Lv}{c^2} . \tag{13.38}$$

as we wanted to show.

6. **Accelerator's point of view**

   (a) **First Solution:** Eq. (13.17) gives the speed of the planet in the accelerating frame of the rocket. Using the results of Exercise 2 to write v in terms of τ, we have (with $c = 1$)

$$\frac{dx}{d\tau} = -(1+gx)\tanh(g\tau). \tag{13.39}$$

   Separating variables and integrating gives

$$\int \frac{dx}{1+gx} = -\int \tanh(g\tau)\,d\tau \quad \Longrightarrow \quad \ln(1+gx) = -\ln(\cosh(g\tau)) + C$$
$$\Longrightarrow \quad 1 + gx = \frac{A}{\cosh(g\tau)}. \tag{13.40}$$

   Since the initial condition is $x = L$ when $\tau = 0$, we must have $A = 1 + gL$, which gives eq. (13.20) as desired.

[8] To derive this, make the substitution $gt \equiv \sinh\theta$.
[9] Again, to derive this, make the substitution $gt \equiv \sinh\theta$.

Second Solution: Eq. (13.4) says that the distance traveled by the rocket (as measured in the original inertial frame), as a function of the time in the inertial frame, is

$$d = \frac{1}{g}\left(\sqrt{1+(gt)^2}-1\right). \tag{13.41}$$

The inertial observer therefore measures the rocket-planet distance to be

$$x = L - \frac{1}{g}\left(\sqrt{1+(gt)^2}-1\right). \tag{13.42}$$

The rocket observer will see this length being contracted by a factor γ. Using the results of Exercise 2, we have $\gamma = \sqrt{1+(gt)^2} = \cosh(g\tau)$. So the rocket-planet distance, as measured in the instantaneous inertial frame of the rocket, is

$$x = \frac{L - \frac{1}{g}\big(\cosh(g\tau)-1\big)}{\cosh(g\tau)} \qquad \Longrightarrow \qquad 1+gx = \frac{1+gL}{\cosh(g\tau)}, \tag{13.43}$$

as desired.

(b) Eq. (13.16) says that the planet's clock runs fast (or slow) according to

$$dt = d\tau\,(1+gx)\sqrt{1-v^2}. \tag{13.44}$$

The results of Exercise 2 yield $\sqrt{1-v^2} = 1/\cosh(g\tau)$. Combining this with the result for $1+gx$ above, and integrating, gives

$$\int dt = \int \frac{(1+gL)\,d\tau}{\cosh^2(g\tau)} \qquad \Longrightarrow \qquad gt = (1+gL)\tanh(g\tau), \tag{13.45}$$

as desired.

7. **Twin paradox times**

(a) As viewed by A, the times of the twins are related by

$$dt_B = \sqrt{1-v^2}\,dt_A. \tag{13.46}$$

Assuming $v_0 \ll c$, we may say that $v(t_A)$ is essentially $v_0 - gt_A$. The outward and backward parts of the trip each take a time of essentially v_0/g in A's frame. Therefore, the total time elapsed on B's clock is

$$\begin{aligned} T_B = \int dt_B &\approx 2\int_0^{v_0/g} \sqrt{1-v^2}\;dt_A \\ &\approx 2\int_0^{v_0/g}\left(1-\frac{v^2}{2}\right) dt_A \\ &\approx 2\int_0^{v_0/g}\left(1-\frac{1}{2}(v_0-gt)^2\right) dt \\ &= 2\left(t+\frac{1}{6g}(v_0-gt)^3\right)\Bigg|_0^{v_0/g} \\ &= \frac{2v_0}{g} - \frac{v_0^3}{3gc^2}, \end{aligned} \tag{13.47}$$

where we have put the c's back in to make the units right. The ratio of B's age to A's age is therefore

$$\frac{T_B}{T_A} = \frac{T_B}{2v_0/g} = 1 - \frac{1}{6}\frac{v_0^2}{c^2}. \tag{13.48}$$

(b) As viewed by B, the relationship between the twins' times is given by eq. (13.12),

$$dt_A = \sqrt{1 - \frac{v^2}{c^2}} \left(1 + \frac{gy}{c^2}\right) dt_B. \tag{13.49}$$

Assuming $v_0 \ll c$, we may say that $v(t_B)$ is essentially $v_0 - gt_B$, and the height is essentially $v_0 t_B - gt_B^2/2$. The outward and backward parts of the trip each take a time of essentially v_0/g in B's frame. Therefore, the total time elapsed on A's clock is (using the approximation in eq. (13.13), and dropping the c's),

$$\begin{aligned} T_A = \int dt_A &\approx 2\int_0^{v_0/g} \left(1 - \frac{v^2}{2} + gy\right) dt_B. \\ &\approx 2\int_0^{v_0/g} \left(1 - \frac{1}{2}(v_0 - gt)^2 + g(v_0 t - gt^2/2)\right) dt. \\ &= 2\left(t + \frac{1}{6g}(v_0 - gt)^3 + g\left(\frac{v_0 t^2}{2} - \frac{gt^3}{6}\right)\right)\Bigg|_0^{v_0/g} \\ &= \frac{2v_0}{g} - \frac{v_0^3}{3g} + g\left(\frac{v_0^3}{g^2} - \frac{v_0^3}{3g^2}\right) \\ &= \frac{2v_0}{g} + \frac{v_0^3}{3gc^2}, \end{aligned} \tag{13.50}$$

where we have put the c's back in to make the units right. The ratio of A's age to B's age is therefore

$$\frac{T_A}{T_B} = \frac{T_A}{2v_0/g} = 1 + \frac{1}{6}\frac{v_0^2}{c^2}. \tag{13.51}$$

Up to corrections of order v_0^4/c^4, this is indeed the reciprocal of the answer for T_B/T_A found in eq. (13.48). So the answers do indeed agree, in accordance with the equivalence principle.

8. **Twin paradox**

(a) In the earth frame, the spaceship travels at speed v (for essentially the whole time). Hence, the traveler ages less by a fraction $\sqrt{1 - v^2/c^2} \approx 1 - v^2/2c^2$. The fractional loss of time is thus $v^2/2c^2$. (The time-dilation effect will be different during the short turning-around period, but this is negligible.)

(b) Let the distance to the star be ℓ (as measured in the earth frame; but the difference of lengths in the two frames is irrelevant in this problem), and let the turn-around take a time T. (Then the given information says that $T \ll (2\ell)/v$.)

During the constant-speed part of the trip, the traveler sees the earth clock running slow by a fraction $\sqrt{1 - v^2/c^2} \approx 1 - v^2/2c^2$. The time for this constant-speed part is $2\ell/v$, so the earth clock loses a time of $(v^2/2c^2)(2\ell/v) = v\ell/c^2$.

However, during the turn-around time, the spaceship is accelerating toward the earth, so the traveler sees the earth clock running fast. The magnitude of the acceleration is $a = 2v/T$ (because the spaceship goes from velocity v to $-v$ in time T). The earth clock therefore runs fast by a fraction $1 + a\ell/c^2 = 1 + 2v\ell/Tc^2$. This happens for a time T, so the earth clock gains a time of $(2v\ell/Tc^2)T = 2v\ell/c^2$.

Combining the results of the previous two paragraphs, we see that the earth clock gains a time of $2v\ell/c^2 - v\ell/c^2 = v\ell/c^2$. This is a fraction $(v\ell/c^2)/(2\ell/v) = v^2/2c^2$ of the total time, in agreement with part (a).

9. **Twin paradox again**

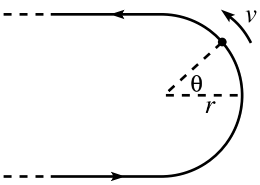

Figure 13.9

(a) The only difference between this problem and the previous one is the nature of the turn-around, so all we need to show here is that the traveler still sees the earth clock gain a time of $2v\ell/c^2$ during the turn-around (where ℓ is the earth-star distance).

Let the radius of the semicircle be r. Then the magnitude of the acceleration is $a = v^2/r$. Let θ be the angle shown in Fig. 13.9. For a given θ, the earth is at a height of $\ell\cos\theta$ in the gravitational field felt by the spaceship. The fractional time that the earth gains while the traveler is at angle θ is therefore $gh/c^2 = (v^2/r)(\ell\cos\theta)/c^2$. Integrating this over the time of the turn-around, and using $dt = r\,d\theta/v$, shows that the earth gains a time of

$$\Delta t = \int_{-\pi/2}^{\pi/2} \left(\frac{v^2\ell\cos\theta}{rc^2}\right)\left(\frac{r\,d\theta}{v}\right) = \frac{2v\ell}{c^2}\,, \tag{13.52}$$

during the turn-around, as was to be shown.

(b) Let the acceleration vector at a given instant be $\mathbf{a}$. And let $\boldsymbol{\ell}$ be the vector from the traveler to the earth. (Note that since the turn-around is done in a small region, $\boldsymbol{\ell}$ is essentially constant in this problem).

The earth is essentially at a height $\hat{\mathbf{a}}\cdot\boldsymbol{\ell}$ in the gravitational field felt by the traveler. (The dot product just gives the cosine term in the above solution to part (a).) The fractional time gain, gh/c^2, is therefore equal to $|\mathbf{a}|(\hat{\mathbf{a}}\cdot\boldsymbol{\ell})/c^2 = \mathbf{a}\cdot\boldsymbol{\ell}/c^2$. Integrating this over the entire duration of the turn-around, we see that the earth gains a time of

$$\begin{aligned}\Delta t = \int_{t_i}^{t_f} \frac{\mathbf{a}\cdot\boldsymbol{\ell}}{c^2}\,dt &= \frac{1}{c^2}\boldsymbol{\ell}\cdot\int_{t_i}^{t_f}\mathbf{a}\,dt \\ &= \frac{1}{c^2}\boldsymbol{\ell}\cdot(\mathbf{v}_f - \mathbf{v}_i) \\ &= \frac{\boldsymbol{\ell}\cdot(2\mathbf{v}_f)}{c^2} \\ &= \frac{2v\ell}{c^2}\,,\end{aligned} \tag{13.53}$$

during the turn-around, as was to be shown. The whole point here is that no matter what complicated motion the traveler undergoes during the turning-around, the total effect is simply a change in velocity from $\mathbf{v}$ outward to $\mathbf{v}$ inward.

Chapter 14

Appendices

14.1 Appendix A: Useful formulas

14.1.1 Taylor series

$$f(x_0+\epsilon) = f(x_0) + f'(x_0)\epsilon + \frac{f''(x_0)}{2!}\epsilon^2 + \frac{f'''(x_0)}{3!}\epsilon^3 + \cdots \tag{14.1}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \tag{14.2}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \tag{14.3}$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \tag{14.4}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{14.5}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{14.6}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{14.7}$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots \tag{14.8}$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} + \cdots \tag{14.9}$$

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots \tag{14.10}$$

14.1.2 Nice formulas

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{14.11}$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \qquad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \tag{14.12}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}, \qquad \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} \tag{14.13}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta} \tag{14.14}$$

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta \tag{14.15}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{14.16}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{14.17}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{14.18}$$

$$\cosh x = \frac{1}{2}(e^{x} + e^{-x}), \qquad \sinh x = \frac{1}{2}(e^{x} - e^{-x}) \tag{14.19}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{14.20}$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \frac{d}{dx}\sinh x = \cosh x \tag{14.21}$$

14.1.3 Integrals

$$\int \ln x\, dx = x\ln x - x \tag{14.22}$$

$$\int x\ln x\, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} \tag{14.23}$$

$$\int xe^{x} = e^{x}(x-1) \tag{14.24}$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x \qquad \text{or} \qquad -\cot^{-1} x \tag{14.25}$$

$$\int \frac{dx}{x(1+x^2)} = \frac{1}{2} \ln\left(\frac{x^2}{1+x^2}\right) \tag{14.26}$$

$$\int \frac{dx}{1-x^2} = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \qquad \text{or} \qquad \tanh^{-1} x \qquad (x^2 < 1) \tag{14.27}$$

$$\int \frac{dx}{1-x^2} = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right) \qquad \text{or} \qquad \coth^{-1} x \qquad (x^2 > 1) \tag{14.28}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \qquad \text{or} \qquad -\cos^{-1} x \tag{14.29}$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \ln(x + \sqrt{x^2+1}\,) \qquad \text{or} \qquad \sinh^{-1} x \tag{14.30}$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \ln(x + \sqrt{x^2-1}\,) \qquad \text{or} \qquad \cosh^{-1} x \tag{14.31}$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x \qquad \text{or} \qquad -\csc^{-1} x \tag{14.32}$$

$$\int \frac{dx}{x\sqrt{1+x^2}} = -\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right) \qquad \text{or} \qquad -\,\mathrm{csch}^{-1} x \tag{14.33}$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \text{or} \qquad -\,\mathrm{sech}^{-1} x \tag{14.34}$$

$$\int \frac{dx}{\cos x} = \ln\left(\frac{1+\sin x}{\cos x}\right) \tag{14.35}$$

$$\int \frac{dx}{\sin x} = \ln\left(\frac{1-\cos x}{\sin x}\right) \tag{14.36}$$

14.2 Appendix B: Units, dimensional analysis

There are two strategies (at least) that you should invoke without hesitation when solving problems. One is the consideration of units (that is, dimensions), which is the subject of this Appendix. The other is the consideration of limiting cases, which is the subject of the next Appendix.

The consideration of units offers two main benefits. First, looking at units before beginning a calculation can tell you roughly what the answer has to look like, up to numerical factors. (In some problems, you can determine the numerical factors by considering a limiting case of a certain parameter. So you may not have to do *any* calculations to solve a problem!) Second, checking units at the end of a calculation (which is something you should *always* do) tells you if your answer has a chance at being correct. It won't tell you that your answer is definitely correct, but it might tell you that your answer is definitely incorrect.

> "Your units are wrong!" cried the teacher.
> "Your church weighs six joules — what a feature!
> And the people inside
> Are four hours wide,
> And eight gauss away from the preacher!"

In practice, the second of the above two benefits is what you will generally make use of. But let's do a few examples relating to the first benefit, since these can be a little more exciting. To solve the following problems exactly, we would need to invoke results derived in earlier chapters in the text. But let's just see how far we can get by using only dimensional analysis. We'll use the "[ ]" notation for units, and we'll let M stand for mass, L for length, and T for time. For example, we will write a speed as $[v] = L/T$, and we will write the gravitational constant as $[G] = L^3/MT^2$ (you can figure this out by noting that Gm_1m_2/r^2 has the dimensions of force).

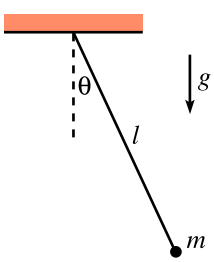

Figure 14.1

Example 1 (The pendulum): A mass m hangs from a massless string of length ℓ (see Fig. 14.1). The acceleration due to gravity is g. What can we say about the frequency of oscillations?

The only dimensionful quantities given in the problem are $[m] = M$, $[\ell] = L$, and $[g] = L/T^2$. There is one more quantity, the maximum angle θ_0, which is dimensionless (this one is easy to forget). We want to find the frequency, which has dimensions $1/T$. Clearly, the only combination of our given dimensionful quantities with the units of $1/T$ is $\sqrt{g/\ell}$. We can't rule out any θ_0 dependence, so the most general possible form for the frequency (in radians per second) is

$$\omega = f(\theta_0)\sqrt{\frac{g}{\ell}}\,, \tag{14.37}$$

where f is a dimensionless function of the dimensionless variable θ_0.

Remarks: It just so happens that for small oscillations, $f(\theta_0)$ is essentially equal to 1, and so the frequency is essentially $\sqrt{g/\ell}$. But there is no way to show this using only dimensional analysis. For larger values of θ_0, the higher-order terms in the expansion of f become important. Exercise 3.1 deals with the leading correction. (The answer is $f(\theta_0) = 1 - \theta_0^2/16 + \cdots$.)

There is only one mass scale in the problem, so there is no way that the frequency, with units $1/T$, can depend on $[m] = M$. ♣

What can we say about the total energy (relative to the lowest point) of the pendulum? Energy has units ML^2/T^2, and the only combination of the given dimensionful constants of this form is $mg\ell$. Therefore, the energy must be of the form $f(\theta_0)mg\ell$, where f is some function. We know, in fact, that the total energy equals the potential energy at the highest point, which is $mg\ell(1 - \cos\theta_0)$. Using the Taylor expansion for $\cos\theta$, we see that $f(\theta_0) = \theta_0^2/2 - \theta_0^4/24 + \cdots$. Unlike in the above case for the frequency, the maximum angle θ_0 plays a critical role in the energy.

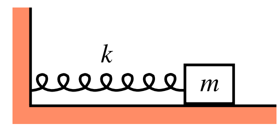

Figure 14.2

Example 2 (The spring): A spring with spring-constant k has a mass m on its end (see Fig. 14.2). The force is $F(x) = -kx$, where x is the displacement from equilibrium. What can we say about the frequency of oscillations?

The only dimensionful quantities in the problem are $[m] = M$, $[k] = M/T^2$ (obtained by noting that kx has the dimensions of force), and the maximum displacement from equilibrium, $[x_0] = L$. (There is also the equilibrium length, but the force doesn't depend on this, so there is no way it can come into the answer.) We want to find the frequency, which has dimensions $1/T$. It is easy to see that the only combination of our given dimensionful quantities with these units is

$$\omega = C\sqrt{\frac{k}{m}}\,, \tag{14.38}$$

where C is a dimensionless number. Note that, in contrast with the case of the pendulum, the frequency cannot have any dependence on the maximum displacement. It just so happens that C is equal to 1, but there is no way to show this using only dimensional analysis.

What can we say about the total energy of the spring? Energy has units ML^2/T^2, and the only combination of the given dimensionful constants of this form is Bkx_0^2, where B is a dimensionless number. It turns out that $B = 1/2$, so the total energy is given by $kx_0^2/2$.

Remark: Real springs don't have perfect parabolic potentials (that is, linear forces), so the force looks more like $F(x) = -kx + bx^2 + \cdots$. If we truncate the series at the second term, then we have one more dimensionful quantity to work with, $[b] = M/LT^2$. To form a quantity with dimensions of frequency, $1/T$, we clearly need the x_0 and b to appear in the combination $(x_0 b)$ (to get rid of the L). It is then easy to see that the frequency must be of the form $f(x_0 b/k)\sqrt{k/m}$. So we can have x_0 dependence in this case. Note that this answer must reduce to $C\sqrt{k/m}$ for $b = 0$. Hence, f must be of the form $f(y) = C + c_1 y + c_2 y^2 + \cdots$. ♣

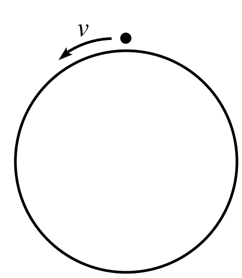

Figure 14.3

Example 3 (Speed of low-orbit satellite): A satellite of mass m travels in an orbit just above the earth's surface (see Fig. 14.3). What can we say about its speed?

Solution: The only dimensionful quantities in the problem are $[m] = M$, $[g] = L/T^2$, and the radius of the earth $[R] = L$. We want to find the speed, which has dimensions L/T. Clearly, the only combination of our dimensionful quantities with these units is

$$v = C\sqrt{gR}. \tag{14.39}$$

It turns out that $C = 1$.

14.2.1 Problems

1. **Escape velocity**

   Find the escape velocity from the earth (that is, the speed above which a particle escapes to infinity), up to factors of order 1.

2. **Mass in tube**

   A tube of mass M and length ℓ is free to swing on a pivot at one end. A mass m is positioned inside the tube at this end. The tube is held horizontal and then released (see Fig. 14.4).

   Let η be the fraction of the tube the mass has traversed by the time the tube becomes vertical. Does η depend on ℓ?

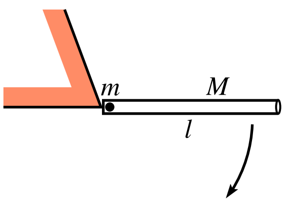

Figure 14.4

3. **Damping** $**$

   A particle with mass M and initial speed V is subject to a velocity-dependent damping force of the form bv^n.

   (a) For $n = 0, 1, 2, \ldots$, determine how the stopping time depends on M, V, and b.

   (b) For $n = 0, 1, 2, \ldots$, determine how the stopping distance depends on M, V, and b.

   (Careful! See if your answers make sense. Dimensional analysis only gives the answer up to a numerical factor. This is a tricky problem.)

14.2.2 Solutions

1. **Escape velocity**

   The reasoning is exactly the same as in the satellite example above. Therefore, the answer is $v = C\sqrt{gR} = C\sqrt{GM/R}$. It turns out that $C = 2$.

2. **Mass in tube**

   The relevant dimensionful constants are $[g] = L/T^2$, $[\ell] = L$, $[m] = M$, and $[M] = M$. We want to produce a dimensionless number η. Since g is the only constant involving time, η cannot depend on g. This then implies that η cannot depend on ℓ, the only length remaining. Therefore, η depends only on the ratio m/M. So the answer to the stated problem is, "No."

   It turns out that you have to solve the problem numerically to find η. Some results are: If $m = M$, then $\eta \approx 0.341$. If $m \ll M$, then $\eta \approx 0.349$. And if $m \gg M$, then $\eta \approx 0$.

3. **Damping**

(a) The constant b has units $[b] = [\text{Force}][v^{-n}] = (ML/T^2)(T^n/L^n)$. The other constants are $[M] = M$ and $[V] = L/T$. There is also n, which is dimensionless. The only combination with units of T is

$$t = f(n)\frac{M}{bV^{n-1}}\,, \tag{14.40}$$

where $f(n)$ is a dimensionless function of n.

For $n = 0$, we have $t = f(0)MV/b$. (This increases with M and V, and decreases with b, as it should.)

For $n = 1$ we have $t = f(1)M/b$. So we *seem* to have $t \sim M/b$. This, however, cannot be correct, because t should definitely grow with V. (A large initial speed V_1 requires some non-zero time to slow down to a smaller speed V_2, after which point we just have the same scenario with initial speed V_2.) Where did we go wrong? After all, the answer *does* have to look like $t = f(1)M/b$, where $f(1)$ is a numerical factor.

The resolution to this puzzle is that $f(1)$ is infinite. (If we wanted to work out the problem exactly, we would encounter an integral that diverges.) So for any V, t is infinite.[1]

Similarly, for $n \geq 2$, there is at least one power of V in the denominator of t. This certainly cannot be correct, because t should not decrease with V. So $f(n)$ must likewise be infinite for all of these cases.

The moral of this exercise is that you have to be careful when using dimensional analysis. The numerical factor in front of your answer generally turns out to be of order 1, but sometimes it turns out to be 0 or ∞.

REMARK: For $n \geq 1$, the expression in eq. (14.40) still has relevance. For example, for $n = 2$, the $M/(Vb)$ expression is relevant if you want to know how long it takes to go from V to some final speed V_f. The answer involves $M/(V_f b)$, which diverges as $V_f \to 0$. ♣

(b) The only combination with units of L is

$$\ell = g(n)\frac{M}{bV^{n-2}}\,, \tag{14.41}$$

where $g(n)$ is a dimensionless function of n.

For $n = 0$, we have $\ell = g(0)MV^2/b$.

For $n = 1$, we have $\ell = g(1)MV/b$.

For $n = 2$ we have $\ell = g(2)M/b$. So we *seem* to have $\ell \sim M/b$. But as in part (a), this cannot be correct, because ℓ should definitely depend on V. (A large initial speed V_1 requires some non-zero distance to slow down to a smaller speed V_2, after which point we just have the same scenario with initial speed V_2.) So, from the reasoning in part (a), the total distance is infinite for $n \geq 2$, because the function g is infinite in these cases.

REMARK: Note that for $n \neq 1$, t and ℓ are either both finite or both infinite. For $n = 1$, however, the total time is infinite, whereas the total distance is finite. (This situation holds for $1 \leq n < 2$, if we want to consider fractional n.) ♣

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[1] The total time t is ill-defined, of course, since the particle never comes to rest. But t does grow with V, in the sense that if t is defined to be the time to attain some given small speed, then t grows with V.

14.3 Appendix C: Approximations, limiting cases

Along with checking units, checking limiting cases is something you should always do at the end of a calculation. As in the case with checking dimensions, this won't tell you that your answer is definitely correct, but it might tell you that your answer is definitely incorrect. It is generally true that your intuition about limiting cases is much better than your intuition about generic values of the parameters. You should use this fact to your advantage.

A main ingredient in checking limiting cases is the Taylor series approximations. The series for many functions are given in Appendix A.

The examples presented below have been taken from various problems throughout the book. For the most part, we'll just repeat here what we've said in the remarks given in the solutions earlier in the text.

Example 1 (Dropped ball): Consider a dropped ball that is subject to an air-resistance drag force of the form $F_d = -m\alpha v$, in addition to the usual mg force downward. Let the initial height be h. We found in Section 2.3 that the ball's speed and position are given by

$$v(t) = -\frac{g}{\alpha}\left(1 - e^{-\alpha t}\right), \qquad \text{and} \qquad y(t) = h - \frac{g}{\alpha}\left(t - \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\right). \tag{14.42}$$

Let's look at some limiting cases. If t is very small (more precisely,[2] if $\alpha t \ll 1$), then we can use the Taylor series $e^{-x} \approx 1 - x + x^2/2$ to make approximations to leading order in αt. Eq. (14.42) gives

$$\begin{aligned} v(t) &= -\frac{g}{\alpha}\left(1 - \left(1 - \alpha t + \frac{(\alpha t)^2}{2} - \cdots\right)\right) \\ &\approx -gt, \end{aligned} \tag{14.43}$$

plus terms of higher order in αt. This answer is expected, because the drag force is negligible at the start, so we almost have a freely falling body. We also have

$$\begin{aligned} y(t) &= h - \frac{g}{\alpha}\left[t - \frac{1}{\alpha}\left(1 - \left(1 - \alpha t + \frac{(\alpha t)^2}{2} - \cdots\right)\right)\right] \\ &\approx h - \frac{gt^2}{2}, \end{aligned} \tag{14.44}$$

plus terms of higher order in αt. Again, this answer is expected.

We may also look at large t (or rather, large αt). In this case, $e^{-\alpha t}$ is essentially 0, so eq. (14.42) gives

$$v(t) \approx -\frac{g}{\alpha}. \tag{14.45}$$

This is the "terminal velocity". This value makes sense, because it is the velocity for which the force $-mg - m\alpha v$ vanishes. Also, eq. (14.42) gives

$$y(t) \approx h - \frac{gt}{\alpha} + \frac{g}{\alpha^2}. \tag{14.46}$$

[2] See the "Remark" following this example.

Apparently, for large t, g/α^2 is the distance our ball lags behind another ball which started out already at the terminal speed, g/α.

Whenever you derive approximate answers as we did above, you gain something and you lose something. You lose some truth, of course, because your new answer is technically not correct. But you gain some aesthetics. Your new answer is invariably much cleaner (sometimes involving only one term), and this makes it a lot easier to see what's going on.

Remark: In the above example, it makes no sense to look at the limit where t is small (or large), because t has dimensions. Is a year a large or small time? How about a hundredth of a second? There is no way to answer this without knowing what problem you're dealing with. A year is short on the time scale of galactic evolution, but a hundredth of a second is long on the time scale of nuclear processes.

It only makes sense to look at the limit of a small (or large) *dimensionless* quantity. In the above example, this quantity is αt. The given constant α has units of $[T^{-1}]$; hence, $1/\alpha$ sets a typical time scale for the system. It therefore makes sense to look at the limit where $t \ll 1/\alpha$ (that is, $\alpha t \ll 1$), or likewise $t \gg 1/\alpha$ (that is, $\alpha t \gg 1$).

In the limit of a small dimensionless quantity, a Taylor series can be used to expand your answer in powers of the small quantity.

We will sometimes get sloppy and say things like, "In the limit of small t." But you know that we really mean, "In the limit of some small dimensionless quantity that has a t in the numerator," or, "In the limit where t is much smaller that a certain quantity that has the dimensions of time." ♣

The results of the limits you check generally fall into two categories. Most of the time you know what the result should be, so this provides a double-check for your answer. But sometimes an interesting limit pops up that you might not expect. Such is the case in the following examples.

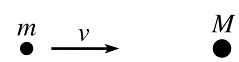

Figure 14.5

Example 2 (Two masses in 1-D): A mass m with speed v approaches a stationary mass M (see Fig. 14.5). The masses bounce off each other elastically. Assume all motion takes place in one dimension. We found in Section 4.6.1 that the final speeds of the particles are

$$v_m = \frac{(m-M)v}{m+M}, \qquad \text{and} \qquad v_M = \frac{2mv}{m+M}. \tag{14.47}$$

Some obvious special cases to check are the following.

- If $m = M$, then m stops, and M picks up a speed of v. This is fairly believable. And it becomes quite obvious once you realize that these final speeds clearly satisfy conservation of energy and momentum with the initial conditions.
- If $M \gg m$, then m bounces backward with speed $\approx v$, and M hardly moves. This is clear, since M basically acts like a brick wall.
- If $m \gg M$, then m keeps plowing along at speed $\approx v$, and M picks up a speed of $\approx 2v$. This $2v$ is an interesting result (it is clear if you consider what is happening in the reference frame of the heavy mass m), and it leads to some neat effects, as in Problem 4.25.

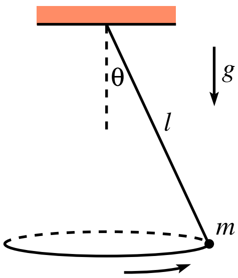

Figure 14.6

Example 3 (Circular pendulum): A mass hangs from a string of length ℓ. Conditions have been set up so that the mass swings around in a horizontal circle, with the string making a constant angle of θ with the vertical (see Fig. 14.6). We found in Section 2.5 that the angular frequency, ω, of this motion is

$$\omega = \sqrt{\frac{g}{\ell \cos\theta}}\,. \tag{14.48}$$

There are two obvious limits to look at.

- If $\theta \approx 90°$, then $\omega \to \infty$. This is clear. The mass has to spin quickly to avoid flopping down.
- If $\theta \approx 0$, then $\omega \approx \sqrt{g/\ell}$, which is the same as the frequency of a plane pendulum of length ℓ. (Convince yourself why this should be true.)

In the above examples, we checked limiting cases of answers that are correct (I hope). This whole process is more useful (and actually a bit more fun) when you check the limits of an answer that is *incorrect*. In this case, you gain the unequivocal information that your answer is wrong. This information is something you should be happy about, considering that the alternative is to carry on in a state of blissful ignorance. Personally, if there's any way I'd want to discover that my answer is garbage, this is it. At any rate, checking limits can often save you a lot of trouble in the long run . . .

The lemmings get set for their race.
With one step and two steps they pace.
They take three and four,
And then head on for more,
Without checking the limiting case.

14.4 Appendix D: Solving differential equations numerically

Sooner or later you will encounter a differential equation that you cannot solve exactly. Having resigned yourself to not getting the exact answer, you should ponder how to obtain a decent approximation to it. In this marvellously advanced technological era (which your children will dismiss with nothing more than a bewildered chuckle), it's easy to write a short program which will give you a very good numerical answer to your problem. Given enough computer time, you can obtain any desired accuracy (assuming the system isn't chaotic; no need to worry about this for the systems we will deal with). We'll demonstrate the procedure using an easy problem, one which we actually do know the answer to. Consider the equation

$$\ddot{x} = -\omega^2 x. \tag{14.49}$$

This is of course the equation for a mass on a spring (with $\omega = \sqrt{k/m}$), and we know the solution can be written, among other ways, in the form

$$x(t) = A\cos(\omega t + \phi). \tag{14.50}$$

But let's pretend we don't know this. If someone comes along and gives you values for $x(0)$ and $\dot{x}(0)$, it seems like somehow you should be able to find $x(t)$ and $\dot{x}(t)$ for any later t, just by using eq. (14.49). Here's how you do it.

Discretize time into intervals of some small unit (call it ϵ), Then determine what happens at each successive point in time. The point is that if you know $x(t)$ and $\dot{x}(t)$, then you can easily find (approximately) what x is at a slightly later time. Similarly, if you know $\dot{x}(t)$ and $\ddot{x}(t)$, then you can easily find (approximately) what $\dot{x}$ is at a slightly later time. Using the definitions of derivatives, the relations are simply

$$\begin{aligned} x(t+\epsilon) &\approx x(t) + \epsilon\dot{x}(t), \\ \dot{x}(t+\epsilon) &\approx \dot{x}(t) + \epsilon\ddot{x}(t). \end{aligned} \tag{14.51}$$

These two equations, combined with (14.49) (which gives you $\ddot{x}$ if you know x), allow you to march along in time, obtaining successive values for x, $\dot{x}$, and $\ddot{x}$.

Here's what a typical program might look like. (This is MAPLE, but even if you aren't familiar with this, the general idea should be clear.) Let's say a particle starts from rest at a position $x = 2$, and let's take $\omega^2 = 5$. We'll use the notation where "x1" stands for $\dot{x}$, and "x2" stands for $\ddot{x}$. And "e" stands for ϵ. Let's say the goal is to find x at $t = 3$.

```
x:=2;     # initial position
x1:=0;     # initial speed
e:=.01;     # a small time interval
for i to 300 do     # do 300 steps (ie, up to 3 seconds)
x2:=-5*x;     # the given equation
x:=x+e*x1;     # how x changes
```

```
x1:=x1+e*x2;     # how x1 changes
od;     # the MAPLE command to stop the do loop
```

This procedure of course won't give the exact values for x, because x and $\dot{x}$ don't really change according to eqs. (14.51). These equations are just first-order approximations to the full Taylor series with higher-order terms. Said differently, there is no way the above procedure can be exactly correct, because there are ambiguities in how the program can be written. Should line 5 come before or after line 7? (That is, in determining the $\dot{x}$ at time $t + \epsilon$, should you use the $\ddot{x}$ at time t or $t + \epsilon$?) And should line 7 come before or after line 6? The point is that for very small ϵ, it doesn't matter much. And in the limit $\epsilon \to 0$, it doesn't matter at all.

If you want to obtain a better approximation, just shorten ϵ down to .001, and increase the number of steps to 3000. If the result looks basically the same as with $\epsilon = .01$, then you know you've pretty much got the right answer.

In the present example, $\epsilon = .01$ yields $x \approx 1.965$ after 3 seconds. If we set $\epsilon = .001$, then we obtain $x \approx 1.836$. And if we set $\epsilon = .0001$, then we get $x \approx 1.823$. The correct answer must therefore be somewhere around $x = 1.82$. And indeed, if we solved the problem exactly, we would obtain $x(t) = 2\cos(\sqrt{5}t)$. Plugging in $t = 3$ gives $x \approx 1.822$.

This is a wonderful procedure, but it shouldn't be abused. It's nice to know that you can always obtain a decent numerical approximation if all else fails. But you should set your initial goal on obtaining the correct algebraic expression, because this allows you to see the overall behavior of the system. And besides, nothing beats the truth. People tend to rely a bit too much on computers and calculators nowadays, without pausing to think about what is actually going on in a problem.

The skill to do math on a page
Has declined to the point of outrage.
Equations quadratica
Are solved on Math'matica,
And on birthdays we don't know our age.

14.5 Appendix E: $F = ma$ vs. $F = dp/dt$

In nonrelativistic mechanics,[3] the equations $F = ma$ and $F = dp/dt$ say exactly the same thing, provided that m is constant. But if m in not constant, then $dp/dt = d(mv)/dt = ma + (dm/dt)v$, which does not equal ma. In this case, should we use $F = ma$ or $F = dp/dt$? Which law correctly describes the physics? The answer to this depends on what exactly you label as the "system" to which you associate the quantities m, p, and a. You can generally do a problem using either $F = ma$ or $F = dp/dt$, but you must be very careful about how you label things and how you treat them. The subtleties are best understood through two examples.

Example 1 (Sand dropping into cart): Consider a cart into which sand is dropped (vertically) at a rate $dm/dt = \sigma$. With what force must you push on the cart to keep it moving (horizontally) at a constant speed v?

First solution: Let $m(t)$ be the mass of the cart-plus-sand-inside (label this system as "C"). If we use $F = ma$ (where a is the acceleration of the cart), then we obtain $F = 0$, because $a = 0$. This is not correct. The correct expression to use is $F = dp/dt$. This gives

$$F = \frac{dp}{dt} = ma + \frac{dm}{dt}v = 0 + \sigma v. \tag{14.52}$$

This makes sense, because your force is what increases the momentum of C, and the momentum increases simply because the mass of C increases.

Second solution: It is possible to solve this problem by using $F = ma$, provided that you let your system be a small piece of mass that is being added to the cart. Your force, of course, is what accelerates this mass from rest to speed v. Consider a mass Δm that falls into the cart during time Δt. Imagine that it falls into the cart in one lump at the start of the Δt, and then accelerates up to speed v after time Δt (and then this procedure repeats itself during each successive Δt interval). Then $F = ma = \Delta m(v/\Delta t)$. Writing this as $(\Delta m/\Delta t)v$ gives the σv result in the first solution.

Example 2 (Sand leaking from cart): Consider a cart which leaks sand out of the bottom at a rate $dm/dt = \sigma$. If you apply a force F to the cart, what is its acceleration?

Solution: Let $m(t)$ be the mass of the cart-plus-sand-inside (label this system as "C"). In this example, we want to use $F = ma$. So the acceleration is simply

$$a = \frac{F}{m}\,. \tag{14.53}$$

Note that since m decreases with time, a increases with time.

[3] We won't bother with relativity in this Appendix, since nonrelativistic mechanics contains all the critical aspects we want to address.

We used $F = ma$, because at any instant, the mass m is what is being accelerated by the force F. If you want, you can imagine the process occurring in discrete steps: The force pushes on the mass for a short period of time, then a little piece instantaneously leaks out; then the force pushes again on the new mass, then another little piece leaks out; and so on. In this scenario, it is clear that $F = ma$ is the appropriate formula, since it holds for each step in the process. (The only ambiguity is whether you use m or $m + dm$ at a certain time, but this yields a negligible error.)

REMARKS: It is still true that $F = dp/dt$ in this problem, if you let F be *total* force, and let p be the *total* momentum. In the present problem, F is the only force. However, the total momentum consists of both the sand in the cart and the sand that has leaked out and is falling through the air.[4] A common mistake is to use $F = dp/dt$, with p being only the cart-plus-sand-inside's momentum.

There is an easy example that demonstrates why $F = dp/dt$ doesn't work when p refers to only C. Imagine that $F = 0$, and let the cart move with speed v. Cut the cart in half, and label the back part as "leaked sand", and the front part as "the cart". If you want the cart's p to have $dp/dt = 0$, then the cart's speed must double if its mass gets cut in half. But this is nonsense. Both halves of the cart simply continue to move at the same rate. ♣

To sum up, $F = dp/dt$ is always valid, provided that you use the *total* force and *total* momentum. This approach, however, can get messy in certain situations. So in some cases it is easier to use an $F = ma$ argument, but you must be careful to correctly identify the system that is being accelerated by the force. In the first example above, it is the small additional piece of sand that the force accelerates; F does not accelerate the cart. In the second example above, it is the cart that the force accelerates; F does not decelerate the small leaked piece of sand.

[4]If there were air resistance, we would have to worry about its effect on the falling sand if we wanted to use $F = dp/dt$ to solve the problem, where p is the total momentum. This is clearly not the best way to do the problem. If complicated things occur with the sand in the air, it would be foolish to consider this part of the sand if we don't have to.

14.6 Appendix F: Existence of principal axes

In this Appendix, we will prove Theorem 8.4. That is, we will prove that an orthonormal set of principal axes does indeed exist, for any object, and for and any choice of origin. It is not crucial that you study this proof. If you want to simply accept the fact that the principal axes exist, that's perfectly fine. But the method used in this proof is one you will see again and again in your physics career, in particular when you study quantum mechanics (see the remark following the proof).

Theorem 14.1 *Given a real, symmetric* 3×3 *matrix,* $\mathbf{I}$*, there exist three orthonormal real vectors,* $\hat{\boldsymbol{\omega}}_k$*, and three real numbers,* I_k*, with the property that*

$$\mathbf{I}\hat{\boldsymbol{\omega}}_k = I_k \hat{\boldsymbol{\omega}}_k. \tag{14.54}$$

Proof: This theorem holds more generally with 3 replaced by N (all the steps below easily generalize), but we'll work with $N = 3$, to be concrete.

Consider a general matrix $\mathbf{I}$ (we don't need to assume yet that it's real or symmetric). Assume that $\mathbf{I}\mathbf{u} = I\mathbf{u}$ for some vector $\mathbf{u}$ and some number I.[5] This may be rewritten as

$$\begin{pmatrix} (I_{xx} - I) & I_{xy} & I_{xz} \\ I_{yx} & (I_{yy} - I) & I_{yz} \\ I_{zx} & I_{zy} & (I_{zz} - I) \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{14.55}$$

In order for there to be a nontrivial solution for the vector $\mathbf{u}$ (that is, one where $\mathbf{u} \neq (0,0,0)$), the determinant of this matrix must be zero.[6] Taking the determinant, we see that we get an equation for I of the form

$$aI^3 + bI^2 + cI + d = 0. \tag{14.56}$$

The constants a, b, c, and d are functions of the matrix entries I_{ij}, but we won't need their precise form to prove this existence theorem. All we need this equation for is to say that there do exist three (generally complex) roots to this equation, since it is of third degree.

We will now show that the roots for I are real. This will imply that there exist three real vectors $\mathbf{u}$ satisfying $\mathbf{I}\mathbf{u} = I\mathbf{u}$ (because we can plug the real I's back into eq. (14.55) to solve for the real components u_x, u_y, and u_z, up to an overall constant). We will then show that these vectors are orthogonal.

- *Proof that the I are real:* This follows from the real and symmetric conditions on $\mathbf{I}$. Start with the equation $\mathbf{I}\mathbf{u} = I\mathbf{u}$, and take the dot product with $\mathbf{u}^*$ to obtain

$$\mathbf{u}^* \cdot \mathbf{I}\mathbf{u} = I\mathbf{u}^* \cdot \mathbf{u}. \tag{14.57}$$

[5] Such a vector $\mathbf{u}$ is called an *eigenvector* of $\mathbf{I}$, and I is the associated *eigenvalue*. But don't let these names scare you. They're just definitions.

[6] Quick proof: If the determinant were not zero, then we could explicitly construct the inverse of the matrix (it involves cofactors divided by the determinant). Multiplying both sides by this inverse would show that $\mathbf{u} = \mathbf{0}$.

The vector $\mathbf{u}^*$ is the vector obtained by simply complex conjugating each component of $\mathbf{u}$ (we don't know yet that $\mathbf{u}$ can be chosen to be real). On the right side, I is a scalar, so we can take it out from between the $\mathbf{u}^*$ and $\mathbf{u}$.

Now complex conjugate the equation $\mathbf{I}\mathbf{u} = I\mathbf{u}$ to obtain $\mathbf{I}\mathbf{u}^* = I^*\mathbf{u}^*$ (we know that $\mathbf{I}$ is real, but we don't know yet that I is real), and take the dot product with $\mathbf{u}$ to obtain

$$\mathbf{u} \cdot \mathbf{I}\mathbf{u}^* = I^*\mathbf{u} \cdot \mathbf{u}^*. \tag{14.58}$$

We now claim that if $\mathbf{I}$ is symmetric, then $\mathbf{a} \cdot \mathbf{I}\mathbf{b} = \mathbf{b} \cdot \mathbf{I}\mathbf{a}$, for any vectors $\mathbf{a}$ and $\mathbf{b}$. (You can show this by simply multiplying each side out). In particular, $\mathbf{u}^* \cdot \mathbf{I}\mathbf{u} = \mathbf{u} \cdot \mathbf{I}\mathbf{u}^*$, so eqs. (14.57) and (14.58) give

$$(I - I^*)\mathbf{u} \cdot \mathbf{u}^* = 0. \tag{14.59}$$

Since $\mathbf{u} \cdot \mathbf{u}^* = |u_1|^2 + |u_2|^2 + |u_3|^2 \neq 0$, we must have $I = I^*$. Therefore, I is real.

- *Proof that the* $\mathbf{u}$ *are orthogonal:* This follows from the symmetric condition on $\mathbf{I}$. Let $\mathbf{I}\mathbf{u}_1 = I_1\mathbf{u}_1$, and $\mathbf{I}\mathbf{u}_2 = I_2\mathbf{u}_2$. Take the dot product of the former equation with $\mathbf{u}_2$ to obtain

$$\mathbf{u}_2 \cdot \mathbf{I}\mathbf{u}_1 = I_1\mathbf{u}_2 \cdot \mathbf{u}_1. \tag{14.60}$$

Take the dot product of the latter equation with $\mathbf{u}_1$ to obtain

$$\mathbf{u}_1 \cdot \mathbf{I}\mathbf{u}_2 = I_2\mathbf{u}_1 \cdot \mathbf{u}_2. \tag{14.61}$$

As above, the symmetric condition on $\mathbf{I}$ implies that the left-hand sides of eqs. (14.60) and (14.61) are equal. Therefore,

$$(I_1 - I_2)\mathbf{u}_1 \cdot \mathbf{u}_2 = 0. \tag{14.62}$$

There are two possibilities here. (1) If $I_1 \neq I_2$, then we are done, because $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, which says that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal. (2) If $I_1 = I_2 \equiv I$, then we have $\mathbf{I}(a\mathbf{u}_1 + b\mathbf{u}_2) = I(a\mathbf{u}_1 + b\mathbf{u}_2)$, so any linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ has the same property that $\mathbf{u}_1$ and $\mathbf{u}_2$ have (namely, applying $\mathbf{I}$ results in simply multiplying by I). We therefore have a whole plane of such vectors, so we can pick any two orthogonal vectors in this plane to be called $\mathbf{u}_1$ and $\mathbf{u}_2$. ■

This theorem proves the existence of principal axes, because the inertia tensor in eq. (8.8) is indeed a real and symmetric matrix.

REMARK: (Warning: This remark has nothing to do with classical mechanics. It is simply an ill-disguised excuse to write down another limerick.) In Quantum Mechanics, it turns out that any observable quantity (such as position, energy, momentum, angular momentum, etc.) can be represented by a *Hermitian* matrix, with the observed value being an eigenvalue of the matrix. A Hermitian matrix is a (generally complex) matrix with the property that the transpose of the matrix equals the complex conjugate of the matrix. For example, a 2×2 Hermitian matrix must be of the form,

$$\begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}, \tag{14.63}$$

for real a, b, c, and d. Now, if observed values are to be given by the eigenvalues of such a matrix, then the eigenvalues had *better* be real, since no one (in this world, at least) is about to go for a jog of $4+3i$ miles, or pay an electric bill for $17-43i$ kilowatt-hours. And indeed, you can show via a slightly modified version of the "Proof that the I are real" procedure above that the eigenvalues of any Hermitian matrix are real. (And likewise, the eigenvectors are orthogonal.) This is, to say the least, very fortunate.

God's first tries were hardly ideal,
For complex worlds have no appeal.
So in the present edition,
He made things Hermitian,
And *this* world, it seems, is quite real. ♣

14.7 Appendix G: Diagonalizing matrices

This appendix is relevant to Section 8.3 (on Principal axes). The process of diagonalizing matrices (that is, finding the *eigenvectors* and *eigenvalues*) has applications in countless types of problems in a wide variety of subjects. We will describe the process here as it applies to principle axes and moments of inertia.

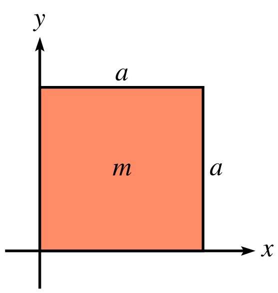

Figure 14.7

Let's find the three principal axes (and moments of inertia) for a square with side length a, mass m, and one corner at the origin. The square lies in the x-y plane, with sides along the x- and y-axes (see Fig. 14.7).

We'll choose the given x- y- and z-axes as our initial basis. From eq. (8.8), the matrix $\mathbf{I}$ (with respect to this initial basis) is easily shown to be

$$\mathbf{I} = \rho \begin{pmatrix} \int y^2 & -\int xy & 0 \\ -\int xy & \int x^2 & 0 \\ 0 & 0 & \int(x^2+y^2) \end{pmatrix} = ma^2 \begin{pmatrix} 1/3 & -1/4 & 0 \\ -1/4 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}. \qquad (14.64)$$

(ρ is the mass per unit area, so that $a^2\rho = m$; and we have not bothered writing the $dx\,dy$ in the integrals; and we have used the fact that $z = 0$.)

We must find the basis in which $\mathbf{I}$ is diagonal. That is, we must find three solutions[7] for $\mathbf{u}$ (and I) in the equation $\mathbf{I}\mathbf{u} = I\mathbf{u}$. Letting $I \equiv \lambda ma^2$ (to make things look a little cleaner), and using the above form of $\mathbf{I}$, the equation $(\mathbf{I} - I)\mathbf{u} = 0$ becomes

$$ma^2 \begin{pmatrix} 1/3-\lambda & -1/4 & 0 \\ -1/4 & 1/3-\lambda & 0 \\ 0 & 0 & 2/3-\lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (14.65)$$

In order for this to have a nonzero solution for the components u_1, u_2, u_3, the determinant of this matrix must be zero. The resulting cubic equation for λ is easy to solve, since the determinant is simply $[(1/3-\lambda)^2 - (1/4)^2](2/3-\lambda) = 0$. The solutions are $\lambda = 1/3 \pm 1/4$ and $\lambda = 2/3$. So our three moments of inertia are

$$I_1 = \frac{7}{12}ma^2, \qquad I_2 = \frac{1}{12}ma^2, \qquad I_3 = \frac{2}{3}ma^2. \qquad (14.66)$$

These are the *eigenvalues* of $\mathbf{I}$.

What are the vectors, $\mathbf{u}_1$, $\mathbf{u_2}$, and $\mathbf{u}_3$, associated with each of these I's? Plugging $\lambda = 7/12$ into (14.65) gives the three equations (one for each component) $-u_1 - u_2 = 0$, $-u_1 - u_2 = 0$, and $u_3 = 0$. These are redundant equations (that was the whole point of setting the determinant equal to zero). So $u_1 = -u_2$, and $u_3 = 0$. Our vector may therefore be written as $(c, -c, 0)$, where c is any constant.[8] If we want a normalized vector, then $c = 1/\sqrt{2}$. In a similar manner, plugging $\lambda = 1/12$ into eq. (14.65) gives $(u_1, u_2, u_3) = (c, c, 0)$. And finally, plugging $\lambda = 1/3$ into eq.

[7]One obvious solution is $\mathbf{u} = \hat{\mathbf{z}}$, because $\mathbf{I}\hat{\mathbf{z}} = (2/3)ma^2\hat{\mathbf{z}}$. From the orthogonality result of Theorem 8.4, we know that the other two vectors must lie in the x-y plane. So we could quickly reduce this problem to a two-dimensional one, but let's forge ahead with the general method.

[8]We can only solve for $\mathbf{u}$ up to an overall constant, because if $\mathbf{I}\mathbf{u} = I\mathbf{u}$ is true for a certain $\mathbf{u}$, then it is also true that $\mathbf{I}(c\mathbf{u}) = I(c\mathbf{u})$, where c is any constant.

(14.65) gives $(u_1, u_2, u_3) = (0, 0, c)$ (as claimed in the above footnote). So our three orthonormal principal axes corresponding to the moments in eq. (14.66) are

$$\hat{\boldsymbol{\omega}}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \qquad \hat{\boldsymbol{\omega}}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \qquad \hat{\boldsymbol{\omega}}_3 = (0, 0, 1). \tag{14.67}$$

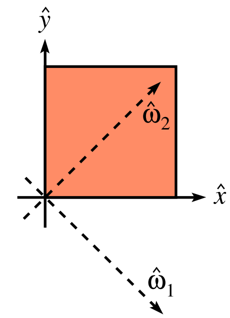

Figure 14.8

These are the *eigenvectors* of **I**. These axes are shown in Fig. 14.8. In our new basis of the principal axes, the matrix **I** takes the form

$$\mathbf{I} = ma^2 \begin{pmatrix} 7/12 & 0 & 0 \\ 0 & 1/12 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}. \tag{14.68}$$

The basic idea is that from now on we should use the principal axes as our basis vectors. We can forget we ever had anything to do with the initial x-y-z-axis basis.

REMARKS: (1) $I_1 + I_2 = I_3$, as the perpendicular axis theorem demands. (2) I_2 is the moment around one diagonal through the center of the square, which of course equals the moment around the other diagonal through the center. But the latter is related to I_1 by the parallel axis theorem; and indeed, $I_1 = I_2 + m(a/\sqrt{2})^2$. (3) Convince yourself why I_2 is simply the moment around the center of a stick of mass m and length a. (Any axis through the center of a square, in the plane of the square, has the same moment.) (4) An application of the parallel and perpendicular axis theorems gives (considering an axis through the center and parallel to the I_3 axis) $I_3 = 2I_2 + m(a/\sqrt{2})^2$. ♣

14.8 Appendix H: Qualitative Relativity Questions

1. Is there such a thing as a perfectly rigid body?

   Answer: No. Since information can move no faster than the speed of light, it takes time for the atoms in the body to communicate with each other. If you push on one end of a rod, then the other end will not move right away.

2. Moving clocks run slow. Does this result have anything to do with the time it takes light to travel from the clock to your eye?

   Answer: No. When we talk about how fast a clock is running in a given frame, we are referring to what the clock actually reads in that frame. It will of course take time for the light from the clock to reach an observer's eye, but it is understood that the observer subtracts off this transit time in order to calculate the time at which the clock actually shows a particular reading.

   Likewise, other relativistic effects, such as length contraction and loss of simultaneity, have nothing to do with the time it takes light to reach your eye. They deal only with what really *is*, in your frame.

3. Does time dilation depend on whether a clock is moving across your vision or directly away from you?

   Answer: No. A moving clock runs slow, no matter which way it is moving.

4. Does the special-relativistic time dilation depend on the acceleration of the moving clock?

   Answer: No. The time-dilation factor is $\gamma = 1/\sqrt{1 - v^2/c^2}$, which does not depend on a. The only relevant quantity is the v at a given instant. It doesn't matter if v is changing.

   Of course, if *you* are accelerating, then the results of special relativity are not valid; general relativity is required. But as long as you represent an inertial frame, then the clock you are viewing can undergo whatever motion it wants, and you will observe it running slow by the simple factor of γ.

5. Someone says, "A stick that is length-contracted isn't *really* shorter, it's just that it *looks* shorter." Do you agree?

   Answer: Hopefully not. The stick really *is* shorter in your frame. Length contraction has nothing to do with how things look. It has to do with where the ends of the stick are at simultaneous times in your frame. (That, after all, is how you measure the length of something.) At a given instant in time (in your frame), the distance between the ends of the stick is indeed less than the proper length of the stick.

6. Consider a stick that is moving in the direction in which it points. Does its length contraction depend on whether this direction is across your vision or directly away from you?

Answer: No. The stick is length-contracted in both cases. Of course, if you look at the stick in the latter case, then all you will see is the end (which will just be a dot). But the stick is indeed shorter in your reference frame.

7. A mirror moves toward you at speed v. You shine a light towards it and the light beam bounces back at you. What is the speed of the reflected beam?

   Answer: c.

8. In relativity, the order of two events in one frame may be reversed in another frame. Does this imply that there exists a frame in which I get off a bus before I get on it?

   Answer: No. The order of two events can be reversed in another frame only if the events are spacelike separated. (That is, $|\Delta x| > c|\Delta t|$. In other words, the events are too far apart even for light to get from one to the other.) The two relevant events here (getting on the bus, and getting off the bus) are clearly not spacelike separated (since the bus travels at a speed less than c, of course). They are timelike separated. Therefore, in all frames I get off the bus after I get on it.

   There would be causality problems, of course, if there existed a frame in which I got off the bus before I got on it. If I broke my ankle getting off the bus, then I wouldn't be able to make the fast dash that I made to catch the bus is the first place (in which case I wouldn't have the opportunity to break my ankle getting off the bus, in which case I could have made the fast dash to catch the bus and get on, and, well, you get the idea).

9. You are in a spaceship sailing along in outer space. Is there any way you can measure your speed without looking outside?

   Answer: There are two points to be made here. First, the question is meaningless, because absolute speed does not exist. The spaceship does not have a speed; it only has a speed relative to something else.

   Second, even if the question asked for the speed with respect to, say, a given star, the answer would be "no". Uniform speed is not measurable from within the spaceship. (Acceleration, on the other hand, is measurable, assuming there is no gravity around to confuse it with.)

10. If you move at the speed of light, what will the universe look like to you?

    Answer: The question is meaningless, because it is impossible for you to move at the speed of light. A meaningful question to ask is: what would the universe look like if you moved at a speed very close to c? The answer is that in your frame everything would be squashed along the direction of your motion. Any given region of the universe would be squashed down to a pancake.

11. Two objects fly toward you, one from the east with speed u, and the other from the west with speed v. Is it correct that their relative speed, as measured by you, is $u + v$? (Or should you use the velocity-addition formula, $V = (u + v)/(1 + uv/c^2)$?) Is it possible for their relative speed, as measured by you, to exceed c?

Answer: Yes, it is legal to simply add the two speeds to obtain $u + v$. There is no need to use the velocity-addition formula, because both speeds here are measured with respect to the *same thing* (namely, you). It is perfectly legal for the result to be greater than c (but it must be less than $2c$).

You need to use the velocity-addition formula when, for example, you are given the speed of a ball with respect to a train, and also the speed of the train with respect to the ground, and the goal is to find the speed of the ball with respect to the ground. The point is that now the two given speeds are measured with respect to *different* things (namely, the train and the ground).

12. Two clocks at the ends of a train are synchronized with respect to the train. If the train moves past you, which clock shows the higher time?

Answer: The rear clock shows the higher time. (It shows Lv/c^2 more than the front clock, where L is the proper length of the train.)

13. A train moves at speed $4c/5$. A clock is thrown from the back of the train to the front. As measured in the ground frame, the time of flight is 1 second. Is the following reasoning correct? "The γ-factor between the train and the ground is $\gamma = 1/\sqrt{1-(4/5)^2} = 5/3$. Moving clocks run slow, so the time elapsed on the clock during the flight is 3/5 of a second."

Answer: No. It is incorrect, because the time-dilation result holds only for two events that happen at the *same place* in the relevant reference frame (the train, here). The clock moves with respect to the train, so the above reasoning is not correct.

Another way of seeing why it must be incorrect is the following. A certainly valid way to calculate the clock's elapsed time is to find the speed of the clock with respect to the ground (more information would have to be given to determine this), and to then perform time dilation with the associated γ-factor to arrive at the answer of γ. Since the clock's v is definitely not $4c/5$, the correct answer is definitely not $3/5\,\text{s}$.

14. Person A is chasing person B. As measured in the ground frame, they have speeds $4c/5$ and $3c/5$, respectively. If they start a distance L apart (as measured in the ground frame), how much time will it take (as measured in the ground frame) for A to catch B?

Answer: As measured in the ground frame, the relative speed is $4c/5 - 3c/5 = c/5$. A must close the initial gap of L, so the time it takes is $L/(c/5) = 5L/c$. There is no need to use any fancy velocity-addition or length-contraction formulas, because all quantities in this problem are measured with respect to the *same* frame. It quickly reduces to a simple "(rate)(time) = (distance)" problem.

15. Is the "the speed of light is the same in all inertial frames" postulate really necessary? That is, is it not already implied by the "the laws of physics are the same in all inertial frames"?

Answer: Yes, it is necessary. It turns out that nearly all of the results in relativity can be deduced using only the "the laws of physics are the same in all inertial frames" postulate; what you can find (with some work) is that there is some limiting speed (which may or may not be infinite). But you still have to postulate that light is the thing that moves with this speed. See Section 10.8.

16. Imagine closing a very large pair of scissors. It is quite possible for the point of intersection of the blades to move faster than the speed of light. Does this violate anything in relativity?

    Answer: No. If the angle between the blades is small enough, then the tips of the blades (and all the other atoms in the scissors) can move at a speed well below c, while the intersection point moves faster than c. But this does not violate anything in relativity. The intersection point is not an actual object, so there is nothing wrong with it moving faster than c.

    We should check that this setup cannot be used to send a signal down the scissors at a speed faster than c. Since there is no such thing as a rigid body, it is impossible to get the far end of the scissors to move right away, when you apply a force with your hand. The scissors would have to already be moving, in which case the motion is independent of any decision you make at the handle to change the motion of the blades.

17. Two twins travel away from each other at relativistic speed. The time-dilation result from relativity says that each twin sees the other's clock running slow, so each says the other has aged less. How would you reply to someone who asks, "But which twin really *is* younger?"

    Answer: It makes no sense to ask which twin really is younger, because the two twins aren't in the same reference frame; they are using different coordinates to measure time. It's as silly as having two people run away from each other into the distance (so that each person sees the other become very small), and then asking: who is really smaller?

18. The momentum of an object with mass m and speed v is $p = \gamma mv$. "A photon has zero mass, so it should have zero momentum." Correct or incorrect?

    Answer: Incorrect. True, m is zero, but the γ factor is infinite because $v = c$. Infinity times zero is undefined. Photons do indeed have momentum. It equals E/c (which equals $h\nu/c$, where ν is the frequency of the light).

19. For a particle at rest, does E equal γmc^2 or mc^2?

    Answer: Both, of course, because $\gamma = 1$ when $v = 0$.

20. It is not necessary to postulate the impossibility of accelerating an object to speed c. It follows as a consequence of the relativistic form of energy. Explain.

    Answer: $E = \gamma mc^2$, so if $v = c$ then $\gamma = \infty$, and the object must have an infinite amount of energy (unless $m = 0$, as for a photon). All the energy in the universe, let alone all the kings horses and all the kings men, can't accelerate something to speed c.

14.9 Appendix I: Lorentz transformations

In this Appendix, we give an alternate derivation of the Lorentz transformations, eqs. (10.13). The goal here is to derive them from scratch, using only the two postulates of relativity; we will *not* use any of the results derived in Section 10.2. Our strategy will be to use the relativity postulate ("all inertial frames are equivalent") to figure out as much as we can, and to then invoke the speed-of-light postulate at the end. The main reason for doing things in this order is that it will allow us to derive an extremely interesting result in Section 10.8.

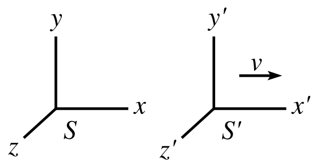

Figure 14.9

As in Section 10.3, consider a coordinate system, S', moving relative to another system, S (see Fig. 14.9). Let the relative speed be v. Let the corresponding axes of S and S' point in the same direction, and let the origin of S' move along the x-axis of S (in the positive direction).

As in Section 10.3, we want to find the constants, A,B,C,D, in the following relations,

$$\begin{aligned} x &= Ax' + Bt', \\ t &= Ct' + Dx'. \end{aligned} \tag{14.69}$$

The four constants here may depend on v (which is constant, given the two inertial frames).

There are four unknowns in eqs. (14.69), so we need four facts. The facts we have at our disposal (using only the two postulates of relativity) are the following.

1. The physical setup: S' travels at a speed v with respect to S.
2. The principle of relativity: S should see things in S' in exactly the same way as S' sees things in S (except for perhaps a minus sign in some relative positions, but this is just convention).
3. The speed-of-light postulate: A light pulse with speed c in S' also has speed c in S.

The second statement here contains two independent bits of information. (It contains at least two, because we will indeed be able to solve for our four unknowns. And it contains no more than two, because then our four unknowns would be over-constrained.) The two bits that are used depend on personal preference. Three that are commonly used are: (A) the relative speed looks the same from either frame, (B) time dilation (if any) looks the same from either frame, and (C) length contraction (if any) looks the same from either frame. It is also common to recast the second statement in the form: The Lorentz transformations are equal to their inverse transformations (up to a possible minus sign). We'll choose (A) and (B). So our four independent facts are:

1. S' travels at a speed v with respect to S.
2. S travels at a speed $-v$ with respect to S'. (The minus sign here is due to the convention that we picked the positive x-axes of the two frames to point in the same direction.)

3. Time dilation (if any) looks the same from either frame.

4. A light pulse with speed c in S' also has speed c in S.

Let's see what these imply.[9]

- (1) says that a given point (say, the origin) in S' moves at speed v with respect to S. Letting $x' = 0$ (that is, looking at two events at the origin of S') in eqs. (14.69) gives $x/t = B/C$. This must be equal to v. Hence, $B = vC$, and the transformations become

$$\begin{aligned} x &= Ax' + vCt', \\ t &= Ct' + Dx'. \end{aligned} \tag{14.70}$$

- (2) says that a given point (say, the origin) in S moves at speed $-v$ with respect to S'. Letting $x = 0$ in the first of eqs. (14.70) gives $x'/t' = -vC/A$. This must be equal to $-v$. Hence, $C = A$, and the transformations become

$$\begin{aligned} x &= Ax' + vAt', \\ t &= At' + Dx'. \end{aligned} \tag{14.71}$$

- (3) can be used in the following way. How fast does a person in S see a clock in S' tick? (The clock is assumed to be at rest with respect to S'.) Let our two events be two successive ticks of the clock. Then $x' = 0$, and the second of eqs. (14.71) gives

$$t = At'. \tag{14.72}$$

In other words, one second on S''s clock takes a time of A seconds in S's frame.

Consider the analogous situation from S''s point of view. How fast does a person in S' see a clock in S tick? (The clock is now assumed to be at rest with respect to S, of course, in order to create the analogous setup. This is important.) Invert eqs. (14.71) to solve for x' and t' in terms of x and t. The result is

$$\begin{aligned} x' &= \frac{x - vt}{A - vD}, \\ t' &= \frac{At - Dx}{A(A - vD)}. \end{aligned} \tag{14.73}$$

Two successive ticks of the clock in S satisfy $x = 0$, so the second of eqs. (14.73) gives

$$t' = \frac{t}{A - vD}. \tag{14.74}$$

In other words, one second on S's clock takes a time of $1/(A - vD)$ seconds in S''s frame.

[9] It would be slightly quicker to invoke the speed-of-light fact prior the time-dilation one. But we'll do things in the above order so that we can easily carry over the results of this Appendix to the discussion in Section 10.8.

Both eqs. (14.72) and (14.74) apply to the same situation (someone looking at a clock flying by). Therefore, the factors on the right-hand sides must be equal, that is,

$$A = \frac{1}{A - vD} \qquad \Longrightarrow \qquad D = \frac{1}{v}\left(A - \frac{1}{A}\right). \tag{14.75}$$

Our transformations in eqs. (14.71) therefore take the form

$$\begin{aligned} x &= A(x' + vt'), \\ t &= A\left(t' + \frac{1}{v}\Big(1 - \frac{1}{A^2}\Big)x'\right). \end{aligned} \tag{14.76}$$

- (4) may now be used to say that if $x' = ct'$, then $x = ct$. In other words, if $x' = ct'$, then

$$c = \frac{x}{t} = \frac{A((ct') + vt')}{A\left(t' + \frac{1}{v}\Big(1 - \frac{1}{A^2}\Big)(ct')\right)} = \frac{c + v}{1 + \frac{c}{v}\Big(1 - \frac{1}{A^2}\Big)}. \tag{14.77}$$

Solving for A gives

$$A = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{14.78}$$

(We have chosen the positive square root so that the positive x- and x'-axes point in the same direction.)

The constant A is commonly denoted by γ, so we may finally write our Lorentz transformations, eqs. (14.76), in the form,

$$\begin{aligned} x &= \gamma(x' + vt'), \\ t &= \gamma(t' + vx'/c^2), \end{aligned} \tag{14.79}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{14.80}$$

in agreement with the results in eq. (10.13).

14.10 Appendix J: Physical constants and data

Earth	
Mass	$M_E = 5.98 \cdot 10^{24}$ kg
Mean radius	$R_E = 6.37 \cdot 10^6$ m
Mean density	5.52 g/cm^3
Surface acceleration	$g = 9.81$ m/s^2
Mean distance from sun	$1.5 \cdot 10^{11}$ m
Orbital speed	29.8 km/s
Period of rotation	23 h 56 min 4 s $= 8.6164 \cdot 10^4$ s
Period of orbit	365 days 6 h $= 3.16 \cdot 10^7$ s
Moon	
Mass	$M_L = 7.35 \cdot 10^{22}$ kg
Radius	$R_L = 1.74 \cdot 10^6$ m
Mean density	3.34 g/cm^3
Surface acceleration	1.62 m/s^2 $\approx g/6$
Mean distance from earth	$3.84 \cdot 10^8$ m
Orbital speed	1.0 km/s
Period of rotation	27.3 days $= 2.36 \cdot 10^6$ s
Period of orbit	27.3 days $= 2.36 \cdot 10^6$ s
Sun	
Mass	$M_S = 1.99 \cdot 10^{30}$ kg
Radius	$R_S = 6.96 \cdot 10^8$ m
Surface acceleration	274 m/s^2 $\approx 28g$
Fundamental constants	
Speed of light	$c = 2.998 \cdot 10^8$ m/s
Gravitational constant	$G = 6.673 \cdot 10^{-11}$ N m^2/kg^2
Planck's constant	$h = 6.63 \cdot 10^{-34}$ J s
Electron charge	$e = 1.602 \cdot 10^{-19}$ C
Electron mass	$m_e = 9.11 \cdot 10^{-31}$ kg $= 0.511$ MeV/c^2
Proton mass	$m_p = 1.673 \cdot 10^{-27}$ kg $= 938.3$ MeV/c^2
Neutron mass	$m_n = 1.675 \cdot 10^{-27}$ kg $= 939.6$ MeV/c^2