Book review:

**A Taste of Jordan Algebras**

by Kevin McCrimmon

*Jordan algebras* owe their name and their existence to physics rather than to mathematics: they are not named after the French mathematician Camille JORDAN (1838-1922) but after the German physicist Pascual JORDAN (1902-1980) who (in two papers 1932 and 1933) proposed a foundation of quantum mechanics based on the commutative product $a \bullet b = \frac{1}{2}(ab+ba)$ instead of the associative product $ab$ of operators, resp. of matrices: in quantum mechanics, the relevant ("observable") operators are the self-adjoint ones, and since the product of two self-adjoint operators is in general no longer self-adjoint, the associative product $ab$ of operators has no physical meaning, whereas the product $a \bullet b$ does. Instead of associativity, the new product satisfies the identity

$$(a \bullet a) \bullet (b \bullet a) = ((a \bullet a) \bullet b) \bullet a, \tag{J}$$

which JORDAN considers a fundamental in defining his new class of algebras. Then, at the end of his second paper, JORDAN comes to the relatively optimistic conclusion ([J], p. 217): "... danach kann die Quantenmechanik im Bereich der $r$-Zahlen genau so formuliert werden, wie im Bereich der assoziativen $q$-Zahlsysteme." [1] Now, after this good news one should expect that the new theory, both mathematically and physically convincing, would have had an easy victory over the old one, and that Jordan algebras would have found their place in everyday-life of mathematics and physics, just like, say, Hilbert spaces, spectral theory or Lie algebras. However, this is not the way things developed.

So what went wrong ? And how does the story continue – can we expect a happy end? If you are curious, read "A Taste of Jordan Algebras" by K. McCrimmon, where, for the first time, a full account both of the mathematical development of Jordan algebra theory and of its historical aspects is given: the book is divided into four parts; the first part (about 40 pages), *A Colloquial Survey of Jordan Theory*, presents the "origin of the species", the "Jordan river" of algebraic structures that had their origin in Jordan algebras, "links with the Real world", "links with the Complex world" and other interactions of Jordan theory with various mathematical theories. Then, in a second circle (about one hundred pages), *An Historical Survey of Jordan Structure Theory*, the author narrates the Jordan story from the beginning to the present: "Jordan Algebras in Physical Antiquity", "Jordan Algebras in the Algebraic Renaissance", "Jordan Algebras in the Enlightenment", "The Classical Theory", up to "The Russian Revolution: 1977-1983", whose leader was the later fields-medaillist Efim ZEL'MANOV. In this part, precise definitions and statements are given, but proofs are left to the following parts *The Classical Theory* (about 250 pages), and *Zel'manov's Exceptional Theorem* (80 pages) which form the third and most important circle; several appendices and detailed indexes (110 pages) complete the book. In each circle, the author re-starts telling the whole story from the beginning, but each time on a higher level, so that in a way this book contains three books in one. The author himself appears in the biggest part, *The Classical Theory*, as one of the major actors of the story, and thus his book is not only a mathematical text but also an outstanding mathematician's account of his life's work (from the Introduction, p. iii): "In keeping with the tone of a private conversation, I give more heuristic material than is common in books at this level (pep talks, philosophical pronouncements on the proper way to think about certain concepts,

[1] "... hence quantum mechanics can be formulated in terms of the $r$-numbers [the later Jordan algebras] in the same way as in terms of the associative $q$-numbers [the tradiational associative algebras]"

random historical anecdotes, offhand mention of some mathematicians who have contributed to our understanding of Jordan algebras, etc.), and employ a few English words which do not standardly appear in mathematical works." Thanks to the very clever organization of the book and to its lively and unconventional style, it is suited both for the very beginner and for the specialist in Jordan theory, for the user who wants to prepare a course on Jordan-theory and for the user who wants to learn Jordan-theory with applications in mind.

A main topic of the book is the problem to find and to describe all *exceptional* Jorden algebras – in fact, this problem has already been raised at the very origins of Jordan algebras in "Physical Antiquity": in his 1932/33-papers, JORDAN briefly mentions two (not yet clearly distinguished) objectives in his approach

(a) to find an "exceptional" setting of quantum mechanics via some new algebra that *cannot* be realized as a subspace of algebras of matrices, and
(b) to give a philosophically satisfying version of the theory of quantum mechanics not involving an un-physical associative structure as background.

However, I do not know many references where this is discussed in detail, and probably it would be worth a separate study in history of sciences to analyze the true intentions of JORDAN and his collaborators at that time (who in subsequent work include John VON NEUMANN and Eugene WIGNER). Anyhow, the commonly accepted version retains basically only objective (a) and explains the abandoning of the new theory by the deception of JORDAN, VON NEUMANN and WIGNER that (a) seemed to be hopeless: their classification of possible algebras contained *only one* exceptional object, a 27-dimensional Jordan algebra, later called the *Albert algebra* $\mathbb{A}$, which, isolated as it was, did not leave much hope of finding an *infinite dimensional* exeptional algebra which was needed for quantum mechanics. I never found this explanation very convincing: how could these leading physicists and mathematicians sacrifice the possibility of a new and philosophically more satisfying foundation of their theory just because of the lack of "exceptional" objects? Why should the failure of (a) lead to an abandon of (b)? Moreover, already the existence of *one* exceptional object raises foundational problems: if JORDAN was right in his conclusion (quoted above) that the whole theory could be based on his basic algebraic laws (commutativity combined with (J)) , then one cannot assume the existence, somewhere in the background, of an associative algebra of "matrices" such that the bullet-product is obtained by $a \bullet b = \frac{ab+ba}{2}$. Or else, JORDAN was wrong, which would mean that we need some additional, "special" properties in order to build a theory of quantum mechanics. We will not go into the mathematical details of this problem here and refer the reader instead to McCrimmon's book: indeed, when "Physical Antiquity" made place for "Algebraic Renaissance", *special identities* that distinguish the classical (hermitian or square) matrix algebras from the exceptional Albert algebra $\mathbb{A}$ were found, and the "Russian Revolution" by E. ZEL'MANOV definitely put an end to the hope of finding an *infinite dimensional* exceptional Jordan algebra by showing that these special identities are intimately related to finite dimensionality.

Unfortunately, these beautiful und deep results are *negative* in nature and seem to confirm JORDAN's disappointment – if all but one are mere matrix algebras, so what is the use of Jordan algebras ? Is there any interesting application that goes beyond "usual" associative algebra ? The answer is "yes, but this is a topic for another book" – in fact, Jordan algebras have found applications in statistics, genetics, probability, differential equations, differential geometry, projective geometry, and finally also returned to physics. Some of these applications are briefly discussed in the "Colloquial Survey" at the beginning of McCrimmon's book. A fairly comprehensive collection of more than 500 references on Jordan theory and its applications can be found in [Io], showing also that this is a rapidly growing area of research, a conclusive overview of which is not yet possible. Some of the most interesting applications are related to the fact that Jordan algebras, much more than associative algebras, automatically lead to *non-linear* structures, and personally, I believe that this non-linear aspect of Jordan theory (unimagined at the time of JORDAN and VON NEUMANN) may provide the key for attacking the objective (b) mentioned above. Non-linear concepts are, on the one hand, the *quadratic Jordan algebras* defined by McCRIMMON in the sixties, and, on the other hand, the use of some natural maps which are not even polynomial – an example familiar to the general mathematician is the *Cayley transform* which defines a bijection between generic Hermitian operators and generic elements of the unitary group; one can say that the unitary group is the "global" or "geometric" object corresponding to the Jordan algebra of Hermitian operators. More generally, one should think that any Jordan algebra corresponds to some geometric object, in a way similar to the correspondence between Lie algebras and Lie groups in Lie theory. This aspect of the theory has strongly been developed by the German school founded by Max KOECHER, and it has led to the invention of *Jordan triple systems* by Kurt MEYBERG and *Jordan pairs* by Ottmar LOOS; these objects generalize Jordan algebras in the same way as square matrices are generalized by rectangular matrices and as vector spaces with an inner product are generalized by pairs of vector spaces in duality. As for Jordan algebras, these more general objects have an important non-linear aspect which is closely related to a construction due to I. KANTOR, M. KOECHER and J. TITS that by now has become classical. They constructed a "big" Lie algebra out of a Jordan algebra, which led to the famous dictum of KANTOR *"There are no Jordan algebras, there are only Lie algebras"* [1]. Of course, one may regret that these important topics are not treated in McCrimmon's book; but, in spite of all generalizations, the heart of the whole theory still is the theory of Jordan *algebras* which finally, after 70 years of development, has found a masterly presentation. Unlike all other monographs on Jordan algebras and related topics (the most important of which are [BK], [Jac], [ZSSS], cf. the Appendix "Index of Collateral Readings" in McCrimmon's book), which rather reflect a particular state in the development of the theory, McCrimmon's book will be the fundamental textbook in this domain for many years to come.

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[1] McCrimmon's answer to KANTOR (p. 14): "Of course, this can be turned around: *nine times out of ten, when you open up a Lie algebra you find a Jordan algebra inside which makes it tick.*"