HANDBOOK OF

NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Andrei D. Polyanin
Valentin F. Zaitsev

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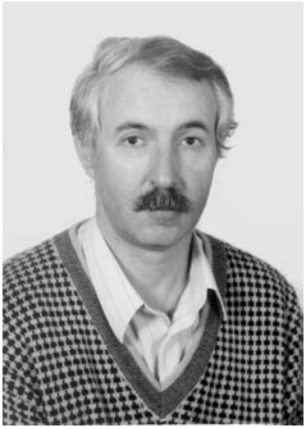

Andrei D. Polyanin, Ph.D., D.Sc., is a noted scientist of broad interests, who works in various areas of mathematics, mechanics, and chemical engineering sciences.

A. D. Polyanin graduated from the Department of Mechanics and Mathematics of the Moscow State University in 1974. He received his Ph.D. degree in 1981 and D.Sc. degree in 1986 at the Institute for Problems in Mechanics of the Russian (former USSR) Academy of Sciences. Since 1975, A. D. Polyanin has been a member of the staff of the Institute for Problems in Mechanics of the Russian Academy of Sciences. He is a member of the Russian National Committee on Theoretical and Applied Mechanics.

Professor Polyanin is an author of 33 books in English, Russian, German, and Bulgarian, as well as over 120 research papers and three patents. He has written a number of fundamental handbooks, including A. D. Polyanin and V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations*, CRC Press, 1995 and 2002; A. D. Polyanin and A. V. Manzhirov, *Handbook of Integral Equations*, CRC Press, 1998; A. D. Polyanin, *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Chapman & Hall/CRC Press, 2002; A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux, *Handbook of First Order Partial Differential Equations*, Taylor & Francis, 2002; and A. D. Polyanin and V. F. Zaitsev, *Handbook of Nonlinear Mathematical Physics Equations*, Fizmatlit, 2002. Professor Polyanin is Editor of the book series *Differential and Integral Equations and Their Applications*, Taylor & Francis, London, and *Physical and Mathematical Reference Literature*, Fizmatlit, Moscow.

In 1991, A. D. Polyanin was awarded a Chaplygin Prize of the Russian Academy of Sciences for his research in mechanics. In 2001, he received an award from the Ministry of Education of the Russian Federation.

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V. F. Zaitsev graduated from the Radio Electronics Faculty of the Leningrad Polytechnical Institute (now Saint-Petersburg Technical University) in 1969 and received his Ph.D. degree in 1983 at the Leningrad State University. His Ph.D. thesis was devoted to the group approach to the study of some classes of ordinary differential equations. In 1992, Professor Zaitsev received his Doctor of Sciences degree; his D.Sc. thesis was dedicated to the discrete-group analysis of ordinary differential equations.

In 1971–1996, V. F. Zaitsev worked in the Research Institute for Computational Mathematics and Control Processes of the St. Petersburg State University. Since 1996, Professor Zaitsev has been a member of the staff of the Russian State Pedagogical University (St. Petersburg).

Professor Zaitsev has made important contributions to new methods in the theory of ordinary and partial differential equations. He is an author of more than 130 scientific publications, including 18 books and one patent.

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FOREWORD

Nonlinear partial differential equations are encountered in various fields of mathematics, physics, chemistry, and biology, and numerous applications.

Exact (closed-form) solutions of differential equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. Exact solutions of nonlinear equations graphically demonstrate and allow unraveling the mechanisms of many complex nonlinear phenomena such as spatial localization of transfer processes, multiplicity or absence steady states under various conditions, existence of peaking regimes and many others. Furthermore, simple solutions are often used in teaching many courses as specific examples illustrating basic tenets of a theory that admit mathematical formulation.

Even those special exact solutions that do not have a clear physical meaning can be used as "test problems" to verify the consistency and estimate errors of various numerical, asymptotic, and approximate analytical methods. Exact solutions can serve as a basis for perfecting and testing computer algebra software packages for solving differential equations. It is significant that many equations of physics, chemistry, and biology contain empirical parameters or empirical functions. Exact solutions allow researchers to design and run experiments, by creating appropriate natural conditions, to determine these parameters or functions.

This book contains more than 1600 nonlinear mathematical physics equations and nonlinear partial differential equations and their solutions. A large number of new exact solutions to nonlinear equations are described. Equations of parabolic, hyperbolic, elliptic, mixed, and general types are discussed. Second-, third-, fourth-, and higher-order nonlinear equations are considered. The book presents exact solutions to equations of heat and mass transfer, wave theory, nonlinear mechanics, hydrodynamics, gas dynamics, plasticity theory, nonlinear acoustics, combustion theory, nonlinear optics, theoretical physics, differential geometry, control theory, chemical engineering sciences, biology, and other fields.

Special attention is paid to general-form equations that depend on arbitrary functions; exact solutions of such equations are of principal value for testing numerical and approximate methods. Almost all other equations contain one or more arbitrary parameters (in fact, this book deals with whole families of partial differential equations), which can be fixed by the reader at will. *In total, the handbook contains significantly more nonlinear PDE's and exact solutions than any other book currently available*.

The supplement of the book presents exact analytical methods for solving nonlinear mathematical physics equations. When selecting the material, the authors have given a pronounced preference to practical aspects of the matter; that is, to methods that allow effectively "constructing" exact solutions. Apart from the classical methods, the book also describes wide-range methods that have been greatly developed over the last decade (the nonclassical and direct methods for symmetry reductions, the differential constraints method, the method of generalized separation of variables, and others). For the reader's better understanding of the methods, numerous examples of solving specific differential equations and systems of differential equations are given throughout the book.

For the convenience of a wide audience with different mathematical backgrounds, the authors tried to do their best, wherever possible, to avoid special terminology. Therefore, some of the methods are outlined in a schematic and somewhat simplified manner, with necessary references made to books where these methods are considered in more detail. Many sections were written so that they could be read independently from each other. This allows the reader to quickly get to the heart of the matter.

The handbook consists of chapters, sections, and subsections. Equations and formulas are numbered separately in each subsection. The equations within subsections are arranged in increasing order of complexity. The extensive table of contents provides rapid access to the desired equations.

Separate parts of the book may be used by lecturers of universities and colleges for practical courses and lectures on nonlinear mathematical physics equations for graduate and postgraduate students. Furthermore, the books may be used as a database of test problems for numerical and approximate methods for solving nonlinear partial differential equations.

We would like to express our deep gratitude to Alexei Zhurov for fruitful discussions and valuable remarks.

The authors hope that this book will be helpful for a wide range of scientists, university teachers, engineers, and students engaged in the fields of mathematics, physics, mechanics, control, chemistry, and engineering sciences.

Andrei D. Polyanin
Valentin F. Zaitsev

SOME NOTATIONS AND REMARKS

Latin Characters

$C_1, C_2, \dots$	are arbitrary constants;
$r,\ \varphi,\ z$	cylindrical coordinates, $r = \sqrt{x^2+y^2}$ and $x = r\cos\varphi$, $y = r\sin\varphi$;
$r,\ \theta,\ \varphi$	spherical coordinates, $r=\sqrt{x^2+y^2+z^2}$ and $x=r\sin\theta\cos\varphi$, $y=\sin\theta\sin\varphi$, $z=r\cos\theta$;
t	time ($t \ge 0$);
w	unknown function (dependent variable);
x, y, z	space (Cartesian) coordinates;
$x_1, \dots, x_n$	Cartesian coordinates in n-dimensional space.

Greek Characters

Δ	Laplace operator; in two-dimensional case, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$; in three-dimensional case, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$; in n-dimensional case, $\Delta = \sum\limits_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$;
$\Delta\Delta$	biharmonic operator; in two-dimensional case, $\Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$.

Brief Notation for Derivatives

$$w_x = \frac{\partial w}{\partial x},\quad w_t = \frac{\partial w}{\partial t},\quad w_{xx} = \frac{\partial^2 w}{\partial x^2},\quad w_{xt} = \frac{\partial^2 w}{\partial x \partial t},\quad w_{tt} = \frac{\partial^2 w}{\partial t^2},\quad \dots \qquad \text{(partial derivatives);}$$

$$f'_x = \frac{df}{dx},\quad f''_{xx} = \frac{d^2 f}{dx^2},\quad f'''_{xxx} = \frac{d^3 f}{dx^3},\quad f''''_{xxxx} = \frac{d^4 f}{dx^4},\quad f^{(n)}_x = \frac{d^n f}{dx^n} \qquad \text{(derivatives for } f = f(x)\text{).}$$

Brief Notation for Differential Operators

$$\partial_x = \frac{\partial}{\partial x},\quad \partial_y = \frac{\partial}{\partial y},\quad \partial_t = \frac{\partial}{\partial t},\quad \partial_w = \frac{\partial}{\partial w} \qquad \text{(differential operators in } x,\ y,\ t, \text{ and } w\text{);}$$

$$\mathrm{D}_x = \frac{\partial}{\partial x} + w_x \frac{\partial}{\partial w} + w_{xx}\frac{\partial}{\partial w_x} + w_{xt}\frac{\partial}{\partial w_t} + \cdots \qquad \text{(total differential operator in } x\text{);}$$

$$\mathrm{D}_t = \frac{\partial}{\partial t} + w_t \frac{\partial}{\partial w} + w_{xt}\frac{\partial}{\partial w_x} + w_{tt}\frac{\partial}{\partial w_t} + \cdots \qquad \text{(total differential operator in } t\text{).}$$

In the last two relations, w is assumed to be dependent on x and t, $w = w(x,t)$.

Remarks

1. The book presents solutions of the following types:

(a) expressible in terms of elementary functions explicitly, implicitly, or parametrically;
(b) expressible in terms of elementary functions and integrals of elementary functions;
(c) expressible in terms of elementary functions, functions involved in the equation (if the equation contains arbitrary functions), and integrals of the equation functions and/or other elementary functions;
(d) expressible in terms of ordinary differential equations or finite systems of ordinary differential equations;
(e) expressible in terms of solutions to nonlinear equations that can be reduced to linear partial differential equations or linear integral equations.

2. The book also deals with solutions described by equations with fewer new variables than those in the original equations. An expression that solves an equations in three independent variables and is determined by an equation in two independent variables will be called a two-dimensional solution.

3. As a rule, the book does not present simple solutions that depend on only one of the variables involved in the original equation.

4. Equations are numbered separately in each subsection. When referencing a particular equation, we use a notation like 3.1.2.5, which implies equation 5 from Subsection 3.1.2.

5. If a formula or a solution contains an expression like $\dfrac{f(x)}{a-2}$, it is often not stated that the assumption $a \neq 2$ is implied.

6. Though incomplete, very simple and graphical classification of solutions by their appearance is used in the book. For equations in two independent variables, x and t, and one unknown, w, the solution name and structure are as follows (x and t in the solutions below can be swapped):

No.	*Solution name*	*Solution structure*
1	Traveling-wave solution*	$w = F(z),\ z = \alpha x + \beta t,\ \alpha\beta \neq 0$
2	Additive separable solution	$w = \varphi(x) + \psi(t)$
3	Multiplicative separable solution	$w = \varphi(x)\psi(t)$
4	Self-similar solution**	$w = t^{\alpha} F(z),\ z = xt^{\beta}$
5	Generalized self-similar solution	$w = \varphi(t)F(z),\ z = x\psi(t)$
6	Generalized traveling-wave solution	$w = F(z),\ z = \varphi(t)x + \psi(t)$
7	Generalized separable solution	$w = \varphi_1(x)\psi_1(t) + \cdots + \varphi_n(x)\psi_n(t)$
8	Functional separable solution	$w = F(z),\ z = \varphi_1(x)\psi_1(t) + \cdots + \varphi_n(x)\psi_n(t)$

* For uniformity of presentation, we also use this term in the cases where the variable t plays the role of a spatial coordinate.
** Sometimes, a solution of the form $w = \bar{t}^{\alpha} F(z),\ z = \bar{x}\bar{t}^{\beta}$, where $\bar{x} = x + C_1$ and $\bar{t} = t + C_2$, will also be called a self-similar solution.

7. The present book does not consider first-order nonlinear partial differential equations. For these equations, see Kamke (1965), Rhee, Aris, and Amundson (1986, 1989), and Polyanin, Zaitsev, and Moussiaux (2002).

8. ODE and PDE are conventional abbreviations for ordinary differential equation and partial differential equation, respectively.

⊙ This symbol indicates references to literature sources whenever:

(*a*) at least one of the solutions was obtained in the cited source (even though the solution contained "correctable" misprints in signs or coefficients);
(*b*) the cited source provides further information on the equation in question and their solutions.

Chapter 1

Parabolic Equations with One Space Variable

1.1. Equations with Power-Law Nonlinearities

1.1.1. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw + cw^2$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw^2.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^2 w(C_1 x + C_2, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution (λ is an arbitrary constant):

$$w = w(z), \qquad z = x + \lambda t,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$aw''_{zz} - \lambda w'_z + bw^2 = 0.$$

3°. Self-similar solution:

$$w = t^{-1}u(\xi), \qquad \xi = xt^{-1/2},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$au''_{\xi\xi} + \tfrac{1}{2}\xi u'_\xi + u + bu^2 = 0.$$

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw(1 - w).$

Fisher equation. This equation arises in heat and mass transfer, combustion theory, biology, and ecology. For example, it describes the mass transfer in a two-component medium at rest with a volume chemical reaction of quasi-first order. The kinetic function $f(w) = aw(1 - w)$ models also an autocatalytic chain reaction in combustion theory.

This is a special case of equation 1.1.3.2 with $m = 2$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm x + C_1, t + C_2),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solutions (C is an arbitrary constant):

$$w(x,t) = \left[1 + C\exp\left(-\tfrac{5}{6}at \pm \tfrac{1}{6}\sqrt{6a}\,x\right)\right]^{-2},$$
$$w(x,t) = \left[-1 + C\exp\left(-\tfrac{5}{6}at \pm \tfrac{1}{6}\sqrt{6a}\,x\right)\right]^{-2},$$
$$w(x,t) = \frac{1 + 2C\exp\left(-\tfrac{5}{6}at \pm \tfrac{1}{6}\sqrt{-6a}\,x\right)}{\left[1 + C\exp\left(-\tfrac{5}{6}at \pm \tfrac{1}{6}\sqrt{-6a}\,x\right)\right]^{2}}.$$

3°. Traveling-wave solutions:

$$w(x,t) = \pm\xi^2\varphi(\xi), \qquad \xi = C_1\exp\left(\tfrac{1}{6}\sqrt{6a}\,x + \tfrac{5}{6}at\right),$$

where the function $\varphi(\xi)$ is defined implicitly by

$$\xi = \int \frac{d\varphi}{\sqrt{\pm(4\varphi^3 - 1)}} - C_2,$$

and C_1 and C_2 are arbitrary constants. For the upper sign, the inversion of this relation corresponds to the classical Weierstrass elliptic function, $\varphi(\xi) = \wp(\xi + C_3,\ 0,\ 1)$.

4°. The substitution $U = 1 - w$ leads to an equation of the similar form

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - aU(1-U).$$

⊙ *References*: M. J. Ablowitz and A. Zeppetella (1978), V. G. Danilov, V. P. Maslov, and K. A. Volosov (1995).

1.1.2. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b_0 + b_1 w + b_2 w^2 + b_3 w^3$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} - bw^3.$

This is a special case of equation 1.1.2.5 with $b_0 = b_1 = b_2 = 0$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(\pm C_1 x + C_2,\, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,t) = \pm\sqrt{\frac{2a}{b}}\,\frac{2C_1 x + C_2}{C_1 x^2 + C_2 x + 6aC_1 t + C_3}.$$

3°. Traveling-wave solution (λ is an arbitrary constant):

$$w = w(z), \qquad z = x + \lambda t,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$aw''_{zz} - \lambda w'_z - bw^3 = 0.$$

4°. Self-similar solution:

$$w = t^{-1/2}u(\xi), \qquad \xi = xt^{-1/2},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$au''_{\xi\xi} + \tfrac{1}{2}\xi u'_\xi + \tfrac{1}{2}u - bu^3 = 0.$$

⊙ *Reference*: P. A. Clarkson and E. L. Mansfield (1994).

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw - bw^3.$

This is a special case of equation 1.1.2.5 with $b_0 = b_2 = 0$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = \pm w(\pm x + C_1, t + C_2),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation (the signs are chosen arbitrarily).

2°. Solutions with $a > 0$ and $b > 0$:

$$w = \sqrt{\frac{a}{b}}\,\frac{C_1 \exp\left(\frac{1}{2}\sqrt{2a}\,x\right) - C_2 \exp\left(-\frac{1}{2}\sqrt{2a}\,x\right)}{C_1 \exp\left(\frac{1}{2}\sqrt{2a}\,x\right) + C_2 \exp\left(-\frac{1}{2}\sqrt{2a}\,x\right) + C_3 \exp\left(-\frac{3}{2}at\right)},$$

$$w = \sqrt{\frac{a}{b}}\left[\frac{2C_1 \exp\left(\sqrt{2a}\,x\right) + C_2 \exp\left(\frac{1}{2}\sqrt{2a}\,x - \frac{3}{2}at\right)}{C_1 \exp\left(\sqrt{2a}\,x\right) + C_2 \exp\left(\frac{1}{2}\sqrt{2a}\,x - \frac{3}{2}at\right) + C_3} - 1\right],$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Solution with $a < 0$ and $b > 0$:

$$w = \sqrt{\frac{|a|}{b}}\,\frac{\sin\left(\frac{1}{2}\sqrt{2|a|}\,x + C_1\right)}{\cos\left(\frac{1}{2}\sqrt{2|a|}\,x + C_1\right) + C_2 \exp\left(-\frac{3}{2}at\right)},$$

where C_1 and C_2 are arbitrary constants.

4°. Solution with $a > 0$ (generalizes the first solution of Item 2°):

$$w = \left[C_1 \exp\left(\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\right) - C_2 \exp\left(-\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\right)\right]U(z),$$
$$z = C_1 \exp\left(\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\right) + C_2 \exp\left(-\tfrac{1}{2}\sqrt{2a}\,x + \tfrac{3}{2}at\right) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $U = U(z)$ is determined by the autonomous ordinary differential equation $aU''_{zz} = 2bU^3$ (whose solution can be written out in implicit form).

5°. Solution with $a < 0$ (generalizes the solution of Item 3°):

$$w = \exp\left(\tfrac{3}{2}at\right)\sin\left(\tfrac{1}{2}\sqrt{2|a|}\,x + C_1\right)V(\xi),$$
$$\xi = \exp\left(\tfrac{3}{2}at\right)\cos\left(\tfrac{1}{2}\sqrt{2|a|}\,x + C_1\right) + C_2,$$

where C_1 and C_2 are arbitrary constants, and the function $V = V(\xi)$ is determined by the autonomous ordinary differential equation $aV''_{\xi\xi} = -2bV^3$ (whose solution can be written out in implicit form).

6°. See also equation 1.1.3.2 with $m = 3$.

⊙ *References*: F. Cariello and M. Tabor (1989), M. C. Nucci and P. A. Clarkson (1992).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} - bw^3 - cw^2.$

This is a special case of equation 1.1.2.5 with $b_1 = b_0 = 0$.

1°. Traveling-wave solutions:

$$w(x,t) = \left(ct \pm \sqrt{\frac{b}{2a}}\,x + C\right)^{-1},$$

where C is an arbitrary constant.

2°. Solutions:

$$w(x,t) = k\sqrt{\frac{2a}{b}}\,\frac{1}{F}\frac{\partial F}{\partial x},$$

where

$$F = C_1\left(x + kc\sqrt{\frac{2a}{b}}\,t\right) + C_2\exp\left(-\frac{kc}{\sqrt{2ab}}x + \frac{c^2}{2b}t\right) + C_3, \qquad k = \pm 1,$$

and C_1, C_2, and C_3 are arbitrary constants.

3°. Solutions:

$$w(x,t) = k\sqrt{\frac{2a}{b}}\,\frac{1}{F}\frac{\partial F}{\partial x} - \frac{c}{b},$$

where

$$F = C_1\exp\left(\frac{kc}{\sqrt{2ab}}x - \frac{c^2}{2b}t\right) + C_2\left(\frac{kc}{\sqrt{2ab}}x + \frac{c^2}{b}t\right)\exp\left(\frac{kc}{\sqrt{2ab}}x - \frac{c^2}{2b}t\right) + C_3, \qquad k = \pm 1.$$

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} - w(1-w)(a-w).$

Fitzhugh–Nagumo equation. This equation arises in population genetics and models the transmission of nerve impulses.

1°. There are three stationary solutions: $w = w_k$, where $w_1 = 0$, $w_2 = 1$, and $w_3 = a$. The linear stability analysis shows that

if $-1 \le a < 0$: the solutions $w = a$, $w = 1$ are stable, $w = 0$ is unstable;

if $0 < a < 1$: the solutions $w = 0$, $w = 1$ are stable, $w = a$ is unstable.

There is a stationary nonhomogeneous solution that can be represented in implicit form (A and B are arbitrary constants):

$$\int \frac{dw}{\sqrt{\frac{1}{4}w^4 - \frac{1}{3}(a+1)w^3 + \frac{1}{2}aw^2 + A}} = \pm x + B.$$

2°. Traveling-wave solutions (A, B, and C are arbitrary constants):

$$w(x,t) = \frac{1}{1 + A\exp\left[\pm\frac{1}{2}\sqrt{2}\,x + \frac{1}{2}(2a-1)t\right]},$$

$$w(x,t) = \frac{a}{1 + A\exp\left[\pm\frac{1}{2}\sqrt{2}\,a\,x + \frac{1}{2}a(2-a)t\right]},$$

$$w(x,t) = \frac{A\exp\left[\pm\frac{1}{2}\sqrt{2}\,(1-a)x + \frac{1}{2}(1-a^2)t\right] + a}{A\exp\left[\pm\frac{1}{2}\sqrt{2}\,(1-a)x + \frac{1}{2}(1-a^2)t\right] + 1},$$

$$w(x,t) = \tfrac{1}{2} + \tfrac{1}{2}\tanh\left[\pm\tfrac{1}{4}\sqrt{2}\,x + \tfrac{1}{4}(1-2a)t + A\right],$$

$$w(x,t) = \tfrac{1}{2}a + \tfrac{1}{2}a\tanh\left[\pm\tfrac{1}{4}\sqrt{2}\,ax + \tfrac{1}{4}a(a-2)t + A\right],$$

$$w(x,t) = \tfrac{1}{2}(1+a) + \tfrac{1}{2}(1-a)\tanh\left[\pm\tfrac{1}{4}\sqrt{2}\,(1-a)x + \tfrac{1}{4}(1-a^2)t + A\right],$$

$$w(x,t) = \frac{2a}{(1+a) - (1-a)\tanh\left[\pm\frac{1}{4}\sqrt{2}\,(1-a)x + \frac{1}{4}(1-a^2)t + A\right]},$$

$$w(x,t) = \tfrac{1}{2} + \tfrac{1}{2}\coth\left[\pm\tfrac{1}{4}\sqrt{2}\,x + \tfrac{1}{4}(1-2a)t + A\right],$$

$$w(x,t) = \tfrac{1}{2}a + \tfrac{1}{2}a\coth\left[\pm\tfrac{1}{4}\sqrt{2}\,ax + \tfrac{1}{4}a(a-2)t + A\right],$$

$$w(x,t) = \tfrac{1}{2}(1+a) + \tfrac{1}{2}(1-a)\coth\left[\pm\tfrac{1}{4}\sqrt{2}\,(1-a)x + \tfrac{1}{4}(1-a^2)t + A\right],$$

$$w(x,t) = \frac{2a}{(1+a) - (1-a)\coth\left[\pm\frac{1}{4}\sqrt{2}\,(1-a)x + \frac{1}{4}(1-a^2)t + A\right]}.$$

3°. "Two-phase" solution:

$$w(x,t) = \frac{A\exp(z_1) + aB\exp(z_2)}{A\exp(z_1) + B\exp(z_2) + C},$$
$$z_1 = \pm\tfrac{\sqrt{2}}{2}\,x + \left(\tfrac{1}{2} - a\right)t, \quad z_2 = \pm\tfrac{\sqrt{2}}{2}\,ax + a\left(\tfrac{1}{2}a - 1\right)t,$$

where A, B, and C are arbitrary constants.

4°. The solutions of Item 2° are special cases of the traveling-wave solution

$$w(x,t) = w(\xi), \qquad \xi = x + \lambda t,$$

where λ is an arbitrary constant, and the function $w(\xi)$ is determined by the autonomous ordinary differential equation

$$w''_{\xi\xi} - \lambda w'_{\xi} = w(1-w)(a-w).$$

The substitution $w'_{\xi} = \lambda y(w)$ leads to an Abel equation of the second kind:

$$yy'_w - y = \lambda^{-2}\left[aw - (a+1)w^2 + w^3\right].$$

The general solution of this equation with $a = -1$ and $\lambda = \pm\frac{3}{\sqrt{2}}$ can be found in Polyanin and Zaitsev (2003).

5°. Let us give two transformations that preserve the form of the original equation.

The substitution $u = 1 - w$ leads to an equation of the similar form with parameter $a_1 = 1 - a$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u(1-u)(1-a-u).$$

The transformation

$$v(z,\tau) = 1 - \frac{1}{a}w(x,t), \quad \tau = a^2 t, \quad z = ax$$

leads to an equation of the similar form with parameter $a_2 = 1 - a^{-1}$:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial z^2} - v(1-v)\left(1 - \frac{1}{a} - v\right).$$

Therefore, if $w = w(x,t;\ a)$ is a solution of the equation in question, then the functions

$$w_1 = 1 - w\bigl(x,t;\ 1-a\bigr),$$
$$w_2 = a - aw\bigl(ax, a^2 t;\ 1 - a^{-1}\bigr)$$

are also solutions of the equation. The abovesaid allows us to "multiply" exact solutions.

6°. See also Example 1 in Subsection S.7.2.

⊙ *References for equation* 1.1.2.4: T. Kawahara and M. Tanaka (1983), M. C. Nucci and P. A. Clarkson (1992), N. H. Ibragimov (1994), V. F. Zaitsev and A. D. Polyanin (1996).

5. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b_0 + b_1 w + b_2 w^2 + b_3 w^3.$

1°. Solutions are given by

$$w(x,t) = \frac{\beta}{F}\frac{\partial F}{\partial x} + \lambda, \qquad \beta = \pm\sqrt{-\frac{2a}{b_3}}, \tag{1}$$

where λ is any of the roots of the cubic equation

$$b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0 \tag{2}$$

and the specific form of $F = F(x,t)$ depends on the equation coefficients.

Introduce the notation

$$p_1 = -3a, \quad p_2 = \beta(b_2 + 3b_3\lambda), \quad q_1 = -\frac{\beta}{2a}(b_2 + 3b_3\lambda), \quad q_2 = -\frac{1}{2a}(3b_3\lambda^2 + 2b_2\lambda + b_1). \tag{3}$$

Four cases are possible.

1.1. For $q_2 \neq 0$ and $q_1^2 \neq 4q_2$, we have

$$F(x,t) = C_1 \exp(k_1x + s_1t) + C_2 \exp(k_2x + s_2t) + C_3,$$
$$k_n = -\tfrac{1}{2}q_1 \pm \tfrac{1}{2}\sqrt{q_1^2 - 4q_2}, \quad s_n = -k_n^2 p_1 - k_n p_2, \tag{4}$$

where C_1, C_2, and C_3 are arbitrary constants; $n = 1, 2$.

1.2. For $q_2 \neq 0$ and $q_1^2 = 4q_2$, we have

$$F(x,t) = C_1 \exp(kx + s_1t) + C_2(kx + s_2t)\exp(kx + s_1t) + C_3,$$
$$k = -\tfrac{1}{2}q_1, \quad s_1 = -\tfrac{1}{4}p_1q_1^2 + \tfrac{1}{2}p_2q_1, \quad s_2 = -\tfrac{1}{2}p_1q_1^2 + \tfrac{1}{2}p_2q_1.$$

1.3. For $q_2 = 0$ and $q_1 \neq 0$,

$$F(x,t) = C_1(x - p_2t) + C_2 \exp[-q_1x + q_1(p_2 - p_1q_1)t] + C_3.$$

1.4. For $q_2 = q_1 = 0$,

$$F(x,t) = C_1(x - p_2t)^2 + C_2(x - p_2t) - 2C_1p_1t + C_3.$$

Example. Let

$$a = 1, \quad b_0 = 0, \quad b_1w + b_2w^2 + b_3w^3 = -bw(w - \lambda_1)(w - \lambda_2).$$

By formulas (1)–(4) with $\lambda = 0$, one can obtain the solution

$$w(x,t) = \frac{C_1\lambda_1 \exp(z_1) + C_2\lambda_2 \exp(z_2)}{C_1 \exp(z_1) + C_2 \exp(z_2) + C_3},$$

where

$$z_1 = \pm\tfrac{1}{2}\sqrt{2b}\,\lambda_1 x + \tfrac{1}{2}b\lambda_1(\lambda_1 - 2\lambda_2)t,$$
$$z_2 = \pm\tfrac{1}{2}\sqrt{2b}\,\lambda_2 x + \tfrac{1}{2}b\lambda_2(\lambda_2 - 2\lambda_1)t.$$

2°. There is a traveling-wave solution, $w = w(x + \gamma t)$.

⊙ *References*: V. G. Danilov and P. Yu. Sybochev (1991), N. A. Kudryashov (1993), P. A. Clarkson and E. L. Mansfield (1994).

1.1.3. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw^k.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{k-1}x + C_2, C_1^{2k-2}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution:

$$w = w(z), \quad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$aw''_{zz} - \lambda w'_z + bw^k = 0.$$

3°. Self-similar solution:

$$w = t^{\frac{1}{1-k}}u(\xi), \quad \xi = \frac{x}{\sqrt{t}},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$au''_{\xi\xi} + \frac{1}{2}\xi u'_\xi + \frac{1}{k-1}u + bu^k = 0.$$

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw + bw^m.$

Kolmogorov–Petrovskii–Piskunov equation. This equation arises in heat and mass transfer, combustion theory, biology, and ecology.

1°. Traveling-wave solutions:

$$w(x,t) = \left[\beta + C\exp(\lambda t \pm \mu x)\right]^{\frac{2}{1-m}}, \tag{1}$$

$$w(x,t) = \left[-\beta + C\exp(\lambda t \pm \mu x)\right]^{\frac{2}{1-m}}, \tag{2}$$

where C is an arbitrary constant and the parameters λ, μ, and β are given by

$$\lambda = \frac{a(1-m)(m+3)}{2(m+1)}, \quad \mu = \sqrt{\frac{a(1-m)^2}{2(m+1)}}, \quad \beta = \sqrt{-\frac{b}{a}}.$$

2°. Solutions (1) and (2) are special cases of a wider class of solutions, the class of traveling-wave solutions:

$$w = w(z), \quad z = \pm\mu x + \lambda t.$$

These are determined by the autonomous equation

$$\mu^2 w''_{zz} - \lambda w'_z + aw + bw^m = 0. \tag{3}$$

For

$$\mu = \sqrt{\frac{a(m+3)^2}{2(m+1)}}, \quad \lambda = \mu^2 \qquad (m \neq \pm 1,\ m \neq -3)$$

the solution of equation (3) can be represented in parametric form as

$$z = \frac{m+3}{m-1}\ln f(\zeta), \quad w = \zeta\left[f(\zeta)\right]^{\frac{2}{m-1}},$$

where the function $f(\zeta)$ is given by

$$f(\zeta) = \pm\int\left[C_1 - \frac{4b}{a(m-1)^2}\zeta^{m+1}\right]^{-1/2} d\zeta + C_2,$$

and C_1 and C_2 are arbitrary constants.

3°. By the change of variable $U(w) = \mu^2\lambda^{-1}w'_z$, equation (3) can be reduced to an Abel equation of the second kind:

$$UU'_w - U = a_1 w + b_1 w^m, \qquad a_1 = -a\mu^2\lambda^{-2}, \quad b_1 = -b\mu^2\lambda^{-2}.$$

The books by Polyanin and Zaitsev (1995, 2003) present exact solutions of this equation for some values of m and a_1 (b_1 is any).

⊙ *References*: P. Kaliappan (1984), V. G. Danilov, V. P. Maslov, and K. A. Volosov (1995), V. F. Zaitsev and A. D. Polyanin (1996).

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw + bw^m + cw^{2m-1}.$

This equation arises in heat and mass transfer, combustion theory, biology, and ecology.

1°. Traveling-wave solutions:

$$w(x,t) = \left[\beta + C\exp(\lambda t + \mu x)\right]^{\frac{1}{1-m}}, \tag{1}$$

where C is an arbitrary constant and the parameters β, λ, and μ are determined by the system of algebraic equations

$$a\beta^2 + b\beta + c = 0, \tag{2}$$

$$\mu^2 - (1-m)\lambda + a(1-m)^2 = 0, \tag{3}$$

$$\mu^2 - \lambda + (1-m)\left[2a + (b/\beta)\right] = 0. \tag{4}$$

The quadratic equation (2) for β can be solved independently. In the general case, system (2)–(4) gives four sets of the parameters, which generate four exact solutions of the original equation.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

2°. Solution (1) is a special case of a wider class of traveling-wave solutions,

$$w = w(z), \quad z = x + \sigma t,$$

that are determined by the autonomous equation

$$w''_{zz} - \sigma w'_z + aw + bw^m + cw^{2m-1} = 0. \tag{5}$$

The substitution $U(w) = w'_z$ brings (5) to the Abel equation

$$UU'_w - \sigma U + aw + bw^m + cw^{2m-1} = 0,$$

whose general solutions for some m (no constraints are imposed on a, b, and c) can be found in the books by Polyanin and Zaitsev (1995, 2003).

3°. The substitution

$$u = w^{1-m}$$

leads to an equation with quadratic nonlinearity:

$$u\frac{\partial u}{\partial t} = u\frac{\partial^2 u}{\partial x^2} + \frac{m}{1-m}\left(\frac{\partial u}{\partial x}\right)^2 + a(1-m)u^2 + b(1-m)u + c(1-m). \tag{6}$$

Solution (1) corresponds to a particular solution of (6) that has the form $u = \beta + C\exp(\omega t + \mu x)$.

For $a = 0$, equation (6) has also other traveling-wave solutions:

$$u(x,t) = (1-m)\left(bt \pm \sqrt{-\frac{c}{m}}\,x\right) + C.$$

4. $\displaystyle \boldsymbol{\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + aw^{m-1} + bmw^m - mb^2 w^{2m-1}}.$

Traveling-wave solution:

$$w = w(z), \quad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$w''_{zz} - \lambda w'_z + aw^{m-1} + bmw^m - mb^2 w^{2m-1} = 0. \tag{1}$$

For $\lambda = 1$, it can be shown that a one-parameter family of solutions to equation (1) satisfies the first-order equation

$$w'_z = w - bw^m + \frac{a}{mb}. \tag{2}$$

Integrating (2) yields a solution in implicit form (A is any):

$$\int \frac{dw}{a + mbw - mb^2 w^m} = \frac{1}{mb}z + A. \tag{3}$$

In the special case $a = 0$, it follows from (3) that

$$w(z) = \left\{C\exp\left[(1-m)z\right] + b\right\}^{\frac{1}{1-m}},$$

where C is an arbitrary constant.

1.1.4. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + s_1(bx+ct)^k + s_2 w^n.$

This is a special case of equation 1.6.1.2 with $f(z,w) = s_1 z^k + s_2 w^n$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + s(w+bx+ct)^k.$

This is a special case of equation 1.6.1.2 with $f(z,w) = s(w+z)^k$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + s(bx+ct)^k w^n.$

This is a special case of equation 1.6.1.2 with $f(z,w) = sz^k w^n$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bt^n x^m w^k.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C^{2n+m+2} w(C^{k-1}x, C^{2k-2}t),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Self-similar solution:

$$w = t^{\frac{2n+m+2}{2(1-k)}} u(\xi), \qquad \xi = \frac{x}{\sqrt{t}},$$

where the function $u = u(\xi)$ is determined by the ordinary differential equation

$$au''_{\xi\xi} + \frac{1}{2}\xi u'_\xi + \frac{2n+m+2}{2(k-1)} u + b\xi^m u^k = 0.$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + se^{bx+ct} w^n.$

This is a special case of equation 1.6.1.2 with $f(z,w) = se^z w^n$.

1.1.5. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(w)\dfrac{\partial w}{\partial x} + g(w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + cw + k_1 w^{n_1} + k_2 w^{n_2}.$

This is a special case of equation 1.6.2.3 with $f(t) = b$. On passing from t, x to the new variables t, $z = x + bt$, one arrives at the simpler equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + cw + k_1 w^{n_1} + k_2 w^{n_2},$$

special cases of which are discussed in Subsections 1.1.1 to 1.1.3.

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + w\dfrac{\partial w}{\partial x}.$

Burgers equation. It is used for describing wave processes in gas dynamics, hydrodynamics, and acoustics.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_1 x + C_1 C_2 t + C_3, C_1^2 t + C_4) + C_2,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t) = \frac{A-x}{B+t},$$

$$w(x,t) = \lambda + \frac{2}{x+\lambda t+A},$$

$$w(x,t) = \frac{4x+2A}{x^2+Ax+2t+B},$$

$$w(x,t) = \frac{6(x^2+2t+A)}{x^3+6xt+3Ax+B},$$

$$w(x,t) = \frac{2\lambda}{1+A\exp(-\lambda^2 t-\lambda x)},$$

$$w(x,t) = -\lambda + A\frac{\exp\bigl[A(x-\lambda t)\bigr]-B}{\exp\bigl[A(x-\lambda t)\bigr]+B},$$

$$w(x,t) = -\lambda + 2A\tanh\bigl[A(x-\lambda t)+B\bigr],$$

$$w(x,t) = \frac{\lambda}{\lambda^2 t+A}\left[2\tanh\left(\frac{\lambda x+B}{\lambda^2 t+A}\right)-\lambda x-B\right],$$

$$w(x,t) = -\lambda + 2A\tan\bigl[A(\lambda t-x)+B\bigr],$$

$$w(x,t) = \frac{2\lambda\cos(\lambda x+A)}{B\exp(\lambda^2 t)+\sin(\lambda x+A)},$$

$$w(x,t) = \frac{2A}{\sqrt{\pi(t+\lambda)}}\exp\left[-\frac{(x+B)^2}{4(t+\lambda)}\right]\left[A\operatorname{erf}\left(\frac{x+B}{2\sqrt{t+\lambda}}\right)+C\right]^{-1},$$

where A, B, C, and λ are arbitrary constants, and $\operatorname{erf} z \equiv \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z \exp(-\xi^2)\,d\xi$ is the error function (also called the probability integral).

3°. Other solutions can be obtained using the following formula (Hopf–Cole transformation):

$$w(x,t) = \frac{2}{u}\frac{\partial u}{\partial x}, \tag{1}$$

where $u = u(x,t)$ is a solution of the linear heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

For details about this equation, see the books Tikhonov and Samarskii (1990) and Polyanin (2002).

⊙ *References*: E. Hopf (1950), J. Cole (1951).

Remark. The transformation (1) and equation 1.6.3.2, which is a generalized Burgers equation, were encountered much earlier in Fortsyth (1906).

4°. *Cauchy problem.* Initial condition:

$$w = f(x) \quad \text{at} \quad t = 0, \qquad -\infty < x < \infty.$$

Solution:

$$w(x,t) = 2\frac{\partial}{\partial x}\ln F(x,t),$$

where

$$F(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x-\xi)^2}{4t}-\frac{1}{2}\int_0^{\xi} f(\xi')\,d\xi'\right]d\xi.$$

⊙ *Reference*: E. Hopf (1950).

5°. The Burgers equation is connected with the linear heat equation (2) by the Bäcklund transformation

$$\frac{\partial u}{\partial x} - \frac{1}{2}uw = 0,$$
$$\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial (uw)}{\partial x} = 0.$$

⊙ *References for equation* 1.1.5.2: J. M. Burgers (1948), O. V. Rudenko and C. I. Soluyan (1975), N. H. Ibragimov (1994), V. F. Zaitsev and A. D. Polyanin (1996).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x}.$

Unnormalized Burgers equation. The scaling of the independent variables $x = \frac{a}{b}z$, $t = \frac{a}{b^2}\tau$ leads to an equation of the form 1.1.5.2:

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial z^2} + w\frac{\partial w}{\partial z}.$$

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x} + c.$

The transformation

$$w = u(z,t) + ct, \qquad z = x + \tfrac{1}{2}bct^2,$$

leads to the Burgers equation 1.1.5.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial z^2} + bu\frac{\partial u}{\partial z}.$$

5. $\dfrac{\partial w}{\partial t} + \sigma w\dfrac{\partial w}{\partial x} = a\dfrac{\partial^2 w}{\partial x^2} + bw.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\big(x - C_1\sigma e^{bt} + C_2,\ t + C_3\big) + Cbe^{bt},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w = w(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$aw''_{zz} - \sigma ww'_z - \lambda w'_z + bw = 0.$$

3°. Degenerate solution:

$$w(x,t) = \frac{b(x + C_1)}{\sigma(1 + C_2 e^{-bt})}.$$

6. $\dfrac{\partial w}{\partial t} + \sigma w\dfrac{\partial w}{\partial x} = a\dfrac{\partial^2 w}{\partial x^2} + b_1 w + b_0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\big(x - C_1\sigma e^{b_1 t} + C_2,\ t + C_3\big) + Cb_1 e^{b_1 t},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. The transformation

$$w = u(z,t) - \frac{b_0}{b_1}, \qquad z = x + \sigma\frac{b_0}{b_1}t,$$

leads to a simpler equation of the form 1.1.5.5:

$$\frac{\partial u}{\partial t} + \sigma u\frac{\partial u}{\partial z} = a\frac{\partial^2 u}{\partial z^2} + b_1 u.$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x} + \dfrac{b^2}{9a}w(w-k)(w+k).$

Solution:

$$w = \frac{k(-1 + C_1 e^{4\lambda x})}{1 + C_1 e^{4\lambda x} + C_2 e^{2\lambda x + bk\lambda t}}, \qquad \lambda = \frac{bk}{12a},$$

where C_1 and C_2 are arbitrary constants.

⊙ *Private communication*: K. A. Volosov (2000).

8. $\dfrac{\partial w}{\partial t} + \sigma w\dfrac{\partial w}{\partial x} = a\dfrac{\partial^2 w}{\partial x^2} + b_0 + b_1 w + b_2 w^2 + b_3 w^3.$

Solutions of the equation are given by

$$w(x,t) = \frac{\beta}{z}\frac{\partial z}{\partial x} + \lambda. \tag{1}$$

Here, β and λ are any of the roots of the respective quadratic and cubic equations

$$b_3\beta^2 + \sigma\beta + 2a = 0,$$
$$b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0,$$

and the specific form of $z = z(x,t)$ depends on the equation coefficients.

1°. *Case* $b_3 \neq 0$. Introduce the notation:

$$p_1 = -\beta\sigma - 3a, \qquad p_2 = \lambda\sigma + \beta b_2 + 3\beta\lambda b_3,$$
$$q_1 = -\frac{\beta b_2 + 3\beta\lambda b_3}{\beta\sigma + 2a}, \quad q_2 = -\frac{3b_3\lambda^2 + 2b_2\lambda + b_1}{\beta\sigma + 2a}.$$

Four cases are possible.

1.1. For $q_2 \neq 0$ and $q_1^2 \neq 4q_2$, we have

$$z(x,t) = C_1 \exp(k_1 x + s_1 t) + C_2 \exp(k_2 x + s_2 t) + C_3,$$
$$k_n = -\tfrac{1}{2}q_1 \pm \tfrac{1}{2}\sqrt{q_1^2 - 4q_2}, \quad s_n = -k_n^2 p_1 - k_n p_2,$$

where C_1, C_2, and C_3 are arbitrary constants; $n = 1,\ 2$.

1.2. For $q_2 \neq 0$ and $q_1^2 = 4q_2$,

$$z(x,t) = C_1 \exp(kx + s_1 t) + C_2(kx + s_2 t)\exp(kx + s_1 t) + C_3,$$
$$k = -\tfrac{1}{2}q_1, \quad s_1 = -\tfrac{1}{4}p_1 q_1^2 + \tfrac{1}{2}p_2 q_1, \quad s_2 = -\tfrac{1}{2}p_1 q_1^2 + \tfrac{1}{2}p_2 q_1.$$

1.3. For $q_2 = 0$ and $q_1 \neq 0$,

$$z(x,t) = C_1(x - p_2 t) + C_2 \exp[-q_1 x + q_1(p_2 - p_1 q_1)t] + C_3.$$

1.4. For $q_2 = q_1 = 0$,

$$z(x,t) = C_1(x - p_2 t)^2 + C_2(x - p_2 t) - 2C_1 p_1 t + C_3.$$

2°. *Case* $b_3 = 0$ *and* $b_2 \neq 0$. Solutions are given by (1) with

$$\beta = -\frac{2a}{\sigma}, \qquad z(x,t) = C_1 + C_2 \exp\left[Ax + A\left(\frac{b_1\sigma}{2b_2} + \frac{2ab_2}{\sigma}\right)t\right], \qquad A = \frac{\sigma(b_1 + 2b_2\lambda)}{2ab_2},$$

where λ is a root of the quadratic equation $b_2\lambda^2 + b_1\lambda + b_0 = 0$.

3°. *Case* $b_3 = b_2 = 0$. See equations 1.1.5.4–1.1.5.6.

⊙ *Reference*: N. A. Kudryashov (1993).

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw^m\dfrac{\partial w}{\partial x}$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w\big(C_1^m x + C_2,\ C_1^{2m} t + C_3\big),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w(x,t) = \left[C\exp\left(-\frac{\lambda m}{a}z\right) + \frac{b}{\lambda(m+1)}\right]^{-1/m}, \qquad z = x + \lambda t,$$

where C and λ are arbitrary constants. A wider family of traveling-wave solutions is presented in 1.6.3.7 for $f(w) = bw^m$.

3°. There is a self-similar solution of the form

$$w(\xi,t) = |t|^{-\frac{1}{2m}}\varphi(\xi), \qquad \xi = x|t|^{-\frac{1}{2}}.$$

1.1.6. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + sw^k$.

This is a special case of equation 1.6.2.1 with $f(w) = sw^k$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1 e^{-bt}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + C_1 e^{-bt},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$aw''_{zz} + bzw'_z + sw^k = 0.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bt^n\dfrac{\partial w}{\partial x} + cw + k_1 w^{m_1} + k_2 w^{m_2}$.

This is a special case of equation 1.6.2.3 with $f(t) = bt^n$. On passing from t, x to the new variables t, $z = x + \dfrac{b}{n+1}t^{n+1}$, one arrives at the simpler equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + cw + k_1 w^{m_1} + k_2 w^{m_2},$$

special cases of which are discussed in Subsections 1.1.1 to 1.1.3.

3. $\dfrac{\partial w}{\partial t} + \dfrac{k}{t}w + bw\dfrac{\partial w}{\partial x} = a\dfrac{\partial^2 w}{\partial x^2}$.

Modified Burgers equation.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$\begin{aligned}
w_1 &= C_1 w(C_1 x + C_2, C_1^2 t),\\
w_2 &= w(x - bC_3 t^{1-k}, t) + C_3(1-k)t^{-k} && \text{if } k \neq 1,\\
w_3 &= w(x - bC_3 \ln|t|, t) + C_3 t^{-1} && \text{if } k = 1,
\end{aligned}$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Degenerate solution linear in x:

$$w(x,t) = \frac{(1-k)x + C_1}{C_2 t^k + bt} \quad \text{if } k \neq 1,$$
$$w(x,t) = \frac{x + C_1}{t(C_2 + b\ln|t|)} \quad \text{if } k = 1,$$

where C_1 and C_2 are arbitrary constants.

3°. Self-similar solution:

$$w(x,t) = u(z)t^{-1/2}, \qquad z = xt^{-1/2},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$au''_{zz} + \left(\tfrac{1}{2}z - bu\right)u'_z + \left(\tfrac{1}{2} - k\right)u = 0.$$

4. $\displaystyle \frac{\partial w}{\partial t} + bw\frac{\partial w}{\partial x} = a\left[\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial w}{\partial x}\right) - \frac{w}{x^2}\right].$

Cylindrical Burgers equation. The variable x plays the role of the radial coordinate.

Solution:

$$w(x,t) = -\frac{2a}{b}\frac{1}{\theta}\frac{\partial \theta}{\partial x},$$

where the function $\theta = \theta(x,t)$ satisfies the linear heat equation with axial symmetry

$$\frac{\partial \theta}{\partial t} = \frac{a}{x}\frac{\partial}{\partial x}\left(x\frac{\partial \theta}{\partial x}\right).$$

⊙ *Reference*: S. Nerney, E. J. Schmahl, and Z. E. Musielak (1996).

5. $\displaystyle \frac{\partial w}{\partial t} + bw\frac{\partial w}{\partial x} = a\frac{\partial^2 w}{\partial x^2} + cx^k\frac{\partial w}{\partial x} + ckx^{k-1}w.$

Solution:

$$w(x,t) = -\frac{2a}{b}\frac{1}{\theta}\frac{\partial \theta}{\partial x},$$

where the function $\theta = \theta(x,t)$ satisfies the linear equation

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2} + cx^k\frac{\partial \theta}{\partial x}.$$

6. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw\frac{\partial w}{\partial x} + c(x+st)^k.$

This is a special case of equation 1.6.3.2 with $f(x,t) = c(x+st)^k$.

7. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw\frac{\partial w}{\partial x} + cx^k + st^n.$

This is a special case of equation 1.6.3.2 with $f(x,t) = cx^k + st^n$.

8. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (bx + cw^k)\frac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1 e^{-bt}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + C_1 e^{-bt},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$aw''_{zz} + (bz + cw^k)w'_z = 0.$$

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bw^m + ct + s)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 1.6.3.11 with $f(w) = bw^m$, $g(t) = ct + s$, and $h(w) = 0$.

On passing from t, x to the new variables t, $z = x + \frac{1}{2}ct^2 + st$, we obtain an equation of the form 1.1.5.9:

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + bw^m\frac{\partial w}{\partial z}.$$

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bw^m + ct^k)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 1.6.3.11 with $f(w) = bw^m$, $g(t) = ct^k$, and $h(w) = 0$.

On passing from t, x to the new variables t, $z = x + \dfrac{c}{k+1}t^{k+1}$, we obtain an equation of the form 1.1.5.9:

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + bw^m\frac{\partial w}{\partial z}.$$

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + s_1(bx + ct)^k w^n\dfrac{\partial w}{\partial x} + s_2(bx + ct)^p w^q.$

This is a special case of equation 1.6.3.13 with $f(z, w) = s_1 z^k w^n$ and $g(z, w) = s_2 z^p w^q$.

1.1.7. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x, t, w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2.$

1°. Solutions:

$$\begin{aligned}
w(x) &= \frac{a}{b}\ln|Ax + B| + C,\\
w(x, t) &= A^2 bt \pm Ax + B,\\
w(x, t) &= -\frac{(x + A)^2}{4bt} - \frac{a}{2b}\ln t + B,\\
w(x, t) &= \frac{a}{b}\ln|x^2 + 2at + Ax + B| + C,\\
w(x, t) &= \frac{a}{b}\ln|x^3 + 6axt + Ax + B| + C,\\
w(x, t) &= \frac{a}{b}\ln|x^4 + 12ax^2 t + 12a^2 t^2 + A| + B,\\
w(x, t) &= -\frac{a^2\lambda^2}{b}t + \frac{a}{b}\ln|\cos(\lambda x + A)| + B,
\end{aligned}$$

where A, B, C, and λ are arbitrary constants.

2°. The substitution

$$w(x, t) = \frac{a}{b}\ln|u(x, t)|$$

leads to the linear heat equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}.$$

For details about this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + a\left(\dfrac{\partial w}{\partial x}\right)^2 + b.$

The substitution $u = e^w$ leads to the constant coefficient linear equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bu.$$

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + s.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, t + C_2) + C_3 e^{ct},$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution quadratic in x:

$$w(x,t) = \frac{c(x + C_2)^2}{C_1 e^{-ct} - 4b} - \frac{2a}{C_1} e^{ct} \ln|C_1 e^{-ct} - 4b| + C_3 e^{ct} - \frac{s}{c}.$$

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \left(\dfrac{\partial w}{\partial x}\right)^2 + aw^2.$

1°. Solutions for $a < 0$:

$$w(x,t) = C_1 \exp\bigl(-at \pm x\sqrt{-a}\,\bigr),$$
$$w(x,t) = \frac{1}{C_1 - at} + \frac{C_2}{(C_1 - at)^2} \exp\bigl(-at \pm x\sqrt{-a}\,\bigr),$$

where C_1 and C_2 are arbitrary constants. The first formula represents a multiplicative separable solution and the second one, a generalized separable solution.

2°. Generalized separable solution for $a < 0$:

$$w(x,t) = \varphi(t) + \psi(t)\bigl[A\exp\bigl(x\sqrt{-a}\,\bigr) + B\exp\bigl(-x\sqrt{-a}\,\bigr)\bigr],$$

where A and B are arbitrary constants, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the autonomous system of first-order ordinary differential equations

$$\varphi'_t = a\bigl(\varphi^2 + 4AB\psi^2\bigr), \tag{1}$$
$$\psi'_t = a(2\varphi - 1)\psi. \tag{2}$$

Dividing equation (1) by (2) termwise yields the first-order equation $(2\varphi - 1)\psi\varphi'_\psi = \varphi^2 + 4AB\psi^2$.

3°. Generalized separable solution for $a > 0$:

$$w(x,t) = \varphi(t) + \psi(t)\cos\bigl(x\sqrt{a} + C\bigr),$$

where C is an arbitrary constant, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the autonomous system of first-order ordinary differential equations

$$\varphi'_t = a\bigl(\varphi^2 + \psi^2\bigr), \tag{3}$$
$$\psi'_t = a(2\varphi - 1)\psi. \tag{4}$$

Dividing equation (3) by (4) termwise yields a first-order equation.

⊙ *References*: V. A. Galaktionov and S. A. Posashkov (1989), V. F. Zaitsev and A. D. Polyanin (1996).

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + bcw^2 + sw + k.$

1°. Generalized separable solution for $c < 0$:

$$w(x,t) = \varphi(t) + \psi(t)\exp\bigl(\pm x\sqrt{-c}\,\bigr), \tag{1}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the autonomous system of first-order ordinary differential equations

$$\varphi'_t = bc\varphi^2 + s\varphi + k, \tag{2}$$
$$\psi'_t = (2bc\varphi + s - ac)\psi. \tag{3}$$

The solution of system (2), (3) is given by

$$\varphi(t) = \lambda + \frac{2bc\lambda + s}{C_1\exp\bigl[-(2bc\lambda + s)t\bigr] - bc},$$
$$\psi(t) = \frac{C_1C_2\exp\bigl[-(2bc\lambda + s + ac)t\bigr]}{\bigl\{C_1\exp\bigl[-(2bc\lambda + s)t\bigr] - bc\bigr\}^2},$$

where C_1 and C_2 are arbitrary constants and $\lambda = \lambda_1$ and $\lambda = \lambda_2$ are roots of the quadratic equation

$$bc\lambda^2 + s\lambda + k = 0.$$

2°. For more complicated generalized separable solutions that involve hyperbolic and trigonometric functions of x, see equation 1.6.6.2 with f, g, h = const.

⊙ *References*: V. A. Galaktionov and S. A. Posashkov (1989), V. F. Zaitsev and A. D. Polyanin (1996).

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^2 + st^n w + kt^m.$

This is a special case of equation 1.6.6.2 with f = const, $g = st^n$, and $h = kt^m$.

1.1.8. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f\Bigl(x, t, w, \dfrac{\partial w}{\partial x}\Bigr)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + a\left(\dfrac{\partial w}{\partial x}\right)^2 + b\dfrac{\partial w}{\partial x} + c.$

The substitution $u = e^w$ leads to the constant coefficient linear equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + a\left(\dfrac{\partial w}{\partial x}\right)^2 + bt^n\dfrac{\partial w}{\partial x} + ct^m.$

This is a special case of equation 1.6.5.4 with $f(x,t) = bt^n$ and $g(x,t) = ct^m$.

The substitution $u = e^w$ leads to the linear equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bt^n\frac{\partial u}{\partial x} + ct^m u.$$

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bt^n\left(\dfrac{\partial w}{\partial x}\right)^2 + ct^m w + st^k.$

This is a special case of equation 1.6.6.1 with $f(t) = bt^n$, $g(t) = ct^m$, and $h(t) = st^k$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda t}\left(\dfrac{\partial w}{\partial x}\right)^2 + ce^{\mu t}w + se^{\nu t}.$

This is a special case of equation 1.6.6.1 with $f(t) = be^{\lambda t}$, $g(t) = ce^{\mu t}$, and $h(t) = se^{\nu t}$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{a}{w}\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 1.6.6.8 with $f(w) = a/w$.

The substitution

$$u = \begin{cases} \dfrac{1}{a+1}w^{a+1} & \text{if } a \neq -1, \\ \ln|w| & \text{if } a = -1 \end{cases}$$

leads to the constant coefficient linear equation $\partial_t u = \partial_{xx} u$.

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw^k\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 1.6.6.8 with $f(w) = aw^k$. For $k = 0$, see equation 1.1.7.1, and for $k = -1$, see equation 1.1.8.5.

The substitution

$$u = \int \exp\left(\frac{a}{k+1}w^{k+1}\right) dw$$

leads to the constant coefficient linear equation $\partial_t u = \partial_{xx} u$.

7. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw^m\left(\dfrac{\partial w}{\partial x}\right)^2 + (bx + ct + s)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 1.6.6.10 with $f(w) = aw^m$, $g(t) = b$, and $h(t) = ct + s$.

1.1.9. Equations of the Form $\dfrac{\partial w}{\partial t} = aw^k\dfrac{\partial^2 w}{\partial x^2} + f\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-2}C_2 w(C_1 x + C_3, C_2 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \frac{x^2 + Ax + B}{C - 2at},$$

where A, B, and C are arbitrary constants.

3°. Traveling-wave solution in implicit form:

$$ak^2 \int \frac{dw}{\lambda \ln|w| + C_1} = kx + \lambda t + C_2,$$

where C_1, C_2, k, and λ are arbitrary constants.

4°. For other exact solutions, see equation 1.1.9.18 with $m = 1$, Items 5° to 8°.

2. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + b.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(C_1 x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t) = Ax + B + bt,$$
$$w(x,t) = \frac{x^2 + Ax + B}{C - 2at} - \frac{b}{4a}(C - 2at),$$

where A, B, and C are arbitrary constants. The first solution is degenerate and the second one is a generalized separable solution.

3°. Traveling-wave solution:

$$w = w(z), \qquad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$ak^2 ww''_{zz} - \lambda w'_z + b = 0.$$

4°. Self-similar solution:

$$w = tU(\xi), \qquad \xi = x/t,$$

where the function $U(\xi)$ is determined by the autonomous ordinary differential equation

$$aUU''_{\xi\xi} + \xi U'_\xi - U + b = 0.$$

3. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + bw + c.$

1°. Generalized separable solutions:

$$w(x,t) = Ae^{bt}x + Be^{bt} - \frac{c}{b},$$
$$w(x,t) = \frac{b(x + A)^2 - Bce^{-bt} - 2act + C}{Bbe^{-bt} - 2a},$$

where A, B, and C are arbitrary constants (the first solution is degenerate).

2°. Traveling-wave solution:

$$w = w(z), \qquad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$ak^2 ww''_{zz} - \lambda w'_z + bw + c = 0.$$

4. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + cw^2 + kw + s.$

This is a special case of equation 1.1.9.9 with $b = 0$.

⦿ *Reference*: V. A. Galaktionov and S. A. Posashkov (1989).

5. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + bw^2 + (ct + d)w + st + k.$

This is a special case of equation 1.6.9.3 with $f(t) = ct + d$ and $g(t) = st + k$.

6. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial w}{\partial x} + (ct + d)w + pt + k.$

This is a special case of equation 1.6.10.2 with $f(t) \equiv 0$, $g(t) = b$, $h(t) = ct + d$, and $s(t) = pt + k$.

7. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} - \dfrac{2}{3}a\left(\dfrac{\partial w}{\partial x}\right)^2 + b.$

Generalized separable solution:

$$w(x,t) = \frac{1}{a}\bigl[3Ax^3 + f_2(t)x^2 + f_1(t)x + f_0(t)\bigr],$$

where A is an arbitrary constant and the functions $f_2(t)$, $f_1(t)$, and $f_0(t)$ are determined by the system of ordinary differential equations

$$f_2' = 6Af_1 - \tfrac{2}{3}f_2^2,$$
$$f_1' = 18Af_0 - \tfrac{2}{3}f_1f_2,$$
$$f_0' = 2f_0f_2 - \tfrac{2}{3}f_1^2 + ab.$$

The general solution of this system with $A \neq 0$ has the form

$$f_2(t) = 3\int\varphi(t)\,dt + 3B, \quad f_1(t) = \frac{1}{A}\left[\int\varphi(t)\,dt + B\right]^2 + \frac{1}{2A}\varphi(t),$$
$$f_0(t) = \frac{1}{9A^2}\left[\int\varphi(t)\,dt + B\right]^3 + \frac{1}{6A^2}\varphi(t)\left[\int\varphi(t)\,dt + B\right] + \frac{1}{36A^2}\varphi_t'(t),$$

where the function $\varphi(t)$ is defined implicitly by

$$\int(C_1 + 72A^2ab\varphi - 8\varphi^3)^{-1/2}\,d\varphi = \pm t + C_2,$$

and B, C_1, and C_2 are arbitrary constants.

⊙ *Reference*: J. R. King (1993), V. A. Galaktionov (1995).

8. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + c\dfrac{\partial w}{\partial x} + pw + q.$

This is a special case of equation 1.6.10.2 with $f(t) = b$, $g(t) = c$, $h(t) = p$, and $s(t) = q$.

9. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^2 + kw + s.$

1°. Generalized separable solutions involving an exponential of x:

$$w(x,t) = \varphi(t) + \psi(t)\exp(\pm\lambda x), \quad \lambda = \left(\frac{-c}{a+b}\right)^{1/2}, \tag{1}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi_t' = c\varphi^2 + k\varphi + s, \tag{2}$$
$$\psi_t' = (a\lambda^2\varphi + 2c\varphi + k)\psi. \tag{3}$$

Integrating (2) yields

$$\int\frac{d\varphi}{c\varphi^2 + k\varphi + s} = t + C_1.$$

On computing the integral, one can find $\varphi = \varphi(t)$ in explicit form. The solution of equation (3) is expressed in terms of $\varphi(t)$ as

$$\psi(t) = C_2\exp\left[\int(a\lambda^2\varphi + 2c\varphi + k)\,dt\right],$$

where C_1 and C_2 are arbitrary constants.

$2°$. There are also generalized separable solutions that involve hyperbolic and trigonometric functions (A is an arbitrary constant):

$$w(x,t) = \varphi(t) + \psi(t)\cosh(\lambda x + A), \quad \lambda = \left(\frac{-c}{a+b}\right)^{1/2};$$

$$w(x,t) = \varphi(t) + \psi(t)\sinh(\lambda x + A), \quad \lambda = \left(\frac{-c}{a+b}\right)^{1/2};$$

$$w(x,t) = \varphi(t) + \psi(t)\cos(\lambda x + A), \quad \lambda = \left(\frac{c}{a+b}\right)^{1/2}.$$

The functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by autonomous systems of first-order ordinary differential equations (these systems can be reduced to a single first-order equation each).

For details about these solutions, see Items $2°$ to $4°$ of equation 1.6.10.1 with $f(t) = k$ and $g(t) = s$.

⊙ *Reference*: V. A. Galaktionov and S. A. Posashkov (1989).

10. $\displaystyle \frac{\partial w}{\partial t} = aw^2 \frac{\partial^2 w}{\partial x^2}.$

The substitution $w = 1/v$ leads to an equation of the form 1.1.10.3:

$$\frac{\partial v}{\partial t} = a\frac{\partial}{\partial x}\left(\frac{1}{v^2}\frac{\partial v}{\partial x}\right).$$

Therefore the solutions of the original equation are expressed via solutions of the linear heat equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial y^2}$$

by the relations

$$w = \frac{\partial u}{\partial y}, \quad x = u.$$

The variable y should be eliminated to obtain $w = w(x,t)$ in explicit form.

⊙ *Reference*: N. H. Ibragimov (1985).

11. $\displaystyle \frac{\partial w}{\partial t} = aw^2 \frac{\partial^2 w}{\partial x^2} + bw^2.$

$1°$. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^2 w(C_1^{-1}x + C_2, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

$2°$. The transformation $w = 1/u$, $\tau = at$ leads to an equation of the form 1.1.11.2:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{u^2}\frac{\partial u}{\partial x}\right) - \frac{b}{a}.$$

12. $\displaystyle \frac{\partial w}{\partial t} = aw^2 \frac{\partial^2 w}{\partial x^2} + bw^{-1}.$

This is a special case of equation 1.1.9.19 with $m = 2$ and $b = -1$.

$1°$. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-1} w(\pm C_1^2 x + C_2, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

$2°$. Functional separable solutions:

$$w(x,t) = \pm\left[C_1(x+C_2)^2 + C_3\exp(2aC_1 t) - \frac{b}{aC_1}\right]^{1/2},$$

where C_1, C_2, and C_3 are arbitrary constants.

13. $\dfrac{\partial w}{\partial t} = aw^2 \dfrac{\partial^2 w}{\partial x^2} + bw + cw^{-1}.$

Functional separable solutions:

$$w(x,t) = \pm\left[bC_1 e^{2bt}(x+C_2)^2 + C_3 F(t) + 2cF(t)\int \frac{dt}{F(t)}\right]^{1/2}, \qquad F(t) = \exp\bigl(aC_1 e^{2bt} + 2bt\bigr),$$

where C_1, C_2, and C_3 are arbitrary constants.

14. $\dfrac{\partial w}{\partial t} = aw^3 \dfrac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 1.1.9.18 with $m = 3$.

Functional separable solution:

$$w(x,t) = a^{-1/3}\bigl[3Ax^3 + f_2(t)x^2 + f_1(t)x + f_0(t)\bigr]^{1/3}.$$

Here,

$$f_2(t) = 3\int \varphi(t)\,dt + 3B, \quad f_1(t) = \frac{1}{A}\left[\int \varphi(t)\,dt + B\right]^2 + \frac{1}{2A}\varphi(t),$$
$$f_0(t) = \frac{1}{9A^2}\left[\int \varphi(t)\,dt + B\right]^3 + \frac{1}{6A^2}\varphi(t)\left[\int \varphi(t)\,dt + B\right] + \frac{1}{36A^2}\varphi'_t(t),$$

where the function $\varphi(t)$ is defined implicitly by

$$\int (C_1 - 8\varphi^3)^{-1/2}\,d\varphi = \pm t + C_2,$$

and A, B, C_1, and C_2 are arbitrary constants. Setting $C_1 = 0$ in the last relation, one obtains the function φ in explicit form: $\varphi = -\frac{1}{2}(t+C_2)^{-2}$.

⊙ *Reference*: G. A. Rudykh and E. I. Semenov (1999).

15. $\dfrac{\partial w}{\partial t} = aw^3 \dfrac{\partial^2 w}{\partial x^2} + bw^{-2}.$

This is a special case of equation 1.1.9.19 with $m = 3$ and $b = -2$.

The substitution $w = u^{1/3}$ leads to an equation of the form 1.1.9.7:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} - \frac{2}{3}a\left(\frac{\partial u}{\partial x}\right)^2 + 3b.$$

Therefore the equation in question has a generalized separable solution of the form

$$w(x,t) = a^{-1/3}\bigl[3Ax^3 + f_2(t)x^2 + f_1(t)x + f_0(t)\bigr]^{1/3}.$$

16. $\dfrac{\partial w}{\partial t} = aw^4 \dfrac{\partial^2 w}{\partial x^2} + bw + cw^{-1}.$

Functional separable solutions:

$$w(x,t) = \pm\sqrt{\varphi(t)x^2 + \psi(t)x + \chi(t)},$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi'_t = \tfrac{1}{2}a\varphi(4\varphi\chi - \psi^2) + 2b\varphi,$$
$$\psi'_t = \tfrac{1}{2}a\psi(4\varphi\chi - \psi^2) + 2b\psi,$$
$$\chi'_t = \tfrac{1}{2}a\chi(4\varphi\chi - \psi^2) + 2b\chi + 2c.$$

It follows from the first two equations that $\varphi = C\psi$, where C is an arbitrary constant.

17. $\dfrac{\partial w}{\partial t} = aw^4 \dfrac{\partial^2 w}{\partial x^2} + bx^m w^5.$

This is a special case of equation 1.6.11.1 with $f(x) = bx^m$.

18. $\dfrac{\partial w}{\partial t} = aw^m \dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-2/m} C_2^{1/m} w(C_1 x + C_3, C_2 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x) = Ax + B,$$
$$w(x,t) = (\pm\beta x + \beta\lambda t + A)^{1/m}, \quad \beta = \frac{m\lambda}{a(1-m)},$$
$$w(x,t) = \left[\frac{m(x-A)^2}{2a(2-m)(B-t)}\right]^{1/m},$$
$$w(x,t) = \left[A|t+B|^{\frac{m}{m-2}} + \frac{m}{2a(m-2)} \frac{(x+C)^2}{t+B}\right]^{1/m},$$
$$w(x,t) = \left[\frac{(x+A)^2}{\varphi(t)} + B(x+A)^m |\varphi(t)|^{\frac{m(m-3)}{2}}\right]^{1/m}, \quad \varphi(t) = C + \frac{2a(m-2)}{m} t,$$

where A, B, C, and λ are arbitrary constants (the first solution is degenerate).

3°. Traveling-wave solution in implicit form:

$$\int \frac{dw}{\lambda w^{1-m} + C_1} = \frac{\beta x + \lambda t + C_2}{a\beta^2(1-m)},$$

where C_1, C_2, β, and λ are arbitrary constants. To $\lambda = 0$ there corresponds a stationary solution, and to $C_1 = 0$ there corresponds the second solution in Item 2°.

4°. Multiplicative separable solution:

$$w(x,t) = (\lambda t + A)^{-1/m} f(x),$$

where λ is an arbitrary constant, and the function $f = f(x)$ is determined by the autonomous ordinary differential equation $amf''_{xx} + \lambda f^{1-m} = 0$ (its solution can be written out in implicit form).

5°. Self-similar solution:

$$w = w(z), \quad z = \frac{x}{\sqrt{t}},$$

where the function $w(z)$ is determined by the ordinary differential equation $2aw^m w''_{zz} zw'_z = 0$.

6°. Self-similar solution of a more general form:

$$w = t^\beta U(\zeta), \quad \zeta = xt^{-\frac{m\beta+1}{2}},$$

where β is an arbitrary constant, and the function $U = U(\zeta)$ is determined by the ordinary differential equation

$$aU^m U''_{\zeta\zeta} = \beta U - \tfrac{1}{2}(m\beta + 1)\zeta U'_\zeta.$$

This equation is generalized homogeneous, and, hence, its order can be reduced.

7°. Generalized self-similar solution:

$$w = e^{-2\lambda t}\varphi(\xi), \quad \xi = xe^{\lambda mt},$$

where λ is an arbitrary constant, and the function $\varphi = \varphi(\xi)$ is determined by the ordinary differential equation

$$a\varphi^m \varphi''_{\xi\xi} = \lambda m\xi\varphi'_\xi - 2\lambda\varphi.$$

This equation is generalized homogeneous, and, hence, its order can be reduced.

8°. Solution:

$$w = (At + B)^{-1/m}\psi(u), \quad u = x + k\ln(At + B),$$

where A, B, and k are arbitrary constants, and the function $\psi = \psi(u)$ is determined by the autonomous ordinary differential equation

$$a\psi^m \psi''_{uu} = Ak\psi'_u - \frac{A}{m}\psi.$$

9°. The substitution $u = w^{1-m}$ leads to the equation

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(u^{\frac{m}{1-m}}\frac{\partial u}{\partial x}\right),$$

which is considered in Subsection 1.1.10.

19. $\displaystyle \frac{\partial w}{\partial t} = aw^m \frac{\partial^2 w}{\partial x^2} + bw^k.$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{k-m-1}x + C_2, C_1^{2k-2}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solutions:

$$w = w(z), \quad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$aw^m w''_{zz} - \lambda w'_z + bw^k = 0.$$

3°. Self-similar solution for $k \neq 1$:

$$w = t^{\frac{1}{1-k}}u(\xi), \quad \xi = xt^{\frac{k-m-1}{2(1-k)}},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$au^m u''_{\xi\xi} + \frac{m-k+1}{2(1-k)}\xi u'_\xi + bu^k - \frac{1}{1-k}u = 0.$$

4°. For $m \neq 1$, the substitution $u = w^{1-m}$ leads to the equation

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(u^{\frac{m}{1-m}}\frac{\partial u}{\partial x}\right) + b(1-m)u^{\frac{k-m}{1-m}},$$

which is considered Subsection 1.1.11.

5°. For $k = 1$, the transformation

$$w(x, t) = e^{bt}U(x, \tau), \quad \tau = \frac{1}{bm}e^{bmt} + \text{const},$$

leads to an equation of the form 1.1.9.18:

$$\frac{\partial U}{\partial \tau} = aU^m \frac{\partial^2 U}{\partial x^2}.$$

20. $\displaystyle \frac{\partial w}{\partial t} = aw^m \frac{\partial^2 w}{\partial x^2} + bxt^n \frac{\partial w}{\partial x} + ct^k w.$

This is a special case of equation 1.6.11.4 with $f(t) = bt^n$ and $g(t) = ct^k$.

1.1.10. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right)$

► *Equations of this form admit traveling-wave solutions* $w = w(kx + \lambda t)$.

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right)$.

This is a special case of equation 1.1.10.7 with $m = 1$.

1°. Solutions:

$$w(x,t) = C_1 x + aC_1^2 t + C_2,$$
$$w(x,t) = -\frac{(x+C_1)^2}{6a(t+C_2)} + \frac{C_3}{|t+C_2|^{1/3}},$$
$$w(x,t) = \frac{(x+C_1)^2}{C_2 - 6at} + C_3|x+C_1|^{1/2}|C_2 - 6at|^{-5/8},$$

where C_1, C_2, and C_3 are arbitrary constants.

⊙ *References*: D. Zwillinger (1989), A. D. Polyanin and V. F. Zaitsev (2002).

2°. Traveling-wave solution in implicit form:

$$w - C_2 \ln|w + C_2| = C_1 x + aC_1^2 t + C_3.$$

3°. Solution in parametric form:

$$x = (6at + C_1)\xi + C_2\xi^2 + C_3,$$
$$w = -(6at + C_1)\xi^2 - 2C_2\xi^3.$$

4°. Solution in parametric form:

$$x = tf(\xi) + g(\xi),$$
$$w = tf'_\xi(\xi) + g'_\xi(\xi),$$

where the functions $f = f(\xi)$ and $g(\xi)$ are determined by the system of ordinary differential equations

$$(f'_\xi)^2 - ff''_{\xi\xi} = af'''_{\xi\xi\xi}, \qquad (1)$$
$$f'_\xi g'_\xi - fg''_{\xi\xi} = ag'''_{\xi\xi\xi}. \qquad (2)$$

The order of equation (1) can be reduced by two. Suppose a solution of equation (1) is known. Equation (2) is linear in g and has two linearly independent particular solutions

$$g_1 = 1, \qquad g_2 = f(\xi).$$

The second particular solution follows from the comparison of (1) and (2). The general solution of equation (1) can be represented in the form (see Polyanin and Zaitsev, 2003):

$$g(\xi) = C_1 + C_2 f + C_3\left(f\int \psi\,d\xi - \int f\psi\,d\xi\right),$$
$$f = f(\xi), \quad \psi = \frac{1}{(f'_\xi)^2}\exp\left(-\frac{1}{a}\int f\,d\xi\right). \qquad (3)$$

It is not difficult to verify that equation (1) has the following particular solutions:

$$f(\xi) = 6a(\xi + C)^{-1},$$
$$f(\xi) = Ce^{\lambda\xi} - a\lambda, \qquad (4)$$

where C and λ are arbitrary constants. One can see, taking into account (1) and (3), that the first solution in (4) leads to the solution of Item 3°. Substituting the second relation of (4) into (1), we obtain another solution.

Remark. The above solution was obtained, with the help of the Mises transformation, from a solution of the hydrodynamic boundary layer equation (see 9.3.1.1, Items 5° and 7°).

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

5°. For other solutions, see Items 4° to 9° of equation 1.1.10.7 with $m = 1$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 1.1.10.7 with $m = -1$.

Solutions:

$$w(x, y) = (C_1 x - aC_1^2 t + C_2)^{-1},$$
$$w(x, y) = (2at + C_1)(x + C_2)^{-2},$$
$$w(x, y) = \frac{2a(t + C_1)}{(x + C_2)^2 + C_3(t + C_1)^2},$$
$$w(x, y) = \frac{C_1^2}{C_2 + C_3 \exp(aC_2 t - C_1 x)},$$
$$w(x, y) = \frac{C_1^2}{at + C_2}\left[C_3 \exp\left(-\frac{C_1 x}{at + C_2}\right) - 1 + \frac{C_1 x}{at + C_2}\right]^{-1},$$
$$w(x, y) = \frac{2aC_1^2 t + C_2}{\sinh^2(C_1 x + C_3)},$$
$$w(x, y) = \frac{C_2 - 2aC_1^2 t}{\cosh^2(C_1 x + C_3)},$$
$$w(x, y) = \frac{2aC_1^2 t + C_2}{\cos^2(C_1 x + C_3)},$$

where C_1, C_2, and C_3 are arbitrary constants.

⊙ *References*: V. V. Pukhnachov (1987), S. N. Aristov (1999).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 1.1.10.7 with $m = -2$.

1°. Solutions:

$$w(x, t) = \pm(2C_1 x - 2aC_1^2 t + C_2)^{-1/2},$$
$$w(x, t) = \pm\frac{\sqrt{2at}}{x}\left[\ln\left(\frac{C_1}{x^2 t}\right)\right]^{-1/2},$$
$$w(x, t) = \pm\left[\frac{C_1(x + C_2)^2}{2a} + C_3 \exp(C_1 t)\right]^{-1/2},$$

where C_1, C_2, and C_3 are arbitrary constants. The first solution is of the traveling-wave type, the second is self-similar, and the third is a functional separable solution.

2°. Introduce a new unknown $z = z(x, t)$ by $w = \dfrac{\partial z}{\partial x}$ and then integrate the resulting equation with respect to x to obtain

$$\frac{\partial z}{\partial t} = a\left(\frac{\partial z}{\partial x}\right)^{-2}\frac{\partial^2 z}{\partial x^2}. \qquad (1)$$

By the hodograph transformation

$$x = u, \quad z = y, \qquad (2)$$

equation (1) can be reduced to a linear heat equation for $u = u(y, t)$:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial y^2}. \qquad (3)$$

Transformation (2) means that the dependent variable z is taken to be the independent variable, and the independent variable x, the dependent one.

Solutions $w = w(x,t)$ of the original equation are expressed via solutions $u = u(y,t)$ of the linear equation (3) according to

$$w = \left(\frac{\partial u}{\partial y}\right)^{-1}, \quad x = u(y,t). \tag{4}$$

The variable y should be eliminated from (4) to obtain $w = w(x,t)$ in explicit form.

3°. The transformation

$$\bar{x} = \int_{x_0}^{x} w(y,t)\,dy + a\int_{t_0}^{t}\left[w^{-2}(x,\tau)\frac{\partial w}{\partial x}(x,\tau)\right]_{x=x_0} d\tau, \quad \bar{t} = t - t_0, \quad \bar{w}(\bar{x},\bar{t}) = \frac{1}{w(x,t)}, \tag{5}$$

where x_0 and t_0 are any numbers, leads to the linear equation

$$\frac{\partial \bar{w}}{\partial \bar{t}} = a\frac{\partial^2 \bar{w}}{\partial \bar{x}^2}.$$

The inversion of transformation (5) is given by

$$x = \int_{\bar{x}_0}^{\bar{x}} \bar{w}(x',\bar{t})\,dx' + \int_{\bar{t}_0}^{\bar{t}}\left(\frac{\partial \bar{w}(\bar{x},t')}{\partial \bar{x}}\right)_{\bar{x}=\bar{x}_0} dt', \quad t = \bar{t} - \bar{t}_0, \quad w(x,t) = \frac{1}{\bar{w}(\bar{x},\bar{t})}.$$

⊙ *References*: M. L. Storm (1951), G. W. Bluman and S. Kumei (1980), A. Munier, J. R. Burgan, J. Gutierres, E. Fijalkow, and M. R. Feix (1981), N. H. Ibragimov (1985).

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 1.1.10.7 with $m = -4/3$ (the equation admits more invariant solutions than for $m \neq -4/3$).

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = \frac{|A_1B_2 - A_2B_1|^{3/2}C_1^{-3/4}}{(A_2x + B_2)^3}\,w\left(\frac{A_1x + B_1}{A_2x + B_2},\ C_1t + C_2\right),$$

where A_1, A_2, B_1, B_2, C_1, and C_2 are arbitrary constants ($A_1B_2 - A_2B_1 \neq 0$), is also a solution of the equation.

⊙ *References*: L. V. Ovsiannikov (1959, 1982).

2°. Solutions:

$$w(x,t) = (\pm 2C_1x - 3aC_1^2t + C_2)^{-3/4},$$
$$w(x,t) = (at + C_1)^{3/4}[(x + C_2)(C_3x + C_2C_3 + 1)]^{-3/2},$$
$$w(x,t) = (\pm 2C_1x^3 + C_2x^4 - 3aC_1^2x^4t)^{-3/4},$$
$$w(x,t) = \left[\frac{(x + C_1)^2}{a(t + C_2)} + C_3(t + C_2)^2\right]^{-3/4},$$
$$w(x,t) = \left[\frac{(x + C_1)^2}{a(t + C_2)} + C_3(t + C_2)^2(x + C_1)^4\right]^{-3/4},$$

where C_1, C_2, and C_3 are arbitrary constants. The first solution is of the traveling-wave type, the second is a solution in multiplicative separable form, and the other are functional separable solutions.

3°. Functional separable solution:

$$w(x,t) = \left[\varphi_4(t)x^4 + \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t)\right]^{-3/4},$$

where the functions $\varphi_k = \varphi_k(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_0' &= -a\tfrac{3}{4}\varphi_1^2 + 2a\varphi_0\varphi_2,\\
\varphi_1' &= -a\varphi_1\varphi_2 + 6a\varphi_0\varphi_3,\\
\varphi_2' &= -a\varphi_2^2 + \tfrac{3}{2}a\varphi_1\varphi_3 + 12a\varphi_0\varphi_4,\\
\varphi_3' &= -a\varphi_2\varphi_3 + 6a\varphi_1\varphi_4,\\
\varphi_4' &= -\tfrac{3}{4}a\varphi_3^2 + 2a\varphi_2\varphi_4.
\end{aligned}$$

The prime denotes a derivative with respect to t.

⊙ *References*: V. A. Galaktionov (1995), G. A. Rudykh and E. I. Semenov (1998).

4°. There are exact solutions of the following forms:

$$w(x,t) = x^{-3}F(y), \quad y = t - \frac{1}{x};$$

$$w(x,t) = x^{-3}G(z), \quad z = \frac{tx^2}{(x+1)^2}.$$

⊙ *Reference*: N. H. Ibragimov (1994).

5°. For other solutions, see equation 1.1.10.7 with $m = -4/3$.

5. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^{-2/3}\frac{\partial w}{\partial x}\right).$

This is a special case of equation 1.1.10.7 with $m = -2/3$.

1°. Solution:

$$w = (C - 4at)^{3/2}\left[(C-4at)^{3/2} - x^2\right]^{-3/2}.$$

2°. The transformation

$$t = \tau, \quad x = v, \quad w = 1/u,$$

where $\frac{\partial v}{\partial \xi} = u$, leads to an equation of the form 1.1.10.4:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial \xi}\left(u^{-4/3}\frac{\partial u}{\partial \xi}\right).$$

⊙ *References*: A. Munier, J. R. Burgan, J. Gutierres, E. Fijalkow, and M. R. Feix (1981), J. R. Burgan, A. Munier, M. R. Feix, and E. Fijalkow (1984), I. Sh. Akhatov, R. K. Gazizov, and N. H. Ibragimov (1989), N. H. Ibragimov (1994).

6. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^{-3/2}\frac{\partial w}{\partial x}\right).$

1°. Functional separable solution:

$$w(x,t) = a^{2/3}\left[3Ax^3 + f_2(t)x^2 + f_1(t)x + f_0(t)\right]^{-2/3}.$$

Here,

$$f_2(t) = 3\int \varphi(t)\,dt + 3B, \quad f_1(t) = \frac{1}{A}\left[\int \varphi(t)\,dt + B\right]^2 + \frac{1}{2A}\varphi(t),$$

$$f_0(t) = \frac{1}{9A^2}\left[\int \varphi(t)\,dt + B\right]^3 + \frac{1}{6A^2}\varphi(t)\left[\int \varphi(t)\,dt + B\right] + \frac{1}{36A^2}\varphi_t'(t),$$

where the function $\varphi(t)$ is defined implicitly by

$$\int (C_1 - 8\varphi^3)^{-1/2}\,d\varphi = \pm t + C_2,$$

and A, B, C_1, and C_2 are arbitrary constants. Setting $C_1 = 0$ in this relation, we find φ in explicit form: $\varphi = -\frac{1}{2}(t + C_2)^{-2}$.

2°. For other solutions, see equation 1.1.10.7 with $m = -3/2$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right).$

This equation often occurs in nonlinear problems of heat and mass transfer, combustion theory, and flows in porous media. For example, it describes unsteady heat transfer in a quiescent medium with the heat diffusivity being a power-law function of temperature. For $m = 1, -1, -2, -4/3, -2/3, -3/2$, see also equations 1.1.7.1 to 1.1.7.6.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_2 x + C_3, C_1^m C_2^2 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:*

$$\begin{aligned}
w(x) &= (Ax + B)^{\frac{1}{m+1}},\\
w(x,t) &= (\pm kx + k\lambda t + A)^{1/m}, \quad k = \lambda m/a,\\
w(x,t) &= \left[\frac{m(x-A)^2}{2a(m+2)(B-t)}\right]^{\frac{1}{m}},\\
w(x,t) &= \left[A|t+B|^{-\frac{m}{m+2}} - \frac{m}{2a(m+2)}\frac{(x+C)^2}{t+B}\right]^{\frac{1}{m}},\\
w(x,t) &= \left[\frac{m(x+A)^2}{\varphi(t)} + B|x+A|^{\frac{m}{m+1}}|\varphi(t)|^{-\frac{m(2m+3)}{2(m+1)^2}}\right]^{\frac{1}{m}}, \quad \varphi(t) = C - 2a(m+2)t,
\end{aligned}$$

where A, B, C, and λ are arbitrary constants. The third solution for $B > 0$ and the fourth solution for $B < 0$ correspond to blow-up regimes (the solution increases without bound on a finite time interval).

Example. A solution satisfying the initial and boundary conditions

$$\begin{aligned}
w &= 0 && \text{at} \quad t = 0 \quad (x > 0),\\
w &= kt^{1/m} && \text{at} \quad x = 0 \quad (t > 0),
\end{aligned}$$

is given by

$$w(x,t) = \begin{cases} k(t - x/\lambda)^{1/m} & \text{for } 0 \le x \le \lambda t,\\ 0 & \text{for } x > \lambda t,\end{cases}$$

where $\lambda = \sqrt{ak^m/m}$.

⊙ *References*: Ya. B. Zel'dovich and A. S. Kompaneets (1950), G. I. Barenblatt (1952), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995), G. A. Rudykh and E. I. Semenov (1998).

3°. Traveling-wave solutions:

$$w = w(z), \quad z = \pm x + \lambda t,$$

where the function $w(z)$ is defined implicitly by

$$a\int \frac{w^m\,dw}{\lambda w + C_1} = C_2 + z,$$

and λ, C_1, and C_2 are arbitrary constants. To $\lambda = 0$ there corresponds a stationary solution, and to $C_1 = 0$ there corresponds the second solution in Item 2°.

* For the sake of brevity, here and henceforth, exact solutions of nonlinear equations are given only for the domain of their spatial localization, where $w \not\equiv 0$.

4°. Multiplicative separable solution:

$$w(x,t) = (\lambda t + A)^{-1/m} f(x), \tag{1}$$

where the function $f = f(x)$ is defined implicitly by

$$\int \frac{f^m\,df}{\sqrt{C_1 - bf^{m+2}}} = \pm x + C_2, \qquad b = \frac{2\lambda}{am(m+2)},$$

and λ, C_1, and C_2 are arbitrary constants.

5°. Self-similar solution:

$$w = w(z), \quad z = \frac{x}{\sqrt{t}} \qquad (0 \le x < \infty),$$

where the function $w(z)$ is determined by the ordinary differential equation

$$2a\big(w^m w'_z\big)'_z + zw'_z = 0. \tag{2}$$

Solution of this sort usually describe situations where the unknown function assumes constant values at the initial and boundary conditions.

To the particular solution of equation (2) with $w(z) = k_2 z^{2/m}$ there corresponds the third solution in Item 2°.

Fujita (1952) obtained the general solution of equation (2) for $m = -1$ and $m = -2$; see also the book by Lykov (1967).

With the boundary conditions

$$w = 1 \quad \text{at} \quad z = 0, \qquad w = 0 \quad \text{at} \quad z = \infty$$

the solution of equation (2) is localized and has the structure

$$w = \begin{cases} (1-Z)^{1/m} \dfrac{P(1-Z,\, m)}{P(1,\, m)} & \text{for } 0 \le Z \le 1, \\ 0 & \text{for } 1 \le Z < \infty, \end{cases}$$

where

$$Z = \frac{z}{z_0}, \quad z_0^2 = \frac{2a}{mP(1,\, m)}, \quad P(\xi,\, m) = \sum_{k=0}^{\infty} b_k \xi^k,$$

$b_0 = 1$, $b_1 = -\frac{1}{2}[m(m+1)]^{-1}, \ldots$; see Samarskii and Sobol' (1963).

6°. Self-similar solution:

$$w = t^{-\frac{1}{m+2}} F(\xi), \quad \xi = xt^{-\frac{1}{m+2}} \qquad (0 \le x < \infty).$$

Here, the function $F = F(\xi)$ is determined by the first-order ordinary differential equation

$$a(m+2)F^m F'_\xi + \xi F = C, \tag{3}$$

where C is an arbitrary constant.

To $C = 0$ in (3) there corresponds the fourth solution in Item 2°, which describes the propagation of a thermal wave coming from a plane source. For details, see the book by Zel'dovich and Raiser (1966).

Performing the change of variable $\varphi = F^m$ in equation (3), one obtains

$$\varphi'_\xi = \alpha\varphi^{-1/m} - \beta\xi, \tag{4}$$

where $\alpha = \frac{mC}{a(m+2)}$ and $\beta = \frac{m}{a(m+2)}$. The books by Polyanin and Zaitsev (1995, 2003) present general solutions of equation (4) for $m = -1$ and $m = 1$.

7°. Self-similar solution of a more general form:

$$w = t^{\beta} g(\zeta), \quad \zeta = xt^{-\frac{m\beta+1}{2}}, \qquad \beta \text{ is any.}$$

Here, the function $g = g(\zeta)$ is determined by the ordinary differential equation

$$G''_{\zeta\zeta} = A_1 \zeta G^{-\frac{m}{m+1}} G'_{\zeta} + A_2 G^{\frac{1}{m+1}}, \qquad G = g^{m+1}, \tag{5}$$

where $A_1 = -(m\beta+1)/(2a)$ and $A_2 = \beta(m+1)/a$. This equation is homogeneous, and, therefore, its order can be reduced (and then it can be transformed to an Abel equation of the second kind). Exact analytical solutions of equation (5) for various values of m can be found in Polyanin and Zaitsev (2003).

8°. Generalized self-similar solution:

$$w = e^{-2\lambda t} \varphi(u), \quad u = xe^{\lambda mt}, \qquad \lambda \text{ is any,}$$

where the function $\varphi = \varphi(u)$ is determined by the ordinary differential equation

$$a(\varphi^m \varphi'_u)'_u = \lambda m u \varphi'_u - 2\lambda\varphi. \tag{6}$$

This equation is homogeneous, and, hence, its order can be reduced (and then it can be transformed to an Abel equation of the second kind). The substitution $\Phi = \varphi^{m+1}$ brings (6) to an equation that coincides, up to notation, with (5).

9°. Solution:

$$w = (t + A)^{-1/m} \psi(u), \quad u = x + b \ln(t + A), \qquad A,\ b \text{ are any,}$$

where the function $\psi = \psi(u)$ is determined by the autonomous ordinary differential equation

$$a(\psi^m \psi'_u)'_u = b\psi'_u - \psi/m. \tag{7}$$

Introduce the new dependent variable $p(\psi) = \dfrac{a}{b}\psi^m \psi'_u$. Taking into account the identity $\dfrac{d}{du} = \dfrac{b}{a}\psi^{-m} p \dfrac{d}{d\psi}$, we arrive at an Abel equation of the second kind:

$$pp'_{\psi} = p - s\psi^{m+1}, \qquad s = a/(mb^2).$$

The general solutions of this equation with $m = -3,\ -2,\ -\frac{3}{2},\ -1$ can be found in Polyanin and Zaitsev (2003).

10°+. Unsteady point source solution with $a = 1$:

$$w(x,t) = \begin{cases} At^{-1/(m+2)} \left(\eta_0^2 - \dfrac{x^2}{t^{2/(m+2)}} \right)^{1/m} & \text{for } \ |x| \le \eta_0 t^{1/(m+2)}, \\ 0 & \text{for } \ |x| > \eta_0 t^{1/(m+2)}, \end{cases}$$

where

$$A = \left[\frac{m}{2(m+2)} \right]^{1/m}, \quad \eta_0 = \left[\frac{\Gamma(1/m + 3/2)}{A\sqrt{\pi}\,\Gamma(1/m+1)} E_0 \right]^{m/(m+2)}, \quad \Gamma(z) = \int_0^{\infty} e^{-\xi} \xi^{z-1}\, d\xi,$$

with $\Gamma(z)$ being the gamma function.

The above solution satisfies the initial condition

$$w(x, 0) = E_0 \delta(x),$$

where $\delta(x)$ is the Dirac delta function, and the condition of conservation of energy

$$\int_{-\infty}^{\infty} w(x,t)\, dx = E_0 > 0.$$

⊙ *Reference*: A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

11°. The transformation

$$\widetilde{t} = t - t_0, \quad \widetilde{x} = \int_{x_0}^{x} w(y,t)\,dy + a\int_{t_0}^{t}\left[w^m(x,\tau)\frac{\partial w}{\partial x}(x,\tau)\right]_{x=x_0} d\tau, \quad \widetilde{w}(\widetilde{x},\widetilde{t}) = \frac{1}{w(x,t)}$$

takes a nonzero solution $w(x,t)$ of the original equation to a solution $\widetilde{w}(\widetilde{x},\widetilde{t})$ of a similar equation

$$\frac{\partial \widetilde{w}}{\partial \widetilde{t}} = a\frac{\partial}{\partial \widetilde{x}}\left(\widetilde{w}^{-m-2}\frac{\partial \widetilde{w}}{\partial \widetilde{x}}\right).$$

⊙ *References for equation* 1.1.10.7: L. V. Ovsiannikov (1959, 1962, 1982), N. H. Ibragimov (1994), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

1.1.11. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + bw^k$

► *Equations of this form admit traveling-wave solutions* $w = w(kx + \lambda t)$.

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + b.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-1} w(\pm C_1 x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solutions linear and quadratic in x:

$$w(x,t) = C_1 x + (aC_1^2 + b)t + C_2,$$
$$w(x,t) = -\frac{(x+C_2)^2}{6a(t+C_1)} + C_3|t+C_1|^{-1/3} + \frac{3}{4}b(t+C_1),$$

where C_1, C_2, and C_3 are arbitrary constants. The first solution is degenerate.

3°. Traveling-wave solution in implicit form:

$$a\int \frac{u\,du}{au^2 - C_1 u + b} = -\ln|\pm x + C_1 t + C_2| + C_3, \qquad u = \frac{w}{\pm x + C_1 t + C_2}.$$

4°. For other solutions, see equation 1.1.11.11 with $m = 1$ and $k = 0$.

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right) + b.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w(\pm C_1 x + C_2, C_1^{-1} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. The transformation

$$x = -\frac{2}{bu}\frac{\partial u}{\partial y}, \quad w(x,t) = -\frac{b}{2}\left[\frac{\partial}{\partial y}\left(\frac{1}{u}\frac{\partial u}{\partial y}\right)\right]^{-1} \tag{1}$$

leads to the equation

$$\frac{\partial}{\partial y}\left(\Phi\frac{\partial \Psi}{\partial y}\right) = 0, \quad \text{where} \quad \Phi = \left[\frac{\partial}{\partial y}\left(\frac{1}{u}\frac{\partial u}{\partial y}\right)\right]^{-1}, \ \ \Psi = \frac{1}{u}\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2}\right).$$

It follows that any solution $u = u(x,t)$ of the linear heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial y^2} = 0 \tag{2}$$

generates a solution (1) of the original nonlinear equation.

⊙ *Reference*: V. A. Dorodnitsyn and S. R. Svirshchevskii (1983).

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right) - bw^3.$

Functional separable solutions:

$$w(x,t) = \pm\left[C_1(x+C_2)^2 + C_3\exp(2aC_1t) - \frac{b}{aC_1}\right]^{-1/2},$$

where C_1, C_2, and C_3 are arbitrary constants.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + bw^{-1/3}.$

1°. For $ab > 0$, the transformation

$$w(x,t) = \exp(\pm 3\lambda x)z(\xi,t), \quad \xi = \frac{1}{2\lambda}\exp(\pm 2\lambda x), \qquad \lambda = \left(\frac{b}{3a}\right)^{1/2},$$

leads to a simpler equation of the form 1.1.10.4:

$$\frac{\partial z}{\partial t} = a\frac{\partial}{\partial \xi}\left(z^{-4/3}\frac{\partial z}{\partial \xi}\right). \tag{1}$$

⊙ *References*: N. H. Ibragimov (1994), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

2°. For $ab < 0$, the transformation

$$w(x,t) = \frac{z(\xi,t)}{\cos^3(\lambda x)}, \quad \xi = \frac{1}{\lambda}\tan(\lambda x), \qquad \lambda = \left(-\frac{b}{3a}\right)^{1/2},$$

also leads to equation 1.1.10.4.

3°. Multiplicative separable solution:

$$w(x,t) = (t+C)^{3/4}u(x),$$

where C is an arbitrary constant, and the function $u = u(x)$ is determined by the autonomous ordinary differential equation

$$a(u^{-4/3}u'_x)'_x + bu^{-1/3} - \tfrac{3}{4}u = 0.$$

4°. See also equation 1.1.12.6 with $b = c = 0$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-3/2}\dfrac{\partial w}{\partial x}\right) + bw^{5/2}.$

1°. Functional separable solution:

$$w(x,t) = a^{2/3}\left[3Ax^3 + f_2(t)x^2 + f_1(t)x + f_0(t)\right]^{-2/3}.$$

Here,

$$f_2(t) = 3\int \varphi(t)\,dt + 3B, \quad f_1(t) = \frac{1}{A}\left[\int \varphi(t)\,dt + B\right]^2 + \frac{1}{2A}\varphi(t),$$

$$f_0(t) = \frac{1}{9A^2}\left[\int \varphi(t)\,dt + B\right]^3 + \frac{1}{6A^2}\varphi(t)\left[\int \varphi(t)\,dt + B\right] + \frac{1}{36A^2}\varphi'_t(t),$$

where the function $\varphi(t)$ is defined implicitly by

$$\int (C_1 - 108A^2ab\varphi - 8\varphi^3)^{-1/2}\,d\varphi = t + C_2,$$

and A, B, C_1, and C_2 are arbitrary constants.

2°. For other solutions, see equations 1.1.11.9 and 1.1.11.11 with $m = -3/2$.

3°. The substitution $w = u^{-2/3}$ leads to an equation of the form 1.1.9.7:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} - \frac{2}{3}a\left(\frac{\partial u}{\partial x}\right)^2 - \frac{3}{2}b.$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + cw^{7/3}.$

This is a special case of equation 1.1.12.5 with $b = 0$. See also equation 1.1.11.11 with $m = -4/3$ and $k = 7/3$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{m}\dfrac{\partial w}{\partial x}\right) + bw.$

1°. Solutions:

$$w(x,t) = e^{bt}(Ax+B)^{\frac{1}{m+1}},$$

$$w(x,t) = e^{bt}\left(\pm\frac{\lambda m}{a}x + \frac{\lambda^2}{ab}e^{bmt} + A\right)^{\frac{1}{m}},$$

$$w(x,t) = e^{bt}\left[\frac{bm^2(x-A)^2}{2a(m+2)(B-e^{bmt})}\right]^{\frac{1}{m}},$$

$$w(x,t) = \left[A\exp\left(\frac{2bmt}{m+2}\right) - \frac{bm^2(x+B)^2}{2a(m+2)}\right]^{\frac{1}{m}},$$

$$w(x,t) = e^{bt}\left[A|e^{bmt}+B|^{-\frac{m}{m+2}} - \frac{bm^2}{2a(m+2)}\frac{(x+C)^2}{e^{bmt}+B}\right]^{\frac{1}{m}},$$

where A, B, C, and λ are arbitrary constants.

2°. By the transformation

$$w(x,t) = e^{bt}v(x,\tau), \qquad \tau = \frac{1}{bm}e^{bmt} + \text{const}$$

the original equation can be reduced to an equation of the form 1.1.10.7:

$$\frac{\partial v}{\partial \tau} = a\frac{\partial}{\partial x}\left(v^m\frac{\partial v}{\partial x}\right).$$

3°. See also equation 1.1.11.11 with $k = 1$.

⊙ *Reference*: L. K. Martinson and K. B. Pavlov (1972).

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{m}\dfrac{\partial w}{\partial x}\right) + bw^{m+1}.$

1°. Multiplicative separable solution ($a = b = 1$, $m > 0$):

$$w(x,t) = \begin{cases} \left[\dfrac{2(m+1)}{m(m+2)}\dfrac{\cos^2(\pi x/L)}{(t_0-t)}\right]^{1/m} & \text{for } |x| \le \dfrac{L}{2}, \\ 0 & \text{for } |x| > \dfrac{L}{2}, \end{cases} \tag{1}$$

where $L = 2\pi(m+1)^{1/2}/m$. Solution (1) describes a blow-up regime that exists on a limited time interval $t \in [0,\, t_0)$. The solution is localized in the interval $|x| < L/2$.

⊙ *References*: N. V. Zmitrenko, S. P. Kurdyumov, A. P. Mikhailov and A. A. Samarskii (1976), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

2°. Multiplicative separable solution:

$$w(x,t) = \left(\frac{Ae^{\mu x} + Be^{-\mu x} + D}{m\lambda t + C}\right)^{1/m},$$

$$B = \frac{\lambda^2(m+1)^2}{4b^2A(m+2)^2}, \qquad D = -\frac{\lambda(m+1)}{b(m+2)}, \qquad \mu = m\sqrt{-\frac{b}{a(m+1)}},$$

where A, C, and λ are arbitrary constants, $ab(m+1) < 0$.

3°. Multiplicative separable solution (C and λ are arbitrary constants):

$$w(x,t) = (m\lambda t + C)^{-1/m}\varphi(x),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$a(\varphi^m \varphi'_x)'_x + b\varphi^{m+1} + \lambda\varphi = 0. \tag{2}$$

Equation (2) has the following solution in implicit form:

$$\int \varphi^m \left[A - \frac{2\lambda}{a(m+2)}\varphi^{m+2} - \frac{b}{a(m+1)}\varphi^{2m+2}\right]^{-1/2} d\varphi = \pm x + B,$$

where A and B are arbitrary constants.

4°. Functional separable solution [it is assumed that $ab(m+1) < 0$]:

$$w(x,t) = \left[f(t) + g(t)e^{\lambda x}\right]^{1/m}, \qquad \lambda = \pm m\sqrt{\frac{-b}{a(m+1)}},$$

where the functions $f = f(t)$ and $g = g(t)$ are determined by the autonomous system of ordinary differential equations

$$f'_t = bmf^2, \qquad g'_t = \frac{bm(m+2)}{m+1} fg.$$

Integrating yields

$$f(t) = (C_1 - bmt)^{-1}, \qquad g(t) = C_2(C_1 - bmt)^{-\frac{m+2}{m+1}},$$

where C_1 and C_2 are arbitrary constants.

5°. Functional separable solution (A and B are arbitrary constants):

$$w(x,t) = \left[f(t) + g(t)(Ae^{\lambda x} + Be^{-\lambda x})\right]^{1/m}, \qquad \lambda = m\sqrt{\frac{-b}{a(m+1)}}, \tag{3}$$

where the functions $f = f(t)$ and $g = g(t)$ are determined by the autonomous system of ordinary differential equations

$$f'_t = bmf^2 + \frac{4bmAB}{m+1} g^2, \qquad g'_t = \frac{bm(m+2)}{m+1} fg. \tag{4}$$

On eliminating t from this system, one obtains a homogeneous first-order equation:

$$f'_g = \frac{m+1}{m+2}\frac{f}{g} + \frac{4AB}{m+2}\frac{g}{f}. \tag{5}$$

The substitution $\zeta = f/g$ leads to a separable equation. Integrating yields the solution of equation (5) in the form

$$f = \pm g\left(4AB + C_1 g^{-\frac{2}{m+2}}\right)^{\frac{1}{2}}, \qquad C_1 \text{ is any}.$$

Substituting this expression into the second equation of system (4), one obtains a separable equation for $g = g(t)$.

6°. The functional separable solutions

$$w(x,t) = \left[f(t) + g(t)\cosh(\lambda x)\right]^{1/m},$$
$$w(x,t) = \left[f(t) + g(t)\sinh(\lambda x)\right]^{1/m}$$

are special cases of formula (3) with $A = \frac{1}{2}$, $B = \frac{1}{2}$ and $A = \frac{1}{2}$, $B = -\frac{1}{2}$, respectively.

7°. Functional separable solution [it is assumed that $ab(m+1) > 0$]:

$$w(x,t) = \big[f(t) + g(t)\cos(\lambda x + C)\big]^{1/m}, \qquad \lambda = m\sqrt{\frac{b}{a(m+1)}},$$

where the functions $f = f(t)$ and $g = g(t)$ are determined by the autonomous system of ordinary differential equations

$$f'_t = bmf^2 + \frac{bm}{m+1}g^2, \quad g'_t = \frac{bm(m+2)}{m+1}fg,$$

which coincides with system (4) for $AB = \frac{1}{4}$.

⊙ *References for equation* 1.1.11.8: M. Bertsch, R. Kersner, and L. A. Peletier (1985), V. A. Galaktionov and S. A. Posashkov (1989), V. F. Zaitsev and A. D. Polyanin (1996).

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + bw^{1-m}.$

This is a special case of equation 1.1.11.11 with $k = 1 - m$.

Functional separable solution:

$$w(x,t) = \left[\frac{1}{F}(x+A)^2 + B|F|^{-\frac{m}{m+2}} - \frac{bm^2}{4a(m+1)}F\right]^{1/m},$$
$$F = F(t) = C - \frac{2a(m+2)}{m}t,$$

where A, B, and C are arbitrary constants.

⊙ *Reference*: R. Kersner (1978).

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{2n} \dfrac{\partial w}{\partial x}\right) + bw^{1-n}.$

This is a special case of equation 1.1.11.11 with $m = 2n$ and $k = 1 - n$.

Generalized traveling-wave solution:

$$w(x,t) = \left[\pm\frac{x + C_1}{\sqrt{C_2 - kt}} - \frac{bn^2}{3a(n+1)}(C_2 - kt)\right]^{1/n}, \qquad k = \frac{2a(n+1)}{n},$$

where C_1 and C_2 are arbitrary constants.

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + bw^k.$

This is a special case of equation 1.6.15.2 with $f(w) = aw^m$ and $g(w) = bw^k$. For $b = 0$, see Subsection 1.1.10.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{k-m-1}x + C_2, C_1^{2k-2}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. A space-homogeneous solution and a stationary solution are given by (the latter is written out in implicit form):

$$w(t) = \begin{cases} \big[(1-k)bt + C\big]^{\frac{1}{1-k}} & \text{if } k \neq 1, \\ Ce^{bt} & \text{if } k = 1, \end{cases}$$

$$\int w^m \left[A - \frac{2b}{a(m+k+1)}w^{m+k+1}\right]^{-1/2} dw = \pm x + B,$$

where A, B, and C are arbitrary constants.

3°. Traveling-wave solutions:

$$w = w(z), \quad z = \pm x + \lambda t,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$a(w^m w'_z)'_z - \lambda w'_z + b w^k = 0. \tag{1}$$

The substitution

$$u(w) = \frac{a}{\lambda} w^m w'_z$$

brings (1) to the Abel equation

$$u u'_w - u = -ab\lambda^{-2} w^{m+k}. \tag{2}$$

The book by Polyanin and Zaitsev (2003) presents exact solutions of equation (2) with $m + k = -2,\ -1,\ -\frac{1}{2},\ 0,\ 1$.

4°. Self-similar solution for $k \neq 1$:

$$w = t^{\frac{1}{1-k}} u(\xi), \quad \xi = x t^{\frac{k-m-1}{2(1-k)}},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$a(u^m u'_\xi)'_\xi + \frac{m-k+1}{2(1-k)} \xi u'_\xi + b u^k - \frac{1}{1-k} u = 0.$$

⊙ *Reference*: V. A. Dorodnitsyn (1982).

1.1.12. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + bw + c_1 w^{k_1} + c_2 w^{k_2} + c_3 w^{k_3}$

► *Equations of this form admit traveling-wave solutions* $w = w(kx + \lambda t)$.

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + bw + c.$

For $c = 0$, see equation 1.1.11.7.

Generalized separable solutions linear and quadratic in x:

$$w(x,t) = C_1 x e^{bt} + C_2 e^{bt} + \frac{aC_1^2}{b} e^{2bt} - \frac{c}{b},$$

$$w(x,t) = \frac{b e^{bt}(x + C_2)^2}{\varphi(t)} + C_3 \frac{e^{bt}}{\varphi^{1/3}(t)} + \frac{c e^{bt}}{\varphi^{1/3}(t)} \int e^{-bt} \varphi^{1/3}(t)\,dt, \qquad \varphi(t) = C_1 - 6ae^{bt},$$

where C_1, C_2, and C_3 are arbitrary constants. The first solution is degenerate.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right) - bw - cw^3.$

Functional separable solutions for $b \neq 0$:

$$w(x,t) = \pm\left[bC_1 e^{2bt}(x + C_2)^2 + C_3 F(t) + 2cF(t)\int \frac{dt}{F(t)}\right]^{-1/2}, \qquad F(t) = \exp\bigl(aC_1 e^{2bt} + 2bt\bigr),$$

where C_1, C_2, and C_3 are arbitrary constants.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + bw^{-1/3} + cw.$

1°. Multiplicative separable solutions:

$$w(x,t) = e^{ct}\left(Ae^{kx} + Be^{-kx}\right)^{-3} \quad \text{if} \quad b/(3a) = k^2 > 0,$$
$$w(x,t) = e^{ct}\left[A\cos(kx) + B\sin(kx)\right]^{-3} \quad \text{if} \quad b/(3a) = -k^2 < 0,$$

where A and B are arbitrary constants.

2°. The transformation

$$w = e^{ct}u(x,\tau), \qquad \tau = -\frac{3}{4c}e^{-\frac{4}{3}ct} + \text{const}$$

leads to a simpler equation of the form 1.1.11.4:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial x}\left(u^{-4/3}\frac{\partial u}{\partial x}\right) + bu^{-1/3}.$$

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + bw + cw^{5/3}.$

Functional separable solution:

$$w(x,t) = \left[\varphi(t)x^2 + \psi(t)x + \chi(t)\right]^{-3/2},$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations

$$\begin{aligned}
\varphi'_t &= \tfrac{1}{2}a\varphi(4\varphi\chi - \psi^2) - \tfrac{2}{3}b\varphi,\\
\psi'_t &= \tfrac{1}{2}a\psi(4\varphi\chi - \psi^2) - \tfrac{2}{3}b\psi,\\
\chi'_t &= \tfrac{1}{2}a\chi(4\varphi\chi - \psi^2) - \tfrac{2}{3}b\chi - \tfrac{2}{3}c.
\end{aligned}$$

It follows from the first two equations that $\varphi = C\psi$, where C is an arbitrary constant.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + bw + cw^{7/3}.$

Functional separable solution:

$$w(x,t) = \left[\varphi_4(t)x^4 + \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t)\right]^{-3/4},$$

where the functions $\varphi_k = \varphi_k(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi'_0 &= -\tfrac{3}{4}a\varphi_1^2 + 2a\varphi_0\varphi_2 - \tfrac{4}{3}b\varphi_0 - \tfrac{4}{3}c,\\
\varphi'_1 &= -a\varphi_1\varphi_2 + 6a\varphi_0\varphi_3 - \tfrac{4}{3}b\varphi_1,\\
\varphi'_2 &= -a\varphi_2^2 + \tfrac{3}{2}a\varphi_1\varphi_3 + 12a\varphi_0\varphi_4 - \tfrac{4}{3}b\varphi_2,\\
\varphi'_3 &= -a\varphi_2\varphi_3 + 6a\varphi_1\varphi_4 - \tfrac{4}{3}b\varphi_3,\\
\varphi'_4 &= -\tfrac{3}{4}a\varphi_3^2 + 2a\varphi_2\varphi_4 - \tfrac{4}{3}b\varphi_4.
\end{aligned}$$

The prime denotes a derivative with respect to t.

⊙ *Reference*: V. A. Galaktionov (1995).

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) - aw^{-1/3} + bw^{7/3} + cw.$

The substitution $u = w^{-4/3}$ leads to an equation with quadratic nonlinearity:

$$\frac{\partial u}{\partial t} = u\frac{\partial^2 u}{\partial x^2} - \frac{3}{4}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{4}{3}\left(au^2 - cu - b\right).$$

1°. For $a = 1$, there is a solution of the form

$$u = \varphi_1(t) + \varphi_2(t)\cos(kx) + \varphi_3(t)\sin(kx) + \varphi_4(t)\cos(2kx) + \varphi_5(t)\sin(2kx), \qquad k = 2\times 3^{-1/2},$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of first-order ordinary differential equations (not written out here).

2°. For $a = -1$, there is a solution of the form

$$u = \varphi_1(t) + \varphi_2(t)\cosh(kx) + \varphi_3(t)\sinh(kx) + \varphi_4(t)\cosh(2kx) + \varphi_5(t)\sinh(2kx), \quad k = 2 \times 3^{-1/2}.$$

⊙ *Reference*: V. A. Galaktionov (1995).

7. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^m \frac{\partial w}{\partial x}\right) + bw^{m+1} + cw.$

1°. Multiplicative separable solutions:

$$w(x,t) = e^{ct}\bigl[A\cos(kx) + B\sin(kx)\bigr]^{\frac{1}{m+1}} \quad \text{if} \quad b(m+1)/a = k^2 > 0,$$

$$w(x,t) = e^{ct}\bigl[A\exp(kx) + B\exp(-kx)\bigr]^{\frac{1}{m+1}} \quad \text{if} \quad b(m+1)/a = -k^2 < 0,$$

where A and B are arbitrary constants.

2°. The transformation

$$w = e^{ct}u(x,\tau), \qquad \tau = \frac{1}{cm}e^{cmt} + \text{const}$$

leads to a simpler equation of the form 1.1.11.8:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right) + bu^{m+1}.$$

Special case. Multiplicative separable solution for $m = -1$:

$$w = A\exp\left(ct - \frac{b}{2a}x^2 + Bx\right),$$

where A and B are arbitrary constants.

8. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^m \frac{\partial w}{\partial x}\right) + b + cw^{-m}.$

Functional separable solution:

$$w = \left[c(m+1)t - \frac{b(m+1)}{2a}x^2 + C_1x + C_2\right]^{\frac{1}{m+1}},$$

where C_1 and C_2 are arbitrary constants.

9. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^m \frac{\partial w}{\partial x}\right) + bw + cw^{1-m}.$

1°. Generalized traveling-wave solution:

$$w(x,t) = \left(C_1e^{bmt}x + \frac{a}{bm^2}e^{2bmt} + C_2e^{bmt} - \frac{c}{b}\right)^{1/m},$$

where C_1 and C_2 are arbitrary constants.

2°. For a more complicated solution, see 1.6.13.4 with $f(t) = b$ and $g(t) = c$.

⊙ *Reference*: V. A. Galaktionov and S. A. Posashkov (1989).

10. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^m \frac{\partial w}{\partial x}\right) + bw^{1+m} + cw + sw^{1-m}.$

This is a special case of equation 1.6.13.5 with $f(t) = c$ and $g(t) = s$.

The substitution $u = w^m$ leads to an equation of the form 1.1.9.9:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} + \frac{a}{m}\left(\frac{\partial u}{\partial x}\right)^2 + bmu^2 + cmu + sm.$$

⊙ *Reference*: V. A. Galaktionov and S. A. Posashkov (1989).

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{2n}\dfrac{\partial w}{\partial x}\right) + bw + cw^{1-n}.$

Generalized traveling-wave solutions:

$$w(x,t) = \left[\varphi(t)(\pm x + C_1) + cn\varphi(t)\int \frac{dt}{\varphi(t)}\right]^{1/n}, \qquad \varphi(t) = \left[C_2 e^{-2bnt} - \frac{a(n+1)}{bn^2}\right]^{-1/2},$$

where C_1 and C_2 are arbitrary constants.

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{n}\dfrac{\partial w}{\partial x}\right) + bw + ck + bkw^{n+1} + cw^{-n}.$

Functional separable solutions:

$$w(x,t) = \left\{\exp\bigl[b(n+1)t\bigr]\bigl[C_1\cos\bigl(x\sqrt{\lambda}\,\bigr) + C_2\sin\bigl(x\sqrt{\lambda}\,\bigr)\bigr] - \frac{c}{b}\right\}^{\frac{1}{n+1}} \qquad \text{if } \lambda > 0,$$

$$w(x,t) = \left\{\exp\bigl[b(n+1)t\bigr]\bigl[C_1\cosh\bigl(x\sqrt{|\lambda|}\,\bigr) + C_2\sinh\bigl(x\sqrt{|\lambda|}\,\bigr)\bigr] - \frac{c}{b}\right\}^{\frac{1}{n+1}} \qquad \text{if } \lambda < 0,$$

where C_1 and C_2 are arbitrary constants and $\lambda = \dfrac{bk}{a}(n+1)$.

⊙ *Reference*: V. A. Galaktionov (1994).

1.1.13. Equations of the Form $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w)$

► *Equations of this form admit traveling-wave solutions* $w = w(kx + \lambda t)$.

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[(aw^2 + bw)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = aw^2 + bw$.

Solutions:

$$w(x,t) = \pm\sqrt{2C_1x + 2aC_1^2t + C_2} - \frac{b}{a},$$

$$w(x,t) = \pm\frac{x + C_1}{\sqrt{C_2 - 4at}} - \frac{b}{2a},$$

where C_1 and C_2 are arbitrary constants. The first solution is of the traveling-wave type and the second one is self-similar.

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(\dfrac{a}{w^2 + b^2}\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 1.6.15.1 with $f(w) = a(w^2 + b^2)^{-1}$.

1°. Solutions (A and B are arbitrary constants):

$$w(x) = b\tan(Ax + B),$$

$$w(x,t) = \pm bx\bigl(A - 2ab^{-2}t - x^2\bigr)^{-1/2},$$

$$w(x,t) = Ab\exp(ab^{-2}t - x)\bigl\{1 - A^2\exp\bigl[2(ab^{-2}t - x)\bigr]\bigr\}^{-1/2}.$$

2°. Traveling-wave solution in implicit form:

$$\lambda(kx + \lambda t) + B = \frac{ak^2}{A^2 + b^2}\left[\ln|w + A| - \frac{1}{2}\ln(w^2 + b^2) + \frac{A}{b}\arctan\frac{w}{b}\right],$$

where A, B, k, and λ are arbitrary constants.

3°. The substitution

$$w = \frac{bu}{\sqrt{1-u^2}} \tag{1}$$

leads to the equation

$$\frac{\partial u}{\partial t} = \frac{a}{b^2}\left[(1-u^2)\frac{\partial^2 u}{\partial x^2} + u\left(\frac{\partial u}{\partial x}\right)^2\right], \tag{2}$$

which is a special case of 8.1.2.12 with $F(t,\xi,\eta) = ab^{-2}(\xi-\eta)$. Equation (2) has multiplicative separable solutions

$$\begin{aligned} u &= \frac{Ae^{\lambda x} + Be^{-\lambda x}}{\sqrt{4AB + Ce^{-kt}}}, &\quad k &= \frac{2a\lambda^2}{b^2}; \\ u &= \frac{A\sin(\lambda x) + B\cos(\lambda x)}{\sqrt{A^2+B^2+Ce^{kt}}}, &\quad k &= \frac{2a\lambda^2}{b^2}, \end{aligned} \tag{3}$$

where A, B, C, and λ are arbitrary constants. Formulas (1) and (3) provide two solutions of the original equation.

⊙ *Reference*: P. W. Doyle and P. J. Vassiliou (1998); see also Example 10 in Subsection S.5.3.

4°. Solution:

$$w = b\tan\left(\pm\frac{1}{2}z \pm \frac{a}{b^2}t + C\right),$$

$$z = x^2\cos^{-2}\left(\pm\frac{1}{2}z - \arctan\bigl(\psi(z)\bigr) \pm \frac{a}{b^2}t + C\right),$$

where C is an arbitrary constant and the function $\psi = \psi(z)$ is determined by the ordinary differential equation

$$\psi'_z = \frac{1}{2}(1+\psi^2)\left(\pm 1 - \frac{\psi}{z}\right).$$

Here the function $z = z(x,t)$ is defined implicitly.

5°. Solution:

$$w = b\tan\left(\varphi(z) + \arctan\bigl(\psi(z)\bigr) + \frac{C}{2}\ln\frac{at}{b^2}\right),$$

$$z = \frac{b^2x^2}{at}\cos^{-2}\left(\varphi(z) + \frac{C}{2}\ln\frac{at}{b^2}\right),$$

where C is an arbitrary constant and the functions $\varphi(z)$ and $\psi(z)$ are determined by the system of ordinary differential equations

$$\varphi'_z = \frac{\psi}{2z}, \quad \psi'_z = \frac{1}{2}(1+\psi^2)\left(\frac{C}{2} - \frac{\psi}{2} - \frac{\psi}{z}\right).$$

Here the function $z = z(x,t)$ is defined implicitly.

⊙ *References*: I. Sh. Akhatov, R. K. Gazizov, and N. H. Ibragimov (1989), N. H. Ibragimov (1994).

3. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[(aw^{2n} + bw^n)\frac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = aw^{2n} + bw^n$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1x + C_2, C_1^2t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w(x,t) = \left(\pm\sqrt{2C_1 nx + 2aC_1^2 nt + C_2} - \frac{b}{a}\right)^{1/n},$$

where C_1 and C_2 are arbitrary constants.

3°. Self-similar solution:

$$w(x,t) = \left[\pm\frac{x+C_1}{\sqrt{C_2 - kt}} - \frac{b}{a(n+1)}\right]^{1/n}, \qquad k = \frac{2a(n+1)}{n}.$$

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[(aw^{2n} + bw^n)\dfrac{\partial w}{\partial x}\right] + cw^{1-n}.$

Generalized traveling-wave solution:

$$w(x,t) = \left[\pm\frac{x+C_1}{\sqrt{C_2 - kt}} - \frac{cn^2}{3a(n+1)}(C_2 - kt) - \frac{b}{a(n+1)}\right]^{1/n}, \qquad k = \frac{2a(n+1)}{n},$$

where C_1 and C_2 are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[(aw^{2n} + bw^n)\dfrac{\partial w}{\partial x}\right] + cw + sw^{1-n}.$

Generalized traveling-wave solutions:

$$w(x,t) = \left[\varphi(t)(\pm x + C_1) + \frac{b}{n}\varphi(t)\int \varphi(t)\,dt + sn\varphi(t)\int \frac{dt}{\varphi(t)}\right]^{1/n},$$

$$\varphi(t) = \left[C_2 e^{-2cnt} - \frac{a(n+1)}{cn^2}\right]^{-1/2},$$

where C_1 and C_2 are arbitrary constants.

1.1.14. Equations of the Form $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + bx^m w^{-1/3}.$

For $m = 0$, see equation 1.1.11.4. For $m \neq 0$, the original equation can be reduced to a simpler equation 1.1.10.4 that corresponds to the case $b = 0$ [see equation 1.6.13.1 with $f(x) = bx^m$].

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial w}{\partial x} + cw.$

This is a special case of equation 1.6.13.8 with $m = -2$, $f(t) = b$, and $g(t) = c$.

The transformation (A and B are arbitrary constants)

$$w(x,t) = e^{ct}u(z,\tau), \quad z = x + bt + A, \quad \tau = B - \frac{1}{2c}e^{-2ct}$$

leads to an equation of the form 1.1.10.3:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial z}\left(u^{-2}\frac{\partial u}{\partial z}\right).$$

⊙ *Reference*: V. A. Dorodnitsyn and S. R. Svirshchevskii (1983); the case $b = 0$ was treated.

3. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[\frac{a}{(w+b)^2}\frac{\partial w}{\partial x}\right] + c\frac{\partial w}{\partial x}.$

The transformation

$$u(z,t) = w(x,t) + b, \qquad z = x + ct$$

leads to an equation of the form 1.1.10.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial z}\left(u^{-2}\frac{\partial u}{\partial z}\right).$$

4. $\displaystyle \frac{\partial w}{\partial t} + aw\frac{\partial w}{\partial x} = b\frac{\partial}{\partial x}\left(w\frac{\partial w}{\partial x}\right).$

This is a special case of equation 1.1.14.16 with $n = 1$.

Degenerate solution linear in x:

$$w(x,t) = \frac{ax + b\ln|t + C_1| + C_2}{a^2(t + C_1)},$$

where C_1 and C_2 are arbitrary constants.

5. $\displaystyle \frac{\partial w}{\partial t} + aw\frac{\partial w}{\partial x} = b\frac{\partial}{\partial x}\left(w^2\frac{\partial w}{\partial x}\right).$

This is a special case of equation 1.1.14.16 with $n = 2$.

1°. Traveling-wave solution in implicit form:

$$2b\int\frac{w^2\,dw}{aw^2 + 2\lambda w + C_1} = x + \lambda t + C_2,$$

where C_1, C_2, and λ are arbitrary constants.

2°. Degenerate solution linear in x:

$$w(x,t) = (x + C_1)f(t).$$

Here, C_1 is an arbitrary constant, and the function $f = f(t)$ is determined by the ordinary differential equation

$$f'_t + af^2 = 2bf^3,$$

whose solution can be represented in implicit form:

$$\frac{1}{af} + \frac{2b}{a^2}\ln\left|\frac{2bf - a}{f}\right| = t + C_2.$$

6. $\displaystyle \frac{\partial w}{\partial t} + aw\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[(bw^2 + cw)\frac{\partial w}{\partial x}\right].$

1°. Degenerate solution linear in x:

$$w(x,t) = f(t)x + g(t),$$

where the functions $f = f(t)$ and $g = g(t)$ are determined by the system of ordinary differential equations

$$f'_t + af^2 = 2bf^3,$$
$$g'_t + afg = 2bf^2g + cf^2.$$

The solution of the first equation can be found in 1.1.14.5, Item 2°. The second equation is easy to integrate, since it is linear in g.

2°. Traveling-wave solution in implicit form:

$$2\int\frac{bw^2 + cw}{aw^2 + 2\lambda w + C_1}\,dw = x + \lambda t + C_2,$$

where C_1, C_2, and λ are arbitrary constants.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + bt^n w.$

This is a special case of equation 1.6.13.2 with $f(t) = bt^n$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + be^{\lambda t} w.$

This is a special case of equation 1.6.13.2 with $f(t) = be^{\lambda t}$.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + bt^n w^{1-m}.$

This is a special case of equation 1.6.13.3 with $f(t) = bt^n$.

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + be^{\lambda t} w^{1-m}.$

This is a special case of equation 1.6.13.3 with $f(t) = be^{\lambda t}$.

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + bt^n w + ct^k w^{1-m}.$

This is a special case of equation 1.6.13.4 with $f(t) = bt^n$ and $g(t) = ct^k$.

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + be^{\lambda t} w + ce^{\mu t} w^{1-m}.$

This is a special case of equation 1.6.13.4 with $f(t) = be^{\lambda t}$ and $g(t) = ce^{\mu t}$.

13. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + bw^{1+m} + ct^n w + st^k w^{1-m}.$

This is a special case of equation 1.6.13.5 with $f(t) = ct^n$ and $g(t) = st^k$.

14. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + bx^n w^{1+m}.$

This is a special case of equation 1.6.13.6 with $f(x) = bx^n$.

15. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) - bx\dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^n x + C_2 e^{bt}, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Generalized traveling-wave solutions:

$$w(x,t) = \left(\pm x + C_1 e^{bt}\right)^{2/n} \left[C_2 e^{2bt} + \frac{a(n+2)}{bn}\right]^{-1/n},$$

where C_1 and C_2 are arbitrary constants.

3°. Generalized traveling-wave solutions:

$$w = w(z), \qquad z = \pm x + Ce^{bt},$$

where C is an arbitrary constant and the function $w(z)$ is determined by the ordinary differential equation

$$a(w^n w'_z)'_z - bzw'_z = 0.$$

16. $\dfrac{\partial w}{\partial t} + aw\dfrac{\partial w}{\partial x} = b\dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(C_1^{1-n}x + C_2, C_1^{2-n}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution in implicit form:

$$2b\int \frac{w^n\,dw}{aw^2 + 2\lambda w + C_1} = x + \lambda t + C_2,$$

where C_1, C_2, and λ are arbitrary constants.

3°. Self-similar solution for $n \neq 2$:

$$w(x,t) = u(z)t^{1/(n-2)}, \quad z = xt^{-(n-1)/(n-2)},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$bu^n u''_{zz} + 2bnu^{n-1}(u'_z)^2 - \left(au - \frac{n-1}{n-2}z\right)u'_z - \frac{1}{n-2}u = 0.$$

17. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + (bw^n + c)\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \left[\frac{C_2 - x}{b(t+C_1)} + \frac{a\ln|t+C_1|}{b^2 n(t+C_1)} - \frac{c}{b}\right]^{1/n},$$

where C_1 and C_2 are arbitrary constants.

18. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + aw^n\dfrac{\partial^2 w}{\partial x^2} + bw^{n-1}\left(\dfrac{\partial w}{\partial x}\right)^2.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(C_2 x + C_3, C_1^n C_2^2 t + C_4),$$

where $C_1, \dots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Functional separable solution for $b = \frac{1}{3}n(a-2) - a - 1$:

$$w(x,t) = \left[n\sum_{k=0}^{3}\varphi_k(t)x^k\right]^{1/n}.$$

Here,

$$\varphi_3(t) = A, \quad \varphi_2(t) = \int \psi(t)\,dt + B, \quad \varphi_1(t) = \frac{1}{3A}\left[\int \psi(t)\,dt + B\right]^2 + \frac{1}{2A\beta n}\psi(t),$$

$$\varphi_0(t) = \frac{1}{27A^2}\left[\int \psi(t)\,dt + B\right]^3 + \frac{1}{6A^2\beta n}\psi(t)\left[\int \psi(t)\,dt + B\right] + \frac{1}{12A^2\beta^2 n^2}\psi'_t(t),$$

where the function $\psi = \psi(t)$ is defined implicitly by

$$\int \left(C_1 - \tfrac{8}{3}\beta n\psi^3\right)^{-1/2} d\psi = C_2 + t,$$

A, B, C_1, and C_2 are arbitrary constants, $\beta = a + 1$; $A \neq 0$, $n \neq 0$, $a > -1$.

⊙ *Reference*: G. A. Rudykh and E. I. Semenov (1998).

3°. Functional separable solution for $b = \frac{1}{4}n(a-3) - a - 1$:

$$w(x,t) = \left[n\sum_{k=0}^{4}\varphi_k(t)x^k\right]^{1/n}.$$

Here, the functions $\varphi_k = \varphi_k(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_0' &= -\tfrac{3}{4}\beta\varphi_1^2 + 2\beta\varphi_2\varphi_0,\\
\varphi_1' &= -\beta\varphi_1\varphi_2 + 6\beta\varphi_3\varphi_0,\\
\varphi_2' &= -\beta\varphi_2^2 + \tfrac{3}{2}\beta\varphi_1\varphi_3 + 12\beta\varphi_4\varphi_0,\\
\varphi_3' &= -\beta\varphi_2\varphi_3 + 6\beta\varphi_1\varphi_4,\\
\varphi_4' &= -\tfrac{3}{4}\beta\varphi_3^2 + 2\beta\varphi_2\varphi_4,
\end{aligned}$$

where $\beta = n(a+1)$; the prime denotes a derivative with respect to t.

⊙ *Reference*: G. A. Rudykh and E. I. Semenov (1998).

4°. There are exact solutions of the following forms:

$$\begin{aligned}
&w(x,t) = F(z), \quad z = Ax + Bt;\\
&w(x,t) = (At+B)^{-1/n}G(x);\\
&w(x,t) = t^{\beta}H(\xi), \quad \xi = xt^{-\frac{\beta n+1}{2}};\\
&w(x,t) = e^{-2t}U(\eta), \quad \eta = xe^{nt};\\
&w(x,t) = (At+B)^{-1/n}V(\zeta), \quad \zeta = x + C\ln(At+B),
\end{aligned}$$

where A, B, C, and β are arbitrary constants. The first solution is of the traveling-wave type, the second is a solution in multiplicative separable form, and the third is self-similar.

19. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[\left(aw^{2n} + bw^n\right)\frac{\partial w}{\partial x}\right] + (cw^n + s)\frac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \left[\varphi(t)x + (st + C_1)\varphi(t) + \frac{b}{n}\varphi(t)\int\varphi(t)\,dt\right]^{1/n},$$

where C_1 is an arbitrary constant and the function $\varphi(t)$ is determined by the first-order separable ordinary differential equation

$$\varphi_t' = \frac{a(n+1)}{n}\varphi^3 + c\varphi^2.$$

1.1.15. Other Equations

1. $\displaystyle \frac{\partial w}{\partial t} = (aw^2 + bw^4)\frac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 1.6.16.3 with $f(w) = aw^2 + bw^4$.

1°. Self-similar solutions:

$$w(x,t) = \pm\left[\frac{(x+C_1)^2}{2a(t+C_2)} - \frac{a}{b}\right]^{1/2},$$

where C_1 and C_2 are arbitrary constants.

2°. Functional separable solutions:

$$w(x,t) = \pm\left[\varphi(t)(x^2 + C_1 x + C_2) - \frac{a}{b}\right]^{1/2},$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi(t)$ is determined by the first-order separable equation

$$\varphi'_t = -2a\varphi^2 + \tfrac{1}{2}b(4C_2 - C_1^2)\varphi^3,$$

whose solution can be written out in implicit form.

2. $\displaystyle \frac{\partial w}{\partial t} = (aw^2 + bw^4)\frac{\partial^2 w}{\partial x^2} + cw + kw^{-1}.$

Functional separable solutions:

$$w(x,t) = \pm\sqrt{\varphi(t)x^2 + \psi(t)x + \chi(t)},$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations

$$\begin{aligned}
\varphi'_t &= \tfrac{1}{2}b\varphi(4\varphi\chi - \psi^2) + 2c\varphi,\\
\psi'_t &= \tfrac{1}{2}b\psi(4\varphi\chi - \psi^2) + 2c\psi,\\
\chi'_t &= \tfrac{1}{2}(b\chi + a)(4\varphi\chi - \psi^2) + 2c\chi + 2k.
\end{aligned}$$

It follows from the first two equations that $\varphi = C\psi$, where C is an arbitrary constant.

Remark. The above remains true if the equation coefficients are arbitrary functions of time: $a = a(t)$, $b = b(t)$, $c = c(t)$, and $k = k(t)$.

3. $\displaystyle \frac{\partial w}{\partial t} = ax^{4-k}w^k\frac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 1.6.16.4 with $f(u) = au^k$.

The transformation

$$w(x,t) = xu(z,t), \quad z = 1/x$$

leads to a simpler equation of the form 1.1.9.18:

$$\frac{\partial u}{\partial t} = au^k\frac{\partial^2 u}{\partial z^2}.$$

4. $\displaystyle \frac{\partial w}{\partial t} = ax^n w^k\frac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(C_2 x, C_1^k C_2^{2-n} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. The substitution $u = w^{1-k}$ leads to an equation of the form 1.1.15.6:

$$\frac{\partial u}{\partial t} = ax^n\frac{\partial}{\partial x}\left(u^{\frac{k}{1-k}}\frac{\partial u}{\partial x}\right).$$

3°. The transformation

$$w(x,t) = xu(z,t), \quad z = 1/x$$

leads to an equation of the similar form

$$\frac{\partial u}{\partial t} = az^{4-n-k}u^k\frac{\partial^2 u}{\partial z^2}.$$

5. $\dfrac{\partial w}{\partial t} = ax^{\frac{3m+4}{m+1}} \dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 1.1.15.6.

The transformation

$$w(x,t) = x^{\frac{1}{m+1}} u(z,t), \quad z = \frac{1}{x}$$

leads to a simpler equation of the form 1.1.10.7:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial z}\left(u^m \frac{\partial u}{\partial z}\right).$$

6. $\dfrac{\partial w}{\partial t} = ax^n \dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right).$

This equation occurs in nonlinear problems of heat and mass transfer and is a special case of equation 1.6.17.16 with $f(w) = aw^m$. For $n = 0$, see equation 1.1.10.7.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_2 x, C_1^m C_2^{2-n} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x) = (Ax + B)^{\frac{1}{m+1}},$$

$$w(x,t) = k(\lambda t + A)^{-\frac{1}{m}} x^{\frac{2-n}{m}}, \quad k = \left[\frac{m\lambda}{a(n-2)(2+m-n-nm)}\right]^{\frac{1}{m}},$$

$$w(x,t) = t^{(1-n)\beta}\left[\frac{m\beta}{a(2-n)}\left(xt^\beta\right)^{2-n} + A\right]^{\frac{1}{m}}, \quad \beta = \frac{1}{nm+n-m-2},$$

$$w(x,t) = \exp(-\lambda t)\left[\frac{\lambda}{a}(m+1)^2 x^{\frac{m}{m+1}} \exp(\lambda m t) + A\right]^{\frac{1}{m}}, \quad n = \frac{m+2}{m+1},$$

where A, B, and λ are arbitrary constants.

3°. Multiplicative separable solution:

$$w(x,t) = (\lambda t + A)^{-1/m} f(x),$$

where the function $f = f(x)$ is expressed via solutions to the Emden–Fowler equation

$$F''_{xx} + \frac{\lambda(m+1)}{am} x^{-n} F^{\frac{1}{m+1}} = 0, \qquad F = f^{m+1}. \tag{1}$$

To the power-law particular solution of this equation there corresponds the second solution of the original equation in Item 1°.

The order of equation (1) can be reduced; the equation is analyzed in detail in Polyanin and Zaitsev (2003), where its exact solutions for 26 different pairs of values of the parameters n and m are presented.

4°. Self-similar solution for $n \neq -2$:

$$w = w(z), \quad z = xt^{\frac{1}{n-2}},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$a(2-n)\left(w^m w'_z\right)'_z + z^{1-n} w'_z = 0. \tag{2}$$

The book by Zaitsev and Polyanin (1993) presents the general solution of equation (2) for $m = -1$ and any n.

5°. Self-similar solution:

$$w = t^{\alpha} g(\zeta), \quad \zeta = x t^{\beta}, \qquad \beta = \frac{m\alpha + 1}{n - 2}, \quad \alpha \text{ is any},$$

where the function $g(\zeta)$ is determined by the ordinary differential equation

$$a\zeta^n \left(g^m g'_\zeta\right)'_\zeta = \beta\zeta g'_\zeta + \alpha g. \tag{3}$$

This equation is homogeneous, and, hence, its order can be reduced (thereafter it can be transformed to an Abel equation of the second kind).

In the special case

$$\alpha = \frac{1 - n}{nm + n - m - 2}, \quad \beta = \frac{1}{nm + n - m - 2},$$

a first integral of equation (3) is given by

$$a g^m g'_\zeta = \beta\zeta^{1-n} g + C. \tag{4}$$

To $C = 0$ in (4) there corresponds the third solution in Item 1°.

In the general case, the change of variable $G = g^{m+1}$ brings (3) to the equation

$$G''_{\zeta\zeta} = A_1 \zeta^{1-n} G^{-\frac{m}{m+1}} G'_\zeta + A_2 \zeta^{-n} G^{\frac{1}{m+1}}, \tag{5}$$

where $A_1 = \beta/a$ and $A_2 = \alpha(m+1)/a$. Exact analytical solutions of equation (5) for various values of the parameters n and m can be found in the books by Polyanin and Zaitsev (1995, 2003).

6°. Generalized self-similar solution:

$$w = e^{\lambda(n-2)t} \varphi(u), \quad u = x e^{\lambda m t}, \qquad \lambda \text{ is any},$$

where the function $\varphi(u)$ is determined by the ordinary differential equation

$$a u^n (\varphi^m \varphi'_u)'_u = \lambda m u \varphi'_u + \lambda(n-2)\varphi. \tag{6}$$

This equation is homogeneous, so its order can be reduced (thereafter it can be transformed to an Abel equation of the second kind).

In the special case $n = \dfrac{m+2}{m+1}$, equation (6) has the first integral

$$a\varphi^m \varphi'_u = \lambda m u^{-\frac{1}{m+1}} \varphi + C.$$

To $C = 0$ there corresponds the last solution in Item 1°.

In the general case, the change of variable $\Phi = \varphi^{m+1}$ brings (6) to an equation that coincides, up to notation, with (5).

7°. For $n = 2$, there are solutions of the form

$$w = w(\xi), \quad \xi = \ln|x| - \lambda t,$$

which are defined implicitly by

$$a(m+1) \int \frac{w^m \, dw}{a w^{m+1} - \lambda(m+1) w + C_1} = \xi + C_2,$$

where λ, C_1, and C_2 are arbitrary constants. To the special case $C_1 = 0$ there corresponds the solution

$$w(x,t) = \left[\frac{\lambda(m+1)}{a} + C|x|^{\frac{m}{m+1}} \exp\left(-\frac{m\lambda}{m+1} t\right)\right]^{\frac{1}{m}},$$

where C is an arbitrary constant.

8°. The transformation

$$w(x,t) = x^{\frac{1}{m+1}} u(z,t), \quad z = \frac{1}{x}$$

leads to an equation of the similar form

$$\frac{\partial u}{\partial t} = a z^{\frac{4+3m-n-nm}{m+1}} \frac{\partial}{\partial z}\left(u^m \frac{\partial u}{\partial z}\right).$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

7. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[\left(\dfrac{ax+b}{cw+k}\right)^2 \dfrac{\partial w}{\partial x}\right].$

The substitution $u = \dfrac{cw+k}{ax+b}$ $(c \neq 0)$ leads to an equation of the form 1.1.10.3:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right).$$

⊙ *Reference*: A. Munier, J. R. Burgan, J. Gutierres, E. Fijalkow, and M. R. Feix (1981).

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(x^n w^m \dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 1.6.17.5 with $f(x) = ax^n$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_2 x, C_1^m C_2^{2-n} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Let $m \neq -1$ and $2m - 2n - nm + 3 \neq 0$. The transformation

$$w(x,t) = x^{\frac{1-n}{m+1}} u(\xi, t), \qquad \xi = x^{\frac{2m-2n-nm+3}{m+1}}$$

leads to an equation of the similar form

$$\frac{\partial u}{\partial t} = A\frac{\partial}{\partial \xi}\left(\xi^{\frac{3m-3n-2nm+4}{2m-2n-nm+3}} u^m \frac{\partial u}{\partial \xi}\right), \tag{1}$$

where $A = a\left(\dfrac{2m-2n-nm+3}{m+1}\right)^2$.

3°. In the special case $n = \dfrac{3m+4}{2m+3}$, the transformed equation is very simple and coincides, up to notation, with equation 1.1.10.7:

$$\frac{\partial u}{\partial t} = A\frac{\partial}{\partial \xi}\left(u^m \frac{\partial u}{\partial \xi}\right).$$

4°. In the special case of $n = 2$ and $m = -2$, the transformed equation becomes

$$\frac{\partial u}{\partial t} = A\frac{\partial}{\partial \xi}\left(u^{-2} \frac{\partial u}{\partial \xi}\right),$$

so it coincides with equation 1.1.10.3 (which can further be reduced to the linear heat equation).

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

9. $\dfrac{\partial w}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n w^m \dfrac{\partial w}{\partial x}\right).$

This equation occurs in nonlinear problems of heat and mass transfer. For $n = 0$, see equation 1.1.10.7. To $n = 1$ there correspond two-dimensional problems with axial symmetry, and to $n = 2$ there correspond spherically symmetric problems. Equation with $n = 5$ are encountered in the theory of static turbulence.

Solutions:

$$w(x) = \left(Ax^{1-n} + B\right)^{\frac{1}{m+1}},$$

$$w(x,t) = \left(\frac{mx^2}{A-kt}\right)^{\frac{1}{m}}, \quad k = 2a(nm+m+2),$$

$$w(x,t) = \left(A|kt+B|^{-\frac{m(n+1)}{nm+m+2}} - \frac{mx^2}{kt+B}\right)^{\frac{1}{m}}, \quad k = 2a(nm+m+2),$$

$$w(x,t) = \left[A\exp\left(-\frac{4a\lambda}{m}t\right) + \lambda x^2\right]^{\frac{1}{m}}, \quad n = -\frac{m+2}{m},$$

where A, B, and λ are arbitrary constants.

⊙ *References*: Ya. B. Zel'dovich and A. S. Kompaneets (1950), G. I. Barenblatt (1952, 1989), Ya. B. Zel'dovich and Yu. P. Raiser (1966), L. I. Sedov (1993), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

10. $\dfrac{\partial w}{\partial t} = k(ax^2+bx+c)^m w^{4-2m}\dfrac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 1.6.16.5 with $f(u) = ku^{-2m}$.

1°. The transformation

$$w(x,t) = u(z,t)\sqrt{ax^2+bx+c}, \qquad z = \int \frac{dx}{ax^2+bx+c} \tag{1}$$

leads to the equation

$$\frac{\partial u}{\partial t} = ku^{4-2m}\frac{\partial^2 u}{\partial z^2} + k(ac - \tfrac{1}{4}b^2)u^{5-2m}, \tag{2}$$

which has a traveling-wave solution $u = u(z+\lambda t)$ and a multiplicative separable solution $u = f(t)g(z)$.

Using the change of variable $\varphi = u^{2m-3}$, one obtains from (2) an equation of the form 1.1.11.8,

$$\frac{\partial \varphi}{\partial t} = k\frac{\partial}{\partial z}\left(\varphi^n \frac{\partial \varphi}{\partial z}\right) + p\varphi^{n+1},$$

$$n = \frac{4-2m}{2m-3}, \quad p = k(2m-3)(ac - \tfrac{1}{4}b^2),$$

which admits a wide class of exact solutions.

2°. By the transformation

$$w(x,t) = \left[v(\xi,t)\right]^{\frac{1}{2m+3}}, \qquad \xi = \int \frac{dx}{(ax^2+bx+c)^m}, \tag{3}$$

the original equation can be reduced to equation 1.6.17.5:

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial \xi}\left[F(\xi)v^{\frac{4-2m}{2m-3}}\frac{\partial v}{\partial \xi}\right], \tag{4}$$

where the function $F(\xi)$ is defined parametrically by

$$F(\xi) = \frac{k}{(ax^2+bx+c)^m}, \qquad \xi = \int \frac{dx}{(ax^2+bx+c)^m}. \tag{5}$$

Note some special cases of equation (4) where the function $F = F(\xi)$ can be written out in explicit form:

$$\frac{\partial v}{\partial t} = k\frac{\partial}{\partial \xi}\left(\frac{\cos^2\xi}{v^2}\frac{\partial v}{\partial \xi}\right), \qquad m=1, \quad a=1, \quad b=0, \quad c=1;$$

$$\frac{\partial v}{\partial t} = k\frac{\partial}{\partial \xi}\left(\frac{\cosh^2\xi}{v^2}\frac{\partial v}{\partial \xi}\right), \qquad m=1, \quad a=-1, \quad b=0, \quad c=1;$$

$$\frac{\partial v}{\partial t} = k\frac{\partial}{\partial \xi}\left(\frac{\xi^{-3/2}}{\cos\xi}\frac{\partial v}{\partial \xi}\right), \qquad m=\frac{1}{2}, \quad a=-1, \quad b=0, \quad c=1.$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

1.2. Equations with Exponential Nonlinearities

1.2.1. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b_0 + b_1 e^{\lambda w} + b_2 e^{2\lambda w}$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda w}$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\lambda} x + C_2, C_1^{2\lambda} t + C_3) + 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution (k and β are arbitrary constants):

$$w = w(z), \qquad z = kx + \beta t,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$ak^2 w''_{zz} - \beta w'_z + be^{\lambda w} = 0.$$

3°. Solution:

$$w = u(\xi) - \frac{1}{\lambda}\ln t, \qquad \xi = \frac{x}{\sqrt{t}},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$au''_{\xi\xi} + \frac{1}{2}\xi u'_\xi + \frac{1}{\lambda} + be^{\lambda u} = 0.$$

⊙ *References*: N. H. Ibragimov (1994), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + a + be^{\lambda w}$.

This equation occurs in heat and mass transfer and combustion theory.

1°. Traveling-wave solutions:

$$\begin{aligned} w(x,t) &= -\frac{2}{\lambda}\ln\bigl[\beta + C\exp\bigl(\pm\mu x - \tfrac{1}{2}a\lambda t\bigr)\bigr],\\ w(x,t) &= -\frac{2}{\lambda}\ln\bigl[-\beta + C\exp\bigl(\pm\mu x - \tfrac{1}{2}a\lambda t\bigr)\bigr], \end{aligned} \qquad \beta = \sqrt{-\frac{b}{a}},\ \mu = \sqrt{\frac{a\lambda}{2}},$$

where C is an arbitrary constant.

2°. The solutions of Item 1° are special cases of the traveling-wave solutions

$$w = w(z), \qquad z = \pm\mu x + \sigma t,$$

that satisfy the autonomous equation

$$\mu^2 w''_{zz} - \sigma w'_z + a + be^{\lambda w} = 0. \tag{1}$$

For

$$\mu = \sqrt{\tfrac{1}{2}a\lambda}, \qquad \sigma = \mu^2,$$

the general solution of equation (1) can be written out in parametric form as

$$z = 2\int \frac{f'_\tau(\tau)\,d\tau}{f(\tau)\bigl[\lambda\tau f(\tau) + 2\bigr]} + C_1, \qquad w = \frac{2}{\lambda}\ln|f(\tau)|,$$

where the function $f(\tau)$ is defined by

$$f(\tau) = \frac{C_2 - 2\ln\bigl|\tau + \sqrt{\tau^2 + k}\bigr|}{\lambda\sqrt{\tau^2 + k}}, \qquad k = \frac{4b}{a\lambda^2},$$

and C_1 and C_2 are arbitrary constants.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + a + be^{\lambda w} + ce^{2\lambda w}$.

Equations of this form are encountered in problems of heat and mass transfer and combustion theory.

1°. Traveling-wave solutions for $a \neq 0$:

$$w(x,t) = -\frac{1}{\lambda}\ln\bigl[\beta + C\exp(\pm\mu x - a\lambda t)\bigr], \qquad \mu = \frac{1}{\beta}\sqrt{-c\lambda}, \tag{1}$$

where C is an arbitrary constant and the parameter β is determined by solving the quadratic equation

$$a\beta^2 + b\beta + c = 0.$$

2°. Traveling-wave solutions for $a = 0$:

$$w(x,t) = -\frac{1}{\lambda}\ln\bigl(\pm\sqrt{-c\lambda}\,x - b\lambda t + C\bigr). \tag{2}$$

3°. The substitution $u = e^{-\lambda w}$ leads to an equation with quadratic nonlinearity:

$$u\frac{\partial u}{\partial t} = u\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial x}\right)^2 - a\lambda u^2 - b\lambda u - c\lambda.$$

The particular solution $u = \beta + C\exp(\lambda t + \mu x)$ of this equation generates a solution (1).

4°. Solutions (1) and (2) are special cases of a wider class of traveling-wave solutions $w = w(x + \sigma t)$.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

1.2.2. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + f(w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right)$.

This equation governs unsteady heat transfer in a quiescent medium in the case where the thermal diffusivity is exponentially dependent on temperature.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1 x + C_2, C_3 t + C_4) + \frac{1}{\lambda}\ln\frac{C_3}{C_1^2},$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t) = \frac{2}{\lambda}\ln\left(\frac{\pm x + A}{\sqrt{B - 2at}}\right),$$

$$w(x,t) = -\frac{1}{\lambda}\ln\bigl(C - 2a\lambda\mu t\bigr) + \frac{1}{\lambda}\ln\bigl(\lambda\mu x^2 + Ax + B\bigr),$$

where A, B, C, and μ are arbitrary constants. The first solution is self-similar and the second one is an additive separable solution.

3°. Traveling-wave solution in implicit form:

$$x + \beta t + C_1 = a\int\frac{e^{\lambda w}\,dw}{\beta w + C_2}.$$

4°. Self-similar solution:

$$w = w(y), \qquad y = x/\sqrt{t},$$

where the function $w(y)$ is determined by the ordinary differential equation

$$a(e^{\lambda w}w'_y)'_y + \tfrac{1}{2}yw'_y = 0.$$

5°. Solution:

$$w(x,t) = U(\xi) + 2kt, \quad \xi = xe^{-k\lambda t},$$

where k is an arbitrary constant, and the function $U = U(\xi)$ is determined by the ordinary differential equation

$$2k - k\lambda\xi U'_\xi = a(e^{\lambda U}U'_\xi)'_\xi.$$

6°. Solution:

$$w(x,t) = F(\zeta) - \frac{1}{\lambda}\ln t, \quad \zeta = x + \beta\ln t,$$

where β is an arbitrary constant, and the function $F = F(\zeta)$ is determined by the first-order ordinary differential equation (C is an arbitrary constant)

$$-\zeta + \beta\lambda F = a\lambda e^{\lambda F}F'_\zeta + C.$$

7°. Solution:

$$w(x,t) = G(\theta) - \frac{2b+1}{\lambda}\ln t, \quad \theta = xt^b,$$

where b is an arbitrary constant, and the function $G = G(\theta)$ is determined by the ordinary differential equation

$$-\frac{2b+1}{\lambda} + b\theta G'_\theta = (ae^{\lambda G}G'_\theta)'_\theta.$$

8°. The substitution $\varphi = e^{\lambda w}$ leads to an equation of the form 1.1.9.1:

$$\frac{\partial\varphi}{\partial t} = a\varphi\frac{\partial^2\varphi}{\partial x^2}.$$

⊙ *References*: L. V. Ovsiannikov (1959, 1982), N. H. Ibragimov (1994), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

2. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(e^w\frac{\partial w}{\partial x}\right) + b.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(C_1x + C_2, t + C_3) - 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solutions:

$$w(x,t) = \ln|C_1x + C_2| + bt + C_3,$$
$$w(x,t) = 2\ln|\pm x + C_1| - \ln\left(C_2e^{-bt} - \frac{2a}{b}\right).$$

The first solution is degenerate.

3°. The transformation

$$w = bt + u(x,\tau), \quad \tau = \frac{1}{b}e^{bt} + \text{const}$$

leads to an equation of the form 1.2.2.1:

$$\frac{\partial u}{\partial\tau} = a\frac{\partial}{\partial x}\left(e^u\frac{\partial u}{\partial x}\right).$$

3. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(e^w\frac{\partial w}{\partial x}\right) - a^2e^w.$

Solutions:

$$w(x,t) = \ln\left|\frac{\pm a\exp[2(\pm ax + B)] + 2\exp(\pm ax + B) + A}{2a^2(t + C)}\right| \mp ax - B,$$

where A, B, and C are arbitrary constants.

4. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(e^w \frac{\partial w}{\partial x}\right) - be^w.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, C_2 t + C_3) + \ln C_2,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Additive separable solution:

$$w(x,t) = u(x) - \ln(aC_1 t + C_2),$$

where C_1 and C_2 are arbitrary constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$u''_{xx} + (u'_x)^2 + C_1 e^{-u} - \frac{b}{a} = 0.$$

Integrating yields the general solution in implicit form:

$$\int \left(C_3 e^{-2u} - 2C_1 e^{-u} + \frac{b}{a}\right)^{-1/2} du = \pm x + C_4.$$

The integral is computable, so the solution can be rewritten in explicit form (if $a = 1$ and $b > 0$, see 1.2.2.3 for a solution).

3°. The substitution $u = e^w$ leads to an equation of the form 1.1.9.9:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} - bu^2.$$

5. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(e^w \frac{\partial w}{\partial x}\right) + ae^w + b, \qquad a \neq 0, \quad b \neq 0.$

1°. Additive separable solution for $a = k^2 > 0$:

$$w(x,t) = \ln\bigl[C_1 \cos(kx) + C_2 \sin(kx)\bigr] + bt + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

2°. Additive separable solution for $a = -k^2 < 0$:

$$w(x,t) = \ln\bigl[C_1 \cosh(kx) + C_2 \sinh(kx)\bigr] + bt + C_3.$$

3°. The transformation

$$w = bt + u(x,\tau), \qquad \tau = \frac{1}{b}e^{bt} + \text{const}$$

leads to an equation of the form 1.2.2.4:

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial x}\left(e^u \frac{\partial u}{\partial x}\right) + ae^u.$$

6. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(e^w \frac{\partial w}{\partial x}\right) + be^{\lambda w}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\lambda-1} x + C_2, C_1^{2\lambda} t + C_3) + 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solution for $\lambda \neq 0$:

$$w(x,t) = u(z) - \frac{1}{\lambda}\ln t, \qquad z = 2\ln x + \frac{1-\lambda}{\lambda}\ln t,$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$2a\lambda e^{-z}\bigl[2(e^u u'_z)'_z - e^u u'_z\bigr] + b\lambda e^{\lambda u} = (1-\lambda)u'_z - 1.$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + b + ce^{-\lambda w}.$

Functional separable solution:

$$w = \frac{1}{\lambda}\ln\left(c\lambda t - \frac{b\lambda}{2a}x^2 + C_1 x + C_2\right),$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{\lambda w} + c + se^{-\lambda w}.$

This is a special case of equation 1.6.14.4 with $f(t) = c$ and $g(t) = s$.

Functional separable solutions:

$$w = \frac{1}{\lambda}\ln\Bigl\{e^{\alpha t}\bigl[C_1\cos(x\sqrt{\beta}\,) + C_2\sin(x\sqrt{\beta}\,)\bigr] + \gamma\Bigr\} \qquad \text{if } ab\lambda > 0,$$
$$w = \frac{1}{\lambda}\ln\Bigl\{e^{\alpha t}\bigl[C_1\cosh(x\sqrt{-\beta}\,) + C_2\sinh(x\sqrt{-\beta}\,)\bigr] + \gamma\Bigr\} \quad \text{if } ab\lambda < 0.$$

Here, C_1 and C_2 are arbitrary constants and

$$\alpha = \lambda(b\gamma + c), \qquad \beta = b\lambda/a,$$

where $\gamma = \gamma_{1,2}$ are roots of the quadratic equation $b\gamma^2 + c\gamma + s = 0$.

⊙ *Reference*: V. A. Galaktionov and S. A. Posashkov (1989).

1.2.3. Equations of the Form $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(we^{\lambda w}\dfrac{\partial w}{\partial x}\right).$

Traveling-wave solution:

$$w(x,t) = \frac{1}{\lambda}\ln\left(C_1 x + \frac{a}{\lambda}C_1^2 t + C_2\right),$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(we^{\lambda w}\dfrac{\partial w}{\partial x}\right) + b.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{\lambda}\ln\left(C_1 e^{b\lambda t}x + \frac{aC_1^2}{b\lambda^2}e^{2b\lambda t} + C_2 e^{b\lambda t}\right),$$

where C_1 and C_2 are arbitrary constants.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(we^{w}\dfrac{\partial w}{\partial x}\right) + b(w + 2).$

Functional separable solution:

$$w(x,t) = \ln\left[C_1 e^{2bt} - \frac{b}{2a}(x + C_2)^2\right],$$

where C_1 and C_2 are arbitrary constants.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(we^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{-\lambda w}.$

Traveling-wave solution:

$$w(x,t) = \frac{1}{\lambda}\ln\left[C_1x + \left(\frac{aC_1^2}{\lambda} + b\lambda\right)t + C_2\right],$$

where C_1 and C_2 are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(we^{\lambda w}\dfrac{\partial w}{\partial x}\right) + b + ce^{-\lambda w}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{\lambda}\ln\left(C_1e^{b\lambda t}x + \frac{aC_1^2}{b\lambda^2}e^{2b\lambda t} + C_2e^{b\lambda t} - \frac{c}{b}\right),$$

where C_1 and C_2 are arbitrary constants.

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[\left(awe^{\lambda w} + be^{\lambda w}\right)\dfrac{\partial w}{\partial x}\right] + b + ce^{-\lambda w}.$

The substitution $w = u - b/a$ leads to an equation of the form 1.2.3.5:

$$\frac{\partial u}{\partial t} = ae^{-b\lambda/a}\frac{\partial}{\partial x}\left(ue^{\lambda u}\frac{\partial u}{\partial x}\right) + b + ce^{b\lambda/a}e^{-\lambda u}.$$

7. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[\left(ae^{2\lambda w} + bwe^{\lambda w}\right)\dfrac{\partial w}{\partial x}\right].$

Self-similar solutions:

$$w(x,t) = \frac{1}{\lambda}\ln\left(\frac{\pm x + C_1}{\sqrt{C_2 - 2at}} - \frac{b}{a\lambda}\right),$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[\left(ae^{2\lambda w} + bwe^{\lambda w}\right)\dfrac{\partial w}{\partial x}\right] + c.$

Generalized traveling-wave solutions:

$$w(x,t) = \frac{1}{\lambda}\ln\left[\pm\varphi(t)x + C_1\varphi(t) + \frac{b}{\lambda}\varphi(t)\int\varphi(t)\,dt\right], \qquad \varphi(t) = \left(C_2e^{-2c\lambda t} - \frac{a}{c\lambda}\right)^{-1/2},$$

where C_1 and C_2 are arbitrary constants.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[w^n\exp\left(\lambda w^n\right)\dfrac{\partial w}{\partial x}\right] + bw^{1-n}.$

Generalized traveling-wave solutions:

$$w(x,t) = \left(\frac{1}{\lambda}\ln z\right)^{1/n}, \qquad z = C_1e^{bn\lambda t}x + \frac{aC_1^2}{bn^2\lambda^2}e^{2bn\lambda t} + C_2e^{bn\lambda t},$$

where C_1 and C_2 are arbitrary constants.

1.2.4. Other Equations Explicitly Independent of x and t

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \beta\dfrac{\partial w}{\partial x} + a + be^{\lambda w} + ce^{2\lambda w}$.

On passing from t, x to the new variables t, $z = x + \beta t$, one arrives at a simpler equation of the form 1.2.1.3:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial z^2} + a + be^{\lambda w} + ce^{2\lambda w}.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda w}\dfrac{\partial w}{\partial x}$.

This is a special case of equation 1.6.3.7 with $f(w) = be^{\lambda w}$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1 x + C_2, C_1^2 t + C_3) + \frac{1}{\lambda}\ln C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Apart from the traveling wave $w = w(x + \lambda t)$, there is also an exact solution of the form

$$w = \varphi(\xi) - \frac{1}{2\lambda}\ln t, \qquad \xi = \frac{x}{\sqrt{t}}.$$

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + ae^{\lambda w}\left(\dfrac{\partial w}{\partial x}\right)^2$.

This is a special case of equation 1.6.6.8 with $f(w) = ae^{\lambda w}$.

The substitution

$$u = \int \exp\left(\frac{a}{\lambda}e^{\lambda w}\right) dw$$

leads to the linear heat equation $\partial_t u = \partial_{xx} u$.

4. $\dfrac{\partial w}{\partial t} + aw\dfrac{\partial w}{\partial x} = b\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right)$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\left(e^{C_1}x + \frac{a}{\lambda}C_1 e^{C_1}t + C_2, e^{C_1}t + C_3\right) - \frac{1}{\lambda}C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution in implicit form:

$$2b\int \frac{e^{\lambda w}\,dw}{aw^2 + 2\beta w + C_1} = x + \beta t + C_2,$$

where C_1, C_2, and β are arbitrary constants.

3°. Solution:

$$w(x,t) = u(z) + \frac{1}{\lambda}\ln t, \qquad z = \frac{x}{t} - \frac{a}{\lambda}\ln t,$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$\left(au - z - \frac{a}{\lambda}\right)u'_z + \frac{1}{\lambda} = b(e^{\lambda}uu'_z)'_z.$$

5. $\dfrac{\partial w}{\partial t} = ae^{\lambda w}\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(C_1x + C_2, C_3t + C_4) + \frac{1}{\lambda}\ln\frac{C_3}{C_1^2},$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w(x,t) = \frac{1}{\lambda}\ln\left[C_2\exp\left(\frac{C_1}{ak^2}z\right) + \frac{\beta}{C_1}\right], \qquad z = kx + \beta t,$$

where C_1, C_2, k, and β are arbitrary constants.

3°. Additive separable solutions:

$$w(x,t) = \frac{1}{\lambda}\ln\left[\frac{\cos^2(C_2x + C_3)}{2C_2^2(at + C_1)}\right],$$
$$w(x,t) = \frac{1}{\lambda}\ln\left[\frac{\sinh^2(C_2x + C_3)}{2C_2^2(at + C_1)}\right],$$
$$w(x,t) = \frac{1}{\lambda}\ln\left[\frac{\cosh^2(C_2x + C_3)}{2C_2^2(C_1 - at)}\right],$$

where C_1, C_2, and C_3 are arbitrary constants; note that $\ln(A/B) = \ln|A| - \ln|B|$ for $AB > 0$.

4°. Self-similar solution:

$$w = w(y), \qquad y = x/\sqrt{t},$$

where the function $w(y)$ is determined by the ordinary differential equation

$$ae^{\lambda w}w''_{yy} + \tfrac{1}{2}yw'_y = 0.$$

5°. Solution:

$$w(x,t) = U(\xi) + 2kt, \qquad \xi = xe^{-k\lambda t},$$

where k is an arbitrary constant, and the function $U = U(\xi)$ is determined by the ordinary differential equation

$$2k - k\lambda\xi U'_\xi = ae^{\lambda U}U''_{\xi\xi}.$$

6°. Solution:

$$w(x,t) = F(\zeta) - \frac{1}{\lambda}\ln t, \qquad \zeta = x + \beta\ln t,$$

where β is an arbitrary constant, and the function $F = F(\zeta)$ is determined by the autonomous ordinary differential equation

$$\beta\lambda F'_\zeta - 1 = a\lambda e^{\lambda F}F''_{\zeta\zeta}.$$

7°. Solution:

$$w(x,t) = G(\theta) - \frac{2b+1}{\lambda}\ln t, \qquad \theta = xt^b,$$

where b is an arbitrary constant, and the function $G = G(\theta)$ is determined by the ordinary differential equation

$$-\frac{2b+1}{\lambda} + b\theta G'_\theta = ae^{\lambda G}G''_{\theta\theta}.$$

6. $\dfrac{\partial w}{\partial t} = ae^{w}\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda w}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\lambda-1}x + C_2,\, C_1^{2\lambda}t + C_3) + 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution:

$$w = w(\xi), \qquad \xi = kx + \beta t,$$

where k and β are arbitrary constants, and the function $w(\xi)$ is determined by the autonomous ordinary differential equation

$$ak^2 e^{w} w''_{\xi\xi} - \beta w'_{\xi} + be^{\lambda w} = 0.$$

3°. Solution for $\lambda \neq 0$:

$$w(x,t) = u(z) - \frac{1}{\lambda}\ln t, \qquad z = 2\ln x + \frac{1-\lambda}{\lambda}\ln t,$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$2a\lambda e^{u-z}\bigl(2u''_{zz} - u'_z\bigr) + b\lambda e^{\lambda u} = (1-\lambda)u'_z - 1.$$

4°. Additive separable solution for $\lambda = 1$:

$$w(x,t) = -\ln(kt + C) + \varphi(x),$$

where the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$a\varphi''_{xx} + b + ke^{-\varphi} = 0.$$

5°. Additive separable solutions for $\lambda = 0$:

$$w(x,t) = \ln\left[\frac{b}{2C_2^2}\,\frac{\cos^2(C_2x + C_3)}{a - C_1e^{-bt}}\right],$$

$$w(x,t) = \ln\left[\frac{b}{2C_2^2}\,\frac{\sinh^2(C_2x + C_3)}{a - C_1e^{-bt}}\right],$$

$$w(x,t) = \ln\left[\frac{b}{2C_2^2}\,\frac{\cosh^2(C_2x + C_3)}{C_1e^{-bt} - a}\right],$$

where C_1, C_2, and C_3 are arbitrary constants; note that $\ln(A/B) = \ln|A| - \ln|B|$ for $AB > 0$.

7. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[\bigl(ae^{2w} + bwe^{w}\bigr)\dfrac{\partial w}{\partial x}\right] + \bigl(ce^{w} + s\bigr)\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \ln\left[\varphi(t)x + (st + C_1)\varphi(t) + b\varphi(t)\int \varphi(t)\,dt\right],$$

where C_1 is an arbitrary constant, and the function $\varphi(t)$ is determined by the first-order separable ordinary differential equation

$$\varphi'_t = a\varphi^3 + c\varphi^2,$$

whose general solution can be written out in implicit form.

In special cases, we have

$$\varphi(t) = (C_2 - 2at)^{-1/2} \quad \text{if } c = 0,$$

$$\varphi(t) = (C_2 - ct)^{-1} \quad \text{if } a = 0.$$

1.2.5. Equations Explicitly Dependent on x and/or t

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + ce^{\lambda w+bx+ct}$.

This is a special case of equation 1.6.1.2 with $f(z,w) = ce^{z+\lambda w}$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + a\lambda\left(\dfrac{\partial w}{\partial x}\right)^2 + be^{\beta x+\mu t-\lambda w}$.

This is a special case of equation 1.6.4.9 with $f(x,t) \equiv 0$ and $g(x,t) = be^{\beta x+\mu t}$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + bt^n$.

This is a special case of equation 1.6.14.1 with $f(t) = bt^n$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{\mu t}$.

This is a special case of equation 1.6.14.1 with $f(t) = be^{\mu t}$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{\lambda w} + ce^{\mu t}$.

This is a special case of equation 1.6.14.4 with $f(t) = ce^{\mu t}$ and $g(t) = 0$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + bt^n e^{-\lambda w}$.

This is a special case of equation 1.6.14.2 with $f(t) = 0$ and $g(t) = bt^n$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{-\lambda w+\mu t}$.

This is a special case of equation 1.6.14.2 with $f(t) = 0$ and $g(t) = be^{\mu t}$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{\mu t} + ce^{-\lambda w+\nu t}$.

This is a special case of equation 1.6.14.2 with $f(t) = be^{\mu t}$ and $g(t) = ce^{\nu t}$.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + (bx+c)e^{\lambda w}$.

Additive separable solution:

$$w = -\frac{1}{\lambda}\ln(\lambda t + C) + \varphi(x),$$

where C is an arbitrary constant and the function $\varphi(x)$ is determined by the second-order linear ordinary differential equation

$$a\psi''_{xx} + \lambda(bx+c)\psi + \lambda = 0, \qquad \psi = e^{\lambda\varphi}.$$

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + be^{\lambda w+\mu x}$.

Additive separable solution:

$$w = -\frac{1}{\lambda}\ln(\lambda t + C) + \varphi(x),$$

where C is an arbitrary constant and the function $\varphi(x)$ is determined by the second-order linear ordinary differential equation

$$a\psi''_{xx} + \lambda be^{\mu x}\psi + \lambda = 0, \qquad \psi = e^{\lambda\varphi}.$$

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(x^n e^{\lambda w}\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 1.6.17.12 with $f(x) = ax^n$.

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w+\mu x}\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 1.6.17.12 with $f(x) = ae^{\mu x}$.

1.3. Equations with Hyperbolic Nonlinearities

1.3.1. Equations Involving Hyperbolic Cosine

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\cosh^k(\lambda w).$

This is a special case of equation 1.6.1.1 with $f(w) = b\cosh^k(\lambda w)$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta\cosh^k(\lambda w + bx + ct).$

This is a special case of equation 1.6.1.2 with $f(z,w) = \beta\cosh^k(z + \lambda w)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + s\cosh^k(\lambda w).$

This is a special case of equation 1.6.2.1 with $f(w) = s\cosh^k(\lambda w)$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\cosh^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 1.6.6.8 with $f(w) = b\cosh^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\cosh^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\cosh^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 1.6.6.10 with $f(w) = b\cosh^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\cosh^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + 2b\cosh(\lambda w) + c\cosh^k(\beta t).$

This is a special case of equation 1.6.14.4 with $f(t) = c\cosh^k(\beta t)$ and $g(t) = b$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\cosh^2(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = a\cosh^2(\beta w)$.

Self-similar solutions:

$$w(x,t) = \frac{1}{\beta}\,\mathrm{arcsinh}\left(\frac{\pm x + C_1}{\sqrt{C_2 - 2at}}\right),$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\cosh^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = a\cosh^k(\beta w)$.

1.3.2. Equations Involving Hyperbolic Sine

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\sinh^k(\lambda w).$

This is a special case of equation 1.6.1.1 with $f(w) = b\sinh^k(\lambda w)$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta\sinh^k(\lambda w + bx + ct).$

This is a special case of equation 1.6.1.2 with $f(z, w) = \beta\sinh^k(z + \lambda w)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + s\sinh^k(\lambda w).$

This is a special case of equation 1.6.2.1 with $f(w) = s\sinh^k(\lambda w)$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\sinh^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 1.6.6.8 with $f(w) = b\sinh^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\sinh^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\sinh^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 1.6.6.10 with $f(w) = b\sinh^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\sinh^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + 2b\sinh(\lambda w) + c\sinh^k(\beta t).$

This is a special case of equation 1.6.14.4 with $f(t) = c\sinh^k(\beta t)$ and $g(t) = -b$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\sinh^2(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = a\sinh^2(\beta w)$.

Self-similar solutions:

$$w(x, t) = \frac{1}{\beta}\operatorname{arccosh}\left(\frac{\pm x + C_1}{\sqrt{C_2 - 2at}}\right),$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\sinh^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = a\sinh^k(\beta w)$.

1.3.3. Equations Involving Hyperbolic Tangent

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\tanh^k(\lambda w).$

This is a special case of equation 1.6.1.1 with $f(w) = b\tanh^k(\lambda w)$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta\tanh^k(\lambda w + bx + ct).$

This is a special case of equation 1.6.1.2 with $f(z, w) = \beta\tanh^k(z + \lambda w)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + s\tanh^k(\lambda w).$

This is a special case of equation 1.6.2.1 with $f(w) = s\tanh^k(\lambda w)$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\tanh^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 1.6.6.8 with $f(w) = b\tanh^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\tanh^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\tanh^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 1.6.6.10 with $f(w) = b\tanh^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\tanh^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\tanh^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = a\tanh^k(\beta w)$.

1.3.4. Equations Involving Hyperbolic Cotangent

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\coth^k(\lambda w).$

This is a special case of equation 1.6.1.1 with $f(w) = b\coth^k(\lambda w)$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta\coth^k(\lambda w + bx + ct).$

This is a special case of equation 1.6.1.2 with $f(z, w) = \beta\coth^k(z + \lambda w)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + s\coth^k(\lambda w).$

This is a special case of equation 1.6.2.1 with $f(w) = s\coth^k(\lambda w)$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\coth^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 1.6.6.8 with $f(w) = b\coth^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\coth^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\coth^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 1.6.6.10 with $f(w) = b\coth^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\coth^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\coth^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = a\coth^k(\beta w)$.

1.4. Equations with Logarithmic Nonlinearities

1.4.1. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x, t, w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\ln w.$

This is a special case of equation 1.6.1.1 with $f(w) = b\ln w$.

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw \ln w.$

This is a special case of equation 1.6.1.1 with $f(w) = aw \ln w$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = \exp(C_1 e^{at}) w(\pm x + C_2, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solutions:

$$w(x,t) = \exp\left(Ae^{at}x + \frac{A^2}{a}e^{2at} + Be^{at}\right),$$
$$w(x,t) = \exp\left[\tfrac{1}{2} - \tfrac{1}{4}a(x+A)^2 + Be^{at}\right],$$
$$w(x,t) = \exp\left[-\frac{a(x+A)^2}{4(1+Be^{-at})} + \frac{1}{2B}e^{at}\ln(1+Be^{-at}) + Ce^{at}\right],$$

where A, B, and C are arbitrary constants.

⊙ *References*: V. A. Dorodnitsyn (1979, 1982), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

3°. Multiplicative separable solution:

$$w(x,t) = \exp\left[Ae^{at} + f(x)\right], \qquad (1)$$

where the function $f(x)$ is defined implicitly by

$$\int \left(Be^{-2f} - af + \tfrac{1}{2}a\right)^{-1/2} df = \pm x + C. \qquad (2)$$

Relations (1) and (2) involve three arbitrary constants, A, B, and C.

4°. There are more complicated solutions of the form

$$w(x,t) = \exp\left[Ae^{at} + f(x+bt)\right],$$

where the function $f(\xi)$ is determined by the autonomous ordinary differential equation

$$f''_{\xi\xi} + (f'_\xi)^2 - bf'_\xi + af = 0.$$

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw \ln w + bw.$

The substitution $w = e^{-b/a}u$ leads to an equation of the form 1.4.1.2:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au \ln u.$$

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw \ln w + (bx + c)w.$

This is a special case of equations 1.6.1.5 and 1.6.1.7.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw \ln w + (bx + ct + k)w.$

This is a special case of equation 1.6.1.7.

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + aw \ln w + (bx^2 + cx + k)w.$

This is a special case of equation 1.6.1.9.

7. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + a(w+b)\ln^2(w+b).$

1°. The substitution $w = e^u - b$ leads to an equation of the form 1.1.7.4:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + au^2. \tag{1}$$

2°. Solutions of equation (1) for $a < 0$:

$$u(x,t) = C_1 \exp\bigl(-at \pm x\sqrt{-a}\,\bigr),$$
$$u(x,t) = \frac{1}{C_1 - at} + \frac{C_2}{(C_1 - at)^2} \exp\bigl(-at \pm x\sqrt{-a}\,\bigr),$$

where C_1 and C_2 are arbitrary constants. The first solution is a traveling-wave solution and the second one is a generalized separable solution.

3°. Equation (1) has also generalized separable solutions of the following forms:

$$u(x,t) = \varphi(t) + \psi(t)\bigl[A\exp(x\sqrt{-a}\,) + A\exp(x\sqrt{-a}\,)\bigr] \quad \text{if} \quad a < 0,$$
$$u(x,t) = \varphi(t) + \psi(t)\bigl[A\sin(x\sqrt{a}\,) + A\cos(x\sqrt{a}\,)\bigr] \qquad \text{if} \quad a > 0.$$

For details, see 1.1.4.4.

⊙ *References*: V. A. Galaktionov and S. A. Posashkov (1989), A. D. Polyanin and V. F. Zaitsev (2002).

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + (1+kw)\bigl[a\ln^2(1+kw) + b\ln(1+kw) + c\bigr].$

This is a special case of equation 1.6.1.10.

⊙ *Reference*: V. A. Galaktionov and S. A. Posashkov (1989).

1.4.2. Other Equations

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial w}{\partial x} + bw\ln w.$

On passing from t, x to the new variables t, $z = x + at$, one obtains a simpler solution of the form 1.4.1.2:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial z^2} + bw\ln w.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bt\dfrac{\partial w}{\partial x} + cw\ln w.$

This is a special case of equation 1.6.2.6 with $f(t) = 0$, $g(t) = bt$, $h(t) = c$, and $p(t) = s(t) = 0$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bx\dfrac{\partial w}{\partial x} + cw\ln w.$

This is a special case of equation 1.6.2.6 with $f(t) = b$, $g(t) = 0$, $h(t) = c$, and $p(t) = s(t) = 0$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{a}{x^k}\dfrac{\partial}{\partial x}\left(x^k\dfrac{\partial w}{\partial x}\right) + bw\ln w.$

The values $k = 1$ and $k = 2$ correspond to problems with axial and central symmetry, respectively.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = \exp(C_1 e^{bt})\, w(\pm x, t + C_2),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution:

$$w(x,t) = \exp\left[-\frac{bx^2}{4a(1+Ae^{-bt})} + Be^{bt} + \frac{1}{2A}(k+1)e^{bt}\ln(1+Ae^{-bt})\right],$$

where A and B are arbitrary constants.

⊙ *Reference*: A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

3°. Multiplicative separable solution:

$$w(x,t) = \exp\bigl(Ae^{bt}\bigr)\theta(x),$$

where A is an arbitrary constant and the function $\theta(x)$ is determined by the second-order ordinary differential equation

$$\frac{a}{x^k}\frac{d}{dx}\left(x^k\frac{d\theta}{dx}\right) + b\theta\ln\theta = 0.$$

5. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (b\ln w + c)\frac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = e^{C_1}w(x + bC_1t + C_2, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w(x,t) = \exp\left[C_1\exp\left(-\frac{b}{a}x + b^2C_2t\right) + 1 - aC_2 - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

3°. Generalized traveling-wave solution:

$$w(x,t) = \exp\left[\frac{C_1 - x}{b(t+C_2)} + \frac{a}{b^2}\frac{\ln|t+C_2|}{t+C_2} - \frac{c}{b}\right].$$

6. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + a\ln^k(bw)\left(\frac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 1.6.6.8 with $f(w) = a\ln^k(bw)$.

7. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[(a\ln w + b)\frac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = a\ln w + b$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1x + C_2, C_1^2t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t) = \exp\left(\pm\sqrt{2C_1x + 2aC_1^2t + C_2} - \frac{b}{a}\right),$$

$$w(x,t) = \exp\left(\frac{C_2 \pm x}{\sqrt{C_1 - 2at}} - \frac{a+b}{a}\right),$$

where C_1 and C_2 are arbitrary constants. The first solution represents a traveling wave and the second one is self-similar.

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[(a \ln w + b)\dfrac{\partial w}{\partial x}\right] + cw.$

Generalized traveling-wave solution:

$$w(x,t) = \exp\left[\frac{C_2 \pm x}{\sqrt{C_1 - 2at}} - \frac{c}{3a}(C_1 - 2at) - \frac{a+b}{a}\right],$$

where C_1 and C_2 are arbitrary constants.

9. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[(a \ln w + b)\dfrac{\partial w}{\partial x}\right] + cw \ln w + sw.$

Generalized traveling-wave solution:

$$w(x,t) = \exp\left[\varphi(t)(C_1 \pm x) + (a+b)\varphi(t)\int \varphi(t)\,dt + s\varphi(t)\int \frac{dt}{\varphi(t)}\right],$$
$$\varphi(t) = \left(C_2 e^{-2ct} - \frac{a}{c}\right)^{-1/2},$$

where C_1 and C_2 are arbitrary constants.

1.5. Equations with Trigonometric Nonlinearities

1.5.1. Equations Involving Cosine

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\cos^k(\lambda w).$

This is a special case of equation 1.6.1.1 with $f(w) = b\cos^k(\lambda w)$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta\cos^k(\lambda w + bx + ct).$

This is a special case of equation 1.6.1.2 with $f(z,w) = \beta\cos^k(z + \lambda w)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + s\cos^k(\lambda w).$

This is a special case of equation 1.6.2.1 with $f(w) = s\cos^k(\lambda w)$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\cos^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 1.6.6.8 with $f(w) = b\cos^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\cos^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\cos^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 1.6.6.10 with $f(w) = b\cos^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\cos^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\cos^2(\lambda w + \beta)\dfrac{\partial^2 w}{\partial x^2}.$

The substitution $u = \tan(\lambda w + \beta)$ leads to an equation of the form 1.1.13.2:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(\frac{1}{u^2+1}\frac{\partial u}{\partial x}\right).$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\cos^2(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = a\cos^2(\beta w)$.

Self-similar solutions:

$$w(x,t) = \frac{1}{\beta}\arcsin\frac{\pm x + C_1}{\sqrt{2at + C_2}},$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\cos^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = a\cos^k(\beta w)$.

1.5.2. Equations Involving Sine

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\sin^k(\lambda w).$

This is a special case of equation 1.6.1.1 with $f(w) = b\sin^k(\lambda w)$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta\sin^k(\lambda w + bx + ct).$

This is a special case of equation 1.6.1.2 with $f(z,w) = \beta\sin^k(z + \lambda w)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + s\sin^k(\lambda w).$

This is a special case of equation 1.6.2.1 with $f(w) = s\sin^k(\lambda w)$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\sin^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 1.6.6.8 with $f(w) = b\sin^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\sin^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\sin^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 1.6.6.10 with $f(w) = b\sin^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\sin^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\sin^2(\lambda w)\dfrac{\partial^2 w}{\partial x^2}.$

The substitution $u = \cot(\lambda w)$ leads to an equation of the form 1.1.13.2:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(\frac{1}{u^2+1}\frac{\partial u}{\partial x}\right).$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\sin^2(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = a\sin^2(\beta w)$.

Self-similar solutions:

$$w(x,t) = \frac{1}{\beta}\arccos\frac{\pm x + C_1}{\sqrt{2at + C_2}},$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\sin^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = a\sin^k(\beta w)$.

1.5.3. Equations Involving Tangent

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\tan^k(\lambda w).$

This is a special case of equation 1.6.1.1 with $f(w) = b\tan^k(\lambda w)$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta\tan^k(\lambda w + bx + ct).$

This is a special case of equation 1.6.1.2 with $f(z, w) = \beta\tan^k(z + \lambda w)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + s\tan^k(\lambda w).$

This is a special case of equation 1.6.2.1 with $f(w) = s\tan^k(\lambda w)$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\tan^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 1.6.6.8 with $f(w) = b\tan^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\tan^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\tan^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 1.6.6.10 with $f(w) = b\tan^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\tan^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\tan^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = a\tan^k(\beta w)$.

1.5.4. Equations Involving Cotangent

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\cot^k(\lambda w).$

This is a special case of equation 1.6.1.1 with $f(w) = b\cot^k(\lambda w)$.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \beta\cot^k(\lambda w + bx + ct).$

This is a special case of equation 1.6.1.2 with $f(z, w) = \beta\cot^k(z + \lambda w)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx + c)\dfrac{\partial w}{\partial x} + s\cot^k(\lambda w).$

This is a special case of equation 1.6.2.1 with $f(w) = s\cot^k(\lambda w)$.

4. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\cot^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 1.6.6.8 with $f(w) = b\cot^k(\lambda w)$.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + b\cot^k(\lambda w)\left(\dfrac{\partial w}{\partial x}\right)^2 + c\cot^k(\beta t)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 1.6.6.10 with $f(w) = b\cot^k(\lambda w)$, $g(t) = 0$, and $h(t) = c\cot^k(\beta t)$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left[\cot^k(\beta w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 1.6.15.1 with $f(w) = a\cot^k(\beta w)$.

1.5.5. Equations Involving Inverse Trigonometric Functions

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(\dfrac{e^{\lambda \arctan w}}{1+w^2}\dfrac{\partial w}{\partial x}\right).$

Solution in parametric form:

$$w = \tan\left(\varphi(z) + \arctan\bigl(2z\varphi'_z\bigr) - \frac{1}{\lambda}\ln t\right),$$
$$x^2 = z\cos^2\left(\varphi(z) + \frac{1}{\lambda}\ln t\right),$$

where z is the parameter and the function $\varphi = \varphi(z)$ is determined by the first-order ordinary differential equation (C is an arbitrary constant)

$$\varphi'_z = \frac{1}{2z}\tan\left(\frac{1}{\lambda}\ln\frac{C-z}{2} - \varphi\right) - \frac{1}{\lambda(C-z)}.$$

⊙ *Reference*: I. Sh. Akhatov, R. K. Gazizov, and N. H. Ibragimov (1989).

2. $\dfrac{\partial w}{\partial t} = \left[\left(\dfrac{\partial w}{\partial x}\right)^2 + 1\right]^{-1} \exp\left[k\arctan\left(\dfrac{\partial w}{\partial x}\right)\right]\dfrac{\partial^2 w}{\partial x^2}, \qquad k \neq 0.$

Solution:

$$w^2 = u(z) - x^2, \qquad z = t\exp\bigl[-k\arctan(x/w)\bigr],$$

where the function $u(z)$ is determined by the ordinary differential equation

$$2k^2z^2uu''_{zz} - k^2z(3zu'_z - 2u)u'_z - 4u^2 - \tfrac{1}{2}(k^2z^2{u'_z}^2 + 4u^2)u'_z\exp\bigl[k\arctan\bigl(\tfrac{1}{2}kzu^{-1}u'_z\bigr)\bigr] = 0.$$

1.6. Equations Involving Arbitrary Functions

1.6.1. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(w).$

Kolmogorov–Petrovskii–Piskunov equation. Equations of this form are often encountered in various problems of heat and mass transfer (with f being the rate of a volume chemical reaction), combustion theory, biology, and ecology. For $f = f(w)$ having power-law, exponential, or logarithmic form, see Subsections 1.1.1 to 1.1.3, equations 1.2.1.1 to 1.2.1.3, or equations 1.4.1.2, 1.4.1.3, 1.4.1.7, and 1.4.1.8, respectively.

1°. The equation has an implicit nonstationary solution independent of the space variable:

$$\int\frac{dw}{f(w)} = t + C, \qquad C \text{ is an arbitrary constant.}$$

2°. Stationary solution in implicit form:

$$\int\left[C_1 - \frac{2}{a}\int f(w)\,dw\right]^{-1/2}dw = C_2 \pm x.$$

3°. Traveling-wave solutions:

$$w = w(z), \quad z = \pm x + \lambda t,$$

where λ is an arbitrary constant. The function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$aw''_{zz} - \lambda w'_z + f(w) = 0. \tag{1}$$

The transformation

$$\xi = (\lambda/a)z, \quad U(w) = w'_\xi$$

brings (1) to the Abel equation

$$UU'_w - U + a\lambda^{-2} f(w) = 0. \tag{2}$$

The book by Polyanin and Zaitsev (2003) presents a considerable number of solutions to equation (2) for various $f = f(w)$.

4°. Subsection S.5.2 (e.g., see Example 1) presents an exact solution of this equation with $f(w)$ defined parametrically.

2. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(bx + ct, w).$

Solution:

$$w = w(\xi), \qquad \xi = bx + ct,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$ab^2 w''_{\xi\xi} - cw'_\xi + f(\xi, w) = 0.$$

3. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \frac{1}{t} f\left(\frac{x}{\sqrt{t}}, w\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(Cx, C^2 t),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. The transformation

$$\tau = \ln t, \quad \xi = \frac{x}{\sqrt{t}}$$

leads to the equation

$$\frac{\partial w}{\partial \tau} = a\frac{\partial^2 w}{\partial \xi^2} + \frac{1}{2}\xi\frac{\partial w}{\partial \xi} + f(\xi,\ w),$$

which admits exact solutions of the form $w = w(\xi)$.

4. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw \ln w + f(t)w.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = \exp\bigl(C_1 e^{bt}\bigr) w\bigl(\pm x + C_2, t\bigr),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Generalized traveling-wave solution:

$$w(x,t) = \exp\left[Ae^{bt}x + Be^{bt} + \frac{a}{b}A^2 e^{2bt} + e^{bt}\int e^{-bt} f(t)\,dt\right],$$

where A and B are arbitrary constants.

3°. Functional separable solution:

$$w(x,t) = \exp\bigl[\varphi(t)x^2 + \psi(t)\bigr].$$

Here, the functions $\varphi(t)$ and $\psi(t)$ are given by

$$\varphi(t) = \frac{be^{bt}}{A - 4ae^{bt}}, \quad \psi(t) = Be^{bt} + e^{bt}\int e^{-bt}\bigl[2a\varphi(t) + f(t)\bigr]\,dt,$$

where A and B are arbitrary constants.

4°. There are also functional separable solutions of the more general form

$$w(x,t) = \exp\bigl[\varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t)\bigr],$$

where the functions $\varphi_2(t)$, $\varphi_1(t)$, and $\varphi_0(t)$ are determined by a system of ordinary differential equations (see equation 1.6.1.9) that can be integrated.

5°. Solution:

$$w(x,t) = \exp\left[Ae^{bt} + e^{bt}\int e^{-bt}f(t)\,dt + \Phi(x+\lambda t)\right],$$

where A and λ are arbitrary constants, and the function $\Phi = \Phi(z)$ is determined by the autonomous ordinary differential equation

$$a\Phi''_{zz} + a(\Phi'_z)^2 - \lambda\Phi'_z + b\Phi = 0,$$

the order of which can be reduced by one.

6°. The substitution

$$w(x,t) = \exp\left[e^{bt}\int e^{-bt}f(t)\,dt\right]u(x,t)$$

leads to a simpler equation of the form 1.4.1.2:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bu\ln u.$$

5. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw\ln w + \bigl[f(x) + g(t)\bigr]w.$

1°. Multiplicative separable solution:

$$w(x,t) = \exp\left[Ce^{bt} + e^{bt}\int e^{-bt}g(t)\,dt\right]\varphi(x),$$

where C is an arbitrary constant, and the function $\varphi(t)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + b\varphi\ln\varphi + f(x)\varphi = 0.$$

2°. With the substitution

$$w(x,t) = \exp\left[e^{bt}\int e^{-bt}g(t)\,dt\right]u(x,t)$$

one arrives at the simpler equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bu\ln u + f(x)u.$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)w\ln w + g(t)w.$

1°. Generalized traveling-wave solution:

$$w(x,t) = \exp\bigl[\Phi(t)x + \Psi(t)\bigr],$$

where the functions $\Phi(t)$ and $\Psi(t)$ are given by

$$\Phi(t) = Ae^F, \quad \Psi(t) = Be^F + e^F \int e^{-F}(aA^2e^{2F} + g)\,dt, \qquad F = \int f\,dt,$$

and A and B are arbitrary constants.

2°. Functional separable solution:

$$w(x,t) = \exp\bigl[\varphi(t)x^2 + \psi(t)\bigr],$$

where the functions $\varphi(t)$ and $\psi(t)$ are given by

$$\varphi(t) = e^F\left(A - 4a\int e^F\,dt\right)^{-1}, \quad F = \int f\,dt,$$
$$\psi(t) = Be^F + e^F \int e^{-F}(2a\varphi + g)\,dt,$$

and A and B are arbitrary constants.

3°. There are also functional separable solutions of the more general form

$$w(x,t) = \exp\bigl[\varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t)\bigr],$$

where the functions $\varphi_2(t)$, $\varphi_1(t)$, and $\varphi_0(t)$ are determined by a system of ordinary differential equations (see equation 1.6.1.9), which can be integrated.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)w\ln w + \bigl[g(t)x + h(t)\bigr]w.$

1°. Generalized traveling-wave solution:

$$w(x,t) = \exp\bigl[\varphi(t)x + \psi(t)\bigr],$$

where the functions $\varphi(t)$ and $\psi(t)$ are given by

$$\varphi(t) = Ae^F + e^F\int e^{-F}g\,dt, \quad F = \int f\,dt,$$
$$\psi(t) = Be^F + e^F \int e^{-F}(a\varphi^2 + h)\,dt,$$

and A and B are arbitrary constants.

2°. There are also functional separable solutions of the form

$$w(x,t) = \exp\bigl[\varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t)\bigr],$$

where the functions $\varphi_2(t)$, $\varphi_1(t)$, and $\varphi_0(t)$ are determined by a system of ordinary differential equations (see equation 1.6.1.9), which can be integrated.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)w\ln w + \big[bf(x)t + g(x)\big]w.$

Multiplicative separable solution:

$$w(x,t) = e^{-bt}\exp\big[\varphi(x)\big],$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + a(\varphi'_x)^2 + f(x)\varphi + g(x) + b = 0.$$

For f, $g = \text{const}$, this equation can be reduced by the substitution $u(\varphi) = (\varphi'_x)^2$ to a first-order linear equation.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)w\ln w + \big[g(t)x^2 + h(t)x + s(t)\big]w.$

Functional separable solution:

$$w(x,t) = \exp\big[\varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t)\big],$$

where the functions $\varphi_n(t)$ $(n = 1,\,2,\,3)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_2 = 4a\varphi_2^2 + f\varphi_2 + g, \qquad (1)$$

$$\varphi'_1 = 4a\varphi_2\varphi_1 + f\varphi_1 + h, \qquad (2)$$

$$\varphi'_0 = f\varphi_0 + a\varphi_1^2 + 2a\varphi_2 + s. \qquad (3)$$

Here, the arguments of the functions f, g, h, and s are not specified, and the prime denotes a derivative with respect to t.

Equation (1) for $\varphi_2 = \varphi_2(t)$ is a Riccati equation, so it can be reduced to a second-order linear equation. The books by Kamke (1977), Polyanin and Zaitsev (2003) present a large number of solutions to this equation for various f and g.

Given a solution of equation (1), the solutions of equations (2) and (3) can be constructed successively, due to the linearity of each of them in the unknown.

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bw+c)\big[k\ln^2(bw+c) + f(t)\ln(bw+c) + g(t)\big].$

The substitution

$$bw + c = \exp u, \qquad u = u(x,t)$$

leads to an equation of the form 1.6.6.2:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + a\left(\frac{\partial u}{\partial x}\right)^2 + bku^2 + bf(t)u + bg(t),$$

which has exponential and sinusoidal solutions with respect to x.

1.6.2. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + (bx+c)\dfrac{\partial w}{\partial x} + f(w).$

This equation governs unsteady mass transfer with a volume chemical reaction in an inhomogeneous fluid flow.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1 e^{-bt}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + C_1 e^{-bt},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$aw''_{zz} + (bz + c)w'_z + f(w) = 0.$$

2. $\displaystyle \frac{\partial w}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial w}{\partial x}\right) + f(t) w \ln w.$

This equation can be rewritten as

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \frac{an}{x}\frac{\partial w}{\partial x} + f(t) w \ln w.$$

Functional separable solution:

$$w(x,t) = \exp\bigl[\varphi(t)x^2 + \psi(t)\bigr],$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = 4a\varphi^2 + f\varphi,$$
$$\psi'_t = 2a(n+1)\varphi + f\psi;$$

the arguments of the functions f and g are not specified. Integrating the first equation and then the second, we obtain

$$\varphi(t) = e^F\left(A - 4a\int e^F\,dt\right)^{-1}, \quad F = \int f\,dt,$$
$$\psi(t) = Be^F + 2a(n+1)e^F\int \varphi e^{-F}\,dt,$$

where A and B are arbitrary constants.

3. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(t)\frac{\partial w}{\partial x} + g(w).$

On passing from t, x to the new variables t, $z = x + \int f(t)\,dt$, one obtains a simpler equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + g(w),$$

which has a traveling-wave solution $w = w(kz + \lambda t)$.

4. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(t)\frac{\partial w}{\partial x} + g(t, w).$

On passing from t, x to the new variables t, $z = x + \int f(t)\,dt$, one obtains a simpler equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + g(t, w).$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\dfrac{\partial w}{\partial x} + bw\ln w + \big[g(x) + h(t)\big]w.$

Multiplicative separable solution:

$$w(x,t) = \exp\left[Ce^{bt} + e^{bt}\int e^{-bt}h(t)\,dt\right]\varphi(x),$$

where C is an arbitrary constant, and the function $\varphi(t)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + f(x)\varphi'_x + b\varphi\ln\varphi + g(x)\varphi = 0.$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[xf(t) + g(t)\big]\dfrac{\partial w}{\partial x} + h(t)w\ln w + \big[xp(t) + s(t)\big]w.$

1°. Generalized traveling-wave solution:

$$w(x,t) = \exp\big[x\varphi(t) + \psi(t)\big],$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = \big[f(t) + h(t)\big]\varphi + p(t), \tag{1}$$
$$\psi'_t = h(t)\psi + a\varphi^2 + g(t)\varphi + s(t). \tag{2}$$

Integrating (1) and then (2), we obtain (C_1 and C_2 are arbitrary constants)

$$\varphi(t) = C_1E(t) + E(t)\int\frac{p(t)}{E(t)}\,dt, \qquad E(t) = \exp\left[\int f(t)\,dt + \int h(t)\,dt\right],$$
$$\psi(t) = C_2H(t) + H(t)\int\frac{a\varphi^2(t) + g(t)\varphi(t) + s(t)}{H(t)}\,dt, \qquad H(t) = \exp\left[\int h(t)\,dt\right].$$

2°. See equation 1.6.2.7 with $r(t) = 0$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[xf(t) + g(t)\big]\dfrac{\partial w}{\partial x} + h(t)w\ln w + \big[x^2r(t) + xp(t) + s(t)\big]w.$

Functional separable solution:

$$w(x,t) = \exp\big[x^2\varphi(t) + x\psi(t) + \chi(t)\big],$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = 4a\varphi^2 + (2f + h)\varphi + r, \tag{1}$$
$$\psi'_t = (4a\varphi + f + h)\psi + 2g\varphi + p, \tag{2}$$
$$\chi'_t = h\chi + 2a\varphi + a\psi^2 + g\psi + s. \tag{3}$$

For $r \equiv 0$, equation (1) is a Bernoulli equation, so it is easy to integrate. In the general case, equation (1) for $\varphi = \varphi(t)$ is a Riccati equation, so it can be reduced to a second-order linear equation. The books by Kamke (1977) and Polyanin and Zaitsev (2003) present a large number of solutions to the Riccati equation for various f, h, and r. With equation (1) solved, the solutions of equations (2) and (3) can be obtained with ease, since these are linear in their unknowns $\psi = \psi(t)$ and $\chi = \chi(t)$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \left[xf(t) + \dfrac{g(t)}{x}\right]\dfrac{\partial w}{\partial x} + h(t)w\ln w + \left[x^2 p(t) + s(t)\right]w.$

Functional separable solution:

$$w(x,t) = \exp\left[\varphi(t)x^2 + \psi(t)\right],$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = 4a\varphi^2 + (2f + h)\varphi + p, \qquad (1)$$

$$\psi'_t = h\psi + 2(a + g)\varphi + s. \qquad (2)$$

For $p \equiv 0$, equation (1) is a Bernoulli equation, so it is easy to integrate. In the general case, equation (1) for $\varphi = \varphi(t)$ is a Riccati equation, so it can be reduced to a second-order linear equation. The books by Kamke (1977) and Polyanin and Zaitsev (2003) present a large number of solutions to the Riccati equation for various f, h, and r. With equation (1) solved, the solution of the linear equation (2) can be obtained with ease.

⊙ *Reference*: A. D. Polyanin (2002).

1.6.3. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x} + f(t).$

The transformation

$$w = u(z,t) + \int_{t_0}^{t} f(\tau)\,d\tau, \qquad z = x + b\int_{t_0}^{t}(t-\tau)f(\tau)\,d\tau,$$

where t_0 is any number, leads to the Burgers equation 1.1.5.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bu\frac{\partial u}{\partial x}.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + bw\dfrac{\partial w}{\partial x} + f(x,t).$

Let us substitute

$$w = \frac{\partial u}{\partial x}$$

and then integrate the resulting equation with respect to x to arrive at an equation of the form 1.6.4.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + \frac{b}{2}\left(\frac{\partial u}{\partial x}\right)^2 + F(x,t),$$

where $F(x,t) = \displaystyle\int f(x,t)\,dx + g(t)$ with $g(t)$ being an arbitrary function.

⊙ *Reference*: A. R. Fortsyth (1906).

3. $\dfrac{\partial w}{\partial t} + bw\dfrac{\partial w}{\partial x} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial x}\left[f(x,t)w\right].$

Solution:

$$w(x,t) = -\frac{2a}{b}\frac{1}{\theta}\frac{\partial \theta}{\partial x},$$

where the function $\theta = \theta(x,t)$ satisfies the linear equation

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2} + f(x,t)\frac{\partial \theta}{\partial x}.$$

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[bw + f(t)\big]\dfrac{\partial w}{\partial x} + g(t).$

The transformation

$$w = u(z,t) + \int_{t_0}^{t} g(\tau)\,d\tau, \qquad z = x + \int_{t_0}^{t} f(\tau)\,d\tau + b\int_{t_0}^{t}(t-\tau)g(\tau)\,d\tau,$$

where t_0 is any number, leads to the Burgers equation 1.1.5.3:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bu\frac{\partial u}{\partial x}.$$

5. $\dfrac{\partial w}{\partial t} + f(t)w\dfrac{\partial w}{\partial x} = a\dfrac{\partial^2 w}{\partial x^2} + g(t)w + h(t).$

Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\big(x + \psi(t),\, t\big) + \varphi(t), \quad \varphi(t) = C\exp\left[\int g(t)\,dt\right], \quad \psi(t) = -\int f(t)\varphi(t)\,dt,$$

where C is an arbitrary constant, is also a solution of the equation.

Remark. This remains true if the equation coefficient a is an arbitrary function of time, $a = a(t)$.

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6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[f(t)\ln w + g(t)\big]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \exp\left\{\varphi(t)(x + C_1) + \varphi(t)\int\big[a\varphi(t) + g(t)\big]\,dt\right\}, \qquad \varphi(t) = -\left[\int f(t)\,dt + C_2\right]^{-1},$$

where C_1 and C_2 are arbitrary constants.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(w)\dfrac{\partial w}{\partial x}.$

Traveling-wave solution:

$$w = w(z), \qquad z = x + \lambda t$$

are defined implicitly by

$$a\int\frac{dw}{\lambda w - F(w) + A} = z + B, \qquad F(w) = \int f(w)\,dw,$$

where A and B are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(w)\dfrac{\partial w}{\partial x} + g(w).$

Traveling-wave solution:

$$w = w(z), \qquad z = x + \lambda t,$$

where the function $w = w(z)$ is determined by the second-order autonomous ordinary differential equation

$$aw''_{zz} + \big[f(w) - \lambda\big]w'_z + g(w) = 0,$$

which can be reduced with the change of variable $w'_z = u(w)$ to a first-order equation. For exact solutions of the above ordinary differential equation with various $f(w)$ and $g(w)$, see Polyanin and Zaitsev (2003).

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[f(w) + bx\big]\dfrac{\partial w}{\partial x} + g(w).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1 e^{-bt}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + C_1 e^{-bt},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$aw''_{zz} + \big[f(w) + bz\big]w'_z + g(w) = 0.$$

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[f(w) + g(t)\big]\dfrac{\partial w}{\partial x}.$

On passing from t, x to the new variables t, $z = x + \int g(t)\,dt$, we obtain an equation of the form 1.6.3.7:

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + f(w)\frac{\partial w}{\partial z}.$$

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[f(w) + g(t)\big]\dfrac{\partial w}{\partial x} + h(w).$

On passing from t, x to the new variables t, $z = x + \int g(t)\,dt$, one obtains a simpler equation of the form 1.6.3.8:

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + f(w)\frac{\partial w}{\partial z} + h(w).$$

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \big[f(w) + g(t) + bx\big]\dfrac{\partial w}{\partial x} + h(w).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + Ce^{-bt}, t),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + Ce^{-bt} + e^{-bt}\int e^{bt} g(t)\,dt,$$

where the function $w(z)$ is determined by the ordinary differential equation

$$aw''_{zz} + \big[f(w) + bz\big]w'_z + h(w) = 0.$$

13. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(bx + ct, w)\dfrac{\partial w}{\partial x} + g(bx + ct, w).$

Solution:

$$w = w(\xi), \qquad \xi = bx + ct,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$ab^2 w''_{\xi\xi} + \big[bf(\xi, w) - c\big]w'_\xi + g(\xi, w) = 0.$$

1.6.4. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + f(x,t,w)$

1. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + f(x) + g(t).$

This is a special case of equation 1.6.4.3.

Additive separable solution:

$$w(x,t) = At + B + \int g(t)\,dt + \varphi(x).$$

Here, A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + b\bigl(\varphi'_x\bigr)^2 + f(x) - A = 0,$$

which can be reduced, with the change of variable $\varphi'_x = \dfrac{a}{b}\dfrac{\psi'_x}{\psi}$, to a second-order linear equation:

$$\psi''_{xx} + ba^{-2}\bigl[f(x) - A\bigr]\psi = 0.$$

2. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + f(t)x^2 + g(t)x + h(t).$

This is a special case of equation 1.6.4.3.

Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = 4b\varphi^2 + f, \qquad (1)$$
$$\psi'_t = 4b\varphi\psi + g, \qquad (2)$$
$$\chi'_t = 2a\varphi + b\psi^2 + h. \qquad (3)$$

Equation (1) for φ is a Riccati equation. In the special case $f = \text{const}$, it can be easily integrated by separation of variables. Having determined φ, one finds the solutions of equation (2) and then (3), which are linear in the unknowns ψ and χ, respectively.

3. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + f(x,t).$

The substitution $u = \exp\left(\dfrac{b}{a}w\right)$ leads to a linear equation for $u = u(x,t)$:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + \frac{b}{a}f(x,t)u.$$

4. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + cw + f(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, t) + C_2 e^{ct},$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution quadratic in x:

$$w(x,t) = \frac{c(x+C_2)^2}{C_1 e^{-ct} - 4b} - \frac{2a}{C_1} e^{ct} \ln |C_1 e^{-ct} - 4b| + C_3 e^{ct} + e^{ct} \int e^{-ct} f(t)\,dt.$$

3°. Solution:

$$w(x,t) = Ae^{ct} + e^{ct} \int e^{-ct} f(t)\,dt + \Theta(\xi), \quad \xi = x + \lambda t,$$

where A and λ are arbitrary constants, and the function $\Theta(\xi)$ is determined by the autonomous ordinary differential equation

$$a\Theta''_{\xi\xi} + b\big(\Theta'_\xi\big)^2 - \lambda\Theta'_\xi + c\Theta = 0.$$

4°. The substitution

$$w = U(x,t) + e^{ct} \int e^{-ct} f(t)\,dt$$

leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^2 U}{\partial x^2} + b\left(\frac{\partial U}{\partial x}\right)^2 + cU.$$

⦿ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

5. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + cw + f(x).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x, t + C_1) + C_2 e^{ct},$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = Ae^{ct} + \varphi(x),$$

where A is an arbitrary constant and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + b(\varphi'_x)^2 + c\varphi + f(x) = 0.$$

6. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + cw + f(x) + g(t).$

Additive separable solution:

$$w(x,t) = \varphi(x) + Ae^{ct} + e^{ct} \int e^{-ct} g(t)\,dt,$$

where A is an arbitrary constant and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + b(\varphi'_x)^2 + c\varphi + f(x) = 0.$$

7. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + g(t)w + h(t).$

This is a special case of equation 1.6.6.1 with $f(t) = b$.

8. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + cw^2 + f(t)w + g(t).$

This is a special case of equation 1.6.6.2.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + a\lambda\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x,t) + g(x,t)e^{-\lambda w}.$

The substitution $u = \exp(\lambda w)$ leads to the linear equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + \lambda f(x,t)u + \lambda g(x,t).$$

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + \dfrac{1}{t}f\left(\dfrac{x}{\sqrt{t}},\, w\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(Cx, C^2 t),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. The transformation $\tau = \ln t$, $\xi = xt^{-1/2}$ leads to the equation

$$\frac{\partial w}{\partial \tau} = a\frac{\partial^2 w}{\partial \xi^2} + b\left(\frac{\partial w}{\partial \xi}\right)^2 + \frac{1}{2}\xi\frac{\partial w}{\partial \xi} + f(\xi, w),$$

which admits an exact solution of the form $w = w(\xi)$.

3°. In the special case $f = f(\xi)$, there is also a exact solution of the form $w = C\tau + \varphi(\xi)$, where C is an arbitrary constant, and the function $\varphi(\xi)$ is determined by the ordinary differential equation

$$a\varphi''_{\xi\xi} + b(\varphi'_\xi)^2 + \tfrac{1}{2}\xi\varphi'_\xi + f(\xi) - C = 0.$$

1.6.5. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x,t,w)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x)\dfrac{\partial w}{\partial x} + kw + g(x) + h(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{kt},$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ce^{kt} + e^{kt}\int e^{-kt}h(t)\,dt,$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + b(\varphi'_x)^2 + f(x)\varphi'_x + k\varphi + g(x) = 0.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(t)\dfrac{\partial w}{\partial x} + cw^2 + g(t)w + h(t).$

This is a special case of equation 1.6.6.5.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw\dfrac{\partial w}{\partial x} + kw^2 + f(t)w + g(t).$

There are generalized separable solutions of the form

$$w(x,t) = \varphi(t) + \psi(t)\exp(\lambda x),$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation $b\lambda^2 + c\lambda + k = 0$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t).$

The substitution $u = \exp\left(\dfrac{b}{a}w\right)$ leads to the linear equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(x,t)\frac{\partial u}{\partial x} + \frac{b}{a}g(x,t)u.$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + a\lambda\left(\dfrac{\partial w}{\partial x}\right)^2 + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t) + h(x,t)e^{-\lambda w}.$

The substitution $u = \exp(\lambda w)$ leads to the linear equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(x,t)\frac{\partial u}{\partial x} + \lambda g(x,t)u + \lambda h(x,t).$$

1.6.6. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x,t,w)\dfrac{\partial w}{\partial x} + h(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(t)w + h(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, t) + C_2 \exp\left[\int g(t)\,dt\right],$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = 4f\varphi^2 + g\varphi, \tag{1}$$
$$\psi'_t = (4f\varphi + g)\psi, \tag{2}$$
$$\chi'_t = g\chi + 2a\varphi + f\psi^2 + h. \tag{3}$$

Equation (1) is a Bernoulli equation for φ, so it can be readily integrated. Having determined φ, we can obtain the solutions of equation (2) and then (3), which are linear in the unknowns ψ and χ, respectively. Finally, we have

$$\begin{aligned}
\varphi &= e^G\left(A_1 - 4\int e^G f\,dt\right)^{-1}, \quad G = \int g\,dt,\\
\psi &= A_2 \exp\left[\int (4f\varphi + g)\,dt\right],\\
\chi &= A_3 e^G + e^G \int e^{-G}(2a\varphi + f\psi^2 + h)\,dt,
\end{aligned} \tag{4}$$

where A_1, A_2, and A_3 are arbitrary constants.

To the limit passage $A_1 \to \infty$ in (4) there corresponds a degenerate solution with $\varphi \equiv 0$.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + bf(t)w^2 + g(t)w + h(t).$

1°. Generalized separable solution quadratic in :

$$w(x,t) = \varphi(t) + \psi(t)\exp\bigl(\pm x\sqrt{-b}\,\bigr), \qquad b < 0, \tag{1}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients (the arguments of f, g, and h are not indicated)

$$\varphi'_t = bf\varphi^2 + g\varphi + h, \tag{2}$$
$$\psi'_t = (2bf\varphi + g - ab)\psi. \tag{3}$$

Equation (2) is a Riccati equation for $\varphi = \varphi(t)$, so it can be reduced to a second-order linear equation. The books by Kamke (1977) and Polyanin and Zaitsev (2003) present a large number of solutions to this equation for various f, g, and h.

Given a solution of (2), one can solve the linear equation (3) for $\psi = \psi(t)$ to obtain

$$\psi(t) = C\exp\left[-abt + \int(2bf\varphi + g)\,dt\right], \tag{4}$$

where C is an arbitrary constant.

Note two special integrable cases of equation (2).

Solution of equation (2) with $h \equiv 0$:

$$\varphi(t) = e^G\left(C_1 - b\int fe^G\,dt\right)^{-1}, \quad G = \int g\,dt,$$

where C_1 is an arbitrary constant.

If the functions f, g, and h are proportional,

$$g = \alpha f, \quad h = \beta f \qquad (\alpha,\ \beta = \text{const}),$$

then the solution of (2) is given by

$$\int\frac{d\varphi}{b\varphi^2 + \alpha\varphi + \beta} = \int f\,dt + C_2, \tag{5}$$

where C_2 is an arbitrary constant. On integrating the left-hand side of equation (5) and solving for φ, one can find $\varphi = \varphi(t)$ in explicit form.

2°. Generalized separable solution of a more general form:

$$w(x,t) = \varphi(t) + \psi(t)\bigl[A\exp\bigl(x\sqrt{-b}\,\bigr) + B\exp\bigl(-x\sqrt{-b}\,\bigr)\bigr], \qquad b < 0, \tag{6}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = bf\bigl(\varphi^2 + 4AB\psi^2\bigr) + g\varphi + h, \tag{7}$$
$$\psi'_t = 2bf\varphi\psi + g\psi - ab\psi. \tag{8}$$

Solving (8) for φ to express it in terms of ψ and then substituting the resulting expression into (7), one arrives at a second-order nonlinear equation for ψ (if $f,\ g,\ h = \text{const}$, this equation is autonomous and, hence, its order can be reduced).

Note two special cases of solution (6) where the exponentials combine to form hyperbolic functions:

$$w(x,t) = \varphi(t) + \psi(t)\cosh\bigl(x\sqrt{-b}\,\bigr), \qquad A = \tfrac{1}{2}, \quad B = \tfrac{1}{2},$$
$$w(x,t) = \varphi(t) + \psi(t)\sinh\bigl(x\sqrt{-b}\,\bigr), \qquad A = \tfrac{1}{2}, \quad B = -\tfrac{1}{2}.$$

3°. Generalized separable solution (c is an arbitrary constant):

$$w(x,t) = \varphi(t) + \psi(t)\cos\bigl(x\sqrt{b} + c\bigr), \qquad b > 0, \tag{9}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = bf\left(\varphi^2 + \psi^2\right) + g\varphi + h, \tag{10}$$

$$\psi'_t = 2bf\varphi\psi + g\psi - ab\psi. \tag{11}$$

Solving (11) for φ to express it in terms of ψ and then substituting the resulting expression into (10), one arrives at a second-order nonlinear equation for ψ (if f, g, h = const, this equation is autonomous and, hence, its order can be reduced).

⦿ *References*: V. A. Galaktionov and S. A. Posashkov (1989, the case f, g, h = const was considered), V. F. Zaitsev and A. D. Polyanin (1996).

3. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(t)\left(\frac{\partial w}{\partial x}\right)^2 + bf(t)w\frac{\partial w}{\partial x} + cf(t)w^2 + g(t)w + h(t).$

There are generalized separable solutions of the form

$$w(x,t) = \varphi(t) + \psi(t)\exp(\lambda x),$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation $\lambda^2 + b\lambda + c = 0$.

4. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(x)\left(\frac{\partial w}{\partial x}\right)^2 + g(x)\frac{\partial w}{\partial x} + bw + h(x) + p(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{bt},$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ce^{bt} + e^{bt}\int e^{-bt}p(t)\,dt,$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + f(x)(\varphi'_x)^2 + g(x)\varphi'_x + b\varphi + h(x) = 0.$$

5. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(t)\left(\frac{\partial w}{\partial x}\right)^2 + g(t)\frac{\partial w}{\partial x} + bf(t)w^2 + h(t)w + p(t).$

On passing from t, x to the new variables t, $z = x + \displaystyle\int g(t)\,dt$, one arrives at an equation of the form 1.6.6.2:

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial z^2} + f(t)\left(\frac{\partial w}{\partial z}\right)^2 + bf(t)w^2 + h(t)w + p(t).$$

6. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + f(t)\left(\frac{\partial w}{\partial x}\right)^2 + \bigl[g_1(t)x + g_0(t)\bigr]\frac{\partial w}{\partial x} + h(t)w + p(t)x^2 + q(t)x + s(t).$

Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = 4f\varphi^2 + (2g_1 + h)\varphi + p, \tag{1}$$

$$\psi'_t = (4f\varphi + g_1 + h)\psi + 2g_0\varphi + q, \tag{2}$$

$$\chi'_t = h\chi + 2a\varphi + f\psi^2 + g_0\psi + s. \tag{3}$$

Equation (1) is a Riccati equation for $\varphi = \varphi(t)$ and, hence, can be reduced to a second-order linear equation. For solution of such equations, see Kamke (1977) and Polyanin and Zaitsev (2003). In the special case $p \equiv 0$, (1) is a Bernoulli equation, so it can be readily integrated.

Given a solution of (1), equations (2) and (3) can be easily solved, since these are linear in their respective unknowns ψ and χ.

7. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} - ak\frac{1}{w}\left(\frac{\partial w}{\partial x}\right)^2 + f(x,t)\frac{\partial w}{\partial x} + g(x,t)w + h(x,t)w^k.$

The substitution $u = w^{1-k}$ leads to the linear equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(x,t)\frac{\partial u}{\partial x} + (1-k)g(x,t)u + (1-k)h(x,t).$$

8. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f(w)\left(\frac{\partial w}{\partial x}\right)^2.$

The substitution

$$u = \int F(w)\,dw, \qquad \text{where} \quad F(w) = \exp\left[\int f(w)\,dw\right],$$

leads to the linear heat equation for $u = u(x,t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

9. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f(w)\left(\frac{\partial w}{\partial x}\right)^2 + g(x)\frac{\partial w}{\partial x}.$

The substitution

$$u = \int F(w)\,dw, \qquad \text{where} \quad F(w) = \exp\left[\int f(w)\,dw\right],$$

leads to a linear equation for $u = u(x,t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x)\frac{\partial u}{\partial x}.$$

Some exact solutions of this equation, for arbitrary g, can be found in Polyanin (2002).

10. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f(w)\left(\frac{\partial w}{\partial x}\right)^2 + \left[xg(t) + h(t)\right]\frac{\partial w}{\partial x}.$

The substitution

$$u = \int F(w)\,dw, \qquad \text{where} \quad F(w) = \exp\left[\int f(w)\,dw\right],$$

leads to a linear equation for $u = u(x,t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \left[xg(t) + h(t)\right]\frac{\partial u}{\partial x}.$$

This equation can be reduced to the linear heat equation (see Polyanin, 2002).

11. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + f(w)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x,t)\dfrac{\partial w}{\partial x}.$

The substitution

$$u = \int F(w)\,dw, \qquad \text{where} \quad F(w) = \exp\left[\int f(w)\,dw\right],$$

leads to the linear equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x,t)\frac{\partial u}{\partial x}.$$

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(w)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(w)\dfrac{\partial w}{\partial x} + h(w).$

Traveling-wave solution:

$$w = w(z), \quad z = x + \lambda t,$$

where the function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$aw''_{zz} + f(w)(w'_z)^2 + [g(w) - \lambda]w'_z + h(w) = 0. \tag{1}$$

The substitution $w'_z = u(w)$ leads to the first-order equation

$$auu'_w + f(w)u^2 + [g(w) - \lambda]u + h(w) = 0. \tag{2}$$

For exact solutions of the ordinary differential equations (1) and (2) for various $f(w)$, $g(w)$, and $h(w)$, see the book by Polyanin and Zaitsev (2003).

Note that in the special case $h \equiv 0$, equation (2) becomes linear and, hence, can be readily integrated.

1.6.7. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^k + g(x)\dfrac{\partial w}{\partial x} + bw + h(x) + p(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{bt},$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ce^{bt} + e^{bt}\int e^{-bt}p(t)\,dt,$$

where the function $\varphi(x)$ is determined by the second-order ordinary differential equation

$$a\varphi''_{xx} + f(x)(\varphi'_x)^k + g(x)\varphi'_x + b\varphi + h(x) = 0.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, \dfrac{\partial w}{\partial x}\right) + bw + g(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{bt},$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ce^{bt} + e^{bt}\int e^{-bt}g(t)\,dt,$$

where the function $\varphi(x)$ is determined by the second-order ordinary differential equation

$$a\varphi''_{xx} + f(x, \varphi'_x) + b\varphi = 0.$$

3. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + wf\left(t, \frac{1}{w}\frac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x + C_2, t),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = A\exp\left[\lambda x + a\lambda^2 t + \int f(t,\lambda)\,dt\right],$$

where A and λ are arbitrary constants.

1.6.8. Equations of the Form $\dfrac{\partial w}{\partial t} = f(x,t)\dfrac{\partial^2 w}{\partial x^2} + g\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\displaystyle \frac{\partial w}{\partial t} = (ax^2 + b)\frac{\partial^2 w}{\partial x^2} + ax\frac{\partial w}{\partial x} + f(w).$

The substitution $z = \displaystyle\int \frac{dx}{\sqrt{ax^2 + b}}$ leads to an equation of the form 1.6.1.1:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial z^2} + f(w).$$

2. $\displaystyle \frac{\partial w}{\partial t} = \frac{f(t)}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + g(t)w\ln w.$

This equation can be rewritten in the form

$$\frac{\partial w}{\partial t} = f(t)\frac{\partial^2 w}{\partial x^2} + \frac{nf(t)}{x}\frac{\partial w}{\partial x} + g(t)w\ln w.$$

Functional separable solution:

$$w(x,t) = \exp\bigl[\varphi(t)x^2 + \psi(t)\bigr],$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = 4f\varphi^2 + g\varphi,$$
$$\psi'_t = 2(n+1)f\varphi + g\psi;$$

the arguments of f and g are omitted. Successively integrating, one obtains

$$\varphi(t) = e^G\left(A - 4\int fe^G dt\right)^{-1}, \quad G = \int g\,dt,$$
$$\psi(t) = Be^G + 2(n+1)e^G\int f\varphi e^{-G}dt,$$

where A and B are arbitrary constants.

3. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial^2 w}{\partial x^2} + \left[xg(t) + \dfrac{h(t)}{x}\right]\dfrac{\partial w}{\partial x} + s(t)w\ln w + \left[x^2 p(t) + q(t)\right]w.$

Functional separable solution:

$$w(x,t) = \exp\left[\varphi(t)x^2 + \psi(t)\right],$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = 4f\varphi^2 + (2g+s)\varphi + p, \tag{1}$$
$$\psi'_t = s\psi + 2(f+h)\varphi + q. \tag{2}$$

For $p \equiv 0$, equation (1) is a Bernoulli equation and, hence, can be easily integrated. In the general case, (1) is a Riccati equation for $\varphi = \varphi(t)$, so it can be reduced to a second-order linear equation. The books by Kamke (1977) and Polyanin and Zaitsev (2003) present a considerable number of solutions to this equation for various f, g, s, and p. Having solved equation (1), one can find $\psi = \psi(t)$ from the linear equation (2).

4. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial}{\partial x}\left(e^{\lambda x}\dfrac{\partial w}{\partial x}\right) + g(t)w\ln w + h(t)w.$

Functional separable solution:

$$w(x,t) = \exp\left[\varphi(t)e^{-\lambda x} + \psi(t)\right],$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the ordinary differential equations

$$\varphi'_t = \lambda^2 f(t)\varphi^2 + g(t)\varphi,$$
$$\psi'_t = g(t)\psi + h(t).$$

Integrating yields

$$\varphi(t) = G(t)\left[A - \lambda^2\int f(t)G(t)\,dt\right]^{-1}, \quad G(t) = \exp\left[\int g(t)\,dt\right],$$
$$\psi(t) = BG(t) + G(t)\int \frac{h(t)}{G(t)}\,dt,$$

where A and B are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right] + aw\ln w.$

This equation can be rewritten in the form

$$\frac{\partial w}{\partial t} = f(x)\frac{\partial^2 w}{\partial x^2} + f'_x(x)\frac{\partial w}{\partial x} + aw\ln w.$$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = \exp(C_1 e^{at})\,w(x, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \exp(Ce^{at})\,\varphi(x),$$

where C is an arbitrary constant, and the function $\varphi(t)$ is determined by the ordinary differential equation

$$(f\varphi'_x)'_x + a\varphi\ln\varphi = 0.$$

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right] + aw\ln w + \left[g(x) + h(t)\right]w.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = \exp(Ce^{at})w(x,t),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \exp\left[Ce^{at} + e^{at}\int e^{-at}h(t)\,dt\right]\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$(f\varphi'_x)'_x + a\varphi\ln\varphi + g(x)\varphi = 0.$$

7. $\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x)\dfrac{\partial w}{\partial x} + aw\ln w + \left[h(x) + s(t)\right]w.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = \exp(Ce^{at})w(x,t),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \exp\left[Ce^{at} + e^{at}\int e^{-at}s(t)\,dt\right]\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi''_{xx} + g(x)\varphi'_x + a\varphi\ln\varphi + h(x)\varphi = 0.$$

8. $\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + h(x)\dfrac{\partial w}{\partial x} + aw + p(x) + q(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{at},$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ce^{at} + e^{at}\int e^{-at}q(t)\,dt,$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi''_{xx} + g(x)(\varphi'_x)^2 + h(x)\varphi'_x + a\varphi + p(x) = 0.$$

9. $\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x)\left(\dfrac{\partial w}{\partial x}\right)^k + h(x)\dfrac{\partial w}{\partial x} + aw + p(x) + q(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{at},$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ce^{at} + e^{at}\int e^{-at}q(t)\,dt,$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi''_{xx} + g(x)(\varphi'_x)^k + h(x)\varphi'_x + a\varphi + p(x) = 0.$$

10. $\dfrac{\partial w}{\partial t} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g\left(x,\ \dfrac{\partial w}{\partial x}\right) + aw + h(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{at},$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ce^{at} + e^{at}\int e^{-at}h(t)\,dt,$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi''_{xx} + g(x,\ \varphi'_x) + a\varphi = 0.$$

1.6.9. Equations of the Form $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(x)w + bx + c.$

Generalized separable solution:

$$w(x,t) = (bx+c)t + Ax + B - \frac{1}{a}\int_{x_0}^{x}(x-\xi)f(\xi)\,d\xi,$$

where A, B, and x_0 are arbitrary constants.

2. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(t)w + g(t).$

1°. Degenerate solution linear in x:

$$w(x,t) = F(t)(Ax+B) + F(t)\int\frac{g(t)}{F(t)}dt, \quad F(t) = \exp\left[\int f(t)\,dt\right],$$

where A and B are arbitrary constants.

2°. Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)(x^2 + Ax + B) + \varphi(t)\int\frac{g(t)}{\varphi(t)}dt,$$

$$\varphi(t) = F(t)\left[C - 2a\int F(t)\,dt\right]^{-1}, \quad F(t) = \exp\left[\int f(t)\,dt\right],$$

where A, B, and C are arbitrary constants.

3. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + cw^2 + f(t)w + g(t).$

This is a special case of equation 1.6.10.1 with $b = 0$.

4. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} - ak^2w^2 + f(x)w + b_1\sinh(kx) + b_2\cosh(kx).$

Generalized separable solution:

$$w(x,t) = t\big[b_1\sinh(kx) + b_2\cosh(kx)\big] + \varphi(x).$$

Here, the function $\varphi(x)$ is determined by the linear nonhomogeneous ordinary differential equation with constant coefficients

$$a\varphi''_{xx} - ak^2\varphi + f(x) = 0,$$

whose general solution is given by

$$\varphi(x) = C_1\sinh(kx) + C_2\cosh(kx) - \frac{1}{ak}\int_{x_0}^{x} f(\xi)\sinh\big[k(x-\xi)\big]\,d\xi,$$

where A, B, and x_0 are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + ak^2w^2 + f(x)w + b_1\sin(kx) + b_2\cos(kx).$

Generalized separable solution:

$$w(x,t) = t\big[b_1\sin(kx) + b_2\cos(kx)\big] + \varphi(x).$$

Here, the function $\varphi(x)$ is determined by the linear nonhomogeneous ordinary differential equation with constant coefficients

$$a\varphi''_{xx} + ak^2\varphi + f(x) = 0,$$

whose general solution is given by

$$\varphi(x) = C_1\sin(kx) + C_2\cos(kx) - \frac{1}{ak}\int_{x_0}^{x} f(\xi)\sin\big[k(x-\xi)\big]\,d\xi,$$

where A, B, and x_0 are arbitrary constants.

6. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(t)\dfrac{\partial w}{\partial x} + g(t)w.$

The transformation

$$w(x,t) = G(t)u(z,\tau), \quad z = x + \int f(t)\,dt, \quad \tau = \int G(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right]$$

leads to a simpler equation of the form 1.1.9.1:

$$\frac{\partial u}{\partial \tau} = au\frac{\partial^2 u}{\partial z^2}.$$

7. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(t)\dfrac{\partial w}{\partial x} + g(t)w + h(t).$

This is a special case of equation 1.6.10.5.

8. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + xf(t)\dfrac{\partial w}{\partial x} + g(t)w.$

The transformation

$$w(x,t) = G(t)u(z,\tau), \quad z = xF(t), \quad \tau = \int F^2(t)G(t)\,dt,$$

where the functions $F(t)$ and $G(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 1.1.9.1:

$$\frac{\partial u}{\partial \tau} = au\frac{\partial^2 u}{\partial z^2}.$$

9. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + \big[xf(t) + g(t)\big]\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = H(t)u(z,\tau), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^2(t)H(t)\,dt,$$

where the functions $F(t)$ and $H(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

leads to a simpler equation of the form 1.1.9.1:

$$\frac{\partial u}{\partial \tau} = au\frac{\partial^2 u}{\partial z^2}.$$

10. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(x)w\dfrac{\partial w}{\partial x} + g(t)w + h(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\Theta(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned} \varphi'_t &= C\varphi^2 + g(t)\varphi, \\ \psi'_t &= \bigl[C\varphi + g(t)\bigr]\psi + h(t), \\ a\Theta''_{xx} &+ f(x)\Theta'_x = C, \end{aligned}$$

where C is an arbitrary constant. Integrating successively, one obtains

$$\begin{aligned} \varphi(t) &= G(t)\left[A_1 - C\int G(t)\,dt\right]^{-1}, \quad G(t) = \exp\left[\int g(t)\,dt\right], \\ \psi(t) &= A_2\varphi(t) + \varphi(t)\int \frac{h(t)}{\varphi(t)}\,dt, \\ \Theta(x) &= B_1\int \frac{dx}{F(x)} + B_2 + \frac{C}{a}\int\left[\int F(x)\,dx\right]\frac{dx}{F(x)}, \quad F(x) = \exp\left[\frac{1}{a}\int f(x)\,dx\right], \end{aligned}$$

where A_1, A_2, B_1, and B_2 are arbitrary constants.

11. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(x)w\dfrac{\partial w}{\partial x} + g(x)w^2 + h(t)w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(x)H(t)\left[A - B\int H(t)\,dt\right]^{-1}, \quad H(t) = \exp\left[\int h(t)\,dt\right].$$

Here, A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the second-order linear ordinary differential equation

$$a\varphi''_{xx} + f(x)\varphi'_x + g(x)\varphi = B.$$

For exact solutions of this equation with various $f(x)$ and $g(x)$, see Kamke (1977) and Polyanin and Zaitsev (2003).

1.6.10. Equations of the Form $\dfrac{\partial w}{\partial t} = (aw+b)\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x,t,w)\dfrac{\partial w}{\partial x} + h(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^2 + f(t)w + g(t).$

1°. Generalized separable solution quadratic in involving an exponential of x:

$$w(x,t) = \varphi(t) + \psi(t)\exp(\pm\lambda x), \quad \lambda = \left(\frac{-c}{a+b}\right)^{1/2}, \tag{1}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients (the arguments of f and g are not indicated)

$$\varphi'_t = c\varphi^2 + f\varphi + g, \tag{2}$$
$$\psi'_t = (a\lambda^2\varphi + 2c\varphi + f)\psi. \tag{3}$$

Equation (2) is a Riccati equation for $\varphi = \varphi(t)$, so it can be reduced to a second-order linear equation. The books by Kamke (1977) and Polyanin and Zaitsev (2003) present a large number of solutions to this equation for various f and g.

In particular, for $g \equiv 0$, equation (2) is a Bernoulli equation, which is easy to integrate. In another special case, f, $g =$ const, a particular solution of (2) is a number, $\varphi = \varphi_0$, which is a root of the quadratic equation $c\varphi_0^2 + f\varphi_0 + g = 0$. The substitution $u = \varphi - \varphi_0$ leads to a Bernoulli equation.

Given a solution of (2), the solution of equation (3) can be obtained in the form

$$\psi(t) = C \exp\left[\int (a\lambda^2\varphi + 2c\varphi + f)\, dt\right], \tag{4}$$

where C is an arbitrary constant.

2°. Generalized separable solution involving hyperbolic cosine (A is an arbitrary constant):

$$w(x,t) = \varphi(t) + \psi(t)\cosh(\lambda x + A), \qquad \lambda = \left(\frac{-c}{a+b}\right)^{1/2}, \tag{5}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients (the arguments of f and g are not specified)

$$\varphi_t' = c\varphi^2 - b\lambda^2\psi^2 + f\varphi + g, \tag{6}$$
$$\psi_t' = (a\lambda^2\varphi + 2c\varphi + f)\psi. \tag{7}$$

Solving equation (7) for φ to express it in terms of ψ and then substituting the resulting expression into (6), one arrives at a second-order nonlinear equation for ψ, which is autonomous if f, $g =$ const and, hence, its order can be reduced.

3°. Generalized separable solution involving hyperbolic sine (A is an arbitrary constant):

$$w(x,t) = \varphi(t) + \psi(t)\sinh(\lambda x + A), \qquad \lambda = \left(\frac{-c}{a+b}\right)^{1/2},$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi_t' = c\varphi^2 + b\lambda^2\psi^2 + f\varphi + g,$$
$$\psi_t' = (a\lambda^2\varphi + 2c\varphi + f)\psi.$$

4°. Generalized separable solution involving a trigonometric function (A is an arbitrary constant):

$$w(x,t) = \varphi(t) + \psi(t)\cos(\lambda x + A), \qquad \lambda = \left(\frac{c}{a+b}\right)^{1/2}, \tag{8}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi_t' = c\varphi^2 + b\lambda^2\psi^2 + f\varphi + g, \tag{9}$$
$$\psi_t' = (-a\lambda^2\varphi + 2c\varphi + f)\psi. \tag{10}$$

Solving equation (10) for φ to express it in terms of ψ and substituting the resulting expression into (9), one arrives at a second-order nonlinear equation for ψ, which is autonomous if f, $g =$ const and, hence, its order can be reduced.

⊙ *References*: V. A. Galaktionov (1995), V. F. Zaitsev and A. D. Polyanin (1996).

2. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(t)\dfrac{\partial w}{\partial x} + h(t)w + s(t).$

Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients (the arguments of f, g, h, and s are not specified)

$$\varphi'_t = 2(2f + a)\varphi^2 + h\varphi, \tag{1}$$
$$\psi'_t = (4f\varphi + 2a\varphi + h)\psi + 2g\varphi, \tag{2}$$
$$\chi'_t = (2a\varphi + h)\chi + f\psi^2 + g\psi + s. \tag{3}$$

Equation (1) is a Bernoulli equation for $\varphi = \varphi(t)$, so it is easy to integrate. After that, equation (2) and then (3) can be solved with ease, since both are linear in their respective unknowns ψ and χ.

⊙ *References*: V. A. Galaktionov (1995), V. F. Zaitsev and A. D. Polyanin (1996).

3. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x)w\dfrac{\partial w}{\partial x} + h(x)w^2 + p(t)w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(x)\psi(t),$$

where $\varphi(x)$ and $\psi(t)$ are determined by the following system of ordinary differential equations (C is an arbitrary constant):

$$a\varphi\varphi''_{xx} + f(x)(\varphi'_x)^2 + g(x)\varphi\varphi'_x + h(x)\varphi^2 = C\varphi, \tag{1}$$
$$\psi'_t = C\psi^2 + p(t)\psi. \tag{2}$$

The general solution of equation (2) is given by

$$\psi(t) = P(t)\left[A - C\int P(t)\,dt\right]^{-1}, \quad P(t) = \exp\left[\int p(t)\,dt\right],$$

where A is an arbitrary constant. In the special case $f \equiv 0$, equation (1) can be reduced, on dividing it by φ, to a second-order linear equation; for exact solutions of this equation with various $g(x)$ and $h(x)$, see Kamke (1997) and Polyanin and Zaitsev (2003).

4. $\dfrac{\partial w}{\partial t} = (aw + b)\dfrac{\partial^2 w}{\partial x^2} + c\left(\dfrac{\partial w}{\partial x}\right)^2 + f(t)\dfrac{\partial w}{\partial x} + kw^2 + g(t)w + h(t).$

The transformation

$$u(z,t) = w(x,t) + \frac{b}{a}, \quad z = x + \int f(t)\,dt$$

leads to an equation of the form 1.6.10.2 for $u = u(z,t)$.

5. $\dfrac{\partial w}{\partial t} = (aw+b)\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + \left[g_1(t)x + g_0(x)\right]\dfrac{\partial w}{\partial x} + h(t)w + p_2(t)x^2 + p_1(t)x + p_0(t).$

There is a generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by a system of first-order ordinary differential equations with variable coefficients, which is not written out here.

1.6.11. Equations of the Form $\frac{\partial w}{\partial t} = aw^m \frac{\partial^2 w}{\partial x^2} + f(x,t)\frac{\partial w}{\partial x} + g(x,t,w)$

1. $\frac{\partial w}{\partial t} = aw^4 \frac{\partial^2 w}{\partial x^2} + f(x)w^5.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^4 t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Suppose $u = u(x)$ is a nontrivial solution of the second-order linear ordinary differential equation

$$au''_{xx} + f(x)u = 0. \qquad (1)$$

Then the transformation

$$\xi = \int \frac{dx}{u^2}, \quad z = \frac{w}{u}$$

simplifies the original equation bringing it to the form

$$\frac{\partial z}{\partial t} = az^4 \frac{\partial^2 z}{\partial \xi^2}.$$

Using the change of variable $v = z^{-3}$, we obtain an equation of the form 1.1.10.4:

$$\frac{\partial v}{\partial t} = a\frac{\partial}{\partial \xi}\left(v^{-4/3}\frac{\partial v}{\partial \xi}\right).$$

3°. Multiplicative separable solution:

$$w(x,t) = (4\lambda t + C)^{-1/4} g(x),$$

where C and λ are arbitrary constants, and the function $g = g(x)$ is determined by Yermakov's equation

$$ag''_{xx} + f(x)g + \lambda g^{-3} = 0. \qquad (2)$$

Given a particular solution, $u = u(x)$, of the linear equation (1), the general solution of the nonlinear equation (2) can be expressed as (e.g., see Polyanin and Zaitsev, 2003)

$$Ag^2 = -\frac{\lambda}{a}u^2 + u^2\left(B + A\int \frac{dx}{u^2}\right)^2,$$

where A and B are arbitrary constants ($A \neq 0$).

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

2. $\frac{\partial w}{\partial t} = aw^m \frac{\partial^2 w}{\partial x^2} + f(t)w.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^m x + C_2, t),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. The transformation

$$w(x,t) = F(t)u(x,\tau), \quad \tau = \int F^m(t)\,dt, \quad F(t) = \exp\left[\int f(t)\,dt\right],$$

leads to a simpler equation of the form 1.1.9.18:

$$\frac{\partial u}{\partial \tau} = au^m \frac{\partial^2 u}{\partial x^2}.$$

3. $\dfrac{\partial w}{\partial t} = aw^m \dfrac{\partial^2 w}{\partial x^2} + f(x)w^{m+1}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^m t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution (C and λ are arbitrary constants):

$$w(x,t) = (m\lambda t + C)^{-1/m}\varphi(x),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$a\varphi^m \varphi''_{xx} + f(x)\varphi^{m+1} + \lambda\varphi = 0.$$

4. $\dfrac{\partial w}{\partial t} = aw^m \dfrac{\partial^2 w}{\partial x^2} + xf(t)\dfrac{\partial w}{\partial x} + g(t)w.$

The transformation

$$w(x,t) = u(z,\tau)G(t), \quad z = xF(t), \quad \tau = \int F^2(t)G^m(t)\,dt,$$

where the functions $F(t)$ and $G(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 1.1.9.18:

$$\frac{\partial u}{\partial \tau} = au^m \frac{\partial^2 u}{\partial z^2}.$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

5. $\dfrac{\partial w}{\partial t} = aw^m \dfrac{\partial^2 w}{\partial x^2} + \big[f(t)x + g(t)\big]\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = u(z,\tau)H(t), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^2(t)H^m(t)\,dt,$$

where the functions $F(t)$ and $H(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

leads to a simpler equation of the form 1.1.9.18:

$$\frac{\partial u}{\partial \tau} = au^m \frac{\partial^2 u}{\partial z^2}.$$

1.6.12. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + f(x,t)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + f(t).$

Generalized separable solutions linear and quadratic in x:

$$w(x,t) = C_1 x + aC_1^2 t + C_2 + \int f(t)\,dt,$$

$$w(x,t) = -\frac{(x+C_2)^2}{6a(t+C_1)} + C_3(t+C_1)^{-1/3} + (t+C_1)^{-1/3}\int (t+C_1)^{1/3} f(t)\,dt,$$

where C_1, C_2, and C_3 are arbitrary constants. The first solution is degenerate.

2. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right)+f(t)w+g(t).$

This is a special case of equation 1.6.13.4 with $m=1$.

3. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right)+bw^2+f(t)w+g(t).$

This is a special case of equation 1.6.13.5 with $m=1$.

4. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right)+f(x)w^2.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1=C_1w(x,C_1t+C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w=(\lambda t+C)^{-1}\varphi(x),$$

where λ and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a(\varphi\varphi'_x)'_x+f(x)\varphi^2+\lambda\varphi=0.$$

5. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right)+f(t)\dfrac{\partial w}{\partial x}+g(t)w.$

This is a special case of equation 1.6.13.8 with $m=1$.

The transformation

$$w(x,t)=G(t)u(z,\tau),\quad z=x+\int f(t)\,dt,\quad \tau=\int G(t)\,dt,\quad G(t)=\exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 1.1.10.1:

$$\frac{\partial u}{\partial t}=a\frac{\partial}{\partial z}\left(u\frac{\partial u}{\partial z}\right).$$

6. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right)+xf(t)\dfrac{\partial w}{\partial x}+g(t)w.$

The transformation

$$w(t,x)=u(z,\tau)G(t),\quad z=xF(t),\quad \tau=\int F^2(t)G(t)\,dt,$$

where the functions $F(t)$ and $G(t)$ are given by

$$F(t)=\exp\left[\int f(t)\,dt\right],\quad G(t)=\exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 1.1.10.1:

$$\frac{\partial u}{\partial \tau}=a\frac{\partial}{\partial z}\left(u\frac{\partial u}{\partial z}\right).$$

7. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right)+\big[xf(t)+g(t)\big]\dfrac{\partial w}{\partial x}+h(t)w.$

This is a special case of equation 1.6.13.10 with $m=1$.

1.6.13. Equations of the Form $\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^m \frac{\partial w}{\partial x}\right) + f(x,t)\frac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-4/3}\dfrac{\partial w}{\partial x}\right) + f(x)w^{-1/3}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^{-4/3} t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. The substitution $w = v^{-3}$ leads to an equation of the form 1.6.11.1:

$$\frac{\partial v}{\partial t} = av^4 \frac{\partial^2 v}{\partial x^2} - \tfrac{1}{3} f(x) v^5.$$

3°. Suppose $u = u(x)$ is any nontrivial particular solution of the second-order linear ordinary differential equation

$$au''_{xx} - \tfrac{1}{3} f(x) u = 0.$$

The transformation

$$\xi = \pm \int \frac{dx}{u^2}, \quad z = wu^3$$

simplifies the original equation, bringing it to equation 1.1.10.4:

$$\frac{\partial z}{\partial t} = a\frac{\partial}{\partial \xi}\left(z^{-4/3}\frac{\partial z}{\partial \xi}\right).$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{m}\dfrac{\partial w}{\partial x}\right) + f(t)w.$

The transformation

$$w(x,t) = u(x,\tau)F(t), \quad \tau = \int F^m(t)\,dt, \qquad F(t) = \exp\left[\int f(t)\,dt\right],$$

leads to a simpler equation of the form 1.1.10.7:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right).$$

If $m = -1$ or $m = -2$, see 1.1.10.2 or 1.1.10.3 for solutions of this equation.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{m}\dfrac{\partial w}{\partial x}\right) + f(t)w^{1-m}.$

The substitution $u = w^m$ leads to an equation of the form 1.6.10.2:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} + \frac{a}{m}\left(\frac{\partial u}{\partial x}\right)^2 + mf(t),$$

which admits a generalized separable solution of the form $u = \varphi(t)x^2 + \psi(t)x + \chi(t)$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{m}\dfrac{\partial w}{\partial x}\right) + f(t)w + g(t)w^{1-m}.$

The substitution $u = w^m$ leads to an equation of the form 1.6.10.2:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} + \frac{a}{m}\left(\frac{\partial u}{\partial x}\right)^2 + mf(t)u + mg(t),$$

which admits a generalized separable solution of the form $u = \varphi(t)x^2 + \psi(t)x + \chi(t)$.

⊙ *References*: V. A. Galaktionov (1995), V. F. Zaitsev and A. D. Polyanin (1996).

5. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^m \frac{\partial w}{\partial x}\right) + bw^{1+m} + f(t)w + g(t)w^{1-m}.$

For $b = 0$, see equation 1.6.13.4:

The substitution $u = w^m$ leads to an equation of the form 1.6.10.1:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} + \frac{a}{m}\left(\frac{\partial u}{\partial x}\right)^2 + bmu^2 + mf(t)u + mg(t),$$

which admits generalized separable solutions of the following forms:

$$\begin{aligned}
u(x,t) &= \varphi(t) + \psi(t)\exp(\pm\lambda x),\\
u(x,t) &= \varphi(t) + \psi(t)\cosh(\lambda x + C),\\
u(x,t) &= \varphi(t) + \psi(t)\sinh(\lambda x + C),\\
u(x,t) &= \varphi(t) + \psi(t)\cos(\lambda x + C),
\end{aligned}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by systems of appropriate first-order ordinary differential equations, the parameter λ is a root of a quadratic equation, and C is an arbitrary constant.

⊙ *References*: V. A. Galaktionov (1995), V. F. Zaitsev and A. D. Polyanin (1996).

6. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^m \frac{\partial w}{\partial x}\right) + f(x)w^{1+m}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^m t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w = (\lambda m t + C)^{-1/m}\varphi(x),$$

where λ and C are arbitrary constants, and the function $\varphi(x)$ is determined by the equation

$$a\psi''_{xx} + (m+1)f(x)\psi + \lambda(m+1)\psi^{\frac{1}{m+1}} = 0, \qquad \psi = \varphi^{m+1}.$$

The book by Polyanin and Zaitsev (2003) presents exact solutions of this equation for various $f(x)$.

7. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^m \frac{\partial w}{\partial x}\right) + g(x)w^{m+1} + f(t)w.$

Multiplicative separable solution:

$$w = \varphi(x)\psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are determined by the ordinary differential equations (C_1 is an arbitrary constant)

$$\begin{aligned}
a(\varphi^m\varphi'_x)'_x + g(x)\varphi^{m+1} + C_1\varphi &= 0,\\
\psi'_t - f(t)\psi + C_1\psi^{m+1} &= 0.
\end{aligned}$$

The general solution of the second equation is given by (C_2 is an arbitrary constant)

$$\psi(t) = e^F\left(C_2 + mC_1\int e^{mF}\,dt\right)^{-1/m}, \qquad F = \int f(t)\,dt.$$

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + f(t)\dfrac{\partial w}{\partial x} + g(t)w.$

The transformation

$$w(x,t) = u(z,\tau)G(t), \quad z = x + \int f(t)\,dt, \quad \tau = \int G^m(t)\,dt, \quad G(t) = \exp\left[\int g(t)dt\right],$$

leads to a simpler equation of the form 1.1.10.7:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial z}\left(u^m\frac{\partial u}{\partial z}\right).$$

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + xf(t)\dfrac{\partial w}{\partial x} + g(t)w.$

The transformation

$$w(t,x) = u(z,\tau)G(t), \quad z = xF(t), \quad \tau = \int F^2(t)G^m(t)\,dt,$$

where the functions $F(t)$ and $G(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 1.1.10.7:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial z}\left(u^m\frac{\partial u}{\partial z}\right).$$

In the special case $m=-2$, this equation can be transformed to the linear heat equation (see 1.1.10.3).

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^m\dfrac{\partial w}{\partial x}\right) + \big[xf(t) + g(t)\big]\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = u(z,\tau)H(t), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^2(t)H^m(t)\,dt,$$

where the functions $F(t)$ and $H(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

leads to a simpler equation of the form 1.1.10.7:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial z}\left(u^m\frac{\partial u}{\partial z}\right).$$

If $m=-1$ or $m=-2$, see 1.1.10.2 or 1.1.10.3 for solutions of this equation.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

1.6.14. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + f(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + f(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(C_1x + C_2, t) - \frac{2}{\lambda}\ln|C_1|,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. The transformation

$$w(x,t) = u(x,\tau) + F(t), \quad \tau = \int \exp\bigl[\lambda F(t)\bigr]\,dt, \qquad F(t) = \int f(t)\,dt,$$

leads to a simpler equation of the form 1.2.2.1:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial}{\partial x}\left(e^{\lambda u}\frac{\partial u}{\partial x}\right).$$

2. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(e^{\lambda w}\frac{\partial w}{\partial x}\right) + f(t) + g(t)e^{-\lambda w}.$

The substitution $u = e^{\lambda w}$ leads to an equation of the form 1.6.9.2:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} + \lambda f(t)u + \lambda g(t).$$

This equation admits a generalized separable solution of the form $u = \varphi(t)x^2 + \psi(t)x + \chi(t)$.

3. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(e^{\lambda w}\frac{\partial w}{\partial x}\right) + f(x) + (bx+c)e^{-\lambda w}.$

The substitution $u = e^{\lambda w}$ leads to an equation of the form 1.6.9.1:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} + \lambda f(x)u + \lambda(bx+c).$$

This equation admits a generalized separable solution of the form $u = \lambda(bx+c)t + \varphi(x)$.

4. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(e^{\lambda w}\frac{\partial w}{\partial x}\right) + be^{\lambda w} + f(t) + g(t)e^{-\lambda w}.$

For $b = 0$, see equation 1.6.14.2.

The substitution $u = e^{\lambda w}$ leads to an equation of the form 1.6.10.1:

$$\frac{\partial u}{\partial t} = au\frac{\partial^2 u}{\partial x^2} + bu^2 + \lambda f(t)u + \lambda g(t).$$

This equation admits generalized separable solutions of the following forms:

$$\begin{aligned}
u(x,t) &= \varphi(t) + \psi(t)\exp(\pm\mu x),\\
u(x,t) &= \varphi(t) + \psi(t)\cosh(\mu x + C),\\
u(x,t) &= \varphi(t) + \psi(t)\sinh(\mu x + C),\\
u(x,t) &= \varphi(t) + \psi(t)\cos(\mu x + C),
\end{aligned}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by systems of appropriate first-order ordinary differential equations, μ is a root of a quadratic equation, and C is an arbitrary constant.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

5. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(e^{\lambda w}\frac{\partial w}{\partial x}\right) + f(x)e^{\lambda w}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x, C_1 t + C_2) + \frac{1}{\lambda}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w = -\frac{1}{\lambda}\ln(\lambda t + C) + \varphi(x),$$

where λ and C are arbitrary constants, and the function $\varphi(x)$ is determined by the second-order linear ordinary differential equation

$$a\psi''_{xx} + \lambda f(x)\psi + \lambda = 0, \qquad \psi = e^{\lambda\varphi}.$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + g(x)e^{\lambda w} + f(t).$

Additive separable solution:

$$w = \varphi(x) + \psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are determined by the ordinary differential equations (C_1 is an arbitrary constant)

$$a(e^{\lambda\varphi}\varphi'_x)'_x + g(x)e^{\lambda\varphi} + C_1 = 0, \tag{1}$$

$$\psi'_t - f(t) + C_1 e^{\lambda\psi} = 0. \tag{2}$$

Equation (1) can be reduced, with the change of variable $U = e^{\lambda\varphi}$, to the linear equation $aU''_{xx} + \lambda g(x)U + \lambda C_1 = 0$. The general solution of equation (2) is given by (C_2 is an arbitrary constant)

$$\psi(t) = F - \frac{1}{\lambda}\ln\left(C_2 + \lambda C_1 \int e^{\lambda F}\,dt\right), \qquad F = \int f(t)\,dt.$$

1.6.15. Equations of the Form $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right].$

This equation is frequently encountered in nonlinear problems of heat and mass transfer (with f being the thermal diffusivity or diffusion coefficient) and the theory of flows through porous media. For $f(w) = aw^m$, see Subsection 1.1.10; for $f(w) = e^{\lambda w}$, see equation 1.2.2.1; and for $f(w) = a\ln w + b$, see equation 1.4.2.7.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1x + C_2, C_1^2t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution in implicit form:

$$k^2\int\frac{f(w)\,dw}{\lambda w + C_1} = kx + \lambda t + C_2, \tag{1}$$

where C_1, C_2, k, and λ are arbitrary constants. To $\lambda = 0$ there corresponds a stationary solution.

3°. Self-similar solution:

$$w = w(z), \quad z = \frac{x}{\sqrt{t}} \qquad (0 \le x < \infty),$$

where the function $w(z)$ is determined by the ordinary differential equation

$$\left[f(w)w'_z\right]'_z + \tfrac{1}{2}zw'_z = 0. \tag{2}$$

Solutions of this form usually correspond to constant w in the initial and boundary conditions for the original partial differential equation (w_0, w_1 = const):

$$\begin{aligned} &w = w_0 &&\text{at}\quad t = 0 &&\text{(initial condition)},\\ &w = w_1 &&\text{at}\quad x = 0 &&\text{(boundary condition)},\\ &w \to w_0 &&\text{as}\quad x \to \infty &&\text{(boundary condition)}.\end{aligned}$$

Then the boundary conditions for the ordinary differential equation (2) are as follows:

$$w = w_1 \quad\text{at}\quad z = 0, \qquad w \to w_0 \quad\text{as}\quad z \to \infty. \tag{3}$$

For $f(w) = aw^{-1}$, $f(w) = aw^{-2}$, and $f(w) = (\alpha w^2 + \beta w + \gamma)^{-1}$, the general solutions of (2) were obtained by Fujita (1952); see also the book by Lykov (1967).

TABLE 1
Solutions equation 1.6.15.1 for various $f = f(w)$, where $z = xt^{-1/2}$.

No.	Function $f = f(w)$	Solution $z = z(w)$	Conditions
1	$\dfrac{n}{2}w^n - \dfrac{n}{2(n+1)}w^{2n}$	$1 - w^n$	$n > 0$
2	$\dfrac{n}{2(n+1)}\big[(1-w)^{n-1} - (1-w)^{2n}\big]$	$(1-w)^n$	$n > 0$
3	$\dfrac{n}{2(1-n)}w^{-2n} - \dfrac{n}{2}w^{-n}$	$w^{-n} - 1$	$0 < n < 1$
4	$\frac{1}{2}\sin^2\big(\frac{1}{2}\pi w\big)$	$\cos\big(\frac{1}{2}\pi w\big)$	
5	$\frac{1}{8}\sin(\pi w)\big[\pi w + \sin(\pi w)\big]$	$\cos^2\big(\frac{1}{2}\pi w\big)$	
6	$\frac{1}{16}\sin^2(\pi w)\big[5 + \cos(\pi w)\big]$	$\cos^3\big(\frac{1}{2}\pi w\big)$	
7	$\frac{1}{2}\cos\big(\frac{1}{2}\pi w\big)\big[\cos\big(\frac{1}{2}\pi w\big) + \frac{1}{2}\pi w - 1\big]$	$1 - \sin(\frac{1}{2}\pi w)$	
8	$\dfrac{w\arccos w + 1}{2\sqrt{1-w^2}} - \dfrac{1}{2}$	$\arccos w$	
9	$\dfrac{\pi - 2(1-w)\arcsin(1-w)}{4\sqrt{2w-w^2}} - \dfrac{1}{2}$	$\arcsin(1-w)$	
10	$\dfrac{w\arcsin w}{4\sqrt{1-w^2}} + \dfrac{1}{4}w^2$	$\sqrt{1-w^2}$	
11	$\frac{1}{2}(1 - \ln w)$	$-\ln w$	

4°. We now describe a simple method for finding an $f(w)$ such that equation (2) admits an exact solution. To this end, we integrate equation (2) with respect to z and then apply the hodograph transformation (where w is regarded as the independent variable and z as the dependent one) to obtain

$$f(w) = -\frac{1}{2}z'_w\left(\int z\,dw + A\right), \qquad A \text{ is an arbitrary constant.} \tag{4}$$

Substituting a specific expression $z = z(w)$ for z on the right-hand side of relation (4), one obtains a one-parameter family of functions $f(w)$ for which $z = z(w)$ solves equation (2). The explicit form of $w = w(z)$ is obtained by the inversion of $z = z(w)$.

The method just outlined was devised by Philip (1960); he obtained a large number of exact solutions to the original equation for various $f = f(w)$. Some of his results, those corresponding to a problem with the initial and boundary conditions of (3) with $w_0 = 0$ and $w_1 = 1$, are listed below in Table 1. All solutions are written out in implicit form, $z = z(w)$, and are valid within the range of their spatial localization $0 \le w \le 1$.

5°. There is another way to find an $f(w)$ for which equation (2) admits exact solutions. By direct substitution, one can verify that equation (2) is satisfied by

$$w = \phi'_z, \quad f(w) = \frac{s + \phi - z\phi'_z}{2\phi''_{zz}}, \tag{5}$$

where $\phi = \phi(z)$ is an arbitrary function, and s is an arbitrary constant. Expressions (5) define a parametric representation of $f = f(w)$; the explicit representation is obtained by eliminating z.

For example, assuming in (5) that

$$\phi(z) = w_0 z + \frac{1}{\lambda}(w_0 - w_1)e^{-\lambda z} \qquad (\lambda > 0,\ w_1 > w_0),$$

and eliminating z, one obtains

$$f(w)=\frac{A}{w-w_0}+B+C\ln(w-w_0),\qquad w=w_0+(w_1-w_0)e^{-\lambda z},$$

where $A=-\frac{1}{2}s\lambda^{-1}$, $B=\frac{1}{2}\lambda^{-2}\bigl[1+\ln(w_1-w_0)\bigr]$, and $C=-\frac{1}{2}\lambda^{-2}$. Note that this solution satisfies the boundary conditions of (3). Likewise, one can construct other $f(w)$.

6°. Here is one more method for constructing an $f(w)$ for which equation (2) admits exact solutions. Suppose $\bar w=\bar w(z)$ is a solution of equation (2) with an $f(w)$. Then $\bar w=\bar w(z)$ is also a solution of the more complicated equation $[F(w)w'_z]'_z+\frac{1}{2}zw'_z=0$ with

$$F(w)=f(w)+Ag(w)\qquad (A\ \text{ is an arbitrary constant}),\tag{6}$$

where the function $g=g(w)$ is defined parametrically by

$$g(w)=\frac{1}{\bar w'_z},\qquad w=\bar w(z).\tag{7}$$

For example, the function $\bar w=bz^{2/m}$, where b is some constant, is a particular solution of equation (2) if $f(w)$ is a power-law function, $f(w)=aw^m$. It follows from (6) and (7) that $\bar w$ is also a solution of equation (2) with $f(w)=aw^m+Aw^{\frac{m-2}{2}}$.

For the first solution presented in Table 1, the method outlined gives the following one-parameter family of functions:

$$f(w)=\frac{n}{2}w^n-\frac{n}{2(n+1)}w^{2n}+Aw^{n-1},$$

for which $z=1-w^n$ is a solution of equation.

7°. The transformation

$$\bar t=t-t_0,\qquad \bar x=\int_{x_0}^{x}w(y,t)\,dy+\int_{t_0}^{t}f\bigl(w(x_0,\tau)\bigr)\left[\frac{\partial w}{\partial x}(x,\tau)\right]_{x=x_0}d\tau,\qquad \bar w(\bar x,\bar t)=\frac{1}{w(x,t)}\tag{8}$$

takes a nonzero solution $w(x,t)$ of the original equation to a solution $\bar w(\bar x,\bar t)$ of a similar equation,

$$\frac{\partial\bar w}{\partial\bar t}=\frac{\partial}{\partial\bar x}\left[\bar f(\bar w)\frac{\partial\bar w}{\partial\bar x}\right],\qquad \bar f(w)=\frac{1}{w^2}f\left(\frac{1}{w}\right).\tag{9}$$

In the special case of power-law dependence, $f(w)=aw^m$, transformation (8) leads to equation (9) where $\bar f(w)=aw^{-m-2}$.

8°. The equation in question is represented in conservative form, i.e., in the form of a conservation law.

Another conservation law:

$$\frac{\partial}{\partial t}(xw)+\frac{\partial}{\partial x}\left[F(w)-xf(w)\frac{\partial w}{\partial x}\right]=0,$$

where $F(w)=\displaystyle\int f(w)\,dw$.

9°. For $f(w)=a(w^2+b)^{-1}$, see equation 1.1.13.2 and Subsection S.5.3 (Example 10).

⊙ *References for equation* 1.6.15.1: L. V. Ovsiannikov (1959, 1962, 1982), V. A. Dorodnitsyn and S. R. Svirshchevskii (1983), W. Strampp (1982), J. R. Burgan, A. Munier, M. R. Feix, and E. Fijalkow (1984), A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995), N. H. Ibragimov (1994), V. F. Zaitsev and A. D. Polyanin (1996), P. W. Doyle and P. J. Vassiliou (1998).

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w).$

This equation governs unsteady heat conduction in a quiescent medium in the case where the thermal diffusivity and the rate of reaction are arbitrary functions of temperature.

1°. Traveling-wave solutions:

$$w = w(z), \quad z = \pm x + \lambda t,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$[f(w)w'_z]'_z - \lambda w'_z + g(w) = 0. \tag{1}$$

The substitution

$$y(w) = \frac{1}{\lambda} f(w) w'_z$$

brings (1) to an Abel equation of the second kind:

$$yy'_w - y = \varphi(w), \qquad \text{where} \quad \varphi(w) = -\lambda^{-2} f(w) g(w). \tag{2}$$

The book by Polyanin and Zaitsev (2003) present a considerable number of solutions to equation (2) for various $\varphi = \varphi(w)$.

2°. Let the function $f = f(w)$ be arbitrary and let $g = g(w)$ be defined by

$$g(w) = \frac{A}{f(w)} + B,$$

where A and B are some numbers. In this case, there is a functional separable solution, which is defined implicitly by

$$\int f(w)\,dw = At - \frac{1}{2}Bx^2 + C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: V. A. Galaktionov (1994).

3°. Let now $g = g(w)$ be arbitrary and let $f = f(w)$ be defined by

$$f(w) = \frac{A_1 A_2 w + B}{g(w)} + \frac{A_2 A_3}{g(w)} \int Z\,dw, \tag{3}$$

$$Z = -A_2 \int \frac{dw}{g(w)}, \tag{4}$$

where A_1, A_2, and A_3 are some numbers. Then there are generalized traveling-wave solutions of the form

$$w = w(Z), \quad Z = \frac{\pm x + C_2}{\sqrt{2A_3 t + C_1}} - \frac{A_1}{A_3} - \frac{A_2}{3A_3}(2A_3 t + C_1),$$

where the function $w(Z)$ is determined by the inversion of (4), and C_1 and C_2 are arbitrary constants.

4°. Let $g = g(w)$ be arbitrary and let $f = f(w)$ be defined by

$$f(w) = \frac{1}{g(w)}\left(A_1 w + A_3 \int Z\,dw\right) \exp\left[-A_4 \int \frac{dw}{g(w)}\right], \tag{5}$$

$$Z = \frac{1}{A_4} \exp\left[-A_4 \int \frac{dw}{g(w)}\right] - \frac{A_2}{A_4}, \tag{6}$$

where A_1, A_2, A_3, and A_4 are some numbers ($A_4 \neq 0$). In this case, there are generalized traveling-wave solutions of the form

$$w = w(Z), \quad Z = \varphi(t)x + \psi(t),$$

where the function $w(Z)$ is determined by the inversion of (6),

$$\varphi(t) = \pm\left(C_1 e^{2A_4 t} - \frac{A_3}{A_4}\right)^{-1/2}, \qquad \psi(t) = -\varphi(t)\left[A_1 \int \varphi(t)\,dt + A_2 \int \frac{dt}{\varphi(t)} + C_2\right],$$

and C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

5°. Let the functions $f(w)$ and $g(w)$ be as follows:

$$f(w) = \varphi'(w), \qquad g(w) = \frac{a\varphi(w) + b}{\varphi'(w)} + c[a\varphi(w) + b],$$

where $\varphi(w)$ is an arbitrary function and a, b, and c are any numbers (the prime denotes a derivative with respect to w). Then there are functional separable solutions defined implicitly by

$$\varphi(w) = e^{at}\big[C_1 \cos(x\sqrt{ac}\,) + C_2 \sin(x\sqrt{ac}\,)\big] - \frac{b}{a} \qquad \text{if } ac > 0,$$
$$\varphi(w) = e^{at}\big[C_1 \cosh(x\sqrt{-ac}\,) + C_2 \sinh(x\sqrt{-ac}\,)\big] - \frac{b}{a} \quad \text{if } ac < 0,$$

and C_1 and C_2 are arbitrary constants.

⊙ *Reference*: V. A. Galaktionov (1994).

6°. Let $f(w)$ and $g(w)$ be as follows:

$$f(w) = w\varphi'_w(w), \qquad g(w) = a\left[w + 2\frac{\varphi(w)}{\varphi'_w(w)}\right],$$

where $\varphi(w)$ is an arbitrary function and a is any number. Then there are functional separable solutions defined implicitly by

$$\varphi(w) = C_1 e^{2at} - \tfrac{1}{2}a(x + C_2)^2,$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: V. A. Galaktionov (1994).

7°. Group classification of solutions to the equation in question was carried out by Dorodnitsyn (1979, 1982); see also Dorodnitsyn and Svirshchevskii (1983), Galaktionov, Dorodnitsyn, Elenin, Kurdyumov, and Samarskii (1986), and Ibragimov (1994). As a result, only a limited number of equations were extracted that possess symmetries other than translations.

8°. If $f = dF(w)/dw$ and $g = aF(w) + bw + c$, where $F(w)$ is an arbitrary function, and a, b, and c are arbitrary constants, then there is a conservation law

$$\big[e^{-bt}p(x)w\big]_t + \big\{e^{-bt}\big[p(x)_x F(w) - p(x)(F(w))_x + \varphi(x)\big]\big\}_x = 0.$$

Here,

$$p(x) = \begin{cases} C_1 \sin(\sqrt{a}\,x) + C_2 \cos(\sqrt{a}\,x) & \text{if } a > 0, \\ C_1 e^{\sqrt{-a}\,x} + C_2 e^{-\sqrt{-a}\,x} & \text{if } a < 0, \\ C_1 x + C_2 & \text{if } a = 0, \end{cases}$$

where $\varphi'_x = cp(x)$; C_1 and C_2 are arbitrary constants.

⊙ *References*: V. A. Dorodnitsyn (1979), V. A. Galaktionov, V. A. Dorodnitsyn, G. G. Elenin, S. P. Kurdyumov, and A. A. Samarskii (1986).

9°. For specific equations of this form, see Subsections 1.1.1 to 1.1.3, 1.1.11 to 1.1.13, 1.2.1 to 1.2.3, and 1.4.1.

3. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] + \frac{g(t)}{f(w)} + h(x).$

Functional separable solution in implicit form:

$$\int f(w)\,dw = \int g(t)\,dt - \int_{x_0}^{x} (x - \xi)h(\xi)\,d\xi + C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants, and x_0 is any number.

4. $\dfrac{\partial w}{\partial t}=\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right]-(ax+b)\dfrac{\partial w}{\partial x}.$

This equation governs unsteady heat and mass transfer in an inhomogeneous fluid flow in the cases where the thermal diffusivity is arbitrarily dependent on temperature.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1=w(x+C_1e^{at},t+C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w=w(z),\qquad z=x+C_1e^{at},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$[f(w)w'_z]'_z-(az+b)w'_z=0.$$

3°. Functional separable solution in implicit form:

$$\int f(w)\,dw=C_1e^{-at}(ax+b)+C_2.$$

4°. On passing from t, x to the new variables

$$\tau=\frac{1}{2a}\left(1-e^{-2at}\right),\qquad \zeta=e^{-at}\left(x+\frac{b}{a}\right),$$

one obtains a simpler equation of the form 1.6.15.1 for $w(\zeta,\tau)$:

$$\frac{\partial w}{\partial \tau}=\frac{\partial}{\partial \zeta}\left[f(w)\frac{\partial w}{\partial \zeta}\right].$$

5. $\dfrac{\partial w}{\partial t}=\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right]+g(w)\dfrac{\partial w}{\partial x}.$

1°. Traveling-wave solution in implicit form:

$$k^2\int\frac{f(w)\,dw}{\lambda w-kG(w)+C_1}=kx+\lambda t+C_2,\qquad G(w)=\int g(w)\,dw,$$

where C_1, C_2, k, and λ are arbitrary constants.

2°. The transformation

$$dz=w\,dx+\left[f(w)w_x+G(w)\right]dt,\quad d\tau=dt,\quad u=1/w\qquad \left(dz=z_x\,dx+z_t\,dt\right)$$

leads to an equation of the similar form

$$\frac{\partial u}{\partial \tau}=\frac{\partial}{\partial z}\left[\Phi(u)\frac{\partial u}{\partial z}\right]+\Psi(u)\frac{\partial u}{\partial z},$$

where

$$\Phi(u)=\frac{1}{u^2}f\left(\frac{1}{u}\right),\quad \Psi(u)=\frac{1}{u}g\left(\frac{1}{u}\right)-G\left(\frac{1}{u}\right),\quad G(w)=\int g(w)\,dw.$$

Example. For $f(w)=a$ and $g(w)=bw$, the original equation is an unnormalized Burgers equation 1.1.5.3. The above transformation brings it to the solvable equation

$$\frac{\partial u}{\partial \tau}=\frac{\partial}{\partial z}\left(\frac{a}{u^2}\frac{\partial u}{\partial z}\right)+\frac{b}{2u^2}\frac{\partial u}{\partial z}.$$

⊙ *Reference*: A. S. Fokas and Y. C. Yortsos (1982).

3°. Let $f(w)$ and $g(w)$ be defined as

$$f(w) = Z'_w\left(A_1 w + A_3 \int Z\,dw\right), \qquad g(w) = A_2 + A_4 Z,$$

where

$$Z = Z(w) \tag{1}$$

is a prescribed function (chosen arbitrarily). Then the original equation has the following generalized traveling-wave solution:

$$w = w(Z), \qquad Z = \varphi(t)x + (A_2 t + C_1)\varphi(t) + A_1\varphi(t)\int \varphi(t)\,dt,$$

where C_1 is an arbitrary constant, the function $w(Z)$ is determined by the inversion of (1), and the function $\varphi(t)$ is determined by the first-order separable ordinary differential equation

$$\varphi'_t = A_3\varphi^3 + A_4\varphi^2, \tag{2}$$

whose general solution can be written out in implicit form.

In special cases, solutions of equation (2) are given by

$$\varphi(t) = (C_2 - 2A_3 t)^{-1/2} \quad \text{if } A_4 = 0,$$
$$\varphi(t) = (C_2 - A_4 t)^{-1} \qquad \text{if } A_3 = 0.$$

4°. Conservation law:

$$D_t(w) + D_x\bigl[-f(w)w_x - G(w)\bigr] = 0, \qquad G(w) = \int g(w)\,dw.$$

6. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] + g(t)\frac{\partial w}{\partial x}.$

This equation governs unsteady heat conduction in a moving medium in the case where the thermal diffusivity is arbitrarily dependent on temperature.

On passing from t, x to the new variables t, $z = x + \int g(t)\,dt$, one obtains a simpler equation of the form 1.6.15.1:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial z}\left[f(w)\frac{\partial w}{\partial z}\right].$$

7. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] + xg(t)\frac{\partial w}{\partial x}.$

On passing from t, x to the new variables (A and B are arbitrary constants)

$$\tau = \int G^2(t)\,dt + A, \quad z = xG(t), \qquad \text{where} \quad G(t) = B\exp\left[\int g(t)\,dt\right],$$

one obtains a simpler equation of the form 1.6.15.1 for $w(\tau, z)$:

$$\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial z}\left[f(w)\frac{\partial w}{\partial z}\right].$$

8. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] + \bigl[xg(t) + h(t)\bigr]\frac{\partial w}{\partial x}.$

The transformation

$$w = U(z,\tau), \quad z = xG(t) + \int h(t)G(t)\,dt, \quad \tau = \int G^2(t)\,dt, \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 1.6.15.1:

$$\frac{\partial U}{\partial \tau} = \frac{\partial}{\partial z}\left[f(U)\frac{\partial U}{\partial z}\right].$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

9. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w)\dfrac{\partial w}{\partial x} + h(w).$

For $g \equiv \text{const}$, this equation governs unsteady heat conduction in a medium moving at a constant velocity in the case where the thermal diffusivity and the reaction rate are arbitrary functions of temperature.

Traveling-wave solution:

$$w = w(z), \quad z = x + \lambda t,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$[f(w)w'_z]'_z + [g(w) - \lambda]w'_z + h(w) = 0. \tag{1}$$

The substitution $y(w) = f(w)w'_z$ brings (1) to the Abel equation

$$yy'_w + [g(w) - \lambda]y + f(w)g(w) = 0. \tag{2}$$

The books by Polyanin and Zaitsev (1995, 2003) present a large number of exact solutions to equation (2) for various $f(w)$, $g(w)$, and $h(w)$.

10. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \big[ax + g(w)\big]\dfrac{\partial w}{\partial x} + h(w).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1e^{-at}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \quad z = x + C_1e^{-at},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$[f(w)w'_z]'_z + [az + g(w)]w'_z + h(w) = 0.$$

1.6.16. Equations of the Form $\dfrac{\partial w}{\partial t} = f(x,w)\dfrac{\partial^2 w}{\partial x^2}$

1. $\dfrac{\partial w}{\partial t} = f(x)w^m\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1w(x, C_1^m t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = (m\lambda t + C)^{-1/m}\varphi(x),$$

where C and λ are arbitrary constants, and the function $\varphi = \varphi(x)$ is determined by the generalized Emden–Fowler equation

$$\varphi''_{xx} + \lambda\big[f(x)\big]^{-1}\varphi^{1-m} = 0. \tag{1}$$

For $m = 1$, a solution of equation (1) is given by

$$\varphi(x) = -\lambda\int_{x_0}^{x}\frac{(x-\xi)}{f(\xi)}\,d\xi + Ax + B,$$

where A, B, and x_0 are arbitrary constants.

The books by Polyanin and Zaitsev (1995, 2003) present a large number of solutions to equation (1) for various $f(x)$.

3°. The transformation $u = w/x$, $\xi = 1/x$ leads to an equation of the similar form

$$\frac{\partial u}{\partial t} = F(\xi)u^m\frac{\partial^2 u}{\partial \xi^2}, \qquad F(\xi) = \xi^{4-m}f(1/\xi).$$

2. $\dfrac{\partial w}{\partial t} = \dfrac{f(x)}{aw+b}\dfrac{\partial^2 w}{\partial x^2}.$

Generalized separable solution linear in t:

$$w(x,t) = \frac{1}{a}\bigl[\varphi(x)t + \psi(x) - b\bigr],$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the system of ordinary differential equations

$$f(x)\varphi''_{xx} - \varphi^2 = 0,$$
$$f(x)\psi''_{xx} - \varphi\psi = 0.$$

The first equation can be treated independently from the second. The second equation has a particular solution $\psi(x) = \varphi(x)$, so its general solution is given by

$$\psi(x) = C_1\varphi(x) + C_2\varphi(x)\int \frac{dx}{\varphi^2(x)},$$

where C_1 and C_2 are arbitrary constants.

3. $\dfrac{\partial w}{\partial t} = f(w)\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(C_1x + C_2, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution in implicit form:

$$k^2\int \frac{dw}{\lambda F(w) + C_1} = kx + \lambda t + C_2, \qquad F(w) = \int \frac{dw}{f(w)},$$

where C_1, C_2, k, and λ are arbitrary constants.

3°. Self-similar solution:

$$w = U(z), \qquad z = \frac{x + C_1}{\sqrt{C_2 t + C_3}},$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $U(z)$ is determined by the ordinary differential equation

$$f(U)U''_{zz} + \tfrac{1}{2}C_2 z U'_z = 0.$$

4°. The substitution $u = \displaystyle\int \frac{dw}{f(w)}$ leads to an equation of the form 1.6.15.1:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[F(u)\frac{\partial u}{\partial x}\right],$$

where the function F is defined parametrically as

$$F(u) = f(w), \qquad u = \int \frac{dw}{f(w)}.$$

To obtain $F = F(u)$ in explicit form, one should eliminate w from the two relations.

5°. Conservation laws:

$$D_t(u) + D_x(-w_x) = 0,$$
$$D_t(xu) + D_x(w - xw_x) = 0,$$

where $D_t = \frac{\partial}{\partial t}$, $D_x = \frac{\partial}{\partial x}$; w_x is the partial derivative of w with respect to x; and u is defined in Item 4°.

4. $\dfrac{\partial w}{\partial t} = x^4 f\left(\dfrac{w}{x}\right)\dfrac{\partial^2 w}{\partial x^2}.$

The transformation $u = w/x$, $\xi = 1/x$ leads to a simpler equation of the form 1.6.16.3:

$$\frac{\partial u}{\partial t} = f(u)\frac{\partial^2 u}{\partial \xi^2}.$$

5. $\dfrac{\partial w}{\partial t} = w^4 f\left(\dfrac{w}{\sqrt{ax^2+bx+c}}\right)\dfrac{\partial^2 w}{\partial x^2}.$

With the transformation

$$w(x,t) = u(z,t)\sqrt{ax^2+bx+c}, \qquad z = \int \frac{dx}{ax^2+bx+c}$$

one arrives at the simpler equation

$$\frac{\partial u}{\partial t} = u^4 f(u)\frac{\partial^2 u}{\partial z^2} + \left(ac - \tfrac{1}{4}b^2\right)u^5 f(u),$$

which has a traveling-wave solution $u = u(z + \lambda t)$.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

1.6.17. Equations of the Form $\dfrac{\partial w}{\partial t} = f(x,t,w)\dfrac{\partial^2 w}{\partial x^2} + g\left(x,t,w,\dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial^2 w}{\partial x^2} + w\left(\dfrac{\partial w}{\partial x}\right)^2 - aw^3.$

1°. Multiplicative separable solutions for $a > 0$:

$$w(x,t) = C\exp\left[\pm x\sqrt{a} + a\int f(t)\,dt\right],$$

where C is an arbitrary constant.

2°. Multiplicative separable solution for $a > 0$:

$$w(x,t) = \left(C_1 e^{x\sqrt{a}} + C_2 e^{-x\sqrt{a}}\right)e^F\left(C_3 + 8aC_1C_2\int e^{2F}\,dt\right)^{-1/2}, \qquad F = a\int f(t)\,dt,$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Multiplicative separable solution for $a < 0$:

$$w(x,t) = \left[C_1\sin\left(x\sqrt{|a|}\right) + C_2\cos\left(x\sqrt{|a|}\right)\right]e^F\left[C_3 + 2a(C_1^2 + C_2^2)\int e^{2F}\,dt\right]^{-1/2},$$

where $F = a\displaystyle\int f(t)\,dt$; C_1, C_2, and C_3 are arbitrary constants.

2. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + g(t)w + h_2(t)x^2 + h_1(t)x + h_0(t).$

Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_t' &= 6f(t)\varphi^2 + g(t)\varphi + h_2(t),\\
\psi_t' &= 6f(t)\varphi\psi + g(t)\psi + h_1(t),\\
\chi_t' &= 2f(t)\varphi\chi + f(t)\psi^2 + g(t)\chi + h_0(t).
\end{aligned}$$

3. $\dfrac{\partial w}{\partial t} + f(t)w\dfrac{\partial w}{\partial x} = g(t)\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right).$

Degenerate solution linear in x:

$$w(x,t) = \frac{1}{F(t)}\left[x + \int \frac{g(t)}{F(t)}\,dt + C_1\right], \qquad F(t) = \int f(t)\,dt + C_2,$$

where C_1 and C_2 are arbitrary constants.

4. $\dfrac{\partial w}{\partial t} + f(t)w\dfrac{\partial w}{\partial x} = g(t)\dfrac{\partial}{\partial x}\left(w^2\dfrac{\partial w}{\partial x}\right).$

Degenerate solution linear in x:

$$w(x,t) = (x + C_1)\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi'_t = 2g(t)\varphi^3 - f(t)\varphi^2.$$

5. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)w^m\dfrac{\partial w}{\partial x}\right].$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^m t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = (m\lambda t + C)^{-1/m}\varphi(x),$$

where C and λ are arbitrary constants, and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$[f(x)\varphi^m\varphi'_x]'_x + \lambda\varphi = 0. \tag{1}$$

The transformation

$$z = \int \frac{dx}{f(x)}, \qquad \Phi = \varphi^{m+1}$$

brings (1) to the generalized Emden–Fowler equation

$$\Phi''_{zz} + F(z)\Phi^{\frac{1}{m+1}} = 0, \tag{2}$$

where the function $F = F(z)$ is defined parametrically by

$$F = \lambda(m+1)f(x), \qquad z = \int \frac{dx}{f(x)}.$$

The book by Polyanin and Zaitsev (2003, Sections 2.3 and 2.7) presents a large number of solutions to equation (2) for various $F = F(z)$.

3°. The transformation

$$w(x,t) = \big[\psi(x)\big]^{\frac{1}{m+1}}u(\xi,t), \quad \xi = -\int \big[\psi(x)\big]^{\frac{m+2}{m+1}}dx, \qquad \psi(x) = \int \frac{dx}{f(x)},$$

leads to an equation of the similar form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial \xi}\left[F(\xi)u^m\frac{\partial u}{\partial \xi}\right],$$

where the function $F = F(\xi)$ is defined parametrically by

$$F = f(x)[\psi(x)]^{\frac{3m+4}{m+1}}, \quad \xi = -\int \big[\psi(x)\big]^{\frac{m+2}{m+1}}dx, \qquad \psi(x) = \int \frac{dx}{f(x)}.$$

6. $\dfrac{\partial w}{\partial t} = f(x)w^m \dfrac{\partial^2 w}{\partial x^2} + g(x)w^{m+1}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^m t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution (C and λ are arbitrary constants):

$$w(x,t) = (m\lambda t + C)^{-1/m}\varphi(x),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi^m \varphi''_{xx} + g(x)\varphi^{m+1} + \lambda\varphi = 0. \tag{1}$$

In the special case of $f(x) = ax^n$ and $g(x) = bx^k$, equation (1) becomes

$$\varphi''_{xx} + (b/a)x^{k-n}\varphi + (\lambda/a)x^{-n}\varphi^{1-m} = 0. \tag{2}$$

The books by Polyanin and Zaitsev (1995, 2003) present a large number of solutions to equation (2) for various values of n, m, and k.

7. $\dfrac{\partial w}{\partial t} = f(t)\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + g(t)w^{1-m}.$

Functional separable solution:

$$w(x,t) = \left[\varphi(t)x^2 + \psi(t)\right]^{1/m},$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are determined by the system of first-order ordinary differential equations

$$\varphi'_t = \frac{2(m+2)}{m} f\varphi^2, \qquad \psi'_t = 2f\varphi\psi + mg.$$

Integrating yields

$$\varphi = \frac{1}{F}, \quad \psi = F^{-\frac{m}{m+2}}\left(A + m\int gF^{\frac{m}{m+2}}\,dt\right),$$
$$F = B - \frac{2(m+2)}{m}\int f\,dt,$$

where A and B are arbitrary constants.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)w^m \dfrac{\partial w}{\partial x}\right] + g(x)w^{m+1}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^m t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution (C and λ are arbitrary constants):

$$w(x,t) = (m\lambda t + C)^{-1/m}\varphi(x),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$[f(x)\varphi^m \varphi'_x]'_x + g(x)\varphi^{m+1} + \lambda\varphi = 0. \tag{1}$$

The transformation

$$z = \int \frac{dx}{f(x)}, \quad \Phi = \varphi^{m+1}$$

brings (1) to the equation

$$\Phi''_{zz} + F(z)\Phi^{\frac{1}{m+1}} + G(z)\Phi = 0, \tag{2}$$

where the functions $F = F(z)$ and $G = G(z)$ are defined parametrically by

$$\begin{cases} F = \lambda(m+1)f(x), \\ z = \displaystyle\int \frac{dx}{f(x)}, \end{cases} \qquad \begin{cases} G = (m+1)f(x)g(x), \\ z = \displaystyle\int \frac{dx}{f(x)}. \end{cases}$$

In the special case of $f(x) = ax^n$ and $g(x) = bx^k$, equation (2) becomes

$$\Phi''_{zz} + Az^{\frac{n}{1-n}}\Phi^{\frac{1}{m+1}} + Bz^{\frac{n+k}{1-n}}\Phi = 0, \qquad n \neq 1, \tag{3}$$

where $A = \lambda a(m+1)\big[a(1-n)\big]^{\frac{n}{1-n}}$ and $B = ab(m+1)\big[a(1-n)\big]^{\frac{n+k}{1-n}}$.

The books by Polyanin and Zaitsev (1995, 2003) present a large number of solutions to equation (3) for various values of n, m, and k.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

9. $\displaystyle \frac{\partial w}{\partial t} = f(t)\frac{\partial}{\partial x}\left(w^m \frac{\partial w}{\partial x}\right) + g(t)\frac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = u(z,\tau)\exp\left[\int h(t)\,dt\right], \quad z = x + \int g(t)\,dt, \quad \tau = \int f(t)\exp\left[m\int h(t)dt\right]dt$$

leads to a simpler equation of the form 1.1.10.7:

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial z}\left(u^m \frac{\partial u}{\partial z}\right).$$

10. $\displaystyle \frac{\partial w}{\partial t} = f(t)\frac{\partial}{\partial x}\left(w^m \frac{\partial w}{\partial x}\right) + \big[xg(t) + h(t)\big]\frac{\partial w}{\partial x} + s(t)w.$

The transformation

$$w(x,t) = u(z,\tau)S(t), \quad z = xG(t) + \int h(t)G(t)\,dt, \quad \tau = \int f(t)G^2(t)S^m(t)\,dt,$$

where the functions $S(t)$ and $G(t)$ are given by

$$S(t) = \exp\left[\int s(t)\,dt\right], \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 1.1.10.7:

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial z}\left(u^m \frac{\partial u}{\partial z}\right).$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

11. $\dfrac{\partial w}{\partial t} = x^k f(t)\dfrac{\partial}{\partial x}\left(w^m \dfrac{\partial w}{\partial x}\right) + xg(t)\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(t,x) = u(z,\tau)H(t), \quad z = xG(t), \quad \tau = \int f(t)G^{2-k}(t)H^m(t)\,dt,$$

where the functions $G(t)$ and $H(t)$ are given by

$$G(t) = \exp\left[\int g(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

leads to a simpler equation of the form 1.1.15.6:

$$\frac{\partial u}{\partial \tau} = z^k \frac{\partial}{\partial z}\left(u^m \frac{\partial u}{\partial z}\right).$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

12. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)e^{\beta w}\dfrac{\partial w}{\partial x}\right].$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x, C_1 t + C_2) + \frac{1}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = -\frac{1}{\beta}\ln(\beta t + C) + \frac{1}{\beta}\ln\left[\int \frac{A - \beta x}{f(x)}\,dx + B\right],$$

where A, B, and C are arbitrary constants.

13. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(x)e^{\beta w}\dfrac{\partial w}{\partial x}\right] + g(x)e^{\beta w}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x, C_1 t + C_2) + \frac{1}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = -\frac{1}{\beta}\ln(\beta t + C) + \varphi(x),$$

where β and C are arbitrary constants, and the function $\varphi(x)$ is determined by the second-order linear ordinary differential equation

$$[f(x)\psi'_x]'_x + \beta g(x)\psi + \beta = 0, \qquad \psi = e^{\beta\varphi}.$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

14. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left\{\left[f_2(t)w^{2n} + f_1(t)w^n\right]\dfrac{\partial w}{\partial x}\right\} + g_1(t)w + g_2(t)w^{1-n}.$

Generalized traveling-wave solution:

$$w(x,t) = \left[\varphi(t)x + \psi(t)\right]^{1/n},$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = \frac{(n+1)}{n}f_2(t)\varphi^3 + ng_1(t)\varphi,$$
$$\psi'_t = \frac{(n+1)}{n}f_2(t)\varphi^2\psi + ng_1(t)\psi + \frac{1}{n}f_1(t)\varphi^2 + ng_2(t),$$

which is easy to integrate (the first equation is a Bernoulli equation and the second one is linear in ψ).

15. $\dfrac{\partial w}{\partial t} = f(w)\dfrac{\partial^2 w}{\partial x^2} + \left[ax + g(w)\right]\dfrac{\partial w}{\partial x} + h(w).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1e^{-at}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + C_1e^{-at},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$f(w)w''_{zz} + [az + g(w)]w'_z + h(w) = 0.$$

16. $\dfrac{\partial w}{\partial t} = x^{1-n}\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right].$

Nonlinear problems of the diffusion boundary layer, defined by equation 1.6.19.2, are reducible to equations of this form. For $n = 1$, see equation 1.6.15.1, and for $f(w) = aw^m$, see equation 1.1.15.6.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(C_1x, C_1^{n+1}t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Self-similar solution for $n \neq -1$:

$$w = w(z), \qquad z = xt^{-\frac{1}{n+1}} \qquad (0 \le x < \infty),$$

where the function $w(z)$ is determined by the ordinary differential equation

$$(n+1)[f(w)w'_z]'_z + z^n w'_z = 0, \tag{1}$$

which is often accompanied by the boundary conditions of (3) in 1.6.15.1.

The general solution of equation (1) with $f(w) = a(w+b)^{-1}$ and arbitrary n can be found in Zaitsev and Polyanin (1993).

3°. We now describe a simple way to find functions $f(w)$ for which equation (1) admits exact solutions. Let us integrate (1) with respect to z and then apply the hodograph transformation (with w regarded as the independent variable and z as the dependent one) to obtain

$$f(w) = -\frac{1}{n+1} z'_w \left(\int z^n \, dw + A \right), \qquad A \text{ is any.} \tag{2}$$

Substituting a specific $z = z(w)$ for z on the right-hand side of (2), one obtains a one-parameter family of functions $f(w)$ for which $z = z(w)$ solves equation (1). An explicit form of the solution, $w = w(z)$, is determined by the inversion of $z = z(w)$.

For example, setting $z = (1-w)^k$, one obtains from (2) the corresponding $f(w)$:

$$f(w) = A(1-w)^{k-1} - \frac{k}{(n+1)(nk+1)}(1-w)^{k(n+1)}, \qquad A \text{ is any.}$$

4°. There is another way to construct $f(w)$ for which equation (1) admits exact solutions. It involves the following. Let $\bar{w} = \bar{w}(z)$ be a solution of equation (1) with some function $f(w)$. Then $\bar{w} = \bar{w}(z)$ is also a solution of the more complicated equation $(n+1)[F(w)w'_z]'_z + z^n w'_z = 0$ with

$$F(w) = f(w) + Ag(w) \qquad (A \text{ is any}), \tag{3}$$

where the function $g = g(w)$ is defined parametrically by

$$g(w) = \frac{1}{\bar{w}'_z}, \quad w = \bar{w}(z). \tag{4}$$

For example, if $f(w)$ is a power-law function of w, $f(w) = aw^m$, then $\bar{w} = bz^{\frac{n+1}{m}}$ is a solution of equation (1), with b being a constant. It follows from (3) and (4) that $\bar{w}$ is also a solution of equation (1) with $f(w) = aw^m + Aw^{\frac{m-n-1}{n+1}}$.

5°. For $n = -1$, there is an exact solution of the form

$$w = w(\xi), \qquad \xi = \ln|x| + \lambda t,$$

where the function $w(\xi)$ is defined implicitly by

$$\int \frac{f(w)\,dw}{\lambda w + F(w) + C_1} = \xi + C_2, \qquad F(w) = \int f(w)\,dw,$$

where λ, C_1, and C_2 are arbitrary constants. To $\lambda = 0$ there corresponds a stationary solution.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

17. $\displaystyle \frac{\partial w}{\partial t} = \frac{1}{x^n}\frac{\partial}{\partial x}\left[x^n f(w)\frac{\partial w}{\partial x}\right] + g(w).$

This is a nonlinear equation of heat and mass transfer in the radial symmetric case ($n = 1$ corresponds to a plane problem and $n = 2$ to a spatial one).

1°. Let $f(w)$ and $g(w)$ be defined by

$$f(w) = w\varphi'_w(w), \qquad g(w) = a(n+1)w + 2a\frac{\varphi(w)}{\varphi'_w(w)},$$

where $\varphi(w)$ is an arbitrary function. In this case, there is a functional separable solution defined implicitly by

$$\varphi(w) = Ce^{2at} - \tfrac{1}{2}ax^2,$$

where C is an arbitrary constant.

⊙ *Reference*: V. A. Galaktionov (1994).

2°. Let $f(w)$ and $g(w)$ be defined as follows:

$$f(w) = a\varphi^{-\frac{n+1}{2}}\varphi' \int \varphi^{\frac{n+1}{2}}\,dw, \qquad g(w) = b\frac{\varphi}{\varphi'},$$

where $\varphi = \varphi(w)$ is an arbitrary function. In this case, there is a functional separable solution defined implicitly by

$$\varphi(w) = \frac{bx^2}{Ce^{-bt} - 4a},$$

where C is an arbitrary constant.

18. $\displaystyle \frac{\partial w}{\partial t} = f(t)\varphi(w)\frac{\partial^2 w}{\partial x^2} + \big[xg(t) + h(t)\big]\frac{\partial w}{\partial x}.$

The transformation

$$z = xG(t) + \int h(t)G(t)\,dt, \quad \tau = \int f(t)G^2(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 1.6.16.3:

$$\frac{\partial w}{\partial \tau} = \varphi(w)\frac{\partial^2 w}{\partial z^2}.$$

19. $\displaystyle \frac{\partial w}{\partial t} = f(t)\frac{\partial}{\partial x}\left[\varphi(w)\frac{\partial w}{\partial x}\right] + \big[xg(t) + h(t)\big]\frac{\partial w}{\partial x}.$

The transformation

$$z = xG(t) + \int h(t)G(t)\,dt, \quad \tau = \int f(t)G^2(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 1.6.15.1:

$$\frac{\partial w}{\partial \tau} = \frac{\partial}{\partial z}\left[\varphi(w)\frac{\partial w}{\partial z}\right].$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

20. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial^2}{\partial x^2}\big[f(x,w)\big] + \frac{g(t)}{f_w(x,w)} + h(x).$

Here, f_w is the partial derivative of f with respect to w.

Functional separable solution in implicit form:

$$f(x,w) = \int g(t)\,dt - \int_{x_0}^{x}(x-\xi)h(\xi)\,d\xi + C_1x + C_2,$$

where C_1 and C_2 are arbitrary constants, and x_0 is any number.

1.6.18. Equations of the Form $\frac{\partial w}{\partial t} = f\left(x, w, \frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial x^2} + g\left(x, t, w, \frac{\partial w}{\partial x}\right)$

1. $\displaystyle \frac{\partial w}{\partial t} = a\left(\frac{\partial w}{\partial x}\right)^k\frac{\partial^2 w}{\partial x^2} + \big[f(t)x + g(t)\big]\frac{\partial w}{\partial x} + h(t)w.$

With the transformation

$$w(x,t) = u(z,\tau)H(t), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^{k+2}(t)H^k(t)\,dt,$$

where the functions $F(t)$ and $H(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

one arrives at the simpler equation

$$\frac{\partial u}{\partial \tau} = a\left(\frac{\partial u}{\partial x}\right)^k \frac{\partial^2 u}{\partial z^2}.$$

See equation 1.6.18.3, the special case 1.

2. $\displaystyle \frac{\partial w}{\partial t} = f(x)\left(\frac{\partial w}{\partial x}\right)^k \frac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_1^k t + C_2) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where the function $\varphi(x)$ is given by

$$\varphi(x) = \int \left[A(k+1)\int \frac{dx}{f(x)} + C_1\right]^{\frac{1}{k+1}} dx + C_2 \quad \text{if } k \neq -1,$$

$$\varphi(x) = C_1 \int \exp\left[A\int \frac{dx}{f(x)}\right] dx + C_2 \quad \text{if } k = -1,$$

A, B, C_1, and C_2 are arbitrary constants.

3°. Solution:

$$w(x,t) = (Akt + B)^{-1/k}\Theta(x) + C,$$

where A, B, and C are arbitrary constants, and the function $\Theta(x)$ is determined by the second-order ordinary differential equation

$$f(x)\bigl(\Theta'_x\bigr)^k \Theta''_{xx} + A\Theta = 0.$$

3. $\displaystyle \frac{\partial w}{\partial t} = f\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial x^2}.$

This equation occurs in the nonlinear theory of flows in porous media; it governs also the motion of a nonlinear viscoplastic medium.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-1} w(C_1 x + C_2, C_1^2 t + C_3) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = At + B + \varphi(z), \qquad z = kx + \lambda t, \tag{1}$$

where A, B, k, and λ are arbitrary constants, and the function $\varphi(z)$ is determined by the ordinary differential equation

$$k^2 f\bigl(k\varphi'_z)\varphi''_{zz} = \lambda\varphi'_z + A. \tag{2}$$

The general solution of equation (2) can be rewritten in parametric form as

$$\varphi = k\int \frac{uf(u)\,du}{\lambda u + Ak} + C_1, \qquad z = k^2 \int \frac{f(u)\,du}{\lambda u + Ak} + C_2, \tag{3}$$

where C_1 and C_2 are arbitrary constants,

Relations (1) and (3) define a traveling-wave solution for $A = 0$ and an additive separable solution for $\lambda = 0$.

3°. Self-similar solution:

$$w(x,t)=\sqrt{t}\,\Theta(\xi),\quad \xi=\frac{x}{\sqrt{t}},$$

where the function $\Theta(\xi)$ is determined by the ordinary differential equation

$$2f\left(\Theta'_\xi\right)\Theta''_{\xi\xi}+\xi\Theta'_\xi-\Theta=0.$$

4°. The substitution $u(x,t)=\dfrac{\partial w}{\partial x}$ leads to an equation of the form 1.6.15.1:

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[f(u)\frac{\partial u}{\partial x}\right].$$

5°. The hodograph transformation

$$\bar{x}=w(x,t),\quad \bar{w}(\bar{x},t)=x$$

leads to an equation of the similar form

$$\frac{\partial \bar{w}}{\partial t}=\bar{f}\left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)\frac{\partial^2 \bar{w}}{\partial \bar{x}^2},\qquad \bar{f}(z)=\frac{1}{z^2}f\left(\frac{1}{z}\right).$$

6°. The transformation

$$\bar{t}=\alpha t+\gamma_1,\quad \bar{x}=\beta_1 x+\beta_2 w+\gamma_2,\quad \bar{w}=\beta_3 x+\beta_4 w+\gamma_3,$$

where α, the β_i, and the γ_i are arbitrary constants such that $\alpha\neq 0$ and $\beta_1\beta_4-\beta_2\beta_3\neq 0$, takes the original equation to an equation with the same form. We have

$$\bar{f}(\bar{w}_{\bar{x}})=\frac{1}{\alpha}(\beta_1+\beta_2 w_x)^2 f(w_x),\qquad w_x=\frac{\beta_1\bar{w}_{\bar{x}}-\beta_3}{\beta_4-\beta_2\bar{w}_{\bar{x}}},$$

where the subscripts x and $\bar{x}$ denote the corresponding partial derivatives.

Special case 1. Equation

$$\frac{\partial w}{\partial t}=a\left(\frac{\partial w}{\partial x}\right)^k\frac{\partial^2 w}{\partial x^2},\quad k\neq 0.$$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1=C_1 w(C_2 x+C_3,\,C_1^k C_2^{k+2}t+C_4)+C_5,$$

where $C_1,\ldots,C_5$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t)=(t+C_1)^{-1/k}u(x)+C_2,$$

where C_1 and C_2 are arbitrary constants, and $u(x)$ is determined by the ordinary differential equation $ak(u'_x)^k u''_{xx}+u=0$, the general solution of which can be written out in the implicit form as

$$\int\left(C_3-\frac{k+2}{2ak}u^2\right)^{-\frac{1}{k+2}}du=x+C_4.$$

Special case 2. Equation

$$\frac{\partial w}{\partial t}=\frac{a}{w_x^2+b^2}\frac{\partial^2 w}{\partial x^2},\qquad w_x=\frac{\partial w}{\partial x}.$$

1. Solution:

$$w(x,t)=\pm\sqrt{C_1-b^2(x+C_2)^2-2at}+C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

2. Solution:

$$w=bx\tan\left(\pm\frac{1}{2}z-\arctan\left(\psi(z)\right)\pm\frac{a}{b^2}t+C\right),$$

$$z=x^2\cos^{-2}\left(\pm\frac{1}{2}z-\arctan\left(\psi(z)\right)\pm\frac{a}{b^2}t+C\right),$$

where C is an arbitrary constant, and the function $\psi = \psi(z)$ is determined by the ordinary differential equation

$$\psi'_z = \frac{1}{2}(1+\psi^2)\left(\pm 1 - \frac{\psi}{z}\right).$$

The function $z = z(x,t)$ in the solution is defined implicitly.

3. Solution:

$$w = bx \tan\left(\varphi(z) + \frac{C}{2}\ln\frac{at}{b^2}\right),$$
$$z = \frac{b^2x^2}{at}\cos^{-2}\left(\varphi(z) + \frac{C}{2}\ln\frac{at}{b^2}\right),$$

where C is an arbitrary constant, and the functions $\varphi(z)$ and $\psi(z)$ are determined by the system of ordinary differential equations

$$\varphi'_z = \frac{\psi}{2z}, \quad \psi'_z = \frac{1}{2}(1+\psi^2)\left(\frac{C}{2} - \frac{\psi}{2} - \frac{\psi}{z}\right).$$

The function $z = z(x,t)$ in the solution is defined implicitly.

Special case 3. Equation

$$\frac{\partial w}{\partial t} = k\exp\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial x^2}.$$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(C_1 x + C_2, C_1^2 e^{C_3} t + C_4) + C_3 x + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = 2A\arctan\left(\frac{x+B}{A}\right) - (x+B)\ln\left|\frac{kt+C}{(x+B)^2+A^2}\right| - (2+\ln 2)x + D,$$

where A, B, C, and D are arbitrary constants.

⊙ *References for equation* 1.6.18.3: E. V. Lenskii (1966), I. Sh. Akhatov, R. K. Gazizov, and N. H. Ibragimov (1989), N. H. Ibragimov (1994).

4. $\dfrac{\partial w}{\partial t} = f(x)g(w)h(w_x)\dfrac{\partial^2 w}{\partial x^2}, \qquad w_x = \dfrac{\partial w}{\partial x}.$

The hodograph transformation, according to which x is taken to be the independent variable and w the dependent one,

$$x = u, \quad w = y,$$

leads to a similar equation for $u = u(y,t)$:

$$\frac{\partial u}{\partial t} = g(y)f(u)\overline{h}(u_y)\frac{\partial^2 u}{\partial y^2}, \qquad \text{where} \quad \overline{h}(z) = z^{-2}h(1/z).$$

5. $\dfrac{\partial w}{\partial t}\dfrac{\partial^2 w}{\partial x^2} = f\left(t, \dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(C_1 x + C_2, t) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. The Euler transformation

$$w(x,t) + u(\xi,\eta) = x\xi, \quad x = \frac{\partial u}{\partial \xi}, \quad t = \eta$$

leads to the linear equation

$$\frac{\partial u}{\partial \eta} = -f(\eta,\xi)\frac{\partial^2 u}{\partial \xi^2};$$

for details, see Subsection S.2.3 (Example 7).

1.6.19. Nonlinear Equations of the Thermal (Diffusion) Boundary Layer

1. $\displaystyle f(x)\frac{\partial w}{\partial x}+g(x)y\frac{\partial w}{\partial y}=\frac{\partial}{\partial y}\left[\varphi(w)\frac{\partial w}{\partial y}\right].$

This equation is encountered in nonlinear problems of the steady diffusion boundary layer (mass exchange between drops or bubbles and a flow); the coordinates x and y are reckoned along and normal to the interphase surface, respectively.

The transformation (A and B are arbitrary constants)

$$t=\int\frac{h^2(x)}{f(x)}dx+A,\quad z=yh(x),\qquad \text{where}\quad h(x)=B\exp\left[-\int\frac{g(x)}{f(x)}dx\right],$$

leads to a simpler equation of the form 1.6.15.1:

$$\frac{\partial w}{\partial t}=\frac{\partial}{\partial z}\left[\varphi(w)\frac{\partial w}{\partial z}\right].$$

⊙ *References*: A. D. Polyanin (1980, 1982), V. F. Zaitsev and A. D. Polyanin (1996).

2. $\displaystyle f(x)y^{n-1}\frac{\partial w}{\partial x}+g(x)y^n\frac{\partial w}{\partial y}=\frac{\partial}{\partial y}\left[\varphi(w)\frac{\partial w}{\partial y}\right].$

This equation is encountered in nonlinear problems of the steady diffusion boundary layer (mass exchange between solid particles, drops, or bubbles and the ambient medium; convective diffusion to a flat plate and that in liquid films); the coordinates x and y are reckoned along and normal to the body surface, respectively. The value $n=2$ corresponds to a solid particle and $n=1$, to a drop or a bubble.

The transformation (A and B are arbitrary constants)

$$t=\int\frac{h^{n+1}(x)}{f(x)}dx+A,\quad z=yh(x),\qquad \text{where}\quad h(x)=B\exp\left[-\int\frac{g(x)}{f(x)}dx\right],$$

leads to a simpler equation of the form 1.6.17.16:

$$\frac{\partial w}{\partial t}=z^{1-n}\frac{\partial}{\partial z}\left[\varphi(w)\frac{\partial w}{\partial z}\right].$$

⊙ *References*: Yu. P. Gupalo, A. D. Polyanin, and Yu. S. Ryazantsev (1985), V. F. Zaitsev and A. D. Polyanin (1996).

3. $\displaystyle f\left(\frac{y}{\sqrt{x}}\right)\frac{\partial w}{\partial x}+\frac{1}{\sqrt{x}}g\left(\frac{y}{\sqrt{x}}\right)\frac{\partial w}{\partial y}=\frac{\partial}{\partial y}\left[\varphi(w)\frac{\partial w}{\partial y}\right].$

This is a generalization of the linear equation of the thermal boundary layer on a flat plate.

1°. Self-similar solution:

$$w=w(\xi),\quad \xi=\frac{y}{\sqrt{x}},\tag{1}$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$[\varphi(w)w'_\xi]'_\xi+\left[\tfrac{1}{2}\xi f(\xi)-g(\xi)\right]w'_\xi=0.\tag{2}$$

2°. Solving the original partial differential equation with simple boundary conditions of the first kind,

$$x=0,\ w=a;\quad y=0,\ w=b;\quad y\to\infty,\ w\to a,$$

where a are b are some constants, is reduced to solving equation (2) with the boundary conditions

$$\xi=0,\ w=b;\quad \xi\to\infty,\ w\to a.$$

Remark. The classical thermal boundary layer equation is defined by

$$f(\xi) = \Pr F'_\xi(\xi), \qquad g(\xi) = \tfrac{1}{2}\Pr\bigl[\xi F'_\xi(\xi) - F(\xi)\bigr],$$

where $F(\xi)$ is the Blasius solution in the hydrodynamic problem on the longitudinal homogeneous translational flow of a viscid incompressible fluid past a flat plane, and Pr is the Prandtl number (x the coordinate along the plate and y the coordinate normal to the plate surface).

⊙ *References*: H. Schlichting (1981), A. D. Polyanin and V. F. Zaitsev (2002).

1.7. Nonlinear Schrödinger Equations and Related Equations

1.7.1. Equations of the Form $i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} + f(|w|)w = 0$ Involving Arbitrary Parameters

► *Throughout this subsection, w is a complex functions of real variables x and t; $i^2 = -1$.*

1. $\displaystyle i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + k|w|^2 w = 0.$

Schrödinger equation with a cubic nonlinearity. Here, k is a real number. This equation occurs in various chapters of theoretical physics, including nonlinear optics, superconductivity, and plasma physics.

1°. Suppose $w(x,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$w_1 = \pm A_1 w(\pm A_1 x + A_2,\, A_1^2 t + A_3),$$
$$w_2 = e^{-i(\lambda x + \lambda^2 t + B)} w(x + 2\lambda t,\, t),$$

where A_1, A_2, A_3, B, and λ are arbitrary real constants, are also solutions of the equation. The plus or minus signs in the expression of w_1 are chosen arbitrarily.

2°. Solutions:

$$w(x,t) = C_1 \exp\bigl\{i\,[C_2 x + (kC_1^2 - C_2^2)t + C_3]\bigr\},$$
$$w(x,t) = \pm C_1 \sqrt{\frac{2}{k}}\,\frac{\exp[i(C_1^2 t + C_2)]}{\cosh(C_1 x + C_3)},$$
$$w(x,t) = \pm A \sqrt{\frac{2}{k}}\,\frac{\exp[iBx + i(A^2 - B^2)t + iC_1]}{\cosh(Ax - 2ABt + C_2)},$$
$$w(x,t) = \frac{C_1}{\sqrt{t}} \exp\left[i\frac{(x + C_2)^2}{4t} + i(kC_1^2 \ln t + C_3)\right],$$

where A, B, C_1, C_2, and C_3 are arbitrary real constants. The second and third solutions are valid for $k > 0$. The third solution describes the motion of a soliton in a rapidly decaying case.

⊙ *Reference*: L. D. Faddeev and L. A. Takhtadjan (1987).

3°. Solution:

$$w(x,t) = (ax + b)\exp\bigl[i(\alpha x^2 + \beta x + \gamma)\bigr],$$

where the functions $a = a(t)$, $b = b(t)$, $\alpha = \alpha(t)$, $\beta = \beta(t)$, and $\gamma = \gamma(t)$ are determined by the autonomous system of ordinary differential equations

$$a'_t = -6a\alpha,$$
$$b'_t = -2a\beta - 2b\alpha,$$
$$\alpha'_t = ka^2 - 4\alpha^2,$$
$$\beta'_t = 2kab - 4\alpha\beta,$$
$$\gamma'_t = kb^2 - \beta^2.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

4°. N-soliton solutions for $k > 0$:

$$w(x,t) = \sqrt{\frac{2}{k}}\,\frac{\det \mathbf{R}(x,t)}{\det \mathbf{M}(x,t)}.$$

Here, $\mathbf{M}(x,t)$ is an $N \times N$ matrix with entries

$$M_{n,k}(x,t) = \frac{1+\overline{g}_n(x,t)g_n(x,t)}{\overline{\lambda}_n - \lambda_k}, \quad g_n(x,t) = \gamma_n e^{i(\lambda_n x - \lambda_n^2 t)}, \qquad n,\ k = 1,\ \ldots,\ N,$$

where the λ_n and γ_n are arbitrary complex numbers that satisfy the constraints $\operatorname{Im}\lambda_n > 0$ $(\lambda_n \neq \lambda_k$ if $n \neq k)$ and $\gamma_n \neq 0$; the bar over a symbol denotes the complex conjugate. The square matrix $\mathbf{R}(x,t)$ is of order $N+1$; it is obtained by augmenting $\mathbf{M}(x,t)$ with a column on the right and a row at the bottom. The entries of $\mathbf{R}$ are defined as

$$\begin{aligned}
&R_{n,k}(x,t) = M_{n,k}(x,t) && \text{for} \quad n,k = 1,\ \ldots,\ N && \text{(bulk of the matrix)},\\
&R_{n,N+1}(x,t) = g_n(x,t) && \text{for} \quad n = 1,\ \ldots,\ N && \text{(rightmost column)},\\
&R_{N+1,n}(x,t) = 1 && \text{for} \quad n = 1,\ \ldots,\ N && \text{(bottom row)},\\
&R_{N+1,N+1}(x,t) = 0 && && \text{(lower right diagonal entry)}.
\end{aligned}$$

The above solution can be represented, for $t \to \pm\infty$, as the sum of N single-soliton solutions.

⊙ *Reference*: L. D. Faddeev and L. A. Takhtadjan (1987).

5°. Self-similar solution:

$$w(x,t) = \frac{1}{\sqrt{C_1 t + C_2}}\,u(z), \qquad z = \frac{x + C_3}{\sqrt{C_1 t + C_2}},$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $u = u(z)$ is determined by the ordinary differential equation

$$u''_{zz} + k|u|^2 u - \tfrac{1}{2} i C_1 (z u'_z + u) = 0.$$

6°. For other exact solutions, see equation 1.7.5.1 with $f(u) = ku^2$.

7°. Auto-Bäcklund transformations preserving the form of the equation (with $k = 1$):

$$\begin{aligned}
\frac{\partial w}{\partial x} - \frac{\partial \widetilde{w}}{\partial x} &= i a f_1 - \frac{i}{2} f_2 g_1,\\
\frac{\partial w}{\partial t} - \frac{\partial \widetilde{w}}{\partial t} &= \frac{1}{2} g_1 \left(\frac{\partial w}{\partial x} + \frac{\partial \widetilde{w}}{\partial x} \right) - a g_2 + \frac{i}{4} f_1 \left(|f_1|^2 + |f_2|^2 \right).
\end{aligned}$$

Here,

$$f_1 = w - \widetilde{w}, \quad f_2 = w + \widetilde{w}, \quad g_1 = i\varepsilon\bigl(b - 2|f_1|^2\bigr)^{1/2}, \quad g_2 = i\bigl(a f_1 - \tfrac{1}{2} f_2 g_1\bigr),$$

where a and b are arbitrary real constants and $\varepsilon = \pm 1$.

⊙ *References*: G. L. Lamb (1974), N. H. Ibragimov (1985).

8°. The Schrödinger equation with a cubic nonlinearity admits infinitely many integrals of motion. The first three integrals for $k = 2$:

$$C_1 = \int_{-\infty}^{\infty} |w|^2\,dx, \quad C_2 = \int_{-\infty}^{\infty} \left(\overline{w}\frac{\partial w}{\partial x} - w\frac{\partial \overline{w}}{\partial x} \right) dx, \quad C_3 = \int_{-\infty}^{\infty} \left(2\left|\frac{\partial w}{\partial x}\right|^2 - |w|^4 \right) dx.$$

It is assumed here that the initial distribution $w(x,0)$ decays quite rapidly as $|x| \to \infty$. The bar over a symbol denotes the complex conjugate.

The first three integrals for $k = -2$:

$$C_1 = \int_{-\infty}^{\infty} \left(1 - |w|^2\right) dx, \quad C_2 = -\int_{-\infty}^{\infty} \left(\overline{w}\frac{\partial w}{\partial x} - w\frac{\partial \overline{w}}{\partial x} \right) dx, \quad C_3 = \int_{-\infty}^{\infty} \left(\left|\frac{\partial w}{\partial x}\right|^2 + |w|^4 - 1 \right) dx.$$

⊙ *References*: V. E. Zakharov and A. B. Shabat (1972), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984).

9°. The Schrödinger equation with a cubic nonlinearity is integrable by the inverse scattering method; see the literature cited below.

⊙ *References*: V. E. Zakharov and A. B. Shabat (1972), M. J. Ablowitz and H. Segur (1981), R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris (1982), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), L. D. Faddeev and L. A. Takhtadjan (1987), V. E. Korepin, N. N. Bogoliubov, and A. G. Izergin (1993), N. N. Akhmediev and A. Ankiewicz (1997), C. Sulem and P.-L. Sulem (1999).

2. $\displaystyle i\frac{\partial w}{\partial t}+\frac{\partial^2 w}{\partial x^2}+(A|w|^2+B)w=0.$

Schrödinger equation with a cubic nonlinearity. The numbers A and B are assumed real.

1°. Solutions:

$$w(x,t)=C_1\exp\bigl\{i\,[C_2x+(AC_1^2+B-C_2^2)t+C_3]\bigr\},$$
$$w(x,t)=\frac{C_1}{\sqrt{t}}\exp\left[i\frac{(x+C_2)^2}{4t}+i(AC_1^2\ln t+Bt+C_3)\right],$$

where C_1, C_2, and C_3 are arbitrary real constants.

2°. Solution:

$$w(x,t)=(ax+b)\exp\bigl[i(\alpha x^2+\beta x+\gamma)\bigr],$$

where the functions $a=a(t)$, $b=b(t)$, $\alpha=\alpha(t)$, $\beta=\beta(t)$, and $\gamma=\gamma(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}a_t'&=-6a\alpha,\\ b_t'&=-2a\beta-2b\alpha,\\ \alpha_t'&=Aa^2-4\alpha^2,\\ \beta_t'&=2Aab-4\alpha\beta,\\ \gamma_t'&=Ab^2-\beta^2+B.\end{aligned}$$

3°. For other exact solutions, see equation 1.7.5.1 with $f(u)=Au^2+B$.

3. $\displaystyle i\frac{\partial w}{\partial t}+\frac{\partial^2 w}{\partial x^2}+(A|w|^2+B|w|+C)w=0.$

Schrödinger equation with a cubic nonlinearity. The numbers A, B, and C are assumed real.

1°. There is an exact solution of the form

$$w(x,t)=(ax+b)\exp\bigl[i(\alpha x^2+\beta x+\gamma)\bigr],$$

where $a=a(t)$, $b=b(t)$, $\alpha=\alpha(t)$, $\beta=\beta(t)$, and $\gamma=\gamma(t)$ are real functions of a real variable.

2°. For other exact solutions, see equation 1.7.5.1 with $f(u)=Au^2+Bu+C$.

4. $\displaystyle i\frac{\partial w}{\partial t}+\frac{\partial^2 w}{\partial x^2}+A|w|^{2n}w=0.$

Schrödinger equation with a power-law nonlinearity. The numbers A and n are assumed real.

1°. Suppose $w(x,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$\begin{aligned}w_1&=\pm B_1w(\pm B_1^nx+B_2,\,B_1^{2n}t+B_3),\\ w_2&=e^{-i(\lambda x+\lambda^2t+C)}w(x+2\lambda t,\,t),\end{aligned}$$

where B_1, B_2, B_3, C, and λ are arbitrary real constants, are also solutions of the equation. The plus or minus signs in the expression of w_1 are chosen arbitrarily.

2°. Solutions:

$$w(x,t) = C_1 \exp\bigl\{i\,[C_2 x + (A|C_1|^{2n} - C_2^2)t + C_3]\bigr\},$$

$$w(x,t) = \pm\left[\frac{(n+1)C_1^2}{A\cosh^2(C_1 nx + C_2)}\right]^{\frac{1}{2n}} \exp[i(C_1^2 t + C_3)],$$

$$w(x,t) = \frac{C_1}{\sqrt{t}} \exp\left[i\frac{(x+C_2)^2}{4t} + i\left(\frac{AC_1^{2n}}{1-n}t^{1-n} + C_3\right)\right],$$

where C_1, C_2, and C_3 are arbitrary real constants.

3°. Solution:

$$w(x,t) = e^{-i(\lambda x + \lambda^2 t + C)} U(x + 2\lambda t),$$

where C and λ are arbitrary constants, and the function $U = U(y)$ is determined by the autonomous ordinary differential equation $U''_{yy} + AU^{2n+1} = 0$. Its solution can be represented in implicit form.

4°. Self-similar solution:

$$w(x,t) = (C_1 t + C_2)^{-\frac{1}{2n}} u(z), \qquad z = \frac{x + C_3}{\sqrt{C_1 t + C_2}},$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $u = u(z)$ is determined by the ordinary differential equation

$$u''_{zz} + k|u|^{2n}u - \frac{1}{2}iC_1\left(zu'_z + \frac{1}{n}u\right) = 0.$$

5°. For other exact solutions, see equation 1.7.5.1 with $f(u) = Au^{2n}$.

1.7.2. Equations of the Form $i\dfrac{\partial w}{\partial t} + \dfrac{1}{x^n}\dfrac{\partial}{\partial x}\left(x^n \dfrac{\partial w}{\partial x}\right) + f(|w|)w = 0$ Involving Arbitrary Parameters

► *Throughout this subsection, w is a complex function of real variables x and t; $i^2 = -1$. To $n = 1$ there corresponds a two-dimensional Schrödinger equation with axial symmetry and to $n = 2$, a three-dimensional Schrödinger equation with central symmetry.*

1. $\displaystyle i\frac{\partial w}{\partial t} + \frac{1}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial w}{\partial x}\right) + A|w|^2 w = 0.$

Schrödinger equation with a cubic nonlinearity.

1°. Suppose $w(x,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$w_1 = C_1 e^{iC_2} w(\pm C_1 x, C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary real constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = u(x) e^{i(C_1 t + C_2)},$$

where C_1 and C_2 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1 u + Au^3 = 0.$$

3°. Solution:

$$w(x,t) = u(x)\exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = C_1 t + C_2 \int \frac{dx}{x^n u^2(x)} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1 u - C_2^2 x^{-2n} u^{-3} + Au^3 = 0.$$

4°. Solution:

$$w(x,t) = C_1(t + C_2)^{-\frac{n+1}{2}} \exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = \frac{x^2}{4(t + C_2)} - \frac{AC_1^2}{n(t + C_2)^n} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants.

2. $\displaystyle i\frac{\partial w}{\partial t} + \frac{1}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial w}{\partial x}\right) + (A|w|^2 + B)w = 0.$

Schrödinger equation with a cubic nonlinearity.

1°. Multiplicative separable solution:

$$w(x,t) = u(x)e^{i(C_1 t + C_2)},$$

where C_1 and C_2 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1 u + (Au^2 + B)u = 0.$$

2°. Solution:

$$w(x,t) = u(x)\exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = C_1 t + C_2 \int \frac{dx}{x^n u^2(x)} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1 u - C_2^2 x^{-2n} u^{-3} + (Au^2 + B)u = 0.$$

3°. Solution:

$$w(x,t) = C_1(t + C_2)^{-\frac{n+1}{2}} \exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = \frac{x^2}{4(t + C_2)} - \frac{AC_1^2}{n(t + C_2)^n} + Bt + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants.

3. $\displaystyle i\frac{\partial w}{\partial t} + \frac{1}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial w}{\partial x}\right) + (A|w|^2 + B|w| + C)w = 0.$

This is a special case of equation 1.7.5.2 with $f(u) = Au^2 + Bu + C$.

4. $\displaystyle i\frac{\partial w}{\partial t} + \frac{1}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial w}{\partial x}\right) + A|w|^k w = 0.$

Schrödinger equation with power-law nonlinearity.

1°. Suppose $w(x,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$w_1 = \pm C_1^2 e^{iC_2} w(\pm C_1^k x, C_1^{2k} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary real constants, are also solutions of the equation. The plus or minus signs are chosen arbitrarily.

2°. Multiplicative separable solution:

$$w(x,t) = u(x)e^{i(C_1 t + C_2)},$$

where C_1 and C_2 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1 u + A|u|^k u = 0.$$

3°. Solution:

$$w(x,t)=u(x)\exp\big[i\varphi(x,t)\big],\qquad \varphi(x,t)=C_1t+C_2\int\frac{dx}{x^nu^2(x)}+C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u=u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^nu'_x)'_x-C_1u-C_2^2x^{-2n}u^{-3}+A|u|^ku=0.$$

4°. Solution:

$$w(x,t)=C_1(t+C_2)^{-\frac{n+1}{2}}\exp\big[i\varphi(x,t)\big],$$
$$\varphi(x,t)=\frac{x^2}{4(t+C_2)}+\frac{2A|C_1|^k}{2-k-nk}(t+C_2)^{\frac{2-k-nk}{2}}+C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants.

1.7.3. Other Equations Involving Arbitrary Parameters

1. $\displaystyle i\frac{\partial w}{\partial t}+\frac{\partial^2 w}{\partial x^2}+a\big(1-e^{-k|w|}\big)w=0.$

This equation is encountered in plasma theory and laser physics. This is a special case of equation 1.7.5.1 with $f(u)=a\big(1-e^{-ku}\big)$.

⊙ *Reference*: R. K. Bullough (1977, 1978).

2. $\displaystyle i\frac{\partial w}{\partial t}+\frac{\partial^2 w}{\partial x^2}+ia\frac{\partial}{\partial x}\big(|w|^2w\big)=0.$

This equation is encountered in plasma physics (propagation of Alfven and radio waves); a is a real number.

1°. Suppose $w(x,t)$ is a solution of the Schrödinger equation in question. Then the function

$$w_1=C_1w(C_1^2x+C_2,\,C_1^4t+C_3),$$

where C_1, C_2, and C_3 are arbitrary real constants, is also a solution of the equation.

2°. Solution:

$$w(x,t)=u(t)\exp[iv(x,t)],\qquad v(x,t)=\varphi(t)x^2+\psi(t)x+\chi(t).$$

Here,

$$u=\frac{C_2}{\sqrt{t+C_1}},\qquad \varphi=\frac{1}{4(t+C_1)},\qquad \psi=\frac{C_3-2aC_2^2\ln|t+C_1|}{4(t+C_1)},\qquad \chi=-\int(\psi^2+a\psi u^2)\,dt+C_4,$$

where $C_1,\,\dots,\,C_4$ are arbitrary real constants.

3°. For another solution, see Item 2° of equation 1.7.5.4 with $f(z)=az^2$. See also Calogero and Degasperis (1982).

3. $\displaystyle i\frac{\partial w}{\partial t}+a\frac{\partial^2}{\partial x^2}\left(\frac{w}{\sqrt{1+|w|^2}}\right)=0.$

This is a special case of equation 1.7.5.5 with $f(z)=a(1+z^2)^{-1/2}$.

⊙ *Reference*: F. Calogero and A. Degasperis (1982).

4. $\displaystyle i\frac{\partial w}{\partial t}+(1+ia)\frac{\partial^2 w}{\partial x^2}+f(x,t)w|w|^2+g(x,t)w=0.$

Exact solutions of this equation for some specific $f(x,t)$ and $g(x,t)$ are listed in Table 2. See also equation 1.7.4.4 with $f_1(t)=1$ and $f_2(t)=a$, and equation 1.7.4.5 $f_1(x)=1$ and $f_2(x)=a$.

⊙ *Reference*: L. Garnon and P. Winternitz (1993).

TABLE 2
Structure of exact solutions for the Schrödinger type
equation $i\frac{\partial w}{\partial t} + (1 + ia)\frac{\partial^2 w}{\partial x^2} + f(x,t)w|w|^2 + g(x,t)w = 0$

a	$f(x,t)$	$g(x,t)$	Solution structure $w(x,t)$
0	$1+ib$	0	Solution 1: $w=\varphi(t)\exp\bigl(\frac{ix^2}{4t}\bigr)$, Solution 2: $w=\psi(z)/x$, $z=x^2/t$
0	$1+ib$	ic/t	Solution 1: $w=\varphi(t)\exp\bigl(\frac{ix^2}{4t}\bigr)$, Solution 2: $w=\psi(z)/x$, $z=x^2/t$
0	$(1+ib)/x$	$(c_1+ic_2)/x^2$	$w=\psi(z)/\sqrt{x}$, $z=x^2/t$
0	$(1+ib)\exp\bigl(\alpha xt^{-1/2}\bigr)$	$\frac{1}{4}\alpha xt^{-3/2}+\beta t^{-1}$	$w=\psi(z)/x$, $z=x^2/t$
0	$\bigl[f_1(t)+if_2(t)\bigr]\exp\bigl[2h(t)x\bigr]$	$ih_t'(t)x$	$w=\varphi(t)\exp\bigl[-h(t)x\bigr]$
arbitrary	$1+ib$	0	$w=\psi(z)/x$, $z=x^2/t$
arbitrary	$(1+ib)e^{-x}$	ic	$w=\varphi(t)\exp\bigl(\frac{1}{2}x\bigr)$
arbitrary	$(1+ib)e^{-kx}$	$x+ic$	$w=\varphi(t)\exp\bigl(\frac{1}{2}kx+ixt\bigr)$
arbitrary	$(1+ib)x^{-k}$	$(c_1+ic_2)x^{-2}$	$w=x^{(k-2)/2}\psi(z)$, $z=x^2/t$
arbitrary	$1+ib$	$c_1xt^{-3/2}-ic_2t^{-1}$	Solution 1: $w=\varphi(t)\exp\bigl(-\frac{2ic_1x}{\sqrt{t}}\bigr)$, Solution 2: $w=\psi(z)/x$, $z=x^2/t$

1.7.4. Equations with Cubic Nonlinearities Involving Arbitrary Functions

► *Throughout this subsection, w is a complex function of real variables x and t; $i^2=-1$.*

1. $\displaystyle i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + \bigl[f(t)|w|^2 + g(t)\bigr]w = 0.$

Schrödinger equation with a cubic nonlinearity. Here, $f(t)$ and $g(t)$ are real functions of a real variable.

1°. Solution:

$$w(x,t) = C_1\exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = C_2x - C_2^2t + \int\bigl[C_1^2f(t) + g(t)\bigr]\,dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants.

2°. Solution:

$$w(x,t) = \frac{C_1}{\sqrt{t}}\exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = \frac{(x+C_2)^2}{4t} + \int\bigl[C_1^2f(t) + tg(t)\bigr]\,\frac{dt}{t} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants.

3°. Solution:

$$w(x,t) = (ax+b)\exp\bigl[i(\alpha x^2 + \beta x + \gamma)\bigr],$$

where the functions $a = a(t)$, $b = b(t)$, $\alpha = \alpha(t)$, $\beta = \beta(t)$, and $\gamma = \gamma(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
a'_t &= -6a\alpha, \\
b'_t &= -2a\beta - 2b\alpha, \\
\alpha'_t &= f(t)a^2 - 4\alpha^2, \\
\beta'_t &= 2f(t)ab - 4\alpha\beta, \\
\gamma'_t &= f(t)b^2 - \beta^2 + g(t).
\end{aligned}$$

2. $\displaystyle i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + [f_1(t) + if_2(t)]w|w|^2 + [g_1(t) + ig_2(t)]w = 0.$

Equations of this form occur in nonlinear optics.

1°. Solutions:

$$w(x,t) = \pm u(t)\exp[i\varphi(x,t)], \quad \varphi(x,t) = C_1x - C_1^2t + \int [f_1(t)u^2(t) + g_1(t)]\,dt + C_2.$$

Here, the function $u = u(t)$ is determined by the Bernoulli equation $u'_t + f_2(t)u^3 + g_2(t)u = 0$, whose general solution is given by

$$u(t) = \left[C_3e^{G(t)} + 2e^{G(t)}\int e^{-G(t)}f_2(t)\,dt\right]^{-1/2}, \quad G(t) = 2\int g_2(t)\,dt.$$

2°. Solutions:

$$w(x,t) = \pm u(t)\exp[i\varphi(x,t)], \quad \varphi(x,t) = \frac{(x+C_1)^2}{4t} + \int [f_1(t)u^2(t) + g_1(t)]\,dt + C_2,$$

where the function $u = u(t)$ is determined by the Bernoulli equation

$$u'_t + f_2(t)u^3 + \left[g_2(t) + \frac{1}{2t}\right]u = 0.$$

Integrating yields

$$u(t) = \left[C_3e^{G(t)} + 2e^{G(t)}\int e^{-G(t)}f_2(t)\,dt\right]^{-1/2}, \quad G(t) = \ln t + 2\int g_2(t)\,dt.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

3. $\displaystyle i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + [f_1(x) + if_2(x)]w|w|^2 + [g_1(x) + ig_2(x)]w = 0.$

Solutions:

$$w(x,t) = \pm u(x)\exp[iC_1t + i\theta(x)],$$

where the functions $u = u(x)$ and $\theta = \theta(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
2u'_x\theta'_x + u\theta''_{xx} + f_2(x)u^3 + g_2(x)u &= 0, \\
u''_{xx} - C_1u - u(\theta'_x)^2 + f_1(x)u^3 + g_1(x)u &= 0.
\end{aligned}$$

4. $\displaystyle i\frac{\partial w}{\partial t} + [f_1(t) + if_2(t)]\frac{\partial^2 w}{\partial x^2} + [g_1(t) + ig_2(t)]w|w|^2 + [h_1(t) + ih_2(t)]w = 0.$

Solutions:

$$w(x,t) = \pm u(t)\exp[i\varphi(x,t)], \quad \varphi(x,t) = C_1x + \int [-C_1^2f_1(t) + g_1(t)u^2(t) + h_1(t)]\,dt + C_2.$$

Here, the function $u = u(t)$ is determined by the Bernoulli equation

$$u'_t + g_2(t)u^3 + [h_2(t) - C_1^2 f_2(t)]u = 0,$$

whose general solution is given by

$$u(t) = \left[C_3 e^{F(t)} + 2e^{F(t)} \int e^{-F(t)} g_2(t)\, dt\right]^{-1/2}, \quad F(t) = 2\int \left[h_2(t) - C_1^2 f_2(t)\right] dt.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

5. $\displaystyle i\frac{\partial w}{\partial t} + [f_1(x) + if_2(x)]\frac{\partial^2 w}{\partial x^2} + [g_1(x) + ig_2(x)]w|w|^2 + [h_1(x) + ih_2(x)]w = 0.$

Solutions:

$$w(x,t) = \pm u(x)\exp[iC_1 t + i\theta(x)],$$

where the functions $u = u(x)$ and $\theta = \theta(x)$ are determined by the system of ordinary differential equations

$$2f_1 u'_x \theta'_x + f_1 u\theta''_{xx} + f_2 u''_{xx} - f_2 u(\theta'_x)^2 + g_2 u^3 + h_2 u = 0,$$
$$f_1 u''_{xx} - C_1 u - f_1 u(\theta'_x)^2 - 2f_2 u'_x \theta'_x - f_2 u\theta''_{xx} + g_1 u^3 + h_1 u = 0.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

6. $\displaystyle \frac{\partial w}{\partial t} + [f_1(t) + if_2(t)]\frac{\partial^2 w}{\partial x^2} + [g_1(t) + ig_2(t)]w|w|^2 + [h_1(t) + ih_2(t)]w = 0.$

With $f_n, g_n, h_n = \text{const}$, this equation is used for describing two-component reaction-diffusion systems near a bifurcation point; see Kuramoto and Tsuzuki (1975).

Solutions:

$$w(x,t) = \pm u(t)\exp[i\varphi(x,t)],$$
$$\varphi(x,t) = C_1 x + \int [C_1^2 f_2(t) - g_2(t)u^2(t) - h_2(t)]\, dt + C_2.$$

Here, the function $u = u(t)$ is determined by the Bernoulli equation

$$u'_t + g_1(t)u^3 + [h_1(t) - C_1^2 f_1(t)]u = 0,$$

whose general solution is given by

$$u(t) = \left[C_3 e^{F(t)} + 2e^{F(t)} \int e^{-F(t)} g_1(t)\, dt\right]^{-1/2},$$
$$F(t) = 2\int \left[h_1(t) - C_1^2 f_1(t)\right] dt.$$

7. $\displaystyle \frac{\partial w}{\partial t} + [f_1(x) + if_2(x)]\frac{\partial^2 w}{\partial x^2} + [g_1(x) + ig_2(x)]w|w|^2 + [h_1(x) + ih_2(x)]w = 0.$

Solutions:

$$w(x,t) = \pm u(x)\exp[iC_1 t + i\theta(x)],$$

where the functions $u = u(x)$ and $\theta = \theta(x)$ are determined by the system of ordinary differential equations

$$f_1 u''_{xx} - f_1 u(\theta'_x)^2 - f_2 u\theta''_{xx} - 2f_2 u'_x \theta'_x + g_1 u^3 + h_1 u = 0,$$
$$f_2 u''_{xx} + C_1 u - f_2 u(\theta'_x)^2 + f_1 u\theta''_{xx} + 2f_1 u'_x \theta'_x + g_2 u^3 + h_2 u = 0.$$

1.7.5. Equations of General Form Involving Arbitrary Functions of a Single Argument

► *Throughout this subsection, w is a complex function of real variables x and t; $i^2 = -1$.*

1. $\displaystyle i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + f(|w|)w = 0.$

Schrödinger equation of general form; $f(u)$ is a real function of a real variable.

1°. Suppose $w(x,t)$ is a solution of the Schrödinger equation in question. Then the function

$$w_1 = e^{-i(\lambda x + \lambda^2 t + C_1)} w(x + 2\lambda t + C_2, t + C_3),$$

where C_1, C_2, C_3, and λ are arbitrary real constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w(x,t) = C_1 \exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = C_2 x - C_2^2 t + f(|C_1|)t + C_3.$$

3°. Multiplicative separable solution:

$$w(x,t) = u(x)e^{i(C_1 t + C_2)},$$

where the function $u = u(x)$ is defined implicitly by

$$\int \frac{du}{\sqrt{C_1 u^2 - 2F(u) + C_3}} = C_4 \pm x, \qquad F(u) = \int u f(|u|)\,du.$$

Here, $C_1, \ldots, C_4$ are arbitrary real constants.

4°. Solution:

$$w(x,t) = U(\xi)e^{i(Ax + Bt + C)}, \qquad \xi = x - 2At, \tag{1}$$

where the function $U = U(\xi)$ is determined by the autonomous ordinary differential equation $U''_{\xi\xi} + f(|U|)U - (A^2 + B)U = 0$. Integrating yields the general solution in implicit form:

$$\int \frac{dU}{\sqrt{(A^2 + B)U^2 - 2F(U) + C_1}} = C_2 \pm \xi, \qquad F(U) = \int U f(|U|)\,dU. \tag{2}$$

Relations (1) and (2) involve arbitrary real constants A, B, C, C_1, and C_2.

5°. Solution (A, B, and C are arbitrary constants):

$$w(x,t) = \psi(z) \exp\bigl[i(Axt - \tfrac{2}{3}A^2 t^3 + Bt + C)\bigr], \qquad z = x - At^2,$$

where the function $\psi = \psi(z)$ is determined by the ordinary differential equation

$$\psi''_{zz} + f(|\psi|)\psi - (Az + B)\psi = 0.$$

6°. Solutions:

$$w(x,t) = \pm\frac{1}{\sqrt{C_1 t}} \exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = \frac{(x + C_2)^2}{4t} + \int f\bigl(|C_1 t|^{-1/2}\bigr)\,dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants.

7°. Solution:

$$w(x,t) = u(x) \exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = C_1 t + C_2 \int \frac{dx}{u^2(x)} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u = u(x)$ is determined by the autonomous ordinary differential equation

$$u''_{xx} - C_1 u - C_2^2 u^{-3} + f(|u|)u = 0.$$

8°. Solution:

$$w(x,t) = u(z)\exp\bigl[iAt + i\varphi(z)\bigr], \qquad z = kx + \lambda t,$$

where A, k, and λ are arbitrary real constants, and the functions $u=u(z)$ and $\varphi=\varphi(z)$ are determined by the system of ordinary differential equations

$$k^2 u\varphi''_{zz} + 2k^2 u'_z\varphi'_z + \lambda u'_z = 0,$$
$$k^2 u''_{zz} - k^2 u(\varphi'_z)^2 - \lambda u\varphi'_z - Au + f(|u|)u = 0.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

2. $\displaystyle i\frac{\partial w}{\partial t} + \frac{1}{x^n}\frac{\partial}{\partial x}\left(x^n\frac{\partial w}{\partial x}\right) + f(|w|)w = 0.$

Schrödinger equation of general form; $f(u)$ is a real function of a real variable. To $n = 1$ there corresponds a two-dimensional Schrödinger equation with axial symmetry and to $n = 2$, a three-dimensional Schrödinger equation with central symmetry.

1°. Multiplicative separable solution:

$$w(x,t) = u(x)e^{i(C_1 t + C_2)},$$

where C_1 and C_2 are arbitrary real constants, and the function $u=u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1 u + f(|u|)u = 0.$$

2°. Solution:

$$w(x,t) = u(x)\exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = C_1 t + C_2\int\frac{dx}{x^n u^2(x)} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^n u'_x)'_x - C_1 u - C_2^2 x^{-2n}u^{-3} + f(|u|)u = 0.$$

3°. Solution:

$$w(x,t) = C_1 t^{-\frac{n+1}{2}}\exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = \frac{x^2}{4t} + \int f\Bigl(|C_1|t^{-\frac{n+1}{2}}\Bigr)\,dt + C_2,$$

where C_1 and C_2 are arbitrary real constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

3. $\displaystyle \frac{\partial w}{\partial t} = (a+ib)\frac{\partial^2 w}{\partial x^2} + \bigl[f(|w|) + ig(|w|)\bigr]w.$

Generalized Landau–Ginzburg equation; $f(u)$ and $g(u)$ are real functions of a real variable, a and b are real numbers. Equations of this form are used for studying second-order phase transitions in superconductivity theory (see Landau and Ginzburg, 1950) and to describe two-component reaction-diffusion systems near a point of bifurcation (Kuramoto and Tsuzuki, 1975).

1°. Suppose $w(x,t)$ is a solution of the generalized Landau–Ginzburg equation. Then the function

$$w_1 = e^{iC_1}w(x + C_2, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary real constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w(x,t) = C_1\exp\bigl[i\varphi(x,t)\bigr], \qquad \varphi(x,t) = \pm x\sqrt{\frac{f(|C_1|)}{a}} + t\left[g(|C_1|) - \frac{b}{a}f(|C_1|)\right] + C_2,$$

where C_1 and C_2 are arbitrary real constants.

3°. Solution:

$$w(x,t) = u(t)\exp\big[i\varphi(x,t)\big], \qquad \varphi(x,t) = C_1x - C_1^2bt + \int g(|u|)\,dt + C_2,$$

where $u = u(t)$ is determined by the ordinary differential equation $u'_t = f(|u|)u - aC_1^2u$, whose general solution can be represented in implicit form as

$$\int \frac{du}{f(|u|)u - aC_1^2u} = t + C_3.$$

4°. Solution:

$$w(x,t) = U(z)\exp\big[iC_1t + i\theta(z)\big], \qquad z = x + \lambda t,$$

where C_1 and λ are arbitrary real constants, and the functions $U = U(z)$ and $\theta = \theta(z)$ are determined by the system of ordinary differential equations

$$aU''_{zz} - aU(\theta'_z)^2 - bU\theta''_{zz} - 2bU'_z\theta'_z - \lambda U'_z + f(|U|)U = 0,$$
$$aU\theta''_{zz} - bU(\theta'_z)^2 + bU''_{zz} + 2aU'_z\theta'_z - \lambda U\theta'_z - C_1U + g(|U|)U = 0.$$

⊙ *References*: V. S. Berman and Yu. A. Danilov (1981), A. D. Polyanin and V. F. Zaitsev (2002).

4. $\displaystyle i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + i\frac{\partial}{\partial x}\big[f(|w|)w\big] = 0.$

1°. Solution:

$$w(x,t) = u(t)\exp[iv(x,t)], \qquad v(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $u = u(t)$, $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$u'_t + 2\varphi u = 0,$$
$$\varphi'_t + 4\varphi^2 = 0,$$
$$\psi'_t + 4\varphi\psi + 2\varphi f(u) = 0,$$
$$\chi'_t + \psi^2 + \psi f(u) = 0.$$

Integrating yields

$$u = \frac{C_2}{\sqrt{t + C_1}}, \qquad \varphi = \frac{1}{4(t + C_1)}, \qquad \psi = -2\varphi\int f(u)\,dt + C_3\varphi, \qquad \chi = -\int\big[\psi^2 + \psi f(u)\big]\,dt + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary real constants.

2°. Solution:

$$w(x,t) = U(z)\exp[i\beta t + iV(z)], \qquad z = kx + \lambda t,$$

where k, β, and λ are arbitrary real constants, and the functions $U = U(z)$ and $V = V(z)$ are determined by the system of ordinary differential equations

$$\lambda U'_z + k^2(UV'_z)'_z + k^2U'_zV'_z + k[f(U)U]'_z = 0,$$
$$-U(\beta + \lambda V'_z) + k^2U''_{zz} - k^2U(V'_z)^2 - kf(U)UV'_z = 0.$$

5. $\displaystyle i\frac{\partial w}{\partial t} + \frac{\partial^2}{\partial x^2}\big[f(|w|)w\big] = 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = \pm e^{iC_1}w(\pm C_2x + C_3, C_2^2t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary real constants, are also solutions of the equation. The plus or minus signs are chosen arbitrarily.

2°. Solution:

$$w(x,t) = u(t)\exp[iv(x,t)], \quad v(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $u = u(t)$, $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$u'_t + 2u\varphi f(u) = 0,$$
$$\varphi'_t + 4\varphi^2 f(u) = 0,$$
$$\psi'_t + 4\varphi\psi f(u) = 0,$$
$$\chi'_t + \psi^2 f(u) = 0.$$

Integrating yields

$$\varphi = C_1 u^2, \quad \psi = C_2 u^2, \quad \chi = -C_2^2 \int u^4 f(u)\,dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u = u(t)$ is defined implicitly as (C_4 is an arbitrary constant)

$$\int \frac{du}{u^3 f(u)} + 2C_1 t + C_4 = 0.$$

3°. There is a solution of the form

$$w(x,t) = U(z)\exp[i\beta t + iV(z)], \quad z = kx + \lambda t,$$

where k, β, and λ are arbitrary real constants, and the functions $U = U(z)$ and $V = V(z)$ are determined by an appropriate system of ordinary differential equations (which is not written out here).

4°. There is a self-similar solution of the form $w(x,t) = V(\xi)$, where $\xi = x^2/t$.

1.7.6. Equations of General Form Involving Arbitrary Functions of Two Arguments

► *Throughout this subsection, w is a complex function of real variables x and t; $i^2 = -1$.*

1. $\displaystyle i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + f(x,|w|)w = 0.$

Schrödinger equation of general form; $f(x,u)$ is a real function of two real variables.

1°. Multiplicative separable solution:

$$w(x,t) = u(x)e^{i(C_1 t + C_2)},$$

where C_1 and C_2 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$u''_{xx} - C_1 u + f(x,|u|)u = 0.$$

2°. Solution:

$$w(x,t) = u(x)\exp\bigl[i\varphi(x,t)\bigr], \quad \varphi(x,t) = C_1 t + C_2 \int \frac{dx}{u^2(x)} + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$u''_{xx} - C_1 u - C_2^2 u^{-3} + f(x,|u|)u = 0.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

2. $\boldsymbol{i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2 w}{\partial x^2}+f(t,|w|)w=0.}$

Schrödinger equation of general form; $f(t,u)$ is a real function of two real variables.

1°. Suppose $w(x,t)$ is a solution of the Schrödinger equation in question. Then the function

$$w_1=e^{-i(\lambda x+\lambda^2 t+C_1)}w(x+2\lambda t+C_2,\ t),$$

where C_1, C_2, and λ are arbitrary real constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t)=C_1\exp\bigl[i\varphi(x,t)\bigr],\qquad \varphi(x,t)=C_2x-C_2^2t+\int f(t,|C_1|)\,dt+C_3;$$

$$w(x,t)=C_1t^{-1/2}\exp\bigl[i\psi(x,t)\bigr],\qquad \psi(x,t)=\frac{(x+C_2)^2}{4t}+\int f\bigl(t,|C_1|t^{-1/2}\bigr)\,dt+C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

3. $\boldsymbol{i\dfrac{\partial w}{\partial t}+\dfrac{1}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right)+f(x,|w|)w=0.}$

Schrödinger equation of general form; $f(x,u)$ is a real function of two real variables. To $n=1$ there corresponds a two-dimensional Schrödinger equation with axial symmetry and to $n=2$, a three-dimensional Schrödinger equation with central symmetry.

1°. Multiplicative separable solution:

$$w(x,t)=u(x)e^{i(C_1t+C_2)},$$

where C_1 and C_2 are arbitrary real constants, and the function $u=u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^nu'_x)'_x-C_1u+f(x,|u|)u=0.$$

2°. Solution:

$$w(x,t)=u(x)\exp\bigl[i\varphi(x,t)\bigr],\qquad \varphi(x,t)=C_1t+C_2\int\frac{dx}{x^nu^2(x)}+C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $u=u(x)$ is determined by the ordinary differential equation

$$x^{-n}(x^nu'_x)'_x-C_1u-C_2^2x^{-2n}u^{-3}+f(x,|u|)u=0.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

4. $\boldsymbol{i\dfrac{\partial w}{\partial t}+\dfrac{1}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right)+f(t,|w|)w=0.}$

Schrödinger equation of general form; $f(t,u)$ is a real function of two real variables.

Solution:

$$w(x,t)=C_1t^{-\frac{n+1}{2}}\exp\bigl[i\varphi(x,t)\bigr],\qquad \varphi(x,t)=\frac{x^2}{4t}+\int f\Bigl(t,|C_1|t^{-\frac{n+1}{2}}\Bigr)\,dt+C_2,$$

where C_1 and C_2 are arbitrary real constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

5. $\boldsymbol{i\dfrac{\partial w}{\partial t}+f(x)\dfrac{\partial^2 w}{\partial x^2}+g(x)\dfrac{\partial w}{\partial x}+\Phi(x,|w|)w=0.}$

Schrödinger equation of general form; $\Phi(x,u)$ is a real function of two real variables. The case $g(x)=f'_x(x)$ corresponds to an anisotropic medium.

1°. Multiplicative separable solution:

$$w(x,t) = u(x)e^{i(C_1 t + C_2)},$$

where C_1 and C_2 are arbitrary real constants, and the function $u = u(x)$ is determined by the ordinary differential equation

$$f(x)u''_{xx} + g(x)u'_x - C_1 u + \Phi(x,|u|)u = 0.$$

2°. Solution:

$$w(x,t) = U(x)\exp\bigl[i\varphi(x,t)\bigr],$$
$$\varphi(x,t) = C_1 t + C_2 \int \frac{R(x)}{U^2(x)}\,dx + C_3, \quad R(x) = \exp\biggl[-\int \frac{g(x)}{f(x)}\,dx\biggr],$$

where C_1, C_2, and C_3 are arbitrary real constants, and the function $U = U(x)$ is determined by the ordinary differential equation

$$f(x)U''_{xx} + g(x)U'_x - C_1 U - C_2^2 f(x)R^2(x)U^{-3} + \Phi(x,|U|)U = 0.$$

6. $\displaystyle \frac{\partial w}{\partial t} = \bigl[f_1(t,|w|) + if_2(t,|w|)\bigr]\frac{\partial^2 w}{\partial x^2} + \bigl[g_1(t,|w|) + ig_2(t,|w|)\bigr]w.$

Solution:

$$w(x,t) = u(t)\exp\bigl[i\varphi(x,t)\bigr], \quad \varphi(x,t) = C_1 x + \int \bigl[g_2(t,|u|) - C_1^2 f_2(t,|u|)\bigr]\,dt + C_2,$$

where C_1 and C_2 are arbitrary real constants, and the function $u = u(t)$ is determined by the ordinary differential equation

$$u'_t = ug_1(t,|u|) - C_1^2 u f_1(t,|u|).$$

7. $\displaystyle \frac{\partial w}{\partial t} = \bigl[f_1(x,|w|) + if_2(x,|w|)\bigr]\frac{\partial^2 w}{\partial x^2} + \bigl[g_1(x,|w|) + ig_2(x,|w|)\bigr]w.$

Solution:

$$w(x,t) = u(x)\exp\bigl[i\varphi(x,t)\bigr], \quad \varphi(x,t) = C_1 t + \theta(x),$$

where the functions $u = u(x)$ and $\theta = \theta(x)$ are determined by the system of ordinary differential equations

$$f_1 u''_{xx} - f_1 u(\theta'_x)^2 - f_2 u\theta''_{xx} - 2f_2 u'_x\theta'_x + g_1(|u|)u = 0,$$
$$f_1 u\theta''_{xx} - f_2 u(\theta'_x)^2 + f_2 u''_{xx} + 2f_1 u'_x\theta'_x - C_1 u + g_2(|u|)u = 0.$$

Here, $f_n = f_n(x,|u|)$, $g_n = g_n(x,|u|)$, $n = 1,\,2$.

8. $\displaystyle i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + i\frac{\partial}{\partial x}\bigl[f(t,|w|)w\bigr] = 0.$

Solution:

$$w(x,t) = u(t)\exp[iv(x,t)], \quad v(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $u = u(t)$, $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$u'_t + 2\varphi u = 0,$$
$$\varphi'_t + 4\varphi^2 = 0,$$
$$\psi'_t + 4\varphi\psi + 2\varphi f(t,u) = 0,$$
$$\chi'_t + \psi^2 + \psi f(t,u) = 0.$$

Integrating yields

$$u = \frac{C_2}{\sqrt{t + C_1}}, \quad \varphi = \frac{1}{4(t + C_1)}, \quad \psi = -2\varphi\int f(t,u)\,dt + C_3\varphi, \quad \chi = -\int \bigl[\psi^2 + \psi f(t,u)\bigr]\,dt + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary real constants.

9. $\boldsymbol{i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} + i\dfrac{\partial}{\partial x}\big[f(x,|w|)w\big] = 0.}$

Solution:

$$w(x,t) = U(x)\exp[i\beta t + iV(x)],$$

where β is an arbitrary real constant, and the real functions $U = U(x)$ and $V = V(x)$ are determined by the system of ordinary differential equations

$$(UV_x')_x' + U_x'V_x' + [f(x,U)U]_x' = 0,$$
$$-\beta U + U_{xx}'' - U(V_x')^2 - f(x,U)UV_x' = 0.$$

10. $\boldsymbol{i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2}{\partial x^2}\big[f(t,|w|)w\big] = 0.}$

Solution:

$$w(x,t) = u(t)\exp[iv(x,t)], \quad v(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $u = u(t)$, $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$u_t' + 2u\varphi f(t,u) = 0,$$
$$\varphi_t' + 4\varphi^2 f(t,u) = 0,$$
$$\psi_t' + 4\varphi\psi f(t,u) = 0,$$
$$\chi_t' + \psi^2 f(t,u) = 0.$$

Integrating yields

$$\varphi = C_1u^2, \quad \psi = C_2u^2, \quad \chi = -C_2^2\int u^4 f(t,u)\,dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and $u = u(t)$ is determined by the ordinary differential equation $u_t' + 2C_1u^3 f(t,u) = 0$.

11. $\boldsymbol{i\dfrac{\partial w}{\partial t} + \dfrac{\partial}{\partial x}\Big[f(t,|w|)\dfrac{\partial w}{\partial x}\Big] = 0.}$

Solution:

$$w(x,t) = u(t)\exp[iv(x,t)], \quad v(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $u = u(t)$, $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$u_t' + 2u\varphi f(t,u) = 0,$$
$$\varphi_t' + 4\varphi^2 f(t,u) = 0,$$
$$\psi_t' + 4\varphi\psi f(t,u) = 0,$$
$$\chi_t' + \psi^2 f(t,u) = 0.$$

Integrating yields

$$\varphi = C_1u^2, \quad \psi = C_2u^2, \quad \chi = -C_2^2\int u^4 f(t,u)\,dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary real constants, and $u = u(t)$ is determined by the ordinary differential equation $u_t' + 2C_1u^3 f(t,u) = 0$.

12. $\boldsymbol{i\dfrac{\partial w}{\partial t} + \dfrac{\partial}{\partial x}\left[f(x,|w|)\dfrac{\partial w}{\partial x}\right] = 0.}$

There is a solution of the form

$$w(x,t) = U(x)\exp[i\beta t + iV(x)],$$

where β is an arbitrary real constant, and the functions $U = U(z)$ and $V = V(z)$ are determined by an appropriate system of ordinary differential equations (which is not written out here).

Chapter 2

Parabolic Equations with Two or More Space Variables

2.1. Equations with Two Space Variables Involving Power-Law Nonlinearities

2.1.1. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] + aw^p$

1. $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m\frac{\partial w}{\partial y}\right) + cw^p.$

This is a special case of equation 2.4.2.1 with $f(w) = cw^p$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\left(C_1^{\frac{p-1}{2-n}} x,\ C_1^{\frac{p-1}{2-m}} y,\ C_1^{p-1} t + C_2\right),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = U(r, t), \quad r^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2};$$
$$w(x, y, t) = t^{\frac{1}{1-p}} V(z_1, z_2), \quad z_1 = xt^{\frac{1}{n-2}}, \quad z_2 = yt^{\frac{1}{m-2}}.$$

2. $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\lambda y}\frac{\partial w}{\partial y}\right) + cw^p.$

This is a special case of equation 2.4.2.3 with $f(w) = cw^p$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\left(C_1^{\frac{p-1}{2-n}} x,\ y + \frac{1-p}{\lambda}\ln C_1,\ C_1^{p-1} t + C_2\right),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = U(r, t), \quad r^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2};$$
$$w(x, y, t) = t^{\frac{1}{1-p}} V(z_1, z_2), \quad z_1 = xt^{\frac{1}{n-2}}, \quad z_2 = y + \frac{1}{\lambda}\ln t.$$

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + cw^p.$

This is a special case of equation 2.4.2.2 with $f(w) = cw^p$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\Big(x + \frac{1-p}{\beta}\ln C_1,\ y + \frac{1-p}{\lambda}\ln C_1,\ C_1^{p-1}t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = U(r,t), \quad r^2 = \frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2};$$

$$w(x,y,t) = t^{\frac{1}{1-p}} V(z_1, z_2), \quad z_1 = x + \frac{1}{\beta}\ln t, \quad z_2 = y + \frac{1}{\lambda}\ln t.$$

2.1.2. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(w^k\dfrac{\partial w}{\partial y}\right)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial}{\partial y}\left(w\dfrac{\partial w}{\partial y}\right).$

This is a special case of equation 2.1.3.1 with $c = 0$.

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(w\dfrac{\partial w}{\partial y}\right).$

Boussinesq equation. It arises in nonlinear heat conduction theory and the theory of unsteady flows through porous media with a free surface (see Polubarinova–Kochina, 1962). This is a special case of equation 2.1.2.4 with $n = 1$.

1°. Solution linear in all independent variables:

$$w(x,y,t) = Ax + By + (A^2 + B^2)t + C,$$

where A, B, and C are arbitrary constants.

2°. Traveling-wave solution (k_1, k_2, and λ are arbitrary constants):

$$w = w(\xi), \quad \xi = k_1 x + k_2 y + \lambda t,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$\lambda w'_\xi = (k_1^2 + k_2^2)(ww'_\xi)'_\xi.$$

The solution of this equation can be written out in implicit form:

$$\xi = B + \frac{k_1^2 + k_2^2}{\lambda^2}\big(\lambda w - A\ln|A + \lambda w|\big),$$

where A and B are arbitrary constants.

3°. Generalized separable solution quadratic in the space variables:

$$w(x,y,t) = f(t)x^2 + g(t)xy + h(t)y^2,$$

where the functions $f(t)$, $g(t)$, and $h(t)$ are determined by the autonomous system of ordinary differential equations

$$f'_t = 6f^2 + 2fh + g^2, \tag{1}$$
$$g'_t = 6(f+h)g, \tag{2}$$
$$h'_t = 6h^2 + 2fh + g^2. \tag{3}$$

It follows from (1) and (3) that $f'_t - h'_t = 6(f+h)(f-h)$. Further, using (2) and assuming $g \not\equiv 0$, we find that $f = h + Ag$, where A is an arbitrary constant. With this relation, we eliminate h from (2) and (3) to obtain a nonlinear ordinary differential equation for $g(t)$:

$$3gg''_{tt} - 5g'^2_t - 36(1+A^2)g^4 = 0.$$

On solving this equation with the change of variable $u(g) = (g'_t)^2$, we obtain (B is an arbitrary constant)

$$g'_t = g\Phi(g), \qquad \Phi(g) = \pm\sqrt{Bg^{4/3} + 36(1+A^2)g^2}, \tag{4}$$
$$h = \tfrac{1}{12}\Phi(g) - \tfrac{1}{2}Ag, \quad f = \tfrac{1}{12}\Phi(g) + \tfrac{1}{2}Ag,$$

where the first equation is separable, and, hence, its solution can be written out in implicit form.

In the special case $B = 0$, the solution can be represented in explicit form (C is an arbitrary constant):

$$f(t) = \frac{\mu + A}{2(C-\mu t)}, \quad g(t) = \frac{1}{C-\mu t}, \quad h(t) = \frac{\mu - A}{2(C-\mu t)}, \qquad \mu = \pm\sqrt{1+A^2}.$$

4°. Generalized separable solution (generalizes the solution of Item 3°):

$$w(x,y,t) = f(t)x^2 + g(t)xy + h(t)y^2 + \varphi(t)x + \psi(t)y + \chi(t),$$

where the functions $f(t)$, $g(t)$, $h(t)$, $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned} f'_t &= 6f^2 + 2fh + g^2, & \varphi'_t &= 2(3f+h)\varphi + 2g\psi, \\ g'_t &= 6(f+h)g, & \psi'_t &= 2g\varphi + 2(f+3h)\psi, \\ h'_t &= 6h^2 + 2fh + g^2, & \chi'_t &= \varphi^2 + \psi^2 + 2(f+h)\chi. \end{aligned}$$

The first three equations for f, g, and h can be solved independently (see Item 3°).

Example. Solution:

$$w(x,t) = -\frac{y^2}{6t} + Cxt^{-1/3} + \frac{3}{2}C^2t^{1/3},$$

where C is an arbitrary constant.

5°. There is a “two-dimensional” solution in multiplicative separable form:

$$w(x,y,t) = (At+B)^{-1}\Theta(x,y),$$

where A and B are arbitrary constants, and the function Θ is determined by the stationary equation written out in Item 4° of equation 2.1.2.4 with $n = \alpha = 1$.

⊙ *References*: S. S. Titov and V. A. Ustinov (1985), V. V. Pukhnachov (1995), A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

3. $$\frac{\partial w}{\partial t} = \alpha\left[\frac{\partial}{\partial x}\left(\frac{1}{w}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{w}\frac{\partial w}{\partial y}\right)\right].$$

This is a special case of equation 2.1.2.4 with $n = -1$.

1°. Traveling-wave solutions:

$$w(\xi) = -\frac{\alpha(k_1^2 + k_2^2)}{\lambda(\xi + A)}, \quad \xi = k_1 x + k_2 y + \lambda t,$$

$$w(\xi) = \left\{ A + B \exp\left[\frac{\lambda \xi}{\alpha A(k_1^2 + k_2^2)} \right] \right\}^{-1},$$

where A, B, k_1, k_2, and λ are arbitrary constants.

2°. Solutions:

$$w(x, y, t) = \frac{2\alpha t + B}{(\sin y + Ae^x)^2},$$

$$w(x, y, t) = \frac{2A^2 \alpha t + C}{e^{2x} \sinh^2\left(Ae^{-x} \sin y + B\right)},$$

$$w(x, y, t) = \frac{C - 2A^2 \alpha t}{e^{2x} \cosh^2\left(Ae^{-x} \sin y + B\right)},$$

$$w(x, y, t) = \frac{2A^2 \alpha t + C}{e^{2x} \cos^2\left(Ae^{-x} \sin y + B\right)},$$

where A, B, and C are arbitrary constants.

3°. The exact solutions specified in Item 2° are special cases of a more general solution having the form of the product of two functions with different arguments:

$$w(x, y, t) = (A\alpha t + B)e^{\Theta(x,y)},$$

where A and B are arbitrary constants, and the function $\Theta(x, y)$ is a solution of the stationary equation

$$\Delta\Theta - Ae^{\Theta} = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

which is encountered in combustion theory. For solutions of this equation, see 5.2.1.1.

⊙ *References*: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983), A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

4°. Other exact solutions:

$$w(x, y, t) = \frac{2 \sinh(\alpha t + C) \cosh(\alpha t + C)}{(x + A)^2 \sinh^2(\alpha t + C) + (y + B)^2 \cosh^2(\alpha t + C)},$$

$$w(x, y, t) = \left[\frac{1}{A + \alpha \mu^2 t} + B(A + \alpha \mu^2 t)e^{\mu x} \pm \frac{\sin(\mu y + \eta_0)}{A + \alpha \mu^2 t} \right]^{-1},$$

$$w(x, y, t) = \left[A \coth \theta(t) + B \sinh \theta(t)\, e^{\mu x} \pm A \frac{\sin(\mu y + \eta_0)}{\sinh \theta(t)} \right]^{-1},$$

$$w(x, y, t) = \left[A \cot \theta(t) + B \sin \theta(t)\, e^{\mu x} \pm A \frac{\sin(\mu y + \eta_0)}{\sin \theta(t)} \right]^{-1},$$

$$w(x, y, t) = \left[\frac{A}{\cos \theta(t)} \pm A \frac{1 + \sin \theta(t)}{2 \cos \theta(t)} \cosh(\mu x + \xi_0) + sA \frac{1 - \sin \theta(t)}{2 \cos \theta(t)} \sin(\mu y + \eta_0) \right]^{-1},$$

$$w(x, y, t) = \left[-\frac{A}{\cos \theta(t)} \pm A \frac{1 - \sin \theta(t)}{2 \cos \theta(t)} \cosh(\mu x + \xi_0) + sA \frac{1 + \sin \theta(t)}{2 \cos \theta(t)} \sin(\mu y + \eta_0) \right]^{-1},$$

$$w(x, y, t) = \left[\frac{A}{\sinh \theta(t)} \pm A \frac{1 - \cosh \theta(t)}{2 \sinh \theta(t)} \cosh(\mu x + \xi_0) + sA \frac{1 + \cosh \theta(t)}{2 \sinh \theta(t)} \sin(\mu y + \eta_0) \right]^{-1},$$

$$w(x, y, t) = \left[-\frac{A}{\sinh \theta(t)} \pm A \frac{1 + \cosh \theta(t)}{2 \sinh \theta(t)} \cosh(\mu x + \xi_0) + sA \frac{1 - \cosh \theta(t)}{2 \sinh \theta(t)} \sin(\mu y + \eta_0) \right]^{-1},$$

where $\theta(t) = \alpha\mu^2 At + \tau_0$; A, B, μ, ξ_0, η_0, and τ_0 are arbitrary constants; and s is a parameter that admits the values 1 or −1 (the first solution was indicated by Pukhnachov, 1995).

By swapping the variables, $x \rightleftarrows y$, in the above relations, one can obtain another group of solutions (not written out here).

5°. Solutions with axial symmetry:

$$w(r,t) = \frac{\lambda^2 r^{\lambda-2}}{r^\lambda + Ce^{\alpha t}},$$
$$w(r,t) = \frac{\lambda\varphi r^{\varphi-2}}{C_1 + r^\varphi(\varphi \ln r - 1)}, \quad \varphi = \frac{\lambda}{\alpha t + C_2},$$

where $r = \sqrt{x^2+y^2}$; C, C_1, C_2, and λ are arbitrary constants.

⊙ *Reference*: S. N. Aristov (1999).

6°. The transformation $w = 1/U$ leads to an equation of the form 2.1.4.3 with $\beta = 0$:

$$\frac{\partial U}{\partial t} = \alpha U \Delta U - \alpha\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2\right].$$

4. $\displaystyle \frac{\partial w}{\partial t} = \alpha\left[\frac{\partial}{\partial x}\left(w^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(w^n \frac{\partial w}{\partial y}\right)\right].$

This is a two-dimensional heat and mass transfer equation with power-law temperature-dependent thermal conductivity (diffusion coefficient), where n can be integer, fractional, and negative. This is a special case of equation 2.4.3.3 with $f(w) = \alpha w^n$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2/n} C_2^{1/n} w(\pm C_1 x + C_3, \pm C_1 y + C_4, C_2 t + C_5),$$
$$w_2 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation. The plus or minus signs can be chosen arbitrarily.

2°. Traveling-wave solution:

$$w = \left[\frac{n\lambda(k_1 x + k_2 y + \lambda t + C)}{\alpha(k_1^2 + k_2^2)}\right]^{1/n},$$

where C, k_1, k_2, and λ are arbitrary constants.

3°. Traveling-wave solution in implicit form (generalizes the solution of Item 2°):

$$\alpha(k_1^2 + k_2^2)\int \frac{w^n\,dw}{\lambda w + C_1} = k_1 x + k_2 y + \lambda t + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

4°. Multiplicative separable solution:

$$w(x,y,t) = f(t)\Theta(x,y), \qquad f(t) = (A\alpha n t + B)^{-1/n}.$$

Here, A and B are arbitrary constants, and the function $\Theta(x,y)$ is a solution of the two-dimensional stationary equation

$$\frac{\partial}{\partial x}\left(\Theta^n \frac{\partial\Theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Theta^n \frac{\partial\Theta}{\partial y}\right) + A\Theta = 0.$$

If $n \neq -1$, this equation can be reduced to

$$\Delta u + A(n+1)u^{\frac{1}{n+1}} = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad u = \Theta^{n+1}.$$

5°. There are "two-dimensional" solutions of the following forms:

$$\begin{aligned}
&w(x,y,t)=F(z,t), && z=k_1x+k_2y;\\
&w(x,y,t)=G(r,t), && r=\sqrt{x^2+y^2};\\
&w(x,y,t)=H(\xi_1,\xi_2), && \xi_1=k_1x+\lambda_1t,\quad \xi_2=k_2y+\lambda_2t;\\
&w(x,y,t)=t^\beta U(\eta_1,\eta_2), && \eta_1=xt^{-\frac{n\beta+1}{2}},\quad \eta_2=yt^{-\frac{n\beta+1}{2}};\\
&w(x,y,t)=e^{2\beta t}V(\zeta_1,\zeta_2), && \zeta_1=xe^{-\beta nt},\quad \zeta_2=ye^{-\beta nt},
\end{aligned}$$

where k_1, k_2, λ_1, λ_2, and β are arbitrary constants.

6°. See also equations 2.5.5.5 and 2.5.5.6 for the case of two space variables.

⦿ *References*: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983), S. S. Titov and V. A. Ustinov (1985), J. R. King (1993), V. V. Pukhnachov (1995).

5. $\displaystyle \frac{\partial w}{\partial t}=a_1\frac{\partial}{\partial x}\left(w^{n_1}\frac{\partial w}{\partial x}\right)+a_2\frac{\partial}{\partial y}\left(w^{n_2}\frac{\partial w}{\partial y}\right).$

This is a special case of equation 2.4.3.4 with $f(w)=a_1w^{n_1}$ and $g(w)=a_2w^{n_2}$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1=A^2w(\pm A^{-n_1}Bx+C_1,\pm A^{-n_2}By+C_2,B^2t+C_3),$$

where A, B, C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs can be chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int\frac{a_1k_1^2w^{n_1}+a_2k_2^2w^{n_2}}{\lambda w+C_1}\,dw=k_1x+k_2y+\lambda t+C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

⦿ *Reference*: A. A. Samarskii and I. M. Sobol' (1963).

3°. "Two-dimensional" solution:

$$w(x,y,t)=t^kU(\xi,\eta),\quad \xi=xt^{-\frac{1}{2}(kn_1+1)},\quad \eta=yt^{-\frac{1}{2}(kn_2+1)},$$

where k is an arbitrary constant and the function $U(\xi,\eta)$ is determined by the differential equation

$$kU-\tfrac{1}{2}(kn_1+1)\xi\frac{\partial U}{\partial\xi}-\tfrac{1}{2}(kn_2+1)\eta\frac{\partial U}{\partial\eta}=a_1\frac{\partial}{\partial x}\left(U^{n_1}\frac{\partial U}{\partial x}\right)+a_2\frac{\partial}{\partial y}\left(U^{n_2}\frac{\partial U}{\partial y}\right).$$

4°. "Two-dimensional" solution:

$$w(x,y,t)=e^{2\beta t}V(z_1,z_2),\quad z_1=xe^{-\beta n_1t},\quad z_2=ye^{-\beta n_2t},$$

where β is an arbitrary constant and the function $V(z_1,z_2)$ is determined by the differential equation

$$2\beta V-\beta n_1z_1\frac{\partial V}{\partial z_1}-\beta n_2z_2\frac{\partial V}{\partial z_2}=a_1\frac{\partial}{\partial z_1}\left(V^{n_1}\frac{\partial V}{\partial z_1}\right)+a_2\frac{\partial}{\partial z_2}\left(V^{n_2}\frac{\partial V}{\partial z_2}\right).$$

5°. There is a "two-dimensional" solution of the form

$$w(x,y,t)=F(\xi_1,\xi_2),\quad \xi_1=\alpha_1x+\beta_1y+\gamma_1t,\quad \xi_2=\alpha_2x+\beta_2y+\gamma_2t,$$

where the α_i, β_i, and γ_i are arbitrary constants.

⦿ *References*: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983), N. H. Ibragimov (1994).

2.1.3. Equations of the Form $\dfrac{\partial w}{\partial t}=\dfrac{\partial}{\partial x}\Big[f(w)\dfrac{\partial w}{\partial x}\Big]+\dfrac{\partial}{\partial y}\Big[g(w)\dfrac{\partial w}{\partial y}\Big]+h(w)$

1. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\Big[(bw+c)\dfrac{\partial w}{\partial y}\Big].$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1=C_1^{-2}w(\pm C_2x+C_3,\pm C_1C_2y+C_4,C_2^2t+C_5)+\frac{c(1-C_1^2)}{bC_1^2},$$

where $C_1,\dots,C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs can be chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$bk_2^2w+(ak_1^2+ck_2^2-C_1bk_2^2)\ln|w+C_1|=\lambda(k_1x+k_2y+\lambda t)+C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. Solution:

$$w=u(z)-4abC_1^2x^2-4abC_1C_2x,\qquad z=y+bC_1x^2+bC_2x+C_3t,$$

where C_1, C_2, and C_3 are arbitrary constants and the function $u(z)$ is determined by the first-order ordinary differential equation

$$(bu+c+ab^2C_2^2)u'_z+(2abC_1-C_3)u=8a^2bC_1^2z+C_4.$$

With appropriate translations in both variables, one can reduce this equation to a homogeneous one, which can be integrated by quadrature.

4°. Generalized separable solution linear in y (a degenerate solution):

$$w=F(x,t)y+G(x,t),$$

where the functions F and G are determined by solving the one-dimensional equations

$$\frac{\partial F}{\partial t}-a\frac{\partial^2 F}{\partial x^2}=0, \tag{1}$$

$$\frac{\partial G}{\partial t}-a\frac{\partial^2 G}{\partial x^2}=bF^2. \tag{2}$$

Equation (1) is a linear homogeneous heat equation. Given $F=F(x,t)$, equation (2) can be treated as a linear nonhomogeneous heat equation. For these equations, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

5°. Generalized separable solution quadratic in y:

$$w=f(x,t)y^2+g(x,t)y+h(x,t),$$

where the functions $f=f(x,t)$, $g=g(x,t)$, and $h=h(x,t)$ are determined by the system of differential equations

$$f_t=af_{xx}+6bf^2,$$
$$g_t=ag_{xx}+6bfg,$$
$$h_t=ah_{xx}+bg^2+2bfh+2cf.$$

Here, the subscripts denote partial derivatives.

6°. "Two-dimensional" solution:

$$w = |y + C|^{1/2}\theta(x,t) - \frac{c}{b},$$

where the function $\theta(x,t)$ is determined by the linear heat equation

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2}.$$

7°. "Two-dimensional" solution:

$$w = U(\xi,t) - \frac{aC_1^2 + cC_2^2}{bC_2^2}, \qquad \xi = C_1 x + C_2 y,$$

where C_1 and C_2 are arbitrary constants and the function $U(\xi,t)$ is determined by a differential equation of the form 1.10.1.1:

$$\frac{\partial U}{\partial t} = bC_2^2 \frac{\partial}{\partial \xi}\left(U\frac{\partial U}{\partial \xi}\right).$$

8°. "Two-dimensional" solution:

$$w = V(\eta,t) - 4abC_1^2x^2 - 4abC_1C_2x, \qquad \eta = y + bC_1x^2 + bC_2x,$$

where C_1 and C_2 are arbitrary constants and the function $V(\eta,t)$ is determined by the differential equation

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial \eta}\left[(bV + c + ab^2C_2^2)\frac{\partial V}{\partial \eta}\right] + 2abC_1\frac{\partial V}{\partial \eta} - 8a^2bC_1^2.$$

2. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[(\alpha w + \beta)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[(\alpha w + \beta)\frac{\partial w}{\partial y}\right].$

This is a two-dimensional heat and mass transfer equation with a linear temperature-dependent thermal conductivity (diffusion coefficient).

The substitution $U = \alpha w + \beta$ leads to an equation of the form 2.1.2.2 for $U = U(x,y,t)$.

3. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{\alpha w + \beta}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\alpha w + \beta}\frac{\partial w}{\partial y}\right).$

This is a two-dimensional heat and mass transfer equation with a hyperbolic temperature-dependent thermal conductivity (diffusion coefficient).

The substitution $U = \alpha w + \beta$ leads to an equation of the form 2.1.2.3 for $U = U(x,y,t)$.

4. $\displaystyle \frac{\partial w}{\partial t} = \alpha\left[\frac{\partial}{\partial x}\left(\frac{1}{w}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{w}\frac{\partial w}{\partial y}\right)\right] + \beta w.$

1°. The transformation

$$w(x,y,t) = e^{\beta t}u(x,y,\tau), \qquad \tau = C - \frac{1}{\beta}e^{-\beta t},$$

where C is an arbitrary constant, leads to a simpler equation of the form 2.1.2.3:

$$\frac{\partial u}{\partial \tau} = \alpha\left[\frac{\partial}{\partial x}\left(\frac{1}{u}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{u}\frac{\partial u}{\partial y}\right)\right].$$

2°. In Zhuravlev (2000), a nonlinear superposition principle is presented that allows the construction of complicated multimodal solutions of the original equation; some exact solutions are also specified there.

5. $\displaystyle \frac{\partial w}{\partial t} = \alpha\left[\frac{\partial}{\partial x}\left(\frac{1}{w}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{w}\frac{\partial w}{\partial y}\right)\right] + \beta w^2.$

The substitution $w = 1/U$ leads to an equation of the form 2.1.4.3 for $U = U(x, y, t)$.

⊙ *References*: V. A. Galaktionov and S. A. Posashkov (1989), N. H. Ibragimov (1994).

6. $\displaystyle \frac{\partial w}{\partial t} = \alpha\left[\frac{\partial}{\partial x}\left(w^n\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(w^n\frac{\partial w}{\partial y}\right)\right] + \beta w.$

The transformation (C is an arbitrary constant)

$$w(x, y, t) = e^{\beta t} U(x, y, \tau), \quad \tau = \frac{1}{\beta n} e^{\beta n t} + C$$

leads to a simpler equation of the form 2.1.2.4:

$$\frac{\partial U}{\partial \tau} = \alpha\left[\frac{\partial}{\partial x}\left(U^n\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(U^n\frac{\partial U}{\partial y}\right)\right].$$

7. $\displaystyle \frac{\partial w}{\partial t} = a_1\frac{\partial}{\partial x}\left(w^{n_1}\frac{\partial w}{\partial x}\right) + a_2\frac{\partial}{\partial y}\left(w^{n_2}\frac{\partial w}{\partial y}\right) + bw^k.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = A^2 w(\pm A^{k-n_1-1}x + B_1, \pm A^{k-n_2-1}y + B_2, A^{2k-2}t + B_3),$$

where A, B_1, B_2, and B_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs can be chosen arbitrarily).

2°. Traveling-wave solution:

$$w(x, y, t) = u(z), \quad z = t - \lambda_1 x - \lambda_2 y,$$

where λ_1 and λ_2 are arbitrary constants, and the function $u = u(z)$ is determined by the ordinary differential equation

$$u'_z = [(a_1\lambda_1^2 u^{n_1} + a_2\lambda_2^2 u^{n_2})u'_z]'_z + bu^k.$$

3°. "Two-dimensional" solution:

$$w(x, y, t) = (\alpha t + \beta)^{\frac{1}{1-k}} F(\xi, \eta), \quad \xi = x(\alpha t + \beta)^{\frac{n_1-k+1}{2(k-1)}}, \quad \eta = y(\alpha t + \beta)^{\frac{n_2-k+1}{2(k-1)}},$$

where the function $F = F(\xi, \eta)$ is determined by the differential equation

$$\frac{\alpha}{1-k}F + \alpha\frac{n_1-k+1}{2(k-1)}\xi\frac{\partial F}{\partial \xi} + \alpha\frac{n_2-k+1}{2(k-1)}\eta\frac{\partial F}{\partial \eta} = a_1\frac{\partial}{\partial \xi}\left(F^{n_1}\frac{\partial F}{\partial \xi}\right) + a_2\frac{\partial}{\partial \eta}\left(F^{n_2}\frac{\partial F}{\partial \eta}\right) + bF^k.$$

⊙ *References*: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983), M. I. Bakirova, S. N. Dimova, V. A. Dorodnitsyn, S. P. Kurdyumov, A. A. Samarskii, and S. R. Svirshchevskii (1988), N. H. Ibragimov (1994).

2.1.4. Other Equations

1. $\displaystyle \frac{\partial w}{\partial t} = aw\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w(C_2 x + C_3, C_2 y + C_4, C_1 C_2^2 t + C_5),$$
$$w_2 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where $C_1, \dots, C_5$ and β are arbitrary constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,y,t) = \varphi(t)(C_1x^2 + C_2xy + C_3y^2 + C_4x + C_5y + C_6), \qquad \varphi(t) = \frac{1}{C_7 - 2a(C_1 + C_3)t},$$

where $C_1, \ldots, C_7$ are arbitrary constants.

3°. The equation admits a more general solution in the form of the product of functions with different arguments:

$$w(x,y,t) = \frac{\Theta(x,y)}{A + Bt},$$

where A and B are arbitrary constants, and the function $\Theta = \Theta(x,y)$ satisfies the two-dimensional Poisson equation

$$a\Delta\Theta + B = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. Traveling-wave solution in implicit form:

$$a(k_1^2 + k_2^2) \int \frac{dw}{\lambda \ln|w| + C_1} = k_1x + k_2y + \lambda t + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

5°. The equation has "two-dimensional" solutions with following the forms:

$$\begin{aligned}
&w(x,y,t) = F(z,t), && z = k_1x + k_2y;\\
&w(x,y,t) = G(r,t), && r = \sqrt{x^2 + y^2};\\
&w(x,y,t) = H(\xi_1,\xi_2), && \xi_1 = k_1x + \lambda_1 t, \quad \xi_2 = k_2y + \lambda_2 t;\\
&w(x,y,t) = t^\beta U(\eta_1,\eta_2), && \eta_1 = x^2t^{-\beta-1}, \quad \eta_2 = y^2t^{-\beta-1};\\
&w(x,y,t) = e^{2t} V(\zeta_1,\zeta_2), && \zeta_1 = xe^{-t}, \quad \zeta_2 = ye^{-t},
\end{aligned}$$

where k_1, k_2, λ_1, λ_2, and β are arbitrary constants.

2. $\displaystyle \boldsymbol{\frac{\partial w}{\partial t} = (\alpha + \beta w)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \gamma w^2 + \delta w + \varepsilon.}$

Generalized separable solution:

$$w(x,y,t) = f(t) + g(t)\Theta(x,y).$$

Here, $\Theta(x,y)$ is any solution of the two-dimensional Helmholtz equation

$$\Delta\Theta + \varkappa\Theta = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{1}$$

where $\varkappa = \gamma/\beta$ ($\beta \neq 0$). For solutions of the linear equation (1), see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

The functions $f(t)$ and $g(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
f'_t &= \gamma f^2 + \delta f + \varepsilon,\\
g'_t &= (\gamma f + \delta - \alpha\varkappa)g.
\end{aligned} \tag{2}$$

The first equation in (2) is independent of $g(t)$ and is separable. On finding $f(t)$, one can solve the second equation in (2), which is linear in $g(t)$.

The functions $f(t)$ and $g(t)$ have different forms depending on the values of the equation parameters. Below are five possible cases; C_1 and C_2 are arbitrary constants.

1°. For $\gamma = \delta = 0$,

$$f(t) = C_1 + \varepsilon t, \quad g(t) = C_2 e^{-\alpha\varkappa t}.$$

2°. For $\gamma = 0$ and $\delta \neq 0$,

$$f(t) = C_1 e^{\delta t} - \frac{\varepsilon}{\delta}, \quad g(t) = C_2 e^{(\delta - \alpha\varkappa)t}.$$

3°. For $\gamma \neq 0$ and $\delta^2 - 4\gamma\varepsilon = \mu^2 > 0$ ($\mu > 0$),

$$f(t) = \frac{s_1 + s_2 C_1 e^{\mu t}}{1 + C_1 e^{\mu t}}, \quad g(t) = \frac{C_2}{1 + C_1 e^{\mu t}} e^{-(\gamma s_2 + \alpha\varkappa)t}, \qquad s_{1,2} = \frac{-\delta \pm \mu}{2\gamma}.$$

4°. For $\gamma \neq 0$ and $\delta^2 - 4\gamma\varepsilon = 0$,

$$f(t) = -\frac{\delta}{2\gamma} - \frac{1}{C_1 + \gamma t}, \quad g(t) = \frac{C_2}{C_1 + \gamma t} \exp\bigl[\bigl(\tfrac{1}{2}\delta - \alpha\varkappa\bigr)t\bigr].$$

5°. For $\gamma \neq 0$ and $\delta^2 - 4\gamma\varepsilon = -\mu^2 < 0$ ($\mu > 0$),

$$f(t) = \frac{\mu}{2\gamma} \tan\bigl(\tfrac{1}{2}\mu t + C_1\bigr) - \frac{\delta}{2\gamma}, \quad g(t) = C_2 \frac{\exp\bigl[\bigl(\tfrac{1}{2}\delta - \alpha\varkappa\bigr)t\bigr]}{\cos\bigl(\tfrac{1}{2}\mu t + C_1\bigr)}.$$

⊙ *References*: V. A. Galaktionov and S. A. Posashkov (1989), N. H. Ibragimov (1994).

3. $\displaystyle \frac{\partial w}{\partial t} = \alpha w \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) - \alpha \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] - \beta.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-1} w(C_1 x + C_2, C_1 y + C_3, C_1 t + C_4),$$
$$w_2 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution:

$$w(x, y, t) = C_1 - \beta t + C_2 \exp\bigl[\alpha(\mu^2 + \nu^2)\bigl(C_1 t - \tfrac{1}{2}\beta t^2\bigr)\bigr] e^{\mu x + \nu y},$$

where μ, ν, C_1, and C_2 are arbitrary constants.

3°. Solutions for $\beta = 0$:

$$w(x, y, t) = \frac{1}{A + \alpha\mu^2 t} + B(A + \alpha\mu^2 t)e^{\mu x} \pm \frac{\sin(\mu y + \eta_0)}{A + \alpha\mu^2 t},$$
$$w(x, y, t) = A\coth\theta(t) + B\sinh\theta(t)\, e^{\mu x} \pm A\frac{\sin(\mu y + \eta_0)}{\sinh\theta(t)},$$
$$w(x, y, t) = A\cot\theta(t) + B\sin\theta(t)\, e^{\mu x} \pm A\frac{\sin(\mu y + \eta_0)}{\sin\theta(t)},$$
$$w(x, y, t) = \frac{A}{\cos\theta(t)} \pm A\frac{1 + \sin\theta(t)}{2\cos\theta(t)}\cosh(\mu x + \xi_0) + sA\frac{1 - \sin\theta(t)}{2\cos\theta(t)}\sin(\mu y + \eta_0),$$
$$w(x, y, t) = -\frac{A}{\cos\theta(t)} \pm A\frac{1 - \sin\theta(t)}{2\cos\theta(t)}\cosh(\mu x + \xi_0) + sA\frac{1 + \sin\theta(t)}{2\cos\theta(t)}\sin(\mu y + \eta_0),$$
$$w(x, y, t) = \frac{A}{\sinh\theta(t)} \pm A\frac{1 - \cosh\theta(t)}{2\sinh\theta(t)}\cosh(\mu x + \xi_0) + sA\frac{1 + \cosh\theta(t)}{2\sinh\theta(t)}\sin(\mu y + \eta_0),$$
$$w(x, y, t) = -\frac{A}{\sinh\theta(t)} \pm A\frac{1 + \cosh\theta(t)}{2\sinh\theta(t)}\cosh(\mu x + \xi_0) + sA\frac{1 - \cosh\theta(t)}{2\sinh\theta(t)}\sin(\mu y + \eta_0),$$

where A, B, μ, ξ_0, η_0, and τ_0 are arbitrary constants, $\theta(t) = \alpha\mu^2 At + \tau_0$, and s is a parameter that can assume the values 1 or -1.

By swapping the variables, $x \rightleftarrows y$, in the above relations, one can obtain another group of solutions (not written out here).

4°. There is a generalized separable solution of the form

$$w(x,y,t) = f(t) + g(t)\varphi(x) + h(t)\psi(y). \qquad (1)$$

In particular, if $\varphi''_{xx} = \nu\varphi$ and $\psi''_{yy} = -\nu\psi$, where ν is an arbitrary constant, we have

$$\varphi(x) = A_1 \cosh\mu x + A_2 \sinh\mu x, \quad \psi(y) = B_1 \cos\mu y + B_2 \sin\mu y \qquad (\nu = \mu^2 > 0),$$
$$\varphi(x) = A_1 \cos\mu x + A_2 \sin\mu x, \quad \psi(y) = B_1 \cosh\mu y + B_2 \sinh\mu y \qquad (\nu = -\mu^2 < 0).$$

Here, A_1, A_2, B_1, and B_2 are arbitrary constants. The functions $f(t)$, $g(t)$, and $h(t)$ in (1) are determined by the system of ordinary differential equations

$$f'_t = \alpha\nu(A_1^2 - sA_2^2)g^2 - \alpha\nu(B_1^2 + sB_2^2)h^2 - \beta,$$
$$g'_t = \alpha\nu f g,$$
$$h'_t = -\alpha\nu f h,$$

where $s = \operatorname{sign}\nu$. The order of this system can be reduced by 2; then the system becomes

$$f = \Phi(h), \quad g = C_2/h, \quad h'_t = -\alpha\nu h\Phi(h),$$

where

$$\Phi(h) = \pm\sqrt{C_1 + (A_1^2 - sA_2^2)\frac{C_2^2}{h^2} + \frac{2\beta}{\alpha\nu}\ln|h| + (B_1^2 + sB_2^2)h^2}\,;$$

C_1 and C_2 are arbitrary constants. For $\beta = 0$, solutions in explicit form may be obtained in some cases (see Item 3°).

5°. There is a generalized separable solution of the form

$$w(x,y,t) = f(t) + g(t)\varphi(x) + h(t)\psi(y) + u(t)\theta(x)\chi(y). \qquad (2)$$

For $\varphi''_{xx} = 4\nu\varphi$, $\psi''_{yy} = -4\nu\psi$, $\theta''_{xx} = \nu\theta$, and $\chi''_{yy} = -\nu\chi$, where ν is an arbitrary constant, one can set in (2)

for $\nu = \mu^2 > 0$	for $\nu = -\mu^2 < 0$
$\varphi(x) = A_1 \cosh 2\mu x + A_2 \sinh 2\mu x$	$\varphi(x) = A_1 \cos 2\mu x + A_2 \sin 2\mu x$
$\psi(y) = B_1 \cos 2\mu y + B_2 \sin 2\mu y$	$\psi(y) = B_1 \cosh 2\mu y + B_2 \sinh 2\mu y$
$\theta(x) = C_1 \cosh\mu x + C_2 \sinh\mu x$	$\theta(x) = C_1 \cos\mu x + C_2 \sin\mu x$
$\chi(y) = D_1 \cos\mu y + D_2 \sin\mu y$	$\chi(y) = D_1 \cosh\mu y + D_2 \sinh\mu y$

The functions $f(t)$, $g(t)$, $h(t)$, and $u(t)$ are determined by the following system of ordinary differential equations ($s = \operatorname{sign}\nu$):

$$f'_t = -4\alpha\nu(A_1^2 - sA_2^2)g^2 + 4\alpha\nu(B_1^2 + sB_2^2)h^2 - \beta,$$
$$g'_t = -4\alpha\nu f g + \alpha\nu a_1(D_1^2 + sD_2^2)u^2,$$
$$h'_t = 4\alpha\nu f h - \alpha\nu a_2(C_1^2 - sC_2^2)u^2,$$
$$u'_t = -2\alpha\nu(a_3 g - a_4 h)u.$$

The arbitrary constants A_1, A_2, B_1, B_2, C_1, C_2, D_1, and D_2 are related by the two constraints

$$2A_1C_1C_2 = A_2(C_1^2 + sC_2^2), \quad 2B_1D_1D_2 = B_2(D_1^2 - sD_2^2).$$

The coefficients a_1, a_2, a_3, and a_4 are defined by

$$a_1 = \frac{C_1^2 + sC_2^2}{2A_1}, \quad a_2 = \frac{D_1^2 - sD_2^2}{2B_1}, \quad a_3 = A_2\frac{C_1^2 - sC_2^2}{C_1C_2}, \quad a_4 = B_2\frac{D_1^2 + sD_2^2}{D_1D_2},$$

provided $A_1 \neq 0$, $B_1 \neq 0$, $C_1C_2 \neq 0$, and $D_1D_2 \neq 0$.

If $A_1 = 0$ ($A_2 \neq 0$), then one should set $a_1 = C_1C_2/A_2$. If $B_1 = 0$ ($B_2 \neq 0$), then $a_2 = D_1D_2/B_2$. If $C_1 = 0$ ($C_2 \neq 0$), then $a_3 = -A_1$. If $C_2 = 0$ ($C_1 \neq 0$), then $a_3 = A_1$. If $D_1 = 0$ ($D_2 \neq 0$), then $a_4 = -B_1$. If $D_2 = 0$ ($D_1 \neq 0$), then $a_4 = B_1$.

6°. The equation admits a traveling-wave solution: $w = w(k_1 x + k_2 y + \lambda t)$, where k_1, k_2, and λ are arbitrary constants.

7°. There is a generalized separable solution of the form

$$w(x, y, t) = f(t)x^2 + g(t)xy + h(t)y^2 + \varphi(t)x + \psi(t)y + \chi(t).$$

In the special case $\varphi(t) = \psi(t) \equiv 0$, the functions $f(t)$, $g(t)$, $h(t)$, and $\chi(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned} f_t' &= \alpha(2fh - 2f^2 - g^2), \qquad & h_t' &= \alpha(2fh - 2h^2 - g^2), \\ g_t' &= -2\alpha g(f + h), & \chi_t' &= 2\alpha(f + h)\chi - \beta, \end{aligned}$$

which can be completely integrated.

⊙ *References for equation* 2.1.4.3: V. A. Galaktionov and S. A. Posashkov (1989), A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998), A. D. Polyanin and V. F. Zaitsev (2002).

4. $\displaystyle \frac{\partial w}{\partial t} = \alpha\left[\frac{\partial}{\partial x}\left(|\nabla w|\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(|\nabla w|\frac{\partial w}{\partial y}\right)\right] + \beta w^2.$

Here, $|\nabla w|^2 = \left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$\begin{aligned} w_1 &= C_1 w(\pm x + C_2, \pm y + C_3, C_1 t + C_4), \\ w_2 &= w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t), \end{aligned}$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs can be chosen arbitrarily).

2°. There are "two-dimensional" solutions of the following forms:

$$\begin{aligned} &w(x, y, t) = U(\xi, \eta), \quad \xi = C_1 x + C_1 y + C_3 t, \quad \eta = C_4 x + C_5 y + C_6 t, \\ &w(x, y, t) = (C_1 t + C_2)^{-1} V(x, y). \end{aligned}$$

3°. "Two-dimensional" generalized separable solution:

$$w(x, y, t) = \frac{1}{C_1 - \beta t} + \frac{C_2}{(C_1 - \beta t)^2} F(x, y),$$

where C_1 and C_2 are arbitrary constants, and the function $F(x, y)$ is any solution of the stationary equation

$$\frac{\partial}{\partial x}\left(|\nabla F|\frac{\partial F}{\partial x}\right) + \frac{\partial}{\partial y}\left(|\nabla F|\frac{\partial F}{\partial y}\right) + \varkappa F^2 = 0, \qquad \varkappa = \frac{\beta}{\alpha}\operatorname{sign} C_2.$$

4°. "Two-dimensional" generalized separable solution:

$$w(x, y, t) = f(t) + g(t)\Theta(x, y).$$

Here, the functions $f(t)$ and $g(t)$ are given by

$$f(t) = \frac{1}{B - \beta t}, \quad g(t) = \frac{\beta}{(B - \beta t)[A + C(B - \beta t)]},$$

where A, B, and C are arbitrary constants, and the function $\Theta(x, y)$ is any solution of the stationary equation

$$\frac{\partial}{\partial x}\left(|\nabla\Theta|\frac{\partial\Theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(|\nabla\Theta|\frac{\partial\Theta}{\partial y}\right) \pm \varkappa\Theta^2 = \mu\Theta, \qquad \varkappa = \frac{\beta}{\alpha}, \quad \mu = \frac{A}{\alpha}.$$

⊙ *References*: V. A. Galaktionov and S. A. Posashkov (1989), N. H. Ibragimov (1994).

2.2. Equations with Two Space Variables Involving Exponential Nonlinearities

2.2.1. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] + ae^{\lambda w}$

1. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m\frac{\partial w}{\partial y}\right) + ce^{\lambda w}.$

This is a special case of equation 2.4.2.1 with $f(w) = ce^{\lambda w}$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = w\Big(C_1^{\frac{1}{2-n}}x,\ C_1^{\frac{1}{2-m}}y,\ C_1t + C_2\Big) + \frac{1}{\lambda}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. There are “two-dimensional” solutions of the following forms:

$$w(x, y, t) = U(r, t), \quad r^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2};$$

$$w(x, y, t) = V(z_1, z_2) - \frac{1}{\lambda}\ln t, \quad z_1 = xt^{\frac{1}{n-2}}, \quad z_2 = yt^{\frac{1}{m-2}}.$$

2. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\lambda y}\frac{\partial w}{\partial y}\right) + ce^{\beta w}.$

This is a special case of equation 2.4.2.3 with $f(w) = ce^{\beta w}$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = w\Big(C_1^{\frac{1}{2-n}}x,\ y - \frac{1}{\lambda}\ln C_1,\ C_1t + C_2\Big) + \frac{1}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. There are “two-dimensional” solutions of the following forms:

$$w(x, y, t) = U(r, t), \quad r^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2};$$

$$w(x, y, t) = V(z_1, z_2) - \frac{1}{\beta}\ln t, \quad z_1 = xt^{\frac{1}{n-2}}, \quad z_2 = y + \frac{1}{\lambda}\ln t.$$

3. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(ae^{\beta x}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\lambda y}\frac{\partial w}{\partial y}\right) + ce^{\mu w}.$

This is a special case of equation 2.4.2.2 with $f(w) = ce^{\mu w}$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = w\Big(x - \frac{1}{\beta}\ln C_1,\ y - \frac{1}{\lambda}\ln C_1,\ C_1t + C_2\Big) + \frac{1}{\mu}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. There are “two-dimensional” solutions of the following forms:

$$w(x, y, t) = U(r, t), \quad r^2 = \frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2};$$

$$w(x, y, t) = V(z_1, z_2) - \frac{1}{\mu}\ln t, \quad z_1 = x + \frac{1}{\beta}\ln t, \quad z_2 = y + \frac{1}{\lambda}\ln t.$$

2.2.2. Equations of the Form $\frac{\partial w}{\partial t}=a\frac{\partial}{\partial x}\left(e^{\beta w}\frac{\partial w}{\partial x}\right)+b\frac{\partial}{\partial y}\left(e^{\lambda w}\frac{\partial w}{\partial y}\right)+f(w)$

1. $\frac{\partial w}{\partial t}=\alpha\left[\frac{\partial}{\partial x}\left(e^{\mu w}\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(e^{\mu w}\frac{\partial w}{\partial y}\right)\right].$

This is a two-dimensional nonstationary heat (diffusion) equation with exponential temperature-dependent thermal diffusivity (diffusion coefficient). This is a special case of equation 2.4.3.3 with $f(w)=\alpha e^{\mu w}$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1=w(C_1x+C_3,C_1y+C_4,C_2t+C_5)+\frac{1}{\mu}\ln\frac{C_2}{C_1^2},$$
$$w_2=w(x\cos\beta-y\sin\beta,x\sin\beta+y\cos\beta,t),$$

where $C_1,\dots,C_5$ and β are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution in implicit form:

$$\alpha(k_1^2+k_2^2)\int\frac{e^{\mu w}\,dw}{\lambda w+C_1}=k_1x+k_2y+\lambda t+C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. "Two-dimensional" additive separable solution:

$$w(x,y,t)=f(t)+\frac{1}{\mu}\ln\Theta(x,y),\qquad f(t)=-\frac{1}{\mu}\ln(A\alpha t+B).$$

Here, A, B, and μ are arbitrary constants, and the function $\Theta(x,y)$ is any solution of the Poisson equation

$$\Delta\Theta+A=0,\qquad\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. There are "two-dimensional" solutions of the following forms:

$$\begin{aligned}
&w(x,y,t)=F(z,t), && z=k_1x+k_2y;\\
&w(x,y,t)=G(r,t), && r=\sqrt{x^2+y^2};\\
&w(x,y,t)=G(\xi_1,\xi_2), && \xi_1=k_1x+\lambda_1t,\quad \xi_2=k_2y+\lambda_2t;\\
&w(x,y,t)=H(\eta_1,\eta_2), && \eta_1=x^2/t,\quad \eta_2=y^2/t;\\
&w(x,y,t)=\frac{2}{\mu}t+U(\zeta_1,\zeta_2), && \zeta_1=xe^{-t},\quad \zeta_2=ye^{-t},
\end{aligned}$$

where k_1, k_2, λ_1, and λ_2 are arbitrary constants.

⊙ *Reference*: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983).

2. $\frac{\partial w}{\partial t}=a\frac{\partial}{\partial x}\left(e^{\beta w}\frac{\partial w}{\partial x}\right)+b\frac{\partial}{\partial y}\left(e^{\lambda w}\frac{\partial w}{\partial y}\right).$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1=w(C_1C_2^{\beta}x+C_3,\pm C_1C_2^{\lambda}y+C_4,C_1^2t+C_5)-2\ln|C_2|,$$

where $C_1,\dots,C_5$ are arbitrary constants, are also solutions of the equation.

2°. "Two-dimensional" solution:

$$w(x,y,t)=u(\theta,t),\quad \theta=c_1x+c_2y,$$

where c_1 and c_2 are arbitrary constants, and the function $u=u(\theta,t)$ is determined by the differential equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial\theta}\left[\left(ac_1^2e^{\beta w}+bc_2^2e^{\lambda w}\right)\frac{\partial w}{\partial\theta}\right].$$

3°. "Two-dimensional" solution:

$$w(x,y,t)=u(\zeta_1,\zeta_2),\quad \zeta_1=k_1x+m_1t,\quad \zeta_2=k_2y+m_2t,$$

where k_1, k_2, m_1, and m_2 are arbitrary constants, and the function $u=u(\zeta_1,\zeta_2)$ is determined by the differential equation

$$m_1\frac{\partial u}{\partial\zeta_1}+m_2\frac{\partial u}{\partial\zeta_2}=ak_1^2\frac{\partial}{\partial\zeta_1}\left(e^{\beta w}\frac{\partial w}{\partial\zeta_1}\right)+bk_2^2\frac{\partial}{\partial\zeta_2}\left(e^{\lambda w}\frac{\partial w}{\partial\zeta_2}\right).$$

4°. "Two-dimensional" solution:

$$w(x,y,t)=U(\xi,\eta)+2kt,\quad \xi=xe^{-k\beta t},\quad \eta=ye^{-k\lambda t},$$

where k is an arbitrary constant, and the function $U=U(\xi,\eta)$ is determined by the differential equation

$$2k-k\beta\xi\frac{\partial U}{\partial\xi}-k\lambda\eta\frac{\partial U}{\partial\eta}=a\frac{\partial}{\partial\xi}\left(e^{\beta U}\frac{\partial U}{\partial\xi}\right)+b\frac{\partial}{\partial\eta}\left(e^{\lambda U}\frac{\partial U}{\partial\eta}\right).$$

5°. "Two-dimensional" solution:

$$w(x,y,t)=V(z_1,z_2),\quad z_1=\frac{x+C_1}{\sqrt{t+C_3}},\quad z_2=\frac{y+C_2}{\sqrt{t+C_3}},$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $V=V(z_1,z_2)$ is determined by the differential equation

$$-\frac{1}{2}z_1\frac{\partial V}{\partial z_1}-\frac{1}{2}z_2\frac{\partial V}{\partial z_2}=a\frac{\partial}{\partial z_1}\left(e^{\beta w}\frac{\partial w}{\partial z_1}\right)+b\frac{\partial}{\partial z_2}\left(e^{\lambda w}\frac{\partial w}{\partial z_2}\right).$$

6°. "Two-dimensional" solution ($\beta=1$):

$$w(x,y,t)=u(z,t)+2\ln|x|,\quad z=x^{-\lambda}y,$$

where the function $u=u(z,t)$ is determined by the differential equation

$$\frac{\partial u}{\partial t}=\left(a\lambda^2z^2e^u+be^{\lambda u}\right)\frac{\partial^2u}{\partial z^2}+\lambda\left(a\lambda z^2e^u+be^{\lambda u}\right)\left(\frac{\partial u}{\partial z}\right)^2+a\lambda(\lambda-3)ze^u\frac{\partial u}{\partial z}+2ae^u.$$

⊙ *References*: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983), N. H. Ibragimov (1994).

3. $\displaystyle \frac{\partial w}{\partial t}=a\frac{\partial}{\partial x}\left(e^{\lambda_1 w}\frac{\partial w}{\partial x}\right)+b\frac{\partial}{\partial y}\left(e^{\lambda_2 w}\frac{\partial w}{\partial y}\right)+ce^{\beta w}.$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1=w(\pm C_1^{\beta-\lambda_1}x+C_2,\pm C_1^{\beta-\lambda_2}y+C_3,C_1^{2\beta}t+C_4)+2\ln|C_1|,$$

where $C_1,\ldots,C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs can be chosen arbitrarily).

2°. There is a "two-dimensional" solution of the form

$$w(x,y,t)=U(\xi,\eta)-\frac{1}{\beta}\ln t,\quad \xi=xt^{\frac{\lambda_1-\beta}{2\beta}},\quad \eta=yt^{\frac{\lambda_2-\beta}{2\beta}}.$$

⊙ *References*: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983), N. H. Ibragimov (1994).

4. $\displaystyle \frac{\partial w}{\partial t} = \alpha\left[\frac{\partial}{\partial x}\left(e^{\mu w}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(e^{\mu w}\frac{\partial w}{\partial y}\right)\right] + \beta e^{\mu w} + \gamma + \delta e^{-\mu w}.$

The substitution $w = \dfrac{1}{\mu}\ln U$ leads to an equation of the form 2.1.4.2:

$$\frac{\partial U}{\partial t} = \alpha U\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \beta\mu U^2 + \mu\gamma U + \mu\delta.$$

⊙ *References*: V. A. Galaktionov and S. A. Posashkov (1989), N. H. Ibragimov (1994).

2.3. Other Equations with Two Space Variables Involving Arbitrary Parameters

2.3.1. Equations with Logarithmic Nonlinearities

1. $\displaystyle \frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - kw\ln w.$

This is a nonstationary equation with a logarithmic source arising in heat and mass transfer theory and combustion theory. This is a special case of equation 2.4.1.1 with $f(w) = -kw\ln w$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = \exp(C_1 e^{-kt})w(x + C_2, \pm y + C_3, t + C_4),$$
$$w_2 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where $C_1, \dots, C_4$ and β are arbitrary constants, are also solutions of the equation.

2°. "Two-dimensional" multiplicative separable solution:

$$w(x, y, t) = \exp(C_1 e^{-kt})\Theta(x, y),$$

where the function $\Theta(x, y)$ is a solution of the stationary equation

$$a\left(\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2}\right) - k\Theta\ln\Theta = 0.$$

This equation has a particular solution of the form $\Theta = \exp(A_1x^2 + A_2xy + A_3y^2 + A_4x + A_5y + A_6)$, where the coefficients A_k are determined by an algebraic system of equations.

3°. "Two-dimensional" solution with incomplete separation of variables (the solution is separable in the space variables x and y, but is not separable in time t):

$$w(x, y, t) = \varphi(x, t)\psi(y, t),$$

where the functions $\varphi(x, t)$ and $\psi(y, t)$ are determined by solving two independent one-dimensional nonlinear parabolic differential equations:

$$\frac{\partial\varphi}{\partial t} = a\frac{\partial^2\varphi}{\partial x^2} - k\varphi\ln\varphi,$$
$$\frac{\partial\psi}{\partial t} = a\frac{\partial^2\psi}{\partial y^2} - k\psi\ln\psi.$$

For solutions of these equations, see 1.6.1.4 with $f(t) = 0$.

4°. There are exact solutions in the form of the product of functions representing two independent traveling waves:

$$w(x, y, t) = \varphi(\xi)\psi(\eta), \qquad \xi = a_1x + b_1t, \qquad \eta = a_2y + b_2t,$$

where a_1, b_1, a_2, and b_2 are arbitrary constants. This solution is a special case of the solution presented in Item 3°.

⊙ *References*: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), A. D. Polyanin and V. F. Zaitsev (2002).

2. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m \frac{\partial w}{\partial y}\right) + cw \ln w + sw.$

This is a special case of equation 2.4.2.1 with $f(w) = cw \ln w + sw$ and a special case of equation 2.4.2.4 in which a should be renamed c and b renamed s and then the functions $f(x)$ and $g(y)$ should be substituted by ax^n and by^m, respectively.

There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = F(\xi,t), \qquad \xi^2 = \frac{4}{a(2-n)^2}x^{2-n} + \frac{4}{b(2-m)^2}y^{2-m};$$
$$w(x,y,t) = \exp(Ae^{ct})G(x,y);$$
$$w(x,y,t) = H_1(x,t)H_2(y,t),$$

where A is an arbitrary constant.

3. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(ae^{\beta x} \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\lambda y} \frac{\partial w}{\partial y}\right) + cw \ln w + sw.$

This is a special case of equation 2.4.2.2 with $f(w) = cw \ln w + sw$ and a special case of equation 2.4.2.4 in which a should be renamed c and b renamed s and then the functions $f(x)$ and $g(y)$ should be substituted by $e^{\beta x}$ and $be^{\lambda y}$, respectively.

There are exact solutions of the following forms:

$$w(x,y,t) = F(\xi,t), \qquad \xi^2 = \frac{4}{a\beta^2}e^{-\beta x} + \frac{4}{b\lambda^2}e^{-\lambda y};$$
$$w(x,y,t) = \exp(Ae^{ct})G(x,y);$$
$$w(x,y,t) = H_1(x,t)H_2(y,t),$$

where A is an arbitrary constant.

4. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\lambda y} \frac{\partial w}{\partial y}\right) + cw \ln w + sw.$

This is a special case of equation 2.4.2.3 with $f(w) = cw \ln w + sw$ and a special case of equation 2.4.2.4 in which a should be renamed c and b renamed s and then the functions $f(x)$ and $g(y)$ should be substituted by ax^n and $be^{\lambda y}$, respectively.

There are exact solutions of the following forms:

$$w(x,y,t) = F(\xi,t), \qquad \xi^2 = \frac{4}{a(2-n)^2}x^{2-n} + \frac{4}{b\lambda^2}e^{-\lambda y};$$
$$w(x,y,t) = \exp(Ae^{ct})G(x,y);$$
$$w(x,y,t) = H_1(x,t)H_2(y,t),$$

where A is an arbitrary constant.

2.3.2. Equations with Trigonometrical Nonlinearities

1. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m \frac{\partial w}{\partial y}\right) + c\sin(kw + s).$

This is a special case of equation 2.4.2.1 with $f(w) = c\sin(kw + s)$.

There is an exact solution of the form

$$w(x,y,t) = U(\xi,t), \qquad \xi^2 = \frac{4}{a(2-n)^2}x^{2-n} + \frac{4}{b(2-m)^2}y^{2-m}.$$

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + c\sin(kw+s).$

This is a special case of equation 2.4.2.2 with $f(w) = c\sin(kw+s)$.

There is an exact solution of the form

$$w(x,y,t) = U(\xi,t), \qquad \xi^2 = \frac{4}{a\beta^2}e^{-\beta x} + \frac{4}{b\lambda^2}e^{-\lambda y}.$$

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + c\sin(kw+s).$

This is a special case of equation 2.4.2.3 with $f(w) = c\sin(kw+s)$.

There is an exact solution of the form

$$w(x,y,t) = U(\xi,t), \qquad \xi^2 = \frac{4}{a(2-n)^2}x^{2-n} + \frac{4}{b\lambda^2}e^{-\lambda y}.$$

2.4. Equations Involving Arbitrary Functions

2.4.1. Heat and Mass Transfer Equations in Quiescent or Moving Media with Chemical Reactions

1. $\dfrac{\partial w}{\partial t} = a\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) + f(w).$

This is a two-dimensional equation of unsteady heat/mass transfer or combustion in a quiescent medium.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm x + C_1, \pm y + C_2, t + C_3),$$
$$w_2 = w(x\cos\beta - y\sin\beta,\, x\sin\beta + y\cos\beta,\, t),$$

where C_1, C_2, C_3, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 can be chosen independently of each other).

2°. Traveling-wave solution:

$$w = w(\xi), \qquad \xi = Ax + By + \lambda t,$$

where A, B, and λ are arbitrary constants, and the function $w(\xi)$ is determined by the autonomous ordinary differential equation

$$a(A^2+B^2)w''_{\xi\xi} - \lambda w'_\xi + f(w) = 0.$$

For solutions of this equation, see Polyanin and Zaitsev (1995, 2003).

3°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = F(z,t), \qquad z = k_1x + k_2y;$$
$$w(x,y,t) = G(r,t), \qquad r = \sqrt{x^2+y^2};$$
$$w(x,y,t) = H(\xi_1,\xi_2), \quad \xi_1 = k_1x + \lambda_1 t, \quad \xi_2 = k_2y + \lambda_2 t,$$

where k_1, k_2, λ_1, and λ_2 are arbitrary constants.

2. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial w}{\partial x} + b\dfrac{\partial w}{\partial y} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} - f(w).$

This is a two-dimensional equation of unsteady heat/mass transfer with a volume chemical reaction in a steady translational fluid flow.

The transformation

$$w = U(\xi, \eta, t), \qquad \xi = x - at, \qquad \eta = y - bt$$

leads to a simpler equation of the form 2.4.1.1:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} - f(U).$$

3. $\dfrac{\partial w}{\partial t} + (a_1 x + b_1 y + c_1)\dfrac{\partial w}{\partial x} + (a_2 x + b_2 y + c_2)\dfrac{\partial w}{\partial y} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} - f(w).$

This is a two-dimensional equation of unsteady heat/mass transfer with a volume chemical reaction in a steady translational-shear fluid flow.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\bigl(x + Cb_1 e^{\lambda t}, y + C(\lambda - a_1)e^{\lambda t}, t\bigr),$$

where C is an arbitrary constant and $\lambda = \lambda_{1,2}$ are roots of the quadratic equation

$$\lambda^2 - (a_1 + b_2)\lambda + a_1 b_2 - a_2 b_1 = 0, \qquad (1)$$

are also solutions of the equation.

2°. Solutions:

$$w = w(z), \qquad z = a_2 x + (\lambda - a_1)y + Ce^{\lambda t},$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation (1), and the function $w(z)$ is determined by the ordinary differential equation

$$\bigl[\lambda z + a_2 c_1 + (\lambda - a_1)c_2\bigr] w'_z = \bigl[a_2^2 + (\lambda - a_1)^2\bigr] w''_{zz} - f(w).$$

3°. "Two-dimensional" solutions:

$$w = U(\zeta, t), \qquad \zeta = a_2 x + (\lambda - a_1)y,$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation (1), and the function $U(\zeta, t)$ is determined by the differential equation

$$\frac{\partial U}{\partial t} + \bigl[\lambda\zeta + a_2 c_1 + (\lambda - a_1)c_2\bigr]\frac{\partial U}{\partial \zeta} = \bigl[a_2^2 + (\lambda - a_1)^2\bigr]\frac{\partial^2 U}{\partial \zeta^2} - f(U).$$

Remark. In the case of an incompressible fluid, the equation coefficients must satisfy the condition $a_1 + b_2 = 0$.

4. $\dfrac{\partial w}{\partial t} + f_1(t)\dfrac{\partial w}{\partial x} + f_2(t)\dfrac{\partial w}{\partial y} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} - g(w).$

This equation describes mass transfer with volume chemical reaction in an unsteady translational fluid flow.

The transformation

$$w = U(\xi, \eta, t), \qquad \xi = x - \int f_1(t)\,dt, \qquad \eta = y - \int f_2(t)\,dt,$$

leads to a simpler equation of the form 2.4.1.1:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} - g(U).$$

2.4.2. Equations of the Form $\frac{\partial w}{\partial t}=\frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right]+h(w)$

1. $\frac{\partial w}{\partial t}=\frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(by^m\frac{\partial w}{\partial y}\right)+f(w).$

This is a two-dimensional equation of unsteady heat (mass) transfer or combustion in an anisotropic case with power-law coordinate-dependent principal thermal diffusivities (diffusion coefficients).

Solution for $n\neq 2$ and $m\neq 2$:

$$w=w(\xi,t),\qquad \xi^2=\frac{4}{a(2-n)^2}x^{2-n}+\frac{4}{b(2-m)^2}y^{2-m},$$

where the function $w(\xi,t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial \xi^2}+\frac{A}{\xi}\frac{\partial w}{\partial \xi}+f(w),\qquad A=\frac{4-nm}{(2-n)(2-m)}.$$

For solutions of this equation with $A=0$ and various $f(w)$, see Subsections 1.1.1 to 1.1.3 and equations 1.2.1.1 to 1.2.1.3, 1.4.1.2, 1.4.1.3, 1.4.1.7, and 1.4.1.8.

2. $\frac{\partial w}{\partial t}=\frac{\partial}{\partial x}\left(ae^{\beta x}\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(be^{\lambda y}\frac{\partial w}{\partial y}\right)+f(w).$

This is a two-dimensional equation of unsteady heat (mass) transfer or combustion in an anisotropic case with exponential coordinate-dependent principal thermal diffusivities (diffusion coefficients).

Solution for $\beta\neq 0$ and $\lambda\neq 0$:

$$w=w(\xi,t),\qquad \xi^2=\frac{4}{a\beta^2}e^{-\beta x}+\frac{4}{b\lambda^2}e^{-\lambda y},$$

where the function $w(\xi,t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial \xi^2}-\frac{1}{\xi}\frac{\partial w}{\partial \xi}+f(w).$$

3. $\frac{\partial w}{\partial t}=\frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(be^{\lambda y}\frac{\partial w}{\partial y}\right)+f(w).$

Solution for $n\neq 2$ and $\lambda\neq 0$:

$$w=w(\xi,t),\qquad \xi^2=\frac{4}{a(2-n)^2}x^{2-n}+\frac{4}{b\lambda^2}e^{-\lambda y},$$

where the function $w(\xi,t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial \xi^2}+\frac{n}{2-n}\frac{1}{\xi}\frac{\partial w}{\partial \xi}+f(w).$$

4. $\frac{\partial w}{\partial t}=\frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right]+aw\ln w+bw.$

This is a two-dimensional equation of unsteady heat (mass) transfer or combustion in an anisotropic case with arbitrary coordinate-dependent principal thermal diffusivities (diffusion coefficients) and a logarithmic source.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the function

$$w_1=\exp(C_1e^{at})w(x,y,t+C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. "Two-dimensional" multiplicative separable solution:

$$w(x,y,t) = \exp(C_1 e^{at}) U(x,y),$$

where the function $U(x,y)$ is determined by the stationary equation

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial U}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial U}{\partial y}\right] + aU\ln U + bU = 0.$$

3°. "Two-dimensional" solution with incomplete separation of variables (the solution is separable in the space variables x and y, but is not separable in time t):

$$w(x,y,t) = \varphi(x,t)\psi(y,t),$$

where the functions $\varphi(x,t)$ and $\psi(y,t)$ are determined from the two independent one-dimensional nonlinear parabolic differential equations

$$\frac{\partial \varphi}{\partial t} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial \varphi}{\partial x}\right] + a\varphi\ln\varphi + C(t)\varphi,$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial y}\left[g(y)\frac{\partial \psi}{\partial y}\right] + a\psi\ln\psi + b\psi - C(t)\psi,$$

and $C(t)$ is an arbitrary function.

⊙ *Reference*: A. D. Polyanin (2000).

2.4.3. Equations of the Form $\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(w)\frac{\partial w}{\partial y}\right] + h(t,w)$

1. $\displaystyle \frac{\partial w}{\partial t} = a\left[\frac{\partial}{\partial x}\left(w^n\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(w^n\frac{\partial w}{\partial y}\right)\right] + f(t)w.$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w\bigl(\pm C_1^n x + C_2,\ \pm C_1^n y + C_3,\ t\bigr),$$

$$w_2 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where C_1, C_2, C_3, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 can be chosen independently of each other).

2°. Multiplicative separable solution:

$$w(x,y,t) = \exp\left[\int f(t)\,dt\right]\bigl[\Theta(x,y)\bigr]^{\frac{1}{n+1}}, \tag{1}$$

where the function $\Theta(x,y)$ is a solution of the Laplace equation

$$\Delta\Theta = 0, \qquad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

For solutions of this linear stationary equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

3°. Multiplicative separable solution:

$$w(x,y,t) = \varphi(t)\bigl[\Theta(x,y)\bigr]^{\frac{1}{n+1}}, \tag{2}$$

where the function $\varphi(t)$ is a solution of the Bernoulli equation

$$\varphi'_t - f(t)\varphi + Aa\varphi^{n+1} = 0, \tag{3}$$

A is an arbitrary constant, and the function $\Theta(x,y)$ is determined by the stationary equation

$$\Delta\Theta + A(n+1)\Theta^{\frac{1}{n+1}} = 0, \qquad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The general solution of equation (3) is given by

$$\varphi(t) = \exp\bigl[F(t)\bigr]\left\{Aan\int \exp\bigl[nF(t)\bigr]\,dt + B\right\}^{-1/n}, \quad F(t) = \int f(t)\,dt,$$

where B is an arbitrary constant.

4°. The transformation

$$w(x,y,t)=F(t)U(x,y,\tau), \quad \tau=\int F^n(t)\,dt, \qquad F(t)=\exp\left[\int f(t)\,dt\right]$$

leads to a simpler equation of the form 2.1.2.4:

$$\frac{\partial U}{\partial \tau}=a\left[\frac{\partial}{\partial x}\left(U^n\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(U^n\frac{\partial U}{\partial y}\right)\right].$$

2. $\displaystyle \frac{\partial w}{\partial t}=a\left[\frac{\partial}{\partial x}\left(e^{\mu w}\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(e^{\mu w}\frac{\partial w}{\partial y}\right)\right]+f(t).$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1=w\bigl(C_1x+C_2,\ \pm C_1y+C_3,\ t\bigr)-\frac{2}{\mu}\ln|C_1|,$$
$$w_2=w(x\cos\beta-y\sin\beta,x\sin\beta+y\cos\beta,t),$$

where C_1, C_2, C_3, and β are arbitrary constants, are also solutions of the equation.

2°. Additive separable solution:

$$w(x,y,t)=\varphi(t)+\frac{1}{\mu}\ln\Theta(x,y),$$

where the function $\varphi=\varphi(t)$ is determined by the ordinary differential equation

$$\varphi'_t+A(a/\mu)\exp(\mu\varphi)-f(t)=0, \tag{1}$$

and the function $\Theta(x,y)$ is a solution of the two-dimensional Poisson equation

$$\Delta\Theta+A=0, \qquad \Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}. \tag{2}$$

The general solution of equation (1) is given by

$$\varphi(t)=F(t)-\frac{1}{\mu}\ln\left\{B+Aa\int\exp\bigl[\mu F(t)\bigr]\,dt\right\}, \quad F(t)=\int f(t)\,dt. \tag{3}$$

For solutions of the linear stationary equation (2), see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

Note that equations (1), (2) and relation (3) involve arbitrary constants A and B.

3°. The transformation

$$w(x,y,t)=U(x,y,\tau)+F(t), \quad \tau=\int\exp[\mu F(t)]\,dt, \qquad F(t)=\int f(t)\,dt,$$

leads to a simpler equation of the form 2.2.2.1:

$$\frac{\partial U}{\partial \tau}=a\left[\frac{\partial}{\partial x}\left(e^{\mu U}\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(e^{\mu U}\frac{\partial U}{\partial y}\right)\right].$$

3. $\displaystyle \frac{\partial w}{\partial t}=\frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[f(w)\frac{\partial w}{\partial y}\right].$

This is a two-dimensional nonlinear heat and mass transfer equation for an anisotropic medium.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1=w(C_1x+C_2,C_1y+C_3,C_1^2t+C_4),$$
$$w_2=w(x\cos\beta-y\sin\beta,x\sin\beta+y\cos\beta,t),$$

where $C_1,\ldots,C_4$ and β are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution in implicit form:

$$(k_1^2+k_2^2)\int\frac{f(w)\,dw}{\lambda w+C_1}=k_1x+k_2y+\lambda t+C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. Solution:

$$w(x,y,t)=U(\xi),\qquad \xi=\frac{x^2+y^2}{t},$$

where the function $U=U(\xi)$ is determined by the ordinary differential equation

$$[\xi f(U)U'_\xi]'_\xi+\tfrac{1}{4}\xi U'_\xi=0.$$

4°. "Two-dimensional" solutions (for the axisymmetric problems):

$$w(x,y,t)=V(r,t),\qquad r=\sqrt{x^2+y^2},$$

where the function $V=V(r,t)$ is determined by the differential equation

$$\frac{\partial V}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[rf(V)\frac{\partial V}{\partial r}\right].$$

5°. For other "two-dimensional" solutions, see equation 2.4.3.4 with $g(w)=f(w)$.

⦿ *Reference*: V. A. Dorodnitsyn, I. V. Knyazeva, and S. R. Svirshchevskii (1983).

4. $\displaystyle\frac{\partial w}{\partial t}=\frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[g(w)\frac{\partial w}{\partial y}\right].$

This is a two-dimensional unsteady heat and mass transfer equation in an anisotropic case with arbitrary coordinate-dependent principal thermal diffusivities (diffusion coefficients).

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1=w(C_1x+C_2,\pm C_1y+C_3,C_1^2t+C_4),$$

where $C_1,\ldots,C_4$ are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution in implicit form:

$$\int\frac{k_1^2f(w)+k_2^2g(w)}{\lambda w+C_1}\,dw=k_1x+k_2y+\lambda t+C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. "Two-dimensional" solution:

$$w(x,y,t)=U(z,t),\qquad z=k_1x+k_2y,$$

where the function $U=U(z,t)$ is determined by a differential equation of the form 1.6.15.1:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial z}\left[\varphi(U)\frac{\partial U}{\partial z}\right],\qquad \varphi(U)=k_1^2f(U)+k_2^2g(U).$$

4°. There are more complicated "two-dimensional" solutions of the form

$$w(x,y,t)=V(\zeta_1,\zeta_2),\qquad \zeta_1=a_1x+a_2y+a_3t,\qquad \zeta_2=b_1x+b_2y+b_3t.$$

5°. "Two-dimensional" solution:

$$w(x,y,t)=W(\xi,\eta),\qquad \xi=\frac{x}{\sqrt{at}},\qquad \eta=\frac{y}{\sqrt{at}},$$

where $a\neq 0$ is any number and the function $W=W(\xi,\eta)$ is determined by the differential equation

$$\frac{\partial}{\partial \xi}\left[f(W)\frac{\partial W}{\partial \xi}\right]+\frac{\partial}{\partial \eta}\left[g(W)\frac{\partial W}{\partial \eta}\right]+\frac{a}{2}\xi\frac{\partial W}{\partial \xi}+\frac{a}{2}\eta\frac{\partial W}{\partial \eta}=0.$$

6°. For group classification of the equation in question, see Dorodnitsyn, Knyazeva, and Svirshchevskii (1983).

2.4.4. Other Equations Linear in the Highest Derivatives

1. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + f(w)\left[\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right].$

The substitution

$$U = \int F(w)\,dw, \qquad \text{where} \quad F(w) = \exp\left[\int f(w)\,dw\right],$$

leads to the linear heat equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}.$$

For solutions of this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

2. $\dfrac{\partial w}{\partial t} = \big[aw + f(t)\big]\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) + bw^2 + g(t)w + h(t), \qquad a \neq 0.$

"Two-dimensional" generalized separable solution:

$$w(x, y, t) = \varphi(t) + \psi(t)\Theta(x, y),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = b\varphi^2 + g(t)\varphi + h(t), \tag{1}$$

$$\psi'_t = \big[b\varphi - \beta f(t) + g(t)\big]\psi, \qquad \beta = b/a, \tag{2}$$

and the function $\Theta(x, y)$ is any solution of the two-dimensional Helmholtz equation

$$\Delta\Theta + \beta\Theta = 0, \qquad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{3}$$

The first equation (1) is independent of ψ and is a Riccati equation for φ. In Polyanin and Zaitsev (2003), many exact solutions of equation (1) for various $g(t)$ and $h(t)$ are presented. Solving equation (1) followed by substituting the expression of $\varphi = \varphi(t)$ into (2), we arrive at a linear equation for $\psi = \psi(t)$, which is easy to integrate.

In the special case $B = 0$, a solution of system (1), (2) is given by

$$\varphi(t) = \exp\big[G(t)\big]\left\{A + \int h(t)\exp\big[-G(t)\big]\,dt\right\}, \quad G(t) = \int g(t)\,dt,$$

$$\psi(t) = B\exp\big[G(t) - \beta F(t)\big], \quad F(t) = \int f(t)\,dt,$$

where A and B are arbitrary constants.

For solutions of the linear stationary equation (3), see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

3. $\dfrac{\partial w}{\partial t} = aw\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) - a\left[\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right] + f(t).$

1°. Generalized separable solution:

$$w(x, y, t) = \varphi(t) + \psi(t)e^{\beta x + \gamma y},$$

where β and γ are arbitrary constants and the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = f(t), \qquad \psi'_t = a(\beta^2 + \gamma^2)\varphi\psi.$$

Solving this system yields the solution

$$w(x, y, t) = \varphi(t) + A\exp\left[\beta x + \gamma y + a(\beta^2 + \gamma^2)\int \varphi(t)\,dt\right], \quad \varphi(t) = \int f(t)\,dt + B,$$

where A and B are arbitrary constants.

2°. There are generalized separable solutions of the following forms:

$$w(x,y,t) = \varphi(t) + \psi(t)(A_1 \cosh \mu x + A_2 \sinh \mu x) + \chi(t)(B_1 \cos \mu y + B_2 \sin \mu y),$$
$$w(x,y,t) = \varphi(t) + \psi(t)(A_1 \cos \mu x + A_2 \sin \mu x) + \chi(t)(B_1 \cosh \mu y + B_2 \sinh \mu y),$$

where A_1, A_2, B_1, B_2, and μ are arbitrary constants, and the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by a system of ordinary differential equations (not written out here).

3°. There is a generalized separable solution of the form

$$w(x,y,t) = \varphi(t) + \psi(t)F(x) + \chi(t)G(y) + \eta(t)H(x)P(y),$$

where

$$F(x) = A_1 \cos 2\mu x + A_2 \sin 2\mu x, \qquad G(y) = B_1 \cosh 2\mu y + B_2 \sinh 2\mu y,$$
$$H(x) = C_1 \cos \mu x + C_2 \sin \mu x, \qquad P(y) = D_1 \cosh \mu y + D_2 \sinh \mu y,$$

where the constants A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, and μ are related by two constraints, and the functions $\varphi(t)$, $\psi(t)$, $\chi(t)$, and $\eta(t)$ are determined by a system of ordinary differential equations (not written out here).

4. $\displaystyle \frac{\partial w}{\partial t} + (a_1 x + b_1 y + c_1)\frac{\partial w}{\partial x} + (a_2 x + b_2 y + c_2)\frac{\partial w}{\partial y} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(w)\frac{\partial w}{\partial y}\right].$

This equation describes unsteady anisotropic heat/mass transfer in a steady translational-shear fluid flow.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_{1,2} = w\bigl(x + Cb_1 e^{\lambda t},\, y + C(\lambda - a_1)e^{\lambda t},\, t\bigr),$$

where C is an arbitrary constant, and $\lambda = \lambda_{1,2}$ are roots of the quadratic equation

$$\lambda^2 - (a_1 + b_2)\lambda + a_1 b_2 - a_2 b_1 = 0, \tag{1}$$

are also solutions of the equation.

2°. Solutions:

$$w = w(z), \qquad z = a_2 x + (\lambda - a_1)y + Ce^{\lambda t}, \tag{2}$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation (1), and the function $w(z)$ is determined by the ordinary differential equation

$$\bigl[\lambda z + a_2 c_1 + (\lambda - a_1)c_2\bigr] w'_z = [\varphi(w) w'_z]'_z, \qquad \varphi(w) = a_2^2 f(w) + (\lambda - a_1)^2 g(w).$$

3°. "Two-dimensional" solutions:

$$w = U(\zeta, t), \qquad \zeta = a_2 x + (\lambda - a_1)y, \tag{3}$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation (1), and the function $U(\zeta,t)$ is determined by the differential equation

$$\frac{\partial U}{\partial t} + [\lambda\zeta + a_2 c_1 + (\lambda - a_1)c_2]\frac{\partial U}{\partial \zeta} = \frac{\partial}{\partial \zeta}\left[\varphi(U)\frac{\partial U}{\partial \zeta}\right], \qquad \varphi(U) = a_2^2 f(U) + (\lambda - a_1)^2 g(U).$$

Remark 1. A more general equation, with an additional term $h(w)$ on the right-hand side, where h is an arbitrary function, also has solutions of the forms (2) and (3).

Remark 2. In the case of an incompressible fluid, the equation coefficients must satisfy the condition $a_1 + b_2 = 0$.

5. $\displaystyle \frac{\partial w}{\partial t} = f(w)\mathbf{L}[w] + g(t)w + h(t).$

Here, $\mathbf{L}$ is an arbitrary linear differential operator with respect to the space variables x, y (the operator is independent of t).

"Two-dimensional" generalized separable solution:

$$w(x,y,t) = \varphi(t) + \psi(t)\Theta(x,y),$$

where the functions $\varphi(t)$ and $\psi(t)$ are given by

$$\varphi(t) = e^{G(t)}\left[A + \int h(t)e^{-G(t)}\,dt\right], \qquad \psi(t) = Be^{G(t)}, \qquad G(t) = \int g(t)\,dt,$$

A is an arbitrary constant, and the function $\Theta(x,y)$ is a solution of the linear stationary equation

$$\mathbf{L}[\Theta] = 0.$$

Remark 1. In the equation under consideration, the order of the linear operator $\mathbf{L}$ and the number of space variables can be any. The coefficients of $\mathbf{L}$ can be dependent on the space variables.

Remark 2. The above remains valid if $f(w)$ in the equation is substituted by a function $f(x,y,t,w)$. In the special case $f(x,y,t,w) = f_1(t) + \alpha w$, $\mathbf{L}[w] = \Delta w + \beta w$, where Δ is the Laplace operator, α and β are some constants, we obtain an equation of the form 2.4.4.2.

6. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(x,y)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(x,y)\frac{\partial w}{\partial y}\right] + kw\ln w.$

This is an equation of unsteady heat (mass) transfer or combustion in an anisotropic case with arbitrary coordinate-dependent principal thermal diffusivities (diffusion coefficients) and a logarithmic source.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the function

$$w_1 = \exp(C_1e^{kt})w(x,y,t+C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,y,t) = \exp(C_1e^{kt})\Theta(x,y),$$

where the function $U(x,y)$ satisfies the stationary equation

$$\frac{\partial}{\partial x}\left[f(x,y)\frac{\partial U}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(x,y)\frac{\partial U}{\partial y}\right] + kU\ln U = 0.$$

7. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f_1(x,t)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[f_2(y,t)\frac{\partial w}{\partial y}\right] + \big[g_1(x,t) + g_2(y,t)\big]w + h(t)w\ln w.$

Exact solution with incomplete separation of variables (the solution is separable in the space variables x and y, but is not separable in time t):

$$w(x,y,t) = \varphi(x,t)\psi(y,t).$$

Here, the functions $\varphi(x,t)$ and $\psi(y,t)$ are determined from the two one-dimensional nonlinear parabolic differential equations

$$\frac{\partial\varphi}{\partial t} = \frac{\partial}{\partial x}\left[f_1(x,t)\frac{\partial\varphi}{\partial x}\right] + g_1(x,t)\varphi + h(t)\varphi\ln\varphi + C(t)\varphi,$$
$$\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial y}\left[f_2(y,t)\frac{\partial\psi}{\partial y}\right] + g_2(y,t)\psi + h(t)\psi\ln\psi - C(t)\psi,$$

where $C(t)$ is an arbitrary function.

8. $\dfrac{\partial w}{\partial t}=f_1(x,y)\dfrac{\partial^2 w}{\partial x^2}+f_2(x,y)\dfrac{\partial^2 w}{\partial x\partial y}+f_3(x,y)\dfrac{\partial^2 w}{\partial y^2}$
$\qquad\qquad +g_1(x,y)\dfrac{\partial w}{\partial x}+g_2(x,y)\dfrac{\partial w}{\partial y}+\big[h(x,y)+s(t)\big]w+kw\ln w.$

Multiplicative separable solution:

$$w(x,y,t)=\exp\left[Ae^{kt}+e^{kt}\int e^{-kt}s(t)\,dt\right]\Theta(x,y),$$

where A is an arbitrary constant, and the function $\Theta(x,y)$ is a solution of the stationary equation

$$f_1(x,y)\frac{\partial^2\Theta}{\partial x^2}+f_2(x,y)\frac{\partial^2\Theta}{\partial x\partial y}+f_3(x,y)\frac{\partial^2\Theta}{\partial y^2}$$
$$+g_1(x,y)\frac{\partial\Theta}{\partial x}+g_2(x,y)\frac{\partial\Theta}{\partial y}+h(x,y)\Theta+k\Theta\ln\Theta=0.$$

9. $\dfrac{\partial w}{\partial t}=\dfrac{\partial}{\partial x}\left[f(x,t)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left\{\big[g(x,t)w+h(x,t)\big]\dfrac{\partial w}{\partial y}\right\}.$

There are "two-dimensional" generalized separable solutions linear and quadratic in y:

$$w(x,y,t)=\varphi(x,t)y+\psi(x,t),$$
$$w(x,y,t)=\varphi(x,t)y^2+\psi(x,t)y+\chi(x,t).$$

2.4.5. Nonlinear Diffusion Boundary Layer Equations

1. $\dfrac{\partial w}{\partial t}+f(x,t)\dfrac{\partial w}{\partial x}+g(x,t)y\dfrac{\partial w}{\partial y}=h(x,t)\dfrac{\partial}{\partial y}\left[k(w)\dfrac{\partial w}{\partial y}\right].$

This equation arises in nonlinear problems of the unsteady diffusion boundary layer (mass exchange of drops and bubbles with a flow, convective diffusion in fluid films), where the coordinates x and y are longitudinal and normal to the interphase surface, respectively.

The transformation

$$w=U(\zeta,\tau,\psi),\quad \zeta=y\varphi(x,t),\quad \tau=\tau(x,t),\quad \psi=\psi(x,t),$$

where the functions $\varphi(x,t)$, $\tau(x,t)$, and $\psi(x,t)$ are determined by the system of first-order partial differential equations

$$\begin{aligned}\frac{\partial\varphi}{\partial t}+f(x,t)\frac{\partial\varphi}{\partial x}&=-g(x,t)\varphi,\\ \frac{\partial\tau}{\partial t}+f(x,t)\frac{\partial\tau}{\partial x}&=h(x,t)\varphi^2,\\ \frac{\partial\psi}{\partial t}+f(x,t)\frac{\partial\psi}{\partial x}&=0,\end{aligned}\tag{1}$$

leads to a simpler equation of the form 1.6.15.1:

$$\frac{\partial U}{\partial\tau}=\frac{\partial}{\partial\zeta}\left[k(U)\frac{\partial U}{\partial\zeta}\right].\tag{2}$$

The cyclic variable ψ does not appear in equation (2); however, it can be involved in the transformed initial and boundary conditions as a parameter.

Integrating system (1) is reduced to solving a single ordinary differential equation: $x'_t=f(x,t)$. In particular, if the functions f, g, and h are only dependent on x, the general solution of system (1) is given by

$$\varphi=\Phi_1(z)E(x),\quad \tau=\Phi_1^2(z)\int\frac{h(x)}{f(x)}E^2(x)\,dx+\Phi_2(z),\quad \psi=\Phi_3(z),$$

where $\Phi_1(z)$, $\Phi_2(z)$, and $\Phi_3(z)$ are arbitrary functions, and

$$z = t - \int \frac{dx}{f(x)}, \qquad E(x) = \exp\left[-\int \frac{g(x)}{f(x)}\,dx\right].$$

⊙ *Reference*: A. D. Polyanin (1982).

2. $\displaystyle f(x,y)z^{n-1}\frac{\partial w}{\partial x} + g(x,y)z^{n-1}\frac{\partial w}{\partial y} + h(x,y)z^n\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left[k(w)\frac{\partial w}{\partial z}\right].$

This equation arises in nonlinear problems of the steady three-dimensional diffusion boundary layer (mass exchange of solid particles, drops, and bubbles with a flow, convective diffusion in fluid films), where z is a normal coordinate to the particle surface. To a solid particle there corresponds $n = 2$ and to drops and bubbles, $n = 1$.

The transformation

$$w = U(\zeta, \tau, \psi), \quad \zeta = z\varphi(x,y), \quad \tau = \tau(x,y), \quad \psi = \psi(x,y),$$

where the functions $\varphi(x,y)$, $\tau(x,y)$, and $\psi(x,y)$ are determined by the system of first-order partial differential equations

$$f(x,y)\frac{\partial \varphi}{\partial x} + g(x,y)\frac{\partial \varphi}{\partial y} = -h(x,y)\varphi, \tag{1}$$

$$f(x,y)\frac{\partial \tau}{\partial x} + g(x,y)\frac{\partial \tau}{\partial y} = \varphi^2, \tag{2}$$

$$f(x,y)\frac{\partial \psi}{\partial x} + g(x,y)\frac{\partial \psi}{\partial y} = 0, \tag{3}$$

leads to a simpler equation of the form 1.6.17.16:

$$\frac{\partial U}{\partial \tau} = \zeta^{1-n}\frac{\partial}{\partial \zeta}\left[k(U)\frac{\partial U}{\partial \zeta}\right]. \tag{4}$$

The cyclic variable ψ does not enter into equation (4); however, it can appear in the transformed initial and boundary conditions as a parameter.

Suppose an integral of the ordinary differential equation $f(x,y)y'_x = g(x,y)$ has the form

$$\Xi(x,y) = C.$$

Then the general solution of equation (3) is given by $\psi = F(\Xi)$, where F is an arbitrary function. On passing in (1)–(2) from x, y to the new variables x, Ξ, one arrives at ordinary differential equations with independent variable x where Ξ appears as a parameter.

2.5. Equations with Three or More Space Variables

2.5.1. Equations of Mass Transfer in Quiescent or Moving Media with Chemical Reactions

1. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} - f(w).$

This equation describes unsteady mass or heat transfer with a volume reaction in a quiescent medium.

The equation admits translations in any of the variables x, y, z, t.

1°. There is a traveling-wave solution, $w = w(k_1x + k_2y + k_3z + \lambda t)$.

2°. For axisymmetric case, the Laplace operator on the right-hand side of the equation takes the following forms in cylindrical and spherical coordinates, respectively:

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial w}{\partial\rho}\right)+\frac{\partial^2 w}{\partial z^2}, \qquad \rho=\sqrt{x^2+y^2};$$

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right), \quad r=\sqrt{x^2+y^2+z^2}.$$

3°. "Three-dimensional" solution:

$$w=u(\xi,\eta,t), \quad \xi=y+\frac{x}{C}, \quad \eta=(C^2-1)x^2-2Cxy+C^2z^2,$$

where C is an arbitrary constant ($C\neq 0$), and the function $u=u(\xi,\eta,t)$ is determined by the equation

$$\frac{\partial u}{\partial t}=\left(1+\frac{1}{C^2}\right)\frac{\partial^2 u}{\partial\xi^2}-4\xi\frac{\partial^2 u}{\partial\xi\partial\eta}+4C^2(\xi^2+\eta)\frac{\partial^2 u}{\partial\eta^2}+2(2C^2-1)\frac{\partial u}{\partial\eta}-f(u).$$

Remark. The solution specified in Item 3° can be used to obtain other "three-dimensional" solutions by means of the cyclic permutations of the space variables.

4°. "Three-dimensional" solution:

$$w=u(\xi,\eta,t), \quad \xi=Ax+By+Cz, \quad \eta=\sqrt{(Bx-Ay)^2+(Cy-Bz)^2+(Az-Cx)^2},$$

where A, B, and C are arbitrary constants and the function $u=u(\xi,\eta,t)$ is determined by the equation

$$\frac{\partial u}{\partial t}=(A^2+B^2+C^2)\left(\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial u}{\partial\eta}\right)-f(u).$$

2. $\displaystyle \frac{\partial w}{\partial t}=a\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+f(t)w\ln w+g(t)w.$

1°. There is a functional separable solution of the form

$$w(x,y,z,t)=\exp\left[\sum_{n,m=1}^{3}\varphi_{nm}(t)x_nx_m+\sum_{n=1}^{3}\psi_n(t)x_n+\chi(t)\right], \qquad x_1=x, \quad x_2=y, \quad x_3=z.$$

2°. There is a incomplete separable solution of the form

$$w(x,y,z,t)=\Phi_1(x,t)\Phi_2(y,t)\Phi_3(z,t).$$

3°. For $f(t)=b=\text{const}$, the equation also has a multiplicative separable solution of the form

$$w(x,y,z,t)=\varphi(t)\Theta(x,y,z),$$

where $\varphi(t)$ is given by

$$\varphi(t)=\exp\left[Ae^{bt}+e^{bt}\int e^{-bt}g(t)\,dt\right],$$

A is an arbitrary constant, and $\Theta(x,y,z)$ is a solution of the stationary equation

$$a\left(\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial y^2}+\frac{\partial^2\Theta}{\partial z^2}\right)+b\Theta\ln\Theta=0.$$

3. $\displaystyle \frac{\partial w}{\partial t}+a_1\frac{\partial w}{\partial x}+a_2\frac{\partial w}{\partial y}+a_3\frac{\partial w}{\partial z}=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}-f(w).$

This equation describes unsteady mass transfer with a volume chemical reaction in a steady translational fluid flow.

The transformation

$$w=U(\xi,\eta,\zeta,t), \quad \xi=x-a_1t, \quad \eta=y-a_2t, \quad \zeta=z-a_3t$$

leads to a simpler equation of the form 2.5.1.1:

$$\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial\xi^2}+\frac{\partial^2 U}{\partial\eta^2}+\frac{\partial^2 U}{\partial\zeta^2}-f(U).$$

4. $\dfrac{\partial w}{\partial t} + f_1(t)\dfrac{\partial w}{\partial x} + f_2(t)\dfrac{\partial w}{\partial y} + f_3(t)\dfrac{\partial w}{\partial z} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2} - g(w).$

This equation describes unsteady mass transfer with a volume chemical reaction in an unsteady translational fluid flow.

The transformation

$$w = U(\xi, \eta, t), \qquad \xi = x - \int f_1(t)\,dt, \qquad \eta = y - \int f_2(t)\,dt, \qquad \zeta = z - \int f_3(t)\,dt$$

leads to a simpler equation of the form 2.5.1.1:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \zeta^2} - g(U).$$

5. $\dfrac{\partial w}{\partial t} + (a_1 x + b_1 y + c_1 z + d_1)\dfrac{\partial w}{\partial x} + (a_2 x + b_2 y + c_2 z + d_2)\dfrac{\partial w}{\partial y}$

$+ (a_3 x + b_3 y + c_3 z + d_3)\dfrac{\partial w}{\partial z} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2} - f(w).$

This equation describes unsteady mass transfer with a volume chemical reaction in a three-dimensional steady translational-shear fluid flow.

1°. Let λ be a root of the cubic equation

$$\begin{vmatrix} a_1 - \lambda & a_2 & a_3 \\ b_1 & b_2 - \lambda & b_3 \\ c_1 & c_2 & c_3 - \lambda \end{vmatrix} = 0, \qquad (1)$$

and let the constants A_1, A_2, and A_3 solve the degenerate system of linear algebraic equations

$$\begin{aligned} (a_1 - \lambda)A_1 + a_2 A_2 + a_3 A_3 &= 0, \\ b_1 A_1 + (b_2 - \lambda)A_2 + b_3 A_3 &= 0, \\ c_1 A_1 + c_2 A_2 + (c_3 - \lambda)A_3 &= 0. \end{aligned} \qquad (2)$$

One of these equations can be omitted, since it is a consequence of the other two.

Suppose $w(x, y, z, t)$ is a solution of the equation in question. Then the function

$$w_1 = w\bigl(x + A_1 C e^{\lambda t}, y + A_2 C e^{\lambda t}, z + A_3 C e^{\lambda t}, t\bigr),$$

where C is an arbitrary constant, λ is a root of the cubic equation (1), and A_1, A_2, and A_3 are the corresponding solution of the algebraic system (2), is also a solution of the equation.

2°. Solution:

$$w = w(\xi), \qquad \xi = A_1 x + A_2 y + A_3 z + C e^{\lambda t},$$

where C is an arbitrary constant, λ is a root of the cubic equation (1), A_1, A_2, and A_3 are the corresponding solution of the algebraic system (2), and the function $w(\xi)$ is determined by the ordinary differential equation

$$(\lambda\xi + A_1 d_1 + A_2 d_2 + A_3 d_3)w'_\xi = (A_1^2 + A_2^2 + A_3^2)w''_{\xi\xi} - f(w).$$

3°. Let λ be a root of the cubic equation (1) and let A_1, A_2, and A_3 be the corresponding solution of the algebraic system (2).

"Two-dimensional" solution:

$$w = U(\zeta, t), \qquad \zeta = A_1 x + A_2 y + A_3 z,$$

where the function $U(\zeta, t)$ is determined by the differential equation

$$\frac{\partial U}{\partial t} + (\lambda\zeta + A_1 d_1 + A_2 d_2 + A_3 d_3)\frac{\partial U}{\partial \zeta} = (A_1^2 + A_2^2 + A_3^2)\frac{\partial^2 U}{\partial \zeta^2} - f(U).$$

Remark. In the case of an incompressible fluid, the equation coefficients must satisfy the condition $a_1 + b_2 + c_3 = 0$.

6. $\dfrac{\partial w}{\partial t}+(a_1x+b_1y+c_1z+d_1)\dfrac{\partial w}{\partial x}+(a_2x+b_2y+c_2z+d_2)\dfrac{\partial w}{\partial y}+(a_3x+b_3y+c_3z+d_3)\dfrac{\partial w}{\partial z}$
$$=\frac{\partial}{\partial x}\left[f_1(w)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[f_2(w)\frac{\partial w}{\partial y}\right]+\frac{\partial}{\partial z}\left[f_3(w)\frac{\partial w}{\partial z}\right].$$

This equation describes unsteady anisotropic mass or heat transfer in a three-dimensional steady translational-shear fluid flow.

1°. Let λ be a root of the cubic equation

$$\begin{vmatrix} a_1-\lambda & a_2 & a_3 \\ b_1 & b_2-\lambda & b_3 \\ c_1 & c_2 & c_3-\lambda \end{vmatrix}=0, \tag{1}$$

and the constants A_1, A_2, and A_3 solve the degenerate system of linear algebraic equations

$$\begin{aligned} (a_1-\lambda)A_1+a_2A_2+a_3A_3&=0,\\ b_1A_1+(b_2-\lambda)A_2+b_3A_3&=0,\\ c_1A_1+c_2A_2+(c_3-\lambda)A_3&=0. \end{aligned} \tag{2}$$

One of these equations can be omitted, since it is a consequence of the other two.

Suppose $w(x,y,z,t)$ is a solution of the equation in question. Then the function

$$w_1=w\bigl(x+A_1Ce^{\lambda t},y+A_2Ce^{\lambda t},z+A_3Ce^{\lambda t},t\bigr),$$

where C is an arbitrary constant, λ is a root of the cubic equation (1), and A_1, A_2, and A_3 are the corresponding solution of the algebraic system (2), is also a solution of the equation.

2°. Solution:
$$w=w(\xi),\qquad \xi=A_1x+A_2y+A_3z+Ce^{\lambda t}, \tag{3}$$
where C is an arbitrary constant, λ is a root of the cubic equation (1), and A_1, A_2, and A_3 are the corresponding solution of the algebraic system (2), and the function $w(\xi)$ is determined by the ordinary differential equation

$$(\lambda\xi+A_1d_1+A_2d_2+A_3d_3)w'_\xi=[\varphi(w)w'_\xi]'_\xi,$$
$$\varphi(w)=A_1^2f_1(w)+A_2^2f_2(w)+A_3^2f_3(w).$$

3°. Let λ be a root of the cubic equation (1) and let A_1, A_2, and A_3 be the corresponding solution of the algebraic system (2).

"Two-dimensional" solution:

$$w=U(\zeta,t),\qquad \zeta=A_1x+A_2y+A_3z, \tag{4}$$

where the function $U(\zeta,t)$ is determined by the differential equation

$$\frac{\partial U}{\partial t}+(\lambda\zeta+A_1d_1+A_2d_2+A_3d_3)\frac{\partial U}{\partial \zeta}=\frac{\partial}{\partial \zeta}\left[\varphi(U)\frac{\partial U}{\partial \zeta}\right],$$
$$\varphi(U)=A_1^2f_1(U)+A_2^2f_2(U)+A_3^2f_3(U).$$

Remark 1. A more general equation, with an additional term $g(w)$ on the right-hand side, where g is an arbitrary function, also has solutions of the forms (3) and (4).

Remark 2. In the case of an incompressible fluid, the equation coefficients must satisfy the condition $a_1+b_2+c_3=0$.

2.5.2. Heat Equations with Power Law or Exponential Temperature Dependent Thermal Diffusivity

► *Throughout this subsection, the symbols* div, ∇, *and* Δ *stand for the divergence operator, gradient operator, and Laplace operator in Cartesian coordinates* x, y, z *(cylindrical, spherical, and other three-dimensional orthogonal systems of coordinates can be used instead of the Cartesian coordinates).*

1. $\dfrac{\partial w}{\partial t} = \Delta(w^m)$.

This is a special case of equation 2.5.5.6.

2. $\dfrac{\partial w}{\partial t} = \alpha\,\mathrm{div}\big(w^n \nabla w\big) + f(t)w$.

1°. Multiplicative separable solution:

$$w(x,y,z,t) = \exp\left[\int f(t)\,dt\right]\big[\Theta(x,y,z)\big]^{\frac{1}{n+1}}, \tag{1}$$

where the function $\Theta(x,y,z)$ satisfies the Laplace equation

$$\Delta\Theta = 0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

2°. Multiplicative separable solution:

$$w(x,y,z,t) = \varphi(t)\big[\Theta(x,y,z)\big]^{\frac{1}{n+1}}, \tag{2}$$

where the function $\varphi(t)$ is determined by the Bernoulli equation

$$\varphi'_t - f(t)\varphi + A\alpha\varphi^{n+1} = 0. \tag{3}$$

Here, A is an arbitrary constant, and $\Theta(x,y,z)$ is a solution of the stationary equation

$$\Delta\Theta + A(n+1)\Theta^{\frac{1}{n+1}} = 0.$$

The general solution of equation (3) is given by

$$\varphi(t) = \exp\big[F(t)\big]\left\{A\alpha n\int \exp\big[nF(t)\big]\,dt + B\right\}^{-1/n}, \quad F(t) = \int f(t)\,dt,$$

where B is an arbitrary constant.

3°. Using the transformation

$$w(x,y,z,t) = F(t)U(x,y,z,\tau), \quad \tau = \int F^n(t)\,dt, \qquad F(t) = \exp\left[\int f(t)\,dt\right],$$

one arrives at the simpler equation $\dfrac{\partial U}{\partial \tau} = \alpha\,\mathrm{div}(U^n \nabla U)$.

3. $\dfrac{\partial w}{\partial t} = \alpha\,\mathrm{div}(w^n \nabla w) + f(t)w + g(t)w^{1-n}$.

The substitution $U = w^n$ leads to a special case of equation 2.5.4.4:

$$\frac{\partial U}{\partial t} = \alpha U\Delta U + \frac{\alpha}{n}|\nabla U|^2 + nf(t)U + ng(t).$$

4. $\dfrac{\partial w}{\partial t} = \alpha \operatorname{div}\bigl(e^{\mu w}\nabla w\bigr) + f(t).$

1°. Additive separable solution:

$$w(x,y,z,t) = \int f(t)\,dt + \frac{1}{\mu}\ln\Theta(x,y,z),$$

where the function $\Theta = \Theta(x,y,z)$ is any solution of the Laplace equation $\Delta\Theta = 0$.

2°. Additive separable solution:

$$w(x,y,z,t) = \varphi(t) + \frac{1}{\mu}\ln\Theta(x,y,z),$$

where the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi'_t + A(\alpha/\mu)\exp(\mu\varphi) - f(t) = 0. \tag{1}$$

Here, A is an arbitrary constant, and the function $\Theta = \Theta(x,y,z)$ is a solution of the Poisson equation

$$\Delta\Theta + A = 0. \tag{2}$$

The general solution of equation (1) is given by

$$\varphi(t) = F(t) - \frac{1}{\mu}\ln\biggl\{B + A\alpha\int \exp\bigl[\mu F(t)\bigr]\,dt\biggr\}, \qquad F(t) = \int f(t)\,dt. \tag{3}$$

For solutions of the linear stationary equation (2), see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

Note that equations (1), (2) and relation (3) contain arbitrary constants A and B.

3°. Using the transformation

$$w(x,y,z,t) = U(x,y,z,\tau) + F(t), \quad \tau = \int \exp[\mu F(t)]\,dt, \qquad F(t) = \int f(t)\,dt,$$

one arrives at the simpler equation $\dfrac{\partial U}{\partial \tau} = \alpha \operatorname{div}(e^{\mu U}\nabla U)$.

5. $\dfrac{\partial w}{\partial t} = a \operatorname{div}\bigl(e^{\mu w}\nabla w\bigr) + be^{\mu w} + g(t) + h(t)e^{-\mu w}.$

The substitution $U = e^{\mu w}$ leads to an equation of the form 2.5.4.5 for $U = U(x,y,z,t)$:

$$\frac{\partial U}{\partial t} = aU\Delta U + b\mu U^2 + \mu g(t)U + \mu h(t).$$

Hence, the original equation has solutions of the form

$$w(x,y,z,t) = \frac{1}{\mu}\ln[\varphi(t) + \psi(t)\Theta(x,y,z)].$$

Note that, with $g(t) \equiv \text{const}$ and $h(t) \equiv \text{const}$, the original equation was studied in Galaktionov and Posashkov (1989) and Ibragimov (1994).

2.5.3. Equations of Heat and Mass Transfer in Anisotropic Media

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(cz^n\dfrac{\partial w}{\partial z}\right) + f(w).$

1°. Solution for $m \neq 2$ and $n \neq 2$:

$$w = w(\xi,t), \qquad \xi^2 = \frac{x^2}{a} + \frac{4\,y^{2-m}}{b(2-m)^2} + \frac{4\,z^{2-n}}{c(2-n)^2},$$

where the function $w(\xi,t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + \frac{A}{\xi}\frac{\partial w}{\partial \xi} + f(w), \qquad A = \frac{2(4-m-n)}{(2-m)(2-n)}.$$

For solutions of this equation with $A = 0$ and various $f(w)$, see Subsections 1.1.1 to 1.1.3 and equations 1.2.1.1 to 1.2.1.3, 1.4.1.2, 1.4.1.3, 1.4.1.7, and 1.4.1.8.

2°. Solution for $m \neq 2$ and $n \neq 2$:

$$w = w(x, \xi, t), \qquad \xi^2 = \frac{4\, y^{2-m}}{b(2-m)^2} + \frac{4\, z^{2-n}}{c(2-n)^2},$$

where the function $w(x, \xi)$ is determined by the two-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial \xi^2} + \frac{A}{\xi}\frac{\partial w}{\partial \xi} + f(w), \qquad A = \frac{4 - mn}{(2-m)(2-n)}.$$

2. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(cz^l \frac{\partial w}{\partial z}\right) + f(w).$

Solution for $n \neq 2$, $m \neq 2$, and $l \neq 2$:

$$w = w(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-l}}{c(2-l)^2}\right],$$

where the function $w(\xi, t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + \frac{A}{\xi}\frac{\partial w}{\partial \xi} + f(w), \qquad A = 2\left(\frac{1}{2-n} + \frac{1}{2-m} + \frac{1}{2-l}\right) - 1.$$

For solutions of this equation with $A = 0$ and various $f(w)$, see Subsections 1.1.1 to 1.1.3 and equations 1.2.1.1 to 1.2.1.3, 1.4.1.2, 1.4.1.3, 1.4.1.7, and 1.4.1.8.

3. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(ae^{\lambda x} \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\mu y} \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z} \frac{\partial w}{\partial z}\right) + f(w).$

Solution for $\lambda \neq 0$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = w(\xi, t), \qquad \xi^2 = 4\left(\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right),$$

where the function $w(\xi, t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} - \frac{1}{\xi}\frac{\partial w}{\partial \xi} + f(w).$$

4. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z} \frac{\partial w}{\partial z}\right) + f(w).$

Solution for $n \neq 2$, $m \neq 2$, and $\nu \neq 0$:

$$w = w(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

where the function $w(\xi, t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + \frac{A}{\xi}\frac{\partial w}{\partial \xi} + f(w), \qquad A = \frac{4 - nm}{(2-n)(2-m)}.$$

For solutions of this equation with $A = 0$ and various $f(w)$, see Subsections 1.1.1 to 1.1.3 and equations 1.2.1.1 to 1.2.1.3, 1.4.1.2, 1.4.1.3, 1.4.1.7, and 1.4.1.8.

5. $\displaystyle \frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\mu y} \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z} \frac{\partial w}{\partial z}\right) + f(w).$

Solution for $n \neq 2$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = w(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

where the function $w(\xi, t)$ is determined by the one-dimensional nonstationary equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \xi^2} + \frac{n}{2-n}\frac{1}{\xi}\frac{\partial w}{\partial \xi} + f(w).$$

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f_1(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f_2(w)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[f_3(w)\dfrac{\partial w}{\partial z}\right] + g(w).$

For group classification and exact solutions of this equation for some $f_n(w)$ and $g(w)$, see Dorodnitsyn, Knyazeva, and Svirshchevskii (1983).

7. $\dfrac{\partial w}{\partial t} + (a_1x + b_1y + c_1z + d_1)\dfrac{\partial w}{\partial x} + (a_2x + b_2y + c_2z + d_2)\dfrac{\partial w}{\partial y} + (a_3x + b_3y + c_3z + d_3)\dfrac{\partial w}{\partial z}$
$= \dfrac{\partial}{\partial x}\left[f_1(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f_2(w)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[f_3(w)\dfrac{\partial w}{\partial z}\right].$

This equation describes unsteady anisotropic heat or mass transfer in a three-dimensional steady translational-shear fluid flow.

1°. Let λ be a root of the cubic equation

$$\begin{vmatrix} a_1-\lambda & a_2 & a_3 \\ b_1 & b_2-\lambda & b_3 \\ c_1 & c_2 & c_3-\lambda \end{vmatrix} = 0, \tag{1}$$

and let the constants A_1, A_2, and A_3 solve the degenerate system of linear algebraic equations

$$\begin{aligned} (a_1-\lambda)A_1 + a_2A_2 + a_3A_3 &= 0, \\ b_1A_1 + (b_2-\lambda)A_2 + b_3A_3 &= 0, \\ c_1A_1 + c_2A_2 + (c_3-\lambda)A_3 &= 0. \end{aligned} \tag{2}$$

One of these equations is redundant and can be omitted.

Suppose $w(x, y, z, t)$ is a solution of the equation in question. Then the function

$$w_1 = w\bigl(x + A_1Ce^{\lambda t}, y + A_2Ce^{\lambda t}, z + A_3Ce^{\lambda t}, t\bigr),$$

where C is an arbitrary constant, λ is a root of the cubic equation (1), and A_1, A_2, and A_3 are the corresponding solution of the algebraic system (2), is also a solution of the equation.

2°. Solution:

$$w = w(\xi), \qquad \xi = A_1x + A_2y + A_3z + Ce^{\lambda t}, \tag{3}$$

where C is an arbitrary constant, λ is a root of the cubic equation (1), and A_1, A_2, and A_3 are the corresponding solution of the algebraic system (2), and the function $w(\xi)$ is determined by the ordinary differential equation

$$(\lambda\xi + A_1d_1 + A_2d_2 + A_3d_3)w'_\xi = [\varphi(w)w'_\xi]'_\xi, \qquad \varphi(w) = A_1^2f_1(w) + A_2^2f_2(w) + A_3^2f_3(w).$$

3°. Let λ be a root of the cubic equation (1) and let A_1, A_2, and A_3 be the corresponding solution of the algebraic system (2).

"Two-dimensional" solutions:

$$w = U(\zeta, t), \qquad \zeta = A_1x + A_2y + A_3z, \tag{4}$$

where the function $U(\zeta, t)$ is determined by the differential equation

$$\frac{\partial U}{\partial t} + (\lambda\zeta + A_1d_1 + A_2d_2 + A_3d_3)\frac{\partial U}{\partial \zeta} = \frac{\partial}{\partial \zeta}\left[\varphi(U)\frac{\partial U}{\partial \zeta}\right], \qquad \varphi(U) = A_1^2f_1(U) + A_2^2f_2(U) + A_3^2f_3(U).$$

Remark 1. A more general equation, with an additional term $g(w)$ on the right-hand side, where g is an arbitrary function, also has solutions of the forms (3) and (4).

Remark 2. In the case of an incompressible fluid, the equation coefficients must satisfy the condition $a_1 + b_2 + c_3 = 0$.

2.5.4. Other Equations with Three Space Variables

► *Throughout this subsection, the symbols* div, ∇, *and* Δ *stand for the divergence operator, gradient operator, and Laplace operator in Cartesian coordinates x, y, z; cylindrical, spherical, and other three-dimensional orthogonal systems of coordinates can be used instead of the Cartesian coordinates.*

1. $\dfrac{\partial w}{\partial t} = a\Delta w + f(t)|\nabla w|^2 + g(t)w + h(t).$

There is a generalized separable solution of the form

$$w(x_1, x_2, x_3, t) = \sum_{k,l=1}^{3} \varphi_{kl}(t)x_k x_l + \sum_{k=1}^{3} \psi_k(t)x_k + \chi(t).$$

Remark. The more general equation

$$\frac{\partial w}{\partial t} = \sum_{n,m=1}^{3} a_{nm}(t)\frac{\partial^2 w}{\partial x_n \partial x_m} + \sum_{n=1}^{3} b_n(t)\left(\frac{\partial w}{\partial x_n}\right)^2 + \sum_{n=1}^{3} c_n(t)\frac{\partial w}{\partial x_n} + g(t)w + h(t)$$

has solutions of the same form.

2. $\dfrac{\partial w}{\partial t} = \Delta w + f(w)|\nabla w|^2.$

The substitution

$$U = \int F(w)\,dw, \qquad \text{where} \quad F(w) = \exp\left[\int f(w)\,dw\right],$$

leads to the linear heat equation

$$\frac{\partial U}{\partial t} = \Delta U.$$

For solutions of this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

3. $\dfrac{\partial w}{\partial t} = \alpha w\Delta w - \alpha|\nabla w|^2 - \beta.$

1°. Solutions:

$$w(x, y, z, t) = Ax + By + Cz - [\alpha(A^2 + B^2 + C^2) + \beta]t + D,$$
$$w(x, y, z, t) = A - \beta t + B\exp\bigl[\alpha(\varkappa^2 + \mu^2 + \nu^2)\bigl(At - \tfrac{1}{2}\beta t^2\bigr)\bigr]e^{\varkappa x + \mu y + \nu z},$$

where A, B, C, D, $\varkappa$, μ, and ν are arbitrary constants.

2°. See 2.5.4.4 with $f(t) = -\alpha$, $g(t) = 0$, and $h(t) = -\beta$.

4. $\dfrac{\partial w}{\partial t} = \alpha w\Delta w + f(t)|\nabla w|^2 + g(t)w + h(t).$

There are generalized separable solutions of the form

$$w(x_1, x_2, x_3, t) = \sum_{k,l=1}^{3} \varphi_{kl}(t)x_k x_l + \sum_{k=1}^{3} \psi_k(t)x_k + \chi(t).$$

Remark. The more general equation

$$\frac{\partial w}{\partial t} = \sum_{n,m=1}^{3} \bigl[a_{nm}(t)w + b_{nm}(t)\bigr]\frac{\partial^2 w}{\partial x_n \partial x_m} + \sum_{n=1}^{3} c_n(t)\left(\frac{\partial w}{\partial x_n}\right)^2 + \sum_{n=1}^{3} s_n(t)\frac{\partial w}{\partial x_n} + g(t)w + h(t)$$

has solutions of the same form.

5. $\dfrac{\partial w}{\partial t} = [aw + f(t)]\Delta w + bw^2 + g(t)w + h(t).$

Here, $f(t)$, $g(t)$, and $h(t)$ are arbitrary functions; a and b are arbitrary parameters ($a \neq 0$). This is a special case of equation 2.5.4.6 with $\mathbf{L}[w] \equiv \Delta w$.

Note that, with $f(t) \equiv \text{const}$, $g(t) \equiv \text{const}$, and $h(t) \equiv \text{const}$, this equation was studied in Galaktionov and Posashkov (1989) and Ibragimov (1994).

6. $\dfrac{\partial w}{\partial t} = [aw + f(t)]\mathbf{L}[w] + bw^2 + g(t)w + h(t).$

Here, $f(t)$, $g(t)$, and $h(t)$ are arbitrary functions; a and b are arbitrary parameters ($a \neq 0$); $\mathbf{L}[w]$ is an arbitrary linear differential operator of the second (or any) order that depends on the space variables $x_1 = x$, $x_2 = y$, $x_3 = z$ only and satisfies the condition $\mathbf{L}[\text{const}] \equiv 0$:

$$\mathbf{L}[w] \equiv \sum_{n,m=1}^{3} p_{nm}(\mathbf{x}) \frac{\partial^2 w}{\partial x_n \partial x_m} + \sum_{n=1}^{3} q_n(\mathbf{x}) \frac{\partial w}{\partial x_n}, \qquad \mathbf{x} = \{x_1, x_2, x_3\}.$$

There is a generalized separable solution of the form

$$w(x_1, x_2, x_3, t) = \varphi(t) + \psi(t)\Theta(x_1, x_2, x_3),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi'_t = b\varphi^2 + g(t)\varphi + h(t), \tag{1}$$

$$\psi'_t = \big[b\varphi - \beta f(t) + g(t)\big]\psi, \qquad \beta = b/a, \tag{2}$$

and the function $\Theta(x_1, x_2, x_3)$ is a solution of the linear stationary equation

$$\mathbf{L}[\Theta] + \beta\Theta = 0. \tag{3}$$

Equation (1) is independent of ψ and represents a Riccati equation for φ. A large number of exact solutions to equation (1) for various $g(t)$ and $h(t)$ can be found in Polyanin and Zaitsev (2003). On solving (1) and substituting the resulting $\varphi = \varphi(t)$ into (2), one obtains a linear equation for $\psi = \psi(t)$, which is easy to integrate.

In the special case $b = 0$, the solution of system (1), (2) is given by

$$\varphi(t) = \exp\big[G(t)\big]\left\{A + \int h(t)\exp\big[-G(t)\big]\,dt\right\}, \quad G(t) = \int g(t)\,dt,$$

$$\psi(t) = B\exp\big[G(t) - \beta F(t)\big], \quad F(t) = \int f(t)\,dt,$$

where A and B are arbitrary constants.

In the special case $\mathbf{L} \equiv \Delta$, see Tikhonov and Samarskii (1990) and Polyanin (2002) for solutions of the linear stationary equation (3).

7. $\dfrac{\partial w}{\partial t} = f(t)\mathbf{N}_\beta[w] + g(t)w.$

Here, $\mathbf{N}_\beta[w]$ is an arbitrary homogeneous nonlinear differential operator of degree β with respect to w (i.e., $\mathbf{N}_\beta[\alpha w] = \alpha^\beta \mathbf{N}_\beta[w]$, $\alpha = \text{const}$) that depends on the space variables x, y, z only (and is independent of t).

Using the transformation

$$w(x, y, z, t) = G(t)U(x, y, z, \tau), \quad \tau = \int f(t)G^{\beta-1}(t)\,dt, \quad G(t) = \exp\left[\int g(t)\,dt\right],$$

one arrives at the simpler equation

$$\frac{\partial U}{\partial \tau} = \mathbf{N}_\beta[U], \tag{1}$$

which has a multiplicative separable solution $U = \varphi(\tau)\Theta(x, y, z)$.

Remark 1. The order of the nonlinear operator $\mathbf{N}_\beta$ (with respect to the derivatives) and the number of the space variables in the original equation can be any. The coefficients of $\mathbf{N}_\beta$ can be dependent on the space variables.

Remark 2. If $\mathbf{N}_\beta$ is independent explicitly of the space variables, then equation (1) has also a traveling-wave solution, $U = U(\xi)$, where $\xi = k_1 x + k_2 y + k_3 z + \lambda\tau$. Below are two examples of such operators:

$$\mathbf{N}_\beta[w] = a\,\mathrm{div}(w^{\beta-1}\nabla w) + b|\nabla w|^\beta + cw^\beta,$$
$$\mathbf{N}_\beta[w] = a\,\mathrm{div}(|\nabla w|^{\beta-1}\nabla w) + bw^\mu|\nabla w|^{\beta-\mu},$$

where a, b, c, and μ are some constants.

8. $\displaystyle \frac{\partial w}{\partial t} + (\vec{v}\cdot\nabla)w = \Delta w + f(w)|\nabla w|^2.$

This is a special case of equation 2.5.5.8 with $n = 3$.

9. $\displaystyle \frac{\partial w}{\partial t} + (\vec{v}\cdot\nabla)w = a\Delta w + a|\nabla w|^2 + f(\vec{x}, t).$

This is a special case of equation 2.5.5.9 with $n = 3$.

10. $\displaystyle \frac{\partial \vec{w}}{\partial t} + (\vec{w}\cdot\nabla)\vec{w} = a\Delta\vec{w}.$

Vector Burgers equation; $\vec{w} = \{w_1, w_2, w_3\}$ and $w_n = w_n(x_1, x_2, x_3)$. The Hamilton operator ∇ and Laplace operator Δ can be represented in any orthogonal system of coordinates.

Solution:

$$\vec{w} = -\frac{2a}{\theta}\nabla\theta,$$

where θ is a solution of the linear heat equation

$$\frac{\partial\theta}{\partial t} = a\Delta\theta.$$

For solutions of this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

⊙ *Reference*: S. Nerney, E. J. Schmahl, and Z. E. Musielak (1996).

2.5.5. Equations with n Space Variables

► *Notation:* $\mathbf{x} = (x_1, \ldots, x_n)$, $\displaystyle \Delta w = \sum_{k=1}^{n}\frac{\partial^2 w}{\partial x_k^2}$, $\displaystyle |\nabla w|^2 = \sum_{k=1}^{n}\Big(\frac{\partial w}{\partial x_k}\Big)^2$, $\displaystyle (\vec{v}\cdot\nabla)w = \sum_{k=1}^{n} v_k\frac{\partial w}{\partial x_k}$, $\displaystyle \nabla\cdot\vec{v} = \sum_{k=1}^{n}\frac{\partial v_k}{\partial x_k}$.

1. $\displaystyle \frac{\partial w}{\partial t} = \Delta w + f(w)|\nabla w|^2.$

The substitution $\displaystyle U = \int F(w)\,dw$, where $\displaystyle F(w) = \exp\Big[\int f(w)\,dw\Big]$, leads to the linear heat equation

$$\frac{\partial U}{\partial t} = \Delta U.$$

For solutions of this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

2. $\dfrac{\partial w}{\partial t} = f(t)\Delta w + g(t)w \ln w + h(t)w.$

There is a functional separable solution of the form

$$w(x_1,\dots,x_n,t) = \exp\left[\sum_{i,j=1}^{n} \varphi_{ij}(t)x_ix_j + \sum_{i=1}^{n} \psi_i(t)x_i + \chi(t)\right].$$

Example 1. Let $f(t) = 1$, $g(t) = 1$, and $h(t) = 0$. Solutions in the radially symmetric case:

$$w = \exp\left(\frac{n}{2} - \frac{1}{4}r^2 + Be^t\right), \qquad r = \sqrt{x_1^2 + \cdots + x_n^2},$$
$$w = \exp\left\{-\frac{1}{4}r^2(1 - Ae^{-t})^{-1} + e^t\left[B - \frac{n}{2A}\ln(1 - Ae^{-t})\right]\right\},$$

where A and B are arbitrary constants, $A < 1$. The first solution is a special case of the second solution as $A \to 0$.

Example 2. Let $f(t) = 1$, $g(t) = -1$, and $h(t) = 0$. Solution in radially symmetric case:

$$w = \exp\left\{-\frac{1}{4}r^2(Ae^t - 1)^{-1} + e^{-t}\left[B - \frac{n}{2A}\ln(Ae^t - 1)\right]\right\},$$

where A and B are arbitrary constants, $A > 1$.

⊙ *Reference*: A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov (1995).

3. $\dfrac{\partial w}{\partial t} = f_1(t)\Delta w + f_2(t)|\nabla w|^2 + f_3(t)w + \sum_{i,j=1}^{n} g_{ij}(t)x_ix_j + \sum_{i=1}^{n} h_i(t)x_i + p(t).$

There are exact solutions of the following forms:

$$w(x_1,\dots,x_n,t) = \sum_{i,j=1}^{n} \varphi_{ij}(t)x_ix_j + \sum_{i=1}^{n} \psi_i(t)x_i + \chi(t).$$

4. $\dfrac{\partial w}{\partial t} = f_1(t)w\Delta w + f_2(t)|\nabla w|^2 + f_3(t)w + \sum_{i,j=1}^{n} g_{ij}(t)x_ix_j + \sum_{i=1}^{n} h_i(t)x_i + p(t).$

There are exact solutions of the following forms:

$$w(x_1,\dots,x_n,t) = \sum_{i,j=1}^{n} \varphi_{ij}(t)x_ix_j + \sum_{i=1}^{n} \psi_i(t)x_i + \chi(t).$$

5. $\dfrac{\partial w}{\partial t} = a\nabla\cdot(w^m\nabla w).$

For $m > 1$, this equation describes the flow of a polytropic gas through a homogeneous porous medium (w is the gas density).

1°. In the radially symmetric case the equation is written as

$$\frac{\partial w}{\partial t} = \frac{a}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}w^m\frac{\partial w}{\partial r}\right), \qquad r = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Its exact solutions are given in 1.1.15.9, where n should be substituted by $n-1$.

2°. Solution of the instantaneous source type for $a = 1$:

$$w = \begin{cases} t^{-n/(nm+2)}\left[\dfrac{m}{2(nm+2)}\left(K_0^2 - \dfrac{r^2}{t^{2/(nm+2)}}\right)\right]^{1/m} & \text{if } r \le K_0 t^{1/(nm+2)}, \\ 0 & \text{if } r > K_0 t^{1/(nm+2)}, \end{cases}$$

where

$$K_0 = \left\{ \pi^{-n/2} \left[\frac{2(nm+2)}{m} \right]^{1/m} \frac{\Gamma(n/m+1+1/m)}{\Gamma(1/m+1)} E_0 \right\}^{m/(nm+2)}, \qquad E_0 = \text{const}.$$

This is the solution of the initial-value problem with initial function

$$w(\mathbf{x}, 0) = E_0 \delta(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n,$$

satisfying the condition of constant energy:

$$\int_{\mathbb{R}^n} w(\mathbf{x}, t)\, d\mathbf{x} = E_0 = \text{const} > 0.$$

3°. See also equation 2.5.5.6, in which m should be substituted by $m+1$.

⊙ *References*: Ya. B. Zel'dovich and A. S. Kompaneets (1950), G. I. Barenblatt (1952).

6. $\boldsymbol{\dfrac{\partial w}{\partial t} = \Delta(w^m).}$

For $m > 1$, this equation describes the flow of a polytropic gas through a homogeneous porous medium (w is the gas density). It can be rewritten in the form of equation 2.5.5.5:

$$\frac{\partial w}{\partial t} = m\nabla \cdot (w^{m-1} \nabla w).$$

1°. Solution for $m > 1$:

$$w = \left(\prod_{k=1}^{n} \varphi_k \right)^{-1} \left(A - \sum_{k=1}^{n} \frac{x_k^2}{\varphi_k^2} \right)^{\frac{1}{m-1}},$$

where A is an arbitrary constant ($A > 0$), and the functions $\varphi_k = \varphi_k(t)$ are determined by the system of ordinary differential equations

$$\varphi_1 \frac{d\varphi_1}{dt} = \cdots = \varphi_n \frac{d\varphi_n}{dt} = \frac{2m}{m-1} \left(\prod_{k=1}^{n} \varphi_k \right)^{1-m}. \tag{1}$$

System (1) admits $n-1$ first integrals:

$$\varphi_j^2 = \varphi_n^2 + C_j, \qquad j = 1, 2, \ldots, n-1, \tag{2}$$

where the C_j are arbitrary constants.

The function $\varphi_n = \varphi_n(t)$ is defined in implicit form by (the C_j are assumed to be positive)

$$\int_B^{\varphi_n} z^m \left[\prod_{j=1}^{n-1} (z^2 + C_j) \right]^{\frac{m-1}{2}} dz = \frac{2mt}{m-1},$$

where B is an arbitrary constant, and the remaining $\varphi_j(t)$ are determined by the positive roots of the quadratic equations (2).

⊙ *References*: S. S. Titov and V. A. Ustinov (1985), J. R. King (1993), V. V. Pukhnachov (1995).

2°. Solution for $0 < m < 1$:

$$w = \left(\prod_{k=1}^{n} \varphi_k \right)^{-1} \left(A + \sum_{k=1}^{n} \frac{x_k^2}{\varphi_k^2} \right)^{\frac{1}{m-1}},$$

where A is an arbitrary constant, and the functions $\varphi_k = \varphi_k(t)$ are determined by the system of ordinary differential equations (1).

⊙ *References*: J. R. King (1993), V. V. Pukhnachov (1995).

3°. There is an exact solution of the form

$$w = \left[\sum_{i,j=1}^{n} a_{ij}(t) x_i x_j + \sum_{i=1}^{n} b_i(t) x_i + c(t) \right]^{\frac{1}{m-1}}.$$

⊙ *Reference*: G. A. Rudykh and E. I. Semenov (2000); other exact solutions are also given there.

7. $\dfrac{\partial w}{\partial t} = [aw + f(t)]\Delta w + bw^2 + g(t)w + h(t).$

Here, $f(t)$, $g(t)$, and $h(t)$ are arbitrary functions; a and b are arbitrary parameters ($a \neq 0$).

There is a generalized separable solution of the form

$$w(x_1, \dots, x_n, t) = \varphi(t) + \psi(t)\Theta(x_1, \dots, x_n),$$

where the functions $\varphi(t)$, $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi_t' = b\varphi^2 + g(t)\varphi + h(t), \tag{1}$$

$$\psi_t' = \bigl[b\varphi - \beta f(t) + g(t)\bigr]\psi, \qquad \beta = b/a, \tag{2}$$

and the function $\Theta(x_1, \dots, x_n)$ is a solution of the Helmholtz equation

$$\Delta\Theta + \beta\Theta = 0. \tag{3}$$

Equation (1) is independent of ψ and represents a Riccati equation for φ. A large number of exact solutions to equation (1) for various $g(t)$ and $h(t)$ can be found in Polyanin and Zaitsev (2003). On solving (1) and substituting the resulting $\varphi = \varphi(t)$ into (2), one obtains a linear equation for $\psi = \psi(t)$, which is easy to integrate.

For solutions of the linear stationary equation (3), see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

8. $\dfrac{\partial w}{\partial t} + (\vec{v} \cdot \nabla)w = \Delta w + f(w)|\nabla w|^2.$

Here, $\vec{v}$ is a prescribed vector function dependent on the space coordinates and time (but independent of w).

The substitution

$$\Theta = \int F(w)\,dw, \qquad \text{where} \quad F(w) = \exp\biggl[\int f(w)\,dw\biggr],$$

leads a linear convective heat and mass transfer equation for $\Theta = \Theta(x_1, \dots, x_n, t)$:

$$\frac{\partial\Theta}{\partial t} + (\vec{v} \cdot \nabla)\Theta = \Delta\Theta.$$

9. $\dfrac{\partial w}{\partial t} + (\vec{v} \cdot \nabla)w = a\Delta w + a|\nabla w|^2 + f(\mathbf{x}, t).$

Here, $\vec{v}$ is a prescribed vector function dependent on the space coordinates and time (but independent of w).

The substitution $\Theta = e^w$ leads to the linear equation

$$\frac{\partial\Theta}{\partial t} + (\vec{v} \cdot \nabla)\Theta = a\Delta\Theta + f(\mathbf{x}, t)\Theta.$$

10. $\dfrac{\partial w}{\partial t} = \alpha\nabla \cdot \bigl(w^m \nabla w\bigr) + f(t)w.$

1°. Multiplicative separable solution:

$$w(\mathbf{x}, t) = \exp\biggl[\int f(t)\,dt\biggr]\bigl[\Theta(\mathbf{x})\bigr]^{\frac{1}{m+1}}, \tag{1}$$

where the function $\Theta(\mathbf{x})$ satisfies the Laplace equation

$$\Delta\Theta = 0.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

2°. Multiplicative separable solution:

$$w(\mathbf{x},t)=\varphi(t)\big[\Theta(\mathbf{x})\big]^{\frac{1}{m+1}}, \tag{2}$$

where the function $\varphi(t)$ is determined by the Bernoulli equation

$$\varphi'_t-f(t)\varphi+A\alpha\varphi^{m+1}=0. \tag{3}$$

Here, A is an arbitrary constant and the function $\Theta(\mathbf{x})$ satisfies the stationary equation

$$\Delta\Theta+A(m+1)\Theta^{\frac{1}{m+1}}=0.$$

The general solution of equation (3) is given by

$$\varphi(t)=\exp\big[F(t)\big]\left\{A\alpha m\int\exp\big[mF(t)\big]\,dt+B\right\}^{-1/m},\qquad F(t)=\int f(t)\,dt,$$

where B is an arbitrary constant.

3°. The transformation

$$w(\mathbf{x},t)=F(t)U(\mathbf{x},\tau),\qquad \tau=\int F^m(t)\,dt,\qquad F(t)=\exp\left[\int f(t)\,dt\right],$$

leads to a simpler equation: $\dfrac{\partial U}{\partial\tau}=\alpha\nabla\cdot(U^m\nabla w)$.

Example. For $\alpha=1$, $f(t)=-\beta<0$, we have

$$\frac{\partial w}{\partial t}=\nabla\cdot(w^m\nabla w)-\beta w.$$

Solution in the radially symmetric case:

$$w=\begin{cases} e^{-\beta t/m}[g(t)]^{-n/(nm+2)}\left[\dfrac{m}{2(nm+2)}\left(\eta_0^2-\dfrac{r^2}{[g(t)]^{2/(nm+2)}}\right)\right]^{1/m} & \text{if } r\le r_*(t),\\ 0 & \text{if } r>r_*(t),\end{cases}$$

where

$$r=\sqrt{x_1^2+\cdots+x_n^2},\quad g(t)=1+\frac{1-e^{-\beta mt}}{\beta m},\quad r_*(t)=\eta_0\left(1+\frac{1-e^{-\beta mt}}{\beta m}\right)^{1/(nm+2)}.$$

The diameter of the support of this solution is monotonically increasing but is bounded now by the constant

$$L=\lim_{t\to\infty}|r_*(t)|=\eta_0\left(1+\frac{1}{\beta m}\right)^{1/(nm+2)}<\infty.$$

The perturbation is localized in a ball of radius L.

⊙ *References*: L. K. Martinson and K. B. Pavlov (1972), A. D. Polyanin and V. F. Zaitsev (2002).

11. $\dfrac{\partial w}{\partial t}=\nabla\cdot(w^m\nabla w)-w^{1-m}$.

Solution in the radially symmetric case for $0<m<1$:

$$w=\begin{cases}\left[\dfrac{2(nm+2)}{m}t\right]^{-1/m}V^{1/m} & \text{if } V\ge0,\\ 0 & \text{if } V<0,\end{cases}$$

where

$$V=At^{2/(nm+2)}-\frac{(nm+2)^2}{nm+1}t^2-r^2,\qquad r=\sqrt{x_1^2+\cdots+x_n^2}\,;$$

A is an arbitrary constant ($A>0$). The solution has a compact support. The diameter of the support increases with t on the time interval $(0,t_*)$, where

$$t_*=\left[\frac{A(nm+1)}{(nm+2)^3}\right]^{\frac{nm+2}{nm+1}}$$

and decreases on the interval (t_*,T_0), where

$$T_0=\left[\frac{A(nm+1)}{(nm+2)^2}\right]^{\frac{nm+2}{2(nm+1)}}.$$

The solution vanishes at $t=T_0$.

⊙ *References*: R. Kersner (1978), L. K. Martinson (1979).

12. $\dfrac{\partial w}{\partial t} = a\nabla \cdot (e^{\lambda w}\nabla w) + be^{\lambda w} + f(t) + g(t)e^{-\lambda w}.$

Functional separable solution:

$$w(\mathbf{x},t) = \frac{1}{\lambda}\ln\bigl[\varphi(t) + \psi(t)\Theta(\mathbf{x})\bigr], \qquad \psi(t) = \exp\biggl\{\lambda\int\bigl[b\varphi(t) + f(t)\bigr]\,dt\biggr\},$$

where the function $\varphi(t)$ is determined by the Riccati equation

$$\varphi'_t = b\lambda\varphi^2 + \lambda f(t)\varphi + \lambda g(t), \tag{1}$$

and the function $\Theta = \Theta(\mathbf{x})$ is a solution of the Helmholtz equation

$$a\Delta\Theta + b\lambda\Theta = 0. \tag{2}$$

For details about the Riccati equation (1), see Kamke (1977) and Polyanin and Zaitsev (2003). For solutions of the linear equation (2), see Tikhonov and Samarskii (1990) and Polyanin (2002).

13. $\dfrac{\partial w}{\partial t} = \nabla \cdot [f(w)\nabla w] + \dfrac{a}{f(w)} + b.$

Solution in implicit form:

$$\int f(w)\,dw = at + U(\mathbf{x}),$$

where the function $U(\mathbf{x})$ is determined by the Poisson equation

$$\Delta U + b = 0.$$

For details about this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

⊙ *Reference*: V. A. Galaktionov (1994).

14. $\dfrac{\partial w}{\partial t} = \nabla \cdot [f(w)\nabla w] + \dfrac{g(t)}{f(w)} + h(\mathbf{x}).$

Solution in implicit form:

$$\int f(w)\,dw = \int g(t)\,dt + U(\mathbf{x}),$$

where the function $U(\mathbf{x})$ is determined by the Poisson equation

$$\Delta U + h(\mathbf{x}) = 0.$$

15. $\dfrac{\partial w}{\partial t} = \Delta f(w) + \dfrac{af(w) + b}{f'(w)} + c[af(w) + b].$

Solution in implicit form:

$$f(w) = e^{at}U(\mathbf{x}) - \frac{b}{a},$$

where the function $U(\mathbf{x})$ is determined by the Helmholtz equation

$$\Delta U + acU = 0.$$

For details about this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

⊙ *Reference*: V. A. Galaktionov (1994).

16. $\dfrac{\partial w}{\partial t} = \mathbf{L}\,[f(w)] + \dfrac{g(t)}{f'(w)} + h(\mathbf{x}).$

Here, $\mathbf{L}$ is an arbitrary linear differential operator of the second (or any) order with respect to the space variables with coefficients independent of t; the operator satisfies the condition $\mathbf{L}\,[\text{const}] = 0$.

Solution in implicit form:

$$f(w) = \int g(t)\,dt + U(\mathbf{x}),$$

where the function $U(\mathbf{x})$ is determined by the linear equation

$$\mathbf{L}\,[U] + h(\mathbf{x}) = 0.$$

17. $\dfrac{\partial w}{\partial t} = \mathbf{L}\,[f(w)] + \dfrac{af(w) + b}{f'(w)} + g(\mathbf{x})[af(w) + b].$

Here, $\mathbf{L}$ is an arbitrary linear differential operator of the second (or any) order with respect to the space variables with coefficients independent of t; the operator satisfies the condition $\mathbf{L}\,[\text{const}] = 0$.

Solution in implicit form:

$$f(w) = e^{at}U(\mathbf{x}) - \frac{b}{a},$$

where the function $U(\mathbf{x})$ is determined by the linear equation

$$\mathbf{L}\,[U] + ag(\mathbf{x})U = 0.$$

18. $\dfrac{\partial w}{\partial t} = \mathbf{L}\,[f(x, w)] + \dfrac{g(t)}{f_w(x, w)} + h(\mathbf{x}).$

Here, $\mathbf{L}$ is an arbitrary linear differential operator of the second (or any) order with respect to the space variables with coefficients independent of t; the operator satisfies the condition $\mathbf{L}\,[\text{const}] = 0$; and f_w stands for the partial derivative of f with respect to w.

Solution in implicit form:

$$f(x, w) = \int g(t)\,dt + U(\mathbf{x}),$$

where the function $U(\mathbf{x})$ is determined by the linear equation

$$\mathbf{L}\,[U] + h(\mathbf{x}) = 0.$$

19. $\dfrac{\partial w}{\partial t} = \mathbf{L}\,[f(x, w)] + \dfrac{af(x, w) + b}{f_w(x, w)} + g(\mathbf{x})[af(x, w) + b].$

Here, $\mathbf{L}$ is an arbitrary linear differential operator of the second (or any) order with respect to the space variables with coefficients independent of t; the operator satisfies the condition $\mathbf{L}\,[\text{const}] = 0$.

Solution in implicit form:

$$f(x, w) = e^{at}U(\mathbf{x}) - \frac{b}{a},$$

where the function $U(\mathbf{x})$ is determined by the linear equation

$$\mathbf{L}\,[U] + ag(\mathbf{x})U = 0.$$

20. $\dfrac{\partial w}{\partial t} = a\nabla \cdot (|\nabla w|\nabla w) + bw^2 + f(t)w + g(t).$

Generalized separable solution:

$$w(\mathbf{x}, t) = \varphi(t) + \exp\left\{\int \bigl[2b\varphi(t) + f(t)\bigr]\,dt\right\}\Theta(\mathbf{x}),$$

where the function $\varphi(t)$ is determined by the Riccati equation

$$\varphi'_t = b\varphi^2 + f(t)\varphi + g(t),$$

and the function $\Theta = \Theta(\mathbf{x})$ is a solution of the stationary equation

$$a\nabla \cdot (|\nabla\Theta|\nabla\Theta) + b\Theta^2 = 0.$$

2.6. Nonlinear Schrödinger Equations

2.6.1. Two Dimensional Equations

1. $\displaystyle i\frac{\partial w}{\partial t}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+A|w|^2 w=0.$

Two-dimensional Schrödinger equation with a cubic nonlinearity. This is a special case of equation 2.6.1.3 with $f(u)=Au^2$.

1°. Suppose $w(x,y,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$
\begin{aligned}
w_1 &= \pm C_1 w(\pm C_1 x + C_2, \pm C_1 y + C_3, C_1^2 t + C_4),\\
w_2 &= e^{-i[\lambda_1 x + \lambda_2 y + (\lambda_1^2+\lambda_2^2)t + C_5]} w(x + 2\lambda_1 t, y + 2\lambda_2 t, t),\\
w_3 &= w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),
\end{aligned}
$$

where $C_1, \dots, C_5$, λ_1, λ_2, and β are arbitrary real constants, are also solutions of the equation. The plus or minus signs in the expression for w_1 are chosen arbitrarily.

2°. Solutions:

$$
\begin{aligned}
w(x,y,t) &= C_1 \exp\bigl\{i\,[C_2 x + C_3 y + (AC_1^2 - C_2^2 - C_3^2)t + C_4]\bigr\},\\
w(x,y,t) &= \frac{C_1}{t}\exp\left[i\frac{(x+C_2)^2 + (y+C_3)^2 - 4AC_1^2}{4t} + iC_4\right],
\end{aligned}
$$

where $C_1, \dots, C_4$ are arbitrary real constants.

3°. "Two-dimensional" solution:

$$
w(x,y,t) = e^{i(C_1 t + C_2)} u(x,y),
$$

where C_1 and C_2 are arbitrary real constants, and the function $u=u(x,y)$ is determined by the stationary equation

$$
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + Au^3 - C_1 u = 0.
$$

4°. Solution:

$$
w(x,y,t) = (f_1 x + f_2 y + f_3)\exp\bigl[i(g_1 x^2 + g_2 xy + g_3 y^2 + h_1 x + h_2 y + h_3)\bigr],
$$

where the functions $f_k = f_k(t)$, $g_k = g_k(t)$, and $h_k = h_k(t)$ are determined by the autonomous system of ordinary differential equations

$$
\begin{aligned}
f_1' + 2(3g_1 + g_3)f_1 + 2f_2 g_2 &= 0,\\
f_2' + 2(g_1 + 3g_3)f_2 + 2f_1 g_2 &= 0,\\
f_3' + 2(g_1 + g_3)f_3 + 2(f_1 h_1 + f_2 h_2) &= 0,\\
g_1' + 4g_1^2 + g_2^2 - Af_1^2 &= 0,\\
g_2' + 4(g_1 + g_3)g_2 - 2Af_1 f_2 &= 0,\\
g_3' + g_2^2 + 4g_3^2 - Af_2^2 &= 0,\\
h_1' + 4g_1 h_1 + 2g_2 h_2 - 2Af_1 f_3 &= 0,\\
h_2' + 2g_2 h_1 + 4g_3 h_2 - 2Af_2 f_3 &= 0,\\
h_3' + h_1^2 + h_2^2 - Af_3^2 &= 0.
\end{aligned}
$$

The prime denotes a derivative with respect to t.

5°. "Two-dimensional" solution:

$$w(x,y,t) = U(\xi_1,\xi_2)e^{i(k_1x+k_2y+at+b)}, \quad \xi_1 = x - 2k_1t, \quad \xi_2 = y - 2k_2t,$$

where k_1, k_2, a, and b are arbitrary constants, and the function $U = U(\xi_1,\xi_2)$ is determined by a differential equation of the form 5.4.1.1:

$$\frac{\partial^2 U}{\partial \xi_1^2} + \frac{\partial^2 U}{\partial \xi_2^2} + A|U|^2U - (k_1^2 + k_2^2 + a)U = 0.$$

6°. "Two-dimensional" solution:

$$w(x,y,t) = \Phi(z_1,z_2)\exp\bigl[i(k_1xt + k_2yt - \tfrac{2}{3}k_1^2t^3 - \tfrac{2}{3}k_2^2t^3 + at + b)\bigr], \quad z_1 = x - k_1t^2, \quad z_2 = y - k_2t^2,$$

where k_1, k_2, a, and b are arbitrary constants, and the function $\Phi = \Phi(z_1,z_2)$ is determined by a differential equation of the form 5.4.1.1:

$$\frac{\partial^2 \Phi}{\partial z_1^2} + \frac{\partial^2 \Phi}{\partial z_2^2} + A|\Phi|^2\Phi - (k_1z_1 + k_2z_2 + a)\Phi = 0.$$

7°. "Two-dimensional" solution:

$$w(x,y,t) = \frac{1}{\sqrt{C_1t + C_2}}u(\xi,\eta), \quad \xi = \frac{x + C_3}{\sqrt{C_1t + C_2}}, \quad \eta = \frac{y + C_4}{\sqrt{C_1t + C_2}},$$

where $C_1, \ldots, C_4$ are arbitrary constants, and the function $u = u(\xi,\eta)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} - \frac{1}{2}iC_1\left(\xi\frac{\partial u}{\partial \xi} + \eta\frac{\partial u}{\partial \eta} + u\right) + A|u|^2u = 0.$$

2. $\displaystyle i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + A|w|^{2n}w = 0.$

Two-dimensional Schrödinger equation with a power-law nonlinearity; A and n are real numbers. This is a special case of equation 2.6.1.3 with $f(u) = Au^{2n}$.

1°. Suppose $w(x,y,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$\begin{aligned} w_1 &= \pm C_1w(\pm C_1^nx + C_2, \pm C_1^ny + C_3, C_1^{2n}t + C_4),\\ w_2 &= e^{-i[\lambda_1x+\lambda_2y+(\lambda_1^2+\lambda_2^2)t+C_5]}w(x + 2\lambda_1t, y + 2\lambda_2t, t),\\ w_3 &= w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t), \end{aligned}$$

where $C_1, \ldots, C_5$, β, λ_1, and λ_2 are arbitrary real constants, are also solutions of the equation. The plus or minus signs in the expression for w_1 are chosen arbitrarily.

2°. Solutions:

$$\begin{aligned} w(x,y,t) &= C_1\exp\bigl\{i\,[C_2x + C_3y + (A|C_1|^{2n} - C_2^2 - C_3^2)t + C_4]\bigr\},\\ w(x,y,t) &= \frac{C_1}{t}\exp\left[i\frac{(x + C_2)^2 + (y + C_3)^2}{4t} + i\frac{AC_1^{2n}}{1 - 2n}t^{1-2n} + iC_4\right], \end{aligned}$$

where $C_1, \ldots, C_4$ are arbitrary real constants.

3°. For other exact solutions, see equation 2.6.1.3 with $f(w) = Aw^{2n}$.

3. $i\dfrac{\partial w}{\partial t} + \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + f(|w|)w = 0.$

Two-dimensional nonlinear Schrödinger equation of general form.

1°. Suppose $w(x, y, t)$ is a solution of the Schrödinger equation in question. Then the functions

$$w_1 = e^{-i[\lambda_1 x + \lambda_2 y + (\lambda_1^2 + \lambda_2^2)t + A]} w(x + 2\lambda_1 t + C_1, y + 2\lambda_2 t + C_2, t + C_3),$$
$$w_2 = w(x \cos\beta - y \sin\beta, x \sin\beta + y \cos\beta, t),$$

where A, C_1, C_2, C_3, λ_1, λ_2, and β are arbitrary real constants, are also solutions of the equation.

2°. Traveling-wave solution:

$$w(x, y, t) = C_1 \exp\bigl[i\,(C_2 x + C_3 y + \lambda t + C_4)\bigr], \qquad \lambda = f(|C_1|) - C_2^2 - C_3^2,$$

where $C_1, \ldots, C_4$ are arbitrary real constants.

3°. Exact solutions depending only on the radial variable $r = \sqrt{x^2 + y^2}$ and time t are determined by the equation

$$i\frac{\partial w}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + f(|w|)w = 0,$$

which is a special case of equation 1.7.5.2 with $n = 1$.

4°. "Two-dimensional" solution:

$$w(x, y, t) = e^{i(At+B)} u(x, y),$$

where A and B are arbitrary real constants, and the function $u = u(x, y)$ is determined by a stationary equation of the form 5.4.1.1:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(|u|)u - Au = 0.$$

5°. "Two-dimensional" solution:

$$w(x, y, t) = U(\xi, \eta) e^{i(A_1 x + A_2 y + Bt + C)}, \qquad \xi = x - 2A_1 t, \qquad \eta = y - 2A_2 t,$$

where A_1, A_2, B, and C are arbitrary constants, and the function $U = U(\xi, \eta)$ is determined by a differential equation of the form 5.4.1.1:

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + f(|U|)U - (A_1^2 + A_2^2 + B)U = 0.$$

6°. "Two-dimensional" solution:

$$w(x, y, t) = \Phi(z_1, z_2) \exp\bigl[i(k_1 x t + k_2 y t - \tfrac{2}{3}k_1^2 t^3 - \tfrac{2}{3}k_2^2 t^3 + at + b)\bigr], \quad z_1 = x - k_1 t^2, \quad z_2 = y - k_2 t^2,$$

where k_1, k_2, a, and b are arbitrary constants, and the function $\Phi = \Phi(z_1, z_2)$ is determined by the differential equation

$$\frac{\partial^2 \Phi}{\partial z_1^2} + \frac{\partial^2 \Phi}{\partial z_2^2} + f(|\Phi|)\Phi - (k_1 z_1 + k_2 z_2 + a)\Phi = 0.$$

7°. There is a "two-dimensional" solution of the form

$$w(x, y, t) = U(z_1, z_2), \qquad z_1 = a_1 x + b_1 y + c_1 t, \qquad z_2 = a_2 x + b_2 y + c_2 t.$$

8°. For group classification of the original equation, see Gagnon and Winternitz (1988) and Ibragimov (1995).

2.6.2. Three and n Dimensional Equations

1. $i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}+A|w|^2 w=0.$

Three-dimensional Schrödinger equation with a cubic nonlinearity. This is a special case of equation 2.6.2.2 with $f(u)=Au^2$.

1°. Suppose $w(x,y,t)$ is a solution of the Schrödinger equation in question. Then the functions

$$w_1=\pm C_1 w(\pm C_1 x+C_2,\pm C_1 y+C_3,\pm C_1 z+C_4,C_1^2 t+C_5),$$
$$w_2=e^{-i[\lambda_1 x+\lambda_2 y+\lambda_3 z+(\lambda_1^2+\lambda_2^2+\lambda_3^2)t+C_6]}w(x+2\lambda_1 t,y+2\lambda_2 t,z+2\lambda_3 t,t),$$

where $C_1,\ldots,C_6$, λ_1, λ_2, and λ_3 are arbitrary real constants, are also solutions of the equation. The plus or minus signs in the expression for w_1 are chosen arbitrarily.

2°. There is an exact solution of the form

$$w=(f_1x+f_2y+f_3z+f_4)\exp\bigl[i(g_1x^2+g_2y^2+g_3z^2+g_4xy+g_5xz+g_6yz+h_1x+h_2y+h_3z+h_4)\bigr],$$

where $f_k=f_k(t)$, $g_k=g_k(t)$, and $h_k=h_k(t)$.

3°. Solution:

$$w(x,y,z,t)=U(\xi_1,\xi_2,\xi_3)e^{i(k_1x+k_2y+k_3z+at+b)},\quad \xi_1=x-2k_1t,\quad \xi_2=y-2k_2t,\quad \xi_3=z-2k_3t,$$

where k_1, k_2, k_3, a, and b are arbitrary constants, and the function $U=U(\xi_1,\xi_2,\xi_3)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial \xi_1^2}+\frac{\partial^2 U}{\partial \xi_2^2}+\frac{\partial^2 U}{\partial \xi_3^2}+A|U|^2U-(k_1^2+k_2^2+k_3^2+a)U=0.$$

4°. "Three-dimensional" solution:

$$w(x,y,z,t)=\frac{1}{\sqrt{C_1t+C_2}}u(\xi,\eta,\zeta),\quad \xi=\frac{x+C_3}{\sqrt{C_1t+C_2}},\quad \eta=\frac{y+C_4}{\sqrt{C_1t+C_2}},\quad \zeta=\frac{z+C_5}{\sqrt{C_1t+C_2}},$$

where C_1, …, C_5 are arbitrary constants, and the function $u=u(\xi,\eta,\zeta)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}+\frac{\partial^2 u}{\partial \zeta^2}-\frac{1}{2}iC_1\left(\xi\frac{\partial u}{\partial \xi}+\eta\frac{\partial u}{\partial \eta}+\zeta\frac{\partial u}{\partial \zeta}+u\right)+A|u|^2u=0.$$

⊙ *References*: L. Gagnon and P. Winternitz (1988, 1989), N. H. Ibragimov (1995), A. M. Vinogradov and I. S. Krasil'shchik (1997).

2. $i\dfrac{\partial w}{\partial t}+\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}+f(|w|)w=0.$

Three-dimensional nonlinear Schrödinger equation of general form. It admits translations in any of the independent variables.

1°. Suppose $w(x,y,z,t)$ is a solution of the Schrödinger equation in question. Then the function

$$w_1=e^{-i[\lambda_1 x+\lambda_2 y+\lambda_3 z+(\lambda_1^2+\lambda_2^2+\lambda_3^2)t+A]}w(x+2\lambda_1 t+C_1,y+2\lambda_2 t+C_2,z+2\lambda_3 t+C_3,t+C_4),$$

where A, $C_1,\ldots,C_4$, λ_1, λ_2, and λ_3 are arbitrary real constants, is also a solution of the equation.

2°. Exact solutions depending only on the radial variable $r=\sqrt{x^2+y^2+z^2}$ and time t are determined by the equation

$$i\frac{\partial w}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right)+f(|w|)w=0,$$

which is a special case of equation 1.7.5.2 with $n=2$.

3°. "Three-dimensional" solution:

$$w(x, y, z, t) = e^{i(At+B)}u(x, y, z),$$

where A and B are arbitrary real constants, and the function $u = u(x, y, z)$ is determined by the stationary equation

$$\Delta u + f(|u|)u - Au = 0.$$

4°. Axisymmetric solutions in cylindrical and spherical coordinates are determined by equations where the Laplace operator has the form

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial w}{\partial \rho}\right) + \frac{\partial^2 w}{\partial z^2}, \qquad \rho = \sqrt{x^2 + y^2};$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial w}{\partial \theta}\right), \quad r = \sqrt{x^2 + y^2 + z^2},$$

respectively.

5°. "Three-dimensional" solution:

$$w = U(\xi, \eta, t), \qquad \xi = y + \frac{x}{C}, \qquad \eta = (C^2 - 1)x^2 - 2Cxy + C^2z^2,$$

where C is an arbitrary constant ($C \neq 0$), and the function $U = U(\xi, \eta, t)$ is determined by the differential equation

$$i\frac{\partial U}{\partial t} + \left(1 + \frac{1}{C^2}\right)\frac{\partial^2 U}{\partial \xi^2} - 4\xi\frac{\partial^2 U}{\partial \xi \partial \eta} + 4C^2(\xi^2 + \eta)\frac{\partial^2 U}{\partial \eta^2} + 2(2C^2 - 1)\frac{\partial U}{\partial \eta} + f(|U|)U = 0.$$

6°. "Three-dimensional" solution:

$$w = V(\xi, \eta, t), \qquad \xi = Ax + By + Cz, \qquad \eta = \sqrt{(Bx - Ay)^2 + (Cy - Bz)^2 + (Az - Cx)^2},$$

where A, B, and C are arbitrary constants and the function $V = V(\xi, \eta, t)$ is determined by the equation

$$i\frac{\partial V}{\partial t} + (A^2 + B^2 + C^2)\left(\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial V}{\partial \eta}\right) + f(|V|)V = 0.$$

⊙ *References*: L. Gagnon and P. Winternitz (1988, 1989), N. H. Ibragimov (1995).

3. $\boldsymbol{i\dfrac{\partial w}{\partial t} = \Delta w + |w|^2 w.}$

This is an n-*dimensional Schrödinger equation with a cubic nonlinearity*.

Conservation laws:

$$\left(|w|^2\right)_t + i\nabla\cdot\left(\bar{w}\nabla w - w\nabla\bar{w}\right)_x = 0,$$

$$\left(|\nabla w|^2 - \tfrac{1}{2}|w|^4\right)_t + i\nabla\cdot\left[(\Delta w + |w|^2 w)\nabla\bar{w} - (\Delta\bar{w} + |w|^2\bar{w})\nabla w\right]_x = 0.$$

The bar over a symbol denotes the complex conjugate.

⊙ *Reference*: A. M. Vinogradov and I. S. Krasil'shchik (1997).

Chapter 3

Hyperbolic Equations with One Space Variable

3.1. Equations with Power-Law Nonlinearities

3.1.1. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + aw + bw^n + cw^{2n-1}$

► *The general properties of equations of this type are outlined in 3.4.1.1; traveling-wave solutions and some other solutions are also presented there.*

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + aw^n.$

This is a special case of equation 3.4.1.1 with $f(w) = aw^n$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w\big(\pm C_1^{n-1}x + C_2,\ \pm C_1^{n-1}t + C_3\big),$$
$$w_2 = w(x\cosh\lambda + t\sinh\lambda,\ x\sinh\lambda + t\cosh\lambda),$$

where C_1, C_2, C_3, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w(x,t) = b\big(x + C_1 t + C_2\big)^{\frac{2}{1-n}}, \qquad b = \left[\frac{2(1+n)(C_1^2-1)}{a(1-n)^2}\right]^{\frac{1}{n-1}};$$
$$w(x,t) = \big[k(t+C_1)^2 - k(x+C_2)^2\big]^{\frac{1}{1-n}}, \qquad k = \tfrac{1}{4}a(1-n)^2,$$

where C_1 and C_2 are arbitrary constants.

3°. The solutions of Item 2° are special cases of solutions of the following forms:

$$w(x,t) = F(z), \qquad z = x + C_1 t + C_2;$$
$$w(x,t) = G(\xi), \qquad \xi = (t+C_1)^2 - (x+C_2)^2.$$

4°. Self-similar solution:

$$w(x,t) = (t+C_1)^{\frac{2}{1-n}} u(\xi), \qquad \xi = \frac{x+C_2}{t+C_1},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$(1-\xi^2)u''_{\xi\xi} + \frac{2(1+n)}{1-n}\xi u'_\xi - \frac{2(1+n)}{(1-n)^2}u + au^n = 0.$$

The transformation

$$u = (\cosh\theta)^{\frac{2}{n-1}} U(\theta), \qquad \xi = \tanh\theta$$

brings this equation to the autonomous form

$$U''_{\theta\theta} - \frac{4}{(1-n)^2}U + aU^n = 0.$$

Integrating yields the general solution in implicit form

$$\int \left[\frac{4}{(n-1)^2}U^2 - \frac{2a}{n+1}U^{n+1} + C_3\right]^{-1/2} dU = C_4 \pm \theta,$$

where C_3 and C_4 are arbitrary constants.

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + aw + bw^n.$

This is a special case of equation 3.4.1.1 with $f(w) = aw + bw^n$.

1°. Traveling-wave solutions for $a > 0$:

$$w(x,t) = \left[\frac{2b\sinh^2 z}{a(n+1)}\right]^{\frac{1}{1-n}}, \qquad z = \tfrac{1}{2}\sqrt{a}\,(1-n)(x\sinh C_1 \pm t\cosh C_1) + C_2 \quad \text{if } b(n+1) > 0,$$

$$w(x,t) = \left[-\frac{2b\cosh^2 z}{a(n+1)}\right]^{\frac{1}{1-n}}, \quad z = \tfrac{1}{2}\sqrt{a}\,(1-n)(x\sinh C_1 \pm t\cosh C_1) + C_2 \quad \text{if } b(n+1) < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. Traveling-wave solutions for $a < 0$ and $b(n+1) > 0$:

$$w(x,t) = \left[-\frac{2b\cos^2 z}{a(n+1)}\right]^{\frac{1}{1-n}}, \qquad z = \tfrac{1}{2}\sqrt{|a|}\,(1-n)(x\sinh C_1 \pm t\cosh C_1) + C_2.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + aw^n + bw^{2n-1}.$

This is a special case of equation 3.4.1.1 with $f(w) = aw^n + bw^{2n-1}$.

Solutions:

$$w(x,t) = \left[\frac{a(1-n)^2}{2(n+1)}(x\sinh C_1 \pm t\cosh C_1 + C_2)^2 - \frac{b(n+1)}{2an}\right]^{\frac{1}{1-n}},$$

$$w(x,t) = \left\{\frac{1}{4}a(1-n)^2\left[(t+C_1)^2 - (x+C_2)^2\right] - \frac{b}{an}\right\}^{\frac{1}{1-n}},$$

where C_1 and C_2 are arbitrary constants.

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + aw - a(n+1)w^n + bw^{2n-1}.$

1°. Traveling-wave solutions:

$$w(x,t) = \left(\lambda + C_1 \exp z\right)^{\frac{1}{1-n}}, \qquad z = \sqrt{a}\,(1-n)(x\sinh C_2 \pm t\cosh C_2),$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation $a\lambda^2 - a(n+1)\lambda + b = 0$, and C_1 and C_2 are arbitrary constants.

2°. See also equation 3.1.1.5, in which b should be renamed $-a(n+1)$ and c renamed b.

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + aw + bw^n + cw^{2n-1}.$

This is a special case of equation 3.4.1.1 with $f(w) = aw + bw^n + cw^{2n-1}$.

1°. Traveling-wave solutions for $a > 0$:

$$w(x,t) = (A + B\cosh z)^{\frac{1}{1-n}}, \quad z = \sqrt{a}\,(1-n)(x\sinh C_1 \pm t\cosh C_1) + C_2,$$

$$A = -\frac{b}{a(n+1)}, \quad B = \pm\left[\frac{b^2}{a^2(n+1)^2} - \frac{c}{an}\right]^{1/2};$$

$$w(x,t) = (A + B\sinh z)^{\frac{1}{1-n}}, \quad z = \sqrt{a}\,(1-n)(x\sinh C_1 \pm t\cosh C_1) + C_2,$$

$$A = -\frac{b}{a(n+1)}, \quad B = \pm\left[\frac{c}{an} - \frac{b^2}{a^2(n+1)^2}\right]^{1/2},$$

where C_1 and C_2 are arbitrary constants (the expressions in square brackets must be nonnegative).

2°. Traveling-wave solutions for $a < 0$:

$$w(x,t) = (A + B\cos z)^{\frac{1}{1-n}}, \quad z = \sqrt{|a|}\,(1-n)(x\sinh C_1 \pm t\cosh C_1) + C_2,$$

$$A = -\frac{b}{a(n+1)}, \quad B = \pm\left[\frac{b^2}{a^2(n+1)^2} - \frac{c}{an}\right]^{1/2},$$

where C_1 and C_2 are arbitrary constants.

3°. The substitution $u = w^{1-n}$ leads to an equation with a quadratic nonlinearity:

$$u\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right) + \frac{n}{1-n}\left[\left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2\right] = a(1-n)u^2 + b(1-n)u + c(1-n).$$

3.1.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + a(x^2 - t^2)w^k.$

This is a special case of equation 3.4.1.2 with $f(w) = aw^k$.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + c(x + bt)^n w^k.$

This is a special case of equation 3.4.1.4 with $f(z,w) = cz^n w^k$. For $b = \pm 1$, see also equations 3.4.1.13 and 3.4.1.14 with $f(\xi) = c\xi^n$ and $g(w) = w^k$.

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + a(x^2 - t^2)(xt)^n w^k.$

This is a special case of equation 3.4.1.5 with $f(z,w) = az^n w^k$.

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + ae^{\beta t} w^k.$

This is a special case of equation 3.4.1.7 with $f(w) = aw^k$.

Functional separable solutions:

$$w(x,t) = \left\{C\exp\left[\frac{\beta(k-1)}{4(k+1)}(t \pm x)\right] + \frac{\sqrt{a}}{\beta}(1-k)e^{\beta t/2}\right\}^{\frac{2}{1-k}},$$

$$w(x,t) = \left\{C\exp\left[\frac{\beta(k-1)}{4(k+1)}(t \pm x)\right] - \frac{\sqrt{a}}{\beta}(1-k)e^{\beta t/2}\right\}^{\frac{2}{1-k}},$$

where C is an arbitrary constant.

5. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + aw + be^{\beta t} w^k.$

Functional separable solutions:

$$w(x,t) = \left\{ C \exp\left[\frac{k-1}{4\beta(k+1)} \left(\pm\sqrt{[\beta^2 - (k-1)^2 a][\beta^2 - (k+3)^2 a]}\, x + [\beta^2 + (k-1)(k+3)a]\, t \right) \right] + (k-1)\sqrt{\frac{b}{\beta^2 - (k-1)^2 a}}\, e^{\beta t/2} \right\}^{\frac{2}{1-k}},$$

$$w(x,t) = \left\{ C \exp\left[\frac{k-1}{4\beta(k+1)} \left(\pm\sqrt{[\beta^2 - (k-1)^2 a][\beta^2 - (k+3)^2 a]}\, x + [\beta^2 + (k-1)(k+3)a]\, t \right) \right] - (k-1)\sqrt{\frac{b}{\beta^2 - (k-1)^2 a}}\, e^{\beta t/2} \right\}^{\frac{2}{1-k}},$$

where C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

6. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + e^{\beta t}(a + be^{\beta t}) w^k.$

Functional separable solutions:

$$w(x,t) = \left\{ C \exp\left[\frac{\beta(k-1)}{2(k+1)} (t \pm x) \right] + \frac{1}{\beta\sqrt{b}} \left[a + \tfrac{1}{2}(1-k) b e^{\beta t} \right] \right\}^{\frac{2}{1-k}},$$

$$w(x,t) = \left\{ C \exp\left[\frac{\beta(k-1)}{2(k+1)} (t \pm x) \right] - \frac{1}{\beta\sqrt{b}} \left[a + \tfrac{1}{2}(1-k) b e^{\beta t} \right] \right\}^{\frac{2}{1-k}},$$

where C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

7. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\beta^2}{k^2+4} w + \left(a^2 e^{2\beta t} + abke^{\beta t} - b^2 \right) w^{-3}, \qquad k \neq 0.$

Functional separable solutions:

$$w(x,t) = \pm\sqrt{C \exp\left(\beta t + \frac{\beta k x}{\sqrt{k^2+4}} \right) + \frac{\sqrt{k^2+4}}{\beta} \left(\frac{2a}{k} e^{\beta t} + b \right)},$$

$$w(x,t) = \pm\sqrt{C \exp\left(\beta t - \frac{\beta k x}{\sqrt{k^2+4}} \right) + \frac{\sqrt{k^2+4}}{\beta} \left(\frac{2a}{k} e^{\beta t} + b \right)},$$

$$w(x,t) = \pm\sqrt{C \exp\left(\beta t + \frac{\beta k x}{\sqrt{k^2+4}} \right) - \frac{\sqrt{k^2+4}}{\beta} \left(\frac{2a}{k} e^{\beta t} + b \right)},$$

$$w(x,t) = \pm\sqrt{C \exp\left(\beta t - \frac{\beta k x}{\sqrt{k^2+4}} \right) - \frac{\sqrt{k^2+4}}{\beta} \left(\frac{2a}{k} e^{\beta t} + b \right)},$$

where C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

8. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} - \dfrac{\beta^2}{k^2-4}w + \left(a^2 e^{2\beta t} + abke^{\beta t} + b^2\right)w^{-3}, \quad |k| > 2.$

Functional separable solutions:

$$w(x,t) = \pm\sqrt{C\exp\left(\beta t + \frac{\beta k x}{\sqrt{k^2-4}}\right) + \frac{\sqrt{k^2-4}}{\beta}\left(\frac{2a}{k}e^{\beta t} + b\right)},$$

$$w(x,t) = \pm\sqrt{C\exp\left(\beta t - \frac{\beta k x}{\sqrt{k^2-4}}\right) + \frac{\sqrt{k^2-4}}{\beta}\left(\frac{2a}{k}e^{\beta t} + b\right)},$$

$$w(x,t) = \pm\sqrt{C\exp\left(\beta t + \frac{\beta k x}{\sqrt{k^2-4}}\right) - \frac{\sqrt{k^2-4}}{\beta}\left(\frac{2a}{k}e^{\beta t} + b\right)},$$

$$w(x,t) = \pm\sqrt{C\exp\left(\beta t - \frac{\beta k x}{\sqrt{k^2-4}}\right) - \frac{\sqrt{k^2-4}}{\beta}\left(\frac{2a}{k}e^{\beta t} + b\right)},$$

where C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

9. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} - ae^{\beta x}w^k.$

This is a special case of equation 3.4.1.6 with $f(w) = -aw^k$.

Functional separable solutions:

$$w(x,t) = \left\{C\exp\left[\frac{\beta(k-1)}{4(k+1)}(x \pm t)\right] + \frac{\sqrt{a}}{\beta}(1-k)e^{\beta x/2}\right\}^{\frac{2}{1-k}},$$

$$w(x,t) = \left\{C\exp\left[\frac{\beta(k-1)}{4(k+1)}(x \pm t)\right] - \frac{\sqrt{a}}{\beta}(1-k)e^{\beta x/2}\right\}^{\frac{2}{1-k}},$$

where C is an arbitrary constant.

10. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} - aw - be^{\beta x}w^k.$

Functional separable solutions:

$$\begin{aligned} w(x,t) = \Biggl\{C\exp\biggl[\frac{k-1}{4\beta(k+1)}\Bigl(&\pm\sqrt{[\beta^2-(k-1)^2a][\beta^2-(k+3)^2a]}\,t \\ &+ [\beta^2+(k-1)(k+3)a]\,x\Bigr)\biggr] + (k-1)\sqrt{\frac{b}{\beta^2-(k-1)^2a}}\,e^{\beta x/2}\Biggr\}^{\frac{2}{1-k}}, \end{aligned}$$

$$\begin{aligned} w(x,t) = \Biggl\{C\exp\biggl[\frac{k-1}{4\beta(k+1)}\Bigl(&\pm\sqrt{[\beta^2-(k-1)^2a][\beta^2-(k+3)^2a]}\,t \\ &+ [\beta^2+(k-1)(k+3)a]\,x\Bigr)\biggr] - (k-1)\sqrt{\frac{b}{\beta^2-(k-1)^2a}}\,e^{\beta x/2}\Biggr\}^{\frac{2}{1-k}}, \end{aligned}$$

where C is an arbitrary constant.

11. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} - e^{\beta x}(a + be^{\beta x})w^k.$

Functional separable solutions:

$$w(x,t) = \left\{ C \exp\left[\frac{\beta(k-1)}{2(k+1)}(x \pm t)\right] + \frac{1}{\beta\sqrt{b}}\left[a + \tfrac{1}{2}(1-k)be^{\beta x}\right]\right\}^{\frac{2}{1-k}},$$

$$w(x,t) = \left\{ C \exp\left[\frac{\beta(k-1)}{2(k+1)}(x \pm t)\right] - \frac{1}{\beta\sqrt{b}}\left[a + \tfrac{1}{2}(1-k)be^{\beta x}\right]\right\}^{\frac{2}{1-k}},$$

where C is an arbitrary constant.

12. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + ce^{ax+bt}w^k.$

This is a special case of equation 3.4.1.8 with $f(w) = cw^k$.

3.1.3. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n \dfrac{\partial w}{\partial x}\right) + bw^m, \qquad a > 0.$

This equation can be rewritten in the equivalent form

$$\frac{\partial^2 w}{\partial t^2} = a\left(\frac{\partial^2 w}{\partial x^2} + \frac{n}{x}\frac{\partial w}{\partial x}\right) + bw^m.$$

For $n = 1$ and $n = 2$, this equation describes nonlinear waves with axial and central symmetry, respectively.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w\left(\pm C_1^{k-1}x,\ \pm C_1^{k-1}t + C_2\right),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Functional separable solution:

$$w(x,t) = \left\{ \frac{b(1-m)^2}{2a(2+n-nm)}\left[a(t+C)^2 - x^2\right]\right\}^{\frac{1}{1-m}}, \tag{1}$$

where C is an arbitrary constant.

3°. Solution (1) is a special case of the wider family of exact solutions

$$w = w(r), \qquad r^2 = A\left[a(t+C)^2 - x^2\right],$$

where the sign of A must coincide with that of the expression in square brackets, and the function $w = w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{n+1}{r}w'_r = \frac{b}{Aa}w^m. \tag{2}$$

The books by Polyanin and Zaitsev (1995, 2003) give more than 20 exact solutions to equation (2) for specific values of the parameters n and m.

4°. There is a self-similar solution of the form $w = t^{\frac{2}{1-m}} f(\xi)$, where $\xi = x/t$.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bw^m \dfrac{\partial w}{\partial x}$**.**

This is a special case of equation 3.4.2.3 with $f(w) = bw^m$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w\bigl(C_1^m x + C_2,\ \pm C_1^m t + C_3\bigr),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution:

$$w(x,t) = \left[\frac{bm(x+\lambda t+C)}{(m+1)(a-\lambda^2)}\right]^{-1/m}, \tag{1}$$

where C and λ are arbitrary constants.

Solution (1) is a special case of the wider class of traveling-wave solutions

$$\int \frac{dw}{A+bw^{m+1}} = \frac{x+\lambda t+C}{(m+1)(\lambda^2-a)},$$

where A, C, and λ are arbitrary constants.

3°. There is a self-similar solution of the form $w = t^{-1/m} f(x/t)$.

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + s$**.**

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$\begin{aligned} w_1 &= w(\pm x + C_1, \pm t + C_2) + C_3\cosh(kt) + C_4\sinh(kt) \quad &&\text{if } c = k^2 > 0,\\ w_2 &= w(\pm x + C_1, \pm t + C_2) + C_3\cos(kt) + C_4\sin(kt) \quad &&\text{if } c = -k^2 < 0, \end{aligned}$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Generalized separable solution quadratic in x:

$$w(x,t) = -\frac{c}{4b}(\pm x + C_1 t + C_2)^2 + \frac{1}{2b}(a - C_1^2) - \frac{s}{c} + U(t),$$
$$U(t) = \begin{cases} C_3\cosh(kt) + C_4\sinh(kt) & \text{if } c = k^2 > 0,\\ C_3\cos(kt) + C_4\sin(kt) & \text{if } c = -k^2 < 0. \end{cases}$$

3°. For other solutions, see 3.4.2.4 with $f(t) = s$.

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + st^n$**.**

This is a special case of equation 3.4.2.4 with $f(t) = st^n$.

5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + sx^n$**.**

This is a special case of equation 3.4.2.5 with $f(x) = sx^n$.

6. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + c\left(\dfrac{\partial w}{\partial x}\right)^2 + bcw^2 + kw + s$**.**

This is a special case of equation 3.4.2.10 with $f(t) = c$, $g(t) = k$, and $h(t) = s$.

Let A be a root of the quadratic equation $bcA^2 + kA + s = 0$.

1°. Suppose that $2Abc + k - ab = \sigma^2 > 0$. Then there are generalized separable solutions

$$w(x,t) = A + \big[C_1 \exp(\sigma t) + C_2 \exp(-\sigma t)\big] \exp\big(\pm x\sqrt{-b}\,\big),$$

where C_1 and C_2 are arbitrary constants.

2°. If $2Abc + k - ab = -\sigma^2 < 0$, there are generalized separable solutions

$$w(x,t) = A + \big[C_1 \cos(\sigma t) + C_2 \sin(\sigma t)\big] \exp\big(\pm x\sqrt{-b}\,\big).$$

For more complicated solutions, see 3.4.2.10.

⊙ *References*: V. A. Galaktionov (1995, the case $a = c$ was considered), V. F. Zaitsev and A. D. Polyanin (1996).

7. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + bx^n\left(\frac{\partial w}{\partial x}\right)^2 + cx^m + st^k.$

This is a special case of equation 3.4.2.8 with $f(x) = bx^n$, $g(x) = cx^m$, and $h(t) = st^k$.

8. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + ct^n\left(\frac{\partial w}{\partial x}\right)^2 + bct^n w^2 + st^m w + pt^k.$

This is a special case of equation 3.4.2.10 with $f(t) = ct^n$, $g(t) = st^m$, and $h(t) = pt^k$.

9. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + bt^n\left(\frac{\partial w}{\partial x}\right)^2 + ct^k x\frac{\partial w}{\partial x}.$

There is a generalized separable solution quadratic in x:

$$w = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

3.1.4. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g\Big(x, t, w, \dfrac{\partial w}{\partial x}\Big)$

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a(x+\beta)^n\frac{\partial^2 w}{\partial x^2} + bw^m, \qquad a > 0.$

This equation describes the propagation of nonlinear waves in an inhomogeneous medium.

1°. Functional separable solution for $n \neq 2$:

$$w(x,t) = \Big\{s\big[a(2-n)^2(t+C)^2 - 4(x+\beta)^{2-n}\big]\Big\}^{\frac{1}{1-m}}, \qquad s = \frac{b(1-m)^2}{2a(2-n)(nm-3n+4)},$$

where C is an arbitrary constant.

2°. Functional separable solution (generalizes the solution of Item 1°):

$$w = w(r), \qquad r^2 = k\left[\frac{1}{4}(t+C)^2 - \frac{(x+\beta)^{2-n}}{a(2-n)^2}\right],$$

where k and the expression in square brackets must have like signs, and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{2(1-n)}{2-n}\frac{1}{r}w'_r = \frac{4b}{k}w^m. \qquad (1)$$

The substitution $\xi = r^{\frac{n}{2-n}}$ leads to the Emden–Fowler equation

$$w''_{\xi\xi} = \frac{4b(2-n)^2}{kn^2}\xi^{\frac{4(1-n)}{n}}w^m. \qquad (2)$$

The books by Polyanin and Zaitsev (1995, 2003) give more than 20 exact solutions to equation (2) for specific values of the parameters n and m.

Special case. For $n = 1$, the general solution of equation (1) is written in explicit form as

$$\int \left[C_1 + \frac{8b}{k(m+1)} w^{m+1} \right]^{-1/2} dw = \pm r + C_2,$$

where C_1 and C_2 are arbitrary constants.

3°. Solution for $n = 2$:

$$w = w(y), \qquad y = At + B \ln |x + \beta|,$$

where A and B are arbitrary constants, and the function $w = w(y)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2) w''_{yy} - aB w'_y + b w^m = 0. \tag{3}$$

The solution of equation (3) with $A = \pm B\sqrt{a}$ is given by

$$w(y) = \left[\frac{b(1-m)}{aB} y + C \right]^{\frac{1}{1-m}},$$

where C is an arbitrary constant.

For $A \neq \pm B\sqrt{a}$, the substitution $U(w) = \dfrac{aB^2 - A^2}{aB} w'_y$ brings (3) to the Abel equation

$$UU'_w - U = \frac{b(A^2 - aB^2)}{a^2 B^2} w^m,$$

whose general solutions for $m = -2,\ -1,\ -\frac{1}{2},\ 0,\ 1$ can be found in Polyanin and Zaitsev (2003).

4°. There is a self-similar solution of the form $w = t^{\frac{2}{1-m}} f(\xi)$, where $\xi = (x + \beta) t^{\frac{2}{n-2}}$.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(a x^n \frac{\partial w}{\partial x} \right) + b w^m, \qquad a > 0.$

This equation describes the propagation of nonlinear waves in an inhomogeneous medium.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w \Big(C_1^{\frac{m-1}{2-n}} x,\ \pm C_1^{\frac{m-1}{2}} t + C_2 \Big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $n \neq 2$:

$$w(x, t) = \Big\{ s \big[a(2-n)^2 (t + C)^2 - 4x^{2-n} \big] \Big\}^{\frac{1}{1-m}}, \qquad s = \frac{b(1-m)^2}{2a(2-n)(4-n-nm)},$$

where C is an arbitrary constant.

3°. Functional separable solution (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = k \left[\frac{1}{4}(t + C)^2 - \frac{x^{2-n}}{a(2-n)^2} \right],$$

where k and the expression in square brackets must have like signs, and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{2}{2-n} \frac{1}{r} w'_r = \frac{4b}{k} w^m.$$

The substitution $\xi = r^{\frac{n}{n-2}}$ leads to the Emden–Fowler equation

$$w''_{\xi\xi} = \frac{4b(2-n)^2}{kn^2} \xi^{-\frac{4}{n}} w^m. \tag{1}$$

The books by Polyanin and Zaitsev (1995, 2003) give more than 20 exact solutions to equation (1) for specific values of the parameters n and m.

4°. Solution for $n=2$:

$$w=w(z), \quad z=At+B\ln|x|,$$

where A and B are arbitrary constants, and the function $w=w(z)$ is determined by the autonomous ordinary differential equation

$$(aB^2-A^2)w''_{zz}+aBw'_z+bw^m=0. \tag{2}$$

The solution of equation (2) with $A=\pm B\sqrt{a}$ is given by

$$w(z)=\left[\frac{b(m-1)}{aB}z+C\right]^{\frac{1}{1-m}},$$

where C is an arbitrary constant.

For $A\neq\pm B\sqrt{a}$, the substitution $U(w)=\dfrac{A^2-aB^2}{aB}w'_z$ brings (2) to the Abel equation

$$UU'_w-U=\frac{b(A^2-aB^2)}{a^2B^2}w^m,$$

whose exact solutions for $m=-2,\,-1,\,-\frac{1}{2},\,0,\,1$ can be found in Polyanin and Zaitsev (1995, 2003).

5°. There is a self-similar solution of the form $w=t^{\frac{2}{1-m}}f(\xi)$, where $\xi=xt^{\frac{2}{n-2}}$.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

3. $\dfrac{\partial^2 w}{\partial t^2}=ax^n\dfrac{\partial^2 w}{\partial x^2}+bx^{n-1}w^m\dfrac{\partial w}{\partial x}, \qquad a>0.$

This is a special case of equation 3.4.3.5 with $f(w)=bw^m$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1=w(C_1^2x,\,\pm C_1^{2-n}t+C_2),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $n\neq 2$:

$$w=w(z), \quad z=\left|a(2-n)^2(t+C)^2-4x^{2-n}\right|^{1/2},$$

where C is an arbitrary constant, and the function $w=w(z)$ is determined by the ordinary differential equation

$$w''_{zz}+\frac{2}{a(2-n)z}\left[a(1-n)+bw^m\right]w'_z=0. \tag{1}$$

The substitution $u(w)=zw'_z$ leads equation (1) to a first-order separable ordinary differential equation. Integrating yields a solution in implicit form:

$$\int\frac{dw}{anw-\dfrac{2b}{m+1}w^{m+1}+C_1}=\frac{1}{a(2-n)}\ln z+C_2,$$

where C_1 and C_2 are arbitrary constants.

3°. There is a self-similar solution of the form

$$w=U(\zeta), \quad \zeta=xt^{\frac{2}{n-2}}.$$

4°. Solution for $n=2$:

$$w=w(\xi), \quad z=At+B\ln|x|+C,$$

where A, B, and C are arbitrary constants, and the function $w=w(\xi)$ is determined by the autonomous ordinary differential equation

$$(aB^2-A^2)w''_{\xi\xi}+B(bw^m-a)w'_\xi=0.$$

Integrating yields

$$\int\frac{dw}{bw^{m+1}-a(m+1)w+C_1}=-\frac{B\xi}{(m+1)(aB^2-A^2)}.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = ax^n \dfrac{\partial^2 w}{\partial x^2} + bx^{n-1} w^m \dfrac{\partial w}{\partial x} + cw^k, \qquad a > 0.$

This is a special case of equation 3.4.3.6 with $f(w) = bw^m$ and $g(w) = cw^k$.

5. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x} \dfrac{\partial^2 w}{\partial x^2} + cw^m, \qquad a > 0.$

This is an equation of the propagation of nonlinear waves in an inhomogeneous medium. This is a special case of equation 3.4.3.9 with $b = 0$ and $f(w) = cw^m$.

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\lambda x} \dfrac{\partial w}{\partial x}\right) + cw^m.$

This is a special case of equation 3.4.3.9 with $b = a\lambda$ and $f(w) = cw^m$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w\Big(x + \frac{1-m}{\lambda} \ln C_1,\ \pm C_1^{\frac{m-1}{2}} t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $m \neq \pm 1$ and $\lambda \neq 0$:

$$w = \left[-\frac{c(m-1)^2}{2k(1+m)}(r+C_1)^2\right]^{\frac{1}{1-m}}, \qquad r^2 = 4k\left[\frac{e^{-\lambda x}}{a\lambda^2} - \frac{1}{4}(t+C_2)^2\right],$$

where C_1, C_2, and k are arbitrary constants.

3°. Functional separable solution for $\lambda \neq 0$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4k\left[\frac{e^{-\lambda x}}{a\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where the function $w(r)$ is determined by the autonomous ordinary differential equation

$$w''_{rr} + ck^{-1} w^m = 0.$$

Integrating yields the general solution in implicit form

$$\int \left[C_1 - \frac{2c}{k(m+1)} w^{m+1}\right]^{-1/2} dw = C_2 \pm r,$$

where C_1 and C_2 are arbitrary constants.

4°. There is an exact solution of the form

$$w(x,t) = |t|^{\frac{2}{1-m}} F(z), \quad z = x + \frac{2}{\lambda} \ln |t|.$$

7. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x} \dfrac{\partial^2 w}{\partial x^2} + be^{\lambda x} \dfrac{\partial w}{\partial x} + cw^m, \qquad a > 0.$

Functional separable solution:

$$w = w(z), \quad z = \left[4ke^{-\lambda x} - ak\lambda^2(t+C)^2\right]^{1/2}, \qquad k = \pm 1,$$

where C is an arbitrary constant, and the function $w = w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2(a\lambda - b)}{a\lambda} \frac{1}{z} w'_z + \frac{c}{ak\lambda^2} w^m = 0. \tag{1}$$

This equation has the exact solution

$$w(z) = \left\{ \frac{2k\lambda\left[a\lambda(m-3) + 2b(1-m)\right]}{c(1-m)^2 z^2} \right\}^{\frac{1}{m-1}}.$$

For $b = a\lambda$, the general solution of equation (1) is given in implicit form by

$$\int \left[C_1 - \frac{2c}{ak\lambda^2(m+1)} w^{m+1} \right]^{-1/2} dw = \pm z + C_2,$$

where C_1 and C_2 are arbitrary constants.

For $b \neq \frac{1}{2}a\lambda$, the substitution $\xi = z^{\frac{2b-a\lambda}{a\lambda}}$ brings (1) to the generalized Emden–Fowler equation

$$w''_{\xi\xi} + \frac{ac}{k(2b-a\lambda)^2} \xi^{\frac{4(a\lambda-b)}{2b-a\lambda}} w^m = 0. \tag{2}$$

The books by Polyanin and Zaitsev (1995, 2003) give more than 20 exact solutions to equation (2) for specific values of the parameter m.

8. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x} \dfrac{\partial^2 w}{\partial x^2} + be^{\lambda x} w^n \dfrac{\partial w}{\partial x}, \qquad a > 0.$

This is a special case of equation 3.4.3.10 with $f(w) = bw^n$.

3.1.5. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = aw^n \dfrac{\partial^2 w}{\partial x^2} + f(x, w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = aw \dfrac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 3.1.5.5 with $n = 1$.

1°. Solutions:

$$w = C_1 xt + C_2 x + C_3 t + C_4,$$

$$w = \frac{3x^2 + C_1 x + C_2}{a(t + C_3)^2} + C_4(x + C_5)(t + C_3)^3,$$

where $C_1, \ldots, C_5$ are arbitrary constants. The first solution is degenerate and the second one is a generalized separable solution.

2°. Solution:

$$w = U(z) + 4aC_1^2 t^2 + 4aC_1C_2 t, \qquad z = x + aC_1 t^2 + aC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$(U - aC_2^2)U''_{zz} - 2C_1 U'_z = 8C_1^2.$$

3°. Generalized separable solution:

$$w = (x^2 + C_1 x + C_2) f(t) + (C_3 x + C_4) f(t) \int \frac{dt}{f^2(t)},$$

where $C_1, \ldots, C_4$ are arbitrary constants, and the function $f = f(t)$ is determined by the autonomous ordinary differential equation $f''_{tt} = 2af^2$.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

2. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^2 w}{\partial x^2} + b.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{-2} w(\pm C_1^2 x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w = (C_1 t + C_2)x + \tfrac{1}{2}bt^2 + C_3 t + C_4,$$

$$w = \frac{3x^2}{at^2} + \left(C_1 t^3 + \frac{C_2}{t^2}\right)x + C_3 t^3 + \frac{C_4}{t^2} - \frac{1}{4}bt^2.$$

The first solution is degenerate and the second one is a generalized separable solution (another arbitrary constant can be added, since the equation is invariant under translation in t).

3°. Self-similar solution:

$$w(x,t) = t^2 u(\xi), \quad \xi = xt^{-2},$$

where the function $u = u(\xi)$ is determined by the ordinary differential equation

$$2u - 2\xi u'_\xi + 4\xi^2 u''_{\xi\xi} = auu''_{\xi\xi} + b.$$

4°. Solution:

$$w = U(z) + 4aC_1^2 t^2 + 4aC_1 C_2 t, \qquad z = x + aC_1 t^2 + aC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$(aU - a^2 C_2^2)U''_{zz} - 2aC_1 U'_z = 8aC_1^2 - b.$$

5°. The second solution in Item 2° is a special case of the generalized separable solution

$$w(x,t) = f(t)x^2 + g(t)x + h(t).$$

3. $\dfrac{\partial^2 w}{\partial t^2} = aw^4\dfrac{\partial^2 w}{\partial x^2} + bx^n w^5.$

This is a special case of equation 3.4.4.2 with $f(x) = bx^n$.

4. $\dfrac{\partial^2 w}{\partial t^2} = aw^4\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda x} w^5.$

This is a special case of equation 3.4.4.2 with $f(x) = be^{\lambda x}$.

5. $\dfrac{\partial^2 w}{\partial t^2} = aw^n\dfrac{\partial^2 w}{\partial x^2}, \qquad a > 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = (C_2/C_1)^{2/n} w(\pm C_1 x + C_3, \pm C_2 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Multiplicative separable solution:

$$w(x,t) = k(x + C_1)^{\frac{2}{n}}(At + C_2)^{-\frac{2}{n}}, \qquad k = \left[\frac{A^2(n+2)}{a(2-n)}\right]^{\frac{1}{n}}, \tag{1}$$

where A, C_1, and C_2 are arbitrary constants.

3°. Expression (1) is a special case of a wider family of multiplicative separable solutions

$$w = f(x)g(t),$$

where the functions $f = f(x)$ and $g = g(t)$ are determined by solving the equations

$$g''_{tt} - a\lambda g^{n+1} = 0, \qquad (2)$$
$$f''_{xx} - \lambda f^{1-n} = 0. \qquad (3)$$

The general solutions of equations (2) and (3) can be written out in implicit form:

$$\int \left(C_1 + \frac{2a\lambda}{n+2} g^{n+2} \right)^{-1/2} dg = C_2 \pm t,$$
$$\int \left(C_3 + \frac{2\lambda}{2-n} f^{2-n} \right)^{-1/2} df = C_4 \pm x,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

In particular, with $C_1 = 0$, it follows that

$$g(t) = (At + C)^{-2/n}, \qquad A = \pm\sqrt{\frac{a\lambda n^2}{2(n+2)}}.$$

4°. There are also solutions with the following forms:

$$w(x,t) = (t+A)^{-\frac{2k+2}{n}} F(z), \quad z = (x+B)(t+A)^k;$$
$$w(x,t) = e^{-2\lambda t} U(y), \quad y = (x+A)e^{\lambda n t};$$
$$w(x,t) = (At+B)^{-2/n} V(\xi), \quad \xi = x + k\ln(At+B) + C,$$

where A, B, C, k, and λ are arbitrary constants.

3.1.6. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + f(w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right)$**.**

This is a special case of equation 3.1.6.5 with $n = 1$.

1°. Solutions:

$$w(x,t) = \tfrac{1}{2}aA^2t^2 + Bt + Ax + C,$$
$$w(x,t) = \tfrac{1}{12}aA^{-2}(At+B)^4 + Ct + D + x(At+B),$$
$$w(x,t) = \frac{1}{a}\left(\frac{x+A}{t+B}\right)^2,$$
$$w(x,t) = (At+B)\sqrt{Cx+D},$$
$$w(x,t) = \pm\sqrt{A(x+a\lambda t)+B} + a\lambda^2,$$

where A, B, C, D, and λ are arbitrary constants.

⊙ *Reference*: S. Tomotika, K. Tamada (1950).

2°. Generalized separable solution quadratic in x:

$$w(x,t) = f(t)x^2 + g(t)x + h(t),$$

where the functions $f = f(t)$, $g = g(t)$, and $h = h(t)$ are determined by the system of ordinary differential equations

$$f''_{tt} = 6af^2,$$
$$g''_{tt} = 6afg,$$
$$h''_{tt} = 2afh + ag^2.$$

A particular solution of this system is given by

$$f = \frac{1}{at^2}, \quad g = \frac{C_1}{t^2} + C_2 t^3, \quad h = \frac{aC_1^2}{4t^2} + \frac{C_3}{t} + C_4 t^2 + \frac{1}{2}aC_1C_2t^3 + \frac{1}{54}aC_2^2t^8,$$

where $C_1, \ldots, C_4$ are arbitrary constants. Another arbitrary constant can be inserted in this solution through the shift in t.

3°. Solution:

$$w = U(z) + 4aC_1^2t^2 + 4aC_1C_2t, \qquad z = x + aC_1t^2 + aC_2t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the first-order ordinary differential equation

$$(U - aC_2^2)U'_z - 2C_1U = 8C_1^2z + C_3.$$

By appropriate translations in both variables, the equation can be made homogeneous, and, hence, the equation is integrable by quadrature.

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(w\frac{\partial w}{\partial x}\right) + b.$

This is a special case of equation 8.2.1.3 with $F(u, v) = au^2 + b$.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2}w(\pm C_1^2x + C_2, C_1t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution in implicit form:

$$ak^2w^2 - 2\lambda^2w = -b(kx + \lambda t)^2 + C_1(kx + \lambda t) + C_2,$$

where C_1, C_2, k, and λ are arbitrary constants.

3°. Self-similar solution:

$$w(x, t) = t^2u(\xi), \quad \xi = xt^{-2},$$

where the function $u = u(\xi)$ is determined by the ordinary differential equation

$$2u - 2\xi u'_\xi + 4\xi^2u''_{\xi\xi} = a(uu'_\xi)'_\xi + b.$$

4°. Solution:

$$w = U(z) + 4aC_1^2t^2 + 4aC_1C_2t, \qquad z = x + aC_1t^2 + aC_2t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the first-order ordinary differential equation

$$(aU - a^2C_2^2)U'_z - 2aC_1U = (8aC_1^2 - b)z + C_3.$$

By appropriate translations in both variables, the equation can be made homogeneous, and, hence, the equation is integrable by quadrature.

5°. Generalized separable solution quadratic in x:

$$w(x,t) = f(t)x^2 + g(t)x + h(t),$$

where the functions $f = f(t)$, $g = g(t)$, and $h = h(t)$ are determined by the system of ordinary differential equations

$$f''_{tt} = 6af^2,$$
$$g''_{tt} = 6afg,$$
$$h''_{tt} = 2afh + ag^2 + b.$$

A particular solution of this system is given by

$$f = \frac{1}{at^2}, \quad g = \frac{C_1}{t^2} + C_2 t^3, \quad h = \frac{aC_1^2}{4t^2} + \frac{C_3}{t} + C_4 t^2 + \frac{1}{2}aC_1C_2t^3 + \frac{1}{54}aC_2^2t^8 + \frac{1}{9}bt^2\bigl(3\ln|t| - 1\bigr),$$

where $C_1, \ldots, C_4$ are arbitrary constants. Another arbitrary constant, C_5, can be inserted in the solution, by substituting $t + C_5$ for t, since the system is translation invariant in t.

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(\frac{1}{w}\frac{\partial w}{\partial x}\right).$

This is a special case of equation 3.1.6.5 with $n = -1$.

1°. Multiplicative separable solutions:

$$w(x,t) = (At + B)e^{Cx},$$
$$w(x,t) = (at^2 + At + B)(x + C)^{-2},$$
$$w(x,t) = (-aA^2t^2 + Bt + C)\cosh^{-2}(Ax + D),$$
$$w(x,t) = (aA^2t^2 + Bt + C)\sinh^{-2}(Ax + D),$$
$$w(x,t) = (aA^2t^2 + Bt + C)\cos^{-2}(Ax + D),$$

where A, B, C, and D are arbitrary constants.

2°. Traveling-wave solution in implicit form:

$$\lambda^2 w = ak^2 \ln|w| + C_1(kx + \lambda t) + C_2,$$

where C_1, C_2, k, and λ are arbitrary constants.

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{w}}\frac{\partial w}{\partial x}\right).$

This is a special case of equation 3.1.6.5 with $n = -1/2$.

1°. Solutions:

$$w(x,t) = \bigl(\tfrac{1}{2}A^2ax^2 + Bx + Aat + C\bigr)^2,$$
$$w(x,t) = \bigl[\tfrac{1}{12}A^{-2}a^{-1}(Ax + B)^4 + Cx + D + t(Ax + B)\bigr]^2,$$
$$w(x,t) = a^2\left(\frac{t + A}{x + B}\right)^4,$$
$$w(x,t) = (Ax + B)^2(Ct + D),$$
$$w(x,t) = \bigl[\pm\sqrt{A(t + \lambda x) + B} + a\lambda^2\bigr]^2,$$

where A, B, C, D, and λ are arbitrary constants.

2°. The substitution $w = u^2$ leads to an equation of the form 3.1.4.2:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a}\frac{\partial}{\partial t}\left(u\frac{\partial u}{\partial t}\right).$$

5. $\dfrac{\partial^2 w}{\partial t^2} = a^2 \dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right).$

This equation is encountered in wave and gas dynamics. This is a special case of equation 3.4.4.6 with $f(w) = a^2 w^n$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = (C_2/C_1)^{2/n} w(\pm C_1 x + C_3, \pm C_2 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Degenerate solution:

$$w = (At + B)(Cx + D)^{\frac{1}{n+1}},$$

where A, B, C, and D are arbitrary constants.

3°. Multiplicative separable solution:

$$w = f(x)g(t),$$

where $f = f(x)$ and $g = g(t)$ are defined implicitly by

$$\int \left(C_1 + \frac{2\lambda}{n+2} f^{n+2}\right)^{-1/2} f^n \, df = C_2 \pm x, \tag{1}$$

$$\int \left(C_3 + \frac{2a^2\lambda}{n+2} g^{n+2}\right)^{-1/2} dg = C_4 \pm t, \tag{2}$$

and $C_1, \ldots, C_4$ and λ are arbitrary constants.

The functions $f = f(x)$ and $g = g(t)$ defined by (1) and (2) can be represented in explicit form if $C_1 = 0$ and $C_3 = 0$. To the special case $C_1 = C_3 = 0$ there corresponds

$$w(x,t) = \left(\frac{\pm bx + c}{abt + s}\right)^{2/n}, \tag{3}$$

where b, c, and s are arbitrary constants.

4°. Traveling-wave solution:

$$w = w(z), \quad z = x \pm \lambda t,$$

where $w = w(z)$ is defined implicitly by (A and B are arbitrary constants)

$$\lambda^2 w - \frac{a^2}{n+1} w^{n+1} = Az + B. \tag{4}$$

If $n = -\frac{1}{2}$, 1, 2, or 3, equation (4) can be solved for w to give an explicit expression of $w = w(z)$.

5°. Self-similar solution:

$$w = w(\xi), \quad \xi = \frac{x + A}{t + B},$$

where the function $w(\xi)$ is determined by the first-order ordinary differential equation (C is an arbitrary constant):

$$\left(\xi^2 - a^2 w^n\right) w'_\xi = C. \tag{5}$$

To the special case $C = 0$ there corresponds the solution $w = (\xi/a)^{2/n}$, see formula (3). If $C \neq 0$, by treating w in (5) as the independent variable, one obtains a Riccati equation for $\xi = \xi(w)$:

$$C\xi'_w = \xi^2 - a^2 w^n. \tag{6}$$

The general solution of equation (6) is expressed in terms of Bessel functions; see Kamke (1977) and Polyanin and Zaitsev (1995, 2003).

6°. There are more complicated self-similar solutions of the form

$$w = (t+\beta)^{2k} F(z), \quad z = \frac{x+\alpha}{(t+\beta)^{nk+1}},$$

where α, β, and k are arbitrary constants, and the function $F = F(z)$ is determined by solving the generalized-homogeneous ordinary differential equation

$$2k(2k-1)F + (nk+1)(nk-4k+2)zF'_z + (nk+1)^2 z^2 F''_{zz} = a^2 (F^n F'_z)'_z.$$

Its order can be reduced.

7°. Generalized self-similar solution (μ is an arbitrary constant):

$$w = e^{-2\mu t} \varphi(y), \quad y = xe^{\mu nt},$$

where the function $\varphi = \varphi(y)$ is determined by solving the generalized-homogeneous ordinary differential equation

$$4\mu^2 \varphi + \mu^2 n(n-4) y\varphi'_y + (\mu n)^2 y^2 \varphi''_{yy} = a^2 (\varphi^n \varphi'_y)'_y.$$

Its order can be reduced.

8°. Solution (A, b, and c are arbitrary constants):

$$w = (\pm t + A)^{-2/n} \psi(u), \quad u = x + b \ln(\pm t + A) + c,$$

where the function $\psi = \psi(u)$ is determined by the autonomous ordinary differential equation

$$\frac{2(n+2)}{n^2}\psi - \frac{b(n+4)}{n}\psi'_u + b^2 \psi''_{uu} = a^2 (\psi^n \psi'_u)'_u. \tag{7}$$

Note two special cases where the equation obtained is integrable by quadrature. For $n = -2$, equation (7) admits a first integral that represents a separable equation. For $n = -4$, with the change of variable $G(\psi) = (\psi'_u)^2$, equation (7) can be reduced to a first-order linear equation.

In the general case, the change of variable $H(\psi) = \psi'_u$ brings (7) to a first-order equation.

9°. For $n \neq -1$, the transformation

$$\tau = x, \quad \zeta = t, \quad V = w^{n+1}$$

brings the original equation to an equation of the similar form

$$\frac{\partial^2 V}{\partial \tau^2} = a^{-2} \frac{\partial}{\partial \zeta}\left(V^{-\frac{n}{n+1}} \frac{\partial V}{\partial \zeta} \right).$$

For $n = -1$, the transformation

$$\tau = x, \quad \zeta = t, \quad V = \ln w$$

brings the original equation to an equation of the form 3.2.4.3:

$$\frac{\partial^2 V}{\partial \tau^2} = a^{-2} \frac{\partial}{\partial \zeta}\left(e^V \frac{\partial V}{\partial \zeta} \right).$$

⊙ *References for equation* 3.1.6.5: W. F. Ames, R. J. Lohner, and E. Adams (1981), N. H. Ibragimov (1994), V. F. Zaitsev and A. D. Polyanin (1996).

6. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + bw^{n+1} + cw.$

1°. Multiplicative separable solutions for $c = \lambda^2 > 0$:

$$w = \left(A_1 e^{\lambda t} + A_2 e^{-\lambda t}\right)\left[B_1 \cos(kx) + B_2 \sin(kx)\right]^{\frac{1}{n+1}} \quad \text{if} \quad b(n+1)/a = k^2 > 0,$$
$$w = \left(A_1 e^{\lambda t} + A_2 e^{-\lambda t}\right)\left(B_1 e^{kx} + B_2 e^{-kx}\right)^{\frac{1}{n+1}} \quad \text{if} \quad b(n+1)/a = -k^2 < 0,$$

where A_1, A_2, B_1, and B_2 are arbitrary constants.

2°. Multiplicative separable solutions for $c = -\lambda^2 < 0$:

$$w = \left[A_1 \cos(\lambda t) + A_2 \sin(\lambda t)\right]\left[B_1 \cos(kx) + B_2 \sin(kx)\right]^{\frac{1}{n+1}} \quad \text{if} \quad b(n+1)/a = k^2 > 0,$$
$$w = \left[A_1 \cos(\lambda t) + A_2 \sin(\lambda t)\right]\left(B_1 e^{kx} + B_2 e^{-kx}\right)^{\frac{1}{n+1}} \quad \text{if} \quad b(n+1)/a = -k^2 < 0,$$

where A_1, A_2, B_1, and B_2 are arbitrary constants.

3°. Multiplicative separable solution:

$$w = \varphi(x)\psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are determined by the autonomous ordinary differential equations (K is an arbitrary constant)

$$a(\varphi^n \varphi'_x)'_x + b\varphi^{n+1} + K\varphi = 0,$$
$$\psi''_{tt} - c\psi + K\psi^{n+1} = 0.$$

7. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + bw^k.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{k-n-1} x + C_2, \pm C_1^{k-1} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There are solutions of the following forms:

$$w(x,t) = U(z), \quad z = \lambda x + \beta t \qquad \text{traveling-wave solution;}$$
$$w(x,t) = t^{\frac{2}{1-k}} V(\xi), \quad \xi = x t^{\frac{k-n-1}{1-k}} \qquad \text{self-similar solution.}$$

3.1.7. Other Equations

1. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^n + c.$

This is a special case of equation 8.2.1.3 with $F(u,v) = bu^n + c$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^2 x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Self-similar solution:

$$w(x,t) = t^2 u(\xi), \quad \xi = xt^{-2},$$

where the function $u = u(\xi)$ is determined by the ordinary differential equation

$$2u - 2\xi u'_\xi + 4\xi^2 u''_{\xi\xi} = a(uu'_\xi)'_\xi + b(u'_\xi)^n + c.$$

3°. Solution:

$$w = U(z) + 4aC_1^2 t^2 + 4aC_1C_2 t, \quad z = x + aC_1 t^2 + aC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$(aU - a^2C_2^2)U''_{zz} + b(U'_z)^n - 2aC_1U'_z = 8aC_1^2 - c.$$

2. $\dfrac{\partial^2 w}{\partial t^2} = ax^n w^m \dfrac{\partial^2 w}{\partial x^2}, \qquad a > 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w\bigl(C_2^2 x,\ \pm C_1^m C_2^{2-n} t + C_3\bigr),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = kx^{\frac{2-n}{m}}(C_1 t + C_2)^{-\frac{2}{m}}, \qquad k = \left[\frac{2C_1^2(m+2)}{a(2-n)(2-n-m)}\right]^{\frac{1}{m}}. \tag{1}$$

3°. Expression (1) is a special case of a wider family of multiplicative separable solutions

$$w = f(x)g(t),$$

where the functions $f = f(x)$ and $g = g(t)$ are determined by solving the equations

$$g''_{tt} - \lambda g^{m+1} = 0, \tag{2}$$

$$f''_{xx} - (\lambda/a)x^{-n} f^{1-m} = 0. \tag{3}$$

The general solution of equation (2) can be written out in implicit form as

$$\int \left(C_1 + \frac{2\lambda}{m+2} g^{m+2}\right)^{-1/2} dg = C_2 \pm t,$$

where C_1 and C_2 are arbitrary constants.

In particular, with $C_1 = 0$, it follows that

$$g(t) = (At + C)^{-2/m}, \qquad A = \pm\sqrt{\frac{\lambda m^2}{2(m+2)}}.$$

The books by Polyanin and Zaitsev (1995, 2003) give more than 20 exact solutions to the Emden–Fowler equation (3) for specific values of the parameter m.

4°. There is a self-similar solution of the form

$$w = t^{\frac{(n-2)k-2}{m}} F(y), \quad y = xt^k,$$

where k is an arbitrary constant.

5°. The transformation

$$u(z,t) = \frac{1}{x} w(x,t), \quad z = \frac{1}{z}$$

leads to an equation of the similar form

$$\frac{\partial^2 u}{\partial t^2} = az^{4-n-m} u^m \frac{\partial^2 u}{\partial z^2}. \tag{4}$$

In the special case $n = 4 - m$, equation (4) is greatly simplified to become

$$\frac{\partial^2 u}{\partial t^2} = au^m \frac{\partial^2 u}{\partial z^2}$$

and admits a traveling-wave solution $u = u(kz + \mu t)$; see also equation 3.1.6.5.

3. $\dfrac{\partial^2 w}{\partial t^2} = a e^{\lambda x} w^m \dfrac{\partial^2 w}{\partial x^2}, \qquad a > 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w\Big(x + \frac{1}{\lambda} \ln \frac{C_1^m}{C_2^2},\ C_2 t + C_3\Big),$$

where C_1, C_2, C_3, and λ are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w(x,t) = U(z), \qquad z = x + \frac{2}{\lambda} \ln |t|,$$

where the function $U(z)$ is determined by the ordinary differential equation

$$(a\lambda^2 e^{\lambda z} U^m - 4) U''_{zz} + 2\lambda U'_z = 0.$$

3°. For other solutions, see equation 3.4.5.1 with $f(x) = ae^{\lambda x}$.

4. $\dfrac{\partial^2 w}{\partial t^2} = a \dfrac{\partial}{\partial x}\Big(x^n w^m \dfrac{\partial w}{\partial x}\Big), \qquad a > 0.$

This is a special case of equation 3.4.5.2 with $f(x) = ax^n$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w\big(C_2^2 x,\ \pm C_1^m C_2^{2-n} t + C_3\big),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = k x^{\frac{2-n}{m}} (C_1 t + C_2)^{-\frac{2}{m}}, \qquad k = \left[\frac{2C_1^2(m+2)}{a(2-n)(m-n+2)}\right]^{\frac{1}{m}}. \tag{1}$$

3°. Expression (1) is a special case of a wider family of multiplicative separable solutions

$$w = f(x) g(t),$$

where the functions $f = f(x)$ and $g = g(t)$ are determined by solving the equations

$$g''_{tt} - \lambda g^{m+1} = 0, \tag{2}$$

$$a(x^n f^{m+1} f'_x)'_x - \lambda f = 0. \tag{3}$$

The general solution of equation (2) can be written out in implicit form as

$$\int \Big(C_1 + \frac{2\lambda}{m+2} f^{m+2}\Big)^{-1/2} df = C_2 \pm t,$$

where C_1 and C_2 are arbitrary constants.

In particular, with $C_1 = 0$, it follows that

$$f(t) = (At + C)^{-2/m}, \qquad A = \pm\sqrt{\frac{\lambda m^2}{2(m+2)}}.$$

For $n \neq 1$ and $m \neq -1$, the transformation

$$z = x^{1-n}, \quad \varphi = u^{m+1}$$

brings (3) to the Emden–Fowler equation

$$\varphi''_{zz} = \frac{\lambda(m+1)}{a(1-n)^2} z^{\frac{n}{1-n}} \varphi^{\frac{1}{m+1}}. \tag{4}$$

The books by Polyanin and Zaitsev (1995, 2003) give more than 20 exact solutions to equation (4) for specific values of the parameter m.

4°. There is a self-similar solution of the form

$$w = (t+b)^{\frac{(n-2)k-2}{m}} F(y), \quad y = xt^k,$$

where b and k are arbitrary constants.

5°. Suppose $m \neq -1$ and $2m - 2n - nm + 3 \neq 0$. The transformation

$$w(x,t) = x^{\frac{1-n}{m+1}} u(\xi, t), \quad \xi = x^{\frac{2m-2n-nm+3}{m+1}}$$

leads to an equation of the similar form

$$\frac{\partial^2 u}{\partial t^2} = A \frac{\partial}{\partial \xi}\left(\xi^{\frac{3m-3n-2nm+4}{2m-2n-nm+3}} u^m \frac{\partial u}{\partial \xi}\right), \tag{5}$$

where $A = a\left(\dfrac{2m-2n-nm+3}{m+1}\right)^2$.

In the special case $n = \dfrac{3m+4}{2m+3}$, equation (5) is greatly simplified and coincides, up to notation, with equation 3.1.6.5:

$$\frac{\partial^2 u}{\partial t^2} = A \frac{\partial}{\partial \xi}\left(u^m \frac{\partial u}{\partial \xi}\right).$$

5. $\displaystyle \frac{\partial^2 w}{\partial t^2} = k(ax^2 + bx + c)^m w^{4-2m} \frac{\partial^2 w}{\partial x^2}.$

This is a special case of equation 3.4.5.4 with $f(u) = ku^{-2m}$.

1°. The transformation

$$w(x,t) = u(z,t)\sqrt{ax^2+bx+c}, \qquad z = \int \frac{dx}{ax^2+bx+c}$$

leads to an equation of the form 3.4.4.8:

$$\frac{\partial^2 u}{\partial t^2} = ku^{4-2m}\frac{\partial^2 u}{\partial z^2} + k(ac - \tfrac{1}{4}b^2)u^{5-2m},$$

which has a traveling-wave solution $u = u(z+\lambda t)$ and a multiplicative separable solution $u = f(t)g(z)$.

2°. By the transformation

$$w(x,t) = \bigl[v(\xi,t)\bigr]^{\frac{1}{2m+3}}, \qquad \xi = \int \frac{dx}{(ax^2+bx+c)^m} \tag{1}$$

the original equation can be reduced to the divergence form

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial \xi}\left[F(\xi) v^{\frac{4-2m}{2m-3}} \frac{\partial v}{\partial \xi}\right], \tag{2}$$

where the function $F(\xi)$ is defined parametrically by

$$F(\xi) = \frac{k}{(ax^2+bx+c)^m}, \qquad \xi = \int \frac{dx}{(ax^2+bx+c)^m}. \tag{3}$$

Note some special cases of equation (2) where $F = F(\xi)$ of (3) can be represented in explicit form:

$$\frac{\partial^2 v}{\partial t^2} = k\frac{\partial}{\partial \xi}\left(\frac{\cos^2\xi}{v^2}\frac{\partial v}{\partial \xi}\right), \qquad m = 1,\ a = 1,\ \ b = 0,\ c = 1;$$

$$\frac{\partial^2 v}{\partial t^2} = k\frac{\partial}{\partial \xi}\left(\frac{\cosh^2\xi}{v^2}\frac{\partial v}{\partial \xi}\right), \qquad m = 1,\ a = -1,\ b = 0,\ c = 1;$$

$$\frac{\partial^2 v}{\partial t^2} = k\frac{\partial}{\partial \xi}\left(\frac{\xi^{-3/2}}{\cos\xi}\frac{\partial v}{\partial \xi}\right), \qquad m = \tfrac{1}{2},\ a = -1,\ b = 0,\ c = 1.$$

3.2. Equations with Exponential Nonlinearities

3.2.1. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + be^{\beta w} + ce^{\gamma w}$

1. $\dfrac{\partial^2 w}{\partial t^2} = a^2 \dfrac{\partial^2 w}{\partial x^2} + be^{\beta w}$.

This is a special case of equation 3.4.1.1 with $f(w) = be^{\beta w}$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm C_1 x + C_2,\ \pm C_1 t + C_3) + \frac{2}{\beta}\ln|C_1|,$$
$$w_2 = w(x\cosh\lambda + at\sinh\lambda,\ t\cosh\lambda + a^{-1}x\sinh\lambda),$$

where C_1, C_2, C_3, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solutions:

$$w(x,t) = \frac{1}{\beta}\ln\left[\frac{2(B^2 - a^2A^2)}{b\beta(Ax + Bt + C)^2}\right],$$
$$w(x,t) = \frac{1}{\beta}\ln\left[\frac{2(a^2A^2 - B^2)}{b\beta\cosh^2(Ax + Bt + C)}\right],$$
$$w(x,t) = \frac{1}{\beta}\ln\left[\frac{2(B^2 - a^2A^2)}{b\beta\sinh^2(Ax + Bt + C)}\right],$$
$$w(x,t) = \frac{1}{\beta}\ln\left[\frac{2(B^2 - a^2A^2)}{b\beta\cos^2(Ax + Bt + C)}\right],$$

where A, B, and C are arbitrary constants.

3°. Functional separable solutions:

$$w(x,t) = \frac{1}{\beta}\ln\left(\frac{8a^2C}{b\beta}\right) - \frac{2}{\beta}\ln\left|(x + A)^2 - a^2(t + B)^2 + C\right|,$$
$$w(x,t) = -\frac{2}{\beta}\ln\left[C_1e^{\lambda x} \pm \frac{\sqrt{2b\beta}}{2a\lambda}\sinh(a\lambda t + C_2)\right],$$
$$w(x,t) = -\frac{2}{\beta}\ln\left[C_1e^{\lambda x} \pm \frac{\sqrt{-2b\beta}}{2a\lambda}\cosh(a\lambda t + C_2)\right],$$
$$w(x,t) = -\frac{2}{\beta}\ln\left[C_1e^{a\lambda t} \pm \frac{\sqrt{-2b\beta}}{2a\lambda}\sinh(\lambda x + C_2)\right],$$
$$w(x,t) = -\frac{2}{\beta}\ln\left[C_1e^{a\lambda t} \pm \frac{\sqrt{2b\beta}}{2a\lambda}\cosh(\lambda x + C_2)\right],$$

where A, B, C, C_1, C_2, and λ are arbitrary constants.

4°. The change of the independent variables

$$z = x - at, \quad y = x + at$$

leads to the Liouville equation 3.5.1.2:

$$\frac{\partial^2 w}{\partial z\,\partial y} = -\tfrac{1}{4}a^{-2}b\exp(\beta w).$$

Hence, the general solution of the original equation is expressed as

$$w(x,t)=\frac{1}{\beta}\bigl[f(z)+g(y)\bigr]-\frac{2}{\beta}\ln\left|k\int\exp\bigl[f(z)\bigr]\,dz-\frac{b\beta}{8a^2k}\int\exp\bigl[g(y)\bigr]\,dy\right|,$$
$$z=x-at,\qquad y=x+at,$$

where $f=f(z)$ and $g=g(y)$ are arbitrary functions and k is an arbitrary constant.

⊙ *References*: J. Liouville (1853), R. K. Bullough and P. J. Caudrey (1980), V. F. Zaitsev and A. D. Polyanin (1996).

2. $\dfrac{\partial^2 w}{\partial t^2}=\dfrac{\partial^2 w}{\partial x^2}+ae^{\beta w}+be^{2\beta w}.$

1°. Traveling-wave solution for $b\beta>0$:

$$w(x,t)=-\frac{1}{\beta}\ln\left\{-\frac{b}{a}+C_1\exp\left[a\sqrt{\frac{\beta}{b}}\,\bigl(x\sinh C_2\pm t\cosh C_2\bigr)\right]\right\},$$

where C_1 and C_2 are arbitrary constants.

2°. Traveling-wave solution (generalizes the solution of Item 1°):

$$w(x,t)=-\frac{1}{\beta}\ln\left[\frac{a\beta}{C_1^2-C_2^2}+C_3\exp(C_1x+C_2t)+\frac{a^2\beta^2+b\beta(C_1^2-C_2^2)}{4C_3(C_1^2-C_2^2)^2}\exp(-C_1x-C_2t)\right],$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Traveling-wave solution:

$$w(x,t)=-\frac{1}{\beta}\ln\left[\frac{a\beta}{C_2^2-C_1^2}+\frac{\sqrt{a^2\beta^2+b\beta(C_2^2-C_1^2)}}{C_2^2-C_1^2}\sin(C_1x+C_2t+C_3)\right].$$

3. $\dfrac{\partial^2 w}{\partial t^2}=\dfrac{\partial^2 w}{\partial x^2}+ae^{\beta w}-be^{-\beta w}.$

The substitution

$$w(x,t)=u(x,t)+k,\qquad k=\frac{1}{2\beta}\ln\frac{b}{a}$$

leads to an equation of the form 3.3.1.1:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+2\sqrt{ab}\,\sinh(\beta u).$$

4. $\dfrac{\partial^2 w}{\partial t^2}=\dfrac{\partial^2 w}{\partial x^2}+ae^{\beta w}-be^{-2\beta w}.$

1°. Functional separable solution:

$$w(x,t)=\frac{1}{\beta}\ln[\varphi(x)+\psi(t)],$$

where the functions $\psi(t)$ and $\varphi(x)$ are determined by the first-order autonomous ordinary differential equations

$$(\varphi'_x)^2=-2a\beta\varphi^3+C_1\varphi^2-C_2\varphi+C_3-b\beta,$$
$$(\psi'_t)^2=2a\beta\psi^3+C_1\psi^2+C_2\psi+C_3,$$

where C_1, C_2, and C_3 are arbitrary constants. Solving these equations for the derivatives, one obtains separable equations.

⊙ *References*: A. M. Grundland and E. Infeld (1992), R. Z. Zhdanov (1994), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

2°. The transformation

$$t = (a^2 b\beta^3)^{-1/6}(\xi + \eta), \quad x = (a^2 b\beta^3)^{-1/6}(\xi - \eta), \quad w = \frac{1}{\beta}U + \frac{1}{3\beta}\ln\frac{b}{a}$$

leads to an equation of the form 3.5.1.3:

$$\frac{\partial^2 U}{\partial\xi\partial\eta} = e^U - e^{-2U}.$$

3°. The equation can be integrated with the inverse scattering method.

⦿ *References*: A. V. Mikhailov (1979), A. P. Fordy and J. A. Gibbons (1980), F. Calogero and A. Degasperis (1982).

3.2.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + f(x, t, w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + ae^{\beta t}e^{\lambda w}$.

This is a special case of equation 3.4.1.7 with $f(w) = ae^{\lambda w}$.

1°. Solutions:

$$w(x,t) = -\frac{\beta}{\lambda}t - \frac{2}{\lambda}\ln\left(C_1 + C_2 x \pm \sqrt{C_2^2 + \tfrac{1}{2}\lambda a\, t}\right),$$
$$w(x,t) = -\frac{\beta}{\lambda}t - \frac{2}{\lambda}\ln\left(C_1 e^{-\sigma t} + C_2 e^{\sigma x} - \frac{\lambda a}{8\sigma^2 C_1}e^{\sigma t}\right),$$

where C_1, C_2, and σ are arbitrary constants.

2°. The substitution $\lambda U = \lambda w + \beta t$ leads to an equation of the form 3.2.1.1:

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + a\lambda e^{\lambda U}.$$

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + ae^{\beta x}e^{\lambda w}$.

This is a special case of equation 3.4.1.6 with $f(w) = ae^{\lambda w}$.

1°. Solutions:

$$w(x,t) = -\frac{\beta}{\lambda}x - \frac{2}{\lambda}\ln\left(C_1 + C_2 t \pm \sqrt{C_2^2 - \tfrac{1}{2}\lambda a\, x}\right),$$
$$w(x,t) = -\frac{\beta}{\lambda}x - \frac{2}{\lambda}\ln\left(C_1 e^{-\sigma x} + C_2 e^{\sigma t} + \frac{\lambda a}{8\sigma^2 C_1}e^{\sigma x}\right),$$

where C_1, C_2, and σ are arbitrary constants.

2°. The substitution $\lambda U = \lambda w + \beta x$ leads to an equation of the form 3.2.1.1:

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + a\lambda e^{\lambda U}.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + ce^{ax+bt}e^{\lambda w}$.

This is a special case of equation 3.4.1.8 with $f(w) = ce^{\lambda w}$.

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \beta e^{\lambda w} + (\alpha t + \gamma) e^{2\lambda w}.$

Functional separable solutions:

$$w(x,t) = -\frac{1}{\lambda} \ln\left\{ C \exp\left[-\frac{\beta^2\lambda}{2\alpha}(t \pm x)\right] - \frac{1}{\beta}(\alpha t + \gamma) + \frac{\alpha^2}{\beta^3\lambda} \right\},$$

where C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \beta e^{\lambda w} + (\alpha x + \gamma) e^{2\lambda w}.$

Functional separable solutions:

$$w(x,t) = -\frac{1}{\lambda} \ln\left\{ C \exp\left[\frac{\beta^2\lambda}{2\alpha}(x \pm t)\right] - \frac{1}{\beta}(\alpha x + \gamma) - \frac{\alpha^2}{\beta^3\lambda} \right\},$$

where C is an arbitrary constant.

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \beta e^{\lambda w} + (\alpha e^{kt} + \gamma) e^{2\lambda w}.$

1°. Functional separable solutions for $k^2\gamma - \beta^2\lambda \neq 0$:

$$w(x,t) = -\frac{1}{\lambda} \ln\left[C \exp\left(\pm\frac{k^2\gamma - \beta^2\lambda}{2k\gamma} x + \frac{k^2\gamma + \beta^2\lambda}{2k\gamma} t \right) + \frac{\alpha\beta\lambda}{k^2\gamma - \beta^2\lambda} e^{kt} - \frac{\gamma}{\beta} \right],$$

where C is an arbitrary constant.

2°. Generalized traveling-wave solutions for $k^2\gamma - \beta^2\lambda = 0$:

$$w(x,t) = -\frac{1}{\lambda} \ln\left[C e^{kt} + \frac{\alpha k}{2\beta}(t \pm x) e^{kt} - \frac{\lambda\beta}{k^2} \right].$$

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + k\beta e^{\lambda w} + (\alpha e^{kt} + \lambda\beta^2) e^{2\lambda w}.$

Generalized traveling-wave solutions:

$$w(x,t) = -\frac{1}{\lambda} \ln\left[C e^{kt} + \frac{\alpha}{2\beta}(t \pm x) e^{kt} - \frac{\lambda\beta}{k} \right],$$

where C is an arbitrary constant.

8. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \beta e^{\lambda w} + (\alpha e^{kx} + \gamma) e^{2\lambda w}.$

1°. Functional separable solutions for $k^2\gamma + \beta^2\lambda \neq 0$:

$$w(x,t) = -\frac{1}{\lambda} \ln\left[C \exp\left(\pm\frac{k^2\gamma + \beta^2\lambda}{2k\gamma} t + \frac{k^2\gamma - \beta^2\lambda}{2k\gamma} x \right) - \frac{\alpha\beta\lambda}{k^2\gamma + \beta^2\lambda} e^{kx} - \frac{\gamma}{\beta} \right],$$

where C is an arbitrary constant.

2°. Generalized traveling-wave solutions for $k^2\gamma + \beta^2\lambda = 0$:

$$w(x,t) = -\frac{1}{\lambda} \ln\left[C e^{kx} + \frac{\alpha k}{2\beta}(x \pm t) e^{kx} + \frac{\lambda\beta}{k^2} \right].$$

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

9. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} - k\beta e^{\lambda w} - (\alpha e^{kx} + \lambda\beta^2)e^{2\lambda w}.$

Functional separable solutions:

$$w(x,t) = -\frac{1}{\lambda}\ln\left[Ce^{kx} + \frac{\alpha}{2\beta}(x \pm t)e^{kx} - \frac{\lambda\beta}{k}\right],$$

where C is an arbitrary constant.

10. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \beta e^{kt}e^{\lambda w} + (\alpha e^{2kt} + \gamma)e^{2\lambda w}.$

Functional separable solutions:

$$w(x,t) = -\frac{1}{\lambda}\ln\left[C\exp\left(\pm\frac{4k^2\alpha - \beta^2\lambda}{4k\alpha}x - \frac{\beta^2\lambda}{4k\alpha}t\right) + \frac{\beta\gamma\lambda}{4k^2\alpha - \beta^2\lambda}e^{-kt} - \frac{\alpha}{\beta}e^{kt}\right],$$

where C is an arbitrary constant.

11. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \beta e^{kx}e^{\lambda w} + (\alpha e^{2kx} + \gamma)e^{2\lambda w}.$

Functional separable solutions:

$$w(x,t) = -\frac{1}{\lambda}\ln\left[C\exp\left(\pm\frac{4k^2\alpha + \beta^2\lambda}{4k\alpha}t + \frac{\beta^2\lambda}{4k\alpha}x\right) - \frac{\beta\gamma\lambda}{4k^2\alpha + \beta^2\lambda}e^{-kx} - \frac{\alpha}{\beta}e^{kx}\right],$$

where C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

3.2.3. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g\Big(x, t, w, \dfrac{\partial w}{\partial x}\Big)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda w}\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.4.2.3 with $f(w) = be^{\lambda w}$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\big(C_1x + C_2, \pm C_1t + C_3\big) + \frac{1}{\lambda}\ln C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solution:

$$w = -\frac{1}{\lambda}\ln\left[\frac{\exp(Ax + A\mu t + B) - b}{A(a - \mu^2)}\right],$$

where μ, A, and B are arbitrary constants.

3°. There is an exact solution of the form

$$w(x,t) = F(z) - \frac{1}{\lambda}\ln|t|, \qquad z = \frac{x}{t}.$$

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + ce^{\lambda w}, \qquad a > 0.$

This is a special case of equation 3.2.3.5 with $b = an$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\bigl(C_1^{\frac{2}{2-n}} x,\ \pm C_1 t + C_2\bigr) + \frac{2}{\lambda} \ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $n \neq 2$ and $\lambda \neq 0$:

$$w = -\frac{1}{\lambda} \ln\left\{ \frac{2c\lambda(2-n)}{n} \left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2 \right] \right\}.$$

3°. Functional separable solution for $n \neq 2$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4k \left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2 \right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r} w'_r + ck^{-1} e^{\lambda w} = 0, \qquad A = \frac{2}{2-n}.$$

4°. There is an exact solution of the form

$$w(x,t) = F(z) - \frac{2}{\lambda} \ln |t|, \quad z = x|t|^{\frac{2}{n-2}}.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{a}{x^n} \dfrac{\partial}{\partial x}\left(x^n \dfrac{\partial w}{\partial x}\right) + ce^{\lambda w}, \qquad a > 0.$

For $n = 1$ and $n = 2$, the equation describes the propagation of nonlinear waves with axial and central symmetry, respectively.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\bigl(C_1 x,\ \pm C_1 t + C_2\bigr) + \frac{2}{\lambda} \ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $n \neq 0$ and $\lambda \neq 0$:

$$w = -\frac{1}{\lambda} \ln\left\{ \frac{c\lambda}{2an} \left[x^2 - a(t+C)^2 \right] \right\}.$$

3°. Functional separable solution (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = k\left[x^2 - a(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{n+1}{r} w'_r + \frac{c}{ak} e^{\lambda w} = 0.$$

4°. There is an exact solution of the form

$$w(x,t) = F(z) - \frac{2}{\lambda} \ln |t|, \quad z = \frac{x}{t}.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = a(x+\beta)^n \dfrac{\partial^2 w}{\partial x^2} + ce^{\lambda w}, \qquad a > 0.$

This is an equation of the propagation of nonlinear waves in an inhomogeneous medium. The substitution $z = x + \beta$ leads to a special case of equation 3.2.3.5 with $b = 0$:

$$\frac{\partial^2 w}{\partial t^2} = az^n \frac{\partial^2 w}{\partial z^2} + ce^{\lambda w}.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = ax^n \dfrac{\partial^2 w}{\partial x^2} + bx^{n-1}\dfrac{\partial w}{\partial x} + ce^{\lambda w}, \qquad a > 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w\bigl(C_1^2 x,\ \pm C_1^{2-n} t + C_2\bigr) + \frac{4-2n}{\lambda} \ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $n \neq 2$:

$$w = w(\xi), \qquad \xi = \tfrac{1}{4} a(2-n)^2 (t+C)^2 - x^{2-n}.$$

Here, C is an arbitrary constant, and the function $w = w(\xi)$ is determined by the ordinary differential equation

$$\xi w''_{\xi\xi} + A w'_\xi = Be^{\lambda w}, \tag{1}$$

where

$$A = \frac{a(4-3n)+2b}{2a(2-n)}, \quad B = \frac{c}{a(2-n)^2}.$$

For $A \neq 1$, an exact solution of equation (1) is given by

$$w(\xi) = \frac{1}{\lambda} \ln\left(\frac{1-A}{\lambda B \xi}\right).$$

For $A = 1$, which corresponds to $b = \frac{1}{2}an$, exact solutions of equation (1) are expressed as

$$w(\xi) = \frac{1}{\lambda} \ln\left[\frac{2a(2-n)^2}{c\lambda\xi(\ln|\xi|+q)^2}\right],$$
$$w(\xi) = \frac{1}{\lambda} \ln\left[\frac{2ap^2(2-n)^2}{c\lambda\xi \cos^2(p\ln|\xi|+q)}\right],$$
$$w(\xi) = \frac{1}{\lambda} \ln\left[\frac{-2ap^2(2-n)^2}{c\lambda\xi \cosh^2(p\ln|\xi|+q)}\right],$$

where p and q are arbitrary constants.

For $A \neq 1$, the substitution $\xi = kz^{\frac{1}{1-A}}$ $(k = \pm 1)$ brings (1) to the generalized Emden–Fowler equation

$$w''_{zz} = \frac{kB}{(1-A)^2} z^{\frac{2A-1}{1-A}} e^{\lambda w}. \tag{2}$$

In the special case $A = \frac{1}{2}$, which corresponds to $b = a(n-1)$, solutions of equation (2) are given by

$$w(z) = \frac{1}{\lambda} \ln\left[\frac{-a(2-n)^2}{2kc\lambda(z+q)^2}\right],$$
$$w(z) = \frac{1}{\lambda} \ln\left[\frac{ap^2(2-n)^2}{2kc\lambda \cosh^2(pz+q)}\right],$$
$$w(z) = \frac{1}{\lambda} \ln\left[\frac{-ap^2(2-n)^2}{2kc\lambda \cos^2(pz+q)}\right],$$

where p and q are arbitrary constants.

3°. Solution for $n = 2$:

$$w = w(y), \quad y = At + B\ln|x| + C,$$

where A, B, and C are arbitrary constants, and the function $w = w(y)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{yy} + (b-a)Bw'_y + ce^{\lambda w} = 0. \tag{3}$$

Solution of equation (3) with $A = \pm B\sqrt{a}$:

$$w(y) = -\frac{1}{\lambda}\ln\left[\frac{c\lambda}{B(b-a)}y + C_1\right].$$

Solutions of equation (3) with $b = a$:

$$w(y) = \frac{1}{\lambda}\ln\left[\frac{2(A^2 - aB^2)}{c\lambda(y+q)^2}\right],$$

$$w(y) = \frac{1}{\lambda}\ln\left[\frac{2p^2(aB^2 - A^2)}{c\lambda\cosh^2(py+q)}\right],$$

$$w(y) = \frac{1}{\lambda}\ln\left[\frac{2p^2(A^2 - aB^2)}{c\lambda\cos^2(py+q)}\right],$$

where p and q are arbitrary constants.

6. $\displaystyle \frac{\partial^2 w}{\partial t^2} = ax^n\frac{\partial^2 w}{\partial x^2} + bx^{n-1}e^{\lambda w}\frac{\partial w}{\partial x}, \qquad a > 0.$

This is a special case of equation 3.4.3.5 with $f(w) = be^{\lambda w}$.

7. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ae^{\lambda x}\frac{\partial w}{\partial x}\right) + ce^{\mu w}, \qquad a > 0.$

This is a special case of equation 3.2.3.9 with $b = a\lambda$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\Big(x - \frac{2}{\lambda}\ln|C_1|,\ \pm C_1 t + C_2\Big) + \frac{2}{\mu}\ln|C_1|,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $\lambda \neq 0$:

$$w = w(r), \qquad r^2 = 4k\left[\frac{e^{-\lambda x}}{a\lambda^2} - \frac{1}{4}(t+C_1)^2\right],$$

where C_1 and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the autonomous ordinary differential equation

$$w''_{rr} + ck^{-1}e^{\mu w} = 0.$$

Its general solution is expressed as

$$w = \begin{cases} -\dfrac{1}{\mu}\ln\left[-\dfrac{c\mu}{2k}(r+C_3)^2\right] & \text{if } ck\mu < 0, \\ -\dfrac{1}{\mu}\ln\left[-\dfrac{c\mu}{2kC_2^2}\sin^2(C_2 r + C_3)\right] & \text{if } ck\mu < 0, \\ -\dfrac{1}{\mu}\ln\left[-\dfrac{c\mu}{2kC_2^2}\sinh^2(C_2 r + C_3)\right] & \text{if } ck\mu < 0, \\ -\dfrac{1}{\mu}\ln\left[\dfrac{c\mu}{2kC_2^2}\cosh^2(C_2 r + C_3)\right] & \text{if } ck\mu > 0, \end{cases}$$

where C_2 and C_3 are arbitrary constants.

3°. There is an exact solution of the form

$$w(x,t) = F(z) - \frac{2}{\mu}\ln|t|, \quad z = x + \frac{2}{\lambda}\ln|t|.$$

8. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + ce^{\mu w}, \qquad a > 0.$

This is an equation of the propagation of nonlinear waves in an inhomogeneous medium. This is a special case of equation 3.2.3.9 with $b = 0$.

9. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda x}\dfrac{\partial w}{\partial x} + ce^{\mu w}, \qquad a > 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w\Big(x - \frac{2}{\lambda}\ln|C_1|,\ \pm C_1 t + C_2\Big) + \frac{2}{\mu}\ln|C_1|,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution:

$$w = w(z), \quad z = \big[4ke^{-\lambda x} - ak\lambda^2(t+C)^2\big]^{1/2}, \qquad k = \pm 1,$$

where C is an arbitrary constant and the function $w = w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2(a\lambda - b)}{a\lambda}\frac{1}{z}w'_z + \frac{c}{ak\lambda^2}e^{\mu w} = 0. \tag{1}$$

A solution of equation (1) has the form

$$w(z) = \frac{1}{\mu}\ln\left[\frac{2k\lambda(a\lambda - 2b)}{c\mu z^2}\right].$$

Note some other exact solutions of equation (1):

$$\begin{aligned}
w(z) &= \frac{1}{\mu}\ln\left[\frac{-2ak\lambda^2}{c\mu(z+B)^2}\right] && \text{if } b = a\lambda,\\
w(z) &= \frac{1}{\mu}\ln\left[\frac{2aA^2k\lambda^2}{c\mu\cosh^2(Az+B)}\right] && \text{if } b = a\lambda,\\
w(z) &= \frac{1}{\mu}\ln\left[\frac{-2aA^2k\lambda^2}{c\mu\sinh^2(Az+B)}\right] && \text{if } b = a\lambda,\\
w(z) &= \frac{1}{\mu}\ln\left[\frac{-2aA^2k\lambda^2}{c\mu\cos^2(Az+B)}\right] && \text{if } b = a\lambda,\\
w(z) &= \frac{1}{\mu}\ln\left[\frac{8ABak\lambda^2}{c\mu(Az^2+B)^2}\right] && \text{if } b = \tfrac{1}{2}a\lambda,
\end{aligned}$$

where A and B are arbitrary constants.

10. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda x + \mu w}\dfrac{\partial w}{\partial x}, \qquad a > 0.$

This is a special case of equation 3.4.3.10 with $f(w) = be^{\mu w}$.

11. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda x + \mu w}\dfrac{\partial w}{\partial x} + ce^{\beta w}, \qquad a > 0.$

This is a special case of equation 3.4.3.11 with $f(w) = be^{\mu w}$, and $g(w) = ce^{\beta w}$.

3.2.4. Other Equations

1. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda w}\dfrac{\partial^2 w}{\partial x^2}, \qquad a > 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(C_1C_2^\lambda x + C_3, \pm C_1 t + C_4,) - 2\ln|C_2|,$$

where $C_1, \dots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,t) = Axt + Bx + Ct + D,$$

$$w(x,t) = \frac{1}{\lambda}\ln\left[\frac{B^2}{a}\frac{(x+A)^2}{\cosh^2(Bt+C)}\right], \qquad w(x,t) = \frac{1}{\lambda}\ln\left[\frac{1}{aA^2}\frac{\cosh^2(Ax+B)}{(t+C)^2}\right],$$

$$w(x,t) = \frac{1}{\lambda}\ln\left[\frac{C^2}{aA^2}\frac{\sinh^2(Ax+B)}{\cosh^2(Ct+D)}\right], \qquad w(x,t) = \frac{1}{\lambda}\ln\left[\frac{C^2}{aA^2}\frac{\cosh^2(Ax+B)}{\sinh^2(Ct+D)}\right],$$

$$w(x,t) = \frac{1}{\lambda}\ln\left[\frac{C^2}{aA^2}\frac{\cos^2(Ax+B)}{\cosh^2(Ct+D)}\right], \qquad w(x,t) = \frac{1}{\lambda}\ln\left[\frac{C^2}{aA^2}\frac{\cosh^2(Ax+B)}{\cos^2(Ct+D)}\right],$$

$$w(x,t) = \frac{1}{\lambda}\ln\left[\frac{4BC\beta^2(x+A)^2}{a(Be^{\beta t}+Ce^{-\beta t})^2}\right], \qquad w(x,t) = \frac{1}{\lambda}\ln\left[\frac{(Ae^{\beta x}+Be^{-\beta x})^2}{4aAB\beta^2(t+C)^2}\right],$$

where A, B, C, D, and β are arbitrary constants. The first solution is degenerate, while the others are representable as the sum of functions with different arguments.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

3°. Self-similar solution:

$$w = w(z), \qquad z = \frac{x+A}{t+B},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$(ae^{\lambda w} - z^2)w''_{zz} - zw'_z = 0,$$

whose order can be reduced with the transformation $\xi = z^{-2}e^{\lambda w}$, $U(\xi) = zw'_z$.

4°. Solution:

$$w = \frac{2(k-1)}{\lambda}\ln(t+C_1) + f(\zeta), \qquad \zeta = \frac{x+C_2}{(t+C_1)^k},$$

where C_1, C_2, and k are arbitrary constants, and the function $f = f(\zeta)$ is determined by the ordinary differential equation

$$k^2\zeta^2 f''_{\zeta\zeta} + k(k+1)\zeta f'_\zeta - \frac{2(k-1)}{\lambda} = ae^{\lambda f} f''_{\zeta\zeta}.$$

5°. There are exact solutions of the following forms:

$$w(x,t) = F(\eta) - \frac{2}{\lambda}\ln|t|, \qquad \eta = x + k\ln|t|;$$

$$w(x,t) = H(\rho) - \frac{2}{\lambda}t, \qquad \rho = xe^t,$$

where k is an arbitrary constant.

2. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda w}\dfrac{\partial^2 w}{\partial x^2} + be^{\beta w}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\beta-\lambda}x + C_2, \pm C_1^\beta t + C_3) + 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution:

$$w = u(z), \qquad z = k_2 x + k_1 t,$$

where k_1 and k_2 are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$(k_1^2 - a k_2^2 e^{\lambda u}) u''_{zz} = b e^{\beta u}.$$

Its solution can be written out in implicit form as

$$\int \frac{du}{\sqrt{F(u)}} = C_1 \pm z, \qquad F(u) = 2b \int \frac{e^{\beta u}\, du}{k_1^2 - a k_2^2 e^{\lambda u}} + C_2,$$

where C_1 and C_2 are arbitrary constants.

3°. Solution:

$$w = U(\xi) - \frac{2}{\beta} \ln|t|, \qquad \xi = x|t|^{\frac{\lambda-\beta}{\beta}},$$

where the function $U(\xi)$ is determined by the ordinary differential equation

$$\frac{2}{\beta} + \frac{(\lambda-\beta)(\lambda-2\beta)}{\beta^2} \xi U'_\xi + \frac{(\lambda-\beta)^2}{\beta^2} \xi^2 U''_{\xi\xi} = a e^{\lambda U} U''_{\xi\xi} + b e^{\beta U}.$$

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(a e^{\lambda w} \frac{\partial w}{\partial x}\right), \qquad a > 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(C_1 C_2^\lambda x + C_3, \pm C_1 t + C_4,) - 2\ln|C_2|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Additive separable solutions:

$$w(x,t) = \frac{1}{\lambda} \ln|Ax + B| + Ct + D, \tag{1}$$

$$w(x,t) = \frac{2}{\lambda} \ln|Ax + B| - \frac{2}{\lambda} \ln|\pm A\sqrt{a}\, t + C|, \tag{2}$$

$$w(x,t) = \frac{1}{\lambda} \ln(aA^2 x^2 + Bx + C) - \frac{2}{\lambda} \ln(aAt + D), \tag{3}$$

$$w(x,t) = \frac{1}{\lambda} \ln(Ax^2 + Bx + C) + \frac{1}{\lambda} \ln\left[\frac{p^2}{aA\cos^2(pt+q)}\right], \tag{4}$$

$$w(x,t) = \frac{1}{\lambda} \ln(Ax^2 + Bx + C) + \frac{1}{\lambda} \ln\left[\frac{p^2}{aA\sinh^2(pt+q)}\right], \tag{5}$$

$$w(x,t) = \frac{1}{\lambda} \ln(Ax^2 + Bx + C) + \frac{1}{\lambda} \ln\left[\frac{-p^2}{aA\cosh^2(pt+q)}\right], \tag{6}$$

where A, B, C, D, p, and q are arbitrary constants. Expressions (1) to (6) exhaust all solutions that can be representable in the form of the sum of functions with different arguments.

3°. Traveling-wave solution:

$$w = w(z), \qquad z = x \pm \mu t,$$

where $w = w(z)$ is defined implicitly by (A and B are arbitrary constants)

$$\lambda \mu^2 w - a e^{\lambda w} = Az + B.$$

4°. Self-similar solution:

$$w = u(\xi), \qquad \xi = \frac{x + A}{t + B}.$$

Here, A and B are arbitrary constants, and the function $u = u(\xi)$ is determined by the ordinary differential equation

$$(\xi^2 u'_\xi)'_\xi = (ae^{\lambda u} u'_\xi)'_\xi,$$

which admits the first integral

$$\bigl(\xi^2 - ae^{\lambda u}\bigr) u'_\xi = C. \tag{7}$$

To the special case $C = 0$ there corresponds a solution of the form (2). For $C \neq 0$, treating u in (7) as the independent variable, one obtains a Riccati equation for $\xi = \xi(u)$,

$$C\xi'_u = \xi^2 - ae^{\lambda u},$$

which is considered in the book by Polyanin and Zaitsev (2003).

5°. Solution:

$$w = \frac{2(k-1)}{\lambda} \ln(t + C_1) + f(\zeta), \qquad \zeta = \frac{x + C_2}{(t + C_1)^k},$$

where C_1, C_2, and k are arbitrary constants, and the function $f = f(\zeta)$ is determined by the ordinary differential equation

$$k^2 \zeta^2 f''_{\zeta\zeta} + k(k+1)\zeta f'_\zeta - \frac{2(k-1)}{\lambda} = a(e^{\lambda f} f'_\zeta)'_\zeta.$$

6°. There are exact solutions of the following forms:

$$w(x,t) = F(\eta) - \frac{2}{\lambda} \ln|t|, \qquad \eta = x + k \ln|t|;$$
$$w(x,t) = H(\zeta) - \frac{2}{\lambda} t, \qquad \eta = xe^t,$$

where k is an arbitrary constant.

7°. For other solutions, see equation 3.4.4.6 with $f(w) = ae^{\lambda w}$.

⊙ *References*: W. F. Ames, R. J. Lohner, and E. Adams (1981), A. D. Polyanin and V. F. Zaitsev (2002).

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ae^{\lambda w} \frac{\partial w}{\partial x}\right) + be^{\beta w}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\beta-\lambda} x + C_2, \pm C_1^{\beta} t + C_3) + 2 \ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution:

$$w = u(z), \qquad z = k_2 x + k_1 t,$$

where k_1 and k_2 are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$k_1^2 u''_{zz} - ak_2^2 (e^{\lambda u} u'_z)'_z = be^{\beta u}.$$

The substitution $\Theta(u) = (u'_z)^2$ leads to the first-order linear equation

$$(k_1^2 - ak_2^2 e^{\lambda u})\Theta'_u - 2ak_2^2 \lambda e^{\lambda u} \Theta = 2be^{\beta u}.$$

3°. Solution:

$$w = U(\xi) - \frac{2}{\beta} \ln|t|, \qquad \xi = x|t|^{\frac{\lambda-\beta}{\beta}},$$

where the function $U(\xi)$ is determined by the ordinary differential equation

$$\frac{2}{\beta} + \frac{(\lambda-\beta)(\lambda-2\beta)}{\beta^2} \xi U'_\xi + \frac{(\lambda-\beta)^2}{\beta^2} \xi^2 U''_{\xi\xi} = (ae^{\lambda U} U'_\xi)'_\xi + be^{\beta U}.$$

5. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(e^{\lambda w}\frac{\partial w}{\partial x}\right) + b - ce^{-2\lambda w}.$

Functional separable solution:

$$w = \frac{1}{\lambda}\ln\left(\sqrt{c\lambda}\,t - \frac{b\lambda}{2a}x^2 + C_1 x + C_2\right),$$

where C_1 and C_2 are arbitrary constants.

6. $\displaystyle \frac{\partial^2 w}{\partial t^2} = ae^{\lambda x+\mu t+\beta w}\frac{\partial^2 w}{\partial x^2}, \qquad a > 0.$

The substitution $\beta u = \lambda x + \mu t + \beta w$ leads to an equation of the form 3.2.4.1:

$$\frac{\partial^2 u}{\partial t^2} = ae^{\beta u}\frac{\partial^2 u}{\partial x^2}.$$

3.3. Other Equations Involving Arbitrary Parameters

3.3.1. Equations with Hyperbolic Nonlinearities

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + b\sinh(\lambda w).$

Sinh-Gordon equation. It arises in some areas of physics. This is a special case of equation 3.4.1.1 with $f(w) = b\sinh(\lambda w)$.

1°. Traveling-wave solutions:

$$w(x,t) = \pm\frac{2}{\lambda}\ln\left[\tan\frac{b\lambda(kx+\mu t+\theta_0)}{2\sqrt{b\lambda(\mu^2 - ak^2)}}\right],$$

$$w(x,t) = \pm\frac{4}{\lambda}\operatorname{arctanh}\left[\exp\frac{b\lambda(kx+\mu t+\theta_0)}{\sqrt{b\lambda(\mu^2 - ak^2)}}\right],$$

where k, μ, and θ_0 are arbitrary constants. It is assumed that $b\lambda(\mu^2 - ak^2) > 0$ in both formulas.

2°. Functional separable solution:

$$w(x,t) = \frac{4}{\lambda}\operatorname{arctanh}\bigl[f(t)g(x)\bigr], \qquad \operatorname{arctanh} z = \frac{1}{2}\ln\frac{1+z}{1-z},$$

where the functions $f = f(t)$ and $g = g(x)$ are determined by the first-order autonomous ordinary differential equations

$$\bigl(f'_t\bigr)^2 = Af^4 + Bf^2 + C,$$

$$a\bigl(g'_x\bigr)^2 = Cg^4 + (B - b\lambda)g^2 + A,$$

where A, B, and C are arbitrary constants.

3°. For other exact solutions of this equation, see equation 3.4.1.1 with $f(w) = b\sinh(\lambda w)$, Item 2°.

⊙ *References*: A. M. Grundland and E. Infeld (1992), R. Z. Zhdanov (1994), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + a\sinh(\beta w) + b\sinh(2\beta w).$

Double sinh-Gordon equation. Denote: $k = \dfrac{a}{2b}$.

Traveling-wave solutions:

$$w = \pm\frac{1}{\beta}\operatorname{arccosh}\frac{1 - k\sin z}{\sin z - k}, \qquad z = \sqrt{2b\beta(1-k^2)}\,(x\sinh C_1 \pm t\cosh C_1 + C_2) \quad \text{if } |k| < 1;$$

$$w = \pm\frac{2}{\beta}\operatorname{arctanh}\left(\sqrt{\frac{k+1}{k-1}}\,\tanh\frac{\xi}{2}\right), \quad \xi = \sqrt{2b\beta(k^2-1)}\,(x\sinh C_1 \pm t\cosh C_1 + C_2) \quad \text{if } |k| > 1,$$

where C_1 and C_2 are arbitrary constants.

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta t} \sinh^k(\lambda w).$

This is a special case of equation 3.4.1.7 with $f(w) = b\sinh^k(\lambda w)$. Hence, for $k = 1$, this equation is reduced to a simpler equation of 3.3.1.1.

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta x} \sinh^k(\lambda w).$

This is a special case of equation 3.4.1.6 with $f(w) = b\sinh^k(\lambda w)$. Hence, for $k = 1$, this equation is reduced to a simpler equation of 3.3.1.1.

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n \dfrac{\partial w}{\partial x}\right) + k\sinh(\lambda w).$

This is a special case of equation 3.4.2.1 with $f(w) = k\sinh(\lambda w)$ and $n = b/a$.

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[a\cosh(\lambda w)\dfrac{\partial w}{\partial x}\right], \qquad a > 0.$

This is a special case of equation 3.4.4.6 with $f(w) = a\cosh(\lambda w)$.

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[a\sinh(\lambda w)\dfrac{\partial w}{\partial x}\right], \qquad a > 0.$

This is a special case of equation 3.4.4.6 with $f(w) = a\sinh(\lambda w)$.

3.3.2. Equations with Logarithmic Nonlinearities

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bw\ln w + kw.$

This is a special case of equation 3.4.1.1 with $f(w) = bw\ln w + kw$.

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the autonomous ordinary differential equations

$$\begin{aligned}\varphi''_{tt} - b\varphi\ln\varphi - k\varphi &= 0,\\ a\psi''_{xx} + b\psi\ln\psi &= 0,\end{aligned}$$

whose general solutions can be represented in implicit form.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bw\ln w + \big(cx^k + st^n\big)w.$

This is a special case of equation 3.4.1.10 with $f(x) = cx^k$ and $g(t) = st^n$.

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bw^k\ln w.$

This is a special case of equation 3.4.1.1 with $f(w) = bw^k\ln w$. For $k = 1$, see also equation 3.4.1.9 with $f(t) = 0$.

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + a(x^2 - t^2)\ln^k(\lambda w).$

This is a special case of equation 3.4.1.2 with $f(w) = a\ln^k(\lambda w)$.

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta t} w \ln w.$

The transformation

$$w = U(z,\tau), \quad z = \tfrac{1}{2}\beta \exp\bigl(\tfrac{1}{2}\beta t\bigr)\sinh\bigl(\tfrac{1}{2}\beta x\bigr), \quad \tau = \tfrac{1}{2}\beta \exp\bigl(\tfrac{1}{2}\beta t\bigr)\cosh\bigl(\tfrac{1}{2}\beta x\bigr)$$

leads to a simpler equation of the form 3.3.2.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + bU\ln U.$$

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta x} w \ln w.$

The transformation

$$w = U(z,\tau), \quad z = \tfrac{1}{2}\exp\bigl(\tfrac{1}{2}\beta x\bigr)\cosh\bigl(\tfrac{1}{2}\beta t\bigr), \quad \tau = \tfrac{1}{2}\exp\bigl(\tfrac{1}{2}\beta x\bigr)\sinh\bigl(\tfrac{1}{2}\beta t\bigr)$$

leads to a simpler equation of the form 3.3.2.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + bU\ln U.$$

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n \dfrac{\partial w}{\partial x}\right) + cw^k \ln w.$

This is a special case of equation 3.4.2.1 with $f(w) = cw^k \ln w$ and $b = an$.

8. $\dfrac{\partial^2 w}{\partial t^2} = ax^n \dfrac{\partial^2 w}{\partial x^2} + b\ln^k(\lambda w).$

This is a special case of equation 3.4.3.2 with $\beta = 0$ and $f(w) = b\ln^k(\lambda w)$.

9. $\dfrac{\partial^2 w}{\partial t^2} = ax^n \dfrac{\partial^2 w}{\partial x^2} + bx^{n-1}\ln^k(\lambda w)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 3.4.3.5 with $f(w) = b\ln^k(\lambda w)$.

10. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left[\ln^k(\lambda w)\dfrac{\partial w}{\partial x}\right].$

This is a special case of equation 3.4.4.6 with $f(w) = a\ln^k(\lambda w)$.

3.3.3. Sine Gordon Equation and Other Equations with Trigonometric Nonlinearities

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\sin(\lambda w).$

Sine-Gordon equation. It arises in differential geometry and various areas of physics (superconductivity, dislocations in crystals, waves in ferromagnetic materials, laser pulses in two-phase media, and others).

1°. Suppose $w = \varphi(x,t)$ is a solution of the sine-Gordon equation. Then the functions

$$w_1 = \frac{2\pi n}{\lambda} \pm \varphi(C_1 \pm x, C_2 \pm t), \qquad n = 0, \pm1, \pm2, \dots;$$

$$w_2 = \pm\varphi\left(x\cosh\sigma + t\sqrt{a}\,\sinh\sigma,\; x\frac{\sinh\sigma}{\sqrt{a}} + t\cosh\sigma\right),$$

where C_1, C_2, and σ are arbitrary constants, are also solutions of the equation. The plus or minus signs in the first expression are chosen in any sequence.

2°. Traveling-wave solutions:

$$w(x,t) = \frac{4}{\lambda}\arctan\left\{\exp\left[\pm\frac{b\lambda(kx+\mu t+\theta_0)}{\sqrt{b\lambda(\mu^2-ak^2)}}\right]\right\} \qquad \text{if}\quad b\lambda(\mu^2-ak^2)>0,$$

$$w(x,t) = -\frac{\pi}{\lambda}+\frac{4}{\lambda}\arctan\left\{\exp\left[\pm\frac{b\lambda(kx+\mu t+\theta_0)}{\sqrt{b\lambda(ak^2-\mu^2)}}\right]\right\} \quad \text{if}\quad b\lambda(\mu^2-ak^2)<0,$$

where k, μ, and θ_0 are arbitrary constants. The first expression corresponds to a single-soliton solution.

3°. Functional separable solution:

$$w(x,t) = \frac{4}{\lambda}\arctan\bigl[f(x)g(t)\bigr], \tag{1}$$

where the functions $f=f(x)$ and $g=g(t)$ are determined by the first-order autonomous separable ordinary differential equations

$$\begin{aligned}\bigl(f'_x\bigr)^2 &= Af^4+Bf^2+C,\\ \bigl(g'_t\bigr)^2 &= -aCg^4+(aB+b\lambda)g^2-aA,\end{aligned} \tag{2}$$

where A, B, and C are arbitrary constants. Note some exact solutions that follow from (1) and (2).

3.1. For $A=0$, $B=k^2>0$, and $C>0$, we have

$$w(x,t) = \frac{4}{\lambda}\arctan\left[\frac{\mu\sinh(kx+A_1)}{k\sqrt{a}\cosh(\mu t+B_1)}\right], \quad \mu^2=ak^2+b\lambda>0, \tag{3}$$

where k, A_1, and B_1 are arbitrary constants. Formula (3) corresponds to the two-soliton solution of Perring–Skyrme (1962).

3.2. For $A=0$, $B=-k^2<0$, and $C>0$,

$$w(x,t) = \frac{4}{\lambda}\arctan\left[\frac{\mu\sin(kx+A_1)}{k\sqrt{a}\cosh(\mu t+B_1)}\right], \quad \mu^2=b\lambda-ak^2>0,$$

where k, A_1, and B_1 are arbitrary constants.

3.3. For $A=k^2>0$, $B=k^2\gamma^2>0$, and $C=0$,

$$w(x,t) = \frac{4}{\lambda}\arctan\left[\frac{\gamma}{\mu}\,\frac{e^{\mu(t+A_1)}+ak^2e^{-\mu(t+A_1)}}{e^{k\gamma(x+B_1)}+e^{-k\gamma(x+B_1)}}\right], \quad \mu^2=ak^2\gamma^2+b\lambda>0,$$

where k, A_1, B_1, and γ are arbitrary constants.

⊙ *Reference*: R. Steuerwald (1936), G. L. Lamb (1980), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984).

4°. An N-soliton solution is given by ($a=1$, $b=-1$, and $\lambda=1$)

$$w(x,t) = \arccos\left[1-2\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2}\right)(\ln F)\right],$$

$$F=\det\bigl[M_{ij}\bigr], \quad M_{ij}=\frac{2}{a_i+a_j}\cosh\left(\frac{z_i+z_j}{2}\right),$$

$$z_i=\pm\frac{x-\mu_i t+C_i}{\sqrt{1-\mu_i^2}}, \quad a_i=\pm\sqrt{\frac{1-\mu_i}{1+\mu_i}},$$

where μ_i and C_i are arbitrary constants.

⊙ *Reference*: R. K. Bullough and P. J. Caudrey (1980), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984).

5°. For other exact solutions of the original equation, see equation 3.4.1.1 with $f(w) = b\sin(\lambda w)$, Item 3°.

6°. The sine-Gordon equation is integrated by the inverse scattering method; see the book by Novikov, Manakov, Pitaevskii, and Zakharov (1984). Belokolos (1995) obtained a general formula for the solution of the sine-Gordon equation with arbitrary initial and boundary conditions.

7°. The transformation

$$z = x - at, \quad y = x + at$$

leads to an equation of the form 3.5.1.5: $\partial_{zy} w = -\frac{1}{4}a^{-2}\sin w$.

⊙ *References for equation* 3.3.3.1: R. Steuerwald (1936), M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur (1973), V. E. Zakharov, L. A. Takhtajan, and L. D. Faddeev (1974), G. B. Whitham (1974), I. M. Krichever (1980), R. K. Bullough and P. J. Caudrey (1980), M. J. Ablowitz and H. Segur (1981), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), J. Weiss (1984), M. J. Ablowitz and P. A. Clarkson (1991).

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + b\sin(\lambda w) + c\sin\left(\tfrac{1}{2}\lambda w\right).$

Double sine-Gordon equation. It arises in nonlinear optics (propagation of ultrashort pulses in a resonance degenerate medium) and low temperature physics (propagation of spin waves in anisotropic spin liquids).

1°. Traveling-wave solutions:

$$w(x,t) = \frac{4}{\lambda}\arctan\left[\frac{\sqrt{4b^2 - c^2}}{2b - c}\tanh\frac{\lambda\sqrt{4b^2 - c^2}(kx + \mu t + \theta_0)}{4\sqrt{b\lambda(ak^2 - \mu^2)}}\right] \quad \text{if} \quad c^2 < 4b^2,$$

$$w(x,t) = \frac{4}{\lambda}\arctan\left[\frac{\sqrt{c^2 - 4b^2}}{c - 2b}\tan\frac{\lambda\sqrt{c^2 - 4b^2}(kx + \mu t + \theta_0)}{4\sqrt{b\lambda(ak^2 - \mu^2)}}\right] \quad \text{if} \quad c^2 > 4b^2.$$

Here, k, μ, and θ_0 are arbitrary constants. It is assumed that $b\lambda(ak^2 - \mu^2) > 0$ in both formulas.

2°. Traveling-wave solutions:

$$w(x,t) = A + \frac{4}{\lambda}\arctan\left(B_1 e^{\theta} + C_1\right) + \frac{4}{\lambda}\arctan\left(B_2 e^{\theta} + C_2\right), \quad \theta = \mu t \pm kx + \theta_0,$$

where the parameters A, B_1, B_2, C_1, C_2, μ, and k are related by algebraic constraints with the parameters a, b, c, and λ of the original equation; θ_0 is an arbitrary constant.

Note some special cases of interest that arise in applications.

2.1. For $a = 1$, $b = -1$, $c = -\frac{1}{2}$, $\lambda = 1$:

$$w(x,t) = 4\arctan\left(e^{\theta-\Delta}\right) + 4\arctan\left(e^{\theta+\Delta}\right); \qquad \Delta = \ln\left(\sqrt{5} + 2\right), \quad k = \mu + \tfrac{5}{4}\mu^{-1}.$$

2.2. For $a = 1$, $b = -1$, $c = -\frac{1}{2}$, $\lambda = 1$:

$$w(x,t) = 2\pi + 4\arctan\left(e^{\theta-\Delta}\right) - 4\arctan\left(e^{\theta+\Delta}\right); \qquad \Delta = \ln\left(\sqrt{3} + 2\right), \quad k = \mu + \tfrac{3}{4}\mu^{-1}.$$

2.3. For $a = 1$, $b = 1$, $c = \frac{1}{2}$, $\lambda = 1$:

$$w(x,t) = \delta - 2\pi + 4\arctan\left(\frac{4}{\sqrt{15}}e^{\theta} + \frac{1}{\sqrt{15}}\right); \qquad \delta \text{ is any}, \quad k = \mu + \tfrac{15}{16}\mu^{-1}.$$

2.4. For $a = 1$, $b = 1$, $c = \frac{1}{2}$, $\lambda = 1$:

$$w(x,t) = 2\pi - \delta + 4\arctan\left(\frac{4}{\sqrt{15}}e^{\theta} - \frac{1}{\sqrt{15}}\right); \qquad \delta \text{ is any}, \quad k = \mu + \tfrac{15}{16}\mu^{-1}.$$

⊙ *References*: R. K. Bullough and P. J. Caudrey (1980), F. Calogero and A. Degasperis (1982).

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\cos(\lambda w).$

The substitution $w = u + \dfrac{\pi}{2\lambda}$ leads to an equation of the form 3.3.3.1:

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^2 u}{\partial x^2} - b\sin(\lambda u).$$

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta t}\sin^k(\lambda w).$

This is a special case of equation 3.4.1.7 with $f(w) = b\sin^k(\lambda w)$. Therefore, for $k = 1$, the equation is reduced to a simpler equation of 3.3.3.1.

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta x}\sin^k(\lambda w).$

This is a special case of equation 3.4.1.6 with $f(w) = b\sin^k(\lambda w)$. Therefore, for $k = 1$, the equation is reduced to a simpler equation of 3.3.3.1.

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta t}\cos^k(\lambda w).$

This is a special case of equation 3.4.1.7 with $f(w) = b\cos^k(\lambda w)$. Therefore, for $k = 1$, the equation is reduced to a simpler equation of 3.3.3.3.

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + be^{\beta x}\cos^k(\lambda w).$

This is a special case of equation 3.4.1.6 with $f(w) = b\cos^k(\lambda w)$. Therefore, for $k = 1$, the equation reduced to a simpler equation of 3.3.3.3.

8. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n\dfrac{\partial w}{\partial x}\right) + k\sin(\lambda w).$

This is a special case of equation 3.4.2.1 with $f(w) = k\sin(\lambda w)$ and $b = an$.

9. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[a\cos^n(\lambda w)\dfrac{\partial w}{\partial x}\right], \qquad a > 0.$

This is a special case of equation 3.4.4.6 with $f(w) = a\cos^n(\lambda w)$.

10. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[a\sin^n(\lambda w)\dfrac{\partial w}{\partial x}\right], \qquad a > 0.$

This is a special case of equation 3.4.4.6 with $f(w) = a\sin^n(\lambda w)$.

3.3.4. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} + a\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right]$

1. $\dfrac{\partial^2 w}{\partial t^2} + \dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^{-2}\dfrac{\partial w}{\partial x}\right).$

The transformation

$$\tau = t + \ln|w|, \quad dz = aw^{-2}w_x dt + (w + w_t)dx, \quad u = 1/w \qquad (dz = z_t\,dt + z_x\,dx),$$

where the subscripts denote the corresponding partial derivatives, leads to the linear telegraph equation

$$\frac{\partial^2 u}{\partial \tau^2} + \frac{\partial u}{\partial \tau} = a\frac{\partial^2 u}{\partial z^2}.$$

⊙ *References*: C. Rogers and T. Ruggeri (1985), C. Rogers and W. F. Ames (1989).

2. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[a(w+b)^n \dfrac{\partial w}{\partial x}\right].$

1°. Solution for $n \neq -1$:

$$w(x,t) = (x + C_2)^{1/(1+n)}(C_1 e^{-kt} + C_2) - b,$$

where C_1 and C_2 are arbitrary constants.

2°. Solution:

$$w(x,t) = (x + C)^{2/n} u(t) - b,$$

where C is an arbitrary constant, and the function $u = u(t)$ is determined by the ordinary differential equation

$$u''_{tt} + k u'_t = \frac{2a(n+2)}{n^2} u^{n+1}.$$

This equation is easy to integrate for $n = -2$ and $n = -1$. For $n = -3/2$ and -3, its exact solutions are given in the handbook by Polyanin and Zaitsev (2003).

3°. Traveling-wave solution in implicit form:

$$\int_0^{w+b} \frac{bu^n - \lambda^2}{k\lambda u + C_1}\, dw = x + \lambda t + C_2,$$

where C_1, C_2, and λ are arbitrary constants.

4°. Solution for $n = -1$:

$$w(x,t) = \frac{2at + C_1 e^{-kt} + C_2}{k(x + C_3)^2} - b,$$

where C_1, C_2, and C_3 are arbitrary constants.

5°. Generalized separable solution for $n = 1$:

$$w(x,t) = f(t)x^2 + g(t)x + h(t) - b,$$

where the functions $f(t)$, $g(t)$, and $h(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned} f''_{tt} + k f'_t &= 6af^2, \\ g''_{tt} + k g'_t &= 6afg, \\ h''_{tt} + k h'_t &= 2afh + ag^2. \end{aligned}$$

⊙ *References*: N. H. Ibragimov (1994), A. D. Polyanin and V. F. Zaitsev (2002).

3. $\dfrac{\partial^2 w}{\partial t^2} + a\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left(be^{\lambda w} \dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\bigl(C_1 x + C_2,\ t + C_3\bigr) - \frac{2}{\lambda} \ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \frac{1}{\lambda} \ln(C_1 x + C_2) + C_3 e^{-at} + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

3°. Additive separable solution:

$$w(x,t) = \frac{1}{\lambda}\ln(\lambda C_1 x^2 + C_2 x + C_3) + u(t),$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $u = u(t)$ is determined by the ordinary differential equation

$$u''_{tt} + au'_t = 2bC_1 e^{\lambda u}.$$

4°. Traveling-wave solution in implicit form:

$$\int \frac{be^{\lambda w} - \lambda^2}{a\lambda w + C_1}\,dw = x + \lambda t + C_2,$$

where C_1, C_2, and λ are arbitrary constants.

⦿ *References*: N. H. Ibragimov (1994), A. D. Polyanin and V. F. Zaitsev (2002).

3.3.5. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} + f(w)\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\Big[g(w)\dfrac{\partial w}{\partial x}\Big]$

► *Equations of this form admit traveling-wave solutions* $w = w(kx + \lambda t)$*; to* $k = 0$ *there corresponds a homogeneous solution dependent on* t *alone and to* $\lambda = 0$*, a stationary solution dependent on* x *alone. For* $g(w) = \text{const}$*, such equations are encountered in the theory of electric field and nonlinear Ohm laws, where* w *is the electric field strength.*

1. $\dfrac{\partial^2 w}{\partial t^2} + w^n \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w(\pm C_1^n x + C_2,\ C_1^n t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solutions in implicit form:

$$\int \frac{(n+1)dw}{w^{n+1} + C_1} = -\frac{a(x - at)}{1 - a^2} + C_2 \quad \text{if} \quad n \neq -1,$$
$$\int \frac{dw}{\ln|w| + C_1} = -\frac{a(x - at)}{1 - a^2} + C_2 \quad \text{if} \quad n = -1,$$

where C_1, C_2, and a are arbitrary constants ($a \neq 0, \pm 1$).

Example 1. Traveling-wave solution for $n \neq 0, -1$:

$$w = \left(\frac{n}{n+1}\xi + C\right)^{-1/n}, \quad \xi = \frac{a(x - at)}{1 - a^2},$$

where C and a are arbitrary constants ($a \neq 0, \pm 1$).

Example 2. Traveling-wave solutions for $n = 1$:

$$w = 2C_1 \tanh(C_1\xi + C_2), \qquad w = -2C_1 \tan(C_1\xi + C_2), \qquad \xi = \frac{a(x - at)}{1 - a^2},$$

where C_1, C_2, and a are arbitrary constants; $a \neq 0, \pm 1$.

3°. Self-similar solution:

$$w = t^{-1/n}\varphi(\xi), \quad \xi = \frac{x}{t},$$

where the function $\varphi = \varphi(\xi)$ is determined by the ordinary differential equation

$$(1 - \xi^2)\varphi''_{\xi\xi} + \left[\varphi^n - \frac{2(n+1)}{n}\right]\xi\varphi'_\xi + \frac{1}{n}\varphi^{n+1} + \frac{n+1}{n^2}\varphi = 0.$$

4°. Generalized separable solution for $n = 1$:

$$w = \varphi(x)\left[t + C_1 + C_2 \int \frac{dx}{\varphi^2(x)}\right].$$

Here, the function $\varphi = \varphi(x)$ is determined by the autonomous ordinary differential equation $\varphi''_{xx} = \varphi^2$, which has a particular solution $\varphi = 6(x + C)^{-2}$.

⊙ *References*: Y. P. Emech and V. B. Taranov (1972), N. H. Ibragimov (1994, 1995), A. D. Polyanin and V. F. Zaitsev (2002).

2. $\dfrac{\partial^2 w}{\partial t^2} + aw^n \dfrac{\partial w}{\partial t} = b\dfrac{\partial^2 w}{\partial x^2}$.

1°. Self-similar solution:

$$w(x,t) = u(z)t^{-1/n}, \quad z = xt^{-1},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$n^2(z^2 - b)u''_{zz} + nz(2n + 2 - nau^n)u'_z + u(1 + n - nau^n) = 0.$$

2°. Passing to the new independent variables $\tau = at$ and $z = a\beta^{-1/2}x$, we arrive at an equation of the form 3.3.5.1:

$$\frac{\partial^2 w}{\partial \tau^2} + w^n \frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial z^2}.$$

3. $\dfrac{\partial^2 w}{\partial t^2} + aw^n \dfrac{\partial w}{\partial t} = b\dfrac{\partial}{\partial x}\left(w^k \dfrac{\partial w}{\partial x}\right)$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{2n-k}x + C_2,\ C_1^{2n}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Self-similar solution:

$$w(x,t) = u(z)t^{-1/n}, \quad z = xt^{(k-2n)/(2n)},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$\begin{aligned}\left[4bn^2u^k - (2n-k)^2z^2\right]u''_{zz} + 4bkn^2u^{k-1}\left(u'_z\right)^2\\ + (2n-k)(k - 4 - 4n + 2nau^n)zu'_z = 4u(1 + n - anu^n).\end{aligned}$$

⊙ *Reference*: N. H. Ibragimov (1994).

4. $\dfrac{\partial^2 w}{\partial t^2} + e^w \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2}$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1x + C_2,\ C_1t + C_3) + \ln C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Traveling-wave solutions:

$$w(x,t) = -\ln\left\{\frac{1}{C_1}\left[\exp\left(\frac{C_1}{a^2-1}(ax - t) + C_2\right) - 1\right]\right\},$$

$$w(x,t) = -\ln\left\{\frac{1}{a^2-1}(ax - t) + C_1\right\},$$

where C_1, C_2, and a are arbitrary constants ($a \neq \pm 1$).

3°. Solutions:

$$w(x,t) = -\ln\left(\frac{t+Cx}{1-C^2} + 2a\sqrt{|t^2-x^2|}\left|\frac{x+t}{x-t}\right|^a\right),$$
$$w(x,t) = -\ln\left(\mp\frac{x}{2} + C(x\pm t) + \frac{1}{4}(x\pm t)\ln\left|\frac{x+t}{x-t}\right|\right),$$

where C and a are arbitrary constants.

⊙ *References*: Y. P. Emech and V. B. Taranov (1973), N. H. Ibragimov (1994, 1995).

5. $\dfrac{\partial^2 w}{\partial t^2} + ae^{\lambda w}\dfrac{\partial w}{\partial t} = b\dfrac{\partial^2 w}{\partial x^2}.$

1°. Solution:

$$w(x,t) = u(z) - \frac{1}{\lambda}\ln t, \quad z = xt^{-1},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$\lambda(z^2-b)u''_{zz} + \lambda z(2-ae^{\lambda u})u'_z + 1 - ae^{\lambda u} = 0.$$

2°. Passing to the new variables $\tau = at$, $z = a\beta^{-1/2}x$, and $u = \lambda w$, we arrive at an equation of the form 3.3.5.4:

$$\frac{\partial^2 u}{\partial \tau^2} + e^u\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial z^2}.$$

6. $\dfrac{\partial^2 w}{\partial t^2} + ae^{\lambda w}\dfrac{\partial w}{\partial t} = b\dfrac{\partial}{\partial x}\left(e^{\mu w}\dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{2\lambda-\mu}x + C_2,\ C_1^{2\lambda}t + C_3) + 2\ln C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solution:

$$w(x,t) = u(z) - \frac{1}{\lambda}\ln t, \quad z = xt^{(\lambda-2\mu)/(2\mu)},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$\begin{aligned}\big[(\mu-2\lambda)^2z^2 - 4b\lambda^2e^{\mu u}\big]u''_{zz} - 4b\mu\lambda^2e^{\mu u}(u'_z)^2&\\ + (\mu-2\lambda)(\mu-4\lambda+2a\lambda e^{\lambda u})zu'_z + 4\lambda(1-ae^{\lambda u}) &= 0.\end{aligned}$$

⊙ *Reference*: N. H. Ibragimov (1994).

3.4. Equations Involving Arbitrary Functions

3.4.1. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x,t,w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = \alpha\dfrac{\partial^2 w}{\partial x^2} + f(w).$

Nonlinear Klein–Gordon equation.

1°. Suppose $w = w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm x + C_1, \pm t + C_2),$$
$$w_2 = w\big(x\cosh\beta + t\alpha^{1/2}\sinh\beta,\ t\cosh\beta + x\alpha^{-1/2}\sinh\beta\big),$$

where C_1, C_2, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int \left[C_1 + \frac{2}{\lambda^2 - \alpha k^2} \int f(w)\, dw \right]^{-1/2} dw = kx + \lambda t + C_2, \tag{1}$$

where C_1, C_2, k, and λ are arbitrary constants.

Nesterov (1978) indicated several cases where solution (1) can be written out in explicit form ($\alpha = \mu = 1$):

$$f(w) = -b^2 \frac{\tanh w}{\cosh^2 w}, \quad w(z) = \operatorname{arcsinh}\left[\sinh k \, \sin\left(\frac{bz + c}{\cosh k \sqrt{\lambda^2 - 1}} \right) \right],$$

$$f(w) = -b^2 \frac{\tan w}{\cos^2 w}, \quad w(z) = \arcsin\left[\sin k \, \sin\left(\frac{bz + c}{\cos k \sqrt{\lambda^2 - 1}} \right) \right],$$

where k and c are arbitrary constants. In these cases, the following relationships between the wave speed, λ, and the amplitude, b, correspond to periodic solutions in z with period 2π:

$$\lambda^2 = 1 + b^2 \cosh^{-2} k,$$

$$\lambda^2 = 1 + b^2 \cos^{-2} k.$$

3°. Functional separable solution:

$$w = w(\xi), \qquad \xi = \tfrac{1}{4}\alpha(t + C_1)^2 - \tfrac{1}{4}(x + C_2)^2,$$

where C_1 and C_2 are arbitrary constants, and the function $w = w(\xi)$ is determined by the ordinary differential equation

$$\xi w''_{\xi\xi} + w'_{\xi} - \frac{1}{\alpha} f(w) = 0.$$

4°. For exact solutions of the nonlinear Klein–Gordon equation with $f(w) = bw^m$, $f(w) = be^{\beta w}$, $f(w) = b\sinh(\lambda w)$, $f(w) = bw \ln w$, and $f(w) = b\sin(\lambda w)$, see equations 3.1.1.1, 3.2.1.1, 3.3.1.1, 3.3.2.1, and 3.3.3.1, respectively. For solutions of the original equation with some other $f = f(w)$, see Example 11 in Subsection S.5.3.

⊙ *References*: A. M. Grundland and E. Infeld (1992), R. Z. Zhdanov (1994), A. D. Polyanin and V. F. Zaitsev (2002).

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + (x^2 - t^2) f(w).$

1°. Suppose $w = w(x, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x, \pm t),$$

$$w_2 = w(x \cosh\beta + t \sinh\beta, \; x \sinh\beta + t \cosh\beta),$$

where β is an arbitrary constant, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Functional separable solution:

$$w = w(\xi), \qquad \xi = \tfrac{1}{2}\left(x^2 - t^2\right),$$

where the function $w = w(\xi)$ is determined by the ordinary differential equation

$$\xi w''_{\xi\xi} + w'_{\xi} + \xi f(w) = 0.$$

3°. Self-similar solution:

$$w = w(\tau), \qquad \tau = xt.$$

Here, the function $w = w(\tau)$ is determined by the autonomous ordinary differential equation

$$w''_{\tau\tau} = f(w),$$

whose general solution can be represented in implicit form as

$$\int \left[C_1 + 2F(w) \right]^{-1/2} dw = C_2 \pm \tau, \qquad F(w) = \int f(w)\, dw,$$

where C_1 and C_2 are arbitrary constants.

4°. Functional separable solution:

$$w = w(z), \qquad z = \tfrac{1}{2}(x^2 + t^2).$$

Here, the function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$w''_{zz} + f(w) = 0,$$

whose general solution can be represented in implicit form as

$$\int \bigl[C_1 - 2F(w)\bigr]^{-1/2} dw = C_2 \pm z, \qquad F(w) = \int f(w)\, dw,$$

where C_1 and C_2 are arbitrary constants.

5°. Functional separable solution (generalizes the solution of Items 3° and 4°):

$$w = w(r), \qquad r = C_1 x^2 + C_2 xt + C_1 t^2 + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $w = w(r)$ is determined by the autonomous ordinary differential equation

$$(C_2^2 - 4C_1^2) w''_{rr} = f(w),$$

6°. The transformation

$$w = U(z, \tau), \quad z = \tfrac{1}{2}(x^2 + t^2), \quad \tau = xt$$

leads to a simpler equation of the form 3.4.1.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + f(U).$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + (t^2 - x^2)^n f(w), \qquad n = 2,\ 3,\ \ldots$

This is a special case of equation 3.4.1.16 with $f(y) = y^n$ and $g(z) = z^n$.

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a \frac{\partial^2 w}{\partial x^2} + f(x + bt, w).$

Solution:

$$w = w(\xi), \qquad \xi = x + bt,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$(a - b^2) w''_{\xi\xi} + f(\xi, w) = 0.$$

5. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + (x^2 - t^2) f(xt, w).$

1°. Self-similar solution:

$$w = w(\tau), \qquad \tau = xt.$$

Here, the function $w = w(\tau)$ is determined by the ordinary differential equation

$$w''_{\tau\tau} = f(\tau, w).$$

2°. The transformation

$$z = \tfrac{1}{2}(x^2 + t^2), \quad \tau = xt$$

leads to the simpler equation

$$\frac{\partial^2 w}{\partial \tau^2} = \frac{\partial^2 w}{\partial z^2} + f(\tau, w).$$

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + e^{\beta x} f(w).$

The transformation

$$w = U(z,\tau), \quad z = \exp\left(\tfrac{1}{2}\beta x\right)\cosh\left(\tfrac{1}{2}\beta t\right), \quad \tau = \exp\left(\tfrac{1}{2}\beta x\right)\sinh\left(\tfrac{1}{2}\beta t\right)$$

leads to a simpler equation of the form 3.4.1.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + 4\beta^{-2} f(U).$$

For arbitrary $f = f(U)$, this equation admits a traveling-wave solution $U = U(kz + \lambda\tau)$ and a solution of the form $U = U(z^2 - \tau^2)$.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + e^{\beta t} f(w).$

The transformation

$$w = U(z,\tau), \quad z = \exp\left(\tfrac{1}{2}\beta t\right)\sinh\left(\tfrac{1}{2}\beta x\right), \quad \tau = \exp\left(\tfrac{1}{2}\beta t\right)\cosh\left(\tfrac{1}{2}\beta x\right)$$

leads to a simpler equation of the form 3.4.1.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + 4\beta^{-2} f(U).$$

For arbitrary $f = f(U)$, this equation admits a traveling-wave solution $U = U(kz + \lambda\tau)$ and a solution of the form $U = U(z^2 - \tau^2)$.

8. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + e^{ax+bt} f(w).$

1°. There is a solution of the form $w = w(ax + bt)$.

2°. For $b \neq \pm a$, the transformation

$$\xi = ax + bt, \quad \tau = bx + at$$

leads to an equation of the form 3.4.1.6:

$$\frac{\partial^2 w}{\partial \tau^2} = \frac{\partial^2 w}{\partial \xi^2} + \frac{1}{a^2 - b^2} e^{\xi} f(w).$$

3°. For $b = a$, see equation 3.4.1.13 with $f(z) = e^{az}$, and for $b = -a$, see equation 3.4.1.14 with $f(z) = e^{-az}$.

9. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bw \ln w + f(t)w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - \left[b\ln\varphi + f(t) + C\right]\varphi = 0,$$
$$a\psi''_{xx} + \left(b\ln\psi - C\right)\psi = 0,$$

and C is an arbitrary constant.

10. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + bw \ln w + \bigl[f(x) + g(t)\bigr]w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - \bigl[b\ln\varphi + g(t) + C\bigr]\varphi = 0,$$
$$a\psi''_{xx} + \bigl[b\ln\psi + f(x) - C\bigr]\psi = 0,$$

and C is an arbitrary constant.

11. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)w \ln w + \bigl[bf(t)x + g(t)\bigr]w.$

Multiplicative separable solution:

$$w(x,t) = e^{-bx}\varphi(t),$$

where the function $\varphi(t)$ is determined by the ordinary differential equation

$$\varphi''_{tt} = f(t)\varphi\ln\varphi + \bigl[g(t) + ab^2\bigr]\varphi.$$

12. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)w \ln w + \bigl[bf(x)t + g(x)\bigr]w.$

Multiplicative separable solution:

$$w(x,t) = e^{-bt}\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + f(x)\varphi\ln\varphi + \bigl[g(x) - b^2\bigr]\varphi = 0.$$

13. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + f(t+x)g(w).$

The transformation

$$w = U(\xi,\tau), \quad z = \frac{1}{2}\int_a^{t+x} f(\lambda)\,d\lambda - \frac{1}{2}(t-x), \quad \tau = \frac{1}{2}\int_a^{t+x} f(\lambda)\,d\lambda + \frac{1}{2}(t-x)$$

where a is an arbitrary constant, leads to an equation of the form 3.4.1.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + g(U).$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

14. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + f(t-x)g(w).$

The transformation

$$w = U(z,\tau), \quad z = \frac{1}{2}(t+x) - \frac{1}{2}\int_a^{t-x} f(\sigma)\,d\sigma, \quad \tau = \frac{1}{2}(t+x) + \frac{1}{2}\int_a^{t-x} f(\sigma)\,d\sigma$$

where a is an arbitrary constant, leads to an equation of the form 3.4.1.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + g(U).$$

15. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + f(t+x)g(t-x)e^{\beta w}.$

The transformation

$$w = U(z,\tau), \quad z = \frac{1}{2}\int_a^{t+x} f(\lambda)\,d\lambda - \frac{1}{2}\int_b^{t-x} g(\sigma)\,d\sigma, \quad \tau = \frac{1}{2}\int_a^{t+x} f(\lambda)\,d\lambda + \frac{1}{2}\int_b^{t-x} g(\sigma)\,d\sigma,$$

where a and b are arbitrary constants, leads to an equation of the form 3.2.1.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + e^{\beta U}.$$

16. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + f(t+x)g(t-x)h(w).$

The transformation

$$w = U(z,\tau), \quad z = \frac{1}{2}\int_a^{t+x} f(\lambda)\,d\lambda - \frac{1}{2}\int_b^{t-x} g(\sigma)\,d\sigma, \quad \tau = \frac{1}{2}\int_a^{t+x} f(\lambda)\,d\lambda + \frac{1}{2}\int_b^{t-x} g(\sigma)\,d\sigma,$$

where a and b are arbitrary constants, leads to an equation of the form 3.4.1.1:

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial^2 U}{\partial z^2} + h(U).$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

3.4.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{b}{x}\dfrac{\partial w}{\partial x} + f(w).$

1°. This is a special case of equation 3.4.3.4 with $n = 0$. This equation can be rewritten in the form

$$\frac{\partial^2 w}{\partial t^2} = \frac{a}{x^m}\frac{\partial}{\partial x}\left(x^m \frac{\partial w}{\partial x}\right) + f(w), \qquad m = \frac{b}{a}.$$

To $m = 1$ and $m = 2$ there correspond nonlinear waves with axial and central symmetry, respectively.

2°. Functional separable solution:

$$w = w(\xi), \quad \xi = \sqrt{ak(t+C)^2 - kx^2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{a+b}{a\xi} w'_\xi = \frac{1}{ak} f(w).$$

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\dfrac{\partial w}{\partial x} + bw\ln w + \big[g(x) + h(t)\big]w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - \big[b\ln\varphi + h(t) + C\big]\varphi = 0,$$
$$a\psi''_{xx} + f(x)\psi'_x + \big[b\ln\psi + g(x) - C\big]\psi = 0,$$

and C is an arbitrary constant.

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(w)\dfrac{\partial w}{\partial x}.$

Traveling-wave solution in implicit form:

$$(\lambda^2 - a)\int \frac{dw}{F(w)+C_1} = x + \lambda t + C_2, \qquad F(w) = \int f(w)\,dw.$$

where C_1, C_2, and λ are arbitrary constants. To the stationary solution there correspond $\lambda = 0$.

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + f(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$\begin{aligned} w_1 &= w(\pm x + C_1, t) + C_2\cosh(kt) + C_3\sinh(kt) \quad &&\text{if } c = k^2 > 0,\\ w_2 &= w(\pm x + C_1, t) + C_2\cos(kt) + C_3\sin(kt) \quad &&\text{if } c = -k^2 < 0, \end{aligned}$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,t) = U(t) + \Theta(\xi), \quad \xi = \pm x + \lambda t,$$

where λ is an arbitrary constant, and the functions $U(t)$ and $\Theta(\xi)$ are determined by the ordinary differential equations

$$U''_{tt} - cU - f(t) = 0, \tag{1}$$

$$(a-\lambda^2)\Theta''_{\xi\xi} + b\big(\Theta'_\xi\big)^2 + c\Theta = 0. \tag{2}$$

The solution of equation (1) is given by

$$\begin{aligned} U(t) &= C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t f(\tau)\sinh\big[k(t-\tau)\big]\,d\tau \quad &&\text{if} \quad c = k^2 > 0,\\ U(t) &= C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t f(\tau)\sin\big[k(t-\tau)\big]\,d\tau \quad &&\text{if} \quad c = -k^2 < 0, \end{aligned} \tag{3}$$

where C_1 and C_2 are arbitrary constants.

Equation (2) can be solved with the change of variable $z(\Theta) = \big(\Theta'_\xi\big)^2$, which leads to first-order linear equation.

Particular solution of equation (2):

$$\Theta = -\frac{c}{4b}(\xi + C_3)^2 + \frac{1}{2b}(a-\lambda^2).$$

3°. Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t), \tag{4}$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = 4b\varphi^2 + c\varphi, \tag{5}$$

$$\psi''_{tt} = (4b\varphi + c)\psi, \tag{6}$$

$$\chi''_{tt} = c\chi + b\psi^2 + 2a\varphi + f(t). \tag{7}$$

Equation (5) has the trivial particular solution $\varphi(t) \equiv 0$, to which there corresponds a solution of (4) linear in x. Another particular solution to equation (5) is given by $\varphi = -\frac{1}{4}c/b$.

The general solution of the autonomous equation (5) can be represented in implicit form:

$$\int \left(\tfrac{8}{3}b\varphi^3 + c\varphi^2 + C_1\right)^{-1/2} d\varphi = C_2 \pm t,$$

where C_1 and C_2 are arbitrary constants.

The functions $\psi = \psi(t)$ and $\chi = \chi(t)$ can be found by successively integrating equations (6) and (7), which are linear in ψ and χ, respectively.

Note that equation (6) has a particular solution $\psi = \bar{\varphi}(t)$, where $\bar{\varphi}(t)$ is any nontrivial particular solution to (5). Hence, the general solution to (6) is expressed as

$$\psi(t) = C_3\bar{\varphi}(t) + C_4\bar{\varphi}(t)\int \frac{dt}{\bar{\varphi}^2(t)},$$

where C_3 and C_4 are arbitrary constants.

4°. The substitution

$$w = z(x,t) + U(t),$$

where the function $U(t)$ is defined by formula (3), leads to the simpler equation

$$\frac{\partial^2 z}{\partial t^2} = a\frac{\partial^2 z}{\partial x^2} + b\left(\frac{\partial z}{\partial x}\right)^2 + cz.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

5. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + cw + f(x).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(x, \pm t + C_1) + C_2\cosh(kt) + C_3\sinh(kt) \quad \text{if } c = k^2 > 0,$$
$$w_2 = w(x, \pm t + C_1) + C_2\cos(kt) + C_3\sin(kt) \quad \text{if } c = -k^2 < 0,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(x) + \psi(t).$$

Here,

$$\psi(t) = \begin{cases} C_1\cosh(kt) + C_2\sinh(kt) & \text{if } c = k^2 > 0, \\ C_1\cos(kt) + C_2\sin(kt) & \text{if } c = -k^2 < 0, \end{cases}$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + b(\varphi'_x)^2 + c\varphi + f(x) = 0.$$

6. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + cw\frac{\partial w}{\partial x} + kw^2 + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t) + \psi(t)\exp(\lambda x),$$

where λ is a root of the quadratic equation $b\lambda^2 + c\lambda + k = 0$, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = k\varphi^2 + f(t)\varphi + g(t), \tag{1}$$
$$\psi''_{tt} = \left[(c\lambda + 2k)\varphi + f(t) + a\lambda^2\right]\psi. \tag{2}$$

In the special case $f(t) = \text{const}$, $g(t) = \text{const}$, equation (1) has exact solutions of the form $\varphi = \text{const}$ and, due to its autonomity, can be integrated by quadrature. Equation (2) is linear in ψ and, hence, with $\varphi = \text{const}$, the general solution to (6) is expressed in terms of exponentials or sine and cosine.

7. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(t)w + h(t).$

Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t), \tag{1}$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of ordinary differential equations of the second order with variable coefficients (the arguments of f, g, and h are not specified)

$$\varphi''_{tt} = 4f\varphi^2 + g\varphi, \tag{2}$$

$$\psi''_{tt} = (4f\varphi + g)\psi, \tag{3}$$

$$\chi''_{tt} = g\chi + f\psi^2 + h + 2a\varphi. \tag{4}$$

Equation (2) has the trivial particular solution $\varphi(t) \equiv 0$, to which there corresponds a solution of (1) linear in x.

If a solution $\varphi = \varphi(t)$ of the nonlinear equation (2) has been found, the functions $\psi = \psi(t)$ and $\chi = \chi(t)$ can be obtained by successively solving equations (3) and (4), which are linear in ψ and χ, respectively.

Note that equation (3) has a particular solution $\psi = \bar{\varphi}(t)$, where $\bar{\varphi}(t)$ is any nontrivial particular solution to (2). Hence, the general solution to (3) is expressed as

$$\psi(t) = C_1\bar{\varphi}(t) + C_2\bar{\varphi}(t)\int \frac{dt}{\bar{\varphi}^2(t)},$$

where C_1 and C_2 are arbitrary constants. If the functions f and g are proportional to each other, then a particular solution to equation (2) is given by $\varphi = -\frac{1}{4}g/f = \text{const}$.

8. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x) + h(t).$

Additive separable solution:

$$w(x,t) = \tfrac{1}{2}At^2 + Bt + C + \int_0^t (t-\tau)h(\tau)\,d\tau + \varphi(x).$$

Here, A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + f(x)\bigl(\varphi'_x\bigr)^2 + g(x) - A = 0.$$

9. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + bw + g(x) + h(t).$

Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x).$$

Here, the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - h(t) = 0,$$

$$a\psi''_{xx} + f(x)(\psi'_x)^2 + b\psi + g(x) = 0.$$

The solution of the first equation for $\varphi(t)$ is expressed as

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t h(\tau)\sinh\bigl[k(t-\tau)\bigr]\,d\tau \quad \text{if} \quad b = k^2 > 0,$$

$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t h(\tau)\sin\bigl[k(t-\tau)\bigr]\,d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

10. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + bf(t)w^2 + g(t)w + h(t).$

1°. Generalized separable solutions:

$$w(x,t) = \varphi(t) + \psi(t)\exp\bigl(\pm x\sqrt{-b}\,\bigr), \qquad b<0, \tag{1}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations of the second order with variable coefficients (the arguments of f, g, and h are not specified)

$$\varphi''_{tt} = bf\varphi^2 + g\varphi + h, \tag{2}$$
$$\psi''_{tt} = (2bf\varphi + g - ab)\psi. \tag{3}$$

If a solution $\varphi = \varphi(t)$ to equation (2) has been found, the function $\psi = \psi(t)$ can be obtained by solving equation (3) linear in ψ.

If the functions f, g, and h are proportional to each other,

$$g = \alpha f, \quad h = \beta f \qquad (\alpha,\ \beta = \text{const}),$$

particular solutions to equation (2) are expressed as

$$\varphi = k_1, \quad \varphi = k_2, \tag{4}$$

where k_1 and k_2 are roots of the quadratic equation $bk^2 + \alpha k + \beta = 0$. In this case, equation (3) can be rewritten in the form

$$\psi''_{tt} = \bigl[(2bk_n + \alpha)f - ab\bigr]\psi, \qquad n = 1,\ 2. \tag{5}$$

Kamke (1977) and Polyanin and Zaitsev (2003) present many exact solutions of the linear equation (5) for various $f = f(t)$. In the special case $f = \text{const}$, the general solution of equation (5) is the sum of exponentials (or sine and cosine).

2°. Generalized separable solution (generalizes the solutions of Item 1°):

$$w(x,t) = \varphi(t) + \psi(t)\bigl[A\exp\bigl(x\sqrt{-b}\,\bigr) + B\exp\bigl(-x\sqrt{-b}\,\bigr)\bigr], \qquad b<0, \tag{6}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the following system of second-order ordinary differential equations with variable coefficients

$$\varphi''_{tt} = bf\bigl(\varphi^2 + 4AB\psi^2\bigr) + g\varphi + h, \tag{7}$$
$$\psi''_{tt} = \bigl(2bf\varphi + g - ab\bigr)\psi. \tag{8}$$

We express φ from (8) in terms of ψ and then substitute into (7) to obtain a nonlinear fourth-order equation for ψ; with $f,\ g,\ h = \text{const}$, the equation is autonomous and, hence, its order can be reduced.

Note two special cases of solutions (6) that can be expressed in terms of hyperbolic functions:

$$w(x,t) = \varphi(t) + \psi(t)\cosh\bigl(x\sqrt{-b}\,\bigr), \qquad A = \tfrac{1}{2},\ B = \tfrac{1}{2},$$
$$w(x,t) = \varphi(t) + \psi(t)\sinh\bigl(x\sqrt{-b}\,\bigr), \qquad A = \tfrac{1}{2},\ B = -\tfrac{1}{2}.$$

3°. Generalized separable solution (c is an arbitrary constant):

$$w(x,t) = \varphi(t) + \psi(t)\cos\bigl(x\sqrt{b} + c\bigr), \qquad b>0, \tag{9}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations of the second order with variable coefficients

$$\varphi''_{tt} = bf\bigl(\varphi^2 + \psi^2\bigr) + g\varphi + h, \tag{10}$$
$$\psi''_{tt} = \bigl(2bf\varphi + g - ab\bigr)\psi. \tag{11}$$

⊙ *References*: V. A. Galaktionov (1995, the case of $f = a$, $g = \text{const}$, and $h = \text{const}$ was considered), V. F. Zaitsev and A. D. Polyanin (1996).

11. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + \big[g_1(t)x + g_0(t)\big]\dfrac{\partial w}{\partial x} + h(t)w + p_2(t)x^2 + p_1(t)x + p_0(t).$

There is a generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

12. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^k + g(x)\dfrac{\partial w}{\partial x} + bw + h_1(t) + h_2(x).$

Additive separable solution:

$$w(x,t) = \varphi(x) + \psi(t).$$

Here,

$$\psi(t) = \begin{cases} C_1\cosh(kt) + C_2\sinh(kt) + \dfrac{1}{k}\displaystyle\int_0^t \sinh\big[k(t-\tau)\big]h_1(\tau)\,d\tau & \text{if} \quad b = k^2 > 0, \\ C_1\cos(kt) + C_2\sin(kt) + \dfrac{1}{k}\displaystyle\int_0^t \sin\big[k(t-\tau)\big]h_1(\tau)\,d\tau & \text{if} \quad b = -k^2 < 0, \\ C_1 + C_2 t + \displaystyle\int_0^t (t-\tau)h_1(\tau)\,d\tau & \text{if} \quad b = 0, \end{cases}$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + f(x)\big(\varphi'_x\big)^k + g(x)\varphi'_x + b\varphi + h_2(x) = 0.$$

13. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x,\ \dfrac{\partial w}{\partial x}\right) + g(t).$

Additive separable solution:

$$w(x,t) = \tfrac{1}{2}At^2 + Bt + C + \int_0^t (t-\tau)g(\tau)\,d\tau + \varphi(x).$$

Here, A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''_{xx} + f\big(x, \varphi'_x\big) - A = 0.$$

14. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x,\ \dfrac{\partial w}{\partial x}\right) + bw + g(t).$

Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x).$$

Here, the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - g(t) = 0,$$
$$a\psi''_{xx} + f\big(x, \psi'_x\big) + b\psi = 0.$$

The solution of the first equation is expressed as

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t g(\tau)\sinh\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t g(\tau)\sin\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

15. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + wf\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial x}\right).$

Multiplicative separable solution:

$$w(x,t) = e^{\lambda x}\varphi(t),$$

where λ is an arbitrary constant, and the function $\varphi(t)$ is determined by the second-order linear ordinary differential equation

$$\varphi''_{tt} = \left[a\lambda^2 + f(t,\lambda)\right]\varphi.$$

3.4.3. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = (ax^2 + b)\dfrac{\partial^2 w}{\partial x^2} + ax\dfrac{\partial w}{\partial x} + f(w).$

The substitution $z = \displaystyle\int \frac{dx}{\sqrt{ax^2+b}}$ leads to an equation of the form 3.4.1.1:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial z^2} + f(w).$$

2. $\dfrac{\partial^2 w}{\partial t^2} = a(x+\beta)^n\dfrac{\partial^2 w}{\partial x^2} + f(w), \qquad a > 0.$

This equation describes the propagation of nonlinear waves in an inhomogeneous medium. For $n = 0$ see equation 3.4.1.1.

1°. The substitution $y = x + \beta$ leads to a special case of equation 3.4.3.4 with $b = 0$.

2°. Functional separable solution for $n \neq 2$:

$$w = w(r), \qquad r^2 = k\left[\frac{1}{4}(t+C)^2 - \frac{(x+\beta)^{2-n}}{a(2-n)^2}\right],$$

where k and the expression in square brackets must have like signs, and $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{2(1-n)}{2-n}\frac{1}{r}w'_r = \frac{4}{k}f(w). \tag{1}$$

The substitution $\xi = r^{\frac{n}{2-n}}$ leads to the generalized Emden–Fowler equation

$$w''_{\xi\xi} = \frac{4(2-n)^2}{kn^2}\xi^{\frac{4(1-n)}{n}}f(w). \tag{2}$$

The book by Polyanin and Zaitsev (2003) presents a number of exact solutions to equation (2) for various $f = f(w)$.

Special case. For $n = 1$, the general solution of equation (1) is written out in implicit form as

$$\int\left[C_1 + \frac{8}{k}F(w)\right]^{-1/2} dw = \pm r + C_2, \qquad F(w) = \int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

3°. Solution for $n = 2$:

$$w = w(y), \quad y = At + B\ln|x+\beta|,$$

where A and B are arbitrary constants, and the function $w = w(y)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{yy} - aBw'_y + f(w) = 0. \tag{3}$$

Solution of equation (3) with $A = \pm B\sqrt{a}$ in implicit form:

$$aB \int \frac{dw}{f(w)} = y + C,$$

where C is an arbitrary constant.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[a(x+\beta)^n \frac{\partial w}{\partial x}\right] + f(w), \qquad a > 0.$

This equation describes the propagation of nonlinear waves in an inhomogeneous medium.

1°. Functional separable solution for $n \neq 2$:

$$w = w(r), \qquad r^2 = k\left[\frac{1}{4}(t+C)^2 - \frac{(x+\beta)^{2-n}}{a(2-n)^2}\right],$$

where k and the expression in square brackets must have like signs, and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{2}{2-n}\frac{1}{r}w'_r = \frac{4}{k}f(w).$$

The substitution $\xi = r^{\frac{n}{n-2}}$ leads to the generalized Emden–Fowler equation

$$w''_{\xi\xi} = \frac{4(2-n)^2}{kn^2}\xi^{-\frac{4}{n}} f(w). \tag{1}$$

The book by Polyanin and Zaitsev (2003) presents a number of exact solutions to equation (1) for various $f = f(w)$.

2°. Solution for $n = 2$:

$$w = w(z), \quad z = At + B\ln|x+\beta|,$$

where A and B are arbitrary constants, and the function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{zz} + aBw'_z + f(w) = 0. \tag{2}$$

Solution of equation (2) with $A = \pm B\sqrt{a}$ in implicit form:

$$aB \int \frac{dw}{f(w)} = -z + C,$$

where C is an arbitrary constant.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = ax^n \frac{\partial^2 w}{\partial x^2} + bx^{n-1}\frac{\partial w}{\partial x} + f(w), \qquad a > 0.$

1°. Functional separable solution for $n \neq 2$:

$$w = w(\xi), \qquad \xi = \tfrac{1}{4}a(2-n)^2(t+C)^2 - x^{2-n}.$$

Here, C is an arbitrary constant, and the function $w = w(\xi)$ is determined by the ordinary differential equation

$$\xi w''_{\xi\xi} + Aw'_\xi - Bf(w) = 0, \tag{1}$$

where

$$A = \frac{a(4-3n)+2b}{2a(2-n)}, \quad B = \frac{1}{a(2-n)^2}.$$

For $A \neq 1$, the substitution $\xi = kz^{\frac{1}{1-A}}$ $(k = \pm 1)$ brings (1) to the generalized Emden–Fowler equation

$$w''_{zz} - \frac{kB}{(1-A)^2} z^{\frac{2A-1}{1-A}} f(w) = 0. \tag{2}$$

In the special case $A = \frac{1}{2}$, which corresponds to $b = a(n-1)$, the general solution of equation (2) is expressed as

$$\int \Big[C_1 + 8kBF(w)\Big]^{-1/2} dw = \pm z + C_2, \qquad F(w) = \int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

The books by Polyanin and Zaitsev (1995, 2003) present a number of exact solutions to equation (2) for some $f = f(w)$.

2°. Solution for $n = 2$:

$$w = w(y), \quad y = At + B\ln|x| + C,$$

where A, B, and C are arbitrary constants, and the function $w(y)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{yy} + (b-a)Bw'_y + f(w) = 0. \tag{3}$$

Solutions of equation (3) with $A = \pm B\sqrt{a}$ in implicit form:

$$(b-a)B\int \frac{dw}{f(w)} = -y + C_1.$$

Solutions of equation (3) with $b = a$:

$$\int \Big[C_1 + \frac{2}{A^2 - aB^2}F(w)\Big]^{-1/2} dw = \pm y + C_2, \qquad F(w) = \int f(w)\,dw.$$

For $A \neq \pm B\sqrt{a}$ and $b \neq a$, the substitution $u(w) = \dfrac{aB^2 - A^2}{B(a-b)} w'_y$ brings (3) to the Abel equation

$$uu'_w - u = \frac{A^2 - aB^2}{B^2(a-b)^2} f(w),$$

whose exact solutions for various $f = f(w)$ can be found in Polyanin and Zaitsev (2003).

5. $\dfrac{\partial^2 w}{\partial t^2} = ax^n \dfrac{\partial^2 w}{\partial x^2} + x^{n-1} f(w) \dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w\big(C_1^2 x,\ \pm C_1^{2-n} t + C_2\big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $n \neq 2$:

$$w = w(z), \quad z = \big|a(2-n)^2(t+C)^2 - 4x^{2-n}\big|^{1/2},$$

where C is an arbitrary constant, and the function $w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2}{a(2-n)z}\big[a(1-n) + f(w)\big]w'_z = 0. \tag{1}$$

The substitution $u(w) = zw'_z$ brings (1) to a separable first-order equation, the integration of which yields the general solution in implicit form:

$$\int \frac{dw}{anw - 2F(w) + C_1} = \frac{1}{a(2-n)}\ln z + C_2, \qquad F(w) = \int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

3°. Self-similar solution for $n \neq 2$:

$$w = w(\xi), \qquad \xi = x|t|^{\frac{2}{n-2}},$$

where the function $w = w(\xi)$ is determined by the ordinary differential equation

$$\left[a\xi^{n-1} - \frac{4}{(n-2)^2}\xi\right] w''_{\xi\xi} + \left[\xi^{n-2} f(w) + \frac{2(n-4)}{(n-2)^2}\right] w'_{\xi} = 0.$$

4°. Solution for $n = 2$:

$$w = w(y), \quad y = At + B\ln|x| + C,$$

where A, B, and C are arbitrary constants, and the function $w(y)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{yy} + B\big[f(w) - a\big]w'_y = 0,$$

whose solution with $A \neq \pm B\sqrt{a}$ is given by

$$\frac{aB^2 - A^2}{B}\int \frac{dw}{F(w) - aw + C_1} = -y, \qquad F(w) = \int f(w)\,dw.$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

6. $\displaystyle \frac{\partial^2 w}{\partial t^2} = ax^n \frac{\partial^2 w}{\partial x^2} + x^{n-1} f(w)\frac{\partial w}{\partial x} + g(w).$

1°. Functional separable solution for $n \neq 2$:

$$w = w(z), \quad z = \big[ka(2-n)^2(t+C)^2 - 4kx^{2-n}\big]^{1/2}, \qquad k = \pm 1,$$

where C is an arbitrary constant, and the function $w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2}{a(2-n)}\big[a(1-n) + f(w)\big]\frac{1}{z}w'_z - \frac{1}{ak(2-n)^2}g(w) = 0.$$

2°. Solution for $n = 2$:

$$w = w(\xi), \quad \xi = At + B\ln|x| + C,$$

where A, B, and C are arbitrary constants, and the function $w(\xi)$ is determined by the autonomous ordinary differential equation

$$(aB^2 - A^2)w''_{\xi\xi} + B\big[f(w) - a\big]w'_{\xi} + g(w) = 0. \tag{1}$$

Solution of equation (1) with $A = \pm B\sqrt{a}$:

$$B\int \big[f(w) - a\big]\frac{dw}{g(w)} = -\xi + C_1.$$

In the general case, the change of variable $u(w) = w'_{\xi}$ brings (1) to an Abel equation, whose exact solutions for various $f = f(w)$ and $g = g(w)$ can be found in Polyanin and Zaitsev (2003).

7. $\displaystyle \frac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\frac{\partial^2 w}{\partial x^2} + f(w), \qquad a > 0.$

This is a special case of equation 3.4.3.9 with $b = 0$.

8. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ae^{\lambda x}\frac{\partial w}{\partial x}\right) + f(w), \qquad a > 0.$

This is a special case of equation 3.4.3.9 with $b = a\lambda$.

9. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda x}\dfrac{\partial w}{\partial x} + f(w), \qquad a > 0.$

Functional separable solution:

$$w = w(z), \quad z = \left[4ke^{-\lambda x} - ak\lambda^2(t+C)^2\right]^{1/2}, \qquad k = \pm 1,$$

where C is an arbitrary constant and the function $w = w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2(a\lambda - b)}{a\lambda}\frac{1}{z}w'_z + \frac{1}{ak\lambda^2}f(w) = 0. \tag{1}$$

For $b = a\lambda$, the solution of equation (1) is expressed as

$$\int \left[C_1 - \frac{2}{ak\lambda^2}F(w)\right]^{-1/2} dw = \pm z + C_2, \qquad F(w) = \int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

For $b \neq \frac{1}{2}a\lambda$, the substitution $\xi = z^{\frac{2b-a\lambda}{a\lambda}}$ brings (1) to the generalized Emden–Fowler equation

$$w''_{\xi\xi} + \frac{a}{k(2b-a\lambda)^2}\xi^{\frac{4(a\lambda-b)}{2b-a\lambda}} f(w) = 0. \tag{2}$$

The books by Polyanin and Zaitsev (1995, 2003) present a number of exact solutions to equation (2) for some $f = f(w)$.

10. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + e^{\lambda x} f(w)\dfrac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w\bigl(x + 2C_1,\ \pm e^{\lambda C_1}t + C_2\bigr),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Functional separable solution for $\lambda \neq 0$:

$$w = w(z), \quad z = \left|4e^{-\lambda x} - a\lambda^2(t+C)^2\right|^{1/2},$$

where C is an arbitrary constant, and the function $w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2}{z}\left[1 - \frac{1}{a\lambda}f(w)\right]w'_z = 0. \tag{1}$$

The substitution $u(w) = zw'_z$ brings (1) to a separable first-order equation, the integration of which yields the general solution in implicit form:

$$\int \frac{dw}{2F(w) - a\lambda w + C_1} = \frac{1}{a\lambda}\ln z + C_2, \qquad F(w) = \int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

For $\lambda = 0$, see equation 3.4.2.3.

3°. Generalized self-similar solution:

$$w = w(\xi), \quad \xi = t^2 e^{\lambda x},$$

where the function $w = w(z)$ is determined by the ordinary differential equation

$$(a\lambda^2\xi^2 - 4\xi)w''_{\xi\xi} + \left[\lambda\xi f(w) + a\lambda^2\xi - 2\right]w'_\xi = 0.$$

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

11. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + e^{\lambda x} f(w)\dfrac{\partial w}{\partial x} + g(w).$

1°. Functional separable solution for $\lambda \neq 0$:

$$w = w(z), \quad z = \left[4ke^{-\lambda x} - ak\lambda^2(t+C)^2\right]^{1/2}, \qquad k = \pm 1,$$

where C is an arbitrary constant, and the function $w(z)$ is determined by the ordinary differential equation

$$w''_{zz} + \frac{2}{z}\left[1 - \frac{1}{a\lambda}f(w)\right]w'_z + \frac{1}{ak\lambda^2}g(w) = 0.$$

2°. For $\lambda = 0$, there is a traveling-wave solution: $w = w(\alpha x + \beta t)$.

12. $\dfrac{\partial^2 w}{\partial t^2} = f(x)\dfrac{\partial^2 w}{\partial x^2} + g(x)\dfrac{\partial w}{\partial x} + aw\ln w + \big[h(x) + p(t)\big]w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(x)\psi(t),$$

where the functions $\varphi(x)$ and $\psi(t)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$f(x)\varphi''_{xx} + g(x)\varphi'_x + a\varphi\ln\varphi + \big[C + h(x)\big]\varphi = 0,$$
$$\psi''_{tt} - a\psi\ln\psi + \big[C - p(t)\big]\psi = 0.$$

3.4.4. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = f(w)\dfrac{\partial^2 w}{\partial x^2} + g\Big(x, t, w, \dfrac{\partial w}{\partial x}\Big)$

1. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^2 w}{\partial x^2} + f(t)\Big(\dfrac{\partial w}{\partial x}\Big)^2 + g(t)w + h_2(t)x^2 + h_1(t)x + h_0(t).$

Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = 2[2f(t) + a]\varphi^2 + g(t)\varphi + h_2(t),$$
$$\psi''_{tt} = 2[2f(t) + a]\varphi\psi + g(t)\psi + h_1(t),$$
$$\chi''_{tt} = 2a\varphi\chi + f(t)\psi^2 + g(t)\chi + h_0(t).$$

⊙ *Reference*: V. A. Galaktionov (1995); the case of $f = \text{const}$ and $h_1 = h_2 = 0$ was treated.

2. $\dfrac{\partial^2 w}{\partial t^2} = aw^4\dfrac{\partial^2 w}{\partial x^2} + f(x)w^5.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\big(x, \pm C_1^2 t + C_2\big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Suppose $u = u(x)$ is any nontrivial solution of the second-order linear ordinary differential equation

$$au''_{xx} + f(x)u = 0. \qquad (1)$$

The transformation

$$\xi = \int\frac{dx}{u^2}, \quad z = \frac{w}{u}$$

brings the original equation to a simpler equation of the form 3.1.5.5 with $n = 4$:

$$\frac{\partial^2 z}{\partial t^2} = az^4 \frac{\partial^2 z}{\partial \xi^2}.$$

For example, this equation has the following solutions (A, B, C, D, and λ are arbitrary constants):

$$z(\xi,t) = A\xi t + B\xi + Ct + D,$$

$$z(\xi,t) = \lambda^{-1/4}(t+C)^{-1/2}\left[\frac{3\lambda}{4A^2 a} + (A\xi + B)^2\right]^{1/2}.$$

The first solution is degenerate and the second one is a special case of a multiplicative separable solution $z(\xi,t) = f(\xi)g(t)$. There are also a traveling-wave solution, $z = z(\alpha\xi + \beta t)$, and a self-similar solution of the form

$$z = t^k \varphi(\zeta), \quad \zeta = \xi t^{-2k-1},$$

where k is an arbitrary constant.

3°. Multiplicative separable solution:

$$w(x,t) = (\pm 2\lambda t + C)^{-1/2} g(x),$$

where C is an arbitrary constant, and the function $g = g(x)$ is determined by the Yermakov's equation

$$ag''_{xx} + f(x)g - 3\lambda^2 g^{-3} = 0. \tag{2}$$

Given a particular solution $u = u(x)$ of the linear equation (1), the general solution of the nonlinear equation (2) is expressed as (e.g., see Polyanin and Zaitsev, 2003):

$$Ag^2 = \frac{3\lambda^2}{a}u^2 + u^2\left(B + A\int \frac{dx}{u^2}\right)^2,$$

where A and B are arbitrary constants ($A \neq 0$).

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

3. $$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(w^{-4/3}\frac{\partial w}{\partial x}\right) + f(x)w^{-1/3}, \quad a > 0.$$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^3 w\big(x, \pm C_1^{-2}t + C_2\big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Suppose $u = u(x)$ is any nontrivial particular solution of the second-order linear ordinary differential equation

$$au''_{xx} - \tfrac{1}{3}f(x)u = 0. \tag{1}$$

The transformation

$$\xi = \int \frac{dx}{u^2}, \quad z = wu^3 \tag{2}$$

brings the original equation to a simpler equation of the form 3.1.6.5 with $n = -4/3$:

$$\frac{\partial^2 z}{\partial t^2} = a\frac{\partial}{\partial \xi}\left(z^{-4/3}\frac{\partial z}{\partial \xi}\right).$$

3°. For $f = b = \text{const}$, the auxiliary equation (1) employed to determine the transformation (2) has the following solution:

$$u(x) = \begin{cases} C_1\exp(\lambda x) + C_2\exp(-\lambda x) & \text{if } ab > 0, \\ C_1\cos(\lambda x) + C_2\sin(\lambda x) & \text{if } ab < 0, \end{cases}$$

where $\lambda = \left|\tfrac{1}{3}b/a\right|^{1/2}$; C_1 and C_2 are arbitrary constants.

For $f(x) = bx^m$ or $f(x) = be^{\beta x}$, the solutions of equation (1) are expressed in terms of Bessel functions.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + f(x)w^{n+1} + g(t)w.$

Multiplicative separable solution:

$$w = \varphi(x)\psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$a(\varphi^n \varphi'_x)'_x + f(x)\varphi^{n+1} + C\varphi = 0,$$
$$\psi''_{tt} - g(t)\psi + C\psi^{n+1} = 0.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w} \dfrac{\partial w}{\partial x}\right) + f(x)e^{\lambda w} + g(t).$

Additive separable solution:

$$w = \varphi(x) + \psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$a(e^{\lambda\varphi} \varphi'_x)'_x + f(x)e^{\lambda\varphi} + C = 0,$$
$$\psi''_{tt} - g(t) + Ce^{\lambda\psi} = 0.$$

By the change of variable $U = e^{\lambda\varphi}$ the first equation is reduced to the linear equation $aU''_{xx} + \lambda f(x)U + \lambda C = 0$.

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right].$

This equation is encountered in wave and gas dynamics.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm C_1 x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. The transformation

$$x = \tau, \quad t = z, \quad u = \int f(w)\,dw$$

leads to an equation of the similar form

$$\frac{\partial^2 u}{\partial \tau^2} = \frac{\partial}{\partial z}\left[g(u)\frac{\partial w}{\partial z}\right],$$

where the function $g = g(u)$ is defined parametrically as

$$u = \int f(w)\,dw, \quad g(u) = \frac{1}{f(w)}.$$

3°. Traveling-wave solution:

$$w = w(z), \quad z = x \pm \lambda t,$$

where $w = w(z)$ is defined implicitly by (A and B are arbitrary constants)

$$\lambda^2 w - \int f(w)\,dw = Az + B.$$

⊙ *Reference*: W. F. Ames, R. J. Lohner, and E. Adams (1981).

4°. Self-similar solution:

$$w = w(\xi), \quad \xi = \frac{x+a}{t+b},$$

where a and b are arbitrary constants, and the function $w(\xi)$ is determined by the ordinary differential equation

$$(\xi^2 w'_\xi)'_\xi = \bigl[f(w)w'_\xi\bigr]'_\xi,$$

which admits the first integral

$$\bigl[\xi^2 - f(w)\bigr]w'_\xi = C. \tag{1}$$

To the special case $C = 0$ there corresponds the solution (in implicit form):

$$\xi^2 = f(w).$$

For $C \neq 0$, treating w in (1) as the independent variable, one obtains a Riccati equation for $\xi = \xi(w)$:

$$C\xi'_w = \xi^2 - f(w). \tag{2}$$

The handbooks by Polyanin and Zaitsev (1995, 2003) present a large number of solutions to equation (2) for various $f = f(w)$.

By the change of variable $\xi = -Cy'_w/y$, equation (1) is reduced to the second-order linear equation $y''_{ww} = C^{-2}f(w)y$.

⊙ *References*: W. F. Ames, R. J. Lohner, and E. Adams (1981), V. F. Zaitsev and A. D. Polyanin (1996).

5°. Solution in parametric form:

$$x = C_1v^2 + C_2v + \int f(w)(2C_1w + C_3)\,dw + C_4,$$
$$t = (2C_1w + C_3)v + C_2w + C_5.$$

Here and henceforth, $C_1, \dots, C_5$ are arbitrary constants.

6°. Solution in parametric form:

$$x = [C_1F(w) + C_2]v + C_3F(w) + C_4, \quad F(w) = \int f(w)\,dw,$$
$$t = \frac{1}{2}C_1v^2 + C_3v + \int [C_1F(w) + C_2]\,dw + C_5.$$

7°. Solution in parametric form:

$$x = [C_1F(w) + C_2]v^2 + C_3F(w) + C_4 + 2\int \biggl\{f(w)\int [C_1F(w) + C_2]\,dw\biggr\}\,dw,$$
$$t = \frac{1}{3}C_1v^3 + C_3v + 2v\int [C_1F(w) + C_2]\,dw + C_5.$$

8°. Solution in parametric form:

$$x = (C_1e^{\lambda v} + C_2e^{-\lambda v})H(w) + C_3,$$
$$t = \frac{1}{\lambda}(C_1e^{\lambda v} - C_2e^{-\lambda v})\frac{1}{f(w)}H'_w(w) + C_4,$$

where $C_1, \dots, C_4$ and λ are arbitrary constants, the function $H = H(w)$ is determined by the ordinary differential equation $\mathbf{L}_f[H] - \lambda^2H = 0$, and the differential operator $\mathbf{L}_f$ is expressed as

$$\mathbf{L}_f[\varphi] \equiv \frac{d}{dw}\left[\frac{1}{f(w)}\frac{d\varphi}{dw}\right]. \tag{3}$$

9°. Solution in parametric form:

$$x = [C_1 \sin(\lambda v) + C_2 \cos(\lambda v)]Z(w) + C_3,$$
$$t = \frac{1}{\lambda}[C_2 \sin(\lambda v) - C_1 \cos(\lambda v)]\frac{1}{f(w)}Z'_w(w) + C_4,$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants, the function $Z = Z(w)$ is determined by the ordinary differential equation $\mathbf{L}_f[Z] + \lambda^2 Z = 0$, and the differential operator $\mathbf{L}_f$ is defined by (3).

10°. Solution in parametric form:

$$x = [2C_1 F(w) + C_3]v + C_2 F(w) + C_5, \quad F(w) = \int f(w)\,dw,$$
$$t = C_1 v^2 + C_2 v + \int [2C_1 F(w) + C_3]\,dw + C_4.$$

11°. Solution in parametric form:

$$x = \frac{1}{2}C_1 v^2 + C_3 v + \int f(w)(C_1 w + C_2)\,dw + C_5,$$
$$t = (C_1 w + C_2)v + C_3 w + C_4.$$

12°. Solution in parametric form:

$$x = \frac{1}{3}C_1 v^3 + C_3 v + 2v \int f(w)(C_1 w + C_2)\,dw + C_5,$$
$$t = (C_1 w + C_2)v^2 + C_3 w + C_4 + 2\int \left\{ \int f(w)(C_1 w + C_2)\,dw \right\} dw.$$

13°. Solution in parametric form:

$$x = \frac{1}{\lambda}(C_1 e^{\lambda v} - C_2 e^{-\lambda v})H'_w(w) + C_3,$$
$$t = (C_1 e^{\lambda v} + C_2 e^{-\lambda v})H(w) + C_4,$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants, and the function $H = H(w)$ is determined by the ordinary differential equation $H''_{ww} - \lambda^2 f(w)H = 0$.

14°. Solution in parametric form:

$$x = \frac{1}{\lambda}[C_2 \sin(\lambda v) - C_1 \cos(\lambda v)]Z'_w(w) + C_3,$$
$$t = [C_1 \sin(\lambda v) + C_2 \cos(\lambda v)]Z(w) + C_4,$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants, and the function $Z = Z(w)$ is determined by the ordinary differential equation $Z''_{ww} + \lambda^2 f(w)Z = 0$.

15°. The original equation can be represented as the system of equations

$$f(w)\frac{\partial w}{\partial x} = \frac{\partial v}{\partial t}, \quad \frac{\partial w}{\partial t} = \frac{\partial v}{\partial x}. \tag{4}$$

The hodograph transformation

$$x = x(w, v), \quad t = t(w, v), \tag{5}$$

where w and v are treated as the independent variables and x and t as the dependent ones, brings (4) to the linear system

$$f(w)\frac{\partial t}{\partial v} = \frac{\partial x}{\partial w}, \quad \frac{\partial x}{\partial v} = \frac{\partial t}{\partial w}. \tag{6}$$

Eliminating t, we obtain a linear equation for $x = x(w, v)$:

$$\frac{\partial}{\partial w}\left[\frac{1}{f(w)}\frac{\partial x}{\partial w}\right] - \frac{\partial^2 x}{\partial v^2} = 0. \tag{7}$$

Likewise, from system (6) we obtain another linear equation for $t = t(w, v)$:

$$\frac{\partial^2 t}{\partial w^2} - f(w)\frac{\partial^2 t}{\partial v^2} = 0. \tag{8}$$

The procedure for constructing exact solutions of the original nonlinear equation consists of the following two stages. First, one finds an exact solution of the linear equation (7) for $x = x(w, v)$. Further this solution is substituted into the linear system (6), which is then solved to obtain $t = t(w, v)$ in the form

$$t = \int_{v_0}^{v} \frac{1}{f(w)}\frac{\partial x}{\partial w}(w, \xi)\, d\xi + \int_{w_0}^{w} \frac{\partial x}{\partial v}(\eta, v_0)\, d\eta, \tag{9}$$

where w_0 and v_0 are any numbers. The thus obtained expressions of (5) will give an exact solution of the original equation in parametric form.

Likewise, one can first construct an exact solution to the linear equation (8) for $t = t(w, v)$ and then determine $x = x(w, v)$ from (6).

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (2001).

16°. Solutions of equation (7) with even powers of v:

$$x = \sum_{k=0}^{n} \varphi_k(w) v^{2k}, \tag{10}$$

where the functions $\varphi_k = \varphi_k(w)$ are determined by the recurrence relations

$$\varphi_n(w) = A_n F(w) + B_n, \qquad F(w) = \int f(w)\, dw,$$
$$\varphi_{k-1}(w) = A_k F(w) + B_k + 2k(2k-1)\int f(w)\left\{\int \varphi_k(w)\, dw\right\} dw,$$

where the A_k and B_k are arbitrary constants ($k = n, \dots, 1$).

The dependence $t = t(w, v)$ is defined by (9) and, together with (10), gives a solution of the original nonlinear equation in parametric form.

17°. Solutions of equation (7) with odd powers of v:

$$x = \sum_{k=0}^{n} \psi_k(w) v^{2k+1}, \tag{11}$$

where the functions $\psi_k = \psi_k(w)$ are determined by the recurrence relations

$$\psi_n(w) = A_n F(w) + B_n, \qquad F(w) = \int f(w)\, dw,$$
$$\psi_{k-1}(w) = A_k F(w) + B_k + 2k(2k+1)\int f(w)\left\{\int \psi_k(w)\, dw\right\} dw,$$

where the A_k and B_k are arbitrary constants ($k = n, \dots, 1$).

The dependence $t = t(w, v)$ is defined by (9) and, together with (11), gives a solution of the original nonlinear equation in parametric form.

⊙ *References for equation* 3.4.4.6: W. F. Ames, R. J. Lohner, and E. Adams (1981), N. H. Ibragimov (1994), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2002).

7. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] - a^2\dfrac{f'(w)}{f^3(w)} + b.$

Functional separable solution in implicit form:

$$\int f(w)\,dw = at - \frac{1}{2}bx^2 + C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial^2 w}{\partial t^2} = f(w)\dfrac{\partial^2 w}{\partial x^2} + g(w)\dfrac{\partial w}{\partial x} + h(w).$

Traveling-wave solution:

$$w = w(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant, and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$[f(w) - \lambda^2]w''_{zz} + g(w)w'_z + h(w) = 0.$$

By the change variable $u(w) = w'_z$ this equation is reduced to the Abel equation

$$[f(w) - \lambda^2]uu'_w + g(w)u + h(w) = 0. \tag{1}$$

The substitution $\xi = -\displaystyle\int \frac{g(w)\,dw}{f(w) - \lambda^2}$ brings (1) to the canonical form

$$uu'_\xi - u = F(\xi), \tag{2}$$

where the function $F = F(\xi)$ is defined parametrically as

$$F(\xi) = \frac{h(w)}{g(w)}, \qquad \xi = -\int \frac{g(w)\,dw}{f(w) - \lambda^2}.$$

A large number of exact solutions to the Abel equation (2) for various $F = F(\xi)$ can be found in Polyanin and Zaitsev (2003).

9. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + g(w)\dfrac{\partial w}{\partial x} + h(w).$

Traveling-wave solution:

$$w = w(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant, and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$\{[f(w) - \lambda^2]w'_z\}'_z + g(w)w'_z + h(w) = 0.$$

With the change of variable

$$u(w) = [f(w) - \lambda^2]w'_z,$$

this equation is reduced to the Abel equation

$$uu'_w + g(w)u + h(w)[f(w) - \lambda^2] = 0. \tag{1}$$

The substitution $\xi = -\displaystyle\int g(w)\,dw$ brings (1) to the canonical form

$$uu'_\xi - u = F(\xi), \tag{2}$$

where the function $F = F(\xi)$ is defined parametrically by

$$F(\xi) = \frac{h(w)}{g(w)}[f(w) - \lambda^2], \qquad \xi = -\int g(w)\,dw.$$

A large number of exact solutions to the Abel equation (2) for various $F = F(\xi)$ can be found in Polyanin and Zaitsev (2003).

10. $\displaystyle \frac{\partial^2 w}{\partial t^2} = f(w)\frac{\partial^2 w}{\partial x^2} + af'_w(w)\left(\frac{\partial w}{\partial x}\right)^2.$

Equations of this form are encountered in the theory of liquid crystals and other applications.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm C_1 x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. In the general case, the equation has exact solutions of the form

$$w(x,t) = w(z), \quad z = kx + \lambda t \quad \text{(traveling-wave solution)},$$

$$w(x,t) = w(\xi), \quad \xi = \frac{x+b}{t+c} \quad \text{(self-similar solution)},$$

where k, λ, b, and c are arbitrary constants.

3°. The structure of other exact solutions for some specific $f(w)$:

$$f(w) = Aw + B, \qquad w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t);$$

$$f(w) = Aw^k, \qquad w(x,t) = \varphi(x)\psi(t);$$

$$f(w) = Ae^{\beta w}, \qquad w(x,t) = \varphi(x) + \psi(t).$$

4°. A qualitative analysis of the structure of solutions to the original equation was undertaken in Glassey, Hunter, and Zheng (1997) and Melikyan (1998).

3.4.5. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = f(x,w)\dfrac{\partial^2 w}{\partial x^2} + g\Big(x,t,w,\dfrac{\partial w}{\partial x}\Big)$

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} = f(x)w^m\frac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w\big(x,\ \pm C_1^m t + C_2\big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = g(t)h(x),$$

where the functions $g = g(t)$ and $h = h(x)$ are determined by the ordinary differential equations

$$g''_{tt} - \lambda g^{m+1} = 0, \tag{1}$$

$$h''_{xx} - \lambda\big[f(x)\big]^{-1}h^{1-m} = 0, \tag{2}$$

where λ is an arbitrary constant.

The general solution of equation (1) is written out in implicit form:

$$\int\left(C_1 + \frac{2\lambda}{m+2}g^{m+2}\right)^{-1/2} dg = C_2 \pm t,$$

where C_1 and C_2 are arbitrary constants.

In particular, if $C_1 = 0$, it follows that

$$g(t) = (at + C)^{-2/m}, \qquad a = \pm\sqrt{\frac{\lambda m^2}{2(m+2)}}.$$

For $m = 1$, the general solution of equation (2) is expressed as

$$h(x) = \lambda\int_{x_0}^{x}\frac{(x-\xi)}{f(\xi)}\,d\xi + Ax + B,$$

where A, B, and x_0 are arbitrary constants.

The book by Polyanin and Zaitsev (2003, Sections 2.3 and 2.7) presents a large number of exact solutions to the generalized Emden–Fowler equation (2) for various $f = f(x)$.

3°. The transformation

$$u(z,t) = \frac{1}{x} w(x,t), \quad z = \frac{1}{x}$$

leads to an equation of the similar form

$$\frac{\partial^2 u}{\partial t^2} = z^{4-m} f\left(\frac{1}{z}\right) u^m \frac{\partial^2 u}{\partial z^2}.$$

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f(x) w^m \frac{\partial w}{\partial x}\right].$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w\bigl(x, \pm C_1^m t + C_2\bigr),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = g(t) h(x),$$

where the functions $g = g(t)$ and $h = h(x)$ are determined by the ordinary differential equations

$$g''_{tt} - \lambda g^{m+1} = 0, \tag{1}$$

$$[f(x) h^m h'_x]'_x - \lambda h = 0, \tag{2}$$

and λ is an arbitrary constant.

The general solution of equation (1) is written out in implicit form:

$$\int \left(C_1 + \frac{2\lambda}{m+2} g^{m+2}\right)^{-1/2} dg = C_2 \pm t,$$

where C_1 and C_2 are arbitrary constants.

In particular, if $C_1 = 0$, it follows that

$$g(t) = (at + C)^{-2/m}, \qquad a = \pm\sqrt{\frac{\lambda m^2}{2(m+2)}}.$$

The transformation

$$z = \int \frac{dx}{f(x)}, \quad \Phi = h^{m+1}$$

brings (2) to the generalized Emden–Fowler equation

$$\Phi''_{zz} - F(z) \Phi^{\frac{1}{m+1}} = 0, \tag{3}$$

where the function $F = F(z)$ is defined parametrically by

$$F = \lambda(m+1) f(x), \qquad z = \int \frac{dx}{f(x)}.$$

The book by Polyanin and Zaitsev (2003, Sections 2.3 and 2.7) presents a large number of exact solutions to equation (2) for various $F = F(z)$.

3°. The transformation

$$w(x,t) = \bigl[\psi(x)\bigr]^{\frac{1}{m+1}} u(\xi, t), \quad \xi = \int \bigl[\psi(x)\bigr]^{\frac{m+2}{m+1}} dx, \qquad \psi(x) = \int \frac{dx}{f(x)},$$

leads to an equation of the similar form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial \xi}\left[\mathcal{F}(\xi) u^m \frac{\partial u}{\partial \xi}\right],$$

where the function $\mathcal{F} = \mathcal{F}(\xi)$ is defined parametrically by

$$\mathcal{F} = f(x)\bigl[\psi(x)\bigr]^{\frac{3m+4}{m+1}}, \quad \xi = \int \bigl[\psi(x)\bigr]^{\frac{m+2}{m+1}} dx, \qquad \psi(x) = \int \frac{dx}{f(x)}.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = w^4 f\left(\dfrac{w}{x}\right)\dfrac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\bigl(C_1^{-1}x,\ \pm C_1 t + C_2\bigr),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. With the transformation

$$u(z,t) = \frac{1}{x}w(x,t), \qquad z = \frac{1}{x}$$

one arrives at the simpler equation

$$\frac{\partial^2 u}{\partial t^2} = u^4 f(u)\frac{\partial^2 u}{\partial z^2},$$

which has a traveling-wave solution $u = u(z+\lambda t)$ and self-similar solutions of the form $u = u(z/t)$.

4. $\dfrac{\partial^2 w}{\partial t^2} = w^4 f\left(\dfrac{w}{\sqrt{ax^2+bx+c}}\right)\dfrac{\partial^2 w}{\partial x^2}.$

The transformation

$$w(x,t) = u(z,t)\sqrt{ax^2+bx+c}, \qquad z = \int \frac{dx}{ax^2+bx+c}$$

leads to an equation of the form 3.4.4.8:

$$\frac{\partial^2 u}{\partial t^2} = u^4 f(u)\frac{\partial^2 u}{\partial z^2} + (ac - \tfrac{1}{4}b^2)u^5 f(u),$$

which has a traveling-wave solution $u = u(z+\lambda t)$.

3.4.6. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = f(t,w)\dfrac{\partial^2 w}{\partial x^2} + g\left(x,t,w,\dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = f(t)\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-2} w\bigl(C_1 x + C_2,\ t\bigr),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solutions:

$$w(x,t) = (C_1 t + C_2)(C_3 x + C_4)^{1/2},$$
$$w(x,t) = (C_1 t + C_2)x + \int_a^t (t-\tau)(C_1\tau + C_2)^2 f(\tau)\,d\tau + C_3 t + C_4,$$

where $C_1, \ldots, C_4$ and a are arbitrary constants.

3°. Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = 6f(t)\varphi^2,$$
$$\psi''_{tt} = 6f(t)\varphi\psi,$$
$$\chi''_{tt} = 2f(t)\varphi\chi + f(t)\psi^2.$$

4°. Multiplicative separable solution:

$$w(x,t) = \Phi(t)\Psi(x),$$

where the functions $\Phi = \Phi(t)$ and $\Psi = \Psi(x)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$\Phi''_{tt} = Cf(t)\Phi^2,$$
$$(\Psi\Psi'_x)'_x = C\Psi.$$

The last equation is autonomous and has a particular solution $\Psi = \frac{1}{6}Cx^2$; in the general case, it is integrable by quadrature.

2. $\dfrac{\partial^2 w}{\partial t^2} = f(t)\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + g(t)w + h_2(t)x^2 + h_1(t)x + h_0(t).$

Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = 6f(t)\varphi^2 + g(t)\varphi + h_2(t),$$
$$\psi''_{tt} = 6f(t)\varphi\psi + g(t)\psi + h_1(t),$$
$$\chi''_{tt} = 2f(t)\varphi\chi + f(t)\psi^2 + g(t)\chi + h_0(t).$$

3. $\dfrac{\partial^2 w}{\partial t^2} = [a(t)w + b(t)]\dfrac{\partial^2 w}{\partial x^2} + c(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + d(t)w + e(t)x^2 + f(t)x + g(t).$

There is a generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

⊙ *Reference*: V. A. Galaktionov (1995); the case of $a = 1$, $b = e = f = 0$, and $c = \text{const}$ was considered.

3.4.7. Other Equations Linear in the Highest Derivatives

1. $\dfrac{\partial^2 w}{\partial t^2} + f(t)\dfrac{\partial w}{\partial t} = g(t)\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\bigl(C_1x + C_2,\ t\bigr) - \frac{2}{\lambda}\ln|C_1|,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = \frac{1}{\lambda}\ln(C_1x + C_2) + C_3\int F(t)\,dt + C_4, \qquad F(t) = \exp\left[-\int f(t)\,dt\right],$$

where $C_1, \dots, C_4$ are arbitrary constants.

3°. Additive separable solution:

$$w(x,t) = \frac{1}{\lambda}\ln(\lambda C_1x^2 + C_2x + C_3) + u(t),$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $u = u(t)$ is determined by the ordinary differential equation

$$u''_{tt} + f(t)u'_t = 2C_1g(t)e^{\lambda u}.$$

2. $\dfrac{\partial^2 w}{\partial t^2} + f(w)\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\left[g(w)\dfrac{\partial w}{\partial x}\right].$

1°. Traveling-wave solution in implicit form:

$$\int \frac{k^2 g(w) - \lambda^2}{\lambda F(w) + C_1}\,dw = kx + \lambda t + C_2, \qquad F(w) = \int f(w)\,dw,$$

where C_1, C_2, k, and λ are arbitrary constants.

2°. For exact solutions of this equation for some specific $f(w)$ and $g(w)$, see Baikov, Gazizov, and Ibragimov (1989) and Ibragimov (1994).

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, \dfrac{\partial w}{\partial x}\right) + bw + g(t).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(x,t) + C_1\cosh(kt) + C_2\sinh(kt) \quad \text{if } b = k^2 > 0,$$
$$w_2 = w(x,t) + C_1\cos(kt) + C_2\sin(kt) \quad \text{if } b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x).$$

Here, the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - g(t) = 0,$$
$$a\psi''_{xx} + f\bigl(x, \psi'_x\bigr) + b\psi = 0.$$

The solution of the first equation is expressed as

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t g(\tau)\sinh\bigl[k(t-\tau)\bigr]\,d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t g(\tau)\sin\bigl[k(t-\tau)\bigr]\,d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

Special case. For $f(x, w_x) = f(w_x)$, there are more complicated solutions of the form $w(x,t) = \varphi(t) + \psi(z)$, where $z = x + \lambda t$.

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, \dfrac{\partial w}{\partial x}\right) + g\left(t, \dfrac{\partial w}{\partial t}\right).$

Additive separable solution:

$$w(x,t) = \varphi(x) + \psi(t),$$

where the functions $\varphi(x)$ and $\psi(t)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$a\varphi''_{xx} + f\bigl(x, \varphi'_x\bigr) = C,$$
$$\psi''_{tt} - g\bigl(t, \psi'_t\bigr) = C.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + f\left(x, \dfrac{\partial w}{\partial x}\right) + g\left(t, \dfrac{\partial w}{\partial t}\right) + bw.$

Additive separable solution:

$$w(x,t) = \varphi(x) + \psi(t),$$

where the functions $\varphi(x)$ and $\psi(t)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$a\varphi''_{xx} + f\bigl(x, \varphi'_x\bigr) + b\varphi = C,$$
$$\psi''_{tt} - g\bigl(t, \psi'_t\bigr) - b\psi = C.$$

6. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + wf\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial x}\right) + wg\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial t}\right) + bw \ln w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(x)\psi(t),$$

where the functions $\varphi(x)$ and $\psi(t)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$a\varphi''_{xx} + \varphi f\bigl(x, \varphi'_x/\varphi\bigr) + b\varphi \ln \varphi + C\varphi = 0,$$
$$\psi''_{tt} - \psi g\bigl(t, \psi'_t/\psi\bigr) - b\psi \ln \psi + C\psi = 0.$$

7. $\dfrac{\partial^2 w}{\partial t^2} = f\left(\dfrac{\partial w}{\partial x}\right)\dfrac{\partial^2 w}{\partial x^2}.$

For $f(z) = -z$, this equation is encountered in aerodynamics (theory of transonic gas flows).

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-1} w(C_1 x + C_2, C_1 t + C_3) + C_4 t + C_5,$$

where $C_1, \dots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = Axt + Bx + Cx + D,$$

where A, B, C, and D are arbitrary constants.

3°. Additive separable solution:

$$w(x,t) = At^2 + Bt + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation $2A = f(\varphi'_x)\varphi''_{xx}$. Its general solution can be represented in parametric form (C_1 and C_2 are arbitrary constants):

$$x = \frac{1}{2A}\int f(\xi)\,d\xi + C_1, \qquad \varphi = \frac{1}{2A}\int \xi f(\xi)\,d\xi + C_2.$$

4°. Solution of the more general form

$$w(x,t) = At^2 + Bt + \varphi(z), \qquad z = x + \lambda t,$$

where A, B, and λ are arbitrary constants, and the function $\varphi = \varphi(z)$ is determined by the ordinary differential equation $2A = \bigl[f(\varphi'_z) - \lambda^2\bigr]\varphi''_{zz}$. Its general solution can be represented in parametric form (C_1 and C_2 are arbitrary constants):

$$z = \frac{1}{2A}\int f(\xi)\,d\xi - \frac{\lambda^2}{2A}\xi + C_1, \qquad \varphi = \frac{1}{2A}\int \xi f(\xi)\,d\xi - \frac{\lambda^2}{4A}\xi^2 + C_2.$$

5°. Self-similar solution:

$$w = x\psi(z), \qquad z = x/t,$$

where the function $\psi = \psi(z)$ is determined by the ordinary differential equation

$$[f(z\psi'_z + \psi) - z^2](z\psi''_{zz} + 2\psi'_z) = 0.$$

Equating the expression in square brackets to zero, we have

$$f(z\psi'_z + \psi) - z^2 = 0.$$

The general solution of this equation in parametric form:

$$z = \pm\sqrt{f(\tau)}, \qquad \psi = \frac{1}{2\sqrt{f(\tau)}}\int \frac{\tau f'_\tau(\tau)}{\sqrt{f(\tau)}}\,d\tau + C.$$

6°. The Legendre transformation

$$u(z,\tau) = tz + x\tau - w(x,t), \quad z = \frac{\partial w}{\partial t}, \quad \tau = \frac{\partial w}{\partial x},$$

where u is the new dependent variable, and z and τ are the new independent variables, leads to the linear equation

$$\frac{\partial^2 u}{\partial \tau^2} = f(\tau)\frac{\partial^2 u}{\partial z^2}.$$

7°. The substitution $v(x,t) = \dfrac{\partial w}{\partial x}$ leads to an equation of the form 3.4.4.6:

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left[f(v)\frac{\partial v}{\partial x}\right].$$

⊙ *Reference*: N. H. Ibragimov (1994).

8°. Below are exact solutions of the equation for some specific $f = f(U)$.

Special case 1. Let $f(U) = aU$.

1°. Generalized separable solution quadratic in x:

$$w = (C_1t + C_2)x^2 + \left[\tfrac{1}{3}aC_1^{-2}(C_1t + C_2)^4 + C_3t + C_4\right]x$$
$$+ \tfrac{1}{63}a^2C_1^{-4}(C_1t + C_2)^7 + \tfrac{1}{6}aC_1C_3t^4 + \tfrac{1}{3}a(C_1C_4 + C_2C_3)t^3 + aC_2C_4t^2 + C_5t + C_6,$$

where $C_1, \dots, C_6$ are arbitrary constants.

2°. Generalized separable solution cubic in x:

$$w = f(t)x^3 + g(t)x^2 + h(t)x + p(t),$$

where the functions $f = f(t)$, $g = g(t)$, $h = h(t)$, $p = p(t)$ are determined by the system of ordinary differential equations

$$f''_{tt} = 18af^2,$$
$$g''_{tt} = 18afg,$$
$$h''_{tt} = 6afh + 4ag^2,$$
$$p''_{tt} = 2agh.$$

A particular solution of the system of the first three equations is given by

$$f = \frac{1}{3a(t + C_1)^2}, \quad g = \frac{C_2}{(t + C_1)^2} + C_3(t + C_1)^3,$$
$$h = \frac{C_4}{t + C_1} + C_5(t + C_1)^2 + \frac{aC_2^2}{(t + C_1)^2} + 2aC_2C_3(t + C_1)^3 + \frac{2aC_3^2}{27}(t + C_1)^8,$$

where $C_1, \dots, C_5$ are arbitrary constants. The function $p = p(t)$ is determined from the last equation by integrating the right-hand side twice.

3°. There is the solution in multiplicative separable form: $w = \varphi(x)\psi(t)$.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

Special case 2. Let $f(U) = aU^k$.

1°. Multiplicative separable solution:

$$w = \varphi(x)\psi(t),$$

where the functions $\varphi(x)$ and $\psi(t)$ are determined by the autonomous ordinary differential equations

$$(\psi'_t)^2 = \frac{2aC_1}{k+2}\psi^{k+2} + C_2, \qquad \frac{2}{k+2}(\varphi'_x)^{k+2} = C_1\varphi^2 + C_3,$$

and C_1, C_2, and C_3 are arbitrary constants. The general solutions to these equations can be written out in implicit form. Below are exact solutions representable in explicit form:

$$\psi(t) = A_1t^{-2/k} \quad \text{if} \quad C_2 = 0, \qquad \varphi(x) = A_2x^{(k+2)/k} \quad \text{if} \quad C_3 = 0.$$

The coefficients A_1 and A_2 are determined by substituting these expressions into the above equations.

2°. Self-similar solution:

$$w = t^\sigma u(\zeta), \quad \zeta = t^\beta x, \quad \sigma = -(k\beta + 2\beta + 2)/k,$$

where β is an arbitrary constant, and the function $u(\zeta)$ is determined by the ordinary differential equation

$$\sigma(\sigma-1)u + \beta(2\sigma+\beta-1)\zeta u'_\zeta + \beta^2\zeta^2 u''_{\zeta\zeta} = a(u'_\zeta)^k u''_{\zeta\zeta}.$$

3°. Conservation laws for $a = 1$:

$$D_t\big(w_t\big) + D_x\left(-\frac{1}{k+1}w_x^{k+1}\right) = 0,$$
$$D_t\big(w_t w_x\big) + D_x\left(-\frac{1}{k+2}w_x^{k+2} - \frac{1}{2}w_t^2\right) = 0,$$
$$D_t\left(\frac{1}{2}w_t^2 + \frac{1}{(k+1)(k+2)}w_x^{k+2}\right) + D_x\left(-\frac{1}{k+1}w_t w_x^{k+1}\right) = 0,$$
$$D_t\big(tw_t - w\big) + D_x\left(-\frac{1}{k+1}tw_x^{k+1}\right) = 0,$$

where $D_t = \frac{\partial}{\partial t}$, $D_x = \frac{\partial}{\partial x}$.

⊙ *References*: V. A. Vinokurov and I. G. Nurgalieva (1985), N. H. Ibragimov (1994).

Special case 3. Let $f(U) = a\exp(\lambda U)$.
Generalized separable solution:

$$w = (x + C_1)\varphi(t) + \psi(x).$$

Here, the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} = aC_2\exp(\lambda\varphi), \qquad (1)$$
$$\exp(\lambda\psi'_x)\psi''_{xx} = C_2(x + C_1), \qquad (2)$$

where C_1 and C_2 are arbitrary constants. The general solution of equation (1) is given by

$$\varphi(t) = -\frac{1}{\lambda}\ln\left[\frac{aC_2\lambda}{2\beta^2}\cos^2(\beta t + C_3)\right] \quad \text{if} \quad aC_2\lambda > 0,$$
$$\varphi(t) = -\frac{1}{\lambda}\ln\left[\frac{aC_2\lambda}{2\beta^2}\sinh^2(\beta t + C_3)\right] \quad \text{if} \quad aC_2\lambda > 0,$$
$$\varphi(t) = -\frac{1}{\lambda}\ln\left[-\frac{aC_2\lambda}{2\beta^2}\cosh^2(\beta t + C_3)\right] \quad \text{if} \quad aC_2\lambda < 0,$$

where C_3 and β are arbitrary constants. The general solution of equation (2) is expressed as

$$\psi(x) = \int \ln\left(\tfrac{1}{2}C_2x^2 + C_1C_2x + C_4\right)dx + \frac{\ln\lambda}{\lambda}x + C_5,$$

where C_4 and C_5 are arbitrary constants.

⊙ *References for equation* 3.4.7.7: N. H. Ibragimov (1994), A. D. Polyanin and V. F. Zaitsev (2002).

8. $\displaystyle a\frac{\partial^2 w}{\partial t^2} + \frac{\partial w}{\partial t} = f\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial x^2}, \qquad a \neq 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1(x,t) = w(t + C_1, x + C_2) + C_3 e^{-t/a} + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = U(z) + C_1 e^{-t/a} + C_2, \qquad z = x + \lambda t,$$

where C_1, C_2, and λ are arbitrary constants, and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$\lambda U'_z = \big[f(U'_z) - a\lambda^2\big]U''_{zz}.$$

Integrating yields its solution in parametric form:

$$U = \frac{1}{\lambda}\int f(\tau)\,d\tau - a\lambda\tau + C_3, \qquad z = \frac{1}{\lambda}\int \frac{f(\tau)}{\tau}\,d\tau - a\lambda\ln|\tau| + C_4,$$

where C_3 and C_4 are arbitrary constants (C_3 can be set equal to zero).

3°. Additive separable solution:

$$w(x,t) = C_1 t + C_2 + C_3 e^{-t/a} + \varphi(x),$$

where C_1, C_2, and C_3 are arbitrary constants, the function $\varphi(x)$ is determined by the autonomous ordinary differential equation $f(\varphi'_x)\varphi''_{xx} = C_1$. Integrating yields its solution in parametric form:

$$\varphi = \frac{1}{C_1}\int \xi f(\xi)\,d\xi + C_4, \qquad x = \frac{1}{C_1}\int f(\xi)\,d\xi + C_5,$$

where C_4 and C_5 are arbitrary constants (C_4 can be set equal to zero).

4°. The solutions of Items 3° and 4° are special cases of the more general solution

$$w(x,t) = C_1 t + C_2 + C_3 e^{-t/a} + \varphi(z), \qquad z = x + \lambda t,$$

where the function $\varphi(z)$ is determined by the autonomous ordinary differential equation

$$\lambda\varphi'_z + C_1 = \bigl[f(\varphi'_z) - a\lambda^2\bigr]\varphi''_{zz}.$$

5°. The contact transformation

$$\bar t = t + a\ln|w_x|, \quad \bar x = w + a w_t, \quad \bar w = x + a w_t/w_x, \quad \bar w_{\bar x} = 1/w_x, \quad \bar w_{\bar t} = -w_t/w_x \tag{1}$$

leads to an equation of the similar form

$$a\frac{\partial^2 \bar w}{\partial \bar t^2} + \frac{\partial \bar w}{\partial \bar t} = F\left(\frac{\partial \bar w}{\partial \bar x}\right)\frac{\partial^2 \bar w}{\partial \bar x^2}, \qquad \text{where} \qquad F(u) = \frac{1}{u^2} f\left(\frac{1}{u}\right).$$

Transformation (1) has an inverse; it is given by

$$t = \bar t + a\ln|\bar w_{\bar x}|, \quad x = \bar w + a\bar w_{\bar t}, \quad w = \bar x + a\bar w_{\bar t}/\bar w_{\bar x}, \quad w_x = 1/\bar w_{\bar x}, \quad w_t = -\bar w_{\bar t}/\bar w_{\bar x}. \tag{2}$$

The formulas of (2) can be used if the Jacobian function $J = \bigl[(\bar w_{\bar x} + a\bar w_{\bar x\bar t})^2 - a\bar w_{\bar x\bar x}(\bar w_{\bar t} + a\bar w_{\bar t\bar t})\bigr]$ is nonzero.

Special case 1. For $f(w_x) = b(w_x)^{-2}$, transformation (1) leads to the linear telegraph equation

$$a\frac{\partial^2 \bar w}{\partial \bar t^2} + \frac{\partial \bar w}{\partial \bar t} = b\frac{\partial^2 \bar w}{\partial \bar x^2}.$$

⊙ *References*: S. R. Svirshchevskii (1986, 1988), N. H. Ibragimov (1994).

6°. Conservation laws:

$$D_t\bigl(a w_t + w\bigr) + D_x\bigl[-\Psi'(w_x)\bigr] = 0,$$
$$D_t\bigl(a e^{t/a} w_t\bigr) + D_x\bigl[-e^{t/a}\Psi'(w_x)\bigr] = 0,$$
$$D_t\bigl(a e^{t/a} w_t w_x\bigr) + D_x\bigl\{e^{t/a}\bigl[\Psi(w_x) - w_x\Psi'(w_x) - \tfrac{1}{2}a(w_t)^2\bigr]\bigr\} = 0,$$

where the prime stands for the differentiation,

$$D_t = \frac{\partial}{\partial t}, \qquad D_x = \frac{\partial}{\partial x}, \qquad \Psi(u) = \int_0^u (u-\zeta) f(\zeta)\,d\zeta + C_1 u + C_2,$$

and C_1 and C_2 are arbitrary constants.

Special case 2. For $f(w_x) = b w_x^n$ ($n \neq 0, -2$), there is an additional conservation law:

$$D_t\left\{a e^{t/a}\left[\frac{3n+4}{2} a w_t^2 + w_t\bigl((n+2)w - n x w_x\bigr) + (3n+4)\Psi\right]\right\}$$
$$+ D_t\left\{e^{t/a}\left[\frac{n}{2} a x w_t^2 + \frac{d\Psi}{dw_x}\bigl(n x w_x - a(3n+4)w_t - (n+2)w\bigr) - n x\Psi + \Phi\right]\right\} = 0,$$

where

$$\Psi = \frac{b}{(n+1)(n+2)} w_x^{n+2}, \quad \Phi = 0 \quad \text{if } n \neq -1;$$
$$\Psi = b w_x(\ln|w_x| - 1), \quad \Phi = 2bw \quad \text{if } n = -1.$$

Special case 3. For $f(w_x) = b e^{k w_x}$, there is an additional conservation law ($k \neq 0$):

$$D_t\left\{a e^{t/a}\left[\frac{3}{2} a k w_t^2 + w_t(kw + 2x - kxw_x) + \frac{3b}{k} e^{k w_x}\right]\right\} + D_x\left\{e^{t/a}\left[\frac{1}{2} a k x w_t^2 - b\left(w + \frac{3}{k}x - x w_x + 3 a w_t\right) e^{k w_x}\right]\right\} = 0.$$

⊙ *References*: S. R. Svirshchevskii (1986, 1988), N. H. Ibragimov (1994).

9. $\dfrac{\partial^2 w}{\partial t^2} = f\left(\dfrac{\partial w}{\partial t}, \dfrac{\partial w}{\partial x}\right)\dfrac{\partial^2 w}{\partial x^2}$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w\big(C_1 x + C_2,\ C_1 t + C_3\big) + C_4,$$

where $C_1, \dots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. The Legendre transformation

$$u(z,\tau) = tz + x\tau - w(x,t), \quad z = \frac{\partial w}{\partial t}, \quad \tau = \frac{\partial w}{\partial x},$$

where u is the new dependent variable, and z and τ are the new independent variables, leads to the linear equation

$$\frac{\partial^2 u}{\partial \tau^2} = f(z,\tau)\frac{\partial^2 u}{\partial z^2}.$$

Exact solutions of this equation for some specific $f(z,\tau)$ can be found in Polyanin (2002).

3.5. Equations of the Form $\dfrac{\partial^2 w}{\partial x \partial y} = F\left(x, y, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right)$

3.5.1. Equations Involving Arbitrary Parameters of the Form $\dfrac{\partial^2 w}{\partial x \partial y} = f(w)$

1. $\dfrac{\partial^2 w}{\partial x \partial y} = aw^n$.

This is a special case of equation 3.5.3.1 with $f(w) = aw^n$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = (C_1C_2)^{\frac{1}{n-1}} w\big(C_1 x + C_3,\ C_2 y + C_4\big),$$

where $C_1, \dots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,y) = \left[\frac{a(1-n)^2}{2(1+n)}\right]^{\frac{1}{1-n}} \left(C_1 x + \frac{y}{C_1} + C_2\right)^{\frac{2}{1-n}},$$

$$w(x,y) = \big[a(1-n)^2\big]^{\frac{1}{1-n}} \big(xy + C_1 x + C_2 y + C_1 C_2\big)^{\frac{1}{1-n}}.$$

3°. Traveling-wave solution in implicit form (generalizes the first solution of Item 2°):

$$\int \left(C_2 + \frac{2a}{n+1} w^{n+1}\right)^{-1/2} dw = C_1 x + \frac{y}{C_1} + C_3.$$

4°. Self-similar solution:

$$w = x^{\frac{\beta-1}{n-1}} U(\xi), \quad \xi = yx^\beta,$$

where β is an arbitrary constant, and the function $U(\xi)$ is determined by the modified Emden–Fowler equation

$$\beta \xi U''_{\xi\xi} + \frac{n\beta - 1}{n-1} U'_\xi = aU^n.$$

For exact solutions of this equation, see the book by Polyanin and Zaitsev (2003).

2. $\displaystyle \frac{\partial^2 w}{\partial x \partial y} = a e^{\lambda w}.$

Liouville equation. This is a special case of equation 3.5.3.1 with $f(w) = ae^{\lambda w}$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = w\big(C_1 x + C_2,\ C_3 y + C_4\big) + \frac{1}{\lambda}\ln(C_1 C_3),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. General solution:

$$w = \frac{1}{\lambda}\big[f(x) + g(y)\big] - \frac{2}{\lambda}\ln\left|k\int \exp\big[f(x)\big]\,dx + \frac{a\lambda}{2k}\int \exp\big[g(y)\big]\,dy\right|,$$

where $f = f(x)$ and $g = g(y)$ are arbitrary functions and k is an arbitrary constant.

3°. The Liouville equation is related to the linear equation $\partial_{xy} u = 0$ by the Bäcklund transformation

$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial x} + \frac{2k}{\lambda}\exp\left[\frac{1}{2}\lambda(w + u)\right],$$
$$\frac{\partial u}{\partial y} = -\frac{\partial w}{\partial y} - \frac{a}{k}\exp\left[\frac{1}{2}\lambda(w - u)\right].$$

4°. The original equation can also be linearized with either of the differential substitutions

$$w = \frac{1}{\lambda}\ln\left(\frac{2}{v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right), \qquad v = v(x, y);$$
$$w = \frac{1}{\lambda}\ln\left(\frac{2}{\cos^2 z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right), \quad z = z(x, y).$$

5°. Solutions (for $a = \lambda = 1$):

$$w = \ln\left[f(x)g(y)\cosh^{-2}\left(C_1 + C_2\int g(y)\,dy - \frac{1}{2C_2}\int f(x)\,dx\right)\right],$$
$$w = \ln\left[f(x)g(y)\sinh^{-2}\left(C_1 + C_2\int g(y)\,dy + \frac{1}{2C_2}\int f(x)\,dx\right)\right],$$
$$w = \ln\left[f(x)g(y)\cos^{-2}\left(C_1 + C_2\int g(y)\,dy + \frac{1}{2C_2}\int f(x)\,dx\right)\right],$$

where $f(x)$ and $g(y)$ are arbitrary functions, and C_1 and C_2 are arbitrary constants.

⊙ *References*: J. Liouville (1853), R. K. Bullough and P. J. Caudrey (1980), S. V. Khabirov (1990), N. H. Ibragimov (1994).

3. $\displaystyle \frac{\partial^2 w}{\partial x \partial y} = e^{w} - e^{-2w}.$

This is a special case of equation 3.5.3.1 with $f(w) = e^{w} - e^{-2w}$.

1°. Solutions:

$$w = \ln\left[1 - 2\frac{\partial^2(\ln \zeta_k)}{\partial x \partial y}\right], \tag{1}$$

where

$$\zeta_1 = 1 + A\exp\left(kx + \frac{3}{k}y\right),$$

$$\zeta_2 = 1 + A_1\exp\left(k_1x + \frac{3}{k_1}y\right) + A_2\exp\left(k_2x + \frac{3}{k_2}y\right)$$

$$+ A_1A_2\frac{(k_1-k_2)^2(k_1^2 - k_1k_2 + k_2^2)}{(k_1+k_2)^2(k_1^2 + k_1k_2 + k_2^2)}\exp\left[(k_1+k_2)x + \left(\frac{3}{k_1} + \frac{3}{k_2}\right)y\right],$$

$$\zeta_3 = 1 + A(k^2x - 3y)\exp\left(kx + \frac{3}{k}y\right) - \frac{A^2k^2}{12}\exp\left(2kx + \frac{6}{k}y\right),$$

$$\zeta_4 = \sin\left(kx - \frac{3}{k}y\right) + \sqrt{3}\left(kx + \frac{3}{k}y\right),$$

and A, A_1, A_2, k, k_1, and k_2 are arbitrary constants.

2°. On passing to the new independent variables $z = x - y$ and $t = x + y$, one obtains an equation of the form 3.2.1.4:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial z^2} + e^w - e^{-2w}.$$

3°. The substitution $u = e^w$ leads to the Tzitzéica equation:

$$\frac{\partial^2(\ln u)}{\partial x \partial y} = u - \frac{1}{u^2}.$$

⊙ *Reference*: S. S. Safin and R. A. Sharipov (1993), O. V. Kaptsov and Yu. V. Shan'ko (1999, other exact solutions are also given there).

4. $\dfrac{\partial^2 w}{\partial x \partial y} = a\sinh w.$

Sinh-Gordon equation. On passing to the new independent variables $z = x - y$ and $t = x + y$, one obtains an equation of the form 3.3.1.1:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial z^2} + a\sinh w.$$

⊙ *References*: S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), A. Grauel (1985).

5. $\dfrac{\partial^2 w}{\partial x \partial y} = a\sin w.$

Sine-Gordon equation. This is a special case of equation 3.5.3.1 with $f(w) = a\sin w$.

1°. Traveling-wave solution:

$$w(x,y) = \begin{cases} 4\arctan\left[\exp\left(\sqrt{\dfrac{a}{AB}}\,(Ax + By + C)\right)\right] & \text{if } aAB > 0, \\ 4\,\mathrm{arctanh}\left[\exp\left(\sqrt{-\dfrac{a}{AB}}\,(Ax + By + C)\right)\right] & \text{if } aAB < 0, \end{cases}$$

where A, B, and C are arbitrary constants.

2°. Solution:

$$w(x,y) = 4\arctan\left[\frac{C_1 + C_2}{C_1 - C_2}\,\frac{\sinh(v_1 - v_2)}{\cosh(v_1 + v_2)}\right], \quad v_k = \frac{1}{2}\left(C_kx - \frac{a}{C_k}y\right), \quad k = 1,\ 2,$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: R. K. Bullough and P. J. Caudrey (1980).

3°. Self-similar solution:

$$w = U(\xi), \qquad \xi = xy,$$

where the function $U = U(\xi)$ is determined by the second-order ordinary differential equation $\xi U''_{\xi\xi} + U'_\xi = a \sin U$.

4°. The Bäcklund transformation

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial w}{\partial x} + 2k \sin\left(\frac{w+u}{2}\right), \\ \frac{\partial u}{\partial y} &= -\frac{\partial w}{\partial y} - \frac{2a}{k} \sin\left(\frac{w-u}{2}\right) \end{aligned} \tag{1}$$

brings the original equation to the identical equation

$$\frac{\partial^2 u}{\partial x \partial y} = a \sin u.$$

Given a single exact solution, the formulas of (1) allow us to successively generate other solutions of the sine-Gordon equation.

5°. The sine-Gordon equation has infinitely many conservation laws. The first three of them read as follows:

$$\begin{aligned} &D_x\bigl(w_y^2\bigr) + D_y\bigl(2a\cos w\bigr) = 0, \\ &D_x\bigl(w_y^4 - 4w_{yy}^2\bigr) + D_y\bigl(4aw_y^2\cos w\bigr) = 0, \\ &D_x\bigl(3w_y^6 - 12w_y^2 w_{yy}^2 + 16w_y^3 w_{yyy} + 24w_{yyy}^2\bigr) + D_y\bigl[a(2w_y^4 - 24w_{yy}^2)\cos w\bigr] = 0, \end{aligned}$$

where $D_x = \frac{\partial}{\partial x}$ and $D_y = \frac{\partial}{\partial y}$ (analogous laws can be obtained by swapping the independent variables $x \rightleftarrows y$).

⊙ *References*: A. C. Scott, F. Y. Chu, and D. W. McLaughlin (1973), J. L. Lamb (1974), R. K. Dodd and R. K. Bullough (1977).

6°. The equation in question is related to the equation

$$\frac{\partial^2 z}{\partial x \partial y} = z\sqrt{a^2 - \left(\frac{\partial z}{\partial y}\right)^2}$$

by the transformation

$$z = \frac{\partial w}{\partial x}, \qquad \frac{\partial z}{\partial y} = a \sin w.$$

⊙ *References for equation* 3.5.1.5: R. Steuerwald (1936), I. M. Krichever (1980), R. K. Bullough and P. J. Caudrey (1980), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), N. H. Ibragimov (1994).

6. $\displaystyle \frac{\partial^2 w}{\partial x \partial y} = a \sin w + b \sin\bigl(\tfrac{1}{2}w\bigr).$

On passing to the new independent variables $z = x - y$ and $t = x + y$, one obtains an equation of the form 3.3.3.2:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial z^2} + a \sin w + b \sin\bigl(\tfrac{1}{2}w\bigr).$$

⊙ *Reference*: F. Calogero and A. Degasperis (1982)

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3.5.2. Other Equations Involving Arbitrary Parameters

1. $\dfrac{\partial^2 w}{\partial x \partial y} = a \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}.$

General solution:

$$w(x,y) = -\frac{1}{a}\ln\bigl[f(x)+g(y)\bigr],$$

where $f(x)$ and $g(y)$ are arbitrary functions.

2. $\dfrac{\partial^2 w}{\partial x \partial y} + a \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial x} + c\dfrac{\partial w}{\partial y} = 0.$

This equation arises in some problems of chemical engineering and chromatography. The substitution $u = e^{aw}$ leads to the linear equation

$$\frac{\partial^2 u}{\partial x \partial y} + b\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} = 0.$$

⊙ *Reference*: H. C. Thomas (1944), G. B. Whitham (1972).

3. $w\dfrac{\partial^2 w}{\partial x \partial y} = \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}.$

General solution:

$$w(x,y) = f(x)g(y),$$

where $f(x)$ and $g(y)$ are arbitrary functions.

4. $\dfrac{\partial^2 w}{\partial x \partial y} = aw^n \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}.$

This is a special case of equation 3.5.3.7 with $f(w) = aw^n$.

5. $\dfrac{\partial^2 w}{\partial x \partial y} = ae^{\beta w} \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}.$

This is a special case of equation 3.5.3.7 with $f(w) = ae^{\beta w}$.

General solution in implicit form:

$$\int \exp\left(-\frac{a}{\beta}e^{\beta w}\right) dw = \varphi(x) + \psi(y),$$

where $\varphi(x)$ and $\psi(y)$ are arbitrary functions.

6. $\dfrac{\partial^2 w}{\partial x \partial y} = a\sqrt{\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}}.$

This is a special case of equation 3.5.3.8 with $f(x,y) = \frac{1}{4}a^2$ (the original equation is reduced to a linear one).

7. $w\dfrac{\partial^2 w}{\partial x \partial y} = \sqrt{1 - \left(\dfrac{\partial w}{\partial x}\right)^2}\sqrt{1 - \left(\dfrac{\partial w}{\partial y}\right)^2}.$

For this and some other integrable nonlinear hyperbolic equations, see Zhiber and Sokolov (2001).

3.5.3. Equations Involving Arbitrary Functions

1. $\dfrac{\partial^2 w}{\partial x \partial y} = f(w).$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = w\bigl(C_1 x + C_2,\ C_1^{-1} y + C_3\bigr),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w = w(z), \qquad z = ax + by,$$

where a and b are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation $abw''_{zz} = f(w)$.

3°. Self-similar solution:

$$w = w(\xi), \qquad \xi = xy,$$

where the function $w(\xi)$ is determined by the second-order ordinary differential equation $\xi w''_{\xi\xi} + w'_\xi = f(w)$.

4°. On passing to the new independent variables $z = x - y$ and $t = x + y$, we obtain an equation of the form 3.4.1.1:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial z^2} + f(w).$$

5°. Conservation laws:

$$D_x\bigl(\tfrac{1}{2} w_y^2\bigr) + D_y\bigl[-F(w)\bigr] = 0,$$
$$D_x\bigl[-F(w)\bigr] + D_y\bigl(\tfrac{1}{2} w_x^2\bigr) = 0,$$

where $D_x = \dfrac{\partial}{\partial x}$, $D_y = \dfrac{\partial}{\partial y}$, and $F(w) = \displaystyle\int f(w)\,dw$.

2. $\dfrac{\partial^2 w}{\partial x \partial y} = f(x) g(y) e^{\beta w}.$

The transformation

$$\xi = \int f(x)\,dx, \qquad \eta = \int g(y)\,dy$$

leads to an equation of the form 3.5.1.2:

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = e^{\beta w}.$$

3. $\dfrac{\partial^2 w}{\partial x \partial y} = f(x) g(y) h(w).$

The transformation

$$\xi = \int f(x)\,dx, \qquad \eta = \int g(y)\,dy$$

leads to an equation of the form 3.5.3.1:

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = h(w).$$

4. $\dfrac{\partial^2 w}{\partial x \partial y} = f(x)g(w)\dfrac{\partial w}{\partial y}.$

1°. Functional separable solution in implicit form:

$$\int \frac{dw}{G(w)} = \varphi(y) + \int f(x)\,dx, \qquad \text{where} \quad G(w) = \int g(w)\,dw.$$

Here, $\varphi(y)$ is an arbitrary function.

2°. Integrating the original equation with respect to y, we arrive at a first-order partial differential equation:

$$\frac{\partial w}{\partial x} = f(x)\int g(w)\,dw + \psi(x),$$

where $\psi(x)$ is an arbitrary function.

5. $\dfrac{\partial^2 w}{\partial x \partial y} = f(x,w)\dfrac{\partial w}{\partial y} + g(x,y).$

Integrating the original equation with respect to y, we arrive at a first-order partial differential equation:

$$\frac{\partial w}{\partial x} = \int_a^w f(x,\tau)\,d\tau + \int_b^y g(x,s)\,ds + \psi(x),$$

where $\psi(x)$ is an arbitrary function, and a and b are arbitrary constants. The equation obtained can be treated as an ordinary differential equation for $w = w(x)$ with parameter y.

6. $\dfrac{\partial^2 w}{\partial x \partial y} = a\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} + f(x,y)\dfrac{\partial w}{\partial x} + g(x,y)\dfrac{\partial w}{\partial y} + h(x,y).$

The substitution $u = e^{-aw}$ leads to the linear equation

$$\frac{\partial^2 u}{\partial x \partial y} = f(x,y)\frac{\partial u}{\partial x} + g(x,y)\frac{\partial u}{\partial y} - ah(x,y)u.$$

7. $\dfrac{\partial^2 w}{\partial x \partial y} = f(w)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}.$

1°. The substitution

$$u = \int F(w)\,dw, \qquad F(w) = \exp\left[-\int f(w)\,dw\right]$$

leads to the constant coefficient linear equation

$$\frac{\partial^2 u}{\partial x \partial y} = 0.$$

2°. General solution in implicit form:

$$\int \exp\left[-\int f(w)\,dw\right]dw = \varphi(x) + \psi(y),$$

where $\varphi(x)$ and $\psi(y)$ are arbitrary functions.

8. $\dfrac{\partial^2 w}{\partial x \partial y} = 2\sqrt{f(x,y)\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}}.$

Goursat equation. Introduce functions $u = u(x,y)$ and $v = v(x,y)$ by the differential relations

$$u = \sqrt{\frac{\partial w}{\partial x}}, \qquad v = \sqrt{\frac{\partial w}{\partial y}}.$$

On differentiating these relations with respect to y and x, respectively, and eliminating w using the original equation, one arrives at the system

$$\frac{\partial u}{\partial y} = v\sqrt{f(x,y)}, \qquad \frac{\partial v}{\partial x} = u\sqrt{f(x,y)}.$$

Eliminating v yields a linear equation for $u = u(x,y)$:

$$\frac{\partial^2 u}{\partial x \partial y} = g(x,y)\frac{\partial u}{\partial y} + f(x,y)u, \qquad \text{where} \quad g(x,y) = \frac{1}{2}\frac{\partial}{\partial x}\ln f(x,y).$$

⊙ *Reference*: E. I. Ganzha (2000).

9. $\displaystyle \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y} = f(x)g(w)\frac{\partial w}{\partial y}.$

1°. Functional separable solution in implicit form:

$$\int \frac{dw}{\sqrt{G(w)}} = \varphi(y) \pm \int \sqrt{2f(x)}\,dx, \qquad \text{where} \quad G(w) = \int g(w)\,dw.$$

Here, $\varphi(y)$ is an arbitrary function.

2°. Integrating the original equation with respect to y, we have

$$\left(\frac{\partial w}{\partial x}\right)^2 = 2f(x)\int g(w)\,dw + \psi(x),$$

where $\psi(x)$ is an arbitrary function. The equation obtained can be treated as a first-order ordinary differential equation in x for which the constant of integration will be dependent on y.

10. $\displaystyle \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y} = f(x,w)\frac{\partial w}{\partial y} + g(x,y).$

Integrating the original equation with respect to y, one arrives at a first-order partial differential equation:

$$\left(\frac{\partial w}{\partial x}\right)^2 = 2\int_a^w f(x,\tau)\,d\tau + 2\int_b^y g(x,s)\,ds + \psi(x),$$

where $\psi(x)$ is an arbitrary function and a and b are arbitrary constants. The equation obtained can be treated as an ordinary differential equation for $w = w(x)$ with parameter y.

11. $\displaystyle f\left(x, \frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial x \partial y} = g(x,w)\frac{\partial w}{\partial y} + h(x,y).$

Integrating the original equation with respect to y, one arrives at a first-order partial differential equation:

$$\int_a^{w_x} f(x,\lambda)\,d\lambda = \int_b^w g(x,\tau)\,d\tau + \int_c^y h(x,s)\,ds + \psi(x),$$

where w_x is the partial derivative of w with respect to x, $\psi(x)$ is an arbitrary function, and a, b, and c are arbitrary constants. The equation obtained can be treated as an ordinary differential equation for $w = w(x)$ with parameter y.

Chapter 4

Hyperbolic Equations with Two or Three Space Variables

4.1. Equations with Two Space Variables Involving Power-Law Nonlinearities

4.1.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2}=\frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right]+aw^p$

1. $\displaystyle \frac{\partial^2 w}{\partial t^2}=\frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(by^m\frac{\partial w}{\partial y}\right)+cw^p.$

This is a special case of equation 4.4.1.2 with $f(w)=cw^p$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1=C_1 w\Bigl(C_1^{\frac{p-1}{2-n}}x,\ C_1^{\frac{p-1}{2-m}}y,\ \pm C_1^{\frac{p-1}{2}}t+C_2\Bigr),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n\neq 2$, $m\neq 2$, and $p\neq 1$:

$$w=\left[\frac{1}{2c(p-1)}\left(\frac{1+p}{1-p}+\frac{2}{2-n}+\frac{2}{2-m}\right)\right]^{\frac{1}{p-1}}\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}-\frac{1}{4}(t+C)^2\right]^{\frac{1}{1-p}}.$$

3°. Solution for $n\neq 2$ and $m\neq 2$ (generalizes the solution of Item 2°):

$$w=w(r),\qquad r^2=4k\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}-\frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k\neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}+\frac{A}{r}w'_r+ck^{-1}w^p=0,\qquad A=\frac{2(4-n-m)}{(2-n)(2-m)}.$$

4°. There are “two-dimensional” solutions of the following forms:

$$w(x,y,t)=U(\xi,t),\quad \xi^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}\right],$$

$$w(x,y,t)=V(x,\eta),\quad \eta^2=\pm 4\left[\frac{y^{2-m}}{b(2-m)^2}-\frac{1}{4}(t+C)^2\right],$$

$$w(x,y,t)=W(y,\zeta),\quad \zeta^2=\pm 4\left[\frac{x^{2-n}}{a(2-n)^2}-\frac{1}{4}(t+C)^2\right],$$

$$w(x,y,t)=|t|^{\frac{2}{1-p}}F(z_1,z_2),\quad z_1=x|t|^{\frac{2}{n-2}},\quad z_2=y|t|^{\frac{2}{m-2}}.$$

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y} \dfrac{\partial w}{\partial y}\right) + cw^p.$

This is a special case of equation 4.4.1.3 with $f(w) = cw^p$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w\Bigl(C_1^{\frac{p-1}{2-n}} x,\ y + \frac{1-p}{\lambda} \ln C_1,\ \pm C_1^{\frac{p-1}{2}} t + C_2\Bigr),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $\lambda \neq 0$, and $p \neq 1$:

$$w = \left[\frac{1}{2c(p-1)}\left(\frac{1+p}{1-p} + \frac{2}{2-n}\right)\right]^{\frac{1}{p-1}} \left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right]^{\frac{1}{1-p}}.$$

3°. Solution for $n \neq 2$ and $\lambda \neq 0$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r} w'_r + ck^{-1} w^p = 0, \qquad A = \frac{2}{2-n}.$$

4°. There are “two-dimensional” solutions of the following forms:

$$w(x,y,t) = U(\xi, t), \quad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right],$$

$$w(x,y,t) = V(x, \eta), \quad \eta^2 = \pm 4\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

$$w(x,y,t) = W(y, \zeta), \quad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right],$$

$$w(x,y,t) = |t|^{\frac{2}{1-p}} F(z_1, z_2), \quad z_1 = x|t|^{\frac{2}{n-2}}, \quad z_2 = y + \frac{2}{\lambda} \ln|t|.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\beta x} \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y} \dfrac{\partial w}{\partial y}\right) + cw^p.$

This is a special case of equation 4.4.1.4 with $f(w) = cw^p$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w\Bigl(x + \frac{1-p}{\beta} \ln C_1,\ y + \frac{1-p}{\lambda} \ln C_1,\ \pm C_1^{\frac{p-1}{2}} t + C_2\Bigr),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $p \neq \pm 1$, $\beta \neq 0$, and $\lambda \neq 0$:

$$w = \left[-\frac{c(p-1)^2}{2k(1+p)}(r + C_1)^2\right]^{\frac{1}{1-p}}, \qquad r^2 = 4k\left[\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C_2)^2\right],$$

where C_1, C_2, and k are arbitrary constants.

3°. Solution for $\beta \neq 0$ and $\lambda \neq 0$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4k\left[\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where the function $w(r)$ is determined by the autonomous ordinary differential equation

$$w''_{rr} + ck^{-1}w^p = 0.$$

Integrating yields its general solution in implicit form:

$$\int\left[C_1 - \frac{2c}{k(p+1)}w^{p+1}\right]^{-1/2} dw = C_2 \pm r,$$

where C_1 and C_2 are arbitrary constants.

4°. There are "two-dimensional" solutions of the following forms:

$$\begin{aligned}
&w(x,y,t) = U(\xi,t), \quad \xi^2 = 4\left(\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right),\\
&w(x,y,t) = V(x,\eta), \quad \eta^2 = \pm 4\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],\\
&w(x,y,t) = W(y,\zeta), \quad \zeta^2 = \pm 4\left[\frac{e^{-\beta x}}{a\beta^2} - \frac{1}{4}(t+C)^2\right],\\
&w(x,y,t) = |t|^{\frac{2}{1-p}}F(z_1,z_2), \quad z_1 = x + \frac{2}{\beta}\ln|t|, \quad z_2 = y + \frac{2}{\lambda}\ln|t|.
\end{aligned}$$

4.1.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(w^k\dfrac{\partial w}{\partial y}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial}{\partial y}\left(w\dfrac{\partial w}{\partial y}\right).$

This is a special case of equation 4.1.3.1 with $c = 0$.

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(w\dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$\begin{aligned}
w_1 &= C_2^2 w(\pm C_1 x + C_3, \pm C_1 y + C_4, \pm C_1 C_2 t + C_5),\\
w_2 &= w(x\cos\beta + y\sin\beta, -x\sin\beta + y\cos\beta,\ t),
\end{aligned}$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solutions:

$$w = \frac{\lambda^2 \pm \sqrt{A(k_1 x + k_2 y + \lambda t) + B}}{k_1^2 + k_2^2},$$

where A, B, k_1, k_2, and λ are arbitrary constants.

3°. Generalized separable solution linear in space variables:

$$w(x,y,t) = (A_1 t + B_1)x + (A_2 t + B_2)y + \tfrac{1}{12}(A_1^2 + A_2^2)t^4 + \tfrac{1}{3}(A_1B_1 + A_2B_2)t^3 + \tfrac{1}{2}(B_1^2 + B_2^2)t^2 + Ct + D,$$

where A_1, A_2, B_1, B_2, C, and D are arbitrary constants.

4°. Solutions:

$$w(x,y,t) = \tfrac{3}{4}t^{-2}\left[(x+C_1)^2 + (y+C_2)^2\right],$$

$$w(x,y,t) = t^{-2}\left(x\sin\lambda + y\cos\lambda + C_1\right)^2,$$

$$w(x,y,t) = \frac{1}{C_1^2+C_2^2}\left(\frac{C_1x+C_2y+C_3}{t+C_4}\right)^2,$$

$$w(x,y,t) = \frac{C_2^2(x+C_4)^2}{(C_1y+C_2t+C_3)^2 + C_1^2(x+C_4)^2},$$

$$w(x,y,t) = t\left[C_1\ln\left(x^2+y^2\right) + C_2\right]^{1/2},$$

$$w(x,y,t) = t\left[C_1\exp\left(\lambda x\right)\sin\left(\lambda y + C_2\right) + C_3\right]^{1/2},$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants.

5°. "Two-dimensional" solution in multiplicative separable form (generalizes the last two solutions of Item 4°):

$$w(x,y,t) = (C_1t + C_2)\sqrt{|U(x,y)|},$$

where the function $U = U(x,y)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0.$$

For this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

6°. There is a generalized separable solution quadratic in space variables:

$$w(x,y,t) = f(t)x^2 + g(t)xy + h(t)y^2,$$

where the functions $f(t)$, $g(t)$, and $h(t)$ are determined by the autonomous system of ordinary differential equations

$$f''_{tt} = 6f^2 + 2fh + g^2, \qquad (1)$$

$$g''_{tt} = 6(f+h)g, \qquad (2)$$

$$h''_{tt} = 6h^2 + 2fh + g^2. \qquad (3)$$

A particular solution of system (1)–(3) is given by

$$h(t) = f(t), \quad g(t) = \pm 2f(t), \quad \text{where} \quad f''_{tt} = 12f^2$$

(the general solution for f can be written out in implicit form).

7°. There is a generalized separable solution of the form

$$w(x,y,t) = f(t)x^2 + g(t)xy + h(t)y^2 + \varphi(t)x + \psi(t)y + \chi(t),$$

where the functions $f(t)$, $g(t)$, $h(t)$, $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned} f''_{tt} &= 6f^2 + 2fh + g^2, & \varphi''_{tt} &= 2(3f+h)\varphi + 2g\psi, \\ g''_{tt} &= 6(f+h)g, & \psi''_{tt} &= 2g\varphi + 2(f+3h)\psi, \\ h''_{tt} &= 6h^2 + 2fh + g^2, & \chi''_{tt} &= \varphi^2 + \psi^2 + 2(f+h)\chi. \end{aligned}$$

The first three equations for f, g, and h are solved independently of the other three (see Item 6°).

8°. There is a "two-dimensional" solution in multiplicative separable form

$$w(x,y,t) = (At+B)^{-2}\Theta(x,y).$$

9°. For other solutions, see equation 4.1.2.6 with $a = b = n = 1$ and equation 4.1.2.7 with $a = b = n = m = 1$.

⊙ *Reference for equation* 4.1.2.2: A. D. Polyanin and V. F. Zaitsev (2002).

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(\dfrac{1}{\sqrt{w}}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(\dfrac{1}{\sqrt{w}}\dfrac{\partial w}{\partial y}\right).$

This is a special case of equation 4.1.2.7 with $n = k = -1/2$ and equation 4.4.2.3 with $f(w) = aw^{-1/2}$ and $g(w) = bw^{-1/2}$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^4 w(\pm C_1C_2x + C_3, \pm C_1C_2y + C_4, \pm C_2t + C_5),$$
$$w_2 = w(x\cos\beta + y\sqrt{a/b}\,\sin\beta, -x\sqrt{b/a}\,\sin\beta + y\cos\beta,\ t),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solutions:

$$w(x, y, t) = \left(\frac{aC_1^2 + bC_2^2}{C_3^2} \pm \sqrt{C_1x + C_2y + C_3t + C_4}\right)^2,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

3°. Solutions:

$$w(x, y, t) = t^4\left(\frac{\sin\lambda}{\sqrt{a}}x + \frac{\cos\lambda}{\sqrt{b}}y + C_1\right)^{-4},$$
$$w(x, y, t) = \left(\tfrac{2}{3}ab\right)^2 t^4(bx^2 + ay^2)^{-2},$$
$$w(x, y, t) = (aC_1^2 + bC_2^2)^2\left(\frac{t + C_4}{C_1x + C_2y + C_3}\right)^4,$$
$$w(x, y, t) = \frac{\left[a(C_1y + C_2t + C_3)^2 + bC_1^2(x + C_4)^2\right]^2}{C_2^4(x + C_4)^4},$$
$$w(x, y, t) = t\left[C_1\ln\left(bx^2 + ay^2\right) + C_2\right]^2,$$
$$w(x, y, t) = t\left[C_1\exp\left(\lambda\sqrt{b}\,x\right)\sin\left(\lambda\sqrt{a}\,y + C_2\right) + C_3\right]^2,$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants.

4°. "Two-dimensional" solution in multiplicative separable form (generalizes the last two solutions of Item 3°):

$$w(x, y, t) = (C_1t + C_2)U^2(\xi, \eta), \quad \xi = \frac{x}{\sqrt{a}}, \quad \eta = \frac{y}{\sqrt{b}},$$

where the function $U = U(\xi, \eta)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} = 0.$$

For this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

5°. "Two-dimensional" generalized separable solution quadratic in t:

$$w(x, y, t) = \left[f(\xi, \eta)t + g(\xi, \eta)\right]^2, \quad \xi = \frac{x}{\sqrt{a}}, \quad \eta = \frac{y}{\sqrt{b}},$$

where the functions $f = f(\xi, \eta)$ and $g = g(\xi, \eta)$ are determined by the system of differential equations

$$\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} = 0, \tag{1}$$
$$\frac{\partial^2 g}{\partial \xi^2} + \frac{\partial^2 g}{\partial \eta^2} = f^2. \tag{2}$$

Equation (1) is the Laplace equation, and (2) is a Helmholtz equation (wherever f is known). For these linear equations, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

6°. There is a "two-dimensional" generalized separable solution of the form

$$w(x,y,t) = \big[f_2(x,y)t^2 + f_1(x,y)t + f_0(x,y)\big]^2.$$

7°. For other solutions, see equation 4.1.2.6 with $n=-1/2$ and equation 4.1.2.7 with $n=m=-1/2$.

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\left[\frac{\partial}{\partial x}\left(\frac{1}{w}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{w}\frac{\partial w}{\partial y}\right)\right].$

This is a special case of equation 4.1.2.7 with $a=b$, $n=k=-1$ and equation 4.4.2.3 with $f(w)=a/w$ and $g(w)=b/w$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$\begin{aligned}
w_1 &= C_1^2 w(\pm C_1C_2x + C_3, \pm C_1C_2y + C_4, \pm C_2t + C_5),\\
w_2 &= w(x\cos\beta + y\sin\beta, -x\sin\beta + y\cos\beta,\ t),
\end{aligned}$$

where $C_1, \dots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$a(k_1^2 + k_2^2)\ln|w| - \lambda^2 w = A(k_1x + k_2y + \lambda t) + B,$$

where A, B, k_1, k_2, and λ are arbitrary constants.

3°. Solutions:

$$\begin{aligned}
w(x,y,t) &= (C_1t + C_2)e^{Ax+By},\\
w(x,y,t) &= (C_1t + C_2)\exp\big[A(x^2 - y^2)\big],\\
w(x,y,t) &= (C_1t + C_2)\exp\big[Ae^{\lambda x}\sin(\lambda y + B)\big],\\
w(x,y,t) &= \frac{a[(Ay + Bt + C_1)^2 + A^2(x + C_2)^2]}{B^2(x + C_2)^2},\\
w(x,y,t) &= \frac{at^2 + At + B}{(x\sin\lambda + y\cos\lambda + C)^2},\\
w(x,y,t) &= \frac{at^2 + At + B}{(\sin y + Ce^x)^2},\\
w(x,y,t) &= \frac{C_1^2(at^2 + At + B)}{e^{2x}\sinh^2(C_1e^{-x}\sin y + C_2)},\\
w(x,y,t) &= \frac{C_1^2(-at^2 + At + B)}{e^{2x}\cosh^2(C_1e^{-x}\sin y + C_2)},\\
w(x,y,t) &= \frac{C_1^2(at^2 + At + B)}{e^{2x}\cos^2(C_1e^{-x}\sin y + C_2)},
\end{aligned}$$

where A, B, C, C_1, C_2, and λ are arbitrary constants.

4°. "Two-dimensional" solution in multiplicative separable form (generalizes the first three solutions of Item 3°):

$$w(x,y,t) = (C_1t + C_2)e^{U(x,y)},$$

where the function $U = U(x,y)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

5°. "Two-dimensional" solution in multiplicative separable form (generalizes the last four solutions of Item 3°):

$$w(x,y,t) = (\tfrac{1}{2}Aat^2 + Bt + C)e^{\Theta(x,y)},$$

where A, B, and C are arbitrary constants, and the function $\Theta(x,y)$ is a solution of the stationary equation

$$\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} = Ae^{\Theta},$$

which occurs in combustion theory. For solutions of this equation, see 5.2.1.1.

6°. For other solutions, see equation 4.1.2.6 with $a = b$, $n = -1$ and equation 4.1.2.7 with $a = b$, $n = m = -1$.

⊙ *References for equation* 4.1.2.4: V. A. Baikov (1990), N. Ibragimov (1994), A. D. Polyanin and V. F. Zaitsev (2002).

5. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + b\frac{\partial}{\partial y}\left(w^n \frac{\partial w}{\partial y}\right).$

This is a special case of equation 4.4.2.1 with $g(w) = bw^n$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2} w(\pm C_2 x + C_3, \pm C_1^n C_2 y + C_4, \pm C_2 t + C_5),$$
$$w_2 = w(x\cosh\lambda + ta^{1/2}\sinh\lambda,\ y,\ xa^{-1/2}\sinh\lambda + t\cosh\lambda),$$

where $C_1, \dots, C_5$ and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Solutions:

$$w(x,y,t) = y^{\frac{1}{n+1}}\left[\varphi(x - t\sqrt{a}\,) + \psi(x + t\sqrt{a}\,)\right],$$
$$w(x,y,t) = \left[y\varphi(x - t\sqrt{a}\,) + \psi(x - t\sqrt{a}\,)\right]^{\frac{1}{n+1}},$$
$$w(x,y,t) = \left[y\varphi(x + t\sqrt{a}\,) + \psi(x + t\sqrt{a}\,)\right]^{\frac{1}{n+1}},$$

where $\varphi(z_1)$ and $\psi(z_2)$ are arbitrary functions.

3°. Solutions:

$$w(x,y,t) = y^{\frac{2}{n}}\left(C_1 x \pm t\sqrt{aC_1^2 + b} + C_2\right)^{-\frac{2}{n}},$$
$$w(x,y,t) = \left[\frac{2a}{b(n+2)}\right]^{\frac{1}{n}} y^{\frac{2}{n}}\left[a(t+C_1)^2 - (x+C_2)^2\right]^{-\frac{1}{n}},$$
$$w(x,y,t) = \left[\frac{1}{bC_2^2}\left(\frac{C_1 x + C_2 y + C_3}{t + C_4}\right)^2 - \frac{aC_1^2}{bC_2^2}\right]^{\frac{1}{n}},$$
$$w(x,y,t) = \left[\frac{C_2^2}{bC_1^2} - \frac{a}{bC_1^2}\left(\frac{C_1 y + C_2 t + C_3}{x + C_4}\right)\right]^{\frac{1}{n}},$$

where $C_1, \dots, C_4$ are arbitrary constants.

4°. Solutions in implicit form:

$$2\lambda\sqrt{a}\,(y + \lambda t) + (t\sqrt{a} \pm x)(bw^n - \lambda^2) = \psi(w),$$

where $\psi(w)$ is an arbitrary function and λ is an arbitrary constant.

5°. Solution:

$$w(x,y,t) = V(z)y^{2/n}, \quad z = x^2 - at^2,$$

where the function $V = V(z)$ is determined by the ordinary differential equation

$$2an^2(zV''_z + V'_z) + b(n+2)V^{n+1} = 0.$$

6°. "Two-dimensional" solution in multiplicative separable form:

$$w(x,y,t) = u(x,t)y^{2/n},$$

where the function $u = u(x,t)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^2 u}{\partial x^2} + \frac{2b(n+2)}{n^2}u^{n+1}.$$

For $n = -1$ and $n = -2$, this equation is linear.

Remark. The first solution of Item 2°, the first two solutions of Item 3°, and the solutions of Items 5° and 6° are special cases of a multiplicative separable solution $w = u(x,t)\theta(y)$, where $\theta = \theta(y)$ is determined by the autonomous ordinary differential equation $(\theta^n \theta'_y)'_y = C\theta$.

7°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = F(y,r), \quad r = x^2 - at^2;$$
$$w(x,y,t) = |t|^{2\lambda}G(\xi,\eta), \quad \xi = \frac{x}{t}, \quad \eta = \frac{y}{|t|^{n\lambda+1}};$$
$$w(x,y,t) = |t|^{-2/n}H(y,z), \quad z = x/t;$$
$$w(x,y,t) = |y|^{2/n}U(z_1,z_2), \quad z_1 = t + k_1 \ln|y|, \quad z_2 = x + k_2 \ln|y|;$$
$$w(x,y,t) = \exp\Bigl(-\frac{2y}{n+1}\Bigr)V(\rho_1,\rho_2), \quad \rho_1 = t\exp\Bigl(-\frac{ny}{n+1}\Bigr), \quad \rho_2 = x\exp\Bigl(-\frac{ny}{n+1}\Bigr),$$

where k_1, k_2, and λ are arbitrary constants.

8°. There is an exact solution of the form

$$w(x,y,t) = W(z), \quad z = (x^2 - at^2)y^{-2}.$$

9°. For other solutions, see equation 4.1.2.7, in which n should be set equal to zero and k should be renamed n.

⊙ *References for equation* 4.1.2.5: N. Ibragimov (1994), A. D. Polyanin and V. F. Zaitsev (2002).

6. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\Bigl(w^n\frac{\partial w}{\partial x}\Bigr) + b\frac{\partial}{\partial y}\Bigl(w^n\frac{\partial w}{\partial y}\Bigr).$

This is a special case of equation 4.1.2.7 with $n = k$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = (C_2/C_1)^{2/n}w(\pm C_1x + C_3, \pm C_1y + C_4, \pm C_2t + C_5),$$
$$w_2 = w\bigl(x\cos\beta + y\sqrt{a/b}\,\sin\beta, -x\sqrt{b/a}\,\sin\beta + y\cos\beta,\ t\bigr),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w(x,y,t) = t^{-\frac{2}{n}} \left(\frac{\sin\lambda}{\sqrt{a}} x + \frac{\cos\lambda}{\sqrt{b}} y + C_1 \right)^{\frac{2}{n}},$$

$$w(x,y,t) = \left[\frac{n+2}{2ab(n+1)} \right]^{\frac{1}{n}} t^{-\frac{2}{n}} (bx^2 + ay^2)^{\frac{1}{n}},$$

$$w(x,y,t) = \frac{1}{(aC_1^2 + bC_2^2)^{1/n}} \left(\frac{C_1 x + C_2 y + C_3}{t + C_4} \right)^{2/n},$$

$$w(x,y,t) = \frac{C_2^{2/n} (x + C_4)^{2/n}}{\left[a(C_1 y + C_2 t + C_3)^2 + bC_1^2 (x + C_4)^2 \right]^{1/n}},$$

$$w(x,y,t) = t \left[C_1 \ln\left(bx^2 + ay^2 \right) + C_2 \right]^{\frac{1}{n+1}},$$

$$w(x,y,t) = t \left[C_1 \exp\left(\lambda\sqrt{b}\, x \right) \sin\left(\lambda\sqrt{a}\, y + C_2 \right) + C_3 \right]^{\frac{1}{n+1}},$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants.

3°. Traveling-wave solution in implicit form:

$$\frac{ak_1^2 + bk_2^2}{n+1} w^{n+1} - \lambda^2 w = C_1 (k_1 x + k_2 y + \lambda t) + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

4°. "Two-dimensional" solution in multiplicative separable form (generalizes the fifth and sixth solutions of Item 2°):

$$w(x,y,t) = (C_1 t + C_2) \left[U(\xi,\eta) \right]^{\frac{1}{n+1}}, \quad \xi = \sqrt{b}\, x, \quad \eta = \sqrt{a}\, y,$$

where the function $U = U(\xi,\eta)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} = 0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

5°. "Two-dimensional" solution in multiplicative separable form (generalizes the first and second solutions of Item 2°):

$$w(x,y,t) = f(t)\Theta(x,y),$$

where the function $f(t)$ is determined by the autonomous ordinary differential equation

$$f''_{tt} = \lambda f^{n+1}, \tag{1}$$

λ is an arbitrary constant, and the function $\Theta = \Theta(x,y)$ is a solution of the two-dimensional stationary equation

$$a \frac{\partial}{\partial x} \left(\Theta^n \frac{\partial \Theta}{\partial x} \right) + b \frac{\partial}{\partial y} \left(\Theta^n \frac{\partial \Theta}{\partial y} \right) - \lambda \Theta = 0. \tag{2}$$

A particular solution to equation (1) is given by (C is an arbitrary constant):

$$f = (C \pm kt)^{-2/n}, \quad k = n\sqrt{\frac{\lambda}{2(n+2)}}.$$

6°. There are solutions of the following forms:

$w(x,y,t) = F(r,t), \quad r = bx^2 + ay^2$ "two-dimensional" solution;

$w(x,y,t) = t^{2\lambda} G(\xi,\eta), \quad \xi = \dfrac{x}{t^{n\lambda+1}}, \quad \eta = \dfrac{y}{t^{n\lambda+1}}$ "two-dimensional" solution;

$w(x,y,t) = y^{2/n} H(z,t), \quad z = y/x$ "two-dimensional" solution;

$w(x,y,t) = |t|^{-2/n} U(z_1,z_2), \quad z_1 = x + k_1 \ln|t|, \quad z_2 = y + k_2 \ln|t|$ "two-dimensional" solution;

$w(x,y,t) = e^{-2t} V(\rho_1,\rho_2), \quad \rho_1 = xe^{nt}, \quad \rho_2 = ye^{nt}$ "two-dimensional" solution;

$w(x,y,t) = W(\theta), \quad \theta = (bx^2 + ay^2)t^{-2}$ "one-dimensional" solution,

where k_1, k_2, and λ are arbitrary constants.

7°. For other solutions, see equation 4.1.2.7 with $k = n$.

⊙ *References for equation* 4.1.2.6: N. Ibragimov (1994), A. D. Polyanin and V. F. Zaitsev (2002).

7. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(w^k \dfrac{\partial w}{\partial y}\right).$

This is a special case of equation 4.4.2.3 with $f(w) = aw^n$ and $g(w) = bw^k$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^n C_2 x + C_3, \pm C_1^k C_2 y + C_4, \pm C_2 t + C_5),$$

where $C_1, \dots, C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\frac{a\beta_1^2}{n+1} w^{n+1} + \frac{b\beta_2^2}{k+1} w^{k+1} - \lambda^2 w = C_1(\beta_1 x + \beta_2 y + \lambda t) + C_2,$$

where C_1, C_2, β_1, β_2, and λ are arbitrary constants.

3°. Solutions in implicit form:

$$\left(\frac{C_1 x + C_2 y + C_3}{t + C_4}\right)^2 = aC_1^2 w^n + bC_2^2 w^k,$$

$$a\left(\frac{C_1 y + C_2 t + C_3}{x + C_4}\right)^2 w^n + bC_1^2 w^k = C_2^2,$$

$$b\left(\frac{C_1 x + C_2 t + C_3}{y + C_4}\right)^2 w^k + aC_1^2 w^n = C_2^2,$$

where $C_1, \dots, C_4$ are arbitrary constants.

4°. "Two-dimensional" solution (c_1 and c_2 are arbitrary constants):

$$w(x,y,t) = u(z,t), \quad z = c_1 x + c_2 y,$$

where the function $u = u(z,t)$ is determined by a differential equation of the form 3.4.4.6:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial z}\left[\varphi(u)\frac{\partial u}{\partial z}\right], \qquad \varphi(u) = ac_1^2 u^n + bc_2^2 u^k,$$

which can be reduced to a linear equation.

5°. “Two-dimensional” solution (s_1 and s_2 are arbitrary constants):

$$w(x,y,t) = v(x,\xi), \quad \xi = s_1 y + s_2 t,$$

where the function $v = v(x,\xi)$ is determined by a differential equation of the form 5.4.4.8:

$$a\frac{\partial}{\partial x}\left(v^n \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left[\psi(v)\frac{\partial v}{\partial z}\right] = 0, \quad \psi(v) = bs_1^2 v^k - s_2^2,$$

which can be reduced to a linear equation.

6°. There is a “two-dimensional” solution of the form (generalize the solutions of Items 3° and 4°):

$$w(x,y,t) = U(z_1,z_2), \quad z_1 = a_1 x + b_1 y + c_1 t, \quad z_2 = a_2 x + b_2 y + c_2 t.$$

7°. There are exact solutions of the following forms:

$$\begin{aligned}
&w(x,y,t) = t^{2\lambda}F(\xi,\eta), \quad \xi = \frac{x}{t^{n\lambda+1}}, \quad \eta = \frac{y}{t^{k\lambda+1}} && \text{“two-dimensional” solution;}\\
&w(x,y,t) = x^{2/n}G(\zeta,t), \quad \zeta = x^{-k/n}y && \text{“two-dimensional” solution;}\\
&w(x,y,t) = e^{-2t}H(z_1,z_2), \quad z_1 = xe^{nt}, \quad z_2 = ye^{kt} && \text{“two-dimensional” solution;}\\
&w(x,y,t) = (x/t)^{2/n}U(\theta), \quad \theta = x^{-k/n}yt^{k/n-1} && \text{“one-dimensional” solution;}
\end{aligned}$$

where λ is an arbitrary constant.

⊙ *References for equation* 4.1.2.7: N. Ibragimov (1994), A. D. Polyanin and V. F. Zaitsev (2002).

4.1.3. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right]$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left[(bw+c)\dfrac{\partial w}{\partial y}\right].$

This is a special case of equation 4.4.2.1 with $g(w) = bw + c$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$\begin{aligned}
w_1 &= C_1^{-2}w(\pm C_2 x + C_3, \pm C_1 C_2 y + C_4, \pm C_2 t + C_5) + \frac{c(1-C_1^2)}{bC_1^2},\\
w_2 &= w(x\cosh\lambda + ta^{1/2}\sinh\lambda,\ y,\ xa^{-1/2}\sinh\lambda + t\cosh\lambda),
\end{aligned}$$

where $C_1, \dots, C_5$ and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$\begin{aligned}
w(x,y,t) &= |y|^{1/2}\left[\varphi(x - t\sqrt{a}\,) + \psi(x + t\sqrt{a}\,)\right] - \frac{c}{b},\\
w(x,y,t) &= \left|y\varphi(x - t\sqrt{a}\,) + \psi(x - t\sqrt{a}\,)\right|^{1/2} - \frac{c}{b},\\
w(x,y,t) &= \left|y\varphi(x + t\sqrt{a}\,) + \psi(x + t\sqrt{a}\,)\right|^{1/2} - \frac{c}{b},
\end{aligned}$$

where $\varphi(z_1)$ and $\psi(z_2)$ are arbitrary functions.

3°. Solutions:

$$w(x,y,t) = A\sqrt{C_1x + C_2y + C_3t + C_4} + \frac{C_3^2 - aC_1^2}{bC_2^2} - \frac{c}{b},$$

$$w(x,y,t) = \frac{(y + C_1)^2}{\left(C_2x \pm t\sqrt{aC_2^2 + b} + C_3\right)^2} - \frac{c}{b},$$

$$w(x,y,t) = \frac{2a(y + C_1)^2}{3b\left[a(t + C_2)^2 - (x + C_3)^2\right]} - \frac{c}{b},$$

$$w(x,y,t) = \frac{1}{bC_2^2}\left(\frac{C_1x + C_2y + C_3}{t + C_4}\right)^2 - \frac{aC_1^2}{bC_2^2} - \frac{c}{b},$$

$$w(x,y,t) = \frac{C_2^2 - aC_1^2}{b}\left(\frac{y + C_4}{C_1x + C_2t + C_3}\right)^2 - \frac{c}{b},$$

$$w(x,y,t) = \frac{C_2^2}{bC_1^2} - \frac{a}{bC_1^2}\left(\frac{C_1y + C_2t + C_3}{x + C_4}\right)^2 - \frac{c}{b},$$

where A, C_1, ..., C_4 are arbitrary constants (the first solution is of the traveling-wave type).

4°. Solutions in implicit form:

$$2\lambda\sqrt{a}\,(y + \lambda t) + (t\sqrt{a} \pm x)(bw + c - \lambda^2) = \varphi(w),$$

where $\varphi(w)$ is an arbitrary function and λ is an arbitrary constant.

5°. Solution:

$$w = u(z) - 4abC_1^2x^2, \qquad z = y + bC_1x^2 + C_2t,$$

where C_1 and C_2 are arbitrary constants and the function $u(z)$ is determined by the first-order ordinary differential equation

$$(bu + c - C_2^2)u_z' + 2abC_1u = 8a^2bC_1^2z + C_3.$$

With appropriate translations in both variables, the equation can be made homogeneous, which means that the equation is integrable by quadrature.

6°. Solution:

$$w = v(r) - 4abC_1^2x^2 + 4bC_2^2t^2, \qquad r = y + bC_1x^2 + bC_2t^2,$$

where C_1 and C_2 are arbitrary constants and the function $v(r)$ is determined by the first-order ordinary differential equation

$$(bv + c)v_r' + 2b(aC_1 - C_2)v = 8b(a^2C_1^2 + C_2^2)r + C_3.$$

With appropriate translations in both variables, the equation can be made homogeneous, which means that the equation is integrable by quadrature.

7°. Solution (generalizes the solutions of Items 5° and 6°):

$$w = U(\xi) + A_1x^2 + A_2t^2 + A_3xt + A_4x + A_5t, \qquad \xi = y + b(B_1x^2 + B_2t^2 + B_3xt + B_4x + B_5t),$$

where B_1, B_2, B_3, B_4, and B_5 are arbitrary constants, and the coefficients A_n are expressed in terms of B_n as

$$A_1 = b(B_3^2 - 4aB_1^2),$$
$$A_2 = b(4B_2^2 - aB_3^2),$$
$$A_3 = 4bB_3(B_2 - aB_1),$$
$$A_4 = 2b(B_3B_5 - 2aB_1B_4),$$
$$A_5 = 2b(2B_2B_5 - aB_3B_4),$$

and the function $U(\xi)$ is determined by the first-order ordinary differential equation

$$(bU + c + ab^2B_4^2 - b^2B_5^2)U_\xi' + 2b(aB_1 - B_2)U = 2(A_2 - aA_1)\xi + C_1.$$

With appropriate translations in both variables, the equation can be made homogeneous, which means that the equation is integrable by quadrature.

8°. Generalized separable solution linear in y:

$$w = F(x,t)y + G(x,t),$$

where the functions F and G are determined by the system of differential equations

$$\frac{\partial^2 F}{\partial t^2} - a\frac{\partial^2 F}{\partial x^2} = 0, \tag{1}$$

$$\frac{\partial^2 G}{\partial t^2} - a\frac{\partial^2 G}{\partial x^2} = bF^2. \tag{2}$$

Equation (1) is a linear homogeneous wave equation. Given $F = F(x,t)$, (2) represents a linear nonhomogeneous wave equation.

The general solution of system (1)–(2) is given by

$$F(x,t) = \varphi_1(\xi) + \varphi_2(\eta),$$
$$G(x,t) = \psi_1(\xi) + \psi_2(\eta) - \frac{b}{4a}\eta\int\varphi_1^2(\xi)\,d\xi - \frac{b}{4a}\xi\int\varphi_2^2(\eta)\,d\eta - \frac{b}{2a}\int\varphi_1(\xi)\,d\xi\int\varphi_2(\eta)\,d\eta,$$
$$\xi = x + t\sqrt{a}, \quad \eta = x - t\sqrt{a},$$

where $\varphi_1(\xi)$, $\varphi_2(\eta)$, $\psi_1(\xi)$, and $\psi_2(\eta)$ are arbitrary functions.

9°. "Two-dimensional" generalized separable solution quadratic in y (generalizes the second and third solutions of Item 2°):

$$w = f(x,t)y^2 + g(x,t)y + h(x,t),$$

where the functions $f = f(x,t)$, $g = g(x,t)$, and $h = h(x,t)$ are determined by the system of differential equations

$$f_{tt} = af_{xx} + 6bf^2,$$
$$g_{tt} = ag_{xx} + 6bfg,$$
$$h_{tt} = ah_{xx} + bg^2 + 2bfh + 2cf.$$

Here, the subscripts denote partial derivatives.

10°. "Two-dimensional" solution:

$$w = V(\eta,t) - 4abC_1^2x^2 - 4abC_1C_2x, \quad \eta = y + bC_1x^2 + bC_2x,$$

where C_1 and C_2 are arbitrary constants and the function $V(\eta,t)$ is determined by the differential equation

$$\frac{\partial^2 V}{\partial t^2} = \frac{\partial}{\partial\eta}\left[(bV + c + ab^2C_2^2)\frac{\partial V}{\partial\eta}\right] + 2abC_1\frac{\partial V}{\partial\eta} - 8a^2bC_1^2.$$

11°+. "Two-dimensional" solution:

$$w = W(x,\zeta) + 4bC_1^2t^2 + 4bC_1C_2t, \quad \zeta = y + bC_1t^2 + bC_2t,$$

where C_1 and C_2 are arbitrary constants and the function $W(\zeta,t)$ is determined by the differential equation

$$a\frac{\partial^2 W}{\partial x^2} + \frac{\partial}{\partial\zeta}\left[(bW + c - b^2C_2^2)\frac{\partial W}{\partial\zeta}\right] - 2bC_1\frac{\partial W}{\partial\zeta} - 8bC_1^2 = 0.$$

12°+. Solution:

$$w = R(\rho) - 4aC_1\varphi(\xi), \quad \rho = y + bC_1\bigl(x - t\sqrt{a}\bigr) + \int\varphi(\xi)\,d\xi, \quad \xi = x + t\sqrt{a},$$

where C_1 is an arbitrary constant, $\varphi(\xi)$ is an arbitrary function, and the function $R(\rho)$ is determined by the simple ordinary differential equation $[(bR+c)R'_\rho]'_\rho = 0$. Integrating yields a solution of the original equation in the form

$$b(w+4aC_1\varphi)^2 + 2c(w+4aC_1\varphi) = C_2 y + bC_1C_2\bigl(x - t\sqrt{a}\,\bigr) + C_2\int \varphi\,d\xi + C_3, \qquad \varphi = \varphi(\xi).$$

13°+. Solution (obtained in the same way as in Item 12°):

$$b(w+4aC_1\psi)^2 + 2c(w+4aC_1\psi) = C_2 y + bC_1C_2\bigl(x + t\sqrt{a}\,\bigr) + C_2\int \psi\,d\eta + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, $\psi = \psi(\eta)$ is an arbitrary function, $\eta = x - t\sqrt{a}$.

14°. Solution:

$$\begin{aligned} w &= U(z) - \frac{A^2}{2\sqrt{a}\,b}xt + \frac{A^2}{2b}t^2 - \frac{2\sqrt{a}\,AB}{b}t - \frac{1}{b}(A\eta + 4aB)\psi(\eta),\\ z &= y + \frac{A}{8a}\bigl(x^2 + 2\sqrt{a}\,xt - 3at^2\bigr) + B(x + \sqrt{a}\,t) + \int \psi(\eta)\,d\eta, \quad \eta = x - t\sqrt{a}, \end{aligned} \tag{3}$$

where A and B are arbitrary constants, $\psi(\eta)$ is an arbitrary function, and the function $U(z)$ is determined by the first-order ordinary differential equation (C is an arbitrary constant)

$$(bU + c)U'_z + AU - \frac{A^2}{b}z + C = 0.$$

With appropriate translations in both variables, the equation can be made homogeneous, which means it is integrable by quadrature.

Another solution can be obtained by substituting $-t$ for t in (3).

15°. There are solutions of the following forms:

$$\begin{aligned} &w(x,y,t) = F(y,r), \quad r = x^2 - at^2 && \text{"two-dimensional" solution;}\\ &w(x,y,t) = t^{2\lambda}G(\xi,\eta) - \frac{c}{b}, \quad \xi = \frac{x}{t}, \quad \eta = \frac{y}{t^{\lambda+1}} && \text{"two-dimensional" solution;}\\ &w(x,y,t) = H(z), \quad z = (x^2 - at^2)y^{-2} && \text{"one-dimensional" solution;} \end{aligned}$$

where λ is an arbitrary constant.

16°. The substitution $u = w + (c/b)$ leads to a special case of equation 4.1.2.5 with $n = 1$.

17°. For other solutions, see equation 4.4.2.3 with $f(w) = a$ and $g(w) = bw + c$.

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[(aw+b)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[(aw+b)\frac{\partial w}{\partial y}\right].$

The substitution $U = aw + b$ leads to an equation of the form 4.1.2.2:

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial x}\left(U\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(U\frac{\partial U}{\partial y}\right).$$

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[(a_1w+b_1)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[(a_2w+b_2)\frac{\partial w}{\partial y}\right].$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1x + C_2, \pm C_1y + C_3, \pm C_1t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution:

$$w(x,y,t) = A\sqrt{k_1x + k_2y + \lambda t + B} + \frac{\lambda^2 - b_1k_1^2 - b_2k_2^2}{a_1k_1^2 + a_2k_2^2},$$

where A, B, k_1, k_2, and λ are arbitrary constants.

3°. Generalized separable solution linear in space variables:

$$w(x,y,t) = (A_1t + B_1)x + (A_2t + B_2)y$$
$$+ \tfrac{1}{12}(a_1A_1^2 + a_2A_2^2)t^4 + \tfrac{1}{3}(a_1A_1B_1 + a_2A_2B_2)t^3 + \tfrac{1}{2}(a_1B_1^2 + a_2B_2^2)t^2 + Ct + D.$$

where A_1, A_2, B_1, B_2, C, and D are arbitrary constants.

4°. Solutions:

$$w(x,y,t) = \frac{1}{a_1C_1^2 + a_2C_2^2}\left(\frac{C_1x + C_2y + C_3}{t + C_4}\right)^2 - \frac{b_1C_1^2 + b_2C_2^2}{a_1C_1^2 + a_2C_2^2},$$
$$w(x,y,t) = \frac{(C_2^2 - b_2C_1^2)(x + C_4)^2 - b_1(C_1y + C_2t + C_3)^2}{a_2C_1^2(x + C_4)^2 + a_1(C_1y + C_2t + C_3)^2},$$
$$w(x,y,t) = \frac{(C_2^2 - b_1C_1^2)(y + C_4)^2 - b_2(C_1x + C_2t + C_3)^2}{a_1C_1^2(y + C_4)^2 + a_2(C_1x + C_2t + C_3)^2},$$

where $C_1, \ldots, C_4$ are arbitrary constants.

5°. There is a generalized separable solution of the form

$$w(x,y,t) = f(t)x^2 + g(t)xy + h(t)y^2 + \varphi(t)x + \psi(t)y + \chi(t).$$

6°. For other solutions, see equation 4.4.2.3 with $f(w) = a_1w + b_1$ and $g(w) = a_2w + b_2$.

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{1}{aw+b}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{aw+b}\frac{\partial w}{\partial y}\right).$

The substitution $U = aw + b$ leads to an equation of the form 4.1.2.4:

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{1}{U}\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{U}\frac{\partial U}{\partial y}\right).$$

5. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[(a_1w^n + b_1)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[(a_2w^n + b_2)\frac{\partial w}{\partial y}\right].$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1x + C_2, \pm C_1y + C_3, \pm C_1t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solutions:

$$w(x,y,t) = \bigl(C_1x + C_2y + \lambda t + C_3\bigr)^{\frac{1}{n+1}}, \qquad \lambda = \pm\sqrt{b_1C_1^2 + b_2C_2^2},$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Solutions:

$$w(x,y,t) = \left[\frac{1}{a_1C_1^2 + a_2C_2^2}\left(\frac{C_1x + C_2y + C_3}{t + C_4}\right)^2 - \frac{b_1C_1^2 + b_2C_2^2}{a_1C_1^2 + a_2C_2^2}\right]^{1/n},$$
$$w(x,y,t) = \left[\frac{(C_2^2 - b_2C_1^2)(x + C_4)^2 - b_1(C_1y + C_2t + C_3)^2}{a_2C_1^2(x + C_4)^2 + a_1(C_1y + C_2t + C_3)^2}\right]^{1/n},$$
$$w(x,y,t) = \left[\frac{(C_2^2 - b_1C_1^2)(y + C_4)^2 - b_2(C_1x + C_2t + C_3)^2}{a_1C_1^2(y + C_4)^2 + a_2(C_1x + C_2t + C_3)^2}\right]^{1/n},$$

where $C_1, \ldots, C_4$ are arbitrary constants.

4°. For other solutions, see equation 4.4.2.3 with $f(w) = a_1w^n + b_1$ and $g(w) = a_2w^n + b_2$.

4.1.4. Other Equations

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} = (\alpha + \beta w)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \gamma w^2 + \delta w + \varepsilon.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, \pm y + C_2, \pm t + C_3),$$
$$w_2 = w(x\cos\beta + y\sin\beta, -x\sin\beta + y\cos\beta,\ t),$$

where C_1, C_2, C_3, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. "Two-dimensional" generalized separable solution:

$$w(x, y, t) = f(t) + g(t)\Theta(x, y). \tag{1}$$

Here, the function $\Theta(x, y)$ satisfies the two-dimensional Helmholtz equation

$$\Delta\Theta + \varkappa\Theta = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

where $\varkappa = \gamma/\beta$ ($\beta \neq 0$). For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002). The functions $f(t)$ and $g(t)$ in (1) are determined from the autonomous system of nonlinear ordinary differential equations

$$f''_{tt} = \gamma f^2 + \delta f + \varepsilon, \tag{2}$$
$$g''_{tt} = (\gamma f + \delta - \alpha\varkappa)g. \tag{3}$$

Equation (2) is independent of $g(t)$. Particular solutions of the equation are given by $f = \text{const}$, where f satisfies the quadratic equation $\gamma f^2 + \delta f + \varepsilon = 0$. For $\gamma = 0$, (2) is a constant-coefficient linear equation. For $\gamma \neq 0$, the general solution of (2) can be written out in implicit form as

$$\int \frac{df}{\sqrt{\frac{2}{3}\gamma f^3 + \delta f^2 + 2\varepsilon f + C_1}} = C_2 \pm t,$$

where C_1 and C_2 are arbitrary constants. Equation (3) is linear in $g(t)$. For particular solutions of the form $f = \text{const}$, it is a constant-coefficient linear equation.

3°. There is a "two-dimensional" solution of the form

$$w(x, y, t) = U(z_1, z_2), \quad z_1 = a_1 x + b_1 y + c_1 t, \quad z_2 = a_2 x + b_2 y + c_2 t,$$

where the a_n, b_n, and c_n are arbitrary constants ($n = 1,\ 2$). To the special case $U = U(z_1)$ there corresponds a traveling wave solution.

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \alpha w\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \alpha\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] - \beta.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^2 x + C_2, \pm C_1^2 y + C_3, C_1 t + C_4),$$
$$w_2 = w(x\cos\beta + y\sin\beta, -x\sin\beta + y\cos\beta,\ t),$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There are generalized separable solutions of the form

$$w(x,y,t) = f(t) + g(t)\varphi(x) + h(t)\psi(y).$$

In particular, if $\varphi''_{xx} = \nu\varphi$ and $\psi''_{yy} = -\nu\psi$, where ν is an arbitrary constant, we have (A_1, A_2, B_1, and B_2 are arbitrary constants)

$$\begin{aligned}
&\varphi(x) = A_1 \cosh \mu x + A_2 \sinh \mu x, && \psi(y) = B_1 \cos \mu y + B_2 \sin \mu y && (\nu = \mu^2 > 0),\\
&\varphi(x) = A_1 \cos \mu x + A_2 \sin \mu x, && \psi(y) = B_1 \cosh \mu y + B_2 \sinh \mu y && (\nu = -\mu^2 < 0).
\end{aligned}$$

The functions $f(t)$, $g(t)$, and $h(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
f''_{tt} &= \alpha\nu(A_1^2 - sA_2^2)g^2 - \alpha\nu(B_1^2 + sB_2^2)h^2 - \beta,\\
g''_{tt} &= \alpha\nu f g,\\
h''_{tt} &= -\alpha\nu f h,
\end{aligned}$$

where $s = \operatorname{sign}\nu$.

3°. There are generalized separable solutions of the form

$$w(x,y,t) = f(t) + g(t)\varphi(x) + h(t)\psi(y) + u(t)\theta(x)\chi(y). \qquad (1)$$

For $\varphi''_{xx} = 4\nu\varphi$, $\psi''_{yy} = -4\nu\psi$, $\theta''_{xx} = \nu\theta$, and $\chi''_{yy} = -\nu\chi$, where ν is an arbitrary constant, one should set in (1):

if $\nu = \mu^2 > 0$	if $\nu = -\mu^2 < 0$
$\varphi(x) = A_1 \cosh 2\mu x + A_2 \sinh 2\mu x$	$\varphi(x) = A_1 \cos 2\mu x + A_2 \sin 2\mu x$
$\psi(y) = B_1 \cos 2\mu y + B_2 \sin 2\mu y$	$\psi(y) = B_1 \cosh 2\mu y + B_2 \sinh 2\mu y$
$\theta(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$	$\theta(x) = C_1 \cos \mu x + C_2 \sin \mu x$
$\chi(y) = D_1 \cos \mu y + D_2 \sin \mu y$	$\chi(y) = D_1 \cosh \mu y + D_2 \sinh \mu y$

The functions $f(t)$, $g(t)$, $h(t)$, and $u(t)$ are determined by the following system of ordinary differential equations ($s = \operatorname{sign}\nu$):

$$\begin{aligned}
f''_{tt} &= -4\alpha\nu(A_1^2 - sA_2^2)g^2 + 4\alpha\nu(B_1^2 + sB_2^2)h^2 - \beta,\\
g''_{tt} &= -4\alpha\nu f g + \alpha\nu a_1(D_1^2 + sD_2^2)u^2,\\
h''_{tt} &= 4\alpha\nu f h - \alpha\nu a_2(C_1^2 - sC_2^2)u^2,\\
u''_{tt} &= -2\alpha\nu(a_3 g - a_4 h)u.
\end{aligned}$$

The arbitrary constants A_1, A_2, B_1, B_2, C_1, C_2, D_1, and D_2 are related by the two constraints

$$2A_1C_1C_2 = A_2(C_1^2 + sC_2^2), \quad 2B_1D_1D_2 = B_2(D_1^2 - sD_2^2).$$

The coefficients a_1, a_2, a_3, and a_4 are expressed as

$$a_1 = \frac{C_1^2 + sC_2^2}{2A_1}, \quad a_2 = \frac{D_1^2 - sD_2^2}{2B_1}, \quad a_3 = A_2\frac{C_1^2 - sC_2^2}{C_1C_2}, \quad a_4 = B_2\frac{D_1^2 + sD_2^2}{D_1D_2},$$

with $A_1 \neq 0$, $B_1 \neq 0$, $C_1C_2 \neq 0$, and $D_1D_2 \neq 0$.

If $A_1 = 0$ ($A_2 \neq 0$), then one should set $a_1 = C_1C_2/A_2$. If $B_1 = 0$ ($B_2 \neq 0$), then $a_2 = D_1D_2/B_2$. If $C_1 = 0$ ($C_2 \neq 0$), then $a_3 = -A_1$. If $C_2 = 0$ ($C_1 \neq 0$), then $a_3 = A_1$. If $D_1 = 0$ ($D_2 \neq 0$), then $a_4 = -B_1$. If $D_2 = 0$ ($D_1 \neq 0$), then $a_4 = B_1$.

4°. There is a generalized separable solution of the form

$$w(x, y, t) = f(t)x^2 + g(t)xy + h(t)y^2 + \varphi(t)x + \psi(t)y + \chi(t).$$

In the special case $\varphi(t) = \psi(t) \equiv 0$, the functions $f(t)$, $g(t)$, $h(t)$, and $\chi(t)$ are determined by the autonomous system of ordinary differential equations

$$f''_{tt} = \alpha(2fh - 2f^2 - g^2), \qquad h''_{tt} = \alpha(2fh - 2h^2 - g^2),$$
$$g''_{tt} = -2\alpha g(f + h), \qquad \chi''_{tt} = 2\alpha(f + h)\chi - \beta.$$

5°. There is a "two-dimensional" solution of the form

$$w(x, y, t) = U(z_1, z_2), \quad z_1 = a_1x + b_1y + c_1t, \quad z_2 = a_2x + b_2y + c_2t,$$

where the a_n, b_n, and c_n are arbitrary constants ($n = 1,\ 2$). To the special case $U = U(z_1)$ there corresponds a traveling wave solution.

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a_1\frac{\partial}{\partial x}\left(w^n\frac{\partial w}{\partial x}\right) + a_2\frac{\partial}{\partial y}\left(w^k\frac{\partial w}{\partial y}\right) + bw^p.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{p-n-1}x + C_2, \pm C_1^{p-k-1}y + C_3, \pm C_1^{p-1}t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = F(\xi, \eta), \quad \xi = \alpha_1x + \alpha_2y + \alpha_3t, \quad \eta = \beta_1x + \beta_2y + \beta_3t;$$
$$w(x, y, t) = t^{\frac{2}{1-p}}U(z_1, z_2), \quad z_1 = xt^{\frac{p-n-1}{1-p}}, \quad z_2 = yt^{\frac{p-k-1}{1-p}}.$$

4.2. Equations with Two Space Variables Involving Exponential Nonlinearities

4.2.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] + ae^{\lambda w}$

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m\frac{\partial w}{\partial y}\right) + ce^{\lambda w}.$

This is a special case of equation 4.4.1.2 with $f(w) = ce^{\lambda w}$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\Big(C_1^{\frac{2}{2-n}}x,\ C_1^{\frac{2}{2-m}}y,\ \pm C_1t + C_2\Big) + \frac{2}{\lambda}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $m \neq 2$, and $\lambda \neq 0$:

$$w = -\frac{1}{\lambda}\ln\left\{\frac{2c\lambda(2-n)(2-m)}{4-nm}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right]\right\}.$$

3°. Solution for $n \neq 2$ and $m \neq 2$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r} w'_r + ck^{-1} e^{\lambda w} = 0, \qquad A = \frac{2(4-n-m)}{(2-n)(2-m)}.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = U(\xi, t), \quad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2}\right],$$

$$w(x,y,t) = V(x, \eta), \quad \eta^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right],$$

$$w(x,y,t) = W(y, \zeta), \quad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right],$$

$$w(x,y,t) = F(z_1, z_2) - \frac{2}{\lambda}\ln|t|, \quad z_1 = x|t|^{\frac{2}{n-2}}, \quad z_2 = y|t|^{\frac{2}{m-2}}.$$

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\lambda y} \frac{\partial w}{\partial y}\right) + ce^{\beta w}.$

This is a special case of equation 4.4.1.3 with $f(w) = ce^{\beta w}$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\left(C_1^{\frac{2}{2-n}} x,\; y - \frac{2}{\lambda}\ln C_1,\; \pm C_1 t + C_2\right) + \frac{2}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $\beta \neq 0$, and $\lambda \neq 0$:

$$w = -\frac{1}{\beta}\ln\left\{\frac{2c\beta(2-n)}{n}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right]\right\}.$$

3°. Solution for $n \neq 2$ and $\lambda \neq 0$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r} w'_r + ck^{-1} e^{\beta w} = 0, \qquad A = \frac{2}{2-n}.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = U(\xi, t), \quad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right],$$

$$w(x,y,t) = V(x, \eta), \quad \eta^2 = \pm 4\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

$$w(x,y,t) = W(y, \zeta), \quad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right],$$

$$w(x,y,t) = F(z_1, z_2) - \frac{2}{\beta}\ln|t|, \quad z_1 = x|t|^{\frac{2}{n-2}}, \quad z_2 = y + \frac{2}{\lambda}\ln|t|.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\lambda y}\dfrac{\partial w}{\partial y}\right) + ce^{\mu w}.$

This is a special case of equation 4.4.1.4 with $f(w) = ce^{\mu w}$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\Big(x - \frac{2}{\beta}\ln C_1,\ y - \frac{2}{\lambda}\ln C_1,\ \pm C_1 t + C_2\Big) + \frac{2}{\mu}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $\beta \neq 0$ and $\lambda \neq 0$:

$$w = w(r), \qquad r^2 = 4k\left[\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t + C_1)^2\right],$$

where C_1 and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the autonomous ordinary differential equation

$$w''_{rr} + ck^{-1}e^{\mu w} = 0.$$

Its general solution is given by

$$w = \begin{cases} -\dfrac{1}{\mu}\ln\left[-\dfrac{c\mu}{2k}(r + C_3)^2\right] & \text{if } ck\mu < 0, \\ -\dfrac{1}{\mu}\ln\left[-\dfrac{c\mu}{2kC_2^2}\sin^2(C_2 r + C_3)\right] & \text{if } ck\mu < 0, \\ -\dfrac{1}{\mu}\ln\left[-\dfrac{c\mu}{2kC_2^2}\sinh^2(C_2 r + C_3)\right] & \text{if } ck\mu < 0, \\ -\dfrac{1}{\mu}\ln\left[\dfrac{c\mu}{2kC_2^2}\cosh^2(C_2 r + C_3)\right] & \text{if } ck\mu > 0, \end{cases}$$

where C_2 and C_3 are arbitrary constants.

3°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = U(\xi, t), \quad \xi^2 = 4\left(\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right),$$

$$w(x, y, t) = V(x, \eta), \quad \eta^2 = \pm 4\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t + C)^2\right],$$

$$w(x, y, t) = W(y, \zeta), \quad \zeta^2 = \pm 4\left[\frac{e^{-\beta x}}{a\beta^2} - \frac{1}{4}(t + C)^2\right],$$

$$w(x, y, t) = F(z_1, z_2) - \frac{2}{\mu}\ln|t|, \quad z_1 = x + \frac{2}{\beta}\ln|t|, \quad z_2 = y + \frac{2}{\lambda}\ln|t|.$$

4.2.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(e^{\beta w}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(e^{\lambda w}\dfrac{\partial w}{\partial y}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial}{\partial y}\left(e^{w}\dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(C_1 x + C_3, C_2 y + C_4, \pm C_1 t + C_5) + \ln\frac{C_1^2}{C_2^2},$$

$$w_2 = w(x\cosh\lambda + ta^{1/2}\sinh\lambda,\ y,\ xa^{-1/2}\sinh\lambda + t\cosh\lambda),$$

where $C_1, \ldots, C_5$ and λ are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,y,t) = \varphi(x - t\sqrt{a}\,) + \psi(x + t\sqrt{a}\,) + \ln|C_1 y + C_2|,$$
$$w(x,y,t) = \ln\bigl[y\varphi(x - t\sqrt{a}\,) + \psi(x - t\sqrt{a}\,)\bigr],$$
$$w(x,y,t) = \ln\bigl[y\varphi(x + t\sqrt{a}\,) + \psi(x + t\sqrt{a}\,)\bigr],$$

where C_1 and C_2 are arbitrary constants and $\varphi(z_1)$ and $\psi(z_2)$ are arbitrary functions.

3°. Solutions:

$$w(x,y,t) = \ln\left[\frac{(B^2 - aA^2)(y+D)^2}{b(Ax+Bt+C)^2}\right],$$
$$w(x,y,t) = \ln\left[\frac{(B^2 - aA^2)(y+D)^2}{2\cos^2(Ax+Bt+C)}\right],$$
$$w(x,y,t) = \ln\left[\frac{(aA^2 - B^2)(y+D)^2}{b\cosh^2(Ax+Bt+C)}\right],$$
$$w(x,y,t) = \ln\left[\frac{(B^2 - aA^2)(y+D)^2}{b\sinh^2(Ax+Bt+C)}\right],$$
$$w(x,y,t) = \ln\left(\frac{4aC}{b}\right) - 2\ln\left|(x+A)^2 - a(t+B)^2 + C\right| + 2\ln|y+D|,$$
$$w(x,y,t) = \ln\left[\frac{1}{bB^2}\left(\frac{Ax+By+C}{t+D}\right)^2 - \frac{aA^2}{bB^2}\right],$$
$$w(x,y,t) = \ln\left[\frac{B^2}{bA^2} - \frac{a}{bA^2}\left(\frac{Ay+Bt+C}{x+D}\right)^2\right],$$

where A, B, C, and D are arbitrary constants.

4°. “Two-dimensional” solution:

$$w(x,y,t) = U(x,t) + 2\ln|y+C|,$$

where the function $U = U(x,t)$ is determined by a differential equation of the form 3.2.1.1:

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^2 U}{\partial x^2} + 2be^U.$$

Integrating yields a solution of the original equation in the form

$$w(x,y,t) = f(\xi) + g(\eta) - 2\ln\left|k\int e^{f(\xi)}\,d\xi - \frac{b}{4ak}\int e^{g(\eta)}\,d\eta\right| + 2\ln|y+C|,$$
$$\xi = x - \sqrt{a}\,t, \qquad \eta = x + \sqrt{a}\,t,$$

where $f = f(\xi)$ and $g = g(\eta)$ are arbitrary functions and k is an arbitrary constant.

5°. Solutions in implicit form:

$$2\lambda\sqrt{a}\,(y + \lambda t) + (t\sqrt{a} \pm x)(be^w - \lambda^2) = \varphi(w),$$

where $\varphi(w)$ is an arbitrary function and λ is an arbitrary constant.

6°. There are “two-dimensional” solutions of the following forms:

$$w(x,y,t) = F(y,r), \quad r = x^2 - at^2;$$
$$w(x,y,t) = G(\xi,\eta) - 2k\ln|t|, \quad \xi = xt^{-1}, \quad \eta = y|t|^{k-1};$$
$$w(x,y,t) = H(\zeta_1,\zeta_2) + 2\ln|y|, \quad \zeta_1 = t + k_1\ln|y|, \quad \zeta_2 = x + k_2\ln|y|;$$
$$w(x,y,t) = U(\rho_1,\rho_2) + 2y, \quad \rho_1 = te^y, \quad \rho_2 = xe^y;$$
$$w(x,y,t) = V(\chi) + 2\ln|y/t|, \quad \chi = x/t,$$

where k, k_1, and k_2 are arbitrary constants.

7°. There is an exact solution of the form

$$w(x,y,t) = W(z), \quad z = (x^2 - at^2)y^{-2}.$$

8°. For other exact solutions, see equation 4.4.2.3 with $f(w) = a$ and $g(w) = be^w$.

⊙ *References for equation* 4.2.2.1: N. Ibragimov (1994), A. D. Polyanin and V. F. Zaitsev (2002).

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(e^{\lambda w}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(e^{\lambda w}\dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = w(C_1x + C_3, \pm C_1 y + C_4, C_2 t + C_5) + \frac{1}{\lambda}\ln\frac{C_2^2}{C_1^2},$$
$$w_2 = w\bigl(x\cos\beta + y\sqrt{a/b}\,\sin\beta, -x\sqrt{b/a}\,\sin\beta + y\cos\beta,\ t\bigr),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,y,t) = C_1t + C_2 + \frac{1}{\lambda}\ln(C_3x + C_4y + C_5);$$
$$w(x,y,t) = C_1t + C_2 + \frac{1}{\lambda}\ln\bigl[C_3(bx^2 - ay^2) + C_4xy + C_5\bigr];$$
$$w(x,y,t) = C_1t + C_2 + \frac{1}{\lambda}\ln\bigl[C_3\ln(bx^2 + ay^2) + C_4\bigr];$$
$$w(x,y,t) = C_1t + C_2 + \sqrt{b}\,C_3x + \frac{1}{\lambda}\ln\cos\bigl(\sqrt{a}\,C_3\lambda y + C_4\bigr);$$
$$w(x,y,t) = C_1t + C_2 + \frac{1}{\lambda}\ln\bigl[C_3\exp\bigl(\sqrt{b}\,C_4x\bigr)\cos\bigl(\sqrt{a}\,C_4y + C_5\bigr) + C_6\bigr];$$
$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{1}{aC_1^2 + bC_2^2}\left(\frac{C_1x + C_2y + C_3}{t + C_4}\right)^2\right];$$
$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{C_2^2(x + C_4)^2}{a(C_1y + C_2t + C_3)^2 + bC_1^2(x + C_4)^2}\right];$$
$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{bC_1x^2 + C_2xy + Ky^2 + C_3x + C_4y + C_5}{\cos^2(C_1t + C_6)}\right], \qquad K = \frac{C_1^2}{b} - aC_1;$$
$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{bC_1x^2 + C_2xy + Ky^2 + C_3x + C_4y + C_5}{\sinh^2(C_1t + C_6)}\right], \qquad K = \frac{C_1^2}{b} - aC_1;$$
$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{bC_1x^2 + C_2xy - Ky^2 + C_3x + C_4y + C_5}{\cosh^2(C_1t + C_6)}\right], \qquad K = \frac{C_1^2}{b} + aC_1;$$
$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{aC_1^2x^2 + C_2\exp\bigl(\sqrt{b}\,C_3x\bigr)\cos\bigl(\sqrt{a}\,C_3y + C_4\bigr)}{\cos^2(aC_1t + C_5)}\right];$$
$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{bC_1^2y^2 + C_2\exp\bigl(\sqrt{b}\,C_3x\bigr)\cos\bigl(\sqrt{a}\,C_3y + C_4\bigr)}{\cos^2(bC_1t + C_5)}\right];$$
$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{aC_1^2x^2 + C_2\exp\bigl(\sqrt{b}\,C_3x\bigr)\cos\bigl(\sqrt{a}\,C_3y + C_4\bigr)}{\sinh^2(aC_1t + C_5)}\right];$$
$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{bC_1^2y^2 + C_2\exp\bigl(\sqrt{b}\,C_3x\bigr)\cos\bigl(\sqrt{a}\,C_3y + C_4\bigr)}{\sinh^2(bC_1t + C_5)}\right];$$
$$w(x,y,t) = \frac{1}{\lambda}\ln\left[\frac{-aC_1^2x^2 + C_2\exp\bigl(\sqrt{b}\,C_3x\bigr)\cos\bigl(\sqrt{a}\,C_3y + C_4\bigr)}{\cosh^2(aC_1t + C_5)}\right];$$

$$w(x,y,t)=\frac{1}{\lambda}\ln\left[\frac{-bC_1^2y^2+C_2\exp\bigl(\sqrt{b}\,C_3x\bigr)\cos\bigl(\sqrt{a}\,C_3y+C_4\bigr)}{\cosh^2(bC_1t+C_5)}\right];$$

where $C_1, \ldots, C_6$ are arbitrary constants.

3°. "Two-dimensional" solution (generalizes the first five solutions of Item 2°):

$$w(x,y,t)=C_1t+C_2+\frac{1}{\lambda}\ln U(\xi,\eta),\qquad \xi=\frac{x}{\sqrt{a}},\qquad \eta=\frac{y}{\sqrt{b}},$$

where C_1 and C_2 are arbitrary constants and the function $U=U(\xi,\eta)$ is determined by the Laplace equation

$$\frac{\partial^2U}{\partial\xi^2}+\frac{\partial^2U}{\partial\eta^2}=0.$$

For this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. "Two-dimensional" solution:

$$w(x,y,t)=f(t)+\frac{1}{\lambda}\ln V(\xi,\eta),\qquad \xi=\frac{x}{\sqrt{a}},\qquad \eta=\frac{y}{\sqrt{b}},$$

where the function $f=f(t)$ is determined by the autonomous ordinary differential equation

$$f''_{tt}=e^{\lambda f}, \tag{1}$$

and the function $V=V(\xi,\eta)$ is a solution of the Poisson equation

$$\Delta V-\lambda=0,\qquad \Delta=\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}. \tag{2}$$

For this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

The general solution of equation (1) is given by

$$f(t)=\begin{cases} -\dfrac{1}{\lambda}\ln\bigl[\tfrac{1}{2}\lambda(t+C_1)^2\bigr] & \text{if } \lambda>0,\\ -\dfrac{1}{\lambda}\ln\left[\dfrac{\lambda}{2C_1^2}\cos^2(C_1t+C_2)\right] & \text{if } \lambda>0,\\ -\dfrac{1}{\lambda}\ln\left[\dfrac{\lambda}{2C_1^2}\sinh^2(C_1t+C_2)\right] & \text{if } \lambda>0,\\ -\dfrac{1}{\lambda}\ln\left[-\dfrac{\lambda}{2C_1^2}\cosh^2(C_1t+C_2)\right] & \text{if } \lambda<0. \end{cases}$$

5°. There are solutions of the following forms:

$w(x,y,t)=F(z,t)+\dfrac{2}{\lambda}\ln x,\quad z=\dfrac{y}{x},$ "two-dimensional" solution;

$w(x,y,t)=G(r,t),\quad r=bx^2+ay^2$ "two-dimensional" solution;

$w(x,y,t)=H(z_1,z_2)-\dfrac{2k}{\lambda}\ln|t|,\quad z_1=x|t|^{k-1},\quad z_2=y|t|^{k-1}$ "two-dimensional" solution;

$w(x,y,t)=U(\xi,\eta)-\dfrac{2}{\lambda}\ln|t|,\quad \xi=x+k_1\ln|t|,\quad \eta=y+k_2\ln|t|$ "two-dimensional" solution;

$w(x,y,t)=V(\rho_1,\rho_2)-\dfrac{2}{\lambda}t,\quad \rho_1=xe^t,\quad \rho_2=ye^t$ "two-dimensional" solution;

$w(x,y,t)=W(z)+\dfrac{2}{\lambda}\ln\left|\dfrac{x}{t}\right|,\quad z=\dfrac{y}{x}$ "one-dimensional" solution,

$w(x,y,t)=R(\zeta),\quad \zeta=\dfrac{bx^2+ay^2}{t^2}$ "one-dimensional" solution,

where k, k_1, and k_2 are arbitrary constants.

6°. For other exact solutions, see equation 4.4.2.3 with $f(w)=ae^{\lambda w}$ and $g(w)=be^{\lambda w}$.

⦿ *References for equation* 4.2.2.2: N. Ibragimov (1994), A. D. Polyanin and V. F. Zaitsev (2002).

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(e^w \frac{\partial w}{\partial x}\right) + b\frac{\partial}{\partial y}\left(e^{\lambda w}\frac{\partial w}{\partial y}\right).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 C_2 x + C_3, \pm C_1 C_2^\lambda y + C_4, \pm C_1 t + C_5) - 2\ln|C_2|,$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$ak_1^2 e^w + bk_2^2 \lambda^{-1} e^{\lambda w} - \beta^2 w = C_1(k_1 x + k_2 y + \beta t) + C_2,$$

where C_1, C_2, k_1, k_2, and β are arbitrary constants.

3°. Solutions in implicit form:

$$\left(\frac{C_1 x + C_2 y + C_3}{t + C_4}\right)^2 = aC_1^2 e^w + bC_2^2 e^{\lambda w},$$

$$a\left(\frac{C_1 y + C_2 t + C_3}{x + C_4}\right)^2 e^w + bC_1^2 e^{\lambda w} = C_2^2,$$

$$b\left(\frac{C_1 x + C_2 t + C_3}{y + C_4}\right)^2 e^{\lambda w} + aC_1^2 e^w = C_2^2,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

4°. "Two-dimensional" solution (c_1 and c_2 are arbitrary constants):

$$w(x, y, t) = u(z, t), \quad z = c_1 x + c_2 y,$$

where the function $u = u(z, t)$ is determined by a differential equation of the form 3.4.4.6:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial z}\left[\varphi(u)\frac{\partial u}{\partial z}\right], \quad \varphi(u) = ac_1^2 e^u + bc_2^2 e^{\lambda u},$$

which can be reduced to a linear equation.

5°. "Two-dimensional" solution (s_1 and s_2 are arbitrary constants):

$$w(x, y, t) = v(x, \xi), \quad \xi = s_1 y + s_2 t,$$

where the function $v = v(x, \xi)$ is determined by a differential equation of the form 5.4.4.8:

$$a\frac{\partial}{\partial x}\left(e^v \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left[\psi(v)\frac{\partial v}{\partial z}\right] = 0, \quad \psi(v) = bs_1^2 e^{\lambda v} - s_2^2,$$

which can be reduced to a linear equation.

6°. There is a "two-dimensional" solution of the form (generalize the solutions of Items 3° and 4°):

$$w(x, y, t) = U(z_1, z_2), \quad z_1 = a_1 x + b_1 y + c_1 t, \quad z_2 = a_2 x + b_2 y + c_2 t.$$

7°. Solution:

$$w(x, y, t) = U(\xi) + 2\ln(x/t), \quad \xi = x^{-\lambda} y t^{\lambda - 1},$$

where the function $U = U(\xi)$ is determined by the ordinary differential equation

$$\bigl[a\lambda^2 \xi^2 e^U + be^{\lambda U} - (\lambda - 1)^2 \xi^2\bigr]U''_{\xi\xi} + \lambda\bigl(a\lambda\xi^2 e^U + be^{\lambda U}\bigr)(U'_\xi)^2 + \xi\bigl[a\lambda(\lambda - 3)e^U - (\lambda - 1)(\lambda - 2)\bigr]U'_\xi + 2(ae^U - 1) = 0.$$

8°. "Two-dimensional" solution:

$$w(x, y, t) = u(z, t) + 2\ln x, \quad z = x^{-\lambda} y,$$

where the function $u = u(z, t)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial t^2} = \bigl(a\lambda^2 z^2 e^u + be^{\lambda u}\bigr)\frac{\partial^2 u}{\partial z^2} + \lambda\bigl(a\lambda z^2 e^u + be^{\lambda u}\bigr)\left(\frac{\partial u}{\partial z}\right)^2 + a\lambda(\lambda - 3)ze^u \frac{\partial u}{\partial z} + 2ae^u.$$

9°. For other exact solutions, see equation 4.4.2.3 with $f(w) = ae^w$ and $g(w) = be^{\lambda w}$.

⊙ *References for equation* 4.2.2.3: N. Ibragimov (1994), A. D. Polyanin and V. F. Zaitsev (2002).

4.2.3. Other Equations

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[(a_1 e^{\lambda w} + b_1)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[(a_2 e^{\lambda w} + b_2)\dfrac{\partial w}{\partial y}\right].$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solutions:

$$w(x, y, t) = \frac{1}{\lambda} \ln\bigl(C_1 x + C_2 y + \beta t + C_3\bigr), \qquad \beta = \pm\sqrt{b_1 C_1^2 + b_2 C_2^2},$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Solutions:

$$w(x, y, t) = \frac{1}{\lambda} \ln\left[\frac{1}{a_1 C_1^2 + a_2 C_2^2}\left(\frac{C_1 x + C_2 y + C_3}{t + C_4}\right)^2 - \frac{b_1 C_1^2 + b_2 C_2^2}{a_1 C_1^2 + a_2 C_2^2}\right],$$
$$w(x, y, t) = \frac{1}{\lambda} \ln\left[\frac{(C_2^2 - b_2 C_1^2)(x + C_4)^2 - b_1 (C_1 y + C_2 t + C_3)^2}{a_2 C_1^2 (x + C_4)^2 + a_1 (C_1 y + C_2 t + C_3)^2}\right],$$
$$w(x, y, t) = \frac{1}{\lambda} \ln\left[\frac{(C_2^2 - b_1 C_1^2)(y + C_4)^2 - b_2 (C_1 x + C_2 t + C_3)^2}{a_1 C_1^2 (y + C_4)^2 + a_2 (C_1 x + C_2 t + C_3)^2}\right],$$

where $C_1, \ldots, C_4$ are arbitrary constants.

4°. For other solutions, see equation 4.4.2.3 with $f(w) = a_1 e^{\lambda w} + b_1$ and $g(w) = a_2 e^{\lambda w} + b_2$.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(e^{\lambda_1 w}\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(e^{\lambda_2 w}\dfrac{\partial w}{\partial y}\right) + c e^{\beta w}.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\beta-\lambda_1} x + C_2, \pm C_1^{\beta-\lambda_2} y + C_3, \pm C_1^{\beta} t + C_4) + 2\ln|C_1|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, t) = U(\xi, \eta) - \frac{2}{\beta}\ln|t|, \qquad \xi = x|t|^{\frac{\lambda_1-\beta}{\beta}}, \qquad \eta = y|t|^{\frac{\lambda_2-\beta}{\beta}};$$
$$w(x, y, t) = V(\eta_1, \eta_2), \qquad \eta_1 = a_1 x + b_1 y + c_1 t, \qquad \eta_2 = a_2 x + b_2 y + c_2 t.$$

4.3. Nonlinear Telegraph Equations with Two Space Variables

4.3.1. Equations Involving Power Law Nonlinearities

1. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left[(bw + c)\dfrac{\partial w}{\partial y}\right].$

1°. "Two-dimensional" generalized separable solution linear in y:

$$w = f(x, t)y + g(x, t),$$

where the functions f and g are determined by the one-dimensional equations

$$\frac{\partial^2 f}{\partial t^2} + k\frac{\partial f}{\partial t} = a\frac{\partial^2 f}{\partial x^2},$$
$$\frac{\partial^2 g}{\partial t^2} + k\frac{\partial g}{\partial t} = a\frac{\partial^2 g}{\partial x^2} + bf^2.$$

The first equation is a linear homogeneous telegraph equation. Given $f = f(x,t)$, the second one represents a linear nonhomogeneous telegraph equation. For these equations, see the book by Polyanin (2002).

2°. There is a "two-dimensional" generalized separable solution quadratic in y:

$$w = f(x,t)y^2 + g(x,t)y + h(x,t).$$

3°. The substitution $u = w + (c/b)$ leads to the special case of equation 4.3.1.4 with $m = 1$.

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[(a_1 w + b_1)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[(a_2 w + b_2)\frac{\partial w}{\partial y}\right].$

1°. Additive separable solution:

$$w(x,y,t) = Akx + Bky + Ce^{-kt} + k(A^2a_1 + B^2a_2)t + D,$$

where A, B, C, and D are arbitrary constants.

2°. Generalized separable solution linear in the space variables:

$$w(x,y,t) = (A_1e^{-kt} + B_1)x + (A_2e^{-kt} + B_2)y + \frac{1}{2k^2}(a_1A_1^2 + a_2A_2^2)e^{-2kt}$$
$$- \frac{2}{k}(a_1A_1B_1 + a_2A_2B_2)te^{-kt} + C_1e^{-kt} + \frac{1}{k}(a_1B_1^2 + a_2B_2^2)t + C_2,$$

where A_1, A_2, B_1, B_2, C_1, and C_2 are arbitrary constants.

3°. Traveling-wave solution in implicit form ($k \neq 0$):

$$k\lambda(a_1\beta_1^2 + a_2\beta_2^2)w + [k\lambda(b_1\beta_1^2 + b_2\beta_2^2 - \lambda^2) - C_1(a_1\beta_1^2 + a_2\beta_2^2)]\ln(k\lambda w + C_1)$$
$$= k^2\lambda^2(\beta_1 x + \beta_2 y + \lambda t) + C_2,$$

where C_1, C_2, β_1, β_2, and λ are arbitrary constants.

4°. There is a generalized separable solution of the form

$$w(x,y,t) = f(t)x^2 + g(t)xy + h(t)y^2 + \varphi(t)x + \psi(t)y + \chi(t).$$

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = a\left[\frac{\partial}{\partial x}\left(\frac{1}{w}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{w}\frac{\partial w}{\partial y}\right)\right].$

This is a special case of equation 4.3.1.6 with $n = m = -1$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w(\pm C_1x + C_2, \pm C_1y + C_3, t + C_4),$$
$$w_2 = w(x\cos\beta + y\sin\beta,\ -x\sin\beta + y\cos\beta,\ t),$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w(x,y,t) = \frac{2at + A + Be^{-kt}}{k(\sin y + Ce^x)^2},$$

$$w(x,y,t) = \frac{C_1^2(2at + A + Be^{-kt})}{ke^{2x}\sinh^2(C_1e^{-x}\sin y + C_2)},$$

$$w(x,y,t) = \frac{C_1^2(-2at + A + Be^{-kt})}{ke^{2x}\cosh^2(C_1e^{-x}\sin y + C_2)},$$

$$w(x,y,t) = \frac{C_1^2(2at + A + Be^{-kt})}{ke^{2x}\cos^2(C_1e^{-x}\sin y + C_2)},$$

where A, B, C, C_1, and C_2 are arbitrary constants.

3°. The exact solutions specified in Item 2° are special cases of a more general solution in the form of the product of functions with different arguments:

$$w(x,y,t) = (Aat + B + Ce^{-kt})e^{\Theta(x,y)},$$

where A, B, and C are arbitrary constants and the function $\Theta(x,y)$ is a solution of the stationary equation

$$\Delta\Theta - Ake^{\Theta} = 0, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

which occurs in combustion theory. For solutions of this equation, see 5.2.1.1.

⊙ *Reference*: N. H. Ibragimov (1994).

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\frac{\partial}{\partial y}\left(w^m\frac{\partial w}{\partial y}\right).$

This is a special case of equation 4.3.1.6 with $n = 0$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2}w(\pm x + C_2, \pm C_1^m y + C_3, t + C_4),$$

where $C_1, \dots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. "Two-dimensional" solution:

$$w(x,y,t) = u(x,t)y^{2/m},$$

where the function $u(x,t)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + \frac{2b(m+2)}{m^2}u^{m+1}.$$

For $m = -2$ and $m = -1$, this equation is linear.

3°. "Two-dimensional" multiplicative separable solution:

$$w(x,y,t) = \begin{cases} U(x,t)|y + C|^{1/(m+1)} & \text{if } m \neq -1, \\ U(x,t)\exp(Cy) & \text{if } m = -1, \end{cases}$$

where C is an arbitrary constant and the function $U(x,t)$ is determined by the telegraph equation

$$\frac{\partial^2 U}{\partial t^2} + k\frac{\partial U}{\partial t} = a\frac{\partial^2 U}{\partial x^2}.$$

For solutions of this linear equation, see the book by Polyanin (2002).

⊙ *Reference*: N. H. Ibragimov (1994).

5. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(w^n \dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^n x + C_2, \pm C_1^n y + C_3, t + C_4),$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a/b}\,\sin\beta, -x\sqrt{b/a}\,\sin\beta + y\cos\beta,\ t\big),$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Multiplicative separable solution:

$$w(x, y, t) = F(t)\Phi(x, y),$$

where the function $F(t)$ is determined by the autonomous ordinary differential equation (C is an arbitrary constant)

$$F''_{tt} + kF'_t = CF^{n+1}, \tag{1}$$

and the function $\Phi(x, y)$ satisfies the stationary equation

$$a\frac{\partial}{\partial x}\left(\Phi^n \frac{\partial \Phi}{\partial x}\right) + b\frac{\partial}{\partial y}\left(\Phi^n \frac{\partial \Phi}{\partial y}\right) = C\Phi. \tag{2}$$

Example. For $C = 0$, it follows from (1) that $F = Ae^{-kt} + B$, where A and B are arbitrary constants. For $C = 0$, equation (2) is reduced to the Laplace equation

$$\frac{\partial^2 \Psi}{\partial \bar{x}^2} + \frac{\partial^2 \Psi}{\partial \bar{y}^2} = 0, \qquad \text{where } \Psi = \Phi^{n+1},\ \bar{x} = \frac{x}{\sqrt{a}},\ \bar{y} = \frac{y}{\sqrt{b}}.$$

3°. "Two-dimensional" solution:

$$w(x, y, t) = u(r, t), \quad r = \sqrt{bx^2 + ay^2},$$

where the function $u(r, t)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} = \frac{ab}{r}\frac{\partial}{\partial r}\left(ru^n \frac{\partial u}{\partial r}\right).$$

4°. Solution:

$$w(x, y, t) = U(t)(bx^2 + ay^2)^{1/n},$$

where the function $U(t)$ is determined by the autonomous ordinary differential equation

$$U''_{tt} + kU'_t = \frac{4ab(n+1)}{n^2}U^{n+1}.$$

⊙ *Reference*: N. H. Ibragimov (1994).

6. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(w^m \dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^n x + C_2, \pm C_1^m y + C_3, t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int \frac{a_1\beta_1^2 w^n + a_2\beta_2^2 w^m - \lambda^2}{k\lambda w + C_1}\,dw = \beta_1 x + \beta_2 y + \lambda t + C_2,$$

where C_1, C_2, β_1, β_2, and λ are arbitrary constants.

3°. “Two-dimensional” solution:

$$w(x, y, t) = U(\xi, t), \quad \xi = \beta x + \mu y,$$

where β and μ are arbitrary constants, and the function $U = U(\xi, t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial t^2} + k\frac{\partial U}{\partial t} = \frac{\partial}{\partial \xi}\left[\left(a\beta^2 U^n + b\mu^2 U^m\right)\frac{\partial U}{\partial \xi}\right].$$

Remark. There is a more general, “two-dimensional” solution of the form

$$w(x, y, t) = V(\xi_1, \xi_2), \quad \xi_1 = \beta_1 x + \mu_1 y + \lambda_1 t, \quad \xi_2 = \beta_2 x + \mu_2 y + \lambda_2 t,$$

where the β_i, μ_i, and λ_i are arbitrary constants.

4°. “Two-dimensional” solution:

$$w(x, y, t) = y^{2/m} u(z, t), \quad z = xy^{-n/m},$$

where the function $u = u(z, t)$ is determined by the differential equation

$$m^2\left(\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t}\right) = \left(am^2u^n + bn^2z^2u^m\right)\frac{\partial^2 u}{\partial z^2} + nm\left(amu^{n-1} + bnz^2u^{m-1}\right)\left(\frac{\partial u}{\partial z}\right)^2 + bn(n - 3m - 4)zu^m\frac{\partial u}{\partial z} + 2b(m + 2)u^{m+1}.$$

⊙ *Reference*: N. H. Ibragimov (1994).

4.3.2. Equations Involving Exponential Nonlinearities

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} + k\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b\frac{\partial}{\partial y}\left(e^w\frac{\partial w}{\partial y}\right).$

This is a special case of equation 4.4.3.10 with $f(t) = k$, $g(t) = a$, $h(t) = b$, and $\lambda = 1$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm x + C_2, \pm C_1 y + C_3, t + C_4) - 2\ln|C_1|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. “Two-dimensional” solution:

$$w(x, y, t) = u(x, t) + \ln|y + C|,$$

where C is an arbitrary constant and the function $u(x, t)$ is determined by the linear telegraph equation

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}.$$

3°. “Two-dimensional” solution:

$$w(x, y, t) = U(x, t) + 2\ln|y + C|,$$

where C is an arbitrary constant and the function $U(x, t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial t^2} + k\frac{\partial U}{\partial t} = a\frac{\partial^2 U}{\partial x^2} + 2be^U.$$

⊙ *Reference*: N. H. Ibragimov (1994).

2. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^w \dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(e^w \dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, t + C_4) - 2\ln|C_1|,$$
$$w_2 = w\bigl(x\cos\beta + y\sqrt{a/b}\,\sin\beta, -x\sqrt{b/a}\,\sin\beta + y\cos\beta,\ t\bigr),$$

where $C_1, \dots, C_4$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. "Two-dimensional" additive separable solution:

$$w(x,y,t) = \varphi(t) + \ln[\psi(x,y)],$$

where the function $u(t)$ is determined by the autonomous ordinary differential equation (C is an arbitrary constant)

$$\varphi''_{tt} + k\varphi'_t = Ce^{\varphi},$$

and the function $\psi(x,y)$ satisfies the Poisson equation

$$\frac{\partial^2\psi}{\partial\bar{x}^2} + \frac{\partial^2\psi}{\partial\bar{y}^2} = C, \qquad \text{where } \bar{x} = \frac{x}{\sqrt{a}},\ \ \bar{y} = \frac{y}{\sqrt{b}}.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

3°. "Two-dimensional" solution:

$$w(x,y,t) = u(r,t), \quad r = \sqrt{bx^2 + ay^2},$$

where the function $u(r,t)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} = \frac{ab}{r}\frac{\partial}{\partial r}\left(re^u \frac{\partial u}{\partial r}\right).$$

4°. Solution:

$$w(x,y,t) = u(t) + \ln(bx^2 + ay^2),$$

where the function $u(t)$ is determined by the autonomous ordinary differential equation

$$u''_{tt} + ku'_t = 4abe^u.$$

5°. There is a "two-dimensional" solution of the form

$$w(x,y,t) = u(z,t) + 2\ln|x|, \quad z = y/x.$$

⊙ *Reference*: N. H. Ibragimov (1994).

3. $\dfrac{\partial^2 w}{\partial t^2} + k\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(e^w \dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(e^{\lambda w} \dfrac{\partial w}{\partial y}\right).$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1^{\lambda} y + C_3, t + C_4) - 2\ln|C_1|,$$

where $C_1, \dots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int \frac{a_1\beta_1^2 e^w + a_2\beta_2^2 e^{\lambda w} - \gamma^2}{k\gamma w + C_1}\,dw = \beta_1 x + \beta_2 y + \gamma t + C_2,$$

where C_1, C_2, β_1, β_2, and γ are arbitrary constants.

3°. "Two-dimensional" solution:

$$w(x,y,t) = U(\xi,t), \quad \xi = \beta x + \mu y,$$

where β and μ are arbitrary constants, and the function $U = U(\xi,t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial t^2} + k\frac{\partial U}{\partial t} = \frac{\partial}{\partial \xi}\left[\left(a\beta^2 e^w + b\mu^2 e^{\lambda w}\right)\frac{\partial U}{\partial \xi}\right].$$

Remark. There is a more general, "two-dimensional" solution of the form

$$w(x,y,t) = V(\xi_1,\xi_2), \quad \xi_1 = \beta_1 x + \mu_1 y + \sigma_1 t, \quad \xi_2 = \beta_2 x + \mu_2 y + \sigma_2 t,$$

where the β_i, μ_i, and σ_i are arbitrary constants.

4°. There is a "two-dimensional" solution of the form

$$w(x,y,t) = u(z,t) + 2\ln|x|, \quad z = y|x|^{-\lambda}.$$

⊙ *Reference*: N. H. Ibragimov (1994).

4.4. Equations with Two Space Variables Involving Arbitrary Functions

4.4.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] + h(w)$

1. $\frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + b\frac{\partial^2 w}{\partial y^2} + f(w).$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$\begin{aligned}
w_1 &= w(\pm x + C_1, \pm y + C_2, \pm t + C_3),\\
w_2 &= w\big(x\cos\beta + y\sqrt{a/b}\,\sin\beta,\ -x\sqrt{b/a}\,\sin\beta + y\cos\beta,\ t\big),\\
w_3 &= w\big(x\cosh\lambda + ta^{1/2}\sinh\lambda,\ y,\ xa^{-1/2}\sinh\lambda + t\cosh\lambda\big),\\
w_4 &= w\big(x,\ y\cosh\lambda + tb^{1/2}\sinh\lambda,\ yb^{-1/2}\sinh\lambda + t\cosh\lambda\big),
\end{aligned}$$

where C_1, C_2, C_3, β, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int\left[C_1 + \frac{2}{\lambda^2 - ak_1^2 - bk_2^2}F(w)\right]^{-1/2} dw = k_1 x + k_2 y + \lambda t + C_2, \qquad F(w) = \int f(w)\,dw,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. Solution (C_1, C_2, and C_3 are arbitrary constants):

$$w = w(r), \qquad r^2 = A\left[\frac{(x+C_1)^2}{a} + \frac{(y+C_2)^2}{b} - (t+C_3)^2\right].$$

Here, A and the expression in square brackets must have like signs, and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + 2r^{-1}w'_r + A^{-1}f(w) = 0.$$

4°. "Two-dimensional" solution:

$$w = U(\xi, \eta), \quad \xi = \frac{x}{\sqrt{a}C} + \frac{y}{\sqrt{b}}, \quad \eta = (C^2 - 1)\frac{x^2}{a} - 2C\frac{xy}{\sqrt{ab}} - C^2t^2, \tag{1}$$

where C is an arbitrary constant ($C \neq 0$), and the function $U = U(\xi, \eta)$ is determined by the equation

$$\left(1 + \frac{1}{C^2}\right)\frac{\partial^2 U}{\partial \xi^2} - 4\xi\frac{\partial^2 U}{\partial \xi \partial \eta} + 4C^2(\xi^2 + \eta)\frac{\partial^2 U}{\partial \eta^2} + 2(2C^2 - 1)\frac{\partial U}{\partial \eta} + f(U) = 0. \tag{2}$$

Remark. Relations (1) and equation (2) can be used to obtain other "two-dimensional" solution by means of the following rename: $(x, a) \rightleftarrows (y, b)$.

5°. There is a "two-dimensional" solution of the form

$$w(x, y, t) = u(z_1, z_2), \quad z_1 = C_1x + C_2y + \lambda_1 t, \quad z_2 = C_3x + C_4y + \lambda_2 t.$$

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m\frac{\partial w}{\partial y}\right) + f(w).$

1°. Solution for $n \neq 2$ and $m \neq 2$:

$$w = w(r), \qquad r^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t + C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + k^{-1}f(w) = 0, \qquad A = \frac{2(4 - n - m)}{(2 - n)(2 - m)}.$$

2°. "Two-dimensional" solution for $n \neq 2$ and $m \neq 2$:

$$w = U(\xi, t), \qquad \xi^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2}\right],$$

where the function $U(\xi, t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial t^2} = k\left(\frac{\partial^2 U}{\partial \xi^2} + \frac{B_1}{\xi}\frac{\partial U}{\partial \xi}\right) + f(U), \qquad B_1 = \frac{4 - nm}{(2 - n)(2 - m)}.$$

3°. "Two-dimensional" solution for $m \neq 2$:

$$w = V(x, \eta), \qquad \eta^2 = 4k\left[\frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t + C)^2\right],$$

where the function $V(x, \eta)$ is determined by the differential equation

$$\frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right) + k\left(\frac{\partial^2 V}{\partial \eta^2} + \frac{B_2}{\eta}\frac{\partial V}{\partial \eta}\right) + f(V) = 0, \qquad B_2 = \frac{2}{2 - m}.$$

4°. "Two-dimensional" solution for $n \neq 2$:

$$w = W(y, \zeta), \qquad \zeta^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t + C)^2\right],$$

where the function $W(y, \zeta)$ is determined by the differential equation

$$\frac{\partial}{\partial y}\left(by^m\frac{\partial w}{\partial y}\right) + k\left(\frac{\partial^2 W}{\partial \zeta^2} + \frac{B_3}{\zeta}\frac{\partial W}{\partial \zeta}\right) + f(W) = 0, \qquad B_3 = \frac{2}{2 - n}.$$

⊙ *References*: A. D. Polyanin and A. I. Zhurov (1998), A. D. Polyanin and V. F. Zaitsev (2002).

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\lambda y} \frac{\partial w}{\partial y}\right) + f(w).$

1°. Solution for $n \neq 2$ and $\lambda \neq 0$:

$$w = w(r), \qquad r^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + k^{-1}f(w) = 0, \qquad A = \frac{2}{2-n}.$$

2°. "Two-dimensional" solution for $n \neq 2$ and $\lambda \neq 0$:

$$w = U(\xi, t), \qquad \xi^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right],$$

where the function $U(\xi, t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial t^2} = k\left(\frac{\partial^2 U}{\partial \xi^2} + \frac{B}{\xi}\frac{\partial U}{\partial \xi}\right) + f(U), \qquad B = \frac{n}{2-n}.$$

3°. "Two-dimensional" solution for $\lambda \neq 0$:

$$w = V(x, \eta), \qquad \eta^2 = 4k\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where the function $V(x, \eta)$ is determined by the differential equation

$$\frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + k\frac{\partial^2 V}{\partial \eta^2} + f(V) = 0.$$

4°. "Two-dimensional" solution for $n \neq 2$:

$$w = W(y, \zeta), \qquad \zeta^2 = 4k\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right],$$

where the function $W(y, \zeta)$ is determined by the differential equation

$$\frac{\partial}{\partial y}\left(be^{\lambda y}\frac{\partial w}{\partial y}\right) + k\left(\frac{\partial^2 W}{\partial \zeta^2} + \frac{A}{\zeta}\frac{\partial W}{\partial \zeta}\right) + f(W) = 0, \qquad A = \frac{2}{2-n}.$$

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ae^{\beta x} \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\lambda y} \frac{\partial w}{\partial y}\right) + f(w).$

1°. Solution for $\beta \neq 0$ and $\lambda \neq 0$:

$$w = w(r), \qquad r^2 = 4k\left[\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where the function $w(r)$ is determined by the autonomous ordinary differential equation

$$w''_{rr} + k^{-1}f(w) = 0.$$

Integrating yields its general solution in implicit form:

$$\int\left[C_1 + 2k^{-1}\int f(w)\,dw\right]^{-1/2} dw = C_2 \pm r,$$

where C_1 and C_2 are arbitrary constants.

2°. "Two-dimensional" solution for $\beta \neq 0$ and $\lambda \neq 0$:

$$w = U(\xi, t), \qquad \xi^2 = 4k\left(\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right),$$

where the function $U(\xi, t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial t^2} = k\left(\frac{\partial^2 U}{\partial \xi^2} - \frac{1}{\xi}\frac{\partial U}{\partial \xi}\right) + f(U).$$

3°. "Two-dimensional" solution for $\lambda \neq 0$:

$$w = V(x, \eta), \qquad \eta^2 = 4k\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where the function $V(x, \eta)$ is determined by the differential equation

$$\frac{\partial}{\partial x}\left(ae^{\beta x}\frac{\partial w}{\partial x}\right) + k\frac{\partial^2 V}{\partial \eta^2} + f(V) = 0.$$

4°. "Two-dimensional" solution for $\beta \neq 0$:

$$w = W(y, \zeta), \qquad \zeta^2 = 4k\left[\frac{e^{-\beta x}}{a\beta^2} - \frac{1}{4}(t+C)^2\right],$$

where the function $W(y, \zeta)$ is determined by the differential equation

$$\frac{\partial}{\partial y}\left(be^{\lambda y}\frac{\partial w}{\partial y}\right) + k\frac{\partial^2 W}{\partial \zeta^2} + f(W) = 0.$$

⊙ *References*: A. D. Polyanin and A. I. Zhurov (1998), A. D. Polyanin and V. F. Zaitsev (2002).

5. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] + aw\ln w + bw.$

This is a special case of equation 4.4.3.6 with $g(t) = b$ and $h_1(x) = h_2(y) = 0$ and a special case of equation 4.4.3.7 with $f(x, y) = f(x)$ and $g(x, y) = g(y)$.

4.4.2. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(w)\frac{\partial w}{\partial y}\right] + h(w)$

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial y}\left[g(w)\frac{\partial w}{\partial y}\right].$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 t + C_4),$$
$$w_2 = w(x\cosh\lambda + ta^{1/2}\sinh\lambda,\ y,\ xa^{-1/2}\sinh\lambda + t\cosh\lambda),$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$k_2^2 \int g(w)\,dw + (ak_1^2 - \lambda^2)w = C_1(k_1 x + k_2 y + \lambda t) + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. Solutions in implicit form:

$$\int g(w)\,dw = y\varphi_1\bigl(x \pm t\sqrt{a}\,\bigr) + \varphi_2\bigl(x \pm t\sqrt{a}\,\bigr),$$

$$2\lambda\sqrt{a}\,(y+\lambda t) + (t\sqrt{a} \pm x)[g(w)-\lambda^2] = \psi(w),$$

where $\varphi_1(z)$, $\varphi_2(z)$, and $\psi(w)$ are arbitrary functions, and λ is an arbitrary constant.

4°. "Two-dimensional" solutions (generalize the solutions of Item 3°):

$$w(x,y,t) = U(\xi,\eta), \quad \xi = y + \lambda t, \quad \eta = x \pm t\sqrt{a},$$

where λ is an arbitrary constant and the function $U = U(\xi,\eta)$ is determined by the first-order partial differential equation

$$\bigl[g(U)-\lambda^2\bigr]\frac{\partial U}{\partial \xi} \mp 2\lambda\sqrt{a}\,\frac{\partial U}{\partial \eta} = \varphi(\eta), \tag{1}$$

with $\varphi(\eta)$ being an arbitrary function.

In the special case $\lambda = 0$, equation (1) is an ordinary differential equation in ξ that can be readily integrated to obtain the first group of solutions specified in Item 3°.

In the general case, the characteristic system corresponding to equation (1) has the form (Polyanin, Zaitsev, and Moussiaux, 2002)

$$\frac{d\xi}{g(U)-\lambda^2} = \mp\frac{d\eta}{2\lambda\sqrt{a}} = \frac{dU}{\varphi(\eta)}.$$

Its independent integrals are given by

$$U \pm \Phi(\eta) = C_1, \quad \xi \pm \frac{1}{2\lambda\sqrt{a}}\int g\bigl(C_1 \mp \Phi(\eta)\bigr)\,d\eta \mp \frac{\lambda}{2\sqrt{a}}\eta = C_2, \tag{2}$$

where

$$\Phi(\eta) = \frac{1}{2\lambda\sqrt{a}}\int \varphi(\eta)\,d\eta.$$

We first calculate the integral in the second relation of (2) and then, in the resulting expression, substitute the left-hand side of the first relation of (2) for C_1.

The general solution of equation (1) has the form

$$F(C_1, C_2) = 0,$$

where $F(C_1,C_2)$ is an arbitrary function of two variables, and C_1 and C_2 are determined by (2).

To the special case $\varphi(\eta) = 0$ in (1) there corresponds the second group of solutions specified in Item 3°.

5°. "Two-dimensional" solution:

$$w(x,y,t) = u(y,z), \quad z = x^2 - at^2,$$

where the function $u = u(y,z)$ is determined by the differential equation

$$4a\left(z\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left[g(u)\frac{\partial u}{\partial y}\right] = 0.$$

6°. Solution:

$$w(x,y,t) = v(\zeta), \quad \zeta = (x^2 - at^2)y^{-2},$$

where the function $v = v(\zeta)$ is determined by the ordinary differential equation

$$2a\zeta v''_{\zeta\zeta} + 2av'_\zeta + 2\zeta^2[g(v)v'_\zeta]'_\zeta + 3\zeta g(v)v'_\zeta = 0.$$

7°. For other exact solutions, see equation 4.4.2.3 with $f(w) = a$.

2. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f(w)\dfrac{\partial w}{\partial y}\right].$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 t + C_4),$$
$$w_2 = w(x\cos\beta - y\sin\beta, x\sin\beta + y\cos\beta, t),$$

where $C_1, \ldots, C_4$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$(k_1^2 + k_2^2)\int f(w)\,dw - \lambda^2 w = C_1(k_1 x + k_2 y + \lambda t) + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. Solution:

$$w(x, y, t) = U(\zeta), \quad \zeta = (x^2 + y^2)t^{-2},$$

where the function $U = U(\zeta)$ is determined by the ordinary differential equation

$$2\zeta^2 U''_{\zeta\zeta} + 3\zeta U'_\zeta = 2[\zeta f(U) U'_\zeta]'_\zeta.$$

4°. "Two-dimensional" solution with axial symmetry:

$$w(x, y, t) = u(r, t), \quad r = \sqrt{x^2 + y^2},$$

where the function $u = u(r, t)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left[r f(u)\frac{\partial u}{\partial r}\right].$$

5°. For other exact solutions, see equation 4.4.2.3 with $f(w) = g(w)$.

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right].$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the function

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int \left[k_1^2 f(w) + k_2^2 g(w)\right] dw - \lambda^2 w = C_1(k_1 x + k_2 y + \lambda t) + C_2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. Solutions in implicit form:

$$\left(\frac{C_1 x + C_2 y + C_3}{t + C_4}\right)^2 = C_1^2 f(w) + C_2^2 g(w),$$
$$\left(\frac{C_1 y + C_2 t + C_3}{x + C_4}\right)^2 f(w) + C_1^2 g(w) = C_2^2,$$
$$\left(\frac{C_1 x + C_2 t + C_3}{y + C_4}\right)^2 g(w) + C_1^2 f(w) = C_2^2,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

4°. Solution:

$$w = w(\xi), \quad \xi = \frac{C_1 x + C_2 y + C_3}{t + C_4},$$

where $C_1, \ldots, C_4$ are arbitrary constants and the function $u(\xi)$ is determined by the ordinary differential equation

$$(\xi^2 w'_\xi)'_\xi = [\varphi(w) w'_\xi]'_\xi, \qquad \varphi(w) = C_1^2 f(w) + C_2^2 g(w),$$

which admits the first integral

$$\bigl[\xi^2 - C_1^2 f(w) - C_2^2 g(w)\bigr] w'_\xi = C_5. \tag{1}$$

To the special case $C_5 = 0$ there corresponds the first solution of Item 3°.

For $C_5 \neq 0$, treating w in (1) as the independent variable, one obtains a Riccati equation for $\xi = \xi(w)$:

$$C_5 \xi'_w = \xi^2 - C_1^2 f(w) - C_2^2 g(w). \tag{2}$$

For exact solutions of equation (2), which is reduced to a second-order linear equation, see the book by Polyanin and Zaitsev (2003).

5°. Solution:

$$w = u(\eta), \quad \eta = \frac{C_1 y + C_2 t + C_3}{x + C_4},$$

where $C_1, \ldots, C_4$ are arbitrary constants and the function $u(\eta)$ is determined by the ordinary differential equation

$$C_2^2 u''_{\eta\eta} = [\eta^2 f(u) u'_\eta]'_\eta + C_1^2 [g(u) u'_\eta]'_\eta,$$

which admits the first integral

$$[\eta^2 f(u) + C_1^2 g(u) - C_2^2] u'_\eta = C_5. \tag{3}$$

To the special case $C_5 = 0$ there corresponds the second solution of Item 3°.

For $C_5 \neq 0$, treating u in (3) as the independent variable, one obtains a Riccati equation for $\eta = \eta(u)$:

$$C_5 \eta'_u = \eta^2 f(u) + C_1^2 g(u) - C_2^2. \tag{4}$$

For exact solutions of equation (4), which is reduced to a second-order linear equation, see the book by Polyanin and Zaitsev (2003).

6°. Solution:

$$w = v(\zeta), \quad \zeta = \frac{C_1 x + C_2 t + C_3}{y + C_4},$$

where $C_1, \ldots, C_4$ are arbitrary constants, and the function $v(\zeta)$ is determined by the first-order ordinary differential equation

$$[\zeta^2 g(v) + C_1^2 f(v) - C_2^2] v'_\zeta = C_5.$$

To the special case $C_5 = 0$, there corresponds the third solution of Item 3°. The inverse function $\zeta = \zeta(v)$ is determined by the Riccati equation that can be obtained from (4) with the renaming $u \to v$, $\eta \to \zeta$, and $f \rightleftarrows g$.

7°. "Two-dimensional" solution (a and b are arbitrary constants):

$$w(x, y, t) = U(z, t), \quad z = ax + by,$$

where the function $U = U(z, t)$ is determined by a differential equation of the form 3.4.4.6:

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial z}\left[\varphi(U)\frac{\partial U}{\partial z}\right], \quad \varphi(U) = a^2 f(U) + b^2 g(U),$$

which can be reduced to a linear equation.

8°. “Two-dimensional” solution (a and b are arbitrary constants):

$$w(x,y,t) = V(x,\xi), \quad \xi = ay + bt,$$

where the function $V = V(x,\xi)$ is determined by a differential equation of the form 5.4.4.8:

$$\frac{\partial}{\partial x}\left[f(V)\frac{\partial V}{\partial x}\right] + \frac{\partial}{\partial \xi}\left[\psi(V)\frac{\partial V}{\partial \xi}\right] = 0, \quad \psi(V) = a^2 g(V) - b^2,$$

which can be reduced to a linear equation.

9°. “Two-dimensional” solution (a and b are arbitrary constants):

$$w(x,y,t) = W(y,\eta), \quad \eta = ax + bt,$$

where the function $W = W(y,\eta)$ is determined by a differential equation of the form 5.4.4.8:

$$\frac{\partial}{\partial y}\left[g(W)\frac{\partial W}{\partial x}\right] + \frac{\partial}{\partial \eta}\left[\chi(W)\frac{\partial W}{\partial \eta}\right] = 0, \quad \chi(W) = a^2 f(W) - b^2,$$

which can be reduced to a linear equation.

10°. There is a “two-dimensional” solution of the form (generalizes the solutions of Items 7°–9°):

$$w(x,y,t) = Q(z_1,z_2), \quad z_1 = a_1x + b_1y + c_1t, \quad z_2 = a_2x + b_2y + c_2t.$$

11°. “Two-dimensional” solution:

$$w(x,y,t) = R(\xi,\eta), \quad \xi = xt^{-1}, \quad \eta = yt^{-1},$$

where the function $R = R(\xi,\eta)$ is determined by the differential equation

$$\xi^2\frac{\partial^2 R}{\partial \xi^2} + 2\xi\eta\frac{\partial^2 R}{\partial \xi\partial \eta} + \eta^2\frac{\partial^2 R}{\partial \eta^2} + 2\xi\frac{\partial R}{\partial \xi} + 2\eta\frac{\partial R}{\partial \eta} = \frac{\partial}{\partial \xi}\left[f(R)\frac{\partial R}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[g(R)\frac{\partial R}{\partial \eta}\right].$$

12°. For results of the group analysis of the original equation, see Ibragimov (1994).

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial y}\left[f(w)\frac{\partial w}{\partial y}\right] + g(w).$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm x + C_1, \pm y + C_2, \pm t + C_3),$$
$$w_2 = w(x\cosh\lambda + ta^{1/2}\sinh\lambda,\ y,\ xa^{-1/2}\sinh\lambda + t\cosh\lambda),$$

where C_1, C_2, C_3, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int h(w)\left[C_1 - 2\int h(w)g(w)\,dw\right]^{-1/2} dw = k_1x + k_2y + \lambda t + C_2, \quad h(w) = k_2^2 f(w) + ak_1^2 - \lambda^2,$$

where C_1, C_2, k_1, k_2, and λ are arbitrary constants.

3°. Solutions in implicit form:

$$\int f(w)\left[\varphi\big(x + t\sqrt{a}\,\big) - 2\int f(w)g(w)\,dw\right]^{-1/2} dw = \psi\big(x + t\sqrt{a}\,\big) \pm y,$$
$$\int f(w)\left[\varphi\big(x - t\sqrt{a}\,\big) - 2\int f(w)g(w)\,dw\right]^{-1/2} dw = \psi\big(x - t\sqrt{a}\,\big) \pm y,$$

where the functions $\varphi(z)$ and $\psi(z)$ are arbitrary functions.

4°. There are “two-dimensional” solutions of the following forms:

$$w(x,y,t) = U(y,z), \quad z = x^2 - at^2;$$
$$w(x,y,t) = V(\xi,\eta), \quad \xi = A_1x + B_1y + C_1t, \quad \eta = A_2x + B_2y + C_2t.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f(w)\dfrac{\partial w}{\partial y}\right] - a^2\dfrac{f'(w)}{f^3(w)} + b.$

Solution in implicit form:

$$\int f(w)\,dw = at + U(x,y),$$

where the function $U = U(x,y)$ is determined by the Poisson equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + b = 0.$$

For this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

Remark. The above remains true if the constant b in the equation is substituted by an arbitrary function $b = b(x,y)$.

4.4.3. Other Equations

1. $\dfrac{\partial^2 w}{\partial t^2} = ax^n\dfrac{\partial^2 w}{\partial x^2} + by^m\dfrac{\partial^2 w}{\partial y^2} + f(w).$

1°. Solution for $n \neq 2$ and $m \neq 2$:

$$w = w(r), \qquad r^2 = \frac{4}{k}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + kf(w) = 0, \qquad A = 2\left(\frac{1-n}{2-n} + \frac{1-m}{2-m}\right).$$

2°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,t) = U(\xi,t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2}\right],$$

$$w(x,y,t) = V(x,\eta), \qquad \eta^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right],$$

$$w(x,y,t) = W(y,\zeta), \qquad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right].$$

2. $\dfrac{\partial^2 w}{\partial t^2} = ax^n\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda y}\dfrac{\partial^2 w}{\partial y^2} + f(w).$

1°. Solution for $n \neq 2$ and $\lambda \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{k}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + kf(w) = 0, \qquad A = \frac{2(3-n)}{2-n}.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi,t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right],$$

$$w = V(x,\eta), \qquad \eta^2 = \pm 4\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

$$w = W(y,\zeta), \qquad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right].$$

3. $\dfrac{\partial^2 w}{\partial t^2} = ae^{\beta x}\dfrac{\partial^2 w}{\partial x^2} + be^{\lambda y}\dfrac{\partial^2 w}{\partial y^2} + f(w).$

1°. Solution for $\beta \neq 0$ and $\lambda \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{k}\left[\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$

where C and k are arbitrary constants ($k \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + 4r^{-1}w'_r + kf(w) = 0.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \qquad \xi^2 = 4\left(\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda y}}{b\lambda^2}\right),$$
$$w = V(x, \eta), \qquad \eta^2 = \pm 4\left[\frac{e^{-\lambda y}}{b\lambda^2} - \frac{1}{4}(t+C)^2\right],$$
$$w = W(y, \zeta), \qquad \zeta^2 = \pm 4\left[\frac{e^{-\beta x}}{a\beta^2} - \frac{1}{4}(t+C)^2\right].$$

4. $\dfrac{\partial^2 w}{\partial t^2} = \big[aw + f(t)\big]\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) + bw^2 + g(t)w + h(t), \qquad a \neq 0.$

Generalized separable solution:

$$w(x, y, t) = \varphi(t) + \psi(t)\Theta(x, y),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = b\varphi^2 + g(t)\varphi + h(t),$$
$$\psi'_{tt} = \big[b\varphi - \beta f(t) + g(t)\big]\psi, \qquad \beta = b/a,$$

and the function $\Theta = \Theta(x, y)$ satisfies the two-dimensional Helmholtz equation

$$\Delta\Theta + \beta\Theta = 0, \qquad \Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

5. $\dfrac{\partial^2 w}{\partial t^2} = aw\left(\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}\right) - a\left[\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right] + f(t).$

1°. Generalized separable solution:

$$w(x, y, t) = \varphi(t) + \psi(t)e^{\beta x + \gamma y},$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = f(t), \qquad \psi''_{tt} = a(\beta^2 + \gamma^2)\varphi\psi.$$

The solution of the first equation is expressed as (C_1 and C_2 are arbitrary constants)

$$\varphi(t) = \int_0^t (t - \tau)f(\tau)\,d\tau + C_1 t + C_2.$$

The solution of the second equation, which is linear in ψ, can be found in Kamke (1977) and Polyanin and Zaitsev (1995, 2003) for many $f(t)$.

2°. There are generalized separable solutions of the following forms:

$$w(x,y,t) = \varphi(t) + \psi(t)(A_1 \cosh \mu x + A_2 \sinh \mu x) + \chi(t)(B_1 \cos \mu y + B_2 \sin \mu y),$$
$$w(x,y,t) = \varphi(t) + \psi(t)(A_1 \cos \mu x + A_2 \sin \mu x) + \chi(t)(B_1 \cosh \mu y + B_2 \sinh \mu y),$$

where A_1, A_2, B_1, B_2, and μ are arbitrary constants, and the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by solving an appropriate system of second-order ordinary differential equations (not written out here).

3°. There are generalized separable solutions of the form

$$w(x,y,t) = \varphi(t) + \psi(t)F(x) + \chi(t)G(y) + \eta(t)H(x)P(y),$$

where

$$F(x) = A_1 \cos 2\mu x + A_2 \sin 2\mu x, \qquad G(y) = B_1 \cosh 2\mu y + B_2 \sinh 2\mu y,$$
$$H(x) = C_1 \cos \mu x + C_2 \sin \mu x, \qquad P(y) = D_1 \cosh \mu y + D_2 \sinh \mu y.$$

The arbitrary constants A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, and μ are related by two constraints, and the functions $\varphi(t)$, $\psi(t)$, $\chi(t)$, and $\eta(t)$ satisfy a system of second-order ordinary differential equations (not written out here).

6. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f_1(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[f_2(y)\frac{\partial w}{\partial y}\right] + aw \ln w + \big[g(t) + h_1(x) + h_2(y)\big]w.$

Multiplicative separable solution:

$$w(x,y,t) = \varphi(x)\psi(y)\chi(t),$$

where the functions $\varphi = \varphi(x)$, $\psi = \psi(y)$, and $\chi = \chi(t)$ are determined by the ordinary differential equations

$$[f_1(x)\varphi'_x]'_x + a\varphi \ln \varphi + [h_1(x) + C_1]\varphi = 0,$$
$$[f_2(y)\psi'_y]'_y + a\psi \ln \psi + [h_2(y) + C_2]\psi = 0,$$
$$\chi''_{tt} - a\chi \ln \chi - [g(t) - C_1 - C_2]\chi = 0,$$

and C_1 and C_2 are arbitrary constants.

7. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f(x,y)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(x,y)\frac{\partial w}{\partial y}\right] + kw \ln w.$

Multiplicative separable solution:

$$w(x,y,t) = \varphi(t)\Theta(x,y),$$

where the function $\varphi(t)$ is determined by the ordinary differential equation

$$\varphi''_{tt} - k\varphi \ln \varphi - A\varphi = 0, \tag{1}$$

A is an arbitrary constant, and the function $\Theta(x,y)$ satisfies the stationary equation

$$\frac{\partial}{\partial x}\left[f(x,y)\frac{\partial \Theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(x,y)\frac{\partial \Theta}{\partial y}\right] + k\Theta \ln \Theta - A\Theta = 0.$$

A particular solution of equation (1) is given by (B is an arbitrary constant)

$$\varphi(t) = \exp\left[\frac{k}{4}(t+B)^2 + \frac{k-2A}{2k}\right],$$

and the general solution can be written out in implicit form as (B and C are arbitrary constants)

$$\int \left[k\varphi^2 \ln \varphi + (A - \tfrac{1}{2}k)\varphi^2 + B\right]^{-1/2} d\varphi = C \pm t.$$

8. $\dfrac{\partial^2 w}{\partial t^2} = f_1(x,y)\dfrac{\partial^2 w}{\partial x^2} + f_2(x,y)\dfrac{\partial^2 w}{\partial x \partial y} + f_3(x,y)\dfrac{\partial^2 w}{\partial y^2} + g_1(x,y)\dfrac{\partial w}{\partial x} + g_2(x,y)\dfrac{\partial w}{\partial y} + \big[h(x,y)+s(t)\big]w + kw\ln w.$

Multiplicative separable solution:

$$w(x,y,t) = \varphi(t)\theta(x,y),$$

where the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi''_{tt} - k\varphi\ln\varphi - \big[s(t)+C\big]\varphi = 0,$$

and the function $\theta = \theta(x,y)$ satisfies the stationary equation

$$f_1(x,y)\frac{\partial^2\theta}{\partial x^2} + f_2(x,y)\frac{\partial^2\theta}{\partial x\partial y} + f_3(x,y)\frac{\partial^2\theta}{\partial y^2} + g_1(x,y)\frac{\partial\theta}{\partial x} + g_2(x,y)\frac{\partial\theta}{\partial y} + \big[h(x,y)-C\big]\theta + k\theta\ln\theta = 0.$$

9. $\dfrac{\partial^2 w}{\partial t^2} + f(t)\dfrac{\partial w}{\partial t} = g(t)\dfrac{\partial^2 w}{\partial x^2} + h(t)\dfrac{\partial}{\partial y}\left(w^m\dfrac{\partial w}{\partial y}\right).$

1°. "Two-dimensional" solution:

$$w(x,y,t) = \begin{cases} u(x,t)|y+C|^{1/(m+1)} & \text{if } m \neq -1, \\ u(x,t)\exp(Cy) & \text{if } m = -1, \end{cases}$$

where C is an arbitrary constant and the function $u(x,t)$ is determined by the linear telegraph equation

$$\frac{\partial^2 u}{\partial t^2} + f(t)\frac{\partial u}{\partial t} = g(t)\frac{\partial^2 u}{\partial x^2}.$$

2°. "Two-dimensional" solution:

$$w(x,y,t) = v(x,t)|y+C|^{2/m},$$

where the function $v(x,t)$ is determined by the differential equation

$$\frac{\partial^2 v}{\partial t^2} + f(t)\frac{\partial v}{\partial t} = g(t)\frac{\partial^2 v}{\partial x^2} + \frac{2(m+2)}{m^2}h(t)v^{m+1}.$$

For $m=-2$ and $m=-1$, this equation is linear.

10. $\dfrac{\partial^2 w}{\partial t^2} + f(t)\dfrac{\partial w}{\partial t} = g(t)\dfrac{\partial^2 w}{\partial x^2} + h(t)\dfrac{\partial}{\partial y}\left(e^{\lambda w}\dfrac{\partial w}{\partial y}\right).$

1°. "Two-dimensional" solution:

$$w(x,y,t) = u(x,t) + \frac{1}{\lambda}\ln|y+C|$$

where C is an arbitrary constant and the function $u(x,t)$ is determined by the linear telegraph equation

$$\frac{\partial^2 u}{\partial t^2} + f(t)\frac{\partial u}{\partial t} = g(t)\frac{\partial^2 u}{\partial x^2}.$$

2°. "Two-dimensional" solution:

$$w(x,y,t) = v(x,t) + \frac{2}{\lambda}\ln|y+C|,$$

where C is an arbitrary constant and the function $v(x,t)$ is determined by the differential equation

$$\frac{\partial^2 v}{\partial t^2} + f(t)\frac{\partial v}{\partial t} = g(t)\frac{\partial^2 v}{\partial x^2} + \frac{2}{\lambda}h(t)e^{\lambda v}.$$

4.5. Equations with Three Space Variables Involving Arbitrary Parameters

4.5.1. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[h(z)\frac{\partial w}{\partial z}\right] + aw^p$

1. $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(cz^k \frac{\partial w}{\partial z}\right) + sw^p.$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\Big(C_1^{\frac{p-1}{2-n}} x,\ C_1^{\frac{p-1}{2-m}} y,\ C_1^{\frac{p-1}{2-k}} z,\ \pm C_1^{\frac{p-1}{2}} t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $m \neq 2$, $k \neq 2$, and $p \neq 1$:

$$w = \left[\frac{A}{2s(p-1)}\right]^{\frac{1}{p-1}} \left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2} - \frac{1}{4}(t+C)^2\right]^{\frac{1}{1-p}},$$
$$A = \frac{1+p}{1-p} + \frac{2}{2-n} + \frac{2}{2-m} + \frac{2}{2-k}.$$

3°. There is a "three-dimensional" solution of the form

$$w(x,y,z,t) = |t|^{\frac{2}{1-p}} F(\rho_1,\rho_2,\rho_3), \quad \rho_1 = x|t|^{\frac{2}{n-2}}, \quad \rho_2 = y|t|^{\frac{2}{m-2}}, \quad \rho_3 = z|t|^{\frac{2}{k-2}}.$$

4°. For other exact solutions, see equation 4.6.1.2 with $f(w) = sw^p$.

2. $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ae^{\lambda x} \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\mu y} \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z} \frac{\partial w}{\partial z}\right) + sw^p.$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\Big(x + \frac{1-p}{\lambda}\ln C_1,\ y + \frac{1-p}{\mu}\ln C_1,\ z + \frac{1-p}{\nu}\ln C_1,\ \pm C_1^{\frac{p-1}{2}} t + C_2\Big),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $p \neq \pm 1$, $\lambda \neq 0$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = \left[-\frac{s(p-1)^2}{2k(1+p)}(r + C_1)^2\right]^{\frac{1}{1-p}}, \qquad r^2 = 4k\left[\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C_2)^2\right],$$

where C_1, C_2, and k are arbitrary constants.

3°. There is a "three-dimensional" solution of the form

$$w(x,y,z,t) = |t|^{\frac{2}{1-p}} F(\rho_1,\rho_2,\rho_3), \quad \rho_1 = x + \frac{2}{\lambda}\ln|t|, \quad \rho_2 = y + \frac{2}{\mu}\ln|t|, \quad \rho_3 = z + \frac{2}{\nu}\ln|t|.$$

4°. For other exact solutions, see equation 4.6.1.3 with $f(w) = sw^p$.

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m \dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\nu z} \dfrac{\partial w}{\partial z}\right) + sw^p.$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\Bigl(C_1^{\frac{p-1}{2-n}} x,\ C_1^{\frac{p-1}{2-m}} y,\ z + \frac{1-p}{\nu} \ln C_1,\ \pm C_1^{\frac{p-1}{2}} t + C_2\Bigr),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $m \neq 2$, $\nu \neq 0$, and $p \neq 1$:

$$w = \left[\frac{A}{2s(p-1)}\right]^{\frac{1}{p-1}} \left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right]^{\frac{1}{1-p}},$$

$$A = \frac{1+p}{1-p} + \frac{2}{2-n} + \frac{2}{2-m}.$$

3°. There is a "three-dimensional" solution of the form

$$w(x, y, z, t) = |t|^{\frac{2}{1-p}} F(\rho_1, \rho_2, \rho_3), \quad \rho_1 = x|t|^{\frac{2}{n-2}}, \quad \rho_2 = y|t|^{\frac{2}{m-2}}, \quad \rho_3 = z + \frac{2}{\nu} \ln|t|.$$

4°. For other exact solutions, see equation 4.6.1.4 with $f(w) = sw^p$.

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\mu y} \dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\nu z} \dfrac{\partial w}{\partial z}\right) + sw^p.$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\Bigl(C_1^{\frac{p-1}{2-n}} x,\ y + \frac{1-p}{\mu} \ln C_1,\ z + \frac{1-p}{\nu} \ln C_1,\ \pm C_1^{\frac{p-1}{2}} t + C_2\Bigr),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $\mu \neq 0$, $\nu \neq 0$, and $p \neq 1$:

$$w = \left[\frac{1}{2s(p-1)}\left(\frac{1+p}{1-p} + \frac{2}{2-n}\right)\right]^{\frac{1}{p-1}} \left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right]^{\frac{1}{1-p}}.$$

3°. There is a "three-dimensional" solution of the form

$$w(x, y, z, t) = |t|^{\frac{2}{1-p}} F(\rho_1, \rho_2, \rho_3), \quad \rho_1 = x|t|^{\frac{2}{n-2}}, \quad \rho_2 = y + \frac{2}{\mu} \ln|t|, \quad \rho_3 = z + \frac{2}{\nu} \ln|t|.$$

4°. For other exact solutions, see equation 4.6.1.5 with $f(w) = sw^p$.

4.5.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[h(z)\dfrac{\partial w}{\partial z}\right] + ae^{\lambda w}$

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m \dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(cz^k \dfrac{\partial w}{\partial z}\right) + se^{\lambda w}.$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = w\Bigl(C_1^{\frac{2}{2-n}} x,\ C_1^{\frac{2}{2-m}} y,\ C_1^{\frac{2}{2-k}} z,\ \pm C_1 t + C_2\Bigr) + \frac{2}{\lambda} \ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $m \neq 2$, $k \neq 2$, and $\lambda \neq 0$:

$$w = -\frac{1}{\lambda} \ln\left\{ \frac{2s\lambda}{A} \left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2} - \frac{1}{4}(t+C)^2 \right] \right\},$$
$$A = \frac{2}{2-n} + \frac{2}{2-m} + \frac{2}{2-k} - 1.$$

3°. There is a "three-dimensional" solution of the form

$$w(x,y,z,t) = F(\rho_1,\rho_2,\rho_3) - \frac{2}{\lambda}\ln|t|, \quad \rho_1 = x|t|^{\frac{2}{n-2}}, \quad \rho_2 = y|t|^{\frac{2}{m-2}}, \quad \rho_3 = z|t|^{\frac{2}{k-2}}.$$

4°. For other exact solutions, see equation 4.6.1.2 with $f(w) = se^{\lambda w}$.

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ae^{\lambda x}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\mu y}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z}\frac{\partial w}{\partial z}\right) + se^{\beta w}.$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = w\Bigl(x - \frac{2}{\lambda}\ln C_1,\ y - \frac{2}{\mu}\ln C_1,\ z - \frac{2}{\nu}\ln C_1,\ \pm C_1 t + C_2\Bigr) + \frac{2}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solutions for $\lambda \neq 0$, $\mu \neq 0$, $\nu \neq 0$, and $\beta \neq 0$:

$$w(x,y,z,t) = -\frac{1}{\beta}\ln\left[-\frac{s\beta}{2k}(r+C_1)^2\right] \qquad \text{if} \quad sk\beta < 0;$$
$$w(x,y,z,t) = -\frac{1}{\beta}\ln\left[-\frac{s\beta}{2kC_1^2}\sin^2(C_1 r + C_2)\right] \qquad \text{if} \quad sk\beta < 0;$$
$$w(x,y,z,t) = -\frac{1}{\beta}\ln\left[-\frac{s\beta}{2kC_1^2}\sinh^2(C_1 r + C_2)\right] \qquad \text{if} \quad sk\beta < 0;$$
$$w(x,y,z,t) = -\frac{1}{\beta}\ln\left[\frac{s\beta}{2kC_1^2}\cosh^2(C_1 r + C_2)\right] \qquad \text{if} \quad sk\beta > 0;$$

where C_1, C_2, and k are arbitrary constants and

$$r^2 = 4k\left[\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C_3)^2\right],$$

where k and the expression in square brackets must have like signs.

3°. There is a "three-dimensional" solution of the form

$$w(x,y,z,t) = F(\rho_1,\rho_2,\rho_3) - \frac{2}{\beta}\ln|t|, \quad \rho_1 = x + \frac{2}{\lambda}\ln|t|, \quad \rho_2 = y + \frac{2}{\mu}\ln|t|, \quad \rho_3 = z + \frac{2}{\nu}\ln|t|.$$

4°. For other exact solutions, see equation 4.6.1.3 with $f(w) = se^{\beta w}$.

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z}\frac{\partial w}{\partial z}\right) + se^{\beta w}.$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = w\Bigl(C_1^{\frac{2}{2-n}}x,\ C_1^{\frac{2}{2-m}}y,\ z - \frac{2}{\nu}\ln C_1,\ \pm C_1 t + C_2\Bigr) + \frac{2}{\beta}\ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $m \neq 2$, $\nu \neq 0$, and $\beta \neq 0$:

$$w = -\frac{1}{\beta} \ln\left\{ \frac{2s\beta}{A} \left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2 \right] \right\},$$
$$A = \frac{2}{2-n} + \frac{2}{2-m} - 1.$$

3°. There is a "three-dimensional" solution of the form

$$w(x,y,z,t) = F(\rho_1, \rho_2, \rho_3) - \frac{2}{\beta} \ln|t|, \quad \rho_1 = x|t|^{\frac{2}{n-2}}, \quad \rho_2 = y|t|^{\frac{2}{m-2}}, \quad \rho_3 = z + \frac{2}{\nu} \ln|t|.$$

4°. For other exact solutions, see equation 4.6.1.4 with $f(w) = se^{\beta w}$.

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\mu y} \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z} \frac{\partial w}{\partial z}\right) + se^{\beta w}.$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = w\Big(C_1^{\frac{2}{2-n}} x,\ y - \frac{2}{\mu} \ln C_1,\ z - \frac{2}{\nu} \ln C_1,\ \pm C_1 t + C_2\Big) + \frac{2}{\beta} \ln C_1,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Solution for $n \neq 2$, $\mu \neq 0$, $\nu \neq 0$, and $\beta \neq 0$:

$$w = -\frac{1}{\beta} \ln\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2 \right] + \frac{1}{\beta} \ln \frac{n}{2s\beta(2-n)}.$$

3°. There is a "three-dimensional" solution of the form

$$w(x,y,z,t) = F(\rho_1, \rho_2, \rho_3) - \frac{2}{\beta} \ln|t|, \quad \rho_1 = x|t|^{\frac{2}{n-2}}, \quad \rho_2 = y + \frac{2}{\mu} \ln|t|, \quad \rho_3 = z + \frac{2}{\nu} \ln|t|.$$

4°. For other exact solutions, see equation 4.6.1.5 with $f(w) = se^{\beta w}$.

4.5.3. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial}{\partial y}\left(w^m \dfrac{\partial w}{\partial y}\right) + c\dfrac{\partial}{\partial z}\left(w^k \dfrac{\partial w}{\partial z}\right) + sw^p$

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a_1 \frac{\partial^2 w}{\partial x^2} + a_2 \frac{\partial^2 w}{\partial y^2} + a_3 \frac{\partial}{\partial z}\left(w^k \frac{\partial w}{\partial z}\right).$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$\begin{aligned}
w_1 &= C_1^{-2} w(\pm C_2 x + C_3, \pm C_2 y + C_4, \pm C_1^k C_2 z + C_5, \pm C_2 t + C_6),\\
w_2 &= w(x\cos\beta + y\sqrt{a_1/a_2}\ \sin\beta, -x\sqrt{a_2/a_1}\ \sin\beta + y\cos\beta,\ z,\ t),\\
w_3 &= w(x\cosh\lambda + ta_1^{1/2}\sinh\lambda,\ y,\ z,\ xa_1^{-1/2}\sinh\lambda + t\cosh\lambda),\\
w_4 &= w(x,\ y\cosh\mu + ta_2^{1/2}\sinh\mu,\ z,\ ya_2^{-1/2}\sinh\mu + t\cosh\mu),
\end{aligned}$$

where $C_1, \dots, C_6$, β, λ, and μ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

⊙ *Reference*: N. H. Ibragimov (1994).

2°. Solutions:

$$w(x,y,z,t) = |z|^{\frac{1}{k+1}}\bigl[Ax^2 + By^2 + (a_1A + a_2B)t^2 + C_1xy + C_2xt + C_3yt + C_4x + C_5y + C_6t + C_7\bigr];$$

$$w(x,y,z,t) = |z|^{\frac{1}{k+1}}\bigl[A(a_2x^2 + a_1y^2 - a_1a_2t^2)^{-1/2} + B\bigr];$$

$$w(x,y,z,t) = A|z|^{\frac{1}{k+1}}\exp\bigl(\lambda_1x + \lambda_2y \pm \gamma t\bigr) + B, \quad \gamma = \sqrt{a_1\lambda_1^2 + a_2\lambda_2^2};$$

$$w(x,y,z,t) = A|z|^{\frac{1}{k+1}}\sin\bigl(\lambda_1x + C_1\bigr)\sin\bigl(\lambda_2y + C_2\bigr)\sin\bigl(\gamma t + C_3\bigr), \quad \gamma = \sqrt{a_1\lambda_1^2 + a_2\lambda_2^2};$$

$$w(x,y,z,t) = \left[\frac{1}{a_3C_3^2}\left(\frac{C_1x + C_2y + C_3z + C_4}{t + C_5}\right)^2 - \frac{a_1C_1^2 + a_2C_2^2}{a_3C_3^2}\right]^{1/k};$$

$$w(x,y,z,t) = \left[\frac{C_3^2 - a_2C_1^2}{a_3C_2^2} - \frac{a_1}{a_3C_2^2}\left(\frac{C_1y + C_2z + C_3t + C_4}{x + C_5}\right)^2\right]^{1/k};$$

$$w(x,y,z,t) = \left[\frac{C_3^2 - a_1C_1^2}{a_3C_2^2} - \frac{a_2}{a_3C_2^2}\left(\frac{C_1x + C_2z + C_3t + C_4}{y + C_5}\right)^2\right]^{1/k};$$

$$w(x,y,z,t) = \left[\frac{C_3^2 - a_1C_1^2 - a_2C_2^2}{a_3}\left(\frac{z + C_5}{C_1x + C_2y + C_3t + C_4}\right)^2\right]^{1/k};$$

where A, B, $C_1, \dots, C_7$, λ_1, and λ_2 are arbitrary constants.

3°. Solutions:

$$w = \bigl|z\varphi(\xi) + \psi(\xi)\bigr|^{\frac{1}{k+1}}, \quad \xi = C_1x + C_2y \pm t\sqrt{a_1C_1^2 + a_2C_2^2},$$

where C_1 and C_2 are arbitrary constants, $\varphi(\xi)$ and $\psi(\xi)$ are arbitrary functions.

4°. "Three-dimensional" solution (generalizes the first four solutions of Item 2°):

$$w(x,y,z,t) = |z|^{\frac{1}{k+1}}u(\widehat{x},\widehat{y},t), \quad \widehat{x} = a_1^{-1/2}x, \quad \widehat{y} = a_2^{-1/2}y,$$

where the function $u = u(\widehat{x},\widehat{y},t)$ is determined by the linear wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \widehat{x}^2} + \frac{\partial^2 u}{\partial \widehat{y}^2}.$$

For this equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

5°. Solutions in implicit form:

$$2\lambda\sqrt{a_1C_1^2 + a_2C_2^2}\,\bigl(z + \lambda t\bigr) \pm \Bigl(C_1x + C_2y \pm t\sqrt{a_1C_1^2 + a_2C_2^2}\,\Bigr)\bigl(a_3w^k - \lambda^2\bigr) = \Phi(w),$$

where $\Phi(w)$ is an arbitrary function, C_1, C_2, and λ are arbitrary constants.

6°. There are solutions of the following forms:

$w(x,y,z,t) = |z|^{2/k}F(x,y,t)$ "three-dimensional" solution;

$w(x,y,z,t) = |t|^{2\lambda}G(\xi,\eta,\zeta), \ \ \xi = xt^{-1}, \ \ \eta = yt^{-1}, \ \ \zeta = z|t|^{-k\lambda-1}$ "three-dimensional" solution;

$w(x,y,z,t) = H(r,z,t), \quad r = a_2x^2 + a_1y^2$ "three-dimensional" solution;

$w(x,y,z,t) = U(\xi,y,z), \quad \xi = x^2 - a_1t^2$ "three-dimensional" solution;

$w(x,y,z,t) = |t|^{2\lambda}V(\rho,\zeta), \ \ \rho = t^{-1}\sqrt{a_2x^2 + a_1y^2}, \ \ \zeta = z|t|^{-k\lambda-1}$ "two-dimensional" solution;

$w(x,y,z,t) = W(\zeta,z), \quad \zeta = a_2x^2 + a_1y^2 - a_1a_2t^2$ "two-dimensional" solution;

$w(x,y,z,t) = R(\eta), \quad \eta = (a_2x^2 + a_1y^2 - a_1a_2t^2)z^{-2}$ "one-dimensional" solution;

$w(x,y,z,t) = |z|^{2/k}Q(p), \quad p = a_2x^2 + a_1y^2 - a_1a_2t^2$ "one-dimensional" solution,

where λ is an arbitrary constant.

7°. For other exact solutions, see equation 4.5.3.6 with $n = m = 0$.

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a_1 \frac{\partial^2 w}{\partial x^2} + a_2 \frac{\partial}{\partial y}\left(w^k \frac{\partial w}{\partial y}\right) + a_3 \frac{\partial}{\partial z}\left(w^k \frac{\partial w}{\partial z}\right).$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_2 x + C_3, \pm C_1^k C_2 y + C_4, \pm C_1^k C_2 z + C_5, \pm C_2 t + C_6),$$
$$w_2 = w\big(x,\ y\cos\beta + z\sqrt{a_2/a_3}\,\sin\beta, -y\sqrt{a_3/a_2}\,\sin\beta + z\cos\beta,\ t\big),$$
$$w_3 = w\big(x\cosh\lambda + t a_1^{1/2}\sinh\lambda,\ y,\ z,\ x a_1^{-1/2}\sinh\lambda + t\cosh\lambda\big),$$

where $C_1, \ldots, C_6$, β, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

⦿ *Reference*: N. H. Ibragimov (1994).

2°. Solutions:

$$w(x,y,z,t) = \left[\frac{(C_1 x + C_2 y + C_3 z + C_4)^2 - a_1 C_1^2 (t + C_5)^2}{(a_2 C_2^2 + a_3 C_3^2)(t + C_5)^2}\right]^{1/k},$$
$$w(x,y,z,t) = \left[\frac{C_3^2 (x + C_5)^2 - a_1 (C_1 y + C_2 z + C_3 t + C_4)^2}{(a_2 C_1^2 + a_3 C_2^2)(x + C_5)^2}\right]^{1/k},$$
$$w(x,y,z,t) = \left[\frac{(C_3^2 - a_1 C_1^2)(y + C_5)^2}{a_2 (C_1 x + C_2 z + C_3 t + C_4)^2 + a_3 C_2^2 (y + C_5)^2}\right]^{1/k},$$
$$w(x,y,z,t) = \left[\frac{(C_3^2 - a_1 C_1^2)(z + C_5)^2}{a_3 (C_1 x + C_2 y + C_3 t + C_4)^2 + a_2 C_2^2 (z + C_5)^2}\right]^{1/k},$$

where $C_1, \ldots, C_5$ are arbitrary constants.

3°. Solution:

$$w(x,y,z,t) = \big[\varphi(x + t\sqrt{a_1}) + \psi(x - t\sqrt{a_1})\big] u^{\frac{1}{k+1}}(\widehat{y}, \widehat{z}), \quad \widehat{y} = a_2^{-1/2} y, \quad \widehat{z} = a_3^{-1/2} z,$$

where $\varphi(\rho_1)$ and $\psi(\rho_2)$ are arbitrary functions and the function $u(\widehat{y}, \widehat{z})$ is determined by the Laplace equation

$$\frac{\partial^2 u}{\partial \widehat{y}^2} + \frac{\partial^2 u}{\partial \widehat{z}^2} = 0.$$

For this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. "Three-dimensional" solutions:

$$w = \big|v(\widehat{y}, \widehat{z}, \zeta)\big|^{\frac{1}{k+1}}, \quad \widehat{y} = a_2^{-1/2} y, \quad \widehat{z} = a_3^{-1/2} z, \quad \zeta = x \pm t\sqrt{a_1},$$

where the function $v(\widehat{y}, \widehat{z}, \zeta)$ is determined by the Laplace equation

$$\frac{\partial^2 v}{\partial \widehat{y}^2} + \frac{\partial^2 v}{\partial \widehat{z}^2} = 0,$$

which is implicitly independent of the cyclic variable ζ (the constants of integration that appear in the solution are arbitrary functions of ζ).

5°. Multiplicative separable solution (generalizes the solution of Item 3°):

$$w(x,y,z,t) = R(x,t) Q(y,z),$$

where the functions $R = R(x,t)$ and $Q = Q(y,z)$ are determined by the differential equations

$$\frac{\partial^2 R}{\partial t^2} = a_1 \frac{\partial^2 R}{\partial x^2} + A R^{k+1},$$
$$a_2 \frac{\partial}{\partial y}\left(Q^k \frac{\partial Q}{\partial y}\right) + a_3 \frac{\partial}{\partial z}\left(Q^k \frac{\partial Q}{\partial z}\right) = AQ,$$

and A is an arbitrary constant.

6°. There are "three-dimensional" solutions of the following forms:

$$w(x,y,z,t)=t^{2\lambda}F(\xi,\eta,\zeta),\quad \xi=xt^{-1},\ \ \eta=yt^{-k\lambda-1},\ \ \zeta=zt^{-k\lambda-1};$$
$$w(x,y,z,t)=G(x,r,t),\qquad r=a_3y^2+a_2z^2;$$
$$w(x,y,z,t)=H(\xi,y,z),\qquad \xi=x^2-a_1t^2,$$

where λ is an arbitrary constant.

7°. There are solutions of the following forms:

$$w(x,y,z,t)=U(\xi,\rho),\quad \xi=xt^{-1},\ \ \rho=t^{-k\lambda-1}\sqrt{a_3y^2+a_2z^2}\quad \text{“two-dimensional” solution;}$$
$$w(x,y,z,t)=V(r,\xi),\quad r=a_3y^2+a_2z^2,\quad \xi=x^2-a_1t^2\quad \text{“two-dimensional” solution;}$$
$$w(x,y,z,t)=W(p,q),\quad p=(a_3y^2+a_2z^2)t^{-2},\quad q=xt^{-1}\quad \text{“two-dimensional” solution;}$$
$$w(x,y,z,t)=\Theta(\eta),\quad \eta=(a_3y^2+a_2z^2)(x^2-a_1t^2)^{-1}\quad \text{“one-dimensional” solution,}$$

where λ is an arbitrary constant.

8°. For other exact solutions, see equation 4.5.3.6 with $n=0$ and $m=k$.

3. $\displaystyle \frac{\partial^2 w}{\partial t^2}=a_1\frac{\partial^2 w}{\partial x^2}+a_2\frac{\partial}{\partial y}\left(w^m\frac{\partial w}{\partial y}\right)+a_3\frac{\partial}{\partial z}\left(w^k\frac{\partial w}{\partial z}\right).$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1=C_1^{-2}w(\pm C_2x+C_3,\pm C_1^mC_2y+C_4,\pm C_1^kC_2z+C_5,\pm C_2t+C_6),$$
$$w_2=w(x\cosh\lambda+ta_1^{1/2}\sinh\lambda,\ y,\ z,xa_1^{-1/2}\sinh\lambda+t\cosh\lambda),$$

where $C_1,\dots,C_6$ and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

⊙ *Reference*: N. H. Ibragimov (1994).

2°. Multiplicative separable solution:

$$w(x,y,z,t)=\left[\varphi(x+t\sqrt{a_1}\,)+\psi(x-t\sqrt{a_1}\,)\right]|y+C_1|^{\frac{1}{m+1}}|z+C_2|^{\frac{1}{k+1}},$$

where $\varphi(\rho_1)$ and $\psi(\rho_2)$ are arbitrary functions and C_1 and C_2 are arbitrary constants.

3°. There are "three-dimensional" solutions of the following forms:

$$w(x,y,z,t)=t^{2\lambda}F(\xi,\eta,\zeta),\ \ \xi=xt^{-1},\ \ \eta=yt^{-m\lambda-1},\ \ \zeta=zt^{-k\lambda-1};$$
$$w(x,y,z,t)=G(r,y,z),\quad r=x^2-a_1t^2;$$
$$w(x,y,z,t)=y^{2/m}H(x,s,t),\quad s=zy^{-k/m},$$

where λ is an arbitrary constant.

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z,t)=U(p,q),\quad p=(x^2-a_1t^2)y^{-2},\quad q=zy^{-1};$$
$$w(x,y,z,t)=y^{2/m}V(r,s),\quad r=x^2-a_1t^2,\quad s=zy^{-k/m}.$$

5°. For other exact solutions, see equation 4.5.3.6 with $n=0$.

$$4. \quad \frac{\partial^2 w}{\partial t^2} = a_1 \frac{\partial}{\partial x}\left(w^k \frac{\partial w}{\partial x}\right) + a_2 \frac{\partial}{\partial y}\left(w^k \frac{\partial w}{\partial y}\right) + a_3 \frac{\partial}{\partial z}\left(w^k \frac{\partial w}{\partial z}\right).$$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^k C_2 x + C_3, \pm C_1^k C_2 y + C_4, \pm C_1^k C_2 z + C_5, \pm C_2 t + C_6),$$
$$w_2 = w\bigl(x \cos\beta + y\sqrt{a_1/a_2}\, \sin\beta, -x\sqrt{a_2/a_1}\, \sin\beta + y \cos\beta, z, t\bigr),$$

where $C_1, \ldots, C_6$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

Reference: N. H. Ibragimov (1994).

2°. Solutions:

$$w(x, y, z, t) = (C_1 t + C_2)\bigl[a_3 C_3 x^2 + a_3 C_4 y^2 - (a_1 C_3 + a_2 C_4) z^2 + C_5\bigr]^{\frac{1}{k+1}},$$
$$w(x, y, z, t) = (C_1 t + C_2)\left(\frac{C_3}{\sqrt{a_2 a_3 x^2 + a_1 a_3 y^2 + a_1 a_2 z^2}} + C_4\right)^{\frac{1}{k+1}},$$
$$w(x, y, z, t) = \left[\frac{1}{a_1 C_1^2 + a_2 C_2^2 + a_3 C_3^2}\left(\frac{C_1 x + C_2 y + C_3 z + C_4}{t + C_5}\right)^2\right]^{1/k},$$
$$w(x, y, z, t) = \left[\frac{C_3^2 (x + C_5)^2}{a_1 (C_1 y + C_2 z + C_3 t + C_4)^2 + (a_2 C_1^2 + a_3 C_2^2)(x + C_5)^2}\right]^{1/k},$$
$$w(x, y, z, t) = \left[\frac{C_3^2 (y + C_5)^2}{a_2 (C_1 x + C_2 z + C_3 t + C_4)^2 + (a_1 C_1^2 + a_3 C_2^2)(y + C_5)^2}\right]^{1/k},$$
$$w(x, y, z, t) = \left[\frac{C_3^2 (z + C_5)^2}{a_3 (C_1 x + C_2 y + C_3 t + C_4)^2 + (a_1 C_1^2 + a_2 C_2^2)(z + C_5)^2}\right]^{1/k},$$

where $C_1, \ldots, C_5$ are arbitrary constants.

3°. Multiplicative separable solution:

$$w(x, y, z, t) = (C_1 t + C_2)\bigl[\Theta(\widehat{x}, \widehat{y}, \widehat{z})\bigr]^{\frac{1}{k+1}}, \quad \widehat{x} = a_1^{-1/2} x, \quad \widehat{y} = a_2^{-1/2} y, \quad \widehat{z} = a_3^{-1/2} z,$$

where the function $\Theta = \Theta(\widehat{x}, \widehat{y}, \widehat{z})$ is determined by the Laplace equation

$$\frac{\partial^2 \Theta}{\partial \widehat{x}^2} + \frac{\partial^2 \Theta}{\partial \widehat{y}^2} + \frac{\partial^2 \Theta}{\partial \widehat{z}^2} = 0.$$

For this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. There are "three-dimensional" solutions of the following forms:

$$w(x, y, z, t) = |t|^{2\lambda} F(\xi, \eta, \zeta), \quad \xi = x|t|^{-k\lambda-1}, \quad \eta = y|t|^{-k\lambda-1}, \quad \zeta = z|t|^{-k\lambda-1};$$
$$w(x, y, z, t) = |z|^{2/k} G(p, q, t), \quad p = xz^{-1}, \quad q = yz^{-1};$$
$$w(x, y, z, t) = H(\rho, z, t), \quad \rho = a_2 x^2 + a_1 y^2,$$

where λ is an arbitrary constant.

5°. There are solutions of the following forms:

$$w(x, y, z, t) = U(r, t), \quad r = a_2 a_3 x^2 + a_1 a_3 y^2 + a_1 a_2 z^2 \qquad \text{“two-dimensional” solution;}$$
$$w(x, y, z, t) = V(\chi), \quad \chi = (a_2 a_3 x^2 + a_1 a_3 y^2 + a_1 a_2 z^2) t^{-2} \qquad \text{“one-dimensional” solution.}$$

6°. For other exact solutions, see equation 4.5.3.6 with $n = m = k$.

5. $\dfrac{\partial^2 w}{\partial t^2} = a_1 \dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + a_2 \dfrac{\partial}{\partial y}\left(w^n \dfrac{\partial w}{\partial y}\right) + a_3 \dfrac{\partial}{\partial z}\left(w^k \dfrac{\partial w}{\partial z}\right).$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^n C_2 x + C_3, \pm C_1^n C_2 y + C_4, \pm C_1^k C_2 z + C_5, \pm C_2 t + C_6),$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a_1/a_2}\ \sin\beta, -x\sqrt{a_2/a_1}\ \sin\beta + y\cos\beta, z, t\big),$$

where $C_1, \ldots, C_6$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

⊙ *Reference*: N. H. Ibragimov (1994).

2°. There are "three-dimensional" solutions of the following forms:

$$w(x, y, z, t) = t^{2\lambda} F(\xi, \eta, \zeta), \quad \xi = x t^{-n\lambda-1}, \quad \eta = y t^{-n\lambda-1}, \quad \zeta = z t^{-k\lambda-1};$$
$$w(x, y, z, t) = G(r, z, t), \quad r = a_2 x^2 + a_1 y^2;$$
$$w(x, y, z, t) = z^{2/k} H(p, q, t), \quad p = x z^{-n/k}, \quad q = y z^{-n/k},$$

where λ is an arbitrary constant.

3°. There are solutions of the following forms:

$$w(x, y, z, t) = U(r, s), \quad r = (a_2 x^2 + a_1 y^2) t^{-2}, \quad s = z t^{-1} \qquad \text{"two-dimensional" solution},$$
$$w(x, y, z, t) = t^{-2/k} z^{2/k} V(\chi), \quad \chi = (a_2 x^2 + a_1 y^2) t^{2n/k-2} z^{-2n/k} \qquad \text{"one-dimensional" solution}.$$

4°. For other exact solutions, see equation 4.5.3.6 with $m = n$.

6. $\dfrac{\partial^2 w}{\partial t^2} = a_1 \dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + a_2 \dfrac{\partial}{\partial y}\left(w^m \dfrac{\partial w}{\partial y}\right) + a_3 \dfrac{\partial}{\partial z}\left(w^k \dfrac{\partial w}{\partial z}\right).$

This is a special case of equation 4.6.2.6 with $f(w) = a_1 w^n$, $g(w) = a_2 w^m$, and $h(w) = a_3 w^k$.

1°. Suppose $w(x, y, z, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^n C_2 x + C_3, \pm C_1^m C_2 y + C_4, \pm C_1^k C_2 z + C_5, \pm C_2 t + C_6),$$

where $C_1, \ldots, C_6$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

⊙ *Reference*: N. H. Ibragimov (1994).

2°. Multiplicative separable solution:

$$w(x, y, z, t) = (C_1 t + C_2)|x + C_3|^{\frac{1}{n+1}} |y + C_4|^{\frac{1}{m+1}} |z + C_5|^{\frac{1}{k+1}}.$$

3°. Traveling-wave solution in implicit form:

$$\frac{a_1 b_1^2}{n+1} w^{n+1} + \frac{a_2 b_2^2}{m+1} w^{m+1} + \frac{a_3 b_3^2}{k+1} w^{k+1} - \lambda^2 w = C_1(b_1 x + b_2 y + b_3 z + \lambda t) + C_2,$$

where C_1, C_2, b_1, b_2, b_3, and λ are arbitrary constants.

4°. Solutions in implicit form:

$$\left(\frac{C_1 x + C_2 y + C_3 z + C_4}{t + C_5}\right)^2 = a_1 C_1^2 w^n + a_2 C_2^2 w^m + a_3 C_3^2 w^k,$$
$$a_1 w^n \left(\frac{C_1 y + C_2 z + C_3 t + C_4}{x + C_5}\right)^2 + a_2 C_1^2 w^m + a_3 C_2^2 w^k = C_3^2,$$
$$a_2 w^m \left(\frac{C_1 x + C_2 z + C_3 t + C_4}{y + C_5}\right)^2 + a_1 C_1^2 w^n + a_3 C_2^2 w^k = C_3^2,$$
$$a_3 w^k \left(\frac{C_1 x + C_2 y + C_3 t + C_4}{z + C_5}\right)^2 + a_1 C_1^2 w^n + a_2 C_2^2 w^m = C_3^2,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

5°. "Two-dimensional" solution (b_1, b_2, and b_3 are arbitrary constants):

$$w(x,y,z,t) = u(\xi,t), \quad \xi = b_1 x + b_2 y + b_3 z,$$

where the function $u = u(\xi,t)$ is determined by a differential equation of the form 3.4.4.6:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial \xi}\left[(a_1 b_1^2 u^n + a_2 b_2^2 u^m + a_3 b_3^2 u^k)\frac{\partial u}{\partial \xi}\right],$$

which can be reduced to a linear equation.

6°. "Two-dimensional" solution (c_1, c_2, and c_3 are arbitrary constants):

$$w(x,y,z,t) = v(x,\eta), \quad \eta = c_1 t + c_2 y + c_3 z, \tag{1}$$

where the function $v = v(x,\eta)$ is determined by a differential equation of the form 5.4.4.8:

$$a_1 \frac{\partial}{\partial x}\left(v^n \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial \eta}\left[\left(a_2 c_2^2 v^m + a_3 c_3^2 v^k - c_1^2\right)\frac{\partial v}{\partial \eta}\right] = 0, \tag{2}$$

which can be reduced to a linear equation.

Formula (1) and equation (2) can be used to obtain two other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x, a_1, n) & \\ \nearrow & & \searrow \\ (z, a_3, k) & \longleftarrow & (y, a_2, m) \end{array}$$

7°. "Two-dimensional" solution (the b_n and c_n are arbitrary constants):

$$w(x,y,z,t) = U(\zeta,\rho), \quad \zeta = b_1 t + b_2 x, \quad \rho = c_1 y + c_2 z,$$

where the function $U = U(\zeta,\rho)$ is determined by a differential equation of the form 5.4.4.8:

$$\frac{\partial}{\partial \zeta}\left[\Phi(U)\frac{\partial U}{\partial \zeta}\right] + \frac{\partial}{\partial \rho}\left[\Psi(U)\frac{\partial U}{\partial \rho}\right] = 0, \quad \Phi(U) = a_1 b_2^2 U^n - b_1^2, \quad \Psi(U) = a_2 c_1^2 U^m + a_3 c_2^2 U^k,$$

which can be reduced to a linear equation.

Remark. The solution specified in Item 7° can be used to obtain other "two-dimensional" solutions by means of cyclic permutations of variables and determining parameters, as shown in Item 6°.

8°. There are "three-dimensional" solutions of the following forms:

$$w(x,y,z,t) = t^{2\lambda} F(\xi,\eta,\zeta), \;\; \xi = x t^{-n\lambda-1}, \;\; \eta = y t^{-m\lambda-1}, \;\; \zeta = z t^{-k\lambda-1};$$

$$w(x,y,z,t) = x^{2/n} G(r,s,t), \quad r = y x^{-m/n}, \quad s = z x^{-k/n},$$

where λ is an arbitrary constant.

9°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z,t) = H(p,q), \;\; p = b_1 x + b_2 y + b_3 z + b_4 t, \;\; q = c_1 x + c_2 y + c_3 z + c_4 t;$$

$$w(x,y,z,t) = t^{-2/n} x^{2/n} U(\rho,\chi), \quad \rho = x^{-m/n} y t^{(m-n)/n}, \quad \chi = x^{-k/n} z t^{(k-n)/n},$$

where the b_n and c_n are arbitrary constants.

7. $\dfrac{\partial^2 w}{\partial t^2} = a_1 \dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + a_2 \dfrac{\partial}{\partial y}\left(w^m \dfrac{\partial w}{\partial y}\right) + a_3 \dfrac{\partial}{\partial z}\left(w^k \dfrac{\partial w}{\partial z}\right) + b w^p.$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{p-n-1} x + C_2, \pm C_1^{p-m-1} y + C_3, \pm C_1^{p-k-1} z + C_4, \pm C_1^{p-1} t + C_5),$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There is a "three-dimensional" solution of the form

$$w(x,y,z,t) = t^{\frac{2}{1-p}} U(\xi,\eta,\zeta), \quad \xi = x t^{\frac{p-n-1}{1-p}}, \quad \eta = y t^{\frac{p-m-1}{1-p}}, \quad \zeta = z t^{\frac{p-k-1}{1-p}}.$$

4.5.4. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2}=a\dfrac{\partial}{\partial x}\Big(e^{\lambda_1 w}\dfrac{\partial w}{\partial x}\Big)+b\dfrac{\partial}{\partial y}\Big(e^{\lambda_2 w}\dfrac{\partial w}{\partial y}\Big)+c\dfrac{\partial}{\partial z}\Big(e^{\lambda_3 w}\dfrac{\partial w}{\partial z}\Big)+se^{\beta w}$

1. $\dfrac{\partial^2 w}{\partial t^2}=a_1\dfrac{\partial^2 w}{\partial x^2}+a_2\dfrac{\partial^2 w}{\partial y^2}+a_3\dfrac{\partial}{\partial z}\Big(e^w\dfrac{\partial w}{\partial z}\Big).$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$\begin{aligned}
w_1&=w(\pm C_1x+C_3,\pm C_1y+C_4,\pm C_1C_2z+C_5,\pm C_1t+C_6)-2\ln|C_2|,\\
w_2&=w\big(x\cos\beta+y\sqrt{a_1/a_2}\,\sin\beta,-x\sqrt{a_2/a_1}\,\sin\beta+y\cos\beta,\ z,\ t\big),\\
w_3&=w\big(x\cosh\lambda+ta_1^{1/2}\sinh\lambda,\ y,\ z,\ xa_1^{-1/2}\sinh\lambda+t\cosh\lambda\big),\\
w_4&=w\big(x,\ y\cosh\mu+ta_2^{1/2}\sinh\mu,\ z,\ ya_2^{-1/2}\sinh\mu+t\cosh\mu\big),
\end{aligned}$$

where $C_1,\dots,C_6$, β, λ, and μ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

⊙ *Reference*: N. H. Ibragimov (1994).

2°. Solutions:

$$\begin{aligned}
w(x,y,z,t)&=C_1x^2+C_2y^2+(a_1C_1+a_2C_2)t^2+C_3xy+C_4xt+C_5yt\\
&\qquad+C_6x+C_7y+C_8t+C_9+\ln|z|;\\
w(x,y,z,t)&=C_1(a_2x^2+a_1y^2-a_1a_2t^2)^{-1/2}+C_2+\ln|z|;\\
w(x,y,z,t)&=C_1\exp\big(\lambda_1x+\lambda_2y\pm\gamma t\big)+C_2+\ln|z|,\qquad \gamma=\sqrt{a_1\lambda_1^2+a_2\lambda_2^2};\\
w(x,y,z,t)&=C_1\sin\big(\lambda_1x+C_2\big)\sin\big(\lambda_2y+C_3\big)\sin\big(\gamma t+C_4\big)+\ln|z|,\qquad \gamma=\sqrt{a_1\lambda_1^2+a_2\lambda_2^2};\\
w(x,y,z,t)&=\ln\left[\frac{1}{a_3C_3^2}\left(\frac{C_1x+C_2y+C_3z+C_4}{t+C_5}\right)^2-\frac{a_1C_1^2+a_2C_2^2}{a_3C_3^2}\right];\\
w(x,y,z,t)&=\ln\left[\frac{C_3^2-a_2C_1^2}{a_3C_2^2}-\frac{a_1}{a_3C_2^2}\left(\frac{C_1y+C_2z+C_3t+C_4}{x+C_5}\right)^2\right];\\
w(x,y,z,t)&=\ln\left[\frac{C_3^2-a_1C_1^2}{a_3C_2^2}-\frac{a_2}{a_3C_2^2}\left(\frac{C_1x+C_2z+C_3t+C_4}{y+C_5}\right)^2\right];\\
w(x,y,z,t)&=\ln\left[\frac{C_3^2-a_1C_1^2-a_2C_2^2}{a_3}\left(\frac{z+C_5}{C_1x+C_2y+C_3t+C_4}\right)^2\right];
\end{aligned}$$

where the C_n are arbitrary constants.

3°. Solutions:

$$w=\ln|z\varphi(\xi)+\psi(\xi)|,\qquad \xi=C_1x+C_2y\pm t\sqrt{a_1C_1^2+a_2C_2^2},$$

where C_1 and C_2 are arbitrary constants, and $\varphi(\xi)$ and $\psi(\xi)$ are arbitrary functions.

4°. "Three-dimensional" solution (generalizes the first four solutions of Item 2°):

$$w(x,y,z,t)=u(\widehat{x},\widehat{y},t)+\ln|z|,\qquad \widehat{x}=a_1^{-1/2}x,\qquad \widehat{y}=a_2^{-1/2}y,$$

where the function $u=u(\widehat{x},\widehat{y},t)$ is determined by the linear wave equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial \widehat{x}^2}+\frac{\partial^2 u}{\partial \widehat{y}^2}.$$

For this equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

5°. "Two-dimensional" solution:

$$w(x,y,z,t) = U(\xi,t) + 2\ln|z|, \quad \xi = C_1x + C_2y,$$

where C_1 and C_2 are arbitrary constants and the function $U = U(\xi,t)$ is determined by a solvable equation of the form 3.2.1.1:

$$\frac{\partial^2 U}{\partial t^2} = (a_1C_1^2 + a_2C_2^2)\frac{\partial^2 U}{\partial \xi^2} + 2a_3e^U.$$

6°. "Two-dimensional" solution:

$$w(x,y,z,t) = v(x,\eta) + 2\ln|z|, \quad \eta = C_1y + C_2t, \tag{1}$$

where C_1 and C_2 are arbitrary constants and the function $v = v(\eta,t)$ is determined by the equation

$$(C_1^2 - a_2C_2^2)\frac{\partial^2 v}{\partial \eta^2} = a_1\frac{\partial^2 v}{\partial x^2} + 2a_3e^v. \tag{2}$$

For $\sigma = C_1^2 - a_2C_2^2 > 0$, on dividing by σ, one obtains a solvable equation of the form 3.2.1.1. For $\sigma = C_1^2 - a_2C_2^2 < 0$, the transformation $\eta = \widetilde{\eta}\sqrt{|\sigma|}$, $x = \widetilde{x}\sqrt{a_1}$ leads to a solvable equation of the form 5.2.1.1:

$$\frac{\partial^2 v}{\partial \widetilde{x}^2} + \frac{\partial^2 v}{\partial \widetilde{\eta}^2} = -2a_3e^v.$$

Remark. Relations (1) and equation (2) can be used to obtain another "two-dimensional" solution by means of the following renaming: $(x,a_1) \rightleftarrows (y,a_2)$.

7°. Solutions in implicit form:

$$2\lambda\sqrt{a_1C_1^2 + a_2C_2^2}\,(z + \lambda t) \pm \left(C_1x + C_2y \pm t\sqrt{a_1C_1^2 + a_2C_2^2}\,\right)\left(a_3e^w - \lambda^2\right) = \Phi(w),$$

where $\Phi(w)$ is an arbitrary function, and C_1, C_2, and λ are arbitrary constants.

8°. There are "three-dimensional" solutions of the following forms:

$$\begin{aligned}
&w(x,y,z,t) = F(x,y,t) + 2\ln|z|;\\
&w(x,y,z,t) = G(\xi_1,\xi_2,\xi_3) - 2k\ln|t|, \ \ \xi_1 = xt^{-1}, \ \ \xi_2 = yt^{-1}, \ \ \xi_3 = z|t|^{k-1};\\
&w(x,y,z,t) = H(\eta_1,\eta_2,\eta_3) + 2\ln|z|, \ \ \eta_1 = t + k_1\ln|z|, \ \ \eta_2 = x + k_2\ln|z|, \ \ \eta_3 = y + k_3\ln|z|;\\
&w(x,y,z,t) = E(\zeta_1,\zeta_2,\zeta_3) + 2z, \ \ \zeta_1 = te^z, \ \ \zeta_2 = xe^z, \ \ \zeta_3 = ye^z;\\
&w(x,y,z,t) = P(r,z,t), \quad r = a_2x^2 + a_1y^2;\\
&w(x,y,z,t) = Q(\rho,y,z), \quad \rho = x^2 - a_1t^2;
\end{aligned}$$

where k, k_1, k_2, and k_3 are arbitrary constants.

9°. There are solutions of the following forms:

$$\begin{aligned}
&w(x,y,z,t) = U(r,t) + 2\ln|z|, \quad r = a_2x^2 + a_1y^2 && \text{"two-dimensional" solution;}\\
&w(x,y,z,t) = V(\rho,y) + 2\ln|z|, \quad \rho = x^2 - a_1t^2 && \text{"two-dimensional" solution;}\\
&w(x,y,z,t) = W(\theta,z), \quad \theta = a_2x^2 + a_1y^2 - a_1a_2t^2 && \text{"two-dimensional" solution;}\\
&w(x,y,z,t) = R(\xi_1,\xi_2) + 2\ln|z/t|, \quad \xi_1 = xt^{-1}, \ \ \xi_2 = yt^{-1} && \text{"two-dimensional" solution;}\\
&w(x,y,z,t) = S(\theta) + 2\ln|z|, \quad \theta = a_2x^2 + a_1y^2 - a_1a_2t^2 && \text{"one-dimensional" solution;}\\
&w(x,y,z,t) = T(\chi), \quad \chi = (a_2x^2 + a_1y^2 - a_1a_2t^2)z^{-2} && \text{"one-dimensional" solution.}
\end{aligned}$$

10°+. For other exact solutions, see equation 4.6.2.6 with $f(w) = a_1$, $g(w) = a_2$, and $h(w) = a_3e^w$.

2. $\dfrac{\partial^2 w}{\partial t^2} = a_1 \dfrac{\partial^2 w}{\partial x^2} + a_2 \dfrac{\partial}{\partial y}\left(e^w \dfrac{\partial w}{\partial y}\right) + a_3 \dfrac{\partial}{\partial z}\left(e^w \dfrac{\partial w}{\partial z}\right).$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$\begin{aligned} w_1 &= w(\pm C_1 x + C_3, \pm C_1 C_2 y + C_4, \pm C_1 C_2 z + C_5, \pm C_1 t + C_6) - 2\ln|C_2|, \\ w_2 &= w\bigl(x, y\cos\beta + z\sqrt{a_2/a_3}\,\sin\beta, -y\sqrt{a_3/a_2}\,\sin\beta + z\cos\beta, t\bigr), \\ w_3 &= w\bigl(x\cosh\lambda + t a_1^{1/2}\sinh\lambda,\ y,\ z, x a_1^{-1/2}\sinh\lambda + t\cosh\lambda\bigr), \end{aligned}$$

where $C_1, \ldots, C_6$, β, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

⦿ *Reference*: N. H. Ibragimov (1994).

2°. Solutions:

$$\begin{aligned} w(x,y,z,t) &= \varphi(x + t\sqrt{a_1}\,) + \psi(x - t\sqrt{a_1}\,) + \ln(a_3 C_1 y^2 + C_2 yz - a_2 C_1 z^2 + C_3 y + C_4 z + C_5), \\ w(x,y,z,t) &= \varphi(x + t\sqrt{a_1}\,) + \psi(x - t\sqrt{a_1}\,) + \ln\bigl[C_1 \exp(C_2\sqrt{a_3}\,y)\sin(C_2\sqrt{a_2}\,z + C_3) + C_4\bigr], \\ w(x,y,z,t) &= \varphi(x + t\sqrt{a_1}\,) + \psi(x - t\sqrt{a_1}\,) + \ln\bigl[C_1 \exp(C_2\sqrt{a_2}\,z)\sin(C_2\sqrt{a_3}\,y + C_3) + C_4\bigr], \\ w(x,y,z,t) &= \ln\left[\frac{(C_1 x + C_2 y + C_3 z + C_4)^2 - a_1 C_1^2 (t + C_5)^2}{(a_2 C_2^2 + a_3 C_3^2)(t + C_5)^2}\right], \\ w(x,y,z,t) &= \ln\left[\frac{C_3^2 (x + C_5)^2 - a_1 (C_1 y + C_2 z + C_3 t + C_4)^2}{(a_2 C_1^2 + a_3 C_2^2)(x + C_5)^2}\right], \\ w(x,y,z,t) &= \ln\left[\frac{(C_3^2 - a_1 C_1^2)(y + C_5)^2}{a_2 (C_1 x + C_2 z + C_3 t + C_4)^2 + a_3 C_2^2 (y + C_5)^2}\right], \\ w(x,y,z,t) &= \ln\left[\frac{(C_3^2 - a_1 C_1^2)(z + C_5)^2}{a_3 (C_1 x + C_2 y + C_3 t + C_4)^2 + a_2 C_2^2 (z + C_5)^2}\right], \end{aligned}$$

where $\varphi(\rho_1)$ and $\psi(\rho_2)$ are arbitrary functions and $C_1, \ldots, C_5$ are arbitrary constants.

3°. Solution (generalizes the first three solutions of Item 2°):

$$w(x,y,z,t) = \varphi(x + t\sqrt{a_1}\,) + \psi(x - t\sqrt{a_1}\,) + \ln u(\widehat{y}, \widehat{z}), \quad \widehat{y} = a_2^{-1/2} y, \quad \widehat{z} = a_3^{-1/2} z,$$

where $\varphi(\rho_1)$ and $\psi(\rho_2)$ are arbitrary functions and the function $u(\widehat{y}, \widehat{z})$ is determined by the Laplace equation

$$\frac{\partial^2 u}{\partial \widehat{y}^2} + \frac{\partial^2 u}{\partial \widehat{z}^2} = 0. \tag{1}$$

For this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. "Three-dimensional" solutions:

$$w = \ln\bigl|v(\widehat{y}, \widehat{z}, \zeta)\bigr|, \quad \widehat{y} = a_2^{-1/2} y, \quad \widehat{z} = a_3^{-1/2} z, \quad \zeta = x \pm t\sqrt{a_1},$$

where the function $v(\widehat{y}, \widehat{z}, \zeta)$ is determined by the Laplace equation

$$\frac{\partial^2 v}{\partial \widehat{y}^2} + \frac{\partial^2 v}{\partial \widehat{z}^2} = 0,$$

which is implicitly independent of the cyclic variable ζ (the constants of integration that appear in the solution will be arbitrary functions of ζ).

5°. Additive separable solution (generalizes the solution of Item 3°):

$$w(x,y,z,t) = R(\widehat{x}, t) + \ln Q(\widehat{y}, \widehat{z}), \quad \widehat{x} = a_1^{-1/2} x, \quad \widehat{y} = a_2^{-1/2} y, \quad \widehat{z} = a_3^{-1/2} z,$$

where the functions $R = R(\widehat{x}, t)$ and $Q = Q(\widehat{y}, \widehat{z})$ are determined by the differential equations

$$\frac{\partial^2 R}{\partial t^2} = \frac{\partial^2 R}{\partial \widehat{x}^2} + Ae^R, \tag{2}$$

$$\frac{\partial^2 Q}{\partial \widehat{y}^2} + \frac{\partial^2 Q}{\partial \widehat{z}^2} = A, \tag{3}$$

and A is an arbitrary constant. The general solution of equation (2) is given in 3.2.1.1. By the substitution $Q = \frac{1}{2}A\widehat{y}^2 + u$, the Helmholtz equation (3) can be reduced to the Laplace equation (1).

6°. There are "three-dimensional" solutions of the following forms:

$$w(x, y, z, t) = F(\xi, \eta, \zeta) - 2\lambda \ln|t|, \quad \xi = xt^{-1}, \quad \eta = y|t|^{\lambda-1}, \quad \zeta = z|t|^{\lambda-1};$$

$$w(x, y, z, t) = G(x, r, t), \quad r = a_3 y^2 + a_2 z^2;$$

$$w(x, y, z, t) = H(\rho, y, z), \quad \rho = x^2 - a_1 t^2,$$

where λ is an arbitrary constant.

7°. There are solutions of the following forms:

$w(x, y, z, t) = E(r, \rho), \quad r = a_3 y^2 + a_2 z^2, \quad \rho = x^2 - a_1 t^2$ "two-dimensional" solution;

$w(x, y, z, t) = U(\chi_1, \chi_2) + 2\ln|y/t|, \quad \chi_1 = x/t, \quad \chi_2 = z/y$ "two-dimensional" solution;

$w(x, y, z, t) = V(p, q), \quad p = (a_3 y^2 + a_2 z^2)t^{-2}, \quad q = xt^{-1}$ "two-dimensional" solution;

$w(x, y, z, t) = W(\eta), \quad \eta = (a_3 y^2 + a_2 z^2)(x^2 - a_1 t^2)^{-1}$ "one-dimensional" solution.

8°. For other exact solutions, see equation 4.6.2.6 with $f(w) = a_1$, $g(w) = a_2 e^w$, and $h(w) = a_3 e^w$.

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a_1 \frac{\partial^2 w}{\partial x^2} + a_2 \frac{\partial}{\partial y}\left(e^w \frac{\partial w}{\partial y}\right) + a_3 \frac{\partial}{\partial z}\left(e^{kw} \frac{\partial w}{\partial z}\right).$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_3, \pm C_1 C_2 y + C_4, \pm C_1 C_2^k z + C_5, \pm C_1 t + C_6) - \ln C_2^2,$$

$$w_2 = w(x \cosh\lambda + t a_1^{1/2} \sinh\lambda,\ y,\ z,\ x a_1^{-1/2} \sinh\lambda + t\cosh\lambda),$$

where $C_1, \ldots, C_6$ and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

⊙ *Reference*: N. H. Ibragimov (1994).

2°. Additive separable solution:

$$w(x, y, z, t) = \varphi(x + t\sqrt{a_1}\,) + \psi(x - t\sqrt{a_1}\,) + \ln|y + C_1| + \frac{1}{k}\ln|z + C_2|,$$

where $\varphi(\rho_1)$ and $\psi(\rho_2)$ are arbitrary functions and C_1 and C_2 are arbitrary constants.

3°. There are "three-dimensional" solutions of the following forms:

$$w(x, y, z, t) = F(\xi, \eta, \zeta) - 2\beta \ln|t|, \quad \xi = xt^{-1}, \quad \eta = y|t|^{\beta-1}, \quad \zeta = z|t|^{k\beta-1};$$

$$w(x, y, z, t) = G(r, y, z), \quad r = x^2 - a_1 t^2,$$

where β is an arbitrary constant.

4°. There are solutions of the following forms:

$w(x, y, z, t) = F(\rho_1, \rho_2) + 2\ln\left|\dfrac{y}{t}\right|, \quad \rho_1 = xt^{-1}, \quad \rho_2 = |t|^{k-1}|y|^{-k}z$ "two-dimensional" solution,

$w(x, y, z, t) = U(p, q), \quad p = (x^2 - a_1 t^2)y^{-2}, \quad q = zy^{-1}$ "two-dimensional" solution,

$w(x, y, z, t) = V(r, s) + 2\ln|y|, \quad r = x^2 - a_1 t^2, \quad s = z|y|^{-k}$ "two-dimensional" solution,

$w(x, y, z, t) = W(\chi) - \dfrac{2}{k-1}\ln\left|\dfrac{y}{z}\right|, \quad \chi = |x^2 - a_1 t^2|^{k-1}|y|^{-2k}z^2$ "one-dimensional" solution.

5°. For other exact solutions, see equation 4.6.2.6 with $f(w) = a_1$, $g(w) = a_2 e^w$, and $h(w) = a_3 e^{kw}$.

4. $\dfrac{\partial^2 w}{\partial t^2} = a_1 \dfrac{\partial}{\partial x}\left(e^w \dfrac{\partial w}{\partial x}\right) + a_2 \dfrac{\partial}{\partial y}\left(e^w \dfrac{\partial w}{\partial y}\right) + a_3 \dfrac{\partial}{\partial z}\left(e^w \dfrac{\partial w}{\partial z}\right).$

This is a special case of equation 4.6.2.4 with $f(w) = e^w$.

1°. Suppose $w(x, y, z, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm C_1 C_2 x + C_3, \pm C_1 C_2 y + C_4, \pm C_1 C_2 z + C_5, \pm C_1 t + C_6) - \ln C_2^2,$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a_1/a_2}\,\sin\beta, -x\sqrt{a_2/a_1}\,\sin\beta + y\cos\beta,\ z,\ t\big),$$

where $C_1, \dots, C_6$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

⊙ *Reference*: N. H. Ibragimov (1994).

2°. Solutions:

$$w(x, y, z, t) = C_1 t + C_2 + \ln\big[a_3 C_3 x^2 + a_3 C_4 y^2 - (a_1 C_3 + a_2 C_4) z^2 + C_5\big],$$
$$w(x, y, z, t) = C_1 t + C_2 + \ln\big[C_3 (a_2 a_3 x^2 + a_1 a_3 y^2 + a_1 a_2 z^2)^{-1/2} + C_4\big],$$
$$w(x, y, z, t) = \ln\left[\frac{1}{a_1 C_1^2 + a_2 C_2^2 + a_3 C_3^2}\left(\frac{C_1 x + C_2 y + C_3 z + C_4}{t + C_5}\right)^2\right],$$
$$w(x, y, z, t) = \ln\left[\frac{C_3^2 (x + C_5)^2}{a_1 (C_1 y + C_2 z + C_3 t + C_4)^2 + (a_2 C_1^2 + a_3 C_2^2)(x + C_5)^2}\right],$$
$$w(x, y, z, t) = \ln\left[\frac{C_3^2 (y + C_5)^2}{a_2 (C_1 x + C_2 z + C_3 t + C_4)^2 + (a_1 C_1^2 + a_3 C_2^2)(y + C_5)^2}\right],$$
$$w(x, y, z, t) = \ln\left[\frac{C_3^2 (z + C_5)^2}{a_3 (C_1 x + C_2 y + C_3 t + C_4)^2 + (a_1 C_1^2 + a_2 C_2^2)(z + C_5)^2}\right],$$

where $C_1, \dots, C_5$ are arbitrary constants.

3°. Additive separable solution:

$$w(x, y, z, t) = C_1 t + C_2 + \ln\Theta(\widehat{x}, \widehat{y}, \widehat{z}), \quad \widehat{x} = a_1^{-1/2} x, \quad \widehat{y} = a_2^{-1/2} y, \quad \widehat{z} = a_3^{-1/2} z,$$

where the function $\Theta = \Theta(\widehat{x}, \widehat{y}, \widehat{z})$ is determined by the Laplace equation

$$\frac{\partial^2 \Theta}{\partial \widehat{x}^2} + \frac{\partial^2 \Theta}{\partial \widehat{y}^2} + \frac{\partial^2 \Theta}{\partial \widehat{z}^2} = 0.$$

For this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. There are “three-dimensional” solutions of the following forms:

$$w(x, y, z, t) = F(\xi, \eta, \zeta) - 2\beta \ln|t|, \quad \xi = x|t|^{\beta-1}, \quad \eta = y|t|^{\beta-1}, \quad \zeta = z|t|^{\beta-1};$$
$$w(x, y, z, t) = G(\rho, z, t), \quad \rho = a_2 x^2 + a_1 y^2;$$
$$w(x, y, z, t) = H(p, q, t) + 2\ln|z|, \quad p = x/z, \quad q = y/z,$$

where β is an arbitrary constant.

5°. There are solutions of the following forms:

$$w(x, y, z, t) = U(r, t), \quad r = a_2 a_3 x^2 + a_1 a_3 y^2 + a_1 a_2 z^2 \qquad \text{“two-dimensional” solution;}$$
$$w(x, y, z, t) = V(\chi), \quad \chi = (a_2 a_3 x^2 + a_1 a_3 y^2 + a_1 a_2 z^2) t^{-2} \qquad \text{“one-dimensional” solution.}$$

6°. For other exact solutions, see equation 4.5.4.6 with $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

5. $\dfrac{\partial^2 w}{\partial t^2} = a_1 \dfrac{\partial}{\partial x}\left(e^w \dfrac{\partial w}{\partial x}\right) + a_2 \dfrac{\partial}{\partial y}\left(e^w \dfrac{\partial w}{\partial y}\right) + a_3 \dfrac{\partial}{\partial z}\left(e^{kw} \dfrac{\partial w}{\partial z}\right).$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 C_2 x + C_3, \pm C_1 C_2 y + C_4, \pm C_1 C_2^k z + C_5, \pm C_1 t + C_6) - \ln C_2^2,$$
$$w_2 = w\bigl(x\cos\beta + y\sqrt{a_1/a_2}\,\sin\beta,\ -x\sqrt{a_2/a_1}\,\sin\beta + y\cos\beta,\ z,\ t\bigr),$$

where $C_1, \ldots, C_6$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

⊙ *Reference*: N. H. Ibragimov (1994).

2°. Additive separable solution:

$$w(x, y, z, t) = C_1 t + C_2 + \frac{1}{k}\ln|z + C_3| + \ln\Theta(\widehat{x}, \widehat{y}), \quad \widehat{x} = a_1^{-1/2} x, \quad \widehat{y} = a_2^{-1/2} y,$$

where the function $\Theta = \Theta(\widehat{x}, \widehat{y})$ is determined by the Laplace equation

$$\frac{\partial^2\Theta}{\partial\widehat{x}^2} + \frac{\partial^2\Theta}{\partial\widehat{y}^2} = 0.$$

For this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

3°. There are solutions of the following forms:

$$w = F(\xi, \eta, \zeta) - 2\beta\ln|t|, \quad \xi = x|t|^{\beta-1}, \quad \eta = y|t|^{\beta-1}, \quad \zeta = z|t|^{k\beta-1} \qquad \text{“three-dimensional” solution;}$$
$$w = G(r, z, t), \quad r = a_2 x^2 + a_1 y^2 \qquad \text{“three-dimensional” solution;}$$
$$w = H(\rho_1, \rho_2) + 2\ln|x/t|, \quad \rho_1 = y/x, \quad \rho_2 = |t|^{k-1}|x|^{-k} z, \qquad \text{“two-dimensional” solution;}$$
$$w = U(\chi) + \ln\bigl[(a_2 x^2 + a_1 y^2)t^{-2}\bigr], \quad \chi = (a_2 x^2 + a_1 y^2)|z|^{-2/k}|t|^{2/k-2} \qquad \text{“one-dimensional” solution;}$$

where β is an arbitrary constant.

4°. For other exact solutions, see equation 4.5.4.6 with $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = k$.

6. $\dfrac{\partial^2 w}{\partial t^2} = a_1 \dfrac{\partial}{\partial x}\left(e^{\lambda_1 w} \dfrac{\partial w}{\partial x}\right) + a_2 \dfrac{\partial}{\partial y}\left(e^{\lambda_2 w} \dfrac{\partial w}{\partial y}\right) + a_3 \dfrac{\partial}{\partial z}\left(e^{\lambda_3 w} \dfrac{\partial w}{\partial z}\right).$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 C_2^{\lambda_1} x + C_3, \pm C_1 C_2^{\lambda_2} y + C_4, \pm C_1 C_2^{\lambda_3} z + C_5, \pm C_1 t + C_6) - \ln C_2^2,$$

where $C_1, \ldots, C_6$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

⊙ *Reference*: N. H. Ibragimov (1994).

2°. Additive separable solution:

$$w(x, y, z, t) = C_1 t + C_2 + \frac{1}{\lambda_1}\ln|x + C_3| + \frac{1}{\lambda_2}\ln|y + C_4| + \frac{1}{\lambda_3}\ln|z + C_5|.$$

3°. Traveling-wave solution in implicit form:

$$\frac{a_1 k_1^2}{\lambda_1} e^{\lambda_1 w} + \frac{a_2 k_2^2}{\lambda_2} e^{\lambda_2 w} + \frac{a_3 k_3^2}{\lambda_3} e^{\lambda_3 w} - \beta^2 w = C_1(k_1 x + k_2 y + k_3 z + \beta t) + C_2,$$

where C_1, C_2, k_1, k_2, k_3, and β are arbitrary constants.

4°. Solutions in implicit form:

$$\left(\frac{C_1x+C_2y+C_3z+C_4}{t+C_5}\right)^2 = a_1C_1^2e^{\lambda_1 w}+a_2C_2^2e^{\lambda_2 w}+a_3C_3^2e^{\lambda_3 w},$$

$$a_1e^{\lambda_1 w}\left(\frac{C_1y+C_2z+C_3t+C_4}{x+C_5}\right)^2 + a_2C_1^2e^{\lambda_2 w}+a_3C_2^2e^{\lambda_3 w} = C_3^2,$$

$$a_2e^{\lambda_2 w}\left(\frac{C_1x+C_2z+C_3t+C_4}{y+C_5}\right)^2 + a_1C_1^2e^{\lambda_1 w}+a_3C_2^2e^{\lambda_3 w} = C_3^2,$$

$$a_3e^{\lambda_3 w}\left(\frac{C_1x+C_2y+C_3t+C_4}{z+C_5}\right)^2 + a_1C_1^2e^{\lambda_1 w}+a_2C_2^2e^{\lambda_2 w} = C_3^2,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

5°. "Two-dimensional" solution (k_1, k_2, and k_3 are arbitrary constants):

$$w(x,y,z,t) = u(\xi,t), \quad \xi = k_1x+k_2y+k_3z,$$

where the function $u = u(\xi,t)$ is determined by a differential equation of the form 3.4.4.6:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial \xi}\left[\varphi(u)\frac{\partial u}{\partial \xi}\right], \quad \varphi(u) = a_1k_1^2e^{\lambda_1 w}+a_2k_2^2e^{\lambda_2 w}+a_3k_3^2e^{\lambda_3 w},$$

which can be reduced to a linear equation.

6°. "Two-dimensional" solution (b_1, b_2, and b_3 are arbitrary constants):

$$w(x,y,z,t) = v(x,\eta), \quad \eta = b_1y+b_2z+b_3t, \tag{1}$$

where the function $v = v(x,\eta)$ is determined by a differential equation of the form 5.4.4.8:

$$a_1\frac{\partial}{\partial x}\left(e^{\lambda_1 v}\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial \eta}\left[\psi(v)\frac{\partial v}{\partial \eta}\right] = 0, \quad \psi(v) = a_2b_1^2e^{\lambda_2 v}+a_3b_2^2e^{\lambda_3 v}-b_3^2, \tag{2}$$

which can be reduced to a linear equation.

Relations (1) and equation (2) can be used to obtain two other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x,a_1,\lambda_1) & \\ \nearrow & & \searrow \\ (z,a_3,\lambda_3) & \longleftarrow & (y,a_2,\lambda_2) \end{array}$$

7°. "Two-dimensional" solution (b_n and c_n are arbitrary constants):

$$w(x,y,z,t) = U(\zeta,\rho), \quad \zeta = b_1t+b_2x, \quad \rho = c_1y+c_2z,$$

where the function $U = U(\zeta,\rho)$ is determined by a differential equation of the form 5.4.4.8:

$$\frac{\partial}{\partial \zeta}\left[\Phi(U)\frac{\partial U}{\partial \zeta}\right] + \frac{\partial}{\partial \rho}\left[\Psi(U)\frac{\partial U}{\partial \rho}\right] = 0, \quad \Phi(U) = a_1b_2^2e^{\lambda_1 U}-b_1^2, \quad \Psi(U) = a_2c_1^2e^{\lambda_2 U}+a_3c_2^2e^{\lambda_3 U},$$

which can be reduced to a linear equation.

Remark. The solution specified in Item 7° can be used to obtain other "two-dimensional" solutions by means of cyclic permutations of variables and determining parameters as shown in Item 6°.

8°. There are more complicated "two-dimensional" solutions of the form

$$w(x,y,z,t) = V(z_1,z_2), \quad z_1 = b_1x+b_2y+b_3z+b_4t, \quad z_2 = c_1x+c_2y+c_3z+c_4t.$$

9°. There is a "two-dimensional" solution of the form

$$w(x,y,z,t) = W(\rho_1,\rho_2) + \frac{2}{\lambda_1}\ln\left|\frac{x}{t}\right|, \quad \rho_1 = |t|^{\lambda_2/\lambda_1-1}|x|^{-\lambda_2/\lambda_1}y, \quad \rho_2 = |t|^{\lambda_3/\lambda_1-1}|x|^{-\lambda_3/\lambda_1}z.$$

10°+. There is a "three-dimensional" solution of the form

$$w(x,y,z,t) = F(\xi,\eta,\zeta) - 2\beta\ln|t|, \quad \xi = x|t|^{\beta\lambda_1-1}, \quad \eta = y|t|^{\beta\lambda_2-1}, \quad \zeta = z|t|^{\beta\lambda_3-1},$$

where β is an arbitrary constant.

7. $\dfrac{\partial^2 w}{\partial t^2} = a_1 \dfrac{\partial}{\partial x}\left(e^{\lambda_1 w}\dfrac{\partial w}{\partial x}\right) + a_2 \dfrac{\partial}{\partial y}\left(e^{\lambda_2 w}\dfrac{\partial w}{\partial y}\right) + a_3 \dfrac{\partial}{\partial z}\left(e^{\lambda_3 w}\dfrac{\partial w}{\partial z}\right) + be^{\beta w}.$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\beta-\lambda_1} x + C_2, \pm C_1^{\beta-\lambda_2} y + C_3, \pm C_1^{\beta-\lambda_3} z + C_4, \pm C_1^{\beta} t + C_5) + 2\ln|C_1|,$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There are "three-dimensional" solutions of the following forms:

$$w(x, y, z, t) = U(\xi, \eta, \zeta) - \frac{2}{\beta}\ln|t|, \quad \xi = x|t|^{\frac{\lambda_1-\beta}{\beta}}, \quad \eta = y|t|^{\frac{\lambda_2-\beta}{\beta}}, \quad \zeta = z|t|^{\frac{\lambda_3-\beta}{\beta}},$$

$$w(x, y, z, t) = V(\eta_1, \eta_2, \eta_3), \quad \eta_n = a_n x + b_n y + c_n z + d_n t \quad (n = 1,\ 2,\ 3).$$

4.6. Equations with Three Space Variables Involving Arbitrary Functions

4.6.1. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f_1(x)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f_2(y)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[f_3(z)\dfrac{\partial w}{\partial z}\right] + g(w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2} + f(w).$

The equation admits translations in any of the variables x, y, z, and t.

1°. Traveling-wave solution in implicit form:

$$\int \left[C_1 + \frac{2}{\lambda^2 - k_1^2 - k_2^2 - k_3^2}\int f(w)\,dw\right]^{-1/2} dw = k_1 x + k_2 y + k_3 z + \lambda t + C_2,$$

where C_1, C_2, k_1, k_2, k_3, and λ are arbitrary constants.

2°. Solution:

$$w(x, y, z, t) = w(\rho), \quad \rho^2 = A\bigl[(x + C_1)^2 + (y + C_2)^2 + (z + C_3)^2 - (t + C_4)^2\bigr],$$

where the arbitrary constant A and the expression in square brackets must have like signs, and the function $w(\rho)$ is determined by the ordinary differential equation

$$w''_{\rho\rho} + 3\rho^{-1} w'_\rho + A^{-1} f(w) = 0.$$

3°. For the case of axisymmetric solutions, the Laplace operator on the right-hand side of the equation is expressed in cylindrical and spherical coordinates as

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}, \qquad r = \sqrt{x^2 + y^2};$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial w}{\partial \theta}\right), \quad r = \sqrt{x^2 + y^2 + z^2}.$$

4°. "Three-dimensional" solution:

$$w = u(\xi, \eta, t), \quad \xi = y + \frac{x}{C}, \quad \eta = (C^2 - 1)x^2 - 2Cxy + C^2 z^2,$$

where C is an arbitrary constant ($C \neq 0$), and the function $u = u(\xi, \eta, t)$ is determined by the equation

$$\frac{\partial^2 u}{\partial t^2} = \left(1 + \frac{1}{C^2}\right)\frac{\partial^2 u}{\partial \xi^2} - 4\xi \frac{\partial^2 u}{\partial \xi \partial \eta} + 4C^2(\xi^2 + \eta)\frac{\partial^2 u}{\partial \eta^2} + 2(2C^2 - 1)\frac{\partial u}{\partial \eta} + f(u).$$

5°. "Three-dimensional" solution:

$$w = v(z, \xi, \zeta), \qquad \xi = y + \frac{x}{C}, \qquad \zeta = (C^2 - 1)x^2 - 2Cxy - C^2 t^2,$$

where C is an arbitrary constant ($C \neq 0$), and the function $v = v(z, \xi, \zeta)$ is determined by the equation

$$\frac{\partial^2 v}{\partial z^2} + \left(1 + \frac{1}{C^2}\right)\frac{\partial^2 v}{\partial \xi^2} - 4\xi \frac{\partial^2 v}{\partial \xi \partial \zeta} + 4C^2(\xi^2 + \zeta)\frac{\partial^2 v}{\partial \zeta^2} + 2(2C^2 - 1)\frac{\partial v}{\partial \zeta} + f(v) = 0.$$

Remark. The solutions specified in Items 4° and 5° can be used to obtain other "three-dimensional" solutions by means of the cyclic permutations of the space variables.

6°. "Three-dimensional" solution:

$$w = U(\xi, \eta, t), \qquad \xi = Ax + By + Cz, \qquad \eta = \sqrt{(Bx - Ay)^2 + (Cy - Bz)^2 + (Az - Cx)^2},$$

where A, B, and C are arbitrary constants and the function $U = U(\xi, \eta, t)$ is determined by the equation

$$\frac{\partial^2 U}{\partial t^2} = (A^2 + B^2 + C^2)\left(\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial U}{\partial \eta}\right) + f(U).$$

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(cz^k \frac{\partial w}{\partial z}\right) + f(w).$

1°. Solution for $n \neq 2$, $m \neq 2$, and $k \neq 2$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2} - \frac{1}{4}(t + C)^2\right],$$

where B and C are arbitrary constants (B and the expression in square brackets must have like signs), and the function $w(r)$ is determined by the ordinary differential equation

$$\frac{d^2 w}{dr^2} + \frac{A}{r}\frac{dw}{dr} + Bf(w) = 0, \qquad A = \frac{2}{2-n} + \frac{2}{2-m} + \frac{2}{2-k}.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2}\right];$$

$$w = V(x, \eta), \qquad \eta^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2} - \frac{1}{4}(t + C)^2\right];$$

$$w = W(\zeta, \rho), \qquad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t + C)^2\right], \qquad \rho^2 = 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2}\right].$$

The second and third solutions can be used to obtain other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x, a, n) & \\ \nearrow & & \searrow \\ (z, c, k) & \longleftarrow & (y, b, m) \end{array}$$

⊙ *Reference*: A. D. Polyanin and A. I. Zhurov (1998).

3. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ae^{\lambda x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\nu z}\dfrac{\partial w}{\partial z}\right) + f(w).$

1°. Solution for $\lambda \neq 0$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t + C_1)^2\right],$$

where B and C_1 are arbitrary constants and the function $w(r)$ is determined by the autonomous ordinary differential equation

$$w''_{rr} + Bf(w) = 0.$$

Integrating yields its solution in implicit form:

$$\int \left[C_2 - 2B \int f(w)\,dw\right]^{-1/2} dw = C_3 \pm r,$$

where C_2 and C_3 are arbitrary constants.

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \qquad \xi^2 = 4\left(\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right);$$
$$w = V(x, \eta), \qquad \eta^2 = \pm 4\left[\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t + C)^2\right];$$
$$w = W(\zeta, \rho), \qquad \zeta^2 = \pm 4\left[\frac{e^{-\lambda x}}{a\lambda^2} - \frac{1}{4}(t + C)^2\right], \qquad \rho^2 = 4\left(\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right).$$

The second and third solutions can be used to obtain other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x, a, \lambda) & \\ \nearrow & & \searrow \\ (z, c, \nu) & \longleftarrow & (y, b, \mu) \end{array}$$

⊙ *Reference*: A. D. Polyanin and A. I. Zhurov (1998).

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(ce^{\nu z}\dfrac{\partial w}{\partial z}\right) + f(w).$

1°. Solution for $n \neq 2$, $m \neq 2$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t + C_1)^2\right],$$

where B and C are arbitrary constants and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + Bf(w) = 0, \qquad A = \frac{2}{2-n} + \frac{2}{2-m}.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$
$$w = V_1(x, \eta_1), \quad \eta_1^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t + C)^2\right];$$

$$w = V_2(y, \eta_2), \qquad \eta_2^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = V_3(z, \eta_3), \qquad \eta_3^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = W_1(\zeta_1, \rho_1), \quad \zeta_1^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_1^2 = 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = W_2(\zeta_2, \rho_2), \quad \zeta_2^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_2^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = W_3(\zeta_3, \rho_3), \quad \zeta_3^2 = \pm 4\left[\frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_3^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2}\right].$$

5. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\mu y} \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\nu z} \frac{\partial w}{\partial z}\right) + f(w).$

1°. Solution for $n \neq 2$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right],$$

where B and C are arbitrary constants and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{2}{2-n}\frac{1}{r}w'_r + Bf(w) = 0.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = V_1(x, \eta_1), \qquad \eta_1^2 = \pm 4\left[\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = V_2(y, \eta_2), \qquad \eta_2^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = V_3(z, \eta_3), \qquad \eta_3^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = W_1(\zeta_1, \rho_1), \quad \zeta_1^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_1^2 = 4\left[\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = W_2(\zeta_2, \rho_2), \quad \zeta_2^2 = \pm 4\left[\frac{e^{-\mu y}}{b\mu^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_2^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = W_3(\zeta_3, \rho_3), \quad \zeta_3^2 = \pm 4\left[\frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_3^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2}\right].$$

6. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[h(z)\frac{\partial w}{\partial z}\right] + aw \ln w + bw.$

1°. Multiplicative separable solution:

$$w(x, y, z, t) = X(x)Y(y)Z(z)\varphi(t),$$

where the functions $X(x)$, $Y(y)$, $Z(z)$, and $\varphi(t)$ are determined by the ordinary differential equations

$$[f(x)X'_x]'_x + aX \ln X + C_1 X = 0,$$
$$[g(y)Y'_y]'_y + aY \ln Y + C_2 Y = 0,$$
$$[h(z)Z'_z]'_z + aZ \ln Z + C_3 Z = 0,$$
$$\varphi''_{tt} - a\varphi \ln \varphi + (C_1 + C_2 + C_3 - b)\varphi = 0,$$

where C_1, C_2, and C_3 are arbitrary constants. A particular and the general solutions of the last equations can be obtained from the formulas of Item 2°, where A should be set equal to $b-C_1-C_2-C_3$.

2°. Multiplicative separable solution:

$$w(x,y,z,t) = \varphi(t)\Theta(x,y,z).$$

Here, the function $\varphi(t)$ is determined by the autonomous ordinary differential equation

$$\varphi''_{tt} - a\varphi\ln\varphi - A\varphi = 0, \tag{1}$$

where A is an arbitrary constant, and the function $\Theta(x,y,z)$ satisfies the stationary equation

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial\Theta}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial\Theta}{\partial y}\right] + \frac{\partial}{\partial z}\left[h(z)\frac{\partial\Theta}{\partial z}\right] + a\Theta\ln\Theta + (b-A)\Theta = 0.$$

A particular solution of equation (1) is given by

$$\varphi(t) = \exp\left[\frac{a}{4}(t+B)^2 + \frac{a-2A}{2a}\right],$$

where B is an arbitrary constant, and the general solution can be written out in implicit form (C is an arbitrary constant):

$$\int\left[a\varphi^2\ln\varphi + (A-\tfrac{1}{2}a)\varphi^2 + B\right]^{-1/2} d\varphi = C \pm t.$$

4.6.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f_1(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f_2(w)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[f_3(w)\dfrac{\partial w}{\partial z}\right] + g(w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a_1\dfrac{\partial^2 w}{\partial x^2} + a_2\dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial}{\partial z}\left[h(w)\dfrac{\partial w}{\partial z}\right].$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$\begin{aligned}
w_1 &= w(\pm C_1x + C_2, \pm C_1y + C_3, \pm C_1z + C_4, \pm C_1t + C_5),\\
w_2 &= w\big(x\cos\beta + y\sqrt{a_1/a_2}\,\sin\beta, -x\sqrt{a_2/a_1}\,\sin\beta + y\cos\beta, z, t\big),\\
w_3 &= w\big(x\cosh\lambda + ta_1^{1/2}\sinh\lambda,\ y,\ z, xa_1^{-1/2}\sinh\lambda + t\cosh\lambda\big),
\end{aligned}$$

where $C_1, \ldots, C_5$, β, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

⊙ *Reference*: N. H. Ibragimov (1994).

2°. Solutions in implicit form:

$$\int h(w)\,dw = z\varphi(\eta) + \psi(\eta), \qquad \eta = C_1x + C_2y \pm t\sqrt{a_1C_1^2 + a_2C_2^2},$$

where C_1 and C_2 are arbitrary constants, $\varphi(\eta)$ and $\psi(\eta)$ are arbitrary functions.

3°. "Two-dimensional" solution (generalizes the solutions of Item 2°):

$$w(x,y,z,t) = U(\xi,\eta), \quad \xi = z + \lambda t, \quad \eta = C_1x + C_2y \pm t\sqrt{a_1C_1^2 + a_2C_2^2},$$

where C_1, C_2, and λ are arbitrary constants, and the function $U = U(\xi,\eta)$ is determined by the first-order partial differential equation

$$\big[h(U) - \lambda^2\big]\frac{\partial U}{\partial\xi} \mp 2\lambda\sqrt{a_1C_1^2 + a_2C_2^2}\,\frac{\partial U}{\partial\eta} = \varphi(\eta), \tag{1}$$

and $\varphi(\eta)$ is an arbitrary function.

In the special case $\lambda = 0$, equation (1) is an ordinary differential equation in ξ and can be easily integrated to obtain solutions of Item 2°.

In the general case, equation (1) can be solved using a characteristic system of ordinary differential equations; see Polyanin, Zaitsev, and Moussiaux (2002). In the special case $\varphi(\eta) = 0$, the general solution of equation (1) can be written out in implicit form:

$$2\lambda\sqrt{a_1C_1^2 + a_2C_2^2}\,\xi \pm \eta[h(U) - \lambda^2] = \Phi(U),$$

where $\Phi(U)$ is an arbitrary function.

4°. "Three-dimensional" solutions:

$$w = u(y, z, \zeta), \qquad \zeta = x \pm t\sqrt{a_1}, \tag{2}$$

where the function $u(y, z, \zeta)$ is determined by a differential equation of the form 5.4.4.8:

$$a_2\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial z}\left[h(u)\frac{\partial u}{\partial z}\right] = 0, \tag{3}$$

which can be reduced to a linear equation. Equation (3) is implicitly independent of the cyclic variable ζ (the constants of integration that appear in the solution will be arbitrary functions of ζ).

Remark 1. Relations (2) and equation (3) can be used to obtain another "three-dimensional" solution by means of the following renaming: $(x, a_1) \rightleftarrows (y, a_2)$.

5°. "Three-dimensional" solution:

$$w = v(z, \xi, \eta), \quad \xi = \frac{x}{\sqrt{a_1}C} + \frac{y}{\sqrt{a_2}}, \quad \eta = (C^2 - 1)\frac{x^2}{a_1} - 2C\frac{xy}{\sqrt{a_1a_2}} - C^2t^2, \tag{4}$$

where C is an arbitrary constant ($C \neq 0$), and the function $v = v(\xi, \eta)$ is determined by the equation

$$\left(1 + \frac{1}{C^2}\right)\frac{\partial^2 v}{\partial \xi^2} - 4\xi\frac{\partial^2 v}{\partial\xi\partial\eta} + 4C^2(\xi^2 + \eta)\frac{\partial^2 v}{\partial\eta^2} + 2(2C^2 - 1)\frac{\partial v}{\partial\eta} + \frac{\partial}{\partial z}\left[h(v)\frac{\partial v}{\partial z}\right] = 0. \tag{5}$$

Remark 2. Relations (4) and equation (5) can be used to obtain another "three-dimensional" solution by means of the following renaming: $(x, a_1) \rightleftarrows (y, a_2)$.

6°. There are solutions of the following forms:

$w(x, y, z, t) = F(r, z, t)$, $\quad r = a_2x^2 + a_1y^2$ — "three-dimensional" solution;

$w(x, y, z, t) = G(\xi, y, z)$, $\quad \xi = x^2 - a_1t^2$ — "three-dimensional" solution;

$w(x, y, z, t) = H(\zeta, z)$, $\quad \zeta = a_2x^2 + a_1y^2 - a_1a_2t^2$ — "two-dimensional" solution;

$w(x, y, z, t) = U(\eta)$, $\quad \eta = (a_2x^2 + a_1y^2 - a_1a_2t^2)z^{-2}$ — "one-dimensional" solution.

7°. For other exact solutions, see equation 4.6.2.6 with $f(w) = a_1$ and $g(w) = a_2$.

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a_1\frac{\partial^2 w}{\partial x^2} + a_2\frac{\partial}{\partial y}\left[g(w)\frac{\partial w}{\partial y}\right] + a_3\frac{\partial}{\partial z}\left[g(w)\frac{\partial w}{\partial z}\right].$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1x + C_2, \pm C_1y + C_3, \pm C_1z + C_4, \pm C_1t + C_5),$$
$$w_2 = w\bigl(x, y\cos\beta + z\sqrt{a_2/a_3}\,\sin\beta, -y\sqrt{a_3/a_2}\,\sin\beta + z\cos\beta, t\bigr),$$
$$w_3 = w\bigl(x\cosh\lambda + ta_1^{1/2}\sinh\lambda,\ y,\ z, xa_1^{-1/2}\sinh\lambda + t\cosh\lambda\bigr),$$

where $C_1, \ldots, C_5$, β, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

⊙ *Reference*: N. H. Ibragimov (1994).

2°. "Three-dimensional" solutions:

$$w = u(y, z, \zeta), \qquad \zeta = x \pm t\sqrt{a_1},$$

where the function $u(y, z, \zeta)$ is determined by the differential equation

$$a_2 \frac{\partial}{\partial y}\left[g(u)\frac{\partial u}{\partial y}\right] + a_3 \frac{\partial}{\partial z}\left[g(u)\frac{\partial u}{\partial z}\right] = 0, \tag{1}$$

which is implicitly independent of the cyclic variable ζ (the constants of integration that appear in the solution are arbitrary functions of ζ). The transformation

$$v = \int g(u)\,du, \qquad \overline{y} = \frac{y}{\sqrt{a_2}}, \qquad \overline{z} = \frac{z}{\sqrt{a_3}}$$

brings (1) to the Laplace equation

$$\frac{\partial^2 v}{\partial \overline{y}^2} + \frac{\partial^2 v}{\partial \overline{z}^2} = 0.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

3°. There are solutions of the following forms:

$w(x, y, z, t) = F(x, r, t)$, $\quad r = a_3 y^2 + a_2 z^2$ — "three-dimensional" solution;

$w(x, y, z, t) = G(\xi, y, z)$, $\quad \xi = x^2 - a_1 t^2$ — "three-dimensional" solution;

$w(x, y, z, t) = H(r, \xi)$, $\quad r = a_3 y^2 + a_2 z^2$, $\quad \xi = x^2 - a_1 t^2$ — "two-dimensional" solution;

$w(x, y, z, t) = U(p, q)$, $\quad p = (a_3 y^2 + a_2 z^2)t^{-2}$, $\quad q = xt^{-1}$ — "two-dimensional" solution;

$w(x, y, z, t) = V(\eta)$, $\quad \eta = (a_3 y^2 + a_2 z^2)(x^2 - a_1 t^2)^{-1}$ — "one-dimensional" solution.

4°. For other exact solutions, see equation 4.6.2.6 with $f(w) = a_1$, in which $g(w)$ should be renamed $a_2 g(w)$ and $h(w)$ renamed $a_3 g(w)$.

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a_1 \frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial y}\left[g(w)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[h(w)\frac{\partial w}{\partial z}\right].$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 z + C_4, \pm C_1 t + C_5),$$
$$w_2 = w(x\cosh\lambda + t a_1^{1/2}\sinh\lambda,\ y,\ z,\ x a_1^{-1/2}\sinh\lambda + t\cosh\lambda),$$

where $C_1, \ldots, C_5$, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. "Three-dimensional" solutions:

$$w = u(y, z, \zeta), \qquad \zeta = x \pm t\sqrt{a_1},$$

where the function $u(y, z, \zeta)$ is determined by a differential equation of the form 5.4.4.8:

$$\frac{\partial}{\partial y}\left[g(u)\frac{\partial u}{\partial y}\right] + \frac{\partial}{\partial z}\left[h(u)\frac{\partial u}{\partial z}\right] = 0,$$

which can be reduced to a linear equation. The equation obtained is implicitly independent of the cyclic variable ζ (the constants of integration that appear in the solution will be arbitrary functions of ζ).

3°. There are solutions of the following forms:

$w(x, y, z, t) = W(\xi, y, z)$, $\quad \xi = x^2 - a_1 t^2$ — "three-dimensional" solution;

$w(x, y, z, t) = U(p, q)$, $\quad p = (x^2 - a_1 t^2)y^{-2}$, $\quad q = zy^{-1}$ — "two-dimensional" solution.

4°. For other exact solutions, see equation 4.6.2.6 with $f(w) = a_1$.

4. $\dfrac{\partial^2 w}{\partial t^2} = a_1 \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + a_2 \dfrac{\partial}{\partial y}\left[f(w)\dfrac{\partial w}{\partial y}\right] + a_3 \dfrac{\partial}{\partial z}\left[f(w)\dfrac{\partial w}{\partial z}\right].$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 z + C_4, \pm C_1 t + C_5),$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a_1/a_2}\,\sin\beta, -x\sqrt{a_2/a_1}\,\sin\beta + y\cos\beta, z, t\big),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

⊙ *Reference*: N. H. Ibragimov (1994).

2°. There are "three-dimensional" solutions of the following forms:

$$w = W(\rho, z, t), \quad \rho = a_2 x^2 + a_1 y^2;$$
$$w = U(\xi, \eta, t), \quad \xi = \frac{x}{\sqrt{a_1}C} + \frac{y}{\sqrt{a_2}}, \quad \eta = (C^2-1)\frac{x^2}{a_1} - 2C\frac{xy}{\sqrt{a_1 a_2}} + C^2\frac{z^2}{a_3};$$
$$w = V(\zeta, \theta, t), \quad \zeta = \frac{Ax}{\sqrt{a_1}} + \frac{By}{\sqrt{a_2}} + \frac{Cz}{\sqrt{a_3}}, \quad \theta = \left(\frac{Bx}{\sqrt{a_1}} - \frac{Ay}{\sqrt{a_2}}\right)^2 + \left(\frac{Cy}{\sqrt{a_2}} - \frac{Bz}{\sqrt{a_3}}\right)^2 + \left(\frac{Az}{\sqrt{a_3}} - \frac{Cx}{\sqrt{a_1}}\right)^2,$$

where A, B, and C are arbitrary constants.

Remark. The first and second solutions specified in Item 2° can be used to obtain other "three-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x, a_1) & \\ \nearrow & & \searrow \\ (z, a_3) & \longleftarrow & (y, a_2) \end{array}$$

3°. There are exact solutions of the following forms:

$$w(x,y,z,t) = \Phi(r,t), \quad r = a_2 a_3 x^2 + a_1 a_3 y^2 + a_1 a_2 z^2 \qquad \text{"two-dimensional" solution;}$$
$$w(x,y,z,t) = \Psi(\chi), \quad \chi = (a_2 a_3 x^2 + a_1 a_3 y^2 + a_1 a_2 z^2)t^{-2} \qquad \text{"one-dimensional" solution.}$$

4°. For other exact solutions, see equation 4.6.2.6, in which $f(w)$, $g(w)$, and $h(w)$ should be renamed $a_1 f(w)$, $a_2 f(w)$, and $a_3 f(w)$, respectively.

5. $\dfrac{\partial^2 w}{\partial t^2} = a_1 \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + a_2 \dfrac{\partial}{\partial y}\left[f(w)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[h(w)\dfrac{\partial w}{\partial z}\right].$

1°. Suppose $w(x,y,z,t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 z + C_4, \pm C_1 t + C_5),$$
$$w_2 = w\big(x\cos\beta + y\sqrt{a_1/a_2}\,\sin\beta, -x\sqrt{a_2/a_1}\,\sin\beta + y\cos\beta, z, t\big),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

⊙ *Reference*: N. H. Ibragimov (1994).

2°. There are solutions of the following forms:

$$w(x,y,z,t) = W(\xi, z, t), \quad \xi = a_2 x^2 + a_1 y^2 \qquad \text{"three-dimensional" solution;}$$
$$w(x,y,z,t) = U(p,q), \quad p = (a_2 x^2 + a_1 y^2)t^{-2}, \quad q = z t^{-1} \qquad \text{"two-dimensional" solution.}$$

3°. For other exact solutions, see equation 4.6.2.6, in which $f(w)$ should be renamed $a_1 f(w)$ and $g(w)$ renamed $a_2 f(w)$.

6. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[h(w)\dfrac{\partial w}{\partial z}\right].$

1°. Suppose $w(x, y, z, t)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 z + C_4, \pm C_1 t + C_5),$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int \left[k_1^2 f(w) + k_2^2 g(w) + k_3^2 h(w)\right] dw - \lambda^2 w = C_1(k_1 x + k_2 y + k_3 z + \lambda t) + C_2,$$

where C_1, C_2, k_1, k_2, k_3, and λ are arbitrary constants.

3°. Solutions in implicit form:

$$\left(\frac{C_1 x + C_2 y + C_3 z + C_4}{t + C_5}\right)^2 = C_1^2 f(w) + C_2^2 g(w) + C_3^2 h(w),$$
$$\left(\frac{C_1 y + C_2 z + C_3 t + C_4}{x + C_5}\right)^2 f(w) + C_1^2 g(w) + C_2^2 h(w) = C_3^2,$$
$$\left(\frac{C_1 x + C_2 z + C_3 t + C_4}{y + C_5}\right)^2 g(w) + C_1^2 f(w) + C_2^2 h(w) = C_3^2,$$
$$\left(\frac{C_1 x + C_2 y + C_3 t + C_4}{z + C_5}\right)^2 h(w) + C_1^2 f(w) + C_2^2 g(w) = C_3^2,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

4°. Solution:

$$w = w(\xi), \quad \xi = \frac{C_1 x + C_2 y + C_3 z + C_4}{t + C_5},$$

where $C_1, \ldots, C_5$ are arbitrary constants, and the function $u(\xi)$ is determined by the ordinary differential equation

$$(\xi^2 w'_\xi)'_\xi = [\varphi(w) w'_\xi]'_\xi, \qquad \varphi(w) = C_1^2 f(w) + C_2^2 g(w) + C_3 h(w),$$

which admits the first integral

$$\left[\xi^2 - C_1^2 f(w) - C_2^2 g(w) - C_3^2 h(w)\right] w'_\xi = C_6. \tag{1}$$

To the special case $C_6 = 0$ there corresponds the first solution in Item 3°.

For $C_6 \neq 0$, treating w in (1) as the independent variable, we obtain a Riccati equation for $\xi = \xi(w)$:

$$C_6 \xi'_w = \xi^2 - C_1^2 f(w) - C_2^2 g(w) - C_3^2 h(w). \tag{2}$$

For exact solutions of equation (2), which can be reduced to a second-order linear equation, see Polyanin and Zaitsev (2003).

5°. Solution:

$$w = u(\eta), \quad \eta = \frac{C_1 y + C_2 z + C_3 t + C_4}{x + C_5}, \tag{3}$$

where $C_1, \ldots, C_5$ are arbitrary constants, and the function $u(\eta)$ is determined by the ordinary differential equation

$$C_3^2 u''_{\eta\eta} = [\eta^2 f(u) u'_\eta]'_\eta + C_1^2 [g(u) u'_\eta]'_\eta + C_2^2 [h(u) u'_\eta]'_\eta,$$

which admits the first integral

$$[\eta^2 f(u) + C_1^2 g(u) + C_2^2 h(u) - C_3^2]u'_\eta = C_6. \tag{4}$$

To the special case $C_6 = 0$ there corresponds the second solution in Item 3°.

For $C_6 \neq 0$, treating u in (4) as the independent variable, we obtain a Riccati equation for $\eta = \eta(u)$:

$$C_6 \eta'_u = \eta^2 f(u) + C_1^2 g(u) + C_2^2 h(u) - C_3^2. \tag{5}$$

For exact solutions of equation (5), which can be reduced to a second-order linear equation, see Polyanin and Zaitsev (2003).

Formula (3) and equation (5) can be used to obtain two other "one-dimensional" solutions by means of the following cyclic permutations of variables and determining functions:

$$\begin{array}{ccc} & (x, f) & \\ \nearrow & & \searrow \\ (z, h) & \longleftarrow & (y, g) \end{array}$$

6°. "Two-dimensional" solution (k_1, k_2, and k_3 are arbitrary constants):

$$w(x, y, z, t) = u(\xi, t), \quad \xi = k_1 x + k_2 y + k_3 z,$$

where the function $u = u(\xi, t)$ is determined by a differential equation of the form 3.4.4.6:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial \xi}\left[\varphi(u)\frac{\partial u}{\partial \xi}\right], \quad \varphi(u) = k_1^2 f(u) + k_2^2 g(u) + k_3^2 h(u),$$

which can be reduced to a linear equation.

7°. "Two-dimensional" solution (a, b, and c are arbitrary constants):

$$w(x, y, z, t) = v(x, \eta), \quad \eta = ay + bz + ct,$$

where the function $v = v(x, \eta)$ is determined by a differential equation of the form 5.4.4.8:

$$\frac{\partial}{\partial x}\left[f(v)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial \eta}\left[\psi(v)\frac{\partial v}{\partial \eta}\right] = 0, \quad \psi(v) = a^2 g(v) + b^2 h(v) - c^2,$$

which can be reduced to a linear equation.

8°. "Two-dimensional" solution (the a_n and b_n are arbitrary constants):

$$w(x, y, z, t) = U(\zeta, \rho), \quad \zeta = a_1 t + a_2 x, \quad \rho = b_1 y + b_2 z,$$

where the function $U = U(\zeta, \rho)$ is determined by a differential equation of the form 5.4.4.8:

$$\frac{\partial}{\partial \zeta}\left[\Phi(U)\frac{\partial U}{\partial \zeta}\right] + \frac{\partial}{\partial \rho}\left[\Psi(U)\frac{\partial U}{\partial \rho}\right] = 0, \quad \Phi(U) = a_2^2 f(U) - a_1^2, \quad \Psi(U) = b_1^2 g(U) + b_2^2 h(U),$$

which can be reduced to a linear equation.

Remark. The solutions specified in Items 7° and 8° can be used to obtain other "two-dimensional" solutions by means of the cyclic permutations of variables and determining functions as shown in Item 5°.

9°. There are more complicated "two-dimensional" solutions of the form

$$w(x, y, z, t) = V(z_1, z_2), \quad z_1 = a_1 x + a_2 y + a_3 z + a_4 t, \quad z_2 = b_1 x + b_2 y + b_3 z + b_4 t.$$

10°+. "Three-dimensional" solution:

$$w(x, y, z, t) = \Theta(p, q, s), \quad p = x/t, \quad q = y/t, \quad s = z/t,$$

where the function $\Theta = \Theta(p, q, s)$ is determined by the differential equation

$$p^2 \frac{\partial^2 \Theta}{\partial p^2} + q^2 \frac{\partial^2 \Theta}{\partial q^2} + r^2 \frac{\partial^2 \Theta}{\partial r^2} + 2pq \frac{\partial^2 \Theta}{\partial p \partial q} + 2pr \frac{\partial^2 \Theta}{\partial p \partial r} + 2rq \frac{\partial^2 \Theta}{\partial r \partial q}$$
$$+ 2p \frac{\partial \Theta}{\partial p} + 2q \frac{\partial \Theta}{\partial q} + 2r \frac{\partial \Theta}{\partial r} = \frac{\partial}{\partial p}\left[f(\Theta)\frac{\partial \Theta}{\partial p}\right] + \frac{\partial}{\partial q}\left[g(\Theta)\frac{\partial \Theta}{\partial q}\right] + \frac{\partial}{\partial r}\left[h(\Theta)\frac{\partial \Theta}{\partial r}\right].$$

11°. For results of the group analysis of the original equation, see Ibragimov (1994).

7. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[f(w)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[f(w)\frac{\partial w}{\partial z}\right] - a^2\frac{f'(w)}{f^3(w)} + b.$

Solution in implicit form:

$$\int f(w)\,dw = at + U(x,y,z),$$

where the function $U = U(x,y,z)$ is determined by the Poisson equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + b = 0.$$

For this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

Remark. The above holds true if the constant b in the equation is replaced by an arbitrary function $b = b(x,y,z)$.

4.6.3. Other Equations

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} = ax^n\frac{\partial^2 w}{\partial x^2} + by^m\frac{\partial^2 w}{\partial y^2} + cz^k\frac{\partial^2 w}{\partial z^2} + f(w).$

1°. Solution for $n \neq 2$, $m \neq 2$, and $k \neq 2$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2} - \frac{1}{4}(t+C)^2\right],$$

where C and B are arbitrary constants ($B \neq 0$) and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + Bf(w) = 0, \qquad A = 2\left(\frac{1-n}{2-n} + \frac{1-m}{2-m} + \frac{1-k}{2-k}\right).$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi,t), \quad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2}\right];$$

$$w = V(x,\eta), \quad \eta^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = W(\zeta,\rho), \quad \zeta^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho^2 = 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2}\right].$$

The second and third solutions can be used to obtain other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x,a,n) & \\ \nearrow & & \searrow \\ (z,c,k) & \longleftarrow & (y,b,m) \end{array}$$

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = ae^{\lambda x}\frac{\partial^2 w}{\partial x^2} + be^{\mu y}\frac{\partial^2 w}{\partial y^2} + ce^{\nu z}\frac{\partial^2 w}{\partial z^2} + f(w).$

1°. Solution for $\lambda \neq 0$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C_1)^2\right],$$

where B and C_1 are arbitrary constants and the function $w(r)$ is determined by the autonomous ordinary differential equation

$$w''_{rr} + 6r^{-1}w'_r + Bf(w) = 0.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \qquad \xi^2 = 4\left(\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right);$$

$$w = V(x, \eta), \qquad \eta^2 = \pm 4\left[\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = W(\zeta, \rho), \quad \zeta^2 = \pm 4\left[\frac{e^{-\lambda x}}{a\lambda^2} - \frac{1}{4}(t+C)^2\right], \quad \rho^2 = 4\left(\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right).$$

The second and third solutions can be used to obtain other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x, a, \lambda) & \\ \nearrow & & \searrow \\ (z, c, \nu) & \longleftarrow & (y, b, \mu) \end{array}$$

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = ax^n \frac{\partial^2 w}{\partial x^2} + by^m \frac{\partial^2 w}{\partial y^2} + ce^{\nu z} \frac{\partial^2 w}{\partial z^2} + f(w).$

1°. Solution for $n \neq 2$, $m \neq 2$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C_1)^2\right],$$

where B and C are arbitrary constants and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r + Bf(w) = 0, \qquad A = 2\left(\frac{1-n}{2-n} + \frac{1-m}{2-m} + 1\right).$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = V_1(x, \eta_1), \qquad \eta_1^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = V_2(y, \eta_2), \qquad \eta_2^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = V_3(z, \eta_3), \qquad \eta_3^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = W_1(\zeta_1, \rho_1), \quad \zeta_1^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_1^2 = 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = W_2(\zeta_2, \rho_2), \quad \zeta_2^2 = \pm 4\left[\frac{y^{2-m}}{b(2-m)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_2^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = W_3(\zeta_3, \rho_3), \quad \zeta_3^2 = \pm 4\left[\frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_3^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2}\right].$$

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = ax^n \frac{\partial^2 w}{\partial x^2} + be^{\mu y} \frac{\partial^2 w}{\partial y^2} + ce^{\nu z} \frac{\partial^2 w}{\partial z^2} + f(w).$

1°. Solution for $n \neq 2$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = \frac{4}{B}\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right],$$

where B and C are arbitrary constants and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{2(5-3n)}{2-n}\frac{1}{r}w'_r + Bf(w) = 0.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(\xi, t), \qquad \xi^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = V_1(x, \eta_1), \qquad \eta_1^2 = \pm 4\left[\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = V_2(y, \eta_2), \qquad \eta_2^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = V_3(z, \eta_3), \qquad \eta_3^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2} - \frac{1}{4}(t+C)^2\right];$$

$$w = W_1(\zeta_1, \rho_1), \quad \zeta_1^2 = \pm 4\left[\frac{x^{2-n}}{a(2-n)^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_1^2 = 4\left[\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = W_2(\zeta_2, \rho_2), \quad \zeta_2^2 = \pm 4\left[\frac{e^{-\mu y}}{b\mu^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_2^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2}\right];$$

$$w = W_3(\zeta_3, \rho_3), \quad \zeta_3^2 = \pm 4\left[\frac{e^{-\nu z}}{c\nu^2} - \frac{1}{4}(t+C)^2\right], \quad \rho_3^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\mu y}}{b\mu^2}\right].$$

Chapter 5

Elliptic Equations with Two Space Variables

5.1. Equations with Power-Law Nonlinearities

5.1.1. Equations of the Form $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = aw + bw^n + cw^{2n-1}$

► *The general properties of this type of equation are listed in 5.4.1.1; traveling-wave solutions and solutions with central symmetry are also treated there.*

1. $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = kw^n.$

This is a steady heat and mass transfer equation with an nth-order volume reaction in two dimensions. This equation arises also in combustion theory and is a special case of equation 5.4.1.1 with $f(w) = kw^n$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{n-1} x + C_2, \pm C_1^{n-1} y + C_3),$$
$$w_2 = w(x \cos\beta - y \sin\beta,\ x \sin\beta + y \cos\beta),$$

where C_1, C_2, C_3, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w(x, y) = (Ax + By + C)^{\frac{2}{1-n}}, \quad B = \pm\sqrt{\frac{k(n-1)^2}{2(n+1)} - A^2};$$
$$w(x, y) = s\left[(x + C_1)^2 + (y + C_2)^2\right]^{\frac{1}{1-n}}, \quad s = \left[\tfrac{1}{4}k(1-n)^2\right]^{\frac{1}{1-n}},$$

where A, C, C_1, and C_2 are arbitrary constants.

3°. Traveling-wave solution in implicit form (generalizes the first solution of Item 2°):

$$\int \left[D + \frac{2kw^{n+1}}{(n+1)(A^2 + B^2)}\right]^{-1/2} dw = Ax + By + C,$$

where A, B, C, and D are arbitrary constants ($n \neq -1$).

4°. Solution (generalizes the second solution of Item 2°):

$$w = w(r), \qquad r = \sqrt{(x + C_1)^2 + (y + C_2)^2},$$

where C_1 and C_2 are arbitrary constants, and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{1}{r} w'_r = kw^n.$$

5°. Self-similar solution:

$$w(x,y) = (x+C_1)^{\frac{2}{1-n}} u(\xi), \qquad \xi = \frac{y+C_2}{x+C_1},$$

where the function $u(\xi)$ is determined by the ordinary differential equation

$$(1+\xi^2)u''_{\xi\xi} - \frac{2(1+n)}{1-n}\xi u'_\xi + \frac{2(1+n)}{(1-n)^2}u - ku^n = 0.$$

6°. Multiplicative separable solution in polar coordinates (another representation of the solution of Item 5°):

$$w(x,y) = r^{\frac{2}{1-n}} U(\theta), \qquad r = \sqrt{(x+C_1)^2 + (y+C_2)^2}, \quad \tan\theta = \frac{y+C_2}{x+C_1},$$

where the function $U = U(\theta)$ is determined by the autonomous ordinary differential equation

$$U''_{\theta\theta} + \frac{4}{(1-n)^2}U = kU^n.$$

Integrating yields the general solution in implicit form:

$$\int \left[\frac{2k}{n+1}U^{n+1} - \frac{4}{(n-1)^2}U^2 + C_3\right]^{-1/2} dU = C_4 \pm \theta,$$

where C_3 and C_4 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

2. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = aw + bw^n.$

This is a special case of equation 5.4.1.1 with $f(w) = aw + bw^n$.

1°. Traveling-wave solutions for $a > 0$:

$$w(x,y) = \left[\frac{2b\sinh^2 z}{a(n+1)}\right]^{\frac{1}{1-n}}, \qquad z = \tfrac{1}{2}\sqrt{a}\,(1-n)(x\sin C_1 + y\cos C_1) + C_2 \quad \text{if } b(n+1) > 0,$$

$$w(x,y) = \left[-\frac{2b\cosh^2 z}{a(n+1)}\right]^{\frac{1}{1-n}}, \quad z = \tfrac{1}{2}\sqrt{a}\,(1-n)(x\sin C_1 + y\cos C_1) + C_2 \quad \text{if } b(n+1) < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. Traveling-wave solutions for $a < 0$ and $b(n+1) > 0$:

$$w(x,y) = \left[-\frac{2b\cos^2 z}{a(n+1)}\right]^{\frac{1}{1-n}}, \quad z = \tfrac{1}{2}\sqrt{|a|}\,(1-n)(x\sin C_1 + y\cos C_1) + C_2.$$

3. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = aw^n + bw^{2n-1}.$

This is a special case of equation 5.4.1.1 with $f(w) = aw^n + bw^{2n-1}$.

Solutions:

$$w(x,y) = \left[\frac{a(1-n)^2}{2(n+1)}(x\sin C_1 + y\cos C_1 + C_2)^2 - \frac{b(n+1)}{2an}\right]^{\frac{1}{1-n}},$$

$$w(x,y) = \left\{\frac{1}{4}a(1-n)^2\left[(x+C_1)^2 + (y+C_2)^2\right] - \frac{b}{an}\right\}^{\frac{1}{1-n}},$$

where C_1 and C_2 are arbitrary constants.

4. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = aw - a(n+1)w^n + bw^{2n-1}.$

1°. Traveling-wave solutions:

$$w(x,y) = \bigl(\lambda + C_1 \exp z\bigr)^{\frac{1}{1-n}}, \qquad z = \sqrt{a}\,(1-n)(x\sin C_2 + y\cos C_2),$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation $a\lambda^2 - a(n+1)\lambda + b = 0$, and C_1 and C_2 are arbitrary constants.

2°. See also equation 5.1.1.5, where the following renaming should be made: $b \to -a(n+1)$ and $c \to b$.

5. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = aw + bw^n + cw^{2n-1}.$

This is a special case of equation 5.4.1.1 with $f(w) = aw + bw^n + cw^{2n-1}$.

1°. Traveling-wave solutions for $a > 0$:

$$w(x,y) = (A + B\cosh z)^{\frac{1}{1-n}}, \qquad z = \sqrt{a}\,(1-n)(x\sin C_1 + y\cos C_1) + C_2,$$
$$A = -\frac{b}{a(n+1)}, \quad B = \pm\left[\frac{b^2}{a^2(n+1)^2} - \frac{c}{an}\right]^{1/2};$$
$$w(x,y) = (A + B\sinh z)^{\frac{1}{1-n}}, \qquad z = \sqrt{a}\,(1-n)(x\sin C_1 + y\cos C_1) + C_2,$$
$$A = -\frac{b}{a(n+1)}, \quad B = \pm\left[\frac{c}{an} - \frac{b^2}{a^2(n+1)^2}\right]^{1/2},$$

where C_1 and C_2 are arbitrary constants (the expressions in square brackets must be nonnegative).

2°. Traveling-wave solutions for $a < 0$:

$$w(x,y) = (A + B\cos z)^{\frac{1}{1-n}}, \qquad z = \sqrt{|a|}\,(1-n)(x\sin C_1 + y\cos C_1) + C_2,$$
$$A = -\frac{b}{a(n+1)}, \quad B = \pm\left[\frac{b^2}{a^2(n+1)^2} - \frac{c}{an}\right]^{1/2},$$

where C_1 and C_2 are arbitrary constants.

3°. The substitution $u = w^{1-n}$ leads to an equation of the form 5.1.6.7:

$$u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{n}{1-n}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] = a(1-n)u^2 + b(1-n)u + c(1-n).$$

5.1.2. Equations of the Form $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = f(x,y,w)$

1. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a(x^2 + y^2)w^n.$

This is a special case of equation 5.4.1.2 with $f(w) = aw^n$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^4 w(\pm C_1^{n-1}x,\ \pm C_1^{n-1}y),$$
$$w_2 = w(x\cos\beta - y\sin\beta,\ x\sin\beta + y\cos\beta),$$

where C_1 and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. The transformation

$$w = U(z, \zeta), \quad z = \tfrac{1}{2}(x^2 - y^2), \quad \zeta = xy$$

leads to a simpler equation of the form 5.1.1.1:

$$\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial \zeta^2} = aU^n.$$

2. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = c(ax + by)^k w^n.$

This is a special case of equation 5.4.1.10 with $f(z, w) = cz^k w^n$.

3. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a(x^2 + y^2)^k w^n.$

This is a special case of equation 5.4.1.3 with $f(w) = aw^n$.

4. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a(x^2 + y^2)(xy)^k w^n.$

This is a special case of equation 5.4.1.12 with $f(z, w) = az^k w^n$.

5. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = ae^{\beta x} w^n.$

This is a special case of equation 5.4.1.4 with $f(w) = aw^n$.

6. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = ke^{ax - by} w^n.$

This is a special case of equation 5.4.1.5 with $f(w) = kw^n$.

7. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = k\bigl(w + A_{11}x^2 + A_{12}xy + A_{22}y^2 + B_1 x + B_2 y\bigr)^n.$

This is a special case of equation 5.4.1.14 with $f(u) = ku^n$.

5.1.3. Equations of the Form $\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial^2 w}{\partial y^2} = F\Bigl(x, y, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\Bigr)$

1. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = (a_1 x + b_1 y + c_1)\dfrac{\partial w}{\partial x} + (a_2 x + b_2 y + c_2)\dfrac{\partial w}{\partial y} + kw^n.$

This is a special case of equation 5.4.2.2 with $f(w) = kw^n$.

2. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{a}{x}\dfrac{\partial w}{\partial x} + \dfrac{b}{y}\dfrac{\partial w}{\partial y} = kw^n.$

This is a special case of equation 5.4.2.4 with $f(\xi, w) = kw^n$.

3. $\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial^2 w}{\partial y^2} = b\Bigl(\dfrac{\partial w}{\partial y}\Bigr)^2 + cw + sx^n.$

This is a special case of equation 5.4.2.6 with $f(x) = b$, $g(x) = c$, and $h(x) = sx^n$.

4. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = \alpha\Bigl(\dfrac{\partial w}{\partial y}\Bigr)^2 + \beta x^n y^2 + \gamma x^m y + \mu x^k.$

This is a special case of equation 5.4.2.8 with $a = b = 1$, $f(x) = \alpha$, $g(x) = h_1(x) = h_0(x) = p(x) = 0$, $q_2(x) = \beta x^n$, $q_1(x) = \gamma x^m$, and $q_0(x) = \mu x^k$.

5. $\dfrac{\partial^2 w}{\partial x^2}+a\dfrac{\partial^2 w}{\partial y^2}=c\left(\dfrac{\partial w}{\partial y}\right)^2+bcw^2+kw+s.$

Let A be a root of the quadratic equation $bcA^2+kA+s=0$.

1°. Suppose the inequality $2Abc+k+ab=\sigma^2>0$ holds. Then the equation has the generalized separable solutions

$$w(x,y)=A+\bigl[C_1\exp(\sigma x)+C_2\exp(-\sigma x)\bigr]\exp\bigl(\pm y\sqrt{-b}\,\bigr),$$

where C_1 and C_2 are arbitrary constants.

2°. If $2Abc+k+ab=-\sigma^2<0$, then the equation has the generalized separable solutions

$$w(x,y)=A+\bigl[C_1\cos(\sigma x)+C_2\sin(\sigma x)\bigr]\exp\bigl(\pm y\sqrt{-b}\,\bigr).$$

3°. For more complicated solutions, see equation 5.4.2.7 with $f(x)=c$, $g(x)=k$, and $h(x)=s$.

6. $\dfrac{\partial^2 w}{\partial x^2}+a\dfrac{\partial^2 w}{\partial y^2}=cx^n\left(\dfrac{\partial w}{\partial y}\right)^2+bcx^n w^2+kx^m w+sx^l.$

This is a special case of equation 5.4.2.7 with $f(x)=cx^n$, $g(x)=kx^m$, and $h(x)=sx^l$.

7. $\dfrac{\partial^2 w}{\partial x^2}+a\dfrac{\partial^2 w}{\partial y^2}=ce^{\beta x}\left(\dfrac{\partial w}{\partial y}\right)^2+bce^{\beta x}w^2+ke^{\mu x}w+se^{\nu x}.$

This is a special case of equation 5.4.2.7 with $f(x)=ce^{\beta x}$, $g(x)=ke^{\mu x}$, and $h(x)=se^{\nu x}$.

8. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=aw^n\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2\right].$

This is a special case of equation 5.4.2.9 with $f(w)=aw^n$. The substitution

$$U=\int\exp\left(-\frac{a}{n+1}w^{n+1}\right)dw$$

leads to the two-dimensional Laplace equation for $U=U(x,y)$:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

9. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=\alpha\left(\dfrac{\partial w}{\partial x}\right)^n+\beta\left(\dfrac{\partial w}{\partial y}\right)^m+kw.$

This is a special case of equation 5.4.2.10 with $a=b=1$, $f(x)=\alpha$, and $g(y)=\beta$.

10. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=(a_1x+b_1y+c_1)\left(\dfrac{\partial w}{\partial x}\right)^k+(a_2x+b_2y+c_2)\left(\dfrac{\partial w}{\partial y}\right)^k.$

This is a special case of equation 5.4.2.12 with $f(w,u,v)=0$.

5.1.4. Equations of the Form $\dfrac{\partial}{\partial x}\left[f_1(x,y)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[f_2(x,y)\dfrac{\partial w}{\partial y}\right]=g(w)$

► *Equations of this form are encountered in stationary problems of heat and mass transfer and combustion theory. Here, f_1 and f_2 are the principal thermal diffusivities (diffusion coefficients) dependent on the space coordinates x and y, and $g=g(w)$ is a source function that defines the law of heat (substance) release or absorption.*

1. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)=cw^k.$

This is a special case of equation 5.4.3.1 with $f(w)=cw^k$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\Big(C_1^{\frac{k-1}{2-n}}\,x,\ C_1^{\frac{k-1}{2-m}}\,y\Big),$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution for $n \neq 2$ and $m \neq 2$:

$$w = w(\xi), \qquad \xi = \big[b(2-m)^2 x^{2-n} + a(2-n)^2 y^{2-m}\big]^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{A}{\xi} w'_\xi = B w^k, \tag{1}$$

where

$$A = \frac{4-nm}{(2-n)(2-m)}, \quad B = \frac{4c}{ab(2-n)^2(2-m)^2}.$$

3°. Below are some exact solutions of equation (1).

3.1. Equation (1) admits an exact solution of the form

$$w = \left[\frac{2(1+k+A-Ak)}{B(1-k)^2}\right]^{\frac{1}{k-1}} \xi^{\frac{2}{1-k}}$$

with $k \neq 1$.

3.2. For $m = 4/n$, the exact solution can be represented in implicit form as

$$\int \left[C_1 + \frac{2cn^2 w^{k+1}}{ab(k+1)(2-n)^4}\right]^{-1/2} dw = C_2 \pm \xi,$$

where C_1 and C_2 are arbitrary constants.

3.3. The substitution $\zeta = \xi^{1-A}$ brings (1) to the Emden–Fowler equation

$$w''_{\zeta\zeta} = \frac{B}{(1-A)^2} \zeta^{\frac{2A}{1-A}} w^k. \tag{2}$$

Over 20 exact solutions to equation (2) for various values of k can be found in Polyanin and Zaitsev (2003).

2. $\displaystyle \frac{\partial}{\partial x}\left(ax^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\mu y} \frac{\partial w}{\partial y}\right) = cw^m.$

This is a special case of equation 5.4.3.8 with $f(w) = cw^m$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\Big(C_1^{\frac{m-1}{2-n}}\,x,\ y + \frac{1-m}{\mu}\ln C_1\Big),$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution for $n \neq 2$ and $\mu \neq 0$:

$$w = w(\xi), \qquad \xi = \big[b\mu^2 x^{2-n} + a(2-n)^2 e^{-\mu y}\big]^{1/2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{n}{2-n}\frac{1}{\xi} w'_\xi = \frac{4c}{ab\mu^2(2-n)^2} w^m.$$

3. $\displaystyle \frac{\partial}{\partial x}\left(ae^{\beta x} \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\mu y} \frac{\partial w}{\partial y}\right) = cw^m.$

This is a special case of equation 5.4.3.6 with $f(w) = cw^m$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\Big(x + \frac{1-m}{\beta} \ln C_1,\ y + \frac{1-m}{\mu} \ln C_1\Big),$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution for $\beta\mu \neq 0$:

$$w = w(\xi), \qquad \xi = \big(b\mu^2 e^{-\beta x} + a\beta^2 e^{-\mu y}\big)^{1/2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} - \frac{1}{\xi} w'_\xi = A w^m, \qquad A = \frac{4c}{ab\beta^2\mu^2}. \tag{1}$$

3°. Below are some exact solutions of equation (1).

3.1. Equation (1) admits a solution of the form

$$w = \left[\frac{abm\beta^2\mu^2}{c(1-m)^2}\right]^{\frac{1}{m-1}} \xi^{\frac{2}{1-m}}.$$

3.2. The substitution $\zeta = \xi^2$ brings (1) to the Emden–Fowler equation

$$w''_{\zeta\zeta} = \tfrac{1}{4} A \zeta^{-1} w^m,$$

whose solutions with $m = -1$ and $m = -2$ can be found in Polyanin and Zaitsev (2003).

4. $\displaystyle \frac{\partial}{\partial x}\left[(ay + c)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[(bx + s)\frac{\partial w}{\partial y}\right] = kw^n.$

This is a special case of equation 5.4.4.1 with $f(w) = kw^n$.

The equation can be rewritten in the simpler form

$$(ay + c)\frac{\partial^2 w}{\partial x^2} + (bx + s)\frac{\partial^2 w}{\partial y^2} = kw^n.$$

5. $\displaystyle \frac{\partial}{\partial x}\left[(a_1 x + b_1 y + c_1)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[(a_2 x + b_2 y + c_2)\frac{\partial w}{\partial y}\right] = kw^n.$

This is a special case of equation 5.4.4.2 with $f(w) = kw^n$.

5.1.5. Equations of the Form $\frac{\partial}{\partial x}\left[f_1(w)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[f_2(w)\frac{\partial w}{\partial y}\right] = g(w)$

► *Equations of this form are encountered in stationary problems of heat and mass transfer and combustion theory. Here, $f_1 = f_1(w)$ and $f_2 = f_2(w)$ are the temperature (concentration) dependent principal thermal diffusivities (diffusion coefficients), and $g = g(w)$ is a source function that defines the law of heat (substance) release or absorption. Simple solutions dependent on a single space variable, $w = w(x)$ and $w = w(y)$, are not considered in this subsection.*

1. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial y}\left[(\alpha w + \beta)\frac{\partial w}{\partial y}\right] = 0.$

Stationary Khokhlov–Zabolotskaya equation (for $\alpha = 1$ and $\beta = 0$). It arises in acoustics, nonlinear mechanics, and heat and mass transfer theory. This is a special case of equation 5.4.4.8 with $f(w) = 1$ and $g(w) = \alpha w + \beta$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = \frac{C_1^2}{C_2^2} w(C_1x + C_3, C_2y + C_4) + \frac{\beta}{\alpha}\left(\frac{C_1^2}{C_2^2} - 1\right),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$\begin{aligned}
w(x,y) &= Ay - \tfrac{1}{2}A^2\alpha x^2 + C_1x + C_2,\\
w(x,y) &= (Ax+B)y - \frac{\alpha}{12A^2}(Ax+B)^4 + C_1x + C_2,\\
w(x,y) &= -\frac{1}{\alpha}\left(\frac{y+A}{x+B}\right)^2 + \frac{C_1}{x+B} + C_2(x+B)^2 - \frac{\beta}{\alpha},\\
w(x,y) &= -\frac{1}{\alpha}\left[\beta + \lambda^2 \pm \sqrt{A(y+\lambda x)+B}\,\right],\\
w(x,y) &= (Ax+B)\sqrt{C_1y+C_2} - \frac{\beta}{\alpha},
\end{aligned}$$

where A, B, C_1, C_2, and λ are arbitrary constants. The first two solutions are linear in y, the third is quadratic in y, and the fourth one is a traveling-wave solution.

3°. Generalized separable solution quadratic in y (generalizes the third solution of Item 2°):

$$w(x,y) = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where the functions $\varphi = \varphi(x)$, $\psi = \psi(x)$, and $\chi = \chi(x)$ are determined by the system of ordinary differential equations

$$\varphi''_{xx} + 6\alpha\varphi^2 = 0, \tag{1}$$
$$\psi''_{xx} + 6\alpha\varphi\psi = 0, \tag{2}$$
$$\chi''_{xx} + 2\alpha\varphi\chi = -2\beta\varphi - \alpha\psi^2. \tag{3}$$

The nonlinear autonomous equation (1) is independent of the others; its solution can be expressed in terms elliptic integrals. Equations (2) and (3) are solved successively (these are linear in the unknowns ψ and χ, respectively).

System (1)–(3) admits the following five-parameter family of solutions:

$$\begin{aligned}
\varphi(x) &= -\frac{1}{\alpha(x+A)^2},\\
\psi(x) &= \frac{B_1}{(x+A)^2} + B_2(x+A)^3,\\
\chi(x) &= \frac{C_1}{x+A} + C_2(x+A)^2 - \frac{\beta}{\alpha} - \frac{\alpha B_1^2}{4(x+A)^2} - \frac{1}{2}\alpha B_1B_2(x+A)^3 - \frac{1}{54}\alpha B_2^2(x+A)^8,
\end{aligned}$$

where A, B_1, B_2, C_1, and C_2 are arbitrary constants.

4°. Solution in parametric form:

$$\begin{aligned}
x &= C_1wt + C_2w + C_3t + C_4,\\
y &= \tfrac{1}{2}C_1t^2 + C_2t - \tfrac{1}{3}\alpha C_1w^3 - \tfrac{1}{2}(\alpha C_3 + \beta C_1)w^2 - \beta C_3w + C_5.
\end{aligned}$$

5°. Solution in parametric form:

$$\begin{aligned}
x &= C_1t^2 + C_2wt + C_3t + C_4w - C_1\left(\tfrac{1}{3}\alpha w^3 + \beta w^2\right) + C_5,\\
y &= \tfrac{1}{2}C_2t^2 + C_4t - C_1t\left(\alpha w^2 + 2\beta w\right) - \tfrac{1}{3}\alpha C_2w^3 - \tfrac{1}{2}(\alpha C_3 + \beta C_2)w^2 - \beta C_3w + C_6.
\end{aligned}$$

6°. Self-similar solution (A and B are arbitrary constants):

$$w = w(\zeta), \quad \zeta = \frac{x + A}{y + B}$$

where the function $w(\zeta)$ is determined by the ordinary differential equation

$$w''_{\zeta\zeta} + [\zeta^2(\alpha w + \beta)w'_\zeta]'_\zeta = 0.$$

On integrating the equation once and taking w to be the independent variable, one obtains a Riccati equation for $\zeta = \zeta(w)$:

$$C\zeta'_w = (\alpha w + \beta)\zeta^2 + 1,$$

where C is an arbitrary constant. The general solution to this equation can be expressed in terms of Bessel functions; see Polyanin and Zaitsev (2003).

7°. Solution (generalizes the last solution of Item 2°):

$$w(x, y) = \frac{1}{\alpha} f(x)g(y) - \frac{\beta}{\alpha}.$$

The functions $f(x)$ and $g(y)$ are determined by the autonomous ordinary differential equations (A is an arbitrary constant)

$$f''_{xx} = Af^2, \qquad (gg'_y)'_y = -Ag, \tag{4}$$

which are independent. Integrating the equations of (4) yields their general solutions in implicit form:

$$C_1 \pm x = \int \left(\tfrac{2}{3}Af^3 + B_1\right)^{-1/2} df,$$

$$C_2 \pm y = \int g\left(-\tfrac{2}{3}Ag^3 + B_2\right)^{-1/2} dg,$$

where B_1, B_2, C_1, and C_2 are arbitrary constants.

8°. Solution (A, B, and k are arbitrary constants):

$$w = \frac{1}{\alpha}(x + A)^{2k} F(z) - \frac{\beta}{\alpha}, \quad z = \frac{y + B}{(x + A)^{k+1}},$$

where the function $F = F(z)$ is determined by solving the generalized-homogeneous ordinary differential equation

$$2k(2k - 1)F - (k + 1)(3k - 2)zF'_z + (k + 1)^2 z^2 F''_{zz} + (FF'_z)'_z = 0.$$

Its order can be reduced.

9°. Solution (A and λ are arbitrary constants):

$$w = \frac{1}{\alpha} e^{-2\lambda x} \Phi(u) - \frac{\beta}{\alpha}, \quad u = (y + A)e^{\lambda x},$$

where the function $\Phi = \Phi(u)$ is determined by solving the generalized-homogeneous ordinary differential equation

$$4\lambda^2\Phi - 3\lambda^2 u\Phi'_u + \lambda^2 u^2 \Phi''_{uu} + (\Phi\Phi'_u)'_u = 0.$$

Its order can be reduced.

10°. Solution (A, B, and C are arbitrary constants):

$$w = \frac{1}{\alpha}(\pm x + A)^{-2}\Psi(\xi) - \frac{\beta}{\alpha}, \quad \xi = y + B\ln(\pm x + A) + C,$$

where the function $\Psi = \Psi(\xi)$ is determined by the autonomous ordinary differential equation

$$6\Psi - 5B\Psi'_\xi + B^2\Psi''_{\xi\xi} + (\Psi\Psi'_\xi)'_\xi = 0.$$

Its order can be reduced.

11°+. Solution:

$$w = U(\eta) - 4\alpha C_1^2 x^2 - 4\alpha C_1 C_2 x, \qquad \eta = y + \alpha C_1 x^2 + \alpha C_2 x,$$

where C_1 and C_2 are arbitrary constants and the function $U(\eta)$ is determined by the first-order ordinary differential equation

$$(\alpha U + \beta + \alpha^2 C_2^2) U'_\eta + 2\alpha C_1 U = 8\alpha C_1^2 \eta + C_3.$$

Through appropriate translations in both variables, one can make the equation homogeneous, which means that the equation is integrable by quadrature.

12°. The original equation can be rewritten as the system of equations

$$\frac{\partial w}{\partial x} = \frac{\partial v}{\partial y}, \quad -(\alpha w + \beta)\frac{\partial w}{\partial y} = \frac{\partial v}{\partial x}.$$

The hodograph transformation $x = x(w, v)$, $y = y(w, v)$ (w and v treated as the independent variables, and x and y, as the dependent ones) brings it to the linear system

$$\frac{\partial y}{\partial v} = \frac{\partial x}{\partial w}, \quad -(\alpha w + \beta)\frac{\partial x}{\partial v} = \frac{\partial y}{\partial w}.$$

On eliminating y, one obtains a linear equation for $x = x(w, v)$:

$$\frac{\partial^2 x}{\partial w^2} + (\alpha w + \beta)\frac{\partial^2 x}{\partial v^2} = 0.$$

13°. Let $w(x, y)$ be any solution of the Khokhlov–Zabolotskaya equation (with $\alpha = 1$ and $\beta = 0$). Then the ordinary differential equation

$$u''_{tt} = F(t, u), \qquad F(t, u) = \frac{1}{9\varphi}\left(\frac{\partial v}{\partial u} + 3\varphi''_{tt} u + 3\psi'_t\right),$$

where

$$v = -\varphi^{1/3} w(x, y) - \varphi^{-1}(\varphi'_t u + \psi)^2, \quad x = \frac{1}{3}\int \varphi^{-2/3}\,dt, \quad y = \varphi^{-1/3} u - \frac{1}{3}\int \varphi^{-4/3}\psi\,dt,$$

and $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are arbitrary functions, has a first integral cubic in u'_t.

⊙ *References for equation* 5.1.5.1: Y. Kodama and J. Gibbons (1989), V. V. Kozlov (1995), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2002).

2. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{1}{\alpha w + \beta}\frac{\partial w}{\partial y}\right) = 0.$

1°. Solutions:

$$w(x, y) = \frac{-A^2 x^2 + Bx + C}{\alpha(Ay + D)^2} - \frac{\beta}{\alpha},$$

$$w(x, y) = \frac{p^2}{A\alpha}\frac{Ax^2 + Bx + C}{\cosh^2(py + q)} - \frac{\beta}{\alpha},$$

$$w(x, y) = -\frac{p^2}{A\alpha}\frac{Ax^2 + Bx + C}{\sinh^2(py + q)} - \frac{\beta}{\alpha},$$

$$w(x, y) = -\frac{p^2}{A\alpha}\frac{Ax^2 + Bx + C}{\cos^2(py + q)} - \frac{\beta}{\alpha},$$

where A, B, C, D, p, and q are arbitrary constants.

2°. Solution in parametric form:

$$x = C_1 wt + C_2 w + C_3 t + C_4,$$
$$y = \frac{1}{2}C_1 t^2 + C_2 t - \frac{C_1}{\alpha} w - \frac{1}{\alpha^2}(\alpha C_3 - \beta C_1) \ln|\alpha w + \beta| + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

3°. For other exact solutions, see equation 5.4.4.8 with $f(w) = 1$ and $g(w) = (\alpha w + \beta)^{-1}$.

4°. The substitution $\alpha w + \beta = e^U$ leads to an equation of the form 5.2.4.1 (with swapped variables, $x \rightleftarrows y$):

$$\frac{\partial}{\partial x}\left(e^U \frac{\partial U}{\partial x}\right) + \frac{\partial^2 U}{\partial y^2} = 0.$$

3. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\alpha}{\sqrt{w+\beta}}\frac{\partial w}{\partial y}\right) = 0.$

The substitution $U = \dfrac{1}{\alpha}\sqrt{w+\beta}$ leads to the equation

$$\frac{\partial}{\partial x}\left(U \frac{\partial U}{\partial x}\right) + \frac{\partial^2 U}{\partial y^2} = 0.$$

Up to the swap of the coordinates ($x \rightleftarrows y$) and renaming the unknown function, this equation coincides with a special case of 5.1.5.1.

4. $\displaystyle \frac{\partial}{\partial x}\left[(\alpha_1 w + \beta_1)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[(\alpha_2 w + \beta_2)\frac{\partial w}{\partial y}\right] = \gamma.$

1°. Traveling-wave solutions linear in the coordinates:

$$w(x,y) = Ax \pm \sqrt{\frac{\gamma - A^2\alpha_1}{\alpha_2}}\, y + B,$$

where A and B are arbitrary constants.

2°. Traveling-wave solution in implicit form:

$$(A^2\alpha_1 + B^2\alpha_2)w^2 + 2(A^2\beta_1 + B^2\beta_2)w = \gamma(Ax+By)^2 + C_1(Ax+By) + C_2,$$

where A, B, C_1, and C_2 are arbitrary constants.

3°. For other solutions with $\gamma = 0$, see 5.4.4.8 with $f(w) = \alpha_1 w + \beta_1$ and $g(w) = \alpha_2 w + \beta_2$.

5. $\displaystyle \frac{\partial}{\partial x}\left(w^m \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(w^m \frac{\partial w}{\partial y}\right) = \alpha w^n.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{n-m-1}x + C_2, \pm C_1^{n-m-1}y + C_3),$$
$$w_2 = w(x\cos\beta - y\sin\beta,\ x\sin\beta + y\cos\beta),$$

where C_1, C_2, C_3, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. For $m \neq -1$, the substitution $U = w^{m+1}$ leads to an equation of the form 5.1.1.1:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \alpha(m+1)U^{\frac{n}{m+1}}.$$

3°. For $m = -1$, the substitution $w = e^V$ leads to an equation of the form 5.2.1.1:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \alpha e^{nV}.$$

6. $\dfrac{\partial}{\partial x}\left(aw^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(bw^m \dfrac{\partial w}{\partial y}\right) = 0.$

1°. Suppose $w(x, y, z)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(\pm C_1^n C_2 x + C_3, \pm C_1^m C_2 y + C_4),$$

where $C_1, \dots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Multiplicative separable solution:

$$w(x, y) = f(x)g(y). \tag{1}$$

The functions $f(x)$ and $g(y)$ are determined by the autonomous ordinary differential equations (A is an arbitrary constant)

$$(f^n f'_x)'_x = Abf^{m+1}, \qquad (g^m g'_y)'_y = -Aag^{n+1}, \tag{2}$$

which are independent. Integrating the equations of (2) yields their general solutions in implicit form:

$$\int f^n \left(\frac{2Ab}{n+m+2} f^{n+m+2} + B_1\right)^{-1/2} df = C_1 \pm x,$$

$$\int g^m \left(-\frac{2Aa}{n+m+2} g^{n+m+2} + B_2\right)^{-1/2} dg = C_2 \pm y,$$

where B_1, B_2, C_1, and C_2 are arbitrary constants; $n + m + 2 \neq 0$.

3°. There are exact solutions of the following forms:

$$w(x, y) = x^{-2k} F(z), \qquad z = yx^{mk-nk-1},$$
$$w(x, y) = x^{\frac{2}{n-m}} G(\xi), \quad \xi = y + k \ln x,$$
$$w(x, y) = e^{2x} H(\eta), \qquad \eta = ye^{(n-m)x},$$

where k is an arbitrary constant.

4°. For other exact solutions of the original equation, see 5.4.4.8 with $f(w) = aw^n$ and $g(w) = bw^m$.

7. $a_1 \dfrac{\partial}{\partial x}\left(w^n \dfrac{\partial w}{\partial x}\right) + a_2 \dfrac{\partial}{\partial y}\left(w^m \dfrac{\partial w}{\partial y}\right) = bw^k.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{k-n-1} x + C_2, \pm C_1^{k-m-1} y + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There are exact solutions of the following forms:

$$w(x, y) = F(\xi), \quad \xi = \alpha_1 x + \alpha_2 y \qquad \text{traveling-wave solution;}$$
$$w(x, y) = x^{\frac{2}{n-k+1}} U(z), \quad z = yx^{\frac{k-m-1}{n-k+1}} \qquad \text{self-similar solution.}$$

5.1.6. Other Equations Involving Arbitrary Parameters

1. $\dfrac{\partial^2 w}{\partial x^2} + aw^4 \dfrac{\partial^2 w}{\partial y^2} = by^n w^5.$

This is a special case of equation 5.4.5.1 with $f(y) = by^n$.

2. $\dfrac{\partial^2 w}{\partial x^2} + aw^4 \dfrac{\partial^2 w}{\partial y^2} = be^{\beta y} w^5.$

This is a special case of equation 5.4.5.1 with $f(y) = be^{\beta y}$.

3. $ax^n \dfrac{\partial^2 w}{\partial x^2} + by^m \dfrac{\partial^2 w}{\partial y^2} = cw^k.$

This is a special case of equation 5.4.5.5 with $k = s = 0$ and $f(w) = cw^k$.

1°. Functional separable solution for $n \neq 2$ and $m \neq 2$:

$$w = w(\xi), \qquad \xi = \left[b(2-m)^2 x^{2-n} + a(2-n)^2 y^{2-m}\right]^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{A}{\xi} w'_\xi = Bw^k, \tag{1}$$

where

$$A = \frac{3nm - 4n - 4m + 4}{(2-n)(2-m)}, \quad B = \frac{4c}{ab(2-n)^2(2-m)^2}.$$

2°. Below are some exact solutions of equation (1).

2.1. For $k \neq 1$, equation (1) admits an exact solution of the form

$$w = \left[\frac{2(1+k+A-Ak)}{B(1-k)^2}\right]^{\frac{1}{k-1}} \xi^{\frac{2}{1-k}}.$$

2.2. For $m = \dfrac{4n-4}{3n-4}$, the general solution of (1) is written out in implicit form as

$$\int \left[C_1 + \frac{2c(3n-4)^2 w^{k+1}}{ab(k+1)(2-n)^4}\right]^{-1/2} dw = C_2 \pm \xi,$$

where C_1 and C_2 are arbitrary constants.

2.3. The substitution $\zeta = \xi^{1-A}$ brings (1) to the Emden–Fowler equation

$$w''_{\zeta\zeta} = \frac{B}{(1-A)^2} \zeta^{\frac{2A}{1-A}} w^k. \tag{2}$$

Over 20 exact solutions to equation (2) for various values of k can be found in Polyanin and Zaitsev (2003).

4. $ax^n \dfrac{\partial^2 w}{\partial x^2} + be^{\beta y} \dfrac{\partial^2 w}{\partial y^2} = cw^m.$

This is a special case of equation 5.4.5.9 with $k = s = 0$ and $f(w) = cw^m$.

5. $ae^{\beta x} \dfrac{\partial^2 w}{\partial x^2} + be^{\mu y} \dfrac{\partial^2 w}{\partial y^2} = cw^m.$

This is a special case of equation 5.4.5.7 with $k = s = 0$ and $f(w) = cw^m$.

1°. Functional separable solution for $\beta\mu \neq 0$:

$$w = w(\xi), \qquad \xi = \left(b\mu^2 e^{-\beta x} + a\beta^2 e^{-\mu y}\right)^{1/2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{3}{\xi} w'_\xi = Aw^m, \qquad A = \frac{4c}{ab\beta^2\mu^2}. \tag{1}$$

2°. Below are some exact solutions of equation (1).

2.1. There is a solution of the form

$$w(\xi) = \left[\frac{ab(2-m)\beta^2\mu^2}{c(1-m)^2}\right]^{\frac{1}{m-1}} \xi^{\frac{2}{1-m}}.$$

2.2. The substitution $\zeta = \xi^{-2}$ brings (1) to the Emden–Fowler equation

$$w''_{\zeta\zeta} = \tfrac{1}{4} A\zeta^{-3} w^m,$$

whose solution for $m = 3$ can be found in Polyanin and Zaitsev (2003).

6. $\boldsymbol{w\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{\partial w}{\partial y}\right)^2 = \alpha w^\beta.}$

The substitution $w = e^U$ leads to an equation of the form 5.2.1.1:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \alpha e^{(\beta-2)U}.$$

7. $\boldsymbol{w\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \sigma\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = \alpha w^2 + \beta w + \gamma.}$

1°. Traveling-wave solutions for $\alpha(1+\sigma) > 0$:

$$w(x,y) = A_1 + B_1 \cosh z, \quad z = \sqrt{\frac{\alpha}{1+\sigma}}\,\frac{k_1 x + k_2 y}{\sqrt{k_1^2 + k_2^2}} + C,$$

$$A_1 = -\frac{\beta}{\alpha}\,\frac{1+\sigma}{1+2\sigma}, \quad B_1 = \pm\sqrt{\frac{\beta^2(1+\sigma)^2}{\alpha^2(1+2\sigma)^2} - \frac{\gamma(1+\sigma)}{\alpha\sigma}};$$

$$w(x,y) = A_2 + B_2 \sinh z, \quad z = \sqrt{\frac{\alpha}{1+\sigma}}\,\frac{k_1 x + k_2 y}{\sqrt{k_1^2 + k_2^2}} + C,$$

$$A_2 = -\frac{\beta}{\alpha}\,\frac{1+\sigma}{1+2\sigma}, \quad B_2 = \pm\sqrt{\frac{\gamma(1+\sigma)}{\alpha\sigma} - \frac{\beta^2(1+\sigma)^2}{\alpha^2(1+2\sigma)^2}},$$

where k_1, k_2, and C are arbitrary constants.

2°. Traveling-wave solutions for $\alpha(1+\sigma) < 0$:

$$w(x,y) = A + B\cos z, \quad z = \sqrt{-\frac{\alpha}{1+\sigma}}\,\frac{k_1 x + k_2 y}{\sqrt{k_1^2 + k_2^2}} + C,$$

$$A = -\frac{\beta}{\alpha}\,\frac{1+\sigma}{1+2\sigma}, \quad B = \pm\sqrt{\frac{\beta^2(1+\sigma)^2}{\alpha^2(1+2\sigma)^2} - \frac{\gamma(1+\sigma)}{\alpha\sigma}},$$

where k_1, k_2, and C are arbitrary constants.

3°. Solution:

$$w = w(r), \qquad r = \sqrt{(x+C_1)^2 + (y+C_2)^2},$$

where C_1 and C_2 are arbitrary constants and the function $w(r)$ is determined by the ordinary differential equation

$$ww''_{rr} + \frac{1}{r} ww'_r + \sigma\left(w'_r\right)^2 = \alpha w^2 + \beta w + \gamma.$$

4°. For $\gamma=0$, apart from the solutions presented in Items 1° to 3°, other solutions can be constructed. To this end, we apply the change of variable $w=u^2$ to the original equation to obtain

$$u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+(1+2\sigma)\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]=\tfrac{1}{2}\alpha u^2+\tfrac{1}{2}\beta.$$

This equation is a special case of the original one. It follows that its solution can be obtained with the formulas given in Items 1° and 2°, where variables and parameters should be renamed as follows: $\sigma\to 1+2\sigma$, $\alpha\to\frac{1}{2}\alpha$, $\beta\to 0$, and $\gamma\to\frac{1}{2}\beta$.

5°. Solutions for $\alpha=0$:

$$w(x,y)=\frac{\beta}{2(1+2\sigma)}\left(\frac{k_1x+k_2y}{\sqrt{k_1^2+k_2^2}}+C\right)^2-\frac{\gamma(1+2\sigma)}{2\beta\sigma},$$

$$w(x,y)=\frac{\beta}{4(1+\sigma)}\left[(x+C_1)^2+(y+C_2)^2\right]-\frac{\gamma(1+\sigma)}{\beta\sigma},$$

where k_1, k_2, C, C_1, and C_2 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

8. $\displaystyle \frac{\partial}{\partial x}\left[(a_1x+b_1y+c_1w+k_1)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[(a_2x+b_2y+c_2w+k_2)\frac{\partial w}{\partial y}\right]=0.$

This is a special case of equation 5.4.4.10 with $f(w)=c_1w+k_1$ and $g(w)=c_2w+k_2$.

9. $\displaystyle \frac{\partial}{\partial x}\left[(a_1x+b_1y+c_1w^n)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[(a_2x+b_2y+c_2w^k)\frac{\partial w}{\partial y}\right]=0.$

This is a special case of equation 5.4.4.10 with $f(w)=c_1w^n$ and $g(w)=c_2w^k$.

10. $\displaystyle a\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0.$

This is an *equation of steady transonic gas flow*.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1=C_1^{-3}C_2^2w(C_1x+C_3,C_2y+C_4)+C_5y+C_6,$$

where $C_1,\dots,C_6$ are arbitrary constants, is also a solution of the equation.

⊙ *Reference*: N. H. Ibragimov (1985).

2°. Solutions:

$$w(x,y)=C_1xy+C_2x+C_3y+C_4,$$

$$w(x,y)=-\frac{(x+C_1)^3}{3a(y+C_2)^2}+C_3y+C_4,$$

$$w(x,y)=\frac{a^2C_1^3}{39}(y+A)^{13}+\frac{2}{3}aC_1^2(y+A)^8(x+B)+3C_1(y+A)^3(x+B)^2-\frac{(x+B)^3}{3a(y+A)^2},$$

$$w(x,y)=-aC_1y^2+C_2y+C_3\pm\frac{4}{3C_1}(C_1x+C_4)^{3/2},$$

$$w(x,y)=-aA^3y^2-\frac{B^2}{aA^2}x+C_1y+C_2\pm\frac{4}{3}(Ax+By+C_3)^{3/2},$$

$$w(x,y)=\frac{1}{3}(Ay+B)(2C_1x+C_2)^{3/2}-\frac{aC_1^3}{12A^2}(Ay+B)^4+C_3y+C_4,$$

$$w(x,y)=-\frac{9aA^2}{y+C_1}+4A\left(\frac{x+C_2}{y+C_1}\right)^{3/2}-\frac{(x+C_2)^3}{3a(y+C_1)^2}+C_3y+C_4,$$

$$w(x,y)=-\frac{3}{7}aA^2(y+C_1)^7+4A(x+C_2)^{3/2}(y+C_1)^{5/2}-\frac{(x+C_2)^3}{3a(y+C_1)^2}+C_3y+C_4,$$

where A, B, $C_1,\dots,C_4$ are arbitrary constants (the first solution is degenerate).

3°. Self-similar solution:

$$w(x, y) = y^{-3k-2} U(z), \qquad z = xy^k,$$

where k is an arbitrary constant, and the function $U = U(z)$ is determined by the ordinary differential equation

$$aU'_z U''_{zz} + k^2 z^2 U''_{zz} - 5k(k+1)zU'_z + 3(k+1)(3k+2)U = 0.$$

4°. Generalized separable solution:

$$w(x, y) = \varphi_1(y) + \varphi_2(y)x^{3/2} + \varphi_3(y)x^3,$$

where the functions $\varphi_k = \varphi_k(y)$ are determined by the autonomous system of ordinary differential equations

$$\varphi_1'' + \tfrac{9}{8}a\varphi_2^2 = 0,$$
$$\varphi_2'' + \tfrac{45}{4}a\varphi_2\varphi_3 = 0,$$
$$\varphi_3'' + 18a\varphi_3^2 = 0,$$

where the prime stands for the differentiation with respect to y. The general solution of the first equation can be written out in implicit form (it is expressed in terms of the Weierstrass function).

5°. Generalized separable solution cubic in x:

$$w(x, y) = \psi_1(y) + \psi_2(y)x + \psi_3(y)x^2 + \psi_4(y)x^3,$$

where the functions $\psi_k = \psi_k(y)$ are determined by the autonomous system of ordinary differential equations

$$\psi_1'' + 2a\psi_2\psi_3 = 0,$$
$$\psi_2'' + 2a(2\psi_3^2 + 3\psi_2\psi_4) = 0,$$
$$\psi_3'' + 18a\psi_3\psi_4 = 0,$$
$$\psi_4'' + 18a\psi_4^2 = 0.$$

A particular solution of the system is given by

$$\psi_1(y) = -2a\int_{y_0}^{y}(y-t)\psi_2(t)\psi_3(t)\,dt + B_1 y + B_2,$$

$$\psi_2(y) = C_3(y+A)^{-1} + C_4(y+A)^2 - aC_1^2(y+A)^{-2} - 2aC_1C_2(y+A)^3 - \frac{2}{27}aC_2^2(y+A)^8,$$

$$\psi_3(y) = C_1(y+A)^{-2} + C_2(y+A)^3, \qquad \psi_4(y) = -\frac{1}{3a}(y+A)^{-2},$$

where A, B_1, B_2, C_1, …, C_4 are arbitrary constants and y_0 is any number.

6°. Generalized separable solution:

$$w(x, y) = \eta(y)\theta(x) - aC_1\int_0^y (y-t)\eta^2(t)\,dt + C_2 y + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, and the functions $\eta(y)$ and $\theta(x)$ satisfy the autonomous ordinary differential equations (C_4 is an arbitrary constant)

$$\eta''_{yy} + aC_4\eta^2 = 0, \qquad (1)$$
$$\theta'_x\theta''_{xx} = C_4\theta + C_1. \qquad (2)$$

The solutions to equations (1) and (2) can be written out in implicit form:

$$\int\left(C_5 - \tfrac{2}{3}aC_4\eta^3\right)^{-1/2} d\eta = C_6 \pm y,$$

$$\int\left(\tfrac{3}{2}C_4\theta^2 + 3C_1\theta + C_7\right)^{-1/3} d\theta = x + C_8,$$

where C_5, C_6, C_7, and C_8 are arbitrary constants.

⊙ *References for equation* 5.1.6.10: S. S. Titov (1988), S. R. Svirshchevskii (1995), A. D. Polyanin and V. F. Zaitsev (2002).

11. $\dfrac{\partial^2 w}{\partial y^2} + \dfrac{a}{y}\dfrac{\partial w}{\partial y} + b\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} = 0.$

For $b < 0$, this equation describes a transonic gas flow.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-3}C_2^2 w(C_1x + C_3, C_2y) + C_4y^{1-a} + C_5,$$

where $C_1, \dots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,y) = -\frac{bC_1}{4(a+1)}y^2 + C_2y^{1-a} + C_3 \pm \frac{2}{3C_1}(C_1x + C_4)^{3/2},$$

where $C_1, \dots, C_4$ are arbitrary constants.

3°. Generalized separable solutions:

$$w(x,y) = -\frac{9A^2b}{16(n+1)(2n+1+a)}(y+C_1)^{2n+2} + A(y+C_1)^n(x+C_2)^{3/2} + \frac{a-3}{9b}\frac{(x+C_2)^3}{(y+C_1)^2},$$

where A, C_1, and C_2 are arbitrary constants, and the $n = n_{1,2}$ are roots of the quadratic equation

$$n^2 + (a-1)n + \tfrac{5}{4}(a-3) = 0.$$

4°. Generalized separable solution:

$$w(x,y) = (Ay^{1-a} + B)(2C_1x + C_2)^{3/2} + 9bC_1^3\theta(y),$$

where A, B, C_1, and C_2 are arbitrary constants, and the function $\theta = \theta(y)$ is determined by the second-order linear ordinary differential equation

$$\theta''_{yy} + \frac{a}{y}\theta'_y + (Ay^{1-a} + B)^2 = 0.$$

Integrating yields

$$\theta(y) = -\frac{B^2}{2(a+1)}y^2 - \frac{AB}{3-a}y^{3-a} - \frac{A^2}{2(2-a)(3-a)}y^{4-2a} + C_3y^{1-a} + C_4.$$

5°. Self-similar solution:

$$w(x,y) = y^{-3k-2}U(z), \qquad z = xy^k,$$

where k is an arbitrary constant, and the function $U = U(z)$ is determined by the ordinary differential equation

$$bU'_zU''_{zz} + k^2z^2U''_{zz} + k(a-5k-5)zU'_z + (3k+2)(3k+3-a)U = 0.$$

6°. Generalized separable solution:

$$w(x,y) = \varphi_1(y) + \varphi_2(y)x^{3/2} + \varphi_3(y)x^3,$$

where the functions $\varphi_k = \varphi_k(y)$ are determined by the system of ordinary differential equations

$$\varphi_1'' + \frac{a}{y}\varphi_1' + \frac{9}{8}b\varphi_2^2 = 0,$$
$$\varphi_2'' + \frac{a}{y}\varphi_2' + \frac{45}{4}b\varphi_2\varphi_3 = 0,$$
$$\varphi_3'' + \frac{a}{y}\varphi_3' + 18b\varphi_3^2 = 0,$$

where the prime stands for the differentiation with respect to y.

7°. Generalized separable solution cubic in x:

$$w(x,y) = \psi_1(y) + \psi_2(y)x + \psi_3(y)x^2 + \psi_4(y)x^3,$$

where the functions $\psi_k = \psi_k(y)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
&\psi_1'' + \frac{a}{y}\psi_1' + 2b\psi_2\psi_3 = 0,\\
&\psi_2'' + \frac{a}{y}\psi_2' + 2b(2\psi_3^2 + 3\psi_2\psi_4) = 0,\\
&\psi_3'' + \frac{a}{y}\psi_3' + 18b\psi_3\psi_4 = 0,\\
&\psi_4'' + \frac{a}{y}\psi_4' + 18b\psi_4^2 = 0.
\end{aligned}$$

8°. Generalized separable solution:

$$w(x,y) = \xi(y) + \eta(y)\theta(x).$$

Here, the functions $\xi(y)$ and $\eta(y)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
&\eta_{yy}'' + \frac{a}{y}\eta_y' + bC_1\eta^2 = 0,\\
&\xi_{yy}'' + \frac{a}{y}\xi_y' + bC_2\eta^2 = 0,
\end{aligned}$$

where C_1 and C_2 are arbitrary constants, and the function $\theta = \theta(x)$ is determined by the autonomous ordinary differential equation

$$\theta_x'\theta_{xx}'' = C_1\theta + C_2.$$

Its solution can be written out in implicit form:

$$\int \left(\tfrac{3}{2}C_1\theta^2 + 3C_2\theta + C_3\right)^{-1/3} d\theta = x + C_4,$$

where C_3 and C_4 are arbitrary constants.

⊙ *References for equation* 5.1.6.11: S. S. Titov (1988), S. R. Svirshchevskii (1995), A. D. Polyanin and V. F. Zaitsev (2002).

5.2. Equations with Exponential Nonlinearities

5.2.1. Equations of the Form $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a + be^{\beta w} + ce^{\gamma w}$

1. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = ae^{\beta w}.$

This equation occurs in combustion theory and is a special case of equation 5.4.1.1 with $f(w) = ae^{\beta w}$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$\begin{aligned}
w_1 &= w(\pm C_1^\beta x + C_2, \pm C_1^\beta y + C_3) + 2\ln|C_1|,\\
w_2 &= w(x\cos\lambda - y\sin\lambda,\ x\sin\lambda + y\cos\lambda),
\end{aligned}$$

where C_1, C_2, C_3, and λ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Solutions:

$$w(x,y)=\frac{1}{\beta}\ln\left[\frac{2(A^2+B^2)}{a\beta(Ax+By+C)^2}\right] \qquad \text{if } a\beta>0,$$

$$w(x,y)=\frac{1}{\beta}\ln\left[\frac{2(A^2+B^2)}{a\beta\sinh^2(Ax+By+C)}\right] \qquad \text{if } a\beta>0,$$

$$w(x,y)=\frac{1}{\beta}\ln\left[\frac{-2(A^2+B^2)}{a\beta\cosh^2(Ax+By+C)}\right] \qquad \text{if } a\beta<0,$$

$$w(x,y)=\frac{1}{\beta}\ln\left[\frac{2(A^2+B^2)}{a\beta\cos^2(Ax+By+C)}\right] \qquad \text{if } a\beta>0,$$

$$w(x,y)=\frac{1}{\beta}\ln\left(\frac{8C}{a\beta}\right)-\frac{2}{\beta}\ln\left|(x+A)^2+(y+B)^2-C\right|,$$

where A, B, and C are arbitrary constants. The first four solutions are of traveling-wave type and the last one is a radial symmetric solution with center at the point $(-A,-B)$.

Example. For $a=\beta=1$, the boundary value problem for the circle $r=\sqrt{x^2+y^2}\le 1$ with the boundary condition $w\big|_{r=1}=0$ has the following two solutions (see the last solution in Item 2° with $a=\beta=1$, $A=B=0$, and $C=k$):

$$w(r)=\ln\frac{8k}{(k-r^2)^2}, \qquad k=5\pm 2\sqrt{6}.$$

The first solution is bounded at every point inside the circle, $r\le 1$, and the second one has a singularity at the circumference $r=\sqrt{k}$.

⦿ *References*: D. A. Frank-Kamenetskii (1987), V. F. Zaitsev and A. D. Polyanin (1996).

3°. Functional separable solutions:

$$w(x,y)=-\frac{2}{\beta}\ln\left[C_1e^{ky}\pm\frac{\sqrt{2a\beta}}{2k}\cos(kx+C_2)\right],$$

$$w(x,y)=\frac{1}{\beta}\ln\frac{2k^2(B^2-A^2)}{a\beta[A\cosh(kx+C_1)+B\sin(ky+C_2)]^2},$$

$$w(x,y)=\frac{1}{\beta}\ln\frac{2k^2(A^2+B^2)}{a\beta[A\sinh(kx+C_1)+B\cos(ky+C_2)]^2},$$

where A, B, C_1, C_2, and k are arbitrary constants (x and y can be swapped to give another three solutions).

⦿ *Reference*: S. N. Aristov (1999).

4°. General solution:

$$w(x,y)=-\frac{2}{\beta}\ln\frac{\sqrt{|a|\beta^2}\,\left[1+\operatorname{sign}(a\beta)\Phi(z)\overline{\Phi(z)}\right]}{4|\Phi'_z(z)|},$$

where $\Phi=\Phi(z)$ is an arbitrary analytic (holomorphic) function of the complex variable $z=x+iy$ with nonzero derivative, and the bar over a symbol denotes the complex conjugate.

⦿ *References*: I. N. Vekua (1960), I. Kh. Sabitov (2001).

5°. The original equation is related to the linear equation

$$\frac{\partial^2U}{\partial x^2}+\frac{\partial^2U}{\partial y^2}=0 \tag{1}$$

by the Bäcklund transformation

$$\frac{\partial U}{\partial x}+\tfrac{1}{2}\beta\frac{\partial w}{\partial y}=\left(\tfrac{1}{2}a\beta\right)^{1/2}\exp\left(\tfrac{1}{2}\beta w\right)\sin U, \tag{2}$$

$$\frac{\partial U}{\partial y}-\tfrac{1}{2}\beta\frac{\partial w}{\partial x}=\left(\tfrac{1}{2}a\beta\right)^{1/2}\exp\left(\tfrac{1}{2}\beta w\right)\cos U. \tag{3}$$

Suppose there is a (particular) solution $U = U(x, y)$ of the Laplace equation (1). Then (2) can be treated as a first-order ordinary differential equation for $w = w(y)$ with parameter x, which can be reduced to a linear equation with the help of the change of variable $z = \exp\left(-\frac{1}{2}\beta w\right)$. Finally, we have

$$w = -\frac{2}{\beta}F - \frac{2}{\beta}\ln\left[\Psi(x) - k\int e^{-F}\sin U\,dy\right], \qquad F = \int\left(\frac{\partial U}{\partial x}\right)dy,$$

where $k = \left(\frac{1}{2}a\beta\right)^{1/2}$; in the integration x is treated as a parameter. The function $\Psi(x)$ is determined after substituting this expression for w into equation (3).

⊙ *Reference*: R. K. Bullough and P. J. Caudrey (1980).

2. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = ae^{\beta w} + be^{2\beta w}.$

1°. Traveling-wave solution for $b\beta > 0$:

$$w(x, y) = -\frac{1}{\beta}\ln\left\{-\frac{b}{a} + C_1\exp\left[a\sqrt{\frac{\beta}{b}}\left(x\sin C_2 + y\cos C_2\right)\right]\right\},$$

where C_1 and C_2 are arbitrary constants.

2°. Traveling-wave solution (generalizes the solution of Item 1°):

$$w(x, y) = -\frac{1}{\beta}\ln\left[-\frac{a\beta}{C_1^2 + C_2^2} + C_3\exp(C_1x + C_2y) + \frac{a^2\beta^2 - b\beta(C_1^2 + C_2^2)}{4C_3(C_1^2 + C_2^2)^2}\exp(-C_1x - C_2y)\right],$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Traveling-wave solution:

$$w(x, y) = -\frac{1}{\beta}\ln\left[\frac{a\beta}{C_1^2 + C_2^2} + \frac{\sqrt{a^2\beta^2 + b\beta(C_1^2 + C_2^2)}}{C_1^2 + C_2^2}\sin(C_1x + C_2y + C_3)\right].$$

3. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = ae^{\beta w} - be^{-\beta w}.$

The transformation

$$w(x, y) = u(x, y) + k, \quad k = \frac{1}{2\beta}\ln\frac{b}{a}$$

leads to an equation of the form 5.3.1.1:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2\sqrt{ab}\,\sinh(\beta u).$$

4. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = ae^{\beta w} + be^{-2\beta w}.$

Functional separable solution:

$$w(x, y) = \frac{1}{\beta}\ln[\varphi(x) + \psi(y)],$$

where the functions $\varphi(x)$ and $\psi(y)$ are determined by the first-order autonomous ordinary differential equations

$$(\varphi_x')^2 = 2a\beta\varphi^3 + C_1\varphi^2 + C_2\varphi + C_3,$$
$$(\psi_y')^2 = 2a\beta\psi^3 - C_1\psi^2 + C_2\psi - C_3 - b\beta,$$

where C_1, C_2, and C_3 are arbitrary constants. Solving these equations for the derivatives, one obtains separable equations.

⊙ *References*: A. M. Grundland and E. Infeld (1992), J. Miller (Jr.) and L. A. Rubel (1993), R. Z. Zhdanov (1994), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

5. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a + be^{\beta w} + ce^{2\beta w}$.

The substitution $u = e^{-\beta w}$ leads to a equation with a quadratic nonlinearity of the form 5.1.6.7:

$$u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 + a\beta u^2 + b\beta u + c\beta = 0.$$

5.2.2. Equations of the Form $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = f(x, y, w)$

1. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = Ae^{\alpha x + \beta y} e^{\mu w}$.

The substitution $U = \alpha x + \beta y + \mu w$ leads to an equation of the form 5.2.1.1:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = A\mu e^U.$$

2. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = Ae^{\alpha xy + \beta x + \gamma y} e^{\mu w}$.

The substitution $U = \alpha xy + \beta x + \gamma y + \mu w$ leads to an equation of the form 5.2.1.1:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = A\mu e^U.$$

3. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = A(x^2 + y^2)e^{\beta w}$.

The transformation

$$w = U(z, \zeta), \quad z = \tfrac{1}{2}(x^2 - y^2), \quad \zeta = xy$$

leads to a simpler equation of the form 5.2.1.1:

$$\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial \zeta^2} = Ae^{\beta U}.$$

4. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = A(x^2 + y^2)^k e^{\beta w}$.

This is a special case of equation 5.4.1.3 with $f(w) = Ae^{\beta w}$.

5.2.3. Equations of the Form $\dfrac{\partial}{\partial x}\left[f_1(x,y)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[f_2(x,y)\dfrac{\partial w}{\partial y}\right] = g(w)$

► *Equations of this form are encountered in stationary problems of heat and mass transfer and combustion theory. Here, f_1 and f_2 are the principal thermal diffusivities (diffusion coefficients) dependent on the space coordinates x and y, and $g = g(w)$ is a source function that defines the law of heat (substance) release or absorption.*

1. $\dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m \dfrac{\partial w}{\partial y}\right) = ce^{\beta w}$.

This is a special case of equation 5.4.3.1 with $f(w) = ce^{\beta w}$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = w\Big(C^{\frac{2}{2-n}}x,\, C^{\frac{2}{2-m}}y\Big) + \frac{2}{\beta}\ln C,$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution for $n \neq 2$ and $m \neq 2$:

$$w = w(\xi), \qquad \xi = \big[b(2-m)^2x^{2-n} + a(2-n)^2y^{2-m}\big]^{1/2}.$$

Here, the function $w = w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{A}{\xi}w'_\xi = Be^{\beta w}, \tag{1}$$

where

$$A = \frac{4-nm}{(2-n)(2-m)}, \quad B = \frac{4c}{ab(2-n)^2(2-m)^2}.$$

3°. Below are some exact solutions of equation (1).

3.1. For $A \neq 1$, there is a solution of the form

$$w(\xi) = -\frac{1}{\beta}\ln\left[\frac{B\beta}{2(1-A)}\xi^2\right].$$

3.2. For $A = 0$, which corresponds to $m = \dfrac{4}{n}$ and $B = \dfrac{cn^2}{ab(2-n)^4}$, we obtain from (1) several more families of exact solutions to the original equation:

$$w(\xi) = \frac{1}{\beta}\ln\left[\frac{2}{\beta B(\xi+C)^2}\right] \qquad \text{if } \beta B > 0,$$

$$w(\xi) = \frac{1}{\beta}\ln\left[\frac{2\lambda^2}{\beta B\cos^2(\lambda\xi+C)}\right] \qquad \text{if } \beta B > 0,$$

$$w(\xi) = \frac{1}{\beta}\ln\left[\frac{2\lambda^2}{\beta B\sinh^2(\lambda\xi+C)}\right] \qquad \text{if } \beta B > 0,$$

$$w(\xi) = \frac{1}{\beta}\ln\left[\frac{-2\lambda^2}{\beta B\cosh^2(\lambda\xi+C)}\right] \qquad \text{if } \beta B < 0,$$

$$w(\xi) = \frac{1}{\beta}\ln\left[\frac{-8\lambda^2C_1C_2}{\beta B\big(C_1e^{\lambda\xi}+C_2e^{-\lambda\xi}\big)^2}\right],$$

where λ, C, C_1, and C_2 are arbitrary constants.

3.3. For $A = 1$, which corresponds to $m = \dfrac{n}{n-1}$, another family of exact solutions follows from (1):

$$w(\xi) = \frac{1}{\beta}\ln\left(-\frac{8C}{\beta B}\right) - \frac{2}{\beta}\ln(\xi^2+C), \qquad B = \frac{4c(n-1)^2}{ab(2-n)^4},$$

where C is an arbitrary constant.

4°. There is an exact solution of the form

$$w(x,y) = U(z) + \frac{n-2}{\beta}\ln x, \qquad z = yx^{\frac{n-2}{2-m}}.$$

2. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)=ce^{\lambda w}.$

This is a special case of equation 5.4.3.6 with $f(w)=ce^{\lambda w}$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1=w\Big(x-\frac{2}{\beta}\ln C,\ y-\frac{2}{\mu}\ln C\Big)+\frac{2}{\lambda}\ln C,$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution:

$$w=-\frac{1}{\lambda}\ln\left[c\lambda\left(\frac{e^{-\beta x}}{a\beta^2}+\frac{e^{-\mu y}}{b\mu^2}\right)\right].$$

3°. Functional separable solution for $\beta\mu\neq 0$ (generalizes the solution of Item 2°):

$$w=w(\xi),\qquad \xi=\big(b\mu^2e^{-\beta x}+a\beta^2e^{-\mu y}\big)^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}-\frac{1}{\xi}w'_\xi=Ae^{\lambda w},\qquad A=\frac{4c}{ab\beta^2\mu^2}.$$

4°. There is an exact solution of the form

$$w(x,y)=U(z)+\frac{\beta}{\lambda}x,\qquad z=y-\frac{\beta}{\mu}x.$$

3. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\beta y}\dfrac{\partial w}{\partial y}\right)=ce^{\lambda w}.$

This is a special case of equation 5.4.3.8 with $f(w)=ce^{\lambda w}$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1=w\Big(C^{\frac{2}{2-n}}x,\ y-\frac{2}{\beta}\ln C\Big)+\frac{2}{\lambda}\ln C,$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution:

$$w=-\frac{1}{\lambda}\ln\left\{\frac{c\lambda(2-n)}{(1-n)}\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2}\right]\right\}.$$

3°. Functional separable solution for $n\neq 2$ and $\beta\neq 0$ (generalizes the solution of Item 2°):

$$w=w(\xi),\qquad \xi^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2}\right],$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}+\frac{A}{\xi}w'_\xi=ce^{\lambda w},\qquad A=\frac{n}{2-n}.$$

4°. There is an exact solution of the form

$$w(x,y)=U(z)+\frac{n-2}{\lambda}\ln x,\qquad z=y+\frac{2-n}{\beta}\ln x.$$

4. $\dfrac{\partial}{\partial x}\left[(ay+c)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[(bx+s)\dfrac{\partial w}{\partial y}\right]=ke^{\lambda w}.$

This is a special case of equation 5.4.4.1 with $f(w)=ke^{\lambda w}$.

The equation can be rewritten in the form

$$(ay+c)\frac{\partial^2 w}{\partial x^2}+(bx+s)\frac{\partial^2 w}{\partial y^2}=ke^{\lambda w}.$$

5. $\dfrac{\partial}{\partial x}\left[(a_1x+b_1y+c_1)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[(a_2x+b_2y+c_2)\dfrac{\partial w}{\partial y}\right]=ke^{\lambda w}.$

This is a special case of equation 5.4.4.2 with $f(w)=ke^{\lambda w}$.

5.2.4. Equations of the Form $\dfrac{\partial}{\partial x}\left[f_1(w)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[f_2(w)\dfrac{\partial w}{\partial y}\right]=g(w)$

► *Equations of this form are encountered in stationary problems of heat and mass transfer and combustion theory. Here, $f_1=f_1(w)$ and $f_2=f_2(w)$ are the principal thermal diffusivities (diffusion coefficients) dependent on the temperature (concentration) w, and $g=g(w)$ is a source function that defines the law of heat (substance) release or absorption. Simple solutions dependent on a single coordinate, $w=w(x)$ and $w=w(y)$, are not treated in this subsection.*

1. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left(ae^{\beta w}\dfrac{\partial w}{\partial y}\right)=0.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1=w(C_1x+C_3,\pm C_1C_2^{\beta}y+C_4)-2\ln|C_2|,$$

where $C_1,\dots,C_4$ are arbitrary constants, are also solutions of the equation.

2°. Additive separable solutions:

$$\begin{aligned}
w(x,y)&=\frac{1}{\beta}\ln(Ay+B)+Cx+D,\\
w(x,y)&=\frac{1}{\beta}\ln(-aA^2y^2+By+C)-\frac{2}{\beta}\ln(-aAx+D),\\
w(x,y)&=\frac{1}{\beta}\ln(Ay^2+By+C)+\frac{1}{\beta}\ln\left[\frac{p^2}{aA\cosh^2(px+q)}\right],\\
w(x,y)&=\frac{1}{\beta}\ln(Ay^2+By+C)+\frac{1}{\beta}\ln\left[\frac{p^2}{-aA\cos^2(px+q)}\right],\\
w(x,y)&=\frac{1}{\beta}\ln(Ay^2+By+C)+\frac{1}{\beta}\ln\left[\frac{p^2}{-aA\sinh^2(px+q)}\right],
\end{aligned}$$

where A, B, C, D, p, and q are arbitrary constants.

3°. Traveling-wave solution in implicit form:

$$k_1^2w+\frac{ak_2^2}{\beta}e^{\beta w}=C_1(k_1x+k_2y)+C_2,$$

where C_1, C_2, k_1, and k_2 are arbitrary constants.

4°. Self-similar solution (A and B are arbitrary constants):

$$u = u(z), \quad z = \frac{x + A}{y + B},$$

where the function $u(z)$ is determined by the ordinary differential equation

$$(z^2 u'_z)'_z + (ae^{\beta u} u'_z)'_z = 0.$$

This equation admits the first integral

$$\bigl(z^2 + ae^{\beta u}\bigr) u'_z = C.$$

Treating u as the independent variable, we get a Riccati equation for $z = z(u)$,

$$Cz'_u = z^2 + ae^{\beta u},$$

whose solution is expressed in terms of Bessel functions.

5°. Solution (generalizes the solution of Item 4°):

$$w = U(\xi) - \frac{2(k+1)}{\beta} \ln |x|, \quad \xi = y|x|^k,$$

where k is an arbitrary constant and the function $U(\xi)$ is determined by the ordinary differential equation

$$\frac{2(k+1)}{\beta} + k(k-1)\xi U'_\xi + k^2 \xi^2 U''_{\xi\xi} + (ae^{\beta U} U'_\xi)'_\xi = 0.$$

6°. There are exact solutions of the following forms:

$$w(x, y) = F(\eta) - \frac{2}{\beta} \ln |x|, \quad \eta = y + k \ln |x|;$$

$$w(x, y) = H(\zeta) - \frac{2}{\beta} x, \quad \zeta = ye^x;$$

where k is an arbitrary constant.

7°. For other solutions, see equation 5.4.4.8 with $f(w) = 1$ and $g(w) = ae^{\beta w}$.

2. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial y}\left(ae^{\beta w} \frac{\partial w}{\partial y}\right) = be^{\lambda w}.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^\lambda x + C_2, \pm C_1^{\lambda-\beta} y + C_3) + 2 \ln |C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution:

$$w = u(z), \quad z = k_1 x + k_2 y,$$

where k_1 and k_2 are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$k_1^2 u''_{zz} + ak_2^2 (e^{\beta u} u'_z)'_z = be^{\lambda u}.$$

The substitution $\Theta(u) = (u'_z)^2$ leads to the first-order linear equation

$$(k_1^2 + ak_2^2 e^{\beta u})\Theta'_u + 2ak_2^2 \beta e^{\beta u} \Theta = 2be^{\lambda u}.$$

3°. Solution:

$$w = U(\xi) - \frac{2}{\lambda} \ln |x|, \quad \xi = yx^{\frac{\beta-\lambda}{\lambda}},$$

where the function $U(\xi)$ is determined by the ordinary differential equation

$$\frac{2}{\lambda} + \frac{(\beta-\lambda)(\beta-2\lambda)}{\lambda^2} \xi U'_\xi + \frac{(\beta-\lambda)^2}{\lambda^2} \xi^2 U''_{\xi\xi} + (ae^{\beta U} U'_\xi)'_\xi = be^{\lambda U}.$$

3. $\dfrac{\partial}{\partial x}\left(ae^{\beta w}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\gamma w}\dfrac{\partial w}{\partial y}\right)=0.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1=w(\pm C_1C_2^{\beta}x+C_3,\pm C_1C_2^{\gamma}y+C_4)-2\ln|C_2|,$$

where $C_1,\ldots,C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$ak_1^2\beta^{-1}e^{\beta w}+bk_2^2\gamma^{-1}e^{\gamma w}=C_1(k_1x+k_2y)+C_2,$$

where C_1, C_2, k_1, and k_2 are arbitrary constants.

3°. Additive separable solution:

$$w(x,y)=\varphi(x)+\psi(y). \qquad (1)$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the autonomous ordinary differential equations (A is an arbitrary constant)

$$\begin{aligned}&\varphi''_{xx}+\beta(\varphi'_x)^2=Abe^{(\gamma-\beta)\varphi},\\&\psi''_{yy}+\gamma(\psi'_y)^2=-Aae^{(\beta-\gamma)\psi},\end{aligned} \qquad (2)$$

which are independent of each other.

Integrating yields the general solutions to the equations of (2) in implicit form:

$$\int e^{\beta\varphi}\left[\frac{2Ab}{\beta+\gamma}e^{(\beta+\gamma)\varphi}+B_1\right]^{-1/2}d\varphi=C_1\pm x,$$

$$\int e^{\gamma\psi}\left[-\frac{2Aa}{\beta+\gamma}e^{(\beta+\gamma)\psi}+B_2\right]^{-1/2}d\psi=C_2\pm y,$$

where B_1, B_2, C_1, and C_2 are arbitrary constants; $\beta+\gamma\neq 0$.

Remark. Particular solutions to equations (2) are given by

$$\varphi(x)=\frac{1}{\beta-\gamma}\ln\left[\frac{Ab(\beta-\gamma)^2}{2(\beta+\gamma)}(x+C_3)^2\right],$$

$$\psi(y)=\frac{1}{\gamma-\beta}\ln\left[-\frac{Aa(\beta-\gamma)^2}{2(\beta+\gamma)}(y+C_4)^2\right],$$

where C_4 and C_2 are arbitrary constants.

4°. There are exact solutions of the following forms:

$$w(x,y)=F(z)+\frac{2k}{\beta-\gamma}\ln|x|,\qquad z=y|x|^{k-1};$$

$$w(x,y)=G(\xi)+\frac{2}{\beta-\gamma}\ln|x|,\qquad \xi=y+k\ln|x|;$$

$$w(x,y)=H(\eta)+2x,\qquad \eta=ye^{(\beta-\gamma)x};$$

where k is an arbitrary constant.

5°. For other exact solutions of the original equation, see 5.4.4.8 with $f(w)=ae^{\beta w}$ and $g(w)=be^{\gamma w}$.

4. $a\dfrac{\partial}{\partial x}\left(e^{\beta w}\dfrac{\partial w}{\partial x}\right)+b\dfrac{\partial}{\partial y}\left(e^{\gamma w}\dfrac{\partial w}{\partial y}\right)=ce^{\lambda w}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1=w(\pm C_1^{\lambda-\beta}x+C_2,\pm C_1^{\lambda-\gamma}y+C_3)+2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There are exact solutions of the following forms:

$$w(x,y) = F(z), \qquad z = k_1 x + k_2 y;$$

$$w(x,y) = G(\xi) + \frac{2}{\beta - \lambda} \ln |x|, \qquad \xi = y|x|^{\frac{\lambda-\gamma}{\beta-\lambda}}.$$

5.2.5. Other Equations Involving Arbitrary Parameters

1. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{a}{x}\frac{\partial w}{\partial x} + \frac{b}{y}\frac{\partial w}{\partial y} = ce^{\beta w}.$

This is a special case of equation 5.4.2.4 with $f(\xi, w) = ce^{\beta w}$.

2. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = ae^{\beta w}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right].$

This is a special case of equation 5.4.2.9 with $f(w) = ae^{\beta w}$.

The substitution $U = \int \exp\left(-\frac{a}{\beta}e^{\beta w}\right) dw$ leads to the two-dimensional Laplace equation for $U = U(x,y)$:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

3. $\displaystyle \frac{\partial^2 w}{\partial x^2} + ae^{\beta w}\frac{\partial^2 w}{\partial y^2} = 0, \qquad a > 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_{\pm} = w(C_1 x + C_3, \pm C_1 C_2^{\beta} y + C_4,) - 2\ln|C_2|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,y) = Axy + By + Cx + D,$$

$$w(x,y) = \frac{1}{\beta}\ln\left[\frac{B^2}{a}\frac{(y+A)^2}{\sinh^2(Bx+C)}\right], \qquad w(x,y) = \frac{1}{\beta}\ln\left[\frac{1}{aA^2}\frac{\sinh^2(Ay+B)}{(x+C)^2}\right],$$

$$w(x,y) = \frac{1}{\beta}\ln\left[\frac{B^2}{a}\frac{(y+A)^2}{\cos^2(Bx+C)}\right], \qquad w(x,y) = \frac{1}{\beta}\ln\left[\frac{1}{aA^2}\frac{\cos^2(Ay+B)}{(x+C)^2}\right],$$

$$w(x,y) = \frac{1}{\beta}\ln\left[\frac{C^2}{aA^2}\frac{\cos^2(Ay+B)}{\sinh^2(Cx+D)}\right], \quad w(x,y) = \frac{1}{\beta}\ln\left[\frac{C^2}{aA^2}\frac{\sinh^2(Ay+B)}{\cos^2(Cx+D)}\right],$$

$$w(x,y) = \frac{1}{\beta}\ln\left[\frac{C^2}{aA^2}\frac{\sinh^2(Ay+B)}{\sinh^2(Cx+D)}\right], \quad w(x,y) = \frac{1}{\beta}\ln\left[\frac{C^2}{aA^2}\frac{\cos^2(Ay+B)}{\cos^2(Cx+D)}\right],$$

where A, B, C, and D are arbitrary constants. The first solution is degenerate and the others are representable as the sum of functions with different arguments.

3°. Self-similar solution:

$$w = w(z), \qquad z = y/x,$$

where the function $w(z)$ is determined by the ordinary differential equation

$$(z^2 + ae^{\beta w})w''_{zz} + 2zw'_z = 0.$$

4°. Solution:

$$w = U(\zeta) - \frac{2(k+1)}{\beta} \ln|x + C_1|, \qquad \zeta = (y + C_2)(x + C_1)^k,$$

where C_1, C_2, and k are arbitrary constants, and the function $U = U(\zeta)$ is determined by the ordinary differential equation

$$(k^2\zeta^2 + ae^{\beta U})U''_{\zeta\zeta} + k(k-1)\zeta U'_\zeta + \frac{2(k+1)}{\beta} = 0.$$

4. $\displaystyle \frac{\partial^2 w}{\partial x^2} + ae^{\beta w}\frac{\partial^2 w}{\partial y^2} = be^{\lambda w}.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\lambda-\beta} y + C_2, \pm C_1^{\lambda} x + C_3) + 2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution:

$$w = u(z), \qquad z = k_1 x + k_2 y,$$

where k_1 and k_2 are arbitrary constants, and the function $u(z)$ is determined by the autonomous ordinary differential equation

$$(k_1^2 + ak_2^2 e^{\beta u})u''_{zz} = be^{\lambda u}.$$

Its solution can be written out in implicit form as

$$\int \frac{du}{\sqrt{F(u)}} = C_1 \pm z, \qquad F(u) = 2b\int \frac{e^{\lambda u}\,du}{k_1^2 + ak_2^2 e^{\beta u}} + C_2,$$

where C_1 and C_2 are arbitrary constants.

3°. Solution:

$$w = U(\xi) - \frac{2}{\lambda}\ln|x|, \qquad \xi = y|x|^{\frac{\beta-\lambda}{\lambda}},$$

where the function $U(\xi)$ is determined by the ordinary differential equation

$$\frac{2}{\lambda} + \frac{(\beta-\lambda)(\beta-2\lambda)}{\lambda^2}\xi U'_\xi + \frac{(\beta-\lambda)^2}{\lambda^2}\xi^2 U''_{\xi\xi} + ae^{\beta U}U''_{\xi\xi} = be^{\lambda U}.$$

5. $\displaystyle ax^n\frac{\partial^2 w}{\partial x^2} + by^m\frac{\partial^2 w}{\partial y^2} = ce^{\beta w}.$

This is a special case of equation 5.4.5.5 with $k = s = 0$ and $f(w) = ce^{\beta w}$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = w\Bigl(C^{\frac{2}{2-n}}x,\, C^{\frac{2}{2-m}}y\Bigr) + \frac{2}{\beta}\ln C,$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution for $n \neq 2$ and $m \neq 2$:

$$w = w(\xi), \qquad \xi = \bigl[b(2-m)^2x^{2-n} + a(2-n)^2y^{2-m}\bigr]^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{A}{\xi}w'_\xi = Be^{\beta w}, \tag{1}$$

where

$$A = \frac{3nm - 4n - 4m + 4}{(2-n)(2-m)}, \quad B = \frac{4c}{ab(2-n)^2(2-m)^2}.$$

3°. Below are some exact solutions of equation (1).

3.1. For $A \neq 1$, equation (1) admits an exact solution of the form

$$w(\xi) = -\frac{1}{\beta} \ln\left[\frac{B\beta}{2(1-A)}\xi^2\right].$$

3.2. For $A = 0$, which corresponds to $m = \dfrac{4n-4}{3n-4}$ and $B = \dfrac{c(3n-4)^2}{ab(2-n)^4}$, we obtain from (1) several more families of exact solutions to the original equation:

$$w(\xi) = \frac{1}{\beta} \ln\left[\frac{2\lambda^2}{\beta B \cos^2(\lambda\xi + C)}\right] \quad \text{if } \beta B > 0,$$

$$w(\xi) = \frac{1}{\beta} \ln\left[\frac{2\lambda^2}{\beta B \sinh^2(\lambda\xi + C)}\right] \quad \text{if } \beta B > 0,$$

$$w(\xi) = \frac{1}{\beta} \ln\left[\frac{-2\lambda^2}{\beta B \cosh^2(\lambda\xi + C)}\right] \quad \text{if } \beta B < 0,$$

where λ and C are arbitrary constants.

3.3. For $A = 1$, which corresponds to $m = \dfrac{n}{n-1}$, another family of exact solutions follows from (1):

$$w(\xi) = \frac{1}{\beta} \ln\left(-\frac{8C}{\beta B}\right) - \frac{2}{\beta} \ln(\xi^2 + C), \qquad B = \frac{4c(n-1)^2}{ab(2-n)^4},$$

where C is an arbitrary constant.

6. $\displaystyle ae^{\beta x}\frac{\partial^2 w}{\partial x^2} + be^{\mu y}\frac{\partial^2 w}{\partial y^2} = ce^{\lambda w}.$

This is a special case of equation 5.4.5.7 with $k = s = 0$ and $f(w) = ce^{\beta w}$.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = w\left(x - \frac{2}{\beta}\ln C,\ y - \frac{2}{\mu}\ln C\right) + \frac{2}{\lambda}\ln C,$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Functional separable solution for $\beta\mu \neq 0$:

$$w = w(\xi), \qquad \xi = \left(b\mu^2 e^{-\beta x} + a\beta^2 e^{-\mu y}\right)^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{3}{\xi}w'_\xi = Ae^{\lambda w}, \qquad A = \frac{4c}{ab\beta^2\mu^2},$$

which admits the exact solution

$$w = -\frac{1}{\lambda}\ln\left(-\frac{1}{4}A\lambda\xi^2\right).$$

7. $\displaystyle ax^n\frac{\partial^2 w}{\partial x^2} + be^{\beta y}\frac{\partial^2 w}{\partial y^2} = ce^{\lambda w}.$

This is a special case of equation 5.4.5.9 with $k = s = 0$ and $f(w) = ce^{\lambda w}$.

5.3. Equations Involving Other Nonlinearities

5.3.1. Equations with Hyperbolic Nonlinearities

1. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a\sinh(\beta w)$.

This is a special case of equation 5.4.1.1 with $f(w) = a\sinh(\beta w)$.

1°. Traveling-wave solution in implicit form:

$$\int \left[D + \frac{2a\cosh(\beta w)}{\beta(A^2+B^2)}\right]^{-1/2} dw = Ax + By + C,$$

where A, B, C, and D are arbitrary constants.

2°. Solution with central symmetry about the point $(-C_1, -C_2)$:

$$w = w(\xi), \qquad \xi = \sqrt{(x+C_1)^2 + (y+C_2)^2},$$

where C_1 and C_2 are arbitrary constants, and the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{1}{\xi}w'_\xi = a\sinh(\beta w).$$

3°. Functional separable solution:

$$w(x,y) = \frac{4}{\beta}\operatorname{arctanh}\bigl[f(x)g(y)\bigr], \qquad \operatorname{arctanh} z = \frac{1}{2}\ln\frac{1+z}{1-z},$$

where the functions $f = f(x)$ and $g = g(y)$ are determined by the first-order autonomous ordinary differential equations

$$\left(f'_x\right)^2 = Af^4 + Bf^2 + C,$$
$$\left(g'_y\right)^2 = -Cg^4 + (a\beta - B)g^2 - A,$$

and A, B, and C are arbitrary constants.

4°. The original equation is related to (see 5.3.3.1)

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = a\sin(\beta U)$$

by the Bäcklund transformation

$$\frac{\partial U}{\partial x} + \frac{\partial w}{\partial y} = 2\sqrt{\frac{a}{\beta}}\sin\bigl(\tfrac{1}{2}\beta U\bigr)\cosh\bigl(\tfrac{1}{2}\beta w\bigr),$$
$$\frac{\partial U}{\partial y} - \frac{\partial w}{\partial x} = 2\sqrt{\frac{a}{\beta}}\cos\bigl(\tfrac{1}{2}\beta U\bigr)\sinh\bigl(\tfrac{1}{2}\beta w\bigr).$$

⊙ *References*: R. K. Bullough and P. J. Caudrey (1980), A. C. Wing, H. H. Cheb, and Y. C. Lee (1987).

2. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a\sinh(\beta w) + b\sinh(2\beta w)$.

Denote $k = \dfrac{a}{2b}$.

Traveling-wave solutions:

$$w = \pm\frac{1}{\beta}\operatorname{arccosh}\frac{1 - k\sin z}{\sin z - k}, \qquad z = \sqrt{2b\beta(1-k^2)}\,(x\sin C_1 + y\cos C_1 + C_2) \quad \text{if } |k| < 1;$$
$$w = \pm\frac{2}{\beta}\operatorname{arctanh}\left(\sqrt{\frac{k+1}{k-1}}\,\tanh\frac{\xi}{2}\right), \quad \xi = \sqrt{2b\beta(k^2-1)}\,(x\sin C_1 + y\cos C_1 + C_2) \quad \text{if } |k| > 1,$$

where C_1 and C_2 are arbitrary constants.

3. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a(x^2 + y^2)\sinh(\beta w).$

This is a special case of equation 5.4.1.2 with $f(w) = a\sinh(\beta w)$.

4. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a(x^2 + y^2)\cosh(\beta w).$

This is a special case of equation 5.4.1.2 with $f(w) = a\cosh(\beta w)$.

5. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = ae^{\beta x}\sinh(\lambda w).$

This is a special case of equation 5.4.1.4 with $f(w) = a\sinh(\lambda w)$.

6. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a\cosh^n(\beta w)\left[\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right].$

This is a special case of equation 5.4.2.9 with $f(w) = a\cosh^n(\beta w)$.

7. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right) = k\sinh(\beta w).$

This is a special case of equation 5.4.3.1 with $f(w) = k\sinh(\beta w)$.

8. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right) = k\sinh(\lambda w).$

This is a special case of equation 5.4.3.6 with $f(w) = k\sinh(\lambda w)$.

9. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left[a\cosh(\beta w)\dfrac{\partial w}{\partial y}\right] = 0.$

This is a special case of equation 5.4.4.8 with $f(w) = 1$ and $g(w) = a\cosh(\beta w)$.

5.3.2. Equations with Logarithmic Nonlinearities

1. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = \alpha w\ln(\beta w).$

This is a special case of equation 5.4.1.1 with $f(w) = \alpha w\ln(\beta w)$.

On making the change of variable $U = \ln(\beta w)$, one obtains an equation with a quadratic nonlinearity:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 = \alpha U. \qquad (1)$$

1°. Equation (1) has exact solutions quadratic in the independent variables:

$$U(x,y) = \tfrac{1}{4}\alpha(x+A)^2 + \tfrac{1}{4}\alpha(y+B)^2 + 1,$$
$$U(x,y) = A(x+B)^2 \pm \sqrt{A\alpha - 4A^2}\,(x+B)(y+C) + (\tfrac{1}{4}\alpha - A)(y+C)^2 + \tfrac{1}{2},$$

where A, B, and C are arbitrary constants.

2°. Equation (1) has a traveling-wave solution:

$$U(x,y) = F(\xi), \qquad \xi = Ax + By + C.$$

Here, the function $F = F(\xi)$ is defined implicitly by

$$\xi = \int\left[De^{-2F} + \frac{\alpha}{A^2+B^2}(F - \tfrac{1}{2})\right]^{-1/2} dF,$$

where A, B, C, and D are arbitrary constants.

3°. Equation (1) has a solution in the form of the sum of functions with different arguments:

$$U(x,y) = f(x) + g(y).$$

Here, the functions $f = f(x)$ and $g = g(y)$ are defined implicitly by

$$A_1 \pm x = \int \left(B_1 e^{-2f} + \alpha f - \tfrac{1}{2}\alpha\right)^{-1/2} df,$$
$$A_2 \pm y = \int \left(B_2 e^{-2g} + \alpha g - \tfrac{1}{2}\alpha\right)^{-1/2} dg,$$

where A_1, B_1, A_2, and B_2 are arbitrary constants.

4°. The original equation admits exact solutions of the form

$$w = w(\zeta), \qquad \zeta = \sqrt{(x + C_1)^2 + (y + C_2)^2},$$

where C_1 and C_2 are arbitrary constants, and the function $w(\zeta)$ is determined by the ordinary differential equation

$$w''_{\zeta\zeta} + \frac{1}{\zeta} w'_\zeta = \alpha w \ln(\beta w).$$

⊙ *References*: J. A. Shercliff (1977), A. D. Polyanin, A. V. Vyazmin, A. I. Zhurov, and D. A. Kazenin (1998).

2. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = aw \ln w + (bx^n + cy^k)w.$

This is a special case of equation 5.4.1.8 with $f(x) = bx^n$ and $g(y) = cy^k$.

3. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = \alpha(x^2 + y^2) \ln(\beta w).$

This is a special case of equation 5.4.1.2 with $f(w) = \alpha \ln(\beta w)$.

4. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = ae^{\beta x} \ln(\lambda w).$

This is a special case of equation 5.4.1.4 with $f(w) = a \ln(\lambda w)$.

5. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = axw + bw \ln |w|.$

This equation is used for describing some flows of ideal stratified fluids. It is a special case of equation 5.3.2.6 with $k = a_2 = a_0 = 0$.

1°. Multiplicative separable solution:

$$w(x,y) = \exp\left[-\frac{a}{b}x + \frac{b}{4}(y + C)^2 + \frac{a^2}{b^3} + \frac{1}{2}\right],$$

where C is an arbitrary constant.

2°. Multiplicative separable solution (generalizes the solution of Item 1°):

$$w(x,y) = \varphi(x)\psi(y),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined by the system of ordinary differential equations

$$\varphi''_{xx} = b\varphi \ln |\varphi| + (ax + C)\varphi,$$
$$\psi''_{yy} = b\psi \ln |\psi| - C\psi,$$

C is an arbitrary constant.

⊙ *Reference*: V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

6. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{k}{x}\dfrac{\partial w}{\partial x} = (a_2 x^2 + a_1 x + a_0)w + bw \ln|w|.$

Grad–Shafranov equation (with $k = -1$ and $a_1 = a_0 = 0$). This equation is used to describe some steady-state axisymmetric (swirling) flows of ideal fluids. It also occurs in plasma physics.

1°. Multiplicative separable solution:

$$w(x,y) = \varphi(x)\psi(y),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined by the system of ordinary differential equations

$$\varphi''_{xx} + \frac{k}{x}\varphi'_x = b\varphi \ln|\varphi| + (a_2 x^2 + a_1 x + a_0 + C)\varphi,$$
$$\psi''_{yy} = b\psi \ln|\psi| - C\psi,$$

C is an arbitrary constant.

2°. Solutions for $a_1 = 0$:

$$w(x,y) = \exp\left[Ax^2 + \frac{b}{4}(y+B)^2 + \frac{2}{b}A(k+1) - \frac{a_0}{b} + \frac{1}{2}\right], \quad A = \frac{1}{8}\left(b \pm \sqrt{b^2 + 16a_2}\,\right),$$

where B is an arbitrary constant.

3°. Solution for $a_1 = a_2 = 0$:

$$w = w(r), \qquad r = \sqrt{x^2 + y^2},$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{k+1}{r}w'_r = a_0 w + bw \ln|w|.$$

⊙ *References*: G. Rosen (1969), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

7. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{a}{x}\dfrac{\partial w}{\partial x} + \dfrac{b}{y}\dfrac{\partial w}{\partial y} = cw^n \ln(\beta w).$

This is a special case of equation 5.4.2.4 with $f(\xi, w) = cw^n \ln(\beta w)$.

8. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a \ln^n(\beta w)\left[\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right].$

This is a special case of equation 5.4.2.9 with $f(w) = a \ln^n(\beta w)$.

9. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left[a \ln^n(\beta w)\dfrac{\partial w}{\partial y}\right] = 0.$

This is a special case of equation 5.4.4.8 with $f(w) = 1$ and $g(w) = a \ln^n(\beta w)$.

10. $\dfrac{\partial}{\partial x}\left[(a_1 x + b_1)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[(a_2 y + b_2)\dfrac{\partial w}{\partial y}\right] = kw \ln(\beta w).$

1°. Traveling-wave solution:

$$w = w(\xi), \qquad \xi = \frac{x}{a_1} + \frac{y}{a_2} + \frac{b_1}{a_1^2} + \frac{b_2}{a_2^2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$(\xi w'_\xi)'_\xi = kw \ln(\beta w).$$

2°. Multiplicative separable solution:

$$w(x,y) = \varphi(x)\psi(y),$$

where the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$[(a_1 x + b_1)\varphi'_x]'_x - k\varphi \ln(\beta\varphi) + C\varphi = 0,$$
$$[(a_2 y + b_2)\psi'_y]'_y - k\psi \ln\psi - C\psi = 0,$$

C is an arbitrary constant.

11. $\dfrac{\partial}{\partial x}\left[(a_1x+b_1y+c_1)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[(a_2x+b_2y+c_2)\dfrac{\partial w}{\partial y}\right]=kw\ln w.$

This is a special case of equation 5.4.4.2 with $f(w)=kw\ln w$.

12. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)=k\ln(\beta w).$

This is a special case of equation 5.4.3.1 with $f(w)=k\ln(\beta w)$.

13. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)=kw\ln w.$

This is a special case of equation 5.4.3.1 with $f(w)=kw\ln w$ and a special case of equation 5.4.3.9 with $f(x)=ax^n$ and $g(y)=by^m$.

14. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)=k\ln(\lambda w).$

This is a special case of equation 5.4.3.6 with $f(w)=k\ln(\lambda w)$.

15. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)=kw\ln w.$

This is a special case of equation 5.4.3.6 with $f(w)=kw\ln w$ and a special case of equation 5.4.3.9 with $f(x)=ae^{\beta x}$ and $g(y)=be^{\mu y}$.

5.3.3. Equations with Trigonometric Nonlinearities

1. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=\alpha\sin(\beta w).$

This is a special case of equation 5.4.1.1 with $f(w)=\alpha\sin(\beta w)$.

1°. Traveling-wave solution:

$$w=w(z),\qquad z=Ax+By+C,$$

where $w(z)$ is defined implicitly by

$$\int\left[D-\frac{2\alpha\cos(\beta w)}{\beta(A^2+B^2)}\right]^{-1/2}dw=z,$$

and A, B, C, and D are arbitrary constants.

2°. Solution with central symmetry about the point $(-C_1,-C_2)$:

$$w=w(\xi),\qquad \xi=\sqrt{(x+C_1)^2+(y+C_2)^2},$$

where C_1 and C_2 are arbitrary constants and the function $w=w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}+\frac{1}{\xi}w'_{\xi}=\alpha\sin(\beta w).$$

3°. Functional separable solution for $\alpha=\beta=1$:

$$w(x,y)=4\arctan\left(\cot A\frac{\cosh F}{\cosh G}\right),$$

$$F=\frac{\cos A}{\sqrt{1+B^2}}(x-By),\quad G=\frac{\sin A}{\sqrt{1+B^2}}(y+Bx),$$

where A and B are arbitrary constants.

4°. Functional separable solution (generalizes the solution of Item 3°):

$$w(x, y) = \frac{4}{\beta} \arctan\bigl[f(x)g(y)\bigr],$$

where the functions $f = f(x)$ and $g = g(y)$ are determined by the first-order autonomous ordinary differential equations

$$\left(f_x'\right)^2 = Af^4 + Bf^2 + C,$$
$$\left(g_y'\right)^2 = Cg^4 + (\alpha\beta - B)g^2 + A,$$

and A, B, and C are arbitrary constants.

5°. Auto-Bäcklund transformations ($\alpha = \beta = 1$):

$$\frac{\partial \widetilde{w}}{\partial x} = -i\frac{\partial w}{\partial y} + k \sin \frac{\widetilde{w} + w}{2} + \frac{1}{k} \sin \frac{\widetilde{w} - w}{2},$$
$$-i\frac{\partial \widetilde{w}}{\partial y} = \frac{\partial w}{\partial x} + k \sin \frac{\widetilde{w} + w}{2} - \frac{1}{k} \sin \frac{\widetilde{w} - w}{2},$$

where $i^2 = -1$.

⊙ *References for equation* 5.3.3.1: R. K. Bullough and P. J. Caudrey (1980), J. Miller (Jr.) and L. A. Rubel (1993), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

2. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a\sin(\beta w) + b\sin(2\beta w).$

Denote $k = \dfrac{a}{2b}$.

Traveling-wave solutions:

$$w = \pm\frac{2}{\beta} \arctan\left(\frac{k+1}{\sqrt{1-k^2}} \coth\frac{z}{2}\right), \quad z = \sqrt{2b\beta(1-k^2)}\,(x\sin C_1 + y\cos C_1 + C_2) \quad \text{if } |k| < 1;$$
$$w = \pm\frac{2}{\beta} \arctan\left(\frac{k+1}{\sqrt{k^2-1}} \tan\frac{\xi}{2}\right), \quad \xi = \sqrt{2b\beta(k^2-1)}\,(x\sin C_1 + y\cos C_1 + C_2) \quad \text{if } |k| > 1,$$

where C_1 and C_2 are arbitrary constants.

3. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = \alpha\cos(\beta w).$

The substitution $\beta w = \beta u + \frac{1}{2}\pi$ leads to an equation of the form 5.3.3.1:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\alpha\sin(\beta u).$$

4. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = \alpha(x^2 + y^2)\sin(\beta w).$

This is a special case of equation 5.4.1.2 with $f(w) = \alpha\sin(\beta w)$.

5. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = \alpha(x^2 + y^2)\cos(\beta w).$

This is a special case of equation 5.4.1.2 with $f(w) = \alpha\cos(\beta w)$.

6. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = ae^{\beta x}\sin(\lambda w).$

This is a special case of equation 5.4.1.4 with $f(w) = a\sin(\lambda w)$.

7. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = a\cos(\beta w)\left[\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right].$

This is a special case of equation 5.4.2.9 with $f(w) = a\cos(\beta w)$.

8. $\dfrac{\partial}{\partial x}\left(ax^n \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(by^m \dfrac{\partial w}{\partial y}\right) = k\sin(\beta w).$

This is a special case of equation 5.4.3.1 with $f(w) = k\sin(\beta w)$.

9. $\dfrac{\partial}{\partial x}\left(ae^{\beta x} \dfrac{\partial w}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(be^{\mu y} \dfrac{\partial w}{\partial y}\right) = k\sin(\lambda w).$

This is a special case of equation 5.4.3.6 with $f(w) = k\sin(\lambda w)$.

10. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left[a\cos^n(\beta w)\dfrac{\partial w}{\partial y}\right] = 0.$

This is a special case of equation 5.4.4.8 with $f(w) = 1$ and $g(w) = a\cos^n(\beta w)$.

5.4. Equations Involving Arbitrary Functions

5.4.1. Equations of the Form $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = F(x, y, w)$

1. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = f(w).$

This is a *stationary heat equation with a nonlinear source*.

1°. Suppose $w = w(x, y)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm x + C_1, \pm y + C_2),$$
$$w_2 = w(x\cos\beta - y\sin\beta,\ x\sin\beta + y\cos\beta),$$

where C_1, C_2, and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int\left[C + \frac{2}{A^2 + B^2}F(w)\right]^{-1/2} dw = Ax + By + D, \qquad F(w) = \int f(w)\,dw,$$

where A, B, C, and D are arbitrary constants.

3°. Solution with central symmetry about the point $(-C_1, -C_2)$:

$$w = w(\zeta), \qquad \zeta = \sqrt{(x + C_1)^2 + (y + C_2)^2},$$

where C_1 and C_2 are arbitrary constants and the function $w = w(\zeta)$ is determined by the ordinary differential equation

$$w''_{\zeta\zeta} + \frac{1}{\zeta}w'_\zeta = f(w).$$

4°. For exact solutions of the original equation for some $f(w)$, see Subsections 5.1.1 and 5.2.1 and equations 5.3.1.1, 5.3.2.1, and 5.3.3.1 (see also Subsection S.5.3, Example 12).

2. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = (x^2 + y^2) f(w).$

1°. Suppose $w = w(x, y)$ is a solution of this equation. Then the function

$$w_1 = w(x \cos\beta - y \sin\beta,\ x \sin\beta + y \cos\beta),$$

where β is an arbitrary constant, is also a solution of the equation.

2°. Solution with central symmetry:

$$w = w(r), \qquad r = \sqrt{x^2 + y^2},$$

where the function $w = w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{1}{r} w'_r = r^2 f(w).$$

3°. Self-similar solution:

$$w = w(\zeta), \qquad \zeta = xy.$$

Here, the function $w = w(\zeta)$ is determined by the autonomous ordinary differential equation

$$w''_{\zeta\zeta} = f(w),$$

whose general solution can be written out in implicit form as

$$\int \bigl[C_1 + 2F(w)\bigr]^{-1/2} dw = C_2 \pm \zeta, \qquad F(w) = \int f(w)\, dw,$$

where C_1 and C_2 are arbitrary constants.

4°. Functional separable solution:

$$w = w(z), \qquad z = \tfrac{1}{2}(x^2 - y^2).$$

Here, the function $w = w(z)$ is determined by the autonomous ordinary differential equation

$$w''_{zz} = f(w),$$

whose general solution can be written out in implicit form as

$$\int \bigl[C_1 + 2F(w)\bigr]^{-1/2} dw = C_2 \pm z, \qquad F(w) = \int f(w)\, dw,$$

where C_1 and C_2 are arbitrary constants.

5°. The transformation

$$w = U(z, \zeta), \quad z = \tfrac{1}{2}(x^2 - y^2), \quad \zeta = xy$$

leads to a simpler equation of the form 5.4.1.1:

$$\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial \zeta^2} = f(U).$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

3. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = (x^2 + y^2)^k f(w).$

1°. Suppose $w = w(x, y)$ is a solution of this equation. Then the function

$$w_1 = w(x \cos\beta - y \sin\beta,\ x \sin\beta + y \cos\beta),$$

where β is an arbitrary constant, is also a solution of the equation.

2°. Solution with central symmetry:

$$w = w(r), \qquad r = \sqrt{x^2 + y^2},$$

where the function $w = w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{1}{r} w'_r = r^{2k} f(w).$$

3°. Suppose $k = \pm 1, \pm 2, \ldots$ The transformation

$$\begin{aligned}
&z = \tfrac{1}{2}(x^2 - y^2), && \zeta = xy && \text{for } k = 1,\\
&z = \tfrac{1}{3}(x^3 - 3xy^2), && \zeta = \tfrac{1}{3}(3x^2 y - y^3) && \text{for } k = 2,\\
&z = \tfrac{1}{2}\ln(x^2 + y^2), && \zeta = \arctan\frac{y}{x} && \text{for } k = -1,\\
&z = -\frac{x}{x^2 + y^2}, && \zeta = \frac{y}{x^2 + y^2} && \text{for } k = -2
\end{aligned}$$

leads to a simpler equation of the form 5.4.1.1:

$$\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial \zeta^2} = f(w). \tag{1}$$

For arbitrary $f = f(w)$, this equation admits a traveling-wave solution $w = w(Az + B\zeta)$, where A and B are arbitrary constants, and a solution of the form $w = w(z^2 + \zeta^2)$.

In the general case, for any integer $k \neq -1$, the transformation

$$z = \frac{(x + iy)^{k+1} + (x - iy)^{k+1}}{2(k+1)}, \quad \zeta = \frac{(x + iy)^{k+1} - (x - iy)^{k+1}}{2(k+1)i}, \quad i^2 = -1 \tag{2}$$

leads to equation (1). It follows from (2) that

$$z^2 + \zeta^2 = \frac{1}{(k+1)^2}(x^2 + y^2)^{k+1}.$$

4°. Suppose k is an arbitrary constant ($k \neq -1$). The transformation

$$z = \frac{1}{k+1} r^{k+1} \cos\bigl[(k+1)\varphi\bigr], \quad \zeta = \frac{1}{k+1} r^{k+1} \sin\bigl[(k+1)\varphi\bigr], \tag{3}$$

where $x = r\cos\varphi$ and $y = r\sin\varphi$, leads to the simpler equation (1). For $k = \pm 1, \pm 2, \ldots$, transformation (3) coincides with transformation (2).

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

4. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = e^{\beta x} f(w).$

The transformation

$$w = W(u, v), \quad u = \exp\bigl(\tfrac{1}{2}\beta x\bigr)\cos\bigl(\tfrac{1}{2}\beta y\bigr), \quad v = \exp\bigl(\tfrac{1}{2}\beta x\bigr)\sin\bigl(\tfrac{1}{2}\beta y\bigr)$$

leads to a simpler equation of the form 5.4.1.1:

$$\frac{\partial^2 W}{\partial u^2} + \frac{\partial^2 W}{\partial v^2} = 4\beta^{-2} f(W).$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

5. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = e^{ax-by} f(w).$

Let us represent the exponent of e in the form

$$ax - by = \beta(x\cos\sigma - y\sin\sigma); \quad \beta = \sqrt{a^2+b^2}, \quad \cos\sigma = a/\beta, \quad \sin\sigma = b/\beta.$$

The transformation

$$\xi = x\cos\sigma - y\sin\sigma, \quad \eta = x\sin\sigma + y\cos\sigma,$$

leads to an equation of the form 5.4.1.4:

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} = e^{\beta\xi} f(w).$$

6. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = f(x,y)e^{\beta w}.$

Suppose $f(x,y) = \varepsilon|F(z)|^2$, where $\varepsilon = \pm 1$ and $F = F(z)$ is a prescribed analytic function of the complex variable $z = x + iy$.

1°. General solution:

$$w(x,y) = -\frac{2}{\beta}\ln\frac{|\beta F(z)|\bigl[1 + \varepsilon\,\mathrm{sign}(\beta)\Phi(z)\overline{\Phi(z)}\bigr]}{4|\Phi'_z(z)|},$$

where $\Phi = \Phi(z)$ is an arbitrary analytic (holomorphic) function of the complex variable $z = x + iy$ with nonzero derivative; the bar over a symbol denotes the complex conjugate.

2°. Another representation of the general solution with $\beta = -2$:

$$w(x,y) = \ln\bigl(|\varphi(z)|^2 + \varepsilon|\psi(z)|^2\bigr).$$

Here, the holomorphic functions $\varphi = \varphi(z)$ and $\psi = \psi(z)$ are given by

$$\varphi^2 = C\Phi\exp\biggl(\frac{a}{2}\int_{z_0}^{z}\frac{F}{\Phi}\,dz\biggr), \quad \psi^2 = \frac{\Phi}{C}\exp\biggl(-\frac{a}{2}\int_{z_0}^{z}\frac{F}{\Phi}\,dz\biggr),$$

where $|a| = 1$, $C \neq 0$ is any, z_0 is an arbitrary point in the complex plane, and $\Phi = \Phi(z)$ is an arbitrary holomorphic function satisfying the condition $\Phi'(z_*) = \pm\frac{1}{2}aF(z_*)$ at any point $z = z_*$ where $\Phi(z_*) = 0$. The condition just mentioned means that the function Φ can only have simple zeros.

⊙ *Reference*: I. Kh. Sabitov (2001).

7. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = aw\ln w + f(x)w.$

Multiplicative separable solution:

$$w(x,y) = \varphi(x)\psi(y),$$

where the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$\varphi''_{xx} - \bigl[a\ln\varphi + f(x) + C\bigr]\varphi = 0,$$
$$\psi''_{yy} - \bigl(a\ln\psi - C\bigr)\psi = 0,$$

and C is an arbitrary constant.

8. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = aw \ln w + \bigl[f(x) + g(y)\bigr]w.$

Multiplicative separable solution:

$$w(x, y) = \varphi(x)\psi(y),$$

where the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$\varphi''_{xx} - \bigl[a \ln \varphi + f(x) + C\bigr]\varphi = 0,$$
$$\psi''_{yy} - \bigl[a \ln \psi + g(y) - C\bigr]\psi = 0,$$

and C is an arbitrary constant.

9. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = f(x)w \ln w + \bigl[af(x)y + g(x)\bigr]w.$

Multiplicative separable solution:

$$w(x, y) = e^{-ay}\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$\varphi''_{xx} = f(x)\varphi \ln \varphi + \bigl[g(x) - a^2\bigr]\varphi.$$

10. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = f(ax + by, w).$

Solution:

$$w = w(\xi), \qquad \xi = ax + by,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$(a^2 + b^2)w''_{\xi\xi} = f(\xi, w).$$

11. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = f(x^2 + y^2, w).$

1°. Suppose $w = w(x, y)$ is a solution of this equation. Then the function

$$w_1 = w(x \cos\beta - y \sin\beta,\ x \sin\beta + y \cos\beta),$$

where β is an arbitrary constant, is also a solution of the equation.

2°. Solution with central symmetry:

$$w = w(\xi), \qquad \xi = \bigl(x^2 + y^2\bigr)^{1/2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{1}{\xi}w'_\xi = f(\xi^2, w).$$

12. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = (x^2 + y^2)f(xy,\ w).$

1°. Self-similar solution:

$$w = w(\zeta), \qquad \zeta = xy,$$

where the function $w(\zeta)$ is determined by the ordinary differential equation

$$w''_{\zeta\zeta} = f(\zeta, w).$$

2°. The transformation

$$w = U(z, \zeta), \quad z = \tfrac{1}{2}(x^2 - y^2), \quad \zeta = xy$$

leads to the simpler equation

$$\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial \zeta^2} = f(\zeta, U).$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

13. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = (x^2 + y^2) f(x^2 - y^2,\ w).$

1°. Functional separable solution:

$$w = w(z), \qquad z = \tfrac{1}{2}(x^2 - y^2),$$

where the function $w(z)$ is determined by the ordinary differential equation

$$w''_{zz} = f(2z, w).$$

2°. The transformation

$$w = U(z, \zeta), \quad z = \tfrac{1}{2}(x^2 - y^2), \quad \zeta = xy$$

leads to the simpler equation

$$\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial \zeta^2} = f(2z, U).$$

14. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = f\big(w + A_{11}x^2 + A_{12}xy + A_{22}y^2 + B_1 x + B_2 y\big).$

The substitution $U = w + A_{11}x^2 + A_{12}xy + A_{22}y^2 + B_1 x + B_2 y$ leads to an equation of the form 5.4.1.1:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f(U) + 2A_{11} + 2A_{22}.$$

5.4.2. Equations of the Form $a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial^2 w}{\partial y^2} = F\Big(x, y, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\Big)$

1. $a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial^2 w}{\partial y^2} = f(w).$

This equation describes steady-state processes of heat/mass transfer or combustion in anisotropic media. Here, a and b are the principal thermal diffusivities (diffusion coefficients) and $f = f(w)$ is a kinetic function that defines the law of heat (substance) release.

The transformation $\xi = x/\sqrt{a}$, $\eta = y/\sqrt{b}$ leads to an equation of the form 5.4.1.1:

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} = f(w).$$

2. $(a_1 x + b_1 y + c_1)\dfrac{\partial w}{\partial x} + (a_2 x + b_2 y + c_2)\dfrac{\partial w}{\partial y} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} - f(w).$

This equation describes steady-state mass transfer with a volume chemical reaction in a translational-shear fluid flow.

Traveling-wave solution:

$$w = w(z), \qquad z = a_2 x + (k - a_1)y,$$

where k is a root of the quadratic equation

$$k^2 - (a_1 + b_2)k + a_1 b_2 - a_2 b_1 = 0,$$

and the function $w(z)$ is determined by the ordinary differential equation

$$\big[kz + a_2 c_1 + (k - a_1)c_2\big] w'_z = \big[a_2^2 + (k - a_1)^2\big] w''_{zz} - f(w).$$

Remark. In the case of an incompressible fluid, the equation coefficients must satisfy the condition $a_1 + b_2 = 0$.

3. $\displaystyle a\frac{\partial^2 w}{\partial x^2}+b\frac{\partial^2 w}{\partial y^2}+cx\frac{\partial w}{\partial x}-cy\frac{\partial w}{\partial y}=(bx^2+ay^2)f(w).$

Solution:

$$w=w(z),\qquad z=xy,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$w''_{zz}=f(w).$$

4. $\displaystyle \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{a}{x}\frac{\partial w}{\partial x}+\frac{b}{y}\frac{\partial w}{\partial y}=f(x^2+y^2,w).$

Solution with central symmetry:

$$w=w(\xi),\qquad \xi=\left(x^2+y^2\right)^{1/2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}+\frac{a+b+1}{\xi}w'_\xi=f(\xi^2,w).$$

5. $\displaystyle a\frac{\partial^2 w}{\partial x^2}+b\frac{\partial^2 w}{\partial y^2}=f_1(x)\frac{\partial w}{\partial x}+f_2(y)\frac{\partial w}{\partial y}+kw\ln w+\bigl[g_1(x)+g_2(y)\bigr]w.$

Multiplicative separable solution:

$$w(x,y)=\varphi(x)\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(x)$ are determined by the ordinary differential equations

$$a\varphi''_{xx}=f_1(x)\varphi'_x+k\varphi\ln\varphi+\bigl[g_1(x)+C\bigr]\varphi,$$
$$b\psi''_{yy}=f_2(y)\psi'_y+k\psi\ln\psi+\bigl[g_2(y)-C\bigr]\psi,$$

where C is an arbitrary constant.

6. $\displaystyle \frac{\partial^2 w}{\partial x^2}+a\frac{\partial^2 w}{\partial y^2}=f(x)\left(\frac{\partial w}{\partial y}\right)^2+g(x)w+h(x).$

Generalized separable solution quadratic in y:

$$w(x,y)=\varphi(x)y^2+\psi(x)y+\chi(x),\tag{1}$$

where the functions $\varphi(x)$, $\psi(x)$, and $\chi(x)$ are determined by the following system of ordinary differential equations (the arguments of f, g, and h are omitted):

$$\varphi''_{xx}=4f\varphi^2+g\varphi,\tag{2}$$
$$\psi''_{xx}=(4f\varphi+g)\psi,\tag{3}$$
$$\chi''_{xx}=g\chi+f\psi^2+h-2a\varphi.\tag{4}$$

Whenever a solution $\varphi=\varphi(x)$ of the nonlinear equation (2) is found, the functions $\psi=\psi(x)$ and $\chi=\chi(x)$ can be determined successively from equations (3) and (4), which are linear in ψ and χ, respectively.

It is apparent from the comparison of equations (2) and (3) that (3) has the particular solution $\psi=\varphi(x)$. Hence, the general solution to (3) is given by (see Polyanin and Zaitsev, 2003)

$$\psi(x)=C_1\varphi(x)+C_2\varphi(x)\int\frac{dx}{\varphi^2(x)},\qquad \varphi\not\equiv 0.$$

Note that equation (2) has the trivial particular solution $\varphi(x)\equiv 0$, to which there corresponds solution (1) linear in y. If the functions f and g are proportional, then a particular solution to equation (2) is given by $\varphi=-\frac{1}{4}g/f=\text{const}$.

7. $\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial^2 w}{\partial y^2} = f(x)\left(\dfrac{\partial w}{\partial y}\right)^2 + bf(x)w^2 + g(x)w + h(x).$

1°. Generalized separable solutions:

$$w(x,y) = \varphi(x) + \psi(x)\exp\bigl(\pm y\sqrt{-b}\bigr), \qquad b < 0, \tag{1}$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the following system of ordinary differential equations (the arguments of f, g, and h are omitted):

$$\varphi''_{xx} = bf\varphi^2 + g\varphi + h, \tag{2}$$

$$\psi''_{xx} = (2bf\varphi + g + ab)\psi. \tag{3}$$

Whenever a solution $\varphi = \varphi(x)$ of equation (2) is found, the functions $\psi = \psi(x)$ can be determined by solving equation (3), which is linear in ψ.

If the functions f, g, and h are proportional, i.e.,

$$g = \alpha f, \quad h = \beta f \qquad (\alpha,\ \beta = \text{const}),$$

then particular solutions of equation (2) are given by

$$\varphi = k_1, \quad \varphi = k_2, \tag{4}$$

where k_1 and k_2 are roots of the quadratic equation $bk^2 + \alpha k + \beta = 0$. In this case, equation (3) can be rewritten as

$$\psi''_{xx} = \bigl[(2bk_n + \alpha)f + ab\bigr]\psi, \qquad n = 1,\ 2. \tag{5}$$

The books by Kamke (1977) and Polyanin and Zaitsev (2003) present a large number of exact solutions to the linear equation (5) for various $f = f(x)$. In the special case $f = \text{const}$, the general solution of equation (5) is the sum of exponentials (or sine and cosine).

2°. Generalized separable solution (generalizes the solutions of Item 1°):

$$w(x,y) = \varphi(x) + \psi(x)\bigl[A\exp\bigl(y\sqrt{-b}\bigr) + B\exp\bigl(-y\sqrt{-b}\bigr)\bigr], \qquad b < 0, \tag{6}$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the system of ordinary differential equations

$$\varphi''_{xx} = bf\bigl(\varphi^2 + 4AB\psi^2\bigr) + g\varphi + h,$$
$$\psi''_{xx} = 2bf\varphi\psi + g\psi + ab\psi.$$

Note two special cases of solution (6) that involve hyperbolic functions. These are:

$$w(x,y) = \varphi(x) + \psi(x)\cosh\bigl(y\sqrt{-b}\bigr), \quad A = \tfrac{1}{2},\ B = \tfrac{1}{2},$$
$$w(x,y) = \varphi(x) + \psi(x)\sinh\bigl(y\sqrt{-b}\bigr), \quad A = \tfrac{1}{2},\ B = -\tfrac{1}{2}.$$

3°. Generalized separable solution (c is an arbitrary constant):

$$w(x,y) = \varphi(x) + \psi(x)\cos\bigl(y\sqrt{b} + c\bigr), \qquad b > 0,$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the system of ordinary differential equations

$$\varphi''_{xx} = bf\bigl(\varphi^2 + \psi^2\bigr) + g\varphi + h,$$
$$\psi''_{xx} = 2bf\varphi\psi + g\psi + ab\psi.$$

⊙ *References*: V. A. Galaktionov (1995), V. F. Zaitsev and A. D. Polyanin (1996).

8. $a\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial^2 w}{\partial y^2} = f(x)\left(\dfrac{\partial w}{\partial y}\right)^2 + g(x)\dfrac{\partial w}{\partial x}$

$\qquad + \bigl[h_1(x)y + h_0(x)\bigr]\dfrac{\partial w}{\partial y} + p(x)w + q_2(x)y^2 + q_1(x)y + q_0(x).$

There is a generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x)y^2 + \psi(x)y + \chi(x).$$

9. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=f(w)\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2\right].$

The substitution

$$U=\int\frac{dw}{F(w)},\qquad \text{where}\quad F(w)=\exp\left[\int f(w)\,dw\right],$$

leads to the two-dimensional Laplace equation for $U=U(x,y)$:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

10. $a\dfrac{\partial^2 w}{\partial x^2}+b\dfrac{\partial^2 w}{\partial y^2}=f(x)\left(\dfrac{\partial w}{\partial x}\right)^n+g(y)\left(\dfrac{\partial w}{\partial y}\right)^m+kw.$

Additive separable solution:

$$w(x,y)=\varphi(x)+\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(x)$ are determined by the ordinary differential equations

$$a\varphi''_{xx}-f(x)\bigl(\varphi'_x\bigr)^n-k\varphi=C,$$
$$b\psi''_{yy}-g(y)\bigl(\psi'_y\bigr)^m-k\psi=-C,$$

where C is an arbitrary constant.

11. $a\dfrac{\partial^2 w}{\partial x^2}+b\dfrac{\partial^2 w}{\partial y^2}=f_1(x)\left(\dfrac{\partial w}{\partial x}\right)^n+f_2(y)\left(\dfrac{\partial w}{\partial y}\right)^m$
$\qquad\qquad+g_1(x)\dfrac{\partial w}{\partial x}+g_2(y)\dfrac{\partial w}{\partial y}+h_1(x)+h_2(y)+kw.$

Additive separable solution:

$$w(x,y)=\varphi(x)+\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(x)$ are determined by the ordinary differential equations

$$a\varphi''_{xx}-f_1(x)\bigl(\varphi'_x\bigr)^n-g_1(x)\varphi'_x-k\varphi-h_1(x)=C,$$
$$b\psi''_{yy}-f_2(y)\bigl(\psi'_y\bigr)^m-g_2(y)\psi'_y-k\psi-h_2(y)=-C,$$

where C is an arbitrary constant.

12. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}=(a_1x+b_1y+c_1)\left(\dfrac{\partial w}{\partial x}\right)^k+(a_2x+b_2y+c_2)\left(\dfrac{\partial w}{\partial y}\right)^k+f\left(w,\,\dfrac{\partial w}{\partial x},\,\dfrac{\partial w}{\partial y}\right).$

Solutions are sought in the traveling-wave form

$$w=w(z),\qquad z=Ax+By+C,$$

where the constants A, B, and C are determined by solving the algebraic system of equations

$$a_1A^k+a_2B^k=A, \tag{1}$$
$$b_1A^k+b_2B^k=B, \tag{2}$$
$$c_1A^k+c_2B^k=C. \tag{3}$$

Equations (1) and (2) are first solved for A and B and then equation (3) is used to evaluate C. The unknown function $w(z)$ is determined by the ordinary differential equation

$$(A^2+B^2)w''_{zz}=z(w'_z)^k+f\bigl(w,\,Aw'_z,\,Bw'_z\bigr).$$

13. $\displaystyle a\frac{\partial^2 w}{\partial x^2} + b\frac{\partial^2 w}{\partial y^2} = f_1\left(x, \frac{\partial w}{\partial x}\right) + f_2\left(y, \frac{\partial w}{\partial y}\right) + kw.$

Additive separable solution:

$$w(x, y) = \varphi(x) + \psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(x)$ are determined by the ordinary differential equations

$$a\varphi''_{xx} - f_1\big(x, \varphi'_x\big) - k\varphi = C,$$
$$b\psi''_{yy} - f_2\big(y, \psi'_y\big) - k\psi = -C,$$

where C is an arbitrary constant.

14. $\displaystyle a\frac{\partial^2 w}{\partial x^2} + b\frac{\partial^2 w}{\partial y^2} = f_1\left(x, \frac{1}{w}\frac{\partial w}{\partial x}\right)w + f_2\left(y, \frac{1}{w}\frac{\partial w}{\partial y}\right)w.$

Multiplicative separable solution:

$$w(x, y) = \varphi(x)\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$a\frac{\varphi''_{xx}}{\varphi} - f_1\left(x, \frac{\varphi'_x}{\varphi}\right) = C,$$
$$b\frac{\psi''_{yy}}{\psi} - f_2\left(y, \frac{\psi'_y}{\psi}\right) = -C,$$

where C is an arbitrary constant.

5.4.3. Heat and Mass Transfer Equations of the Form $\displaystyle \frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] = h(w)$

► *Equations of this form describe steady-state heat/mass transfer or combustion processes in inhomogeneous anisotropic media. Here, $f = f(x)$ and $g = g(y)$ are the principal thermal diffusivities (diffusion coefficients) dependent on coordinates; $h = h(w)$ is the kinetic function (source function), which defines the law of heat (substance) release of absorption. The simple solutions dependent on a single coordinate, $w = w(x)$ or $w = w(y)$, are not considered in this subsection.*

1. $\displaystyle \frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m\frac{\partial w}{\partial y}\right) = f(w).$

1°. Functional separable solution for $n \neq 2$ and $m \neq 2$:

$$w = w(\xi), \qquad \xi = \left[b(2-m)^2x^{2-n} + a(2-n)^2y^{2-m}\right]^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi} + \frac{A}{\xi}w'_\xi = Bf(w), \tag{1}$$

where

$$A = \frac{4-nm}{(2-n)(2-m)}, \quad B = \frac{4}{ab(2-n)^2(2-m)^2}.$$

For $m = 4/n$, a family of exact solutions to the original equation with arbitrary $f = f(w)$ follows from (1). It is given by

$$\int\left[C_1 + \frac{2n^2}{ab(2-n)^4}F(w)\right]^{-1/2} dw = C_2 \pm \xi, \qquad F(w) = \int f(w)\,dw,$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution $\zeta=\xi^{1-A}$ brings (1) to the generalized Emden–Fowler equation

$$w''_{\zeta\zeta}=\frac{B}{(1-A)^2}\zeta^{\frac{2A}{1-A}}f(w). \qquad (2)$$

A large number of exact solutions to equation (2) for various $f=f(w)$ can be found in Polyanin and Zaitsev (2003).

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

2. $\dfrac{\partial}{\partial x}\left[a(x+c)^n\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[b(y+s)^m\dfrac{\partial w}{\partial y}\right]=f(w).$

The transformation $\zeta=x+c$, $\eta=y+s$ leads to an equation of the form 5.4.3.1:

$$\frac{\partial}{\partial\zeta}\left(a\zeta^n\frac{\partial w}{\partial\zeta}\right)+\frac{\partial}{\partial\eta}\left(b\eta^m\frac{\partial w}{\partial\eta}\right)=f(w).$$

3. $\dfrac{\partial}{\partial x}\left[a\big(|x|+c\big)^n\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[b\big(|y|+s\big)^m\dfrac{\partial w}{\partial y}\right]=f(w).$

The transformation $\zeta=|x|+c$, $\eta=|y|+s$ leads to an equation of the form 5.4.3.1:

$$\frac{\partial}{\partial\zeta}\left(a\zeta^n\frac{\partial w}{\partial\zeta}\right)+\frac{\partial}{\partial\eta}\left(b\eta^m\frac{\partial w}{\partial\eta}\right)=f(w).$$

4. $a\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)=f(w).$

Functional separable solution for $\mu\neq 0$:

$$w=w(\xi),\qquad \xi=\left[b\mu^2(x+C_1)^2+4ae^{-\mu y}\right]^{1/2},$$

where C_1 is an arbitrary constant and the function $w(\xi)$ is determined by the autonomous ordinary differential equation

$$w''_{\xi\xi}=\frac{1}{ab\mu^2}f(w).$$

The general solution of this equation with arbitrary kinetic function $f=f(w)$ is defined implicitly by

$$\int\left[C_2+\frac{2}{ab\mu^2}F(w)\right]^{-1/2}dw=C_3\pm\xi,\qquad F(w)=\int f(w)\,dw,$$

where C_2 and C_3 are arbitrary constants.

5. $a\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left(be^{\mu|y|}\dfrac{\partial w}{\partial y}\right)=f(w).$

The substitution $\zeta=|y|$ leads to an equation of the form 5.4.3.4.

6. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)=f(w).$

Functional separable solution for $\beta\mu\neq 0$:

$$w=w(\xi),\qquad \xi=\left(b\mu^2e^{-\beta x}+a\beta^2e^{-\mu y}\right)^{1/2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}-\frac{1}{\xi}w'_{\xi}=Af(w),\qquad A=\frac{4}{ab\beta^2\mu^2}. \qquad (1)$$

The substitution $\zeta=\xi^2$ brings (1) to the generalized Emden–Fowler equation

$$w''_{\zeta\zeta}=\tfrac{1}{4}A\zeta^{-1}f(w),$$

whose solutions with $f(w)=(kw+s)^{-1}$ and $f(w)=(kw+s)^{-2}$ ($k,s=\text{const}$) can be found in Polyanin and Zaitsev (2003).

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

7. $\dfrac{\partial}{\partial x}\left(ae^{\beta|x|}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu|y|}\dfrac{\partial w}{\partial y}\right)=f(w).$

The transformation $\zeta=|x|$, $\eta=|y|$ leads to an equation of the form 5.4.3.6.

8. $\dfrac{\partial}{\partial x}\left(ax^{n}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\mu y}\dfrac{\partial w}{\partial y}\right)=f(w).$

Functional separable solution for $n\neq 2$ and $\mu\neq 0$:

$$w=w(\xi),\qquad \xi=\left[b\mu^2x^{2-n}+a(2-n)^2e^{-\mu y}\right]^{1/2},$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$w''_{\xi\xi}+\frac{n}{2-n}\frac{1}{\xi}w'_{\xi}=\frac{4}{ab\mu^2(2-n)^2}f(w).$$

9. $\dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right]=kw\ln w.$

Multiplicative separable solution:

$$w(x,y)=\varphi(x)\psi(y),$$

where the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$[f(x)\varphi'_x]'_x=k\varphi\ln\varphi+C\varphi,$$
$$[g(y)\psi'_y]'_y=k\psi\ln\psi-C\psi,$$

and C is an arbitrary constant.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

10. $\dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right]=aw\ln w+bw.$

This is a special case of equation 5.4.4.6 with $k=a$, $h_1(x)=b$, and $h_2(y)=0$.

5.4.4. Equations of the Form $\dfrac{\partial}{\partial x}\left[f(x,y,w)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[g(x,y,w)\dfrac{\partial w}{\partial y}\right]=h(x,y,w)$

1. $(ay+c)\dfrac{\partial^2 w}{\partial x^2}+(bx+s)\dfrac{\partial^2 w}{\partial y^2}=f(w).$

This equation can be rewritten in the divergence form

$$\frac{\partial}{\partial x}\left[(ay+c)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[(bx+s)\frac{\partial w}{\partial y}\right]=f(w).$$

For $ab\neq 0$, there is an exact solution of the form

$$w=w(\xi),\qquad \xi=(a^2b)^{-1/3}x+(ab^2)^{-1/3}y+(a^2b)^{-2/3}c+(ab^2)^{-2/3}s,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$\xi w''_{\xi\xi}=f(w).$$

2. $\dfrac{\partial}{\partial x}\left[(a_1x+b_1y+c_1)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[(a_2x+b_2y+c_2)\dfrac{\partial w}{\partial y}\right]=f(w).$

Solutions are sought in the traveling wave form

$$w=w(z),\qquad z=Ax+By+C,$$

where the constants A, B, and C are determined by solving the algebraic system of equations

$$a_1A^2+a_2B^2=A, \tag{1}$$
$$b_1A^2+b_2B^2=B, \tag{2}$$
$$c_1A^2+c_2B^2=C. \tag{3}$$

Equations (1) and (2) are first solved for A and B and then equation (3) is used to evaluate C. The unknown function $w(z)$ is determined by the ordinary differential equation

$$zw''_{zz}+(Aa_1+Bb_2)w'_z=f(w).$$

3. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left\{\left[f(x)w+g(x)\right]\dfrac{\partial w}{\partial y}\right\}=0.$

1°. Generalized separable solution linear in y:

$$w(x,y)=(Ax+B)y-\int_{x_0}^{x}(x-t)(At+B)^2f(t)\,dt+C_1x+C_2,$$

where A, B, C_1, C_2, and x_0 are arbitrary constants. This solution is degenerate.

2°. Generalized separable solution quadratic in y:

$$w(x,y)=\varphi(x)y^2+\psi(x)y+\chi(x),$$

where the functions $\varphi=\varphi(x)$, $\psi=\psi(x)$, and $\chi=\chi(x)$ are determined by the system of ordinary differential equations

$$\varphi''_{xx}+6f\varphi^2=0, \tag{1}$$
$$\psi''_{xx}+6f\varphi\psi=0, \tag{2}$$
$$\chi''_{xx}+2f\varphi\chi+2\varphi g+f\psi^2=0. \tag{3}$$

The nonlinear equation (1) is treated independently from the others. For $f\equiv\text{const}$, its solution can be expressed in terms of elliptic integrals. For $f=ae^{\lambda x}$, a particular solution to (1) is given by $\varphi=-\frac{\lambda^2}{6a}e^{-\lambda x}$. Equations (2) and (3) are solved successively (these are linear in their respective unknowns). Since $\psi=\varphi(x)$ is a particular solution to equation (2), the general solution is expressed as (see Polyanin and Zaitsev, 2003)

$$\psi(x)=C_1\varphi(x)+C_2\varphi(x)\int\frac{dx}{\varphi^2(x)},$$

where C_1 and C_2 are arbitrary constants.

4. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial}{\partial y}\left[\dfrac{f(y)}{\sqrt{w+a}}\dfrac{\partial w}{\partial y}\right]=0.$

The substitution $U=\sqrt{w+a}$ leads to the equation

$$\frac{\partial}{\partial x}\left(U\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left[f(y)\frac{\partial U}{\partial y}\right]=0,$$

which has a generalized separable solution of the form

$$U(x,y)=\varphi(y)x+\psi(y),$$
$$U(x,y)=\varphi(y)x^2+\psi(y)x+\chi(y).$$

5. $\dfrac{\partial^2 w}{\partial x^2} + f(w)\dfrac{\partial^2 w}{\partial y^2} = 0.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = w(C_1 x + C_2, \pm C_1 y + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Degenerate solution: $w = C_1 xy + C_2 x + C_3 y + C_4$.

3°. Self-similar solution:

$$w = w(z), \qquad z = y/x,$$

where the function $w(z)$ is determined by the ordinary differential equation

$$[z^2 + f(w)]w''_{zz} + 2zw'_z = 0.$$

6. $\dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(y)\dfrac{\partial w}{\partial y}\right] = kw\ln w + \bigl[h_1(x) + h_2(y)\bigr]w.$

Multiplicative separable solution:

$$w(x,y) = \exp\bigl[\varphi(x) + \psi(y)\bigr].$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$e^{-\varphi}[fe^{\varphi}\varphi'_x]'_x - k\varphi - h_1(x) = C,$$
$$e^{-\psi}[ge^{\psi}\psi'_y]'_y - k\psi - h_2(y) = -C,$$

where C is an arbitrary constant.

7. $\dfrac{\partial}{\partial x}\left[f(x)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left\{\bigl[g(x)w + h(x)\bigr]\dfrac{\partial w}{\partial y}\right\} = 0.$

There are generalized separable solutions linear and quadratic in y:

$$w(x,y) = \varphi(x)y + \psi(x),$$
$$w(x,y) = \varphi(x)y^2 + \psi(x)y + \chi(x).$$

8. $\dfrac{\partial}{\partial x}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[g(w)\dfrac{\partial w}{\partial y}\right] = 0.$

This is a *stationary anisotropic heat (diffusion) equation*; $f(w)$ and $g(w)$ are the principal thermal diffusivities (diffusion coefficients).

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1 x + C_2, \pm C_1 y + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution in implicit form:

$$\int \bigl[A^2 f(w) + B^2 g(w)\bigr]\,dw = C_1(Ax + By) + C_2,$$

where A, B, C_1, and C_2 are arbitrary constants.

3°. Self-similar solution (α and β are arbitrary constants):

$$w = w(\zeta), \quad \zeta = \frac{x+\alpha}{y+\beta},$$

where the function $w(\zeta)$ is determined by the ordinary differential equation

$$[f(w)w'_\zeta]'_\zeta + [\zeta^2 g(w)w'_\zeta]'_\zeta = 0. \tag{1}$$

Integrating (1) and taking w to be the independent variable, one obtains a Riccati equation for $\zeta = \zeta(w)$:

$$C\zeta'_w = g(w)\zeta^2 + f(w), \tag{2}$$

where C is an arbitrary constant. A large number of exact solutions to equation (2) for various $f = f(w)$ and $g = g(w)$ can be found in Polyanin and Zaitsev (2003).

4°. Solution in parametric form:

$$x = C_1 v^2 + C_2 v - \int f(w)[2C_1 G(w) + C_3]\,dw + C_4,$$
$$y = -[2C_1 G(w) + C_3]v - C_2 G(w) + C_5, \quad G(w) = \int g(w)\,dw,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

5°. Solution in parametric form:

$$x = [C_1 F(w) + C_2]v + C_3 F(w) + C_4, \quad F(w) = \int f(w)\,dw,$$
$$y = \frac{1}{2}C_1 v^2 + C_3 v - \int g(w)[C_1 F(w) + C_2]\,dw + C_5.$$

6°. Solution in parametric form:

$$x = [C_1 F(w) + C_2]v^2 + C_3 F(w) + C_4 - 2\int \left\{ f(w) \int g(w)[C_1 F(w) + C_2]\,dw \right\} dw,$$
$$y = \frac{1}{3}C_1 v^3 + C_3 v - 2v \int g(w)[C_1 F(w) + C_2]\,dw + C_5.$$

7°. Solution in parametric form:

$$x = (C_1 e^{\lambda v} + C_2 e^{-\lambda v})H(w) + C_3,$$
$$y = \frac{1}{\lambda}(C_1 e^{\lambda v} - C_2 e^{-\lambda v})\frac{1}{f(w)}H'_w(w) + C_4,$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants, the function $H = H(w)$ is determined by the ordinary differential equation $\mathbf{L}_f[H] + \lambda^2 g(w)H = 0$, and the differential operator $\mathbf{L}_f$ is defined as

$$\mathbf{L}_f[\varphi] \equiv \frac{d}{dw}\left[\frac{1}{f(w)}\frac{d\varphi}{dw}\right]. \tag{3}$$

8°. Solution in parametric form:

$$x = [C_1 \sin(\lambda v) + C_2 \cos(\lambda v)]Z(w) + C_3,$$
$$y = \frac{1}{\lambda}[C_2 \sin(\lambda v) - C_1 \cos(\lambda v)]\frac{1}{f(w)}Z'_w(w) + C_4,$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants, and the function $Z = Z(w)$ is determined by the ordinary differential equation $\mathbf{L}_f[Z] - \lambda^2 g(w)Z = 0$.

9°. Solution in parametric form:

$$x = [2C_1 F(w) + C_3]v + C_2 F(w) + C_5, \quad F(w) = \int f(w)\,dw,$$

$$y = C_1 v^2 + C_2 v - \int g(w)[2C_1 F(w) + C_3]\,dw + C_4.$$

10°. Solution in parametric form:

$$x = \frac{1}{2}C_1 v^2 + C_3 v - \int f(w)[C_1 G(w) + C_2]\,dw + C_5,$$

$$y = -[C_1 G(w) + C_2]v - C_3 G(w) + C_4, \quad G(w) = \int g(w)\,dw.$$

11°. Solution in parametric form:

$$x = \frac{1}{3}C_1 v^3 + C_3 v - 2v \int f(w)[C_1 G(w) + C_2]\,dw + C_5,$$

$$y = -[C_1 G(w) + C_2]v^2 - C_3 G(w) + C_4 + 2\int \left\{ g(w) \int f(w)[C_1 G(w) + C_2]\,dw \right\} dw.$$

12°. Solution in parametric form:

$$x = -\frac{1}{\lambda}(C_1 e^{\lambda v} - C_2 e^{-\lambda v})\frac{1}{g(w)}H'_w(w) + C_3,$$

$$y = (C_1 e^{\lambda v} + C_2 e^{-\lambda v})H(w) + C_4,$$

where C_1, C_2, C_3, and λ are arbitrary constants, the function $H = H(w)$ is determined by the ordinary differential equation $\mathbf{L}_g[H] + \lambda^2 f(w)H = 0$, and the differential operator $\mathbf{L}_g$ is defined by (3) with $f(w) = g(w)$.

13°. Solution in parametric form:

$$x = -\frac{1}{\lambda}[C_2 \sin(\lambda v) - C_1 \cos(\lambda v)]\frac{1}{g(w)}Z'_w(w) + C_3,$$

$$y = [C_1 \sin(\lambda v) + C_2 \cos(\lambda v)]Z(w) + C_4,$$

where C_1, C_2, C_3, and λ are arbitrary constants, the function $Z = Z(w)$ is determined by the ordinary differential equation $\mathbf{L}_g[Z] - \lambda^2 f(w)Z = 0$, and the differential operator $\mathbf{L}_g$ is defined by (3) with $f(w) = g(w)$.

14°. The original equation can be represented as the sum of the equations

$$f(w)\frac{\partial w}{\partial x} = \frac{\partial v}{\partial y}, \quad -g(w)\frac{\partial w}{\partial y} = \frac{\partial v}{\partial x}. \tag{4}$$

The hodograph transformation

$$x = x(w, v), \quad y = y(w, v), \tag{5}$$

where w, v are treated as the independent variables and x, y as the dependent ones, brings (4) to the linear system

$$f(w)\frac{\partial y}{\partial v} = \frac{\partial x}{\partial w}, \quad -g(w)\frac{\partial x}{\partial v} = \frac{\partial y}{\partial w}. \tag{6}$$

Eliminating y yields the following linear equation for $x = x(w, v)$:

$$\frac{\partial}{\partial w}\left[\frac{1}{f(w)}\frac{\partial x}{\partial w}\right] + g(w)\frac{\partial^2 x}{\partial v^2} = 0. \tag{7}$$

Likewise, we can obtain another linear equation for $y = y(w, v)$ from system (6). We have

$$\frac{\partial}{\partial w}\left[\frac{1}{g(w)}\frac{\partial y}{\partial w}\right] + f(w)\frac{\partial^2 y}{\partial v^2} = 0. \tag{8}$$

The procedure for constructing exact solutions to the original equation consists of the following two stages. First, one finds an exact solution to the linear equation (7) for $x = x(w, v)$. Then, this solution is substituted into the linear system (6), from which the function $y = y(w, v)$ is found in the form

$$y = \int_{v_0}^{v} \frac{1}{f(w)}\frac{\partial x}{\partial w}(w, \xi)\, d\xi - \int_{w_0}^{w} g(\eta)\frac{\partial x}{\partial v}(\eta, v_0)\, d\eta, \tag{9}$$

where w_0 and v_0 are any numbers. The thus obtained expressions of (5) define a solution to the original equation in parametric form.

Likewise, one can first construct an exact solution to the linear equation (8) for $y = y(w, v)$ and then find $x = x(w, v)$ from (6) in the form

$$x = -\int_{v_0}^{v} \frac{1}{g(w)}\frac{\partial y}{\partial w}(w, \xi)\, d\xi + \int_{w_0}^{w} f(\eta)\frac{\partial y}{\partial v}(\eta, v_0)\, d\eta,$$

where w_0 and v_0 are any numbers.

Remark 1. Let $x = \Phi(w, v; f, g)$ be a solution to equation (7). Then $y = \Phi(w, v; g, f)$ solves equation (8).

Remark 2. Let $x = \Phi(w, v; f, g)$, $y = \Psi(w, v; f, g)$ be a solution to system of equations (6). Then the functions $x = \Psi(w, v; -g, -f)$ and $y = \Phi(w, v; -g, -f)$ also solve this system.

15°. Solutions to equation (7) with even powers of v:

$$x = \sum_{k=0}^{n} \varphi_k(w) v^{2k}, \tag{10}$$

where the functions $\varphi_k = \varphi_k(w)$ are determined by the recurrence relations

$$\varphi_n(w) = A_n F(w) + B_n, \qquad F(w) = \int f(w)\, dw,$$
$$\varphi_{k-1}(w) = A_k F(w) + B_k - 2k(2k-1)\int f(w)\left\{\int g(w)\varphi_k(w)\, dw\right\} dw,$$

where the A_k and B_k are arbitrary constants ($k = n, \dots, 1$).

The function $y = y(w, v)$ is defined by (9) and, in conjunction with relation (10), gives a solution to the original nonlinear equation in parametric form.

16°. Solutions to equation (7) with odd powers of v:

$$x = \sum_{k=0}^{n} \psi_k(w) v^{2k+1}, \tag{11}$$

where the functions $\psi_k = \psi_k(w)$ are determined by the recurrence relations

$$\psi_n(w) = A_n F(w) + B_n, \qquad F(w) = \int f(w)\, dw,$$
$$\psi_{k-1}(w) = A_k F(w) + B_k - 2k(2k+1)\int f(w)\left\{\int g(w)\psi_k(w)\, dw\right\} dw,$$

where the A_k and B_k are arbitrary constants ($k = n, \dots, 1$).

The function $y = y(w, v)$ is defined by (9) and, in conjunction with relation (11), gives a solution to the original nonlinear equation in parametric form.

17°. In the special case $g(w)=k^2 f(w)$, the transformation

$$\bar{x}=kx, \quad u=\int f(w)\,dw$$

leads to the Laplace equation

$$\frac{\partial^2 u}{\partial \bar{x}^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

For solutions of this linear equation, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

⊙ *References for equation* 5.4.4.8: V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2002).

9. $$\frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[g(w)\frac{\partial w}{\partial y}\right]=(a_1x+b_1y+c_1)\frac{\partial w}{\partial x}+(a_2x+b_2y+c_2)\frac{\partial w}{\partial y}.$$

This equation describes steady-state anisotropic heat/mass transfer in a translational-shear fluid flow.

Traveling-wave solution:

$$w=w(z), \qquad z=a_2x+(k-a_1)y,$$

where k is a root of the quadratic equation

$$k^2-(a_1+b_2)k+a_1b_2-a_2b_1=0,$$

and the function $w(z)$ is determined by the ordinary differential equation

$$[\varphi(w)w'_z]'_z=[kz+a_2c_1+(k-a_1)c_2]w'_z, \qquad \varphi(w)=a_2^2f(w)+(k-a_1)^2g(w).$$

Remark 1. The above remains the same if an arbitrary function, $h(w)$, is added to the right-hand side of the original equation.

Remark 2. In the case of an incompressible fluid, equation coefficients must satisfy the condition $a_1+b_2=0$.

10. $$\frac{\partial}{\partial x}\left\{\left[a_1x+b_1y+f(w)\right]\frac{\partial w}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\left[a_2x+b_2y+g(w)\right]\frac{\partial w}{\partial y}\right\}=0.$$

Solutions are sought in the traveling-wave form

$$w=w(\xi), \qquad \xi=Ax+By,$$

where the constants A and B are determined by solving the algebraic system of equations

$$a_1A^2+a_2B^2=A, \qquad b_1A^2+b_2B^2=B.$$

The desired function $w(\xi)$ is determined by the first-order ordinary differential equation (C is an arbitrary constant):

$$\left[\xi+A^2f(w)+B^2g(w)\right]w'_\xi=C.$$

Taking w to be the independent variable, we obtain a first-order linear equation for $\xi=\xi(w)$:

$$C\xi'_w=\xi+A^2f(w)+B^2g(w).$$

5.4.5. Other Equations

1. $$\frac{\partial^2 w}{\partial x^2}+aw^4\frac{\partial^2 w}{\partial y^2}=f(y)w^5.$$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1=C_1w\bigl(\pm C_1^2x+C_2,\, y\bigr),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

$2°$. Let $u = u(y)$ be any nontrivial particular solution of the second-order linear ordinary differential equation

$$au''_{yy} - f(y)u = 0. \qquad (1)$$

The transformation

$$\zeta = \int \frac{dy}{u^2}, \quad \xi = \frac{w}{u}$$

simplifies the original equation considerably, bringing it to the form

$$\frac{\partial^2 \xi}{\partial x^2} + a\xi^4 \frac{\partial^2 \xi}{\partial \zeta^2} = 0. \qquad (2)$$

This equation is independent of f explicitly and has a degenerate solution

$$\xi(x, \zeta) = Ax\zeta + B\zeta + Cx + D,$$

where A, B, C, and D are arbitrary constants. Furthermore, equation (2) has exact solutions with the following structures, for example:

$$\begin{aligned}
&\xi(x, \zeta) = \xi(kx + \lambda\zeta) && \text{(traveling-wave solution)},\\
&\xi(x, \zeta) = g(x)h(\zeta) && \text{(multiplicative separable solution)},\\
&\xi(x, \zeta) = x^\beta \varphi(\eta), \quad \eta = \zeta x^{-2\beta-1} && \text{(self-similar solution)},
\end{aligned}$$

where k, λ, and β are arbitrary constants.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

2. $\displaystyle \frac{\partial^2 w}{\partial x^2} + a\frac{\partial}{\partial y}\left(w^n \frac{\partial w}{\partial y}\right) = f(y)w^{n+1} + g(x)w.$

Multiplicative separable solution:

$$w = \varphi(x)\psi(y),$$

where the functions $\psi = \psi(y)$ and $\varphi = \varphi(x)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$\varphi''_{xx} - g(x)\varphi + C\varphi^{n+1} = 0,$$
$$a(\psi^n \psi'_y)'_y - f(y)\psi^{n+1} - C\psi = 0.$$

3. $\displaystyle \frac{\partial^2 w}{\partial x^2} + a\frac{\partial}{\partial y}\left(e^{\lambda w} \frac{\partial w}{\partial y}\right) = f(y)e^{\lambda w} + g(x).$

Additive separable solution:

$$w = \varphi(x) + \psi(y),$$

where the functions $\psi = \psi(y)$ and $\varphi = \varphi(x)$ are determined by the ordinary differential equations

$$\varphi''_{xx} - g(x) + Ce^{\lambda\varphi} = 0,$$
$$a(e^{\lambda\psi}\psi'_y)'_y - f(y)e^{\lambda\psi} - C = 0,$$

and C is an arbitrary constant. The second equation can be reduced, with the change of variable $U = e^{\lambda\psi}$, to the linear equation $aU''_{yy} - \lambda f(y)U - \lambda C = 0$.

4. $\displaystyle \frac{\partial^2 w}{\partial x^2} + \bigl[f_3(x)w + f_2(x)y^2 + f_1(x)y + f_0(x)\bigr]\frac{\partial^2 w}{\partial y^2} = g_2(x)\left(\frac{\partial w}{\partial y}\right)^2$
$\displaystyle \qquad + g_1(x)\frac{\partial w}{\partial x} + \bigl[h_1(x)y + h_0(x)\bigr]\frac{\partial w}{\partial y} + s_3(x)w + s_2(x)y^2 + s_1(x)y + s_0(x).$

There is a generalized separable solution quadratic in y:

$$w(x, y) = \varphi(x)y^2 + \psi(x)y + \chi(x).$$

5. $ax^n \dfrac{\partial^2 w}{\partial x^2} + by^m \dfrac{\partial^2 w}{\partial y^2} + kx^{n-1} \dfrac{\partial w}{\partial x} + sy^{m-1} \dfrac{\partial w}{\partial y} = f(w).$

Functional separable solution for $n \neq 2$ and $m \neq 2$:

$$w = w(\xi), \qquad \xi = \big[b(2-m)^2 x^{2-n} + a(2-n)^2 y^{2-m}\big]^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$A w''_{\xi\xi} + \frac{B}{\xi} w'_\xi = f(w), \tag{1}$$

where

$$A = \tfrac{1}{4} ab(2-n)^2(2-m)^2,$$
$$B = \tfrac{1}{4}(2-n)(2-m)\big[ab(3nm-4n-4m+4) + 2bk(2-m) + 2as(2-n)\big].$$

Solution of equation (1) with $B = 0$ and arbitrary $f = f(w)$ in implicit form:

$$\int \left[C_1 + \frac{2}{A} F(w)\right]^{-1/2} dw = C_2 \pm \xi, \qquad F(w) = \int f(w)\, dw,$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

6. $ax^n \dfrac{\partial^2 w}{\partial x^2} + by^m \dfrac{\partial^2 w}{\partial y^2} + kx^{n-1} f(w) \dfrac{\partial w}{\partial x} + sy^{m-1} f(w) \dfrac{\partial w}{\partial y} = g(w).$

For $n \neq 2$ and $m \neq 2$, there is a functional separable solution of the form

$$w = w(\xi), \qquad \xi = \big[b(2-m)^2 x^{2-n} + a(2-n)^2 y^{2-m}\big]^{1/2}.$$

7. $ae^{\beta x} \dfrac{\partial^2 w}{\partial x^2} + be^{\mu y} \dfrac{\partial^2 w}{\partial y^2} + ke^{\beta x} \dfrac{\partial w}{\partial x} + se^{\mu y} \dfrac{\partial w}{\partial y} = f(w).$

Functional separable solution for $\beta\mu \neq 0$:

$$w = w(\xi), \qquad \xi = \big(b\mu^2 e^{-\beta x} + a\beta^2 e^{-\mu y}\big)^{1/2}.$$

Here, the function $w = w(\xi)$ is determined by the ordinary differential equation

$$A w''_{\xi\xi} + \frac{B}{\xi} w'_\xi = f(w), \tag{1}$$

where

$$A = \tfrac{1}{4} ab\beta^2\mu^2, \quad B = \tfrac{1}{4}\beta\mu(3ab\beta\mu - 2bk\mu - 2as\beta).$$

Solution of equation (1) with $B = 0$ and arbitrary $f = f(w)$ in implicit form:

$$\int \left[C_1 + \frac{2}{A} F(w)\right]^{-1/2} dw = C_2 \pm \xi, \qquad F(w) = \int f(w)\, dw,$$

where C_1 and C_2 are arbitrary constants.

8. $ae^{\beta x} \dfrac{\partial^2 w}{\partial x^2} + be^{\mu y} \dfrac{\partial^2 w}{\partial y^2} + ke^{\beta x} f(w) \dfrac{\partial w}{\partial x} + se^{\mu y} f(w) \dfrac{\partial w}{\partial y} = g(w).$

For $\beta\mu \neq 0$, there is a functional separable solution of the form

$$w = w(\xi), \qquad \xi = \big(b\mu^2 e^{-\beta x} + a\beta^2 e^{-\mu y}\big)^{1/2}.$$

9. $\quad ax^n \dfrac{\partial^2 w}{\partial x^2} + be^{\beta y} \dfrac{\partial^2 w}{\partial y^2} + kx^{n-1} \dfrac{\partial w}{\partial x} + se^{\beta y} \dfrac{\partial w}{\partial y} = f(w).$

Functional separable solution $\beta \neq 0$ and $n \neq 2$:

$$w = w(\xi), \qquad \xi = \left[b\beta^2 x^{2-n} + a(2-n)^2 e^{-\beta y}\right]^{1/2}.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$Aw''_{\xi\xi} + \frac{B}{\xi} w'_\xi = f(w), \tag{1}$$

where

$$A = \tfrac{1}{4} ab\beta^2 (2-n)^2, \quad B = \tfrac{1}{4} \beta(2-n)\left[ab\beta(4-3n) + 2bk\beta - 2as(2-n)\right].$$

Solution of equation (1) with $B = 0$ and arbitrary $f = f(w)$ in implicit form:

$$\int \left[C_1 + \frac{2}{A} F(w)\right]^{-1/2} dw = C_2 \pm \xi, \qquad F(w) = \int f(w)\, dw,$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: V. F. Zaitsev and A. D. Polyanin (1996).

10. $\quad ax^n \dfrac{\partial^2 w}{\partial x^2} + be^{\beta y} \dfrac{\partial^2 w}{\partial y^2} + kx^{n-1} f(w) \dfrac{\partial w}{\partial x} + se^{\beta y} f(w) \dfrac{\partial w}{\partial y} = g(w).$

For $\beta \neq 0$ and $n \neq 2$, there is a functional separable solution of the form

$$w = w(\xi), \qquad \xi = \left[b\beta^2 x^{2-n} + a(2-n)^2 e^{-\beta y}\right]^{1/2}.$$

11. $\quad (ay + c) \dfrac{\partial^2 w}{\partial x^2} + (bx + s) \dfrac{\partial^2 w}{\partial y^2} = f\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right).$

Functional separable solution for $ab \neq 0$:

$$w = w(\xi), \qquad \xi = (a^2 b)^{-1/3} x + (ab^2)^{-1/3} y + (a^2 b)^{-2/3} c + (ab^2)^{-2/3} s.$$

Here, the function $w(\xi)$ is determined by the ordinary differential equation

$$\xi w''_{\xi\xi} = f\left(w, \beta w'_\xi, \mu w'_\xi\right),$$

where $\beta = (a^2 b)^{-1/3}$, $\mu = (ab^2)^{-1/3}$.

12. $\quad (a_1 x + b_1 y + c_1) \dfrac{\partial^2 w}{\partial x^2} + (a_2 x + b_2 y + c_2) \dfrac{\partial^2 w}{\partial y^2} = f\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right).$

Solutions are sought in the traveling-wave form

$$w = w(\xi), \qquad \xi = Ax + By + C,$$

where the constants A, B, and C are determined by solving the algebraic system of equations

$$a_1 A^2 + a_2 B^2 = A, \tag{1}$$
$$b_1 A^2 + b_2 B^2 = B, \tag{2}$$
$$c_1 A^2 + c_2 B^2 = C. \tag{3}$$

Equations (1) and (2) are first solved for A and B and then equation (3) is used to evaluate C.

The desired function $w(\xi)$ is determined by the ordinary differential equation

$$\xi w''_{\xi\xi} = f\left(w, Aw'_\xi, Bw'_\xi\right).$$

13. $f_1(x)\dfrac{\partial^2 w}{\partial x^2}+f_2(y)\dfrac{\partial^2 w}{\partial y^2}=g_1(x)\dfrac{\partial w}{\partial x}+g_2(y)\dfrac{\partial w}{\partial y}+kw\ln w+\bigl[h_1(x)+h_2(y)\bigr]w.$

Multiplicative separable solution:

$$w(x,y)=\varphi(x)\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$f_1(x)\varphi''_{xx}=g_1(x)\varphi'_x+k\varphi\ln\varphi+\bigl[h_1(x)+C\bigr]\varphi,$$
$$f_2(y)\psi''_{yy}=g_2(y)\psi'_y+k\psi\ln\psi+\bigl[h_2(y)-C\bigr]\psi,$$

where C is an arbitrary constant.

14. $\bigl[a_1x+b_1y+f(w)\bigr]\dfrac{\partial^2 w}{\partial x^2}+\bigl[a_2x+b_2y+g(w)\bigr]\dfrac{\partial^2 w}{\partial y^2}=h\Bigl(w,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial w}{\partial y}\Bigr).$

Traveling-wave solution:

$$w=w(\xi),\qquad \xi=Ax+By,$$

where the constants A, B, and C are determined by solving the algebraic system of equations

$$a_1A^2+a_2B^2=A,\qquad b_1A^2+b_2B^2=B,$$

and the function $w(\xi)$ is determined by the ordinary differential equation

$$\bigl[\xi+A^2f(w)+B^2g(w)\bigr]w''_{\xi\xi}=h\bigl(w,\ Aw'_\xi,\ Bw'_\xi\bigr).$$

15. $\dfrac{\partial}{\partial x}\Bigl\{\bigl[a_1x+b_1y+f(w)\bigr]\dfrac{\partial w}{\partial x}\Bigr\}+\dfrac{\partial}{\partial y}\Bigl\{\bigl[a_2x+b_2y+g(w)\bigr]\dfrac{\partial w}{\partial y}\Bigr\}=h\Bigl(w,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial w}{\partial y}\Bigr).$

Traveling-wave solution:

$$w=w(\xi),\qquad \xi=Ax+By,$$

where the constants A, B, and C are determined by solving the algebraic system of equations

$$A^2a_1+B^2a_2=A,\qquad A^2b_1+B^2b_2=B,$$

and the function $w(\xi)$ is determined by the ordinary differential equation

$$\bigl[\varphi(\xi,w)w'_\xi\bigr]'_\xi=h\bigl(w,\ Aw'_\xi,\ Bw'_\xi\bigr),$$
$$\varphi(\xi,w)=\xi+A^2f(w)+B^2g(w).$$

16. $\dfrac{\partial^2 w}{\partial x^2}+f(x)\dfrac{\partial w}{\partial x}+g(x)\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial y^2}+h(x)w=0.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1=C_1^{-3}w\bigl(x,\ C_1y+C_2\bigr)+\phi(x),$$

where C_1 and C_2 are arbitrary constants, and $\phi(x)$ is determined by the second-order linear ordinary differential equation $\phi''_{xx}+f(x)\phi'_x+h(x)\phi=0$, is also a solution of the equation.

2°. Generalized separable solution:

$$w(x,y)=\varphi_1(x)+\varphi_2(x)y^{3/2}+\varphi_3(x)y^3,$$

where the functions $\varphi_k=\varphi_k(x)$ are determined by the system of ordinary differential equations

$$\varphi_1''+f(x)\varphi_1'+\tfrac{9}{8}g(x)\varphi_2^2+h(x)\varphi_1=0,$$
$$\varphi_2''+f(x)\varphi_2'+\tfrac{45}{4}g(x)\varphi_2\varphi_3+h(x)\varphi_2=0,$$
$$\varphi_3''+f(x)\varphi_3'+18g(x)\varphi_3^2+h(x)\varphi_3=0,$$

where the prime stands for the differentiation with respect to x.

3°. Generalized separable solution cubic in y:

$$w(x,y) = \psi_1(x) + \psi_2(x)y + \psi_3(x)y^2 + \psi_4(x)y^3,$$

where the functions $\psi_k = \psi_k(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
&\psi_1'' + f(x)\psi_1' + 2g(x)\psi_2\psi_3 + h(x)\varphi_1 = 0,\\
&\psi_2'' + f(x)\psi_2' + 2g(x)(2\psi_3^2 + 3\psi_2\psi_4) + h(x)\varphi_2 = 0,\\
&\psi_3'' + f(x)\psi_3' + 18g(x)\psi_3\psi_4 + h(x)\varphi_3 = 0,\\
&\psi_4'' + f(x)\psi_4' + 18g(x)\psi_4^2 + h(x)\varphi_4 = 0.
\end{aligned}$$

4°. Generalized separable solution:

$$w(x,y) = \xi(x) + \eta(x)\theta(y).$$

Here, the functions $\xi = \xi(x)$ and $\eta = \eta(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\eta''_{xx} + f(x)\eta'_x + ag(x)\eta^2 + h(x)\eta = 0,\\
\xi''_{xx} + f(x)\xi'_x + bg(x)\eta^2 + h(x)\xi = 0,
\end{aligned}$$

where a and b are arbitrary constants, and the function $\theta = \theta(y)$ is determined by the autonomous ordinary differential equation

$$\theta'_y\theta''_{yy} = a\theta + b,$$

whose solution can be written out in implicit form.

17. $\displaystyle \frac{\partial^2 w}{\partial x^2} + f\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right)\frac{\partial^2 w}{\partial y^2} = 0.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1}w\bigl(C_1x + C_2,\ C_1y + C_3\bigr) + C_4,$$

where C_1, C_2, C_4, and C_4 are arbitrary constants, is also a solution of the equation.

2°. The Legendre transformation

$$u(\xi,\eta) = x\xi + y\eta - w(x,y), \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y},$$

where u is the new dependent variable, and ξ and η are the new independent variables, leads to the linear equation

$$\frac{\partial^2 u}{\partial \eta^2} + f(\xi,\eta)\frac{\partial^2 u}{\partial \xi^2} = 0.$$

Exact solutions of this equation for some $f(\xi,\eta)$ can be found in Polyanin (2002).

Chapter 6

Elliptic Equations with Three or More Space Variables

6.1. Equations with Three Space Variables Involving Power-Law Nonlinearities

6.1.1. Equations of the Form $\frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[h(z)\frac{\partial w}{\partial z}\right] = aw^p$

1. $\frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(cz^k\frac{\partial w}{\partial z}\right) = sw^p.$

This is a special case of equation 6.3.1.3 with $f(w) = sw^p$.

1°. Suppose $w(x, y, z)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\Big(C_1^{\frac{p-1}{2-n}}\,x,\ C_1^{\frac{p-1}{2-m}}\,y,\ C_1^{\frac{p-1}{2-k}}\,z\Big),$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $n \neq 2$, $m \neq 2$, $k \neq 2$, and $p \neq 1$:

$$w = \left[\frac{1}{s(1-p)}\left(\frac{p}{1-p} + \frac{1}{2-n} + \frac{1}{2-m} + \frac{1}{2-k}\right)\right]^{\frac{1}{p-1}} \left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2}\right]^{\frac{1}{1-p}}.$$

3°. Functional separable solution for $n \neq 2$, $m \neq 2$, and $k \neq 2$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r = sw^p, \qquad A = \frac{2}{2-n} + \frac{2}{2-m} + \frac{2}{2-k} - 1.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, z) = U(\xi, z), \quad \xi^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2},$$

$$w(x, y, z) = V(x, \eta), \quad \eta^2 = \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2},$$

$$w(x, y, z) = W(y, \zeta), \quad \zeta^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{z^{2-k}}{c(2-k)^2},$$

$$w(x, y, z) = x^{\frac{n-2}{p-1}} F(\rho_1, \rho_2), \quad \rho_1 = yx^{\frac{n-2}{2-m}}, \quad \rho_2 = zx^{\frac{n-2}{2-k}}.$$

2. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right)=sw^p.$

This is a special case of equation 6.3.1.5 with $f(w)=sw^p$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1=C_1w\Bigl(C_1^{\frac{p-1}{2-n}}x,\ C_1^{\frac{p-1}{2-m}}y,\ z+\frac{1-p}{\lambda}\ln C_1\Bigr),$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $n\neq 2$, $m\neq 2$, $\lambda\neq 0$, and $p\neq 1$:

$$w=\left[\frac{1}{s(p-1)}\left(\frac{p}{1-p}+\frac{1}{2-n}+\frac{1}{2-m}\right)\right]^{\frac{1}{p-1}}\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right]^{\frac{1}{1-p}}.$$

3°. Functional separable solution for $n\neq 2$, $m\neq 2$, and $\lambda\neq 0$ (generalizes the solution of Item 2°):

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}+\frac{A}{r}w'_r=sw^p,\qquad A=\frac{2}{2-n}+\frac{2}{2-m}-1.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z)=U(\xi,z),\quad \xi^2=\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2},$$
$$w(x,y,z)=V(x,\eta),\quad \eta^2=\frac{y^{2-m}}{b(2-m)^2}+\frac{e^{-\lambda z}}{c\lambda^2},$$
$$w(x,y,z)=W(y,\zeta),\quad \zeta^2=\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\lambda z}}{c\lambda^2},$$
$$w(x,y,z)=x^{\frac{n-2}{p-1}}F(\rho_1,\rho_2),\quad \rho_1=yx^{\frac{n-2}{2-m}},\quad \rho_2=z+\frac{2-n}{\lambda}\ln x.$$

3. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\beta y}\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right)=sw^p.$

This is a special case of equation 6.3.1.6 with $f(w)=sw^p$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1=C_1w\Bigl(C_1^{\frac{p-1}{2-n}}x,\ y+\frac{1-p}{\beta}\ln C_1,\ z+\frac{1-p}{\lambda}\ln C_1\Bigr),$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $n\neq 2$, $\beta\neq 0$, $\lambda\neq 0$, and $p\neq 1$:

$$w=\left[\frac{1}{s(p-1)}\left(\frac{p}{1-p}+\frac{1}{2-n}\right)\right]^{\frac{1}{p-1}}\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right]^{\frac{1}{1-p}}.$$

3°. Functional separable solution for $n\neq 2$, $\beta\neq 0$, and $\lambda\neq 0$ (generalizes the solution of Item 2°):

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r} w'_r = s w^p, \qquad A = \frac{n}{2-n}.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z) = U(\xi, z), \quad \xi^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\beta y}}{b\beta^2},$$
$$w(x,y,z) = V(x, \eta), \quad \eta^2 = \frac{e^{-\beta y}}{b\beta^2} + \frac{e^{-\lambda z}}{c\lambda^2},$$
$$w(x,y,z) = W(y, \zeta), \quad \zeta^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda z}}{c\lambda^2},$$
$$w(x,y,z) = x^{\frac{n-2}{p-1}} F(\rho_1, \rho_2), \quad \rho_1 = y + \frac{2-n}{\beta} \ln x, \quad \rho_2 = z + \frac{2-n}{\lambda} \ln x.$$

4. $\displaystyle \frac{\partial}{\partial x}\left(a e^{\beta x} \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(b e^{\gamma y} \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(c e^{\lambda z} \frac{\partial w}{\partial z}\right) = s w^p.$

This is a special case of equation 6.3.1.4 with $f(w) = s w^p$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\Big(x + \frac{1-p}{\beta} \ln C_1,\ y + \frac{1-p}{\gamma} \ln C_1,\ z + \frac{1-p}{\lambda} \ln C_1\Big),$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $p \neq 1$, $\beta \neq 0$, $\gamma \neq 0$, and $\lambda \neq 0$:

$$w = \left[\frac{p}{b(1-p)^2}\right]^{\frac{1}{p-1}} \left(\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\gamma y}}{b\gamma^2} + \frac{e^{-\lambda z}}{c\lambda^2}\right)^{\frac{1}{1-p}}.$$

3°. Functional separable solution for $\beta \neq 0$, $\gamma \neq 0$, and $\lambda \neq 0$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4\left(\frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\gamma y}}{b\gamma^2} + \frac{e^{-\lambda z}}{c\lambda^2}\right),$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} - \frac{1}{r} w'_r = s w^p.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z) = U(\xi, z), \quad \xi^2 = \frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\gamma y}}{b\gamma^2},$$
$$w(x,y,z) = V(x, \eta), \quad \eta^2 = \frac{e^{-\gamma y}}{b\gamma^2} + \frac{e^{-\lambda z}}{c\lambda^2},$$
$$w(x,y,z) = W(y, \zeta), \quad \zeta^2 = \frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda z}}{c\lambda^2},$$
$$w(x,y,z) = \exp\Big(\frac{\beta x}{p-1}\Big) F(\rho_1, \rho_2), \quad \rho_1 = y - \frac{\beta}{\gamma} x, \quad \rho_2 = z - \frac{\beta}{\lambda} x.$$

6.1.2. Equations of the Form $\frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[g(w)\frac{\partial w}{\partial y}\right]+\frac{\partial}{\partial z}\left[g(w)\frac{\partial w}{\partial z}\right]=0$

1. $\displaystyle \frac{\partial^2 w}{\partial x^2}+a\frac{\partial^2 w}{\partial y^2}+\frac{\partial}{\partial z}\left[(bw+c)\frac{\partial w}{\partial z}\right]=0.$

1°. Suppose $w(x,y,z)$ is a solution of this equation. Then the functions

$$w_1=C_1^{-2}w(\pm C_2x+C_3,\pm C_2y+C_4,\pm C_1C_2z+C_5)+\frac{c(1-C_1^2)}{bC_1^2},$$
$$w_2=w(x\cos\beta+ya^{-1/2}\ \sin\beta,-xa^{1/2}\ \sin\beta+y\cos\beta,\ z),$$

where $C_1,\ldots,C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Solutions:

$$w(x,y,z)=A\sqrt{C_1x+C_2y+C_3z+C_4}-\frac{C_1^2+aC_2^2}{bC_3^2}-\frac{c}{b},$$
$$w(x,y,z)=(C_1x+C_2)z+C_3x+C_4y+C_5-\tfrac{1}{12}bC_1^{-2}(C_1x+C_2)^4,$$
$$w(x,y,z)=(C_1x+C_2)z+C_3(ax^2-y^2)-\tfrac{1}{12}bC_1^{-2}(C_1x+C_2)^4,$$
$$w(x,y,z)=|z|^{1/2}\left[C_1(ax^2-y^2)+C_2x+C_3+C_4)\right]-\frac{c}{b},$$
$$w(x,y,z)=C_1|z|^{1/2}\exp\left(\sqrt{a}C_2x\right)\sin(C_2y+C_3)-\frac{c}{b},$$
$$w(x,y,z)=C_1|z|^{1/2}\sin(\sqrt{a}C_2x+C_3)\exp(C_2y)-\frac{c}{b},$$

where A, $C_1,\ldots,C_5$ are arbitrary constants (the first solution is of the traveling-wave type).

3°. Solution:

$$w=u(\xi)-4bC_1^2x^2,\qquad \xi=z+bC_1x^2+C_2y,$$

where C_1 and C_2 are arbitrary constants and the function $u(\xi)$ is determined by the first-order ordinary differential equation

$$(bu+c+aC_2^2)u'_\xi+2bC_1u=8bC_1^2\xi+C_3.$$

With appropriate translations in both variables, one can make the equation homogeneous, which means it is integrable by quadrature.

4°. Solution:

$$w=v(r)-4bC_2^2x^2-4abC_1^2y^2,\qquad r=z+bC_1x^2+bC_2y^2,$$

where C_1 and C_2 are arbitrary constants and the function $v(r)$ is determined by the first-order ordinary differential equation

$$(bv+c)v'_r+2b(aC_2+C_1)v=8b(a^2C_1^2+C_2^2)r+C_3.$$

With appropriate translations in both variables, one can make the equation homogeneous.

5°. Solution (generalizes the solution of Items 3° and 4°):

$$w=U(\zeta)+A_1x^2+A_2y^2+A_3xy+A_4x+A_5y,\qquad \zeta=z+b(B_1x^2+B_2y^2+B_3xy+B_4x+B_5y),$$

where B_1, B_2, B_3, B_4, and B_5 are arbitrary constants, the coefficients A_n are expressed in terms of B_n by

$$A_1 = -b(4B_1^2 + aB_3^2),$$
$$A_2 = -b(B_3^2 + 4aB_2^2),$$
$$A_3 = -4bB_3(B_1 + aB_2),$$
$$A_4 = -2b(2B_1B_4 + aB_3B_5),$$
$$A_5 = -2b(B_3B_4 + 2aB_2B_5),$$

and the function $U(\zeta)$ is determined by the first-order ordinary differential equation

$$(bU + c + ab^2B_5^2 + b^2B_4^2)U'_\zeta + 2b(aB_2 + B_1)U + 2(aA_2 + A_1)\zeta = C_1.$$

With appropriate translations in both variables, one can make the equation homogeneous, which means it is integrable by quadrature.

6°. "Two-dimensional" generalized separable solution linear in z (generalizes the second and third solutions of Item 2°):

$$w = f(x,\eta)z + g(x,\eta), \qquad \eta = a^{-1/2}y,$$

where the functions f and g are determined by the system of differential equations

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial \eta^2} = 0, \tag{1}$$

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial \eta^2} = -bf^2. \tag{2}$$

Equation (1) is the Laplace equation. Given $f = f(x,\eta)$, (2) represents a Helmholtz equation. For solutions of these linear equations, see Tikhonov and Samarskii (1990) and Polyanin (2002).

7°. "Two-dimensional" generalized separable solution quadratic in z:

$$w = f(x,y)z^2 + g(x,y)z + h(x,y),$$

where the functions $f = f(x,y)$, $g = g(x,y)$, and $h = h(x,y)$ are determined by the system of differential equations

$$f_{xx} + af_{yy} + 6bf^2 = 0,$$
$$g_{xx} + ag_{yy} + 6bfg = 0,$$
$$h_{xx} + ah_{yy} + bg^2 + 2bfh + 2cf = 0.$$

Here, the subscripts denote the corresponding partial derivatives.

8°. "Two-dimensional" solution (generalizes the last three solutions of Item 2°):

$$w(x,y,z) = |z|^{1/2}U(x,\eta) - \frac{c}{b}, \qquad \eta = a^{-1/2}y,$$

where the function $U = U(x,\eta)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial \eta^2} = 0.$$

9°. There are solutions of the following forms:

$$w(x,y,z) = F(z,r), \quad r = ax^2 + y^2 \qquad \text{"two-dimensional" solution;}$$
$$w(x,y,z) = x^{2\lambda}G(\xi,\eta) - \frac{c}{b}, \quad \xi = \frac{y}{x}, \quad \eta = \frac{z}{x^{\lambda+1}} \qquad \text{"two-dimensional" solution;}$$
$$w(x,y,z) = H(\zeta), \quad \zeta = (ax^2 + y^2)z^{-2} \qquad \text{"one-dimensional" solution;}$$

where λ is an arbitrary constant.

10°+. The substitution $u = w + (c/b)$ leads to a special case of equation 6.1.2.3 with $n = 1$.

Remark. In the special case $a = 1$, $b < 0$, and $c > 0$, the equation in question describes transonic flows of ideal polytropic gases (Pokhozhaev, 1989).

2. $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial y}\left[(a_1 w + b_1)\dfrac{\partial w}{\partial y}\right] + \dfrac{\partial}{\partial z}\left[(a_2 w + b_2)\dfrac{\partial w}{\partial z}\right] = 0.$

1°. Suppose $w(x, y, z)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 z + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution:

$$w(x, y, z) = A\sqrt{k_1 x + k_2 y + k_3 z + B} - \frac{k_1^2 + b_1 k_2^2 + b_2 k_3^2}{a_1 k_2^2 + a_2 k_3^2},$$

where A, B, k_1, k_2, and k_3 are arbitrary constants.

3°. Solution linear in y and z:

$$\begin{aligned} w(x, y, z) &= (A_1 x + B_1)y + (A_2 x + B_2)z \\ &\quad - \tfrac{1}{12}(a_1 A_1^2 + a_2 A_2^2)x^4 - \tfrac{1}{3}(a_1 A_1 B_1 + a_2 A_2 B_2)x^3 - \tfrac{1}{2}(a_1 B_1^2 + a_2 B_2^2)x^2 + Cx + D, \end{aligned}$$

where A_1, A_2, B_1, B_2, C, and D are arbitrary constants.

4°. There is a generalized separable solution of the form

$$w(x, y, z) = f(x)y^2 + g(x)yz + h(x)z^2 + \varphi(x)y + \psi(x)z + \chi(x).$$

5°. For other solutions, see equation 6.3.2.3 with $f(w) = 1$, $g(w) = a_1 w + b_1$, and $h(w) = a_2 w + b_2$.

3. $\dfrac{\partial^2 w}{\partial x^2} + a\dfrac{\partial^2 w}{\partial y^2} + b\dfrac{\partial}{\partial z}\left(w^n \dfrac{\partial w}{\partial z}\right) = 0.$

1°. Suppose $w(x, y, z)$ is a solution of this equation. Then the functions

$$\begin{aligned} w_1 &= C_1^{-2} w(\pm C_2 x + C_3, \pm C_2 y + C_4, \pm C_1^n C_2 z + C_5), \\ w_2 &= w(x\cos\beta + y a^{-1/2}\,\sin\beta, -x a^{1/2}\,\sin\beta + y\cos\beta,\ z), \end{aligned}$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Solutions:

$$\begin{aligned} w(x, y, z) &= z^{\frac{1}{n+1}}\,[C_1(ax^2 - y^2) + C_2 x + C_3 y + C_4], \\ w(x, y, z) &= z^{\frac{1}{n+1}}\,[C_1 \ln(ax^2 + y^2) + C_2], \\ w(x, y, z) &= C_1 z^{\frac{1}{n+1}}\,\exp(\sqrt{a}C_2 x)\cos(C_2 y + C_3), \end{aligned}$$

where $C_1, \ldots, C_4$ are arbitrary constants.

3°. "Two-dimensional" solution (generalizes the solutions of Item 2°):

$$w(x, y, z) = z^{\frac{1}{n+1}}\,U(x, \eta), \qquad \eta = a^{-1/2} y,$$

where the function $U = U(x, \eta)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial \eta^2} = 0.$$

For this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. "Two-dimensional" solution:

$$w(x, y, z) = u(x, \eta)z^{2/n}, \qquad \eta = a^{-1/2}y,$$

where the function $u = u(x, \eta)$ is determined by a differential equation of the form 5.1.1:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{2b(n+2)}{n^2}u^{n+1} = 0.$$

For $n = -1$ and $n = -2$, this equation is linear.

Remark. The solutions of Items 2° and 3° are special cases of a multiplicative separable solution $w = u(x, y)\theta(z)$, where $\theta = \theta(z)$ is determined by the autonomous ordinary differential equation $(\theta^n\theta'_z)'_z = C\theta$.

5°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, z) = F(z, r), \qquad r = ax^2 + y^2;$$
$$w(x, y, z) = x^{2\lambda}G(\xi, \eta), \qquad \xi = \frac{y}{x}, \qquad \eta = \frac{z}{x^{n\lambda+1}};$$
$$w(x, y, z) = |x|^{-2/n}H(z, \zeta), \qquad \zeta = y/x;$$
$$w(x, y, z) = |z|^{2/n}U(t_1, t_2), \qquad t_1 = x + k_1 \ln|z|, \qquad t_2 = y + k_2 \ln|z|;$$
$$w(x, y, z) = \exp\Bigl(-\frac{2z}{n+1}\Bigr)V(\rho_1, \rho_2), \qquad \rho_1 = x\exp\Bigl(-\frac{nz}{n+1}\Bigr), \qquad \rho_2 = y\exp\Bigl(-\frac{nz}{n+1}\Bigr),$$

where k_1, k_2, and λ are arbitrary constants.

6°. There are solutions of the following forms:

$$w(x, y, z) = W(\zeta), \qquad \zeta = (ax^2 + y^2)z^{-2};$$
$$w(x, y, z) = S(r)z^{2/n}, \qquad r = ax^2 + y^2.$$

7°. For other solutions, see equation 6.1.2.5, where n should be set equal to zero and then k should be renamed n.

⊙ *Reference*: N. Ibragimov (1994).

4. $\displaystyle \frac{\partial^2 w}{\partial x^2} + a\frac{\partial}{\partial y}\Bigl(w^n\frac{\partial w}{\partial y}\Bigr) + b\frac{\partial}{\partial z}\Bigl(w^n\frac{\partial w}{\partial z}\Bigr) = 0.$

1°. Suppose $w(x, y, z)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2}w(\pm C_2x + C_3, \pm C_1^nC_2y + C_4, \pm C_1^nC_2z + C_5),$$
$$w_2 = w\bigl(x,\ y\cos\beta + z\sqrt{a/b}\,\sin\beta, -y\sqrt{b/a}\,\sin\beta + z\cos\beta\bigr),$$

where $C_1, \ldots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Degenerate solutions:

$$w(x, y, z) = x\bigl[C_1(by^2 - ax^2) + C_2x + C_3y + C_4\bigr]^{\frac{1}{n+1}},$$
$$w(x, y, z) = x\bigl[C_1\ln\bigl(by^2 + az^2\bigr) + C_2\bigr]^{\frac{1}{n+1}},$$
$$w(x, y, z) = x\bigl[C_1\exp\bigl(\lambda\sqrt{b}\,y\bigr)\sin\bigl(\lambda\sqrt{a}\,z + C_2\bigr) + C_3\bigr]^{\frac{1}{n+1}},$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants.

3°. "Two-dimensional" solution (generalizes the solutions of Item 2°):

$$w(x,y,z) = (C_1 x + C_2)\big[U(\xi,\eta)\big]^{\frac{1}{n+1}}, \quad \xi = \sqrt{b}\,y, \quad \eta = \sqrt{a}\,z,$$

where the function $U = U(\xi,\eta)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} = 0.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. "Two-dimensional" solution:

$$w(x,y,z) = x^{-2/n}\Theta(y,z),$$

where the function $\Theta = \Theta(y,z)$ is determined by the differential equation

$$a\frac{\partial}{\partial y}\left(\Theta^n \frac{\partial \Theta}{\partial y}\right) + b\frac{\partial}{\partial z}\left(\Theta^n \frac{\partial \Theta}{\partial z}\right) + \frac{2(n+2)}{n^2}\Theta = 0.$$

For $n = -2$, the equation obtained can be reduced, with the transformation $u = 1/\Theta$, $\xi = \sqrt{b}\,y$, $\eta = \sqrt{a}\,z$, to the Laplace equation.

Remark. The solutions of Items 2° and 3° are special cases of a multiplicative separable solution $w = \varphi(x)u(y,z)$, where $\varphi = \varphi(x)$ is determined by the autonomous ordinary differential equation $\varphi''_{zz} = C\varphi^{n+1}$.

5°. There are solutions of the following forms:

$w(x,y,z) = F(x,r), \quad r = by^2 + az^2$	"two-dimensional" solution;
$w(x,y,z) = x^{2\lambda}G(\xi,\eta), \quad \xi = \dfrac{y}{x^{n\lambda+1}}, \quad \eta = \dfrac{z}{x^{n\lambda+1}}$	"two-dimensional" solution;
$w(x,y,z) = z^{2/n}H(x,\zeta), \quad \zeta = z/y$	"two-dimensional" solution;
$w(x,y,z) = \lvert x\rvert^{-2/n}U(z_1,z_2), \quad z_1 = y + k_1 \ln\lvert x\rvert, \quad z_2 = z + k_2 \ln\lvert x\rvert$	"two-dimensional" solution;
$w(x,y,z) = e^{-2x}V(\rho_1,\rho_2), \quad \rho_1 = ye^{nx}, \quad \rho_2 = ze^{nx}$	"two-dimensional" solution;
$w(x,y,z) = W(\theta), \quad \theta = (by^2 + az^2)x^{-2}$	"one-dimensional" solution,

where k_1, k_2, and λ are arbitrary constants.

6°. For other solutions, see equation 6.1.2.5 with $k = n$.

⊙ *Reference*: N. Ibragimov (1994).

5. $\displaystyle \frac{\partial^2 w}{\partial x^2} + a\frac{\partial}{\partial y}\left(w^n \frac{\partial w}{\partial y}\right) + b\frac{\partial}{\partial z}\left(w^k \frac{\partial w}{\partial z}\right) = 0.$

This is a special case of equation 6.3.2.3 with $f(w) = 1$, $g(w) = aw^n$, and $h(w) = bw^k$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2} w(\pm C_2 x + C_3, \pm C_1^n C_2 y + C_4, \pm C_1^k C_2 z + C_5),$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\beta_1^2 w + \frac{a\beta_2^2}{n+1}w^{n+1} + \frac{b\beta_3^2}{k+1}w^{k+1} = C_1(\beta_1 x + \beta_2 y + \beta_3 z) + C_2,$$

where C_1, C_2, β_1, β_2, and β_3 are arbitrary constants.

3°. "Two-dimensional" solution (c_1 and c_2 are arbitrary constants):

$$w(x,y,z) = u(x,\xi), \quad \xi = c_1 y + c_2 z,$$

where the function $u = u(x,\xi)$ is determined by a differential equation of the form 5.4.4.8:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial \xi}\left[\varphi(u)\frac{\partial u}{\partial \xi}\right] = 0, \qquad \varphi(u) = ac_1^2 u^n + bc_2^2 u^k,$$

which can be reduced to a linear equation.

4°. "Two-dimensional" solution (s_1 and s_2 are arbitrary constants):

$$w(x,y,z) = v(y,\eta), \quad \eta = s_1 x + s_2 z,$$

where the function $v = v(y,\eta)$ is determined by a differential equation of the form 5.4.4.8:

$$a\frac{\partial}{\partial y}\left(v^n \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial \eta}\left[\psi(v)\frac{\partial v}{\partial \eta}\right] = 0, \qquad \psi(v) = s_1^2 + bs_2^2 v^k,$$

which can be reduced to a linear equation.

5°. There is a "two-dimensional" solution of the form (generalize the solutions of Items 3° and 4°):

$$w(x,y,z) = U(z_1,z_2), \quad z_1 = a_1 y + b_1 z + c_1 x, \quad z_2 = a_2 y + b_2 z + c_2 x.$$

6°. There are exact solutions of the following forms:

$$w(x,y,z) = x^{2\lambda} F(\xi,\eta), \quad \xi = \frac{y}{x^{n\lambda+1}}, \quad \eta = \frac{z}{x^{k\lambda+1}} \qquad \text{"two-dimensional" solution;}$$
$$w(x,y,z) = y^{2/n} G(\zeta,x), \quad \zeta = y^{-k/n} z \qquad \text{"two-dimensional" solution;}$$
$$w(x,y,z) = e^{-2x} H(z_1,z_2), \quad z_1 = ye^{nx}, \quad z_2 = ze^{kx} \qquad \text{"two-dimensional" solution;}$$
$$w(x,y,z) = (y/x)^{2/n} U(\theta), \quad \theta = x^{k/n-1} y^{-k/n} z \qquad \text{"one-dimensional" solution;}$$

where λ is an arbitrary constant.

⊙ *References*: N. Ibragimov (1994), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2002).

6. $\displaystyle a_1\frac{\partial}{\partial y}\left(w^{n_1}\frac{\partial w}{\partial y}\right) + a_2\frac{\partial}{\partial y}\left(w^{n_2}\frac{\partial w}{\partial y}\right) + a_3\frac{\partial}{\partial z}\left(w^{n_3}\frac{\partial w}{\partial z}\right) = 0.$

This is a special case of equation 6.3.2.3 with $f(w) = a_1 w^{n_1}$, $g(w) = a_2 w^{n_2}$, and $h(w) = a_3 w^{n_3}$.

6.2. Equations with Three Space Variables Involving Exponential Nonlinearities

6.2.1. Equations of the Form $\frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[h(z)\frac{\partial w}{\partial z}\right] = ae^{\lambda w}$

1. $\displaystyle \frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(cz^k\frac{\partial w}{\partial z}\right) = se^{\lambda w}.$

This is a special case of equation 6.3.1.3 with $f(w) = se^{\lambda w}$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1 = w\Big(C_1^{\frac{2}{2-n}} x,\, C_1^{\frac{2}{2-m}} y,\, C_1^{\frac{2}{2-k}} y\Big) + \frac{2}{\lambda}\ln C_1,$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $n \neq 2$, $m \neq 2$, and $k \neq 2$:

$$w = -\frac{1}{\lambda} \ln\left\{ \frac{s\lambda}{A} \left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2} \right] \right\}, \qquad A = 1 - \frac{1}{2-n} - \frac{1}{2-m} - \frac{1}{2-k}.$$

3°. Functional separable solution for $n \neq 2$, $m \neq 2$, and $k \neq 2$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2} \right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{B}{r} w'_r = s e^{\lambda w}, \qquad B = \frac{2}{2-n} + \frac{2}{2-m} + \frac{2}{2-k} - 1.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z) = U(\xi, z), \quad \xi^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2},$$

$$w(x,y,z) = V(x, \eta), \quad \eta^2 = \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2},$$

$$w(x,y,z) = W(y, \zeta), \quad \zeta^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{z^{2-k}}{c(2-k)^2},$$

$$w(x,y,z) = F(\rho_1, \rho_2) + \frac{n-2}{\lambda} \ln x, \quad \rho_1 = y x^{\frac{n-2}{2-m}}, \quad \rho_2 = z x^{\frac{n-2}{2-k}}.$$

2. $\displaystyle \frac{\partial}{\partial x}\left(a x^n \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(b y^m \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(c e^{\lambda z} \frac{\partial w}{\partial z}\right) = s e^{\sigma w}.$

This is a special case of equation 6.3.1.5 with $f(w) = s e^{\sigma w}$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1 = w\left(C_1^{\frac{2}{2-n}} x,\ C_1^{\frac{2}{2-m}} y,\ z - \frac{2}{\lambda} \ln C_1 \right) + \frac{2}{\sigma} \ln C_1,$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $n \neq 2$, $m \neq 2$, and $\lambda \neq 0$:

$$w = -\frac{1}{\sigma} \ln\left\{ \frac{s\sigma}{A} \left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\lambda z}}{c\lambda^2} \right] \right\}, \qquad A = 1 - \frac{1}{2-n} - \frac{1}{2-m}.$$

3°. Functional separable solution for $n \neq 2$, $m \neq 2$, and $\lambda \neq 0$ (generalizes the solution of Item 2°):

$$w = w(r), \qquad r^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\lambda z}}{c\lambda^2} \right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{B}{r} w'_r = s e^{\sigma w}, \qquad B = \frac{2}{2-n} + \frac{2}{2-m} - 1.$$

4°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z) = U(\xi, z), \quad \xi^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2},$$

$$w(x,y,z) = V(x, \eta), \quad \eta^2 = \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\lambda z}}{c\lambda^2},$$

$$w(x,y,z) = W(y, \zeta), \quad \zeta^2 = \frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda z}}{c\lambda^2},$$

$$w(x,y,z) = F(\rho_1, \rho_2) + \frac{n-2}{\sigma} \ln x, \quad \rho_1 = y x^{\frac{n-2}{2-m}}, \quad \rho_2 = z + \frac{2-n}{\lambda} \ln x.$$

3. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\beta y}\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right)=se^{\sigma w}.$

This is a special case of equation 6.3.1.6 with $f(w)=se^{\sigma w}$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1=w\Big(C_1^{\frac{2}{2-n}}x,\ y-\frac{2}{\beta}\ln C_1,\ z-\frac{2}{\lambda}\ln C_1\Big)+\frac{2}{\sigma}\ln C_1,$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $n\neq 2$, $\beta\neq 0$, and $\lambda\neq 0$:

$$w=-\frac{1}{\sigma}\ln\left\{\frac{s\sigma(2-n)}{1-n}\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right]\right\}.$$

3°. Functional separable solution for $n\neq 2$, $\beta\neq 0$, and $\lambda\neq 0$ (generalizes the solution of Item 2°):

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}+\frac{A}{r}w'_r=se^{\sigma w},\qquad A=\frac{n}{2-n}.$$

4°. There are “two-dimensional” solutions of the following forms:

$$\begin{aligned}
&w(x,y,z)=U(\xi,z),\quad \xi^2=\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\beta y}}{b\beta^2},\\
&w(x,y,z)=V(x,\eta),\quad \eta^2=\frac{e^{-\beta y}}{b\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2},\\
&w(x,y,z)=W(y,\zeta),\quad \zeta^2=\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\lambda z}}{c\lambda^2},\\
&w(x,y,z)=F(\rho_1,\rho_2)+\frac{n-2}{\sigma}\ln x,\quad \rho_1=y+\frac{2-n}{\beta}\ln x,\quad \rho_2=z+\frac{2-n}{\lambda}\ln x.
\end{aligned}$$

4. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\gamma y}\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right)=se^{\sigma w}.$

This is a special case of equation 6.3.1.4 with $f(w)=se^{\sigma w}$.

1°. Suppose $w(x,y,z)$ is a solution of the equation in question. Then the function

$$w_1=w\Big(x-\frac{2}{\beta}\ln C_1,\ y-\frac{2}{\gamma}\ln C_1,\ z-\frac{2}{\lambda}\ln C_1\Big)+\frac{2}{\sigma}\ln C_1,$$

where C_1 is an arbitrary constant, is also a solution of the equation.

2°. Solution for $\beta\neq 0$, $\gamma\neq 0$, and $\lambda\neq 0$:

$$w=-\frac{1}{\sigma}\ln\left[s\sigma\left(\frac{e^{-\beta x}}{a\beta^2}+\frac{e^{-\gamma y}}{b\gamma^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right)\right].$$

3°. Functional separable solution for $\beta\neq 0$, $\gamma\neq 0$, and $\lambda\neq 0$ (generalizes the solution of Item 2°):

$$w=w(r),\qquad r^2=4\left(\frac{e^{-\beta x}}{a\beta^2}+\frac{e^{-\gamma y}}{b\gamma^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right),$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} - \frac{1}{r} w'_r = s e^{\sigma w}.$$

4°. There are "two-dimensional" solutions of the following forms:

$$\begin{aligned}
w(x,y,z) &= U(\xi,z), \quad \xi^2 = \frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\gamma y}}{b\gamma^2},\\
w(x,y,z) &= V(x,\eta), \quad \eta^2 = \frac{e^{-\gamma y}}{b\gamma^2} + \frac{e^{-\lambda z}}{c\lambda^2},\\
w(x,y,z) &= W(y,\zeta), \quad \zeta^2 = \frac{e^{-\beta x}}{a\beta^2} + \frac{e^{-\lambda z}}{c\lambda^2},\\
w(x,y,z) &= F(\rho_1,\rho_2) + \frac{\beta}{\sigma}x, \quad \rho_1 = y - \frac{\beta}{\gamma}x, \quad \rho_2 = z - \frac{\beta}{\lambda}x.
\end{aligned}$$

6.2.2. Equations of the Form $a_1 \frac{\partial}{\partial x}\left(e^{\lambda_1 w}\frac{\partial w}{\partial x}\right) + a_2 \frac{\partial}{\partial y}\left(e^{\lambda_2 w}\frac{\partial w}{\partial y}\right) + a_3 \frac{\partial}{\partial y}\left(e^{\lambda_2 w}\frac{\partial w}{\partial y}\right) = be^{\beta w}$

1. $\displaystyle \frac{\partial^2 w}{\partial x^2} + a\frac{\partial^2 w}{\partial y^2} + b\frac{\partial}{\partial z}\left(e^w \frac{\partial w}{\partial z}\right) = 0.$

1°. Suppose $w(x,y,z)$ is a solution of this equation. Then the functions

$$\begin{aligned}
w_1 &= w(C_1 x + C_3, \pm C_1 y + C_4, C_2 z + C_5) + \ln\frac{C_1^2}{C_2^2},\\
w_2 &= w(x\cos\beta + ya^{-1/2}\sin\beta,\ -xa^{1/2}\sin\beta + y\cos\beta,\ z),
\end{aligned}$$

where $C_1, \dots, C_5$ and β are arbitrary constants, are also solutions of the equation (the plus or minus signs in w_1 are chosen arbitrarily).

2°. Solutions:

$$\begin{aligned}
w(x,y,z) &= C_1(ax^2 - y^2) + C_2 xy + C_3 x + C_4 y + C_5 + \ln(C_6 z + C_7),\\
w(x,y,z) &= C_1 \exp(\sqrt{a}\,C_2 x)\sin(C_2 y + C_3) + \ln(C_4 z + C_5),\\
w(x,y,z) &= C_1 \exp(C_2 y)\sin(\sqrt{a}\,C_2 x + C_3) + \ln(C_4 z + C_5),\\
w(x,y,z) &= \ln\left[\frac{(C_1^2 + aC_2^2)(z + C_4)^2}{b\cosh^2(C_1 x + C_2 y + C_3)}\right],\\
w(x,y,z) &= \ln\left(\frac{4aC_3}{b}\right) - 2\ln\bigl|(y + C_1)^2 + a(x + C_2)^2 + C_3\bigr| + 2\ln|z + C_4|,
\end{aligned}$$

where $C_1, \dots, C_7$ are arbitrary constants.

3°. "Two-dimensional" solution (generalizes the first three solutions of Item 2°):

$$w(x,y,z) = U(x,\eta) + \ln(C_1 z + C_2), \qquad \eta = a^{-1/2} y,$$

where C_1 and C_2 are arbitrary constants and the function $U(x,\eta)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial \eta^2} = 0.$$

For this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. "Two-dimensional" solution:

$$w(x, y, z) = V(x, \eta) + 2\ln|z + C|, \qquad \eta = a^{-1/2}y,$$

where the function $V = V(x, \eta)$ is determined by a solvable differential equation of the form 5.2.1.1:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial \eta^2} = -2be^V.$$

5°. There are solutions of the following forms:

$$\begin{aligned}
&w(x, y, z) = F(r, z), \quad r = ax^2 + y^2 && \text{"two-dimensional";}\\
&w(x, y, z) = G(\xi_1, \xi_2) - 2k\ln|x|, \quad \xi_1 = yx^{-1}, \quad \xi_2 = z|x|^{k-1} && \text{"two-dimensional";}\\
&w(x, y, z) = H(\eta_1, \eta_2) + 2k\ln|z|, \quad \eta_1 = x|z|^{k-1}, \quad \eta_2 = y|z|^{k-1} && \text{"two-dimensional";}\\
&w(x, y, z) = U(\zeta_1, \zeta_2) + 2\ln|z|, \quad \zeta_1 = x + k_1\ln|z|, \quad \zeta_2 = y + k_2\ln|z| && \text{"two-dimensional";}\\
&w(x, y, z) = V(\rho_1, \rho_2) + 2z, \quad \rho_1 = xe^z, \quad \rho_2 = ye^z && \text{"two-dimensional";}\\
&w(x, y, z) = W(\chi), \quad \chi = (ax^2 + y^2)z^{-2} && \text{"one-dimensional";}
\end{aligned}$$

where k, k_1, and k_2 are arbitrary constants.

6°. For other exact solutions, see equation 6.3.2.3 with $f(w) = 1$, $g(w) = a$, and $h(w) = be^w$.

2. $\displaystyle \frac{\partial^2 w}{\partial x^2} + a\frac{\partial}{\partial y}\left(e^{\lambda w}\frac{\partial w}{\partial y}\right) + b\frac{\partial}{\partial z}\left(e^{\lambda w}\frac{\partial w}{\partial z}\right) = 0.$

1°. Suppose $w(x, y, z)$ is a solution of this equation. Then the functions

$$\begin{aligned}
w_1 &= w(C_1x + C_3, C_2y + C_4, C_2z + C_5) + \frac{1}{\lambda}\ln\frac{C_1^2}{C_2^2},\\
w_2 &= w\bigl(x,\ y\cos\beta + z\sqrt{a/b}\,\sin\beta,\ -y\sqrt{b/a}\,\sin\beta + z\cos\beta\bigr),
\end{aligned}$$

where $C_1, \ldots, C_5$ and β are arbitrary constants,

2°. Solutions:

$$\begin{aligned}
w(x, y, z) &= C_1x + C_2 + \frac{1}{\lambda}\ln(C_3y + C_4z + C_5);\\
w(x, y, z) &= C_1x + C_2 + \frac{1}{\lambda}\ln\bigl[C_3(by^2 - az^2) + C_4yz + C_5\bigr];\\
w(x, y, z) &= C_1x + C_2 + \frac{1}{\lambda}\ln\bigl[C_3\ln(by^2 + az^2) + C_4\bigr];\\
w(x, y, z) &= C_1x + C_2 + \sqrt{b}\,C_3y + \frac{1}{\lambda}\ln\cos\bigl(\sqrt{a}\,C_3\lambda z + C_4\bigr);\\
w(x, y, z) &= C_1x + C_2 + \frac{1}{\lambda}\ln\bigl[C_3\exp\bigl(\sqrt{b}\,C_4y\bigr)\cos\bigl(\sqrt{a}\,C_4z + C_5\bigr) + C_6\bigr];\\
w(x, y, z) &= \frac{1}{\lambda}\ln\left[\frac{-aC_1^2y^2 + C_2\exp\bigl(\sqrt{b}\,C_3y\bigr)\cos\bigl(\sqrt{a}\,C_3z + C_4\bigr)}{\cos^2(aC_1x + C_5)}\right];\\
w(x, y, z) &= \frac{1}{\lambda}\ln\left[\frac{-bC_1^2z^2 + C_2\exp\bigl(\sqrt{b}\,C_3y\bigr)\cos\bigl(\sqrt{a}\,C_3z + C_4\bigr)}{\cos^2(bC_1x + C_5)}\right];\\
w(x, y, z) &= \frac{1}{\lambda}\ln\left[\frac{-aC_1^2y^2 + C_2\exp\bigl(\sqrt{b}\,C_3y\bigr)\cos\bigl(\sqrt{a}\,C_3z + C_4\bigr)}{\sinh^2(aC_1x + C_5)}\right];\\
w(x, y, z) &= \frac{1}{\lambda}\ln\left[\frac{-bC_1^2z^2 + C_2\exp\bigl(\sqrt{b}\,C_3y\bigr)\cos\bigl(\sqrt{a}\,C_3z + C_4\bigr)}{\sinh^2(bC_1x + C_5)}\right];
\end{aligned}$$

$$w(x,y,z) = \frac{1}{\lambda} \ln \left[\frac{aC_1^2 y^2 + C_2 \exp\bigl(\sqrt{b}\, C_3 y\bigr) \cos\bigl(\sqrt{a}\, C_3 z + C_4\bigr)}{\cosh^2(aC_1 x + C_5)} \right];$$

$$w(x,y,z) = \frac{1}{\lambda} \ln \left[\frac{bC_1^2 z^2 + C_2 \exp\bigl(\sqrt{b}\, C_3 y\bigr) \cos\bigl(\sqrt{a}\, C_3 z + C_4\bigr)}{\cosh^2(bC_1 x + C_5)} \right];$$

where $C_1, \ldots, C_6$ are arbitrary constants.

3°. "Two-dimensional" solution (generalizes the first five solutions of Item 2°):

$$w(x,y,z) = C_1 x + C_2 + \frac{1}{\lambda} \ln U(\xi, \eta), \qquad \xi = \frac{y}{\sqrt{a}}, \qquad \eta = \frac{z}{\sqrt{b}},$$

where C_1 and C_2 are arbitrary constants and the function $U = U(\xi, \eta)$ is determined by the Laplace equation

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} = 0.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

4°. "Two-dimensional" solution:

$$w(x,y,z) = f(x) + \frac{1}{\lambda} \ln V(\xi, \eta), \qquad \xi = \frac{y}{\sqrt{a}}, \qquad \eta = \frac{z}{\sqrt{b}},$$

where the function $f = f(x)$ is determined by the autonomous ordinary differential equation (k is an arbitrary constant)

$$f''_{xx} + ke^{\lambda f} = 0, \tag{1}$$

and the function $V = V(\xi, \eta)$ is a solution of the Poisson equation

$$\Delta V - k\lambda = 0, \qquad \Delta = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}. \tag{2}$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

The general solution of equation (1) is expressed as

$$f(x) = \begin{cases} -\dfrac{1}{\lambda} \ln\bigl[-\tfrac{1}{2} k\lambda (x + C_1)^2\bigr] & \text{if } k\lambda < 0, \\ -\dfrac{1}{\lambda} \ln\left[-\dfrac{k\lambda}{2C_1^2} \cos^2(C_1 x + C_2)\right] & \text{if } k\lambda < 0, \\ -\dfrac{1}{\lambda} \ln\left[-\dfrac{k\lambda}{2C_1^2} \sinh^2(C_1 x + C_2)\right] & \text{if } k\lambda < 0, \\ -\dfrac{1}{\lambda} \ln\left[\dfrac{k\lambda}{2C_1^2} \cosh^2(C_1 x + C_2)\right] & \text{if } k\lambda > 0. \end{cases}$$

5°. There are solutions of the following forms:

$$w(x,y,z) = F(x,\tau) + \frac{2}{\lambda} \ln |y|, \quad \tau = \frac{z}{y}, \qquad \text{"two-dimensional" solution;}$$

$$w(x,y,z) = G(x,r), \quad r = by^2 + az^2 \qquad \text{"two-dimensional" solution;}$$

$$w(x,y,z) = H(z_1, z_2) - \frac{2k}{\lambda} \ln |x|, \quad z_1 = y|x|^{k-1}, \quad z_2 = z|x|^{k-1} \qquad \text{"two-dimensional" solution;}$$

$$w(x,y,z) = U(\xi, \eta) - \frac{2}{\lambda} \ln |x|, \quad \xi = y + k_1 \ln |x|, \quad \eta = z + k_2 \ln |x| \qquad \text{"two-dimensional" solution;}$$

$$w(x,y,z) = V(\rho_1, \rho_2) - \frac{2}{\lambda} x, \quad \rho_1 = ye^x, \quad \rho_2 = ze^x \qquad \text{"two-dimensional" solution;}$$

$$w(x,y,z) = W(\zeta), \quad \zeta = \frac{by^2 + az^2}{x^2} \qquad \text{"one-dimensional" solution.}$$

6°. For other exact solutions, see equation 6.3.2.3 with $f(w) = 1$, $g(w) = ae^{\lambda w}$, and $h(w) = be^{\lambda w}$.

3. $\dfrac{\partial^2 w}{\partial x^2}+a\dfrac{\partial}{\partial y}\left(e^{w}\dfrac{\partial w}{\partial y}\right)+b\dfrac{\partial}{\partial z}\left(e^{\lambda w}\dfrac{\partial w}{\partial z}\right)=0.$

1°. Suppose $w(x,y,z)$ is a solution of this equation. Then the function

$$w_1=w(C_1x+C_3,\,C_1C_2y+C_4,\,C_1C_2^{\lambda}z+C_5)-2\ln|C_2|,$$

where $C_1,\,\dots,\,C_5$ are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution in implicit form:

$$k_1^2w+ak_2^2e^w+bk_3^2\lambda^{-1}e^{\lambda w}=C_1(k_1x+k_2y+k_3z)+C_2,$$

where C_1, C_2, k_1, k_2, and k_3 are arbitrary constants.

3°. "Two-dimensional" solution (c_1 and c_2 are arbitrary constants):

$$w(x,y,z)=u(x,\xi),\qquad \xi=c_1y+c_2z,$$

where the function $u=u(x,\xi)$ is determined by a differential equation of the form 5.4.4.8:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial \xi}\left[\varphi(u)\frac{\partial u}{\partial \xi}\right]=0,\qquad \varphi(u)=ac_1^2e^u+bc_2^2e^{\lambda u},$$

which can be reduced to a linear equation.

4°. "Two-dimensional" solution (s_1 and s_2 are arbitrary constants):

$$w(x,y,z)=v(y,\eta),\qquad \eta=s_1x+s_2z,$$

where the function $v=v(y,\eta)$ is determined by a differential equation of the form 5.4.4.8:

$$a\frac{\partial}{\partial y}\left(e^v\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial \eta}\left[\psi(v)\frac{\partial v}{\partial \eta}\right]=0,\qquad \psi(v)=bs_2^2e^{\lambda v}+s_1^2,$$

which can be reduced to a linear equation.

5°. There is a "two-dimensional" solution of the form (generalize the solutions of Items 3° and 4°):

$$w(x,y,z)=U(z_1,z_2),\qquad z_1=a_1x+b_1y+c_1z,\qquad z_2=a_2x+b_2y+c_2z.$$

6°. There are exact solutions of the following forms:

$$\begin{aligned}
&w(x,y,z)=F(\xi_1,\xi_2)-2k\ln|x|, \quad \xi_1=y|x|^{k-1}, \quad \xi_2=z|x|^{k\lambda-1} && \text{"two-dimensional";}\\
&w(x,y,z)=G(x,\eta)+2\ln|y|, \quad \eta=|y|^{-\lambda}z && \text{"two-dimensional";}\\
&w(x,y,z)=H(\zeta_1,\zeta_2)-2kx, \quad \zeta_1=ye^{kx}, \quad \zeta_2=ze^{k\lambda x} && \text{"two-dimensional";}\\
&w(x,y,z)=V(\rho)+2\ln|y/x|, \quad \rho=|x|^{\lambda-1}|y|^{-\lambda}z && \text{"one-dimensional";}
\end{aligned}$$

where k is an arbitrary constant.

7°. For other exact solutions, see equation 6.3.2.2 with $f(w)=1$, $g(w)=ae^w$, and $h(w)=be^{\lambda w}$.

⊙ *Reference*: N. Ibragimov (1994).

4. $a_1\dfrac{\partial}{\partial x}\left(e^{\lambda_1 w}\dfrac{\partial w}{\partial x}\right)+a_2\dfrac{\partial}{\partial y}\left(e^{\lambda_2 w}\dfrac{\partial w}{\partial y}\right)+a_3\dfrac{\partial}{\partial z}\left(e^{\lambda_3 w}\dfrac{\partial w}{\partial z}\right)=0.$

1°. Suppose $w(x,y,z)$ is a solution of this equation. Then the functions

$$w_1=w(\pm C_1C_2^{\lambda_1}x+C_3,\,\pm C_1C_2^{\lambda_2}y+C_4,\,\pm C_1C_2^{\lambda_3}z+C_5)-2\ln|C_2|,$$

where $C_1,\,\dots,\,C_5$ are arbitrary constants, 2°. There is an exact solution of the form

$$w(x,y,z) = U(\xi) - \frac{2}{\lambda_1 - \lambda_2} \ln\left|\frac{y}{x}\right|, \qquad \xi = |x|^{\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_2}} |y|^{\frac{\lambda_3-\lambda_1}{\lambda_1-\lambda_2}} z.$$

3°. There are "two-dimensional" solutions of the following forms:

$$w(x,y,z) = U(\eta_1,\eta_2) - 2k\ln|x|, \quad \eta_1 = y|x|^{k(\lambda_2-\lambda_1)-1}, \quad \eta_2 = z|x|^{k(\lambda_3-\lambda_1)-1},$$
$$w(x,y,z) = V(\zeta_1,\zeta_2) - 2kx, \quad \zeta_1 = y\exp\bigl[k(\lambda_2-\lambda_1)x\bigr], \quad \zeta_2 = z\exp\bigl[k(\lambda_3-\lambda_1)x\bigr],$$

where k is an arbitrary constant.

4°. For other exact solutions, see equation 6.3.2.3 with $f(w) = a_1 e^{\lambda_1 w}$, $g(w) = a_2 e^{\lambda_2 w}$, and $h(w) = a_3 e^{\lambda_3 w}$.

5. $\displaystyle a_1 \frac{\partial}{\partial x}\left(e^{\lambda_1 w}\frac{\partial w}{\partial x}\right) + a_2 \frac{\partial}{\partial y}\left(e^{\lambda_2 w}\frac{\partial w}{\partial y}\right) + a_3 \frac{\partial}{\partial z}\left(e^{\lambda_3 w}\frac{\partial w}{\partial z}\right) = be^{\beta w}.$

1°. Suppose $w(x,y,z)$ is a solution of this equation. Then the functions

$$w_1 = w(\pm C_1^{\beta-\lambda_1}x + C_2, \pm C_1^{\beta-\lambda_2}y + C_3, \pm C_1^{\beta-\lambda_3}z + C_4) + 2\ln|C_1|,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs are chosen arbitrarily).

2°. There is a "two-dimensional" solution of the form

$$w(x,y,z) = U(\xi,\eta) + \frac{2}{\lambda_1 - \beta}\ln|x|, \quad \xi = y|x|^{\frac{\beta-\lambda_2}{\lambda_1-\beta}}, \quad \eta = z|x|^{\frac{\beta-\lambda_3}{\lambda_1-\beta}}.$$

6.3. Three-Dimensional Equations Involving Arbitrary Functions

6.3.1. Heat and Mass Transfer Equations of the Form $\frac{\partial}{\partial x}\left[f_1(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[f_2(y)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[f_3(z)\frac{\partial w}{\partial z}\right] = g(w)$

► *Equations of this type describe steady-state heat/mass transfer or combustion processes in inhomogeneous anisotropic media. Here, $f_1(x)$, $f_2(y)$, and $f_3(z)$ are the principal thermal diffusivities (diffusion coefficients) dependent on coordinates, and $g = g(w)$ is the kinetic function, which defines the law of heat (substance) release or absorption.*

1. $\displaystyle a\frac{\partial^2 w}{\partial x^2} + b\frac{\partial^2 w}{\partial y^2} + c\frac{\partial^2 w}{\partial z^2} = f(w).$

1°. Traveling-wave solution:

$$w = w(\theta), \qquad \theta = Ax + By + Cz.$$

The function $w(\theta)$ is defined implicitly by

$$\int\left[C_1 + \frac{2}{aA^2 + bB^2 + cC^2}F(w)\right]^{-1/2} dw = C_2 \pm \theta, \qquad F(w) = \int f(w)\,dw,$$

where A, B, C, C_1, and C_2 are arbitrary constants.

2°. Solution:

$$w = w(r), \qquad r^2 = \frac{(x+C_1)^2}{a} + \frac{(y+C_2)^2}{b} + \frac{(z+C_3)^2}{c},$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{2}{r}w'_r = f(w).$$

3°. “Two-dimensional” solution:

$$w = U(\xi, \eta), \quad \xi = \frac{y}{\sqrt{b}} + \frac{x}{\sqrt{a}C}, \quad \eta = (C^2 - 1)\frac{x^2}{a} - 2C\frac{xy}{\sqrt{ab}} + C^2\frac{z^2}{c},$$

where C is an arbitrary constant ($C \neq 0$), and the function $U = U(\xi, \eta)$ is determined by the equation

$$\left(1 + \frac{1}{C^2}\right)\frac{\partial^2 U}{\partial \xi^2} - 4\xi\frac{\partial^2 U}{\partial \xi \partial \eta} + 4C^2(\xi^2 + \eta)\frac{\partial^2 U}{\partial \eta^2} + 2(2C^2 - 1)\frac{\partial U}{\partial \eta} = f(U).$$

Remark. The solution specified in Item 3° can be used to obtain other “two-dimensional” solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x,a) & \\ \nearrow & & \searrow \\ (z,c) & \longleftarrow & (y,b) \end{array}$$

4°. “Two-dimensional” solution:

$$w = V(\zeta, \rho), \quad \zeta = \frac{Ax}{\sqrt{a}} + \frac{By}{\sqrt{b}} + \frac{Cz}{\sqrt{c}}, \quad \rho^2 = \left(\frac{Bx}{\sqrt{a}} - \frac{Ay}{\sqrt{b}}\right)^2 + \left(\frac{Cy}{\sqrt{b}} - \frac{Bz}{\sqrt{c}}\right)^2 + \left(\frac{Az}{\sqrt{c}} - \frac{Cx}{\sqrt{a}}\right)^2,$$

where A, B, and C are arbitrary constants and the function $V = V(\zeta, \rho)$ is determined by the equation

$$\frac{\partial^2 V}{\partial \zeta^2} + \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} = \frac{1}{A^2 + B^2 + C^2} f(V).$$

5°. The transformation $x = \sqrt{a}\,\bar{x}$, $y = \sqrt{b}\,\bar{y}$, $z = \sqrt{c}\,\bar{z}$ brings the original equation to the form $\Delta w = f(w)$.

2. $\displaystyle a\frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial y}\left(by^n\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(cz^m\frac{\partial w}{\partial z}\right) = f(w).$

1°. For $n = m = 0$, see equation 6.3.1.1.

2°. Functional separable solution for $n \neq 2$ and $m \neq 2$:

$$w = w(r), \qquad r^2 = 4\left[\frac{x^2}{4a} + \frac{y^{2-n}}{b(2-n)^2} + \frac{z^{2-m}}{c(2-m)^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r}w'_r = f(w), \qquad A = \frac{2(4-n-m)}{(2-n)(2-m)}.$$

3°. “Two-dimensional” solution for $n \neq 2$ and $m \neq 2$:

$$w = U(x, \xi), \qquad \xi^2 = 4\left[\frac{y^{2-n}}{b(2-n)^2} + \frac{z^{2-m}}{c(2-m)^2}\right],$$

where the function $U(x, \xi)$ is determined by the differential equation

$$a\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial \xi^2} + \frac{B}{\xi}\frac{\partial U}{\partial \xi} = f(U), \qquad B = \frac{4-nm}{(2-n)(2-m)}.$$

4°. There are “two-dimensional” solutions of the following forms:

$$w = V(y, \eta), \quad \eta^2 = 4\left[\frac{x^2}{4a} + \frac{z^{2-m}}{c(2-m)^2}\right],$$

$$w = W(z, \zeta), \quad \zeta^2 = 4\left[\frac{x^2}{4a} + \frac{y^{2-n}}{b(2-n)^2}\right].$$

⊙ *Reference*: A. D. Polyanin and A. I. Zhurov (1998).

3. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(cz^k\dfrac{\partial w}{\partial z}\right)=f(w).$

1°. Functional separable solution for $n\neq 2$, $m\neq 2$, and $k\neq 2$:

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{z^{2-k}}{c(2-k)^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}+\frac{A}{r}w'_r=f(w),\qquad A=2\left(\frac{1}{2-n}+\frac{1}{2-m}+\frac{1}{2-k}\right)-1.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w=U(x,\xi),\quad \xi^2=4\left[\frac{y^{2-m}}{b(2-m)^2}+\frac{z^{2-k}}{c(2-k)^2}\right],$$
$$w=V(y,\eta),\quad \eta^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{z^{2-k}}{c(2-k)^2}\right],$$
$$w=W(z,\zeta),\quad \zeta^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}\right].$$

⊙ *Reference*: A. D. Polyanin and A. I. Zhurov (1998).

4. $\dfrac{\partial}{\partial x}\left(ae^{\beta x}\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(be^{\gamma y}\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right)=f(w).$

1°. Functional separable solution for $\beta\neq 0$, $\gamma\neq 0$, and $\lambda\neq 0$:

$$w=w(r),\qquad r^2=4\left(\frac{e^{-\beta x}}{a\beta^2}+\frac{e^{-\gamma y}}{b\gamma^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right),$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}-\frac{1}{r}w'_r=f(w).$$

2°. There are "two-dimensional" solutions of the following forms:

$$w=U(x,\xi),\quad \xi^2=4\left(\frac{e^{-\gamma y}}{b\gamma^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right),$$
$$w=V(y,\eta),\quad \eta^2=4\left(\frac{e^{-\beta x}}{a\beta^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right),$$
$$w=W(z,\zeta),\quad \zeta^2=4\left(\frac{e^{-\beta x}}{a\beta^2}+\frac{e^{-\gamma y}}{b\gamma^2}\right).$$

⊙ *Reference*: A. D. Polyanin and A. I. Zhurov (1998).

5. $\dfrac{\partial}{\partial x}\left(ax^n\dfrac{\partial w}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(by^m\dfrac{\partial w}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(ce^{\lambda z}\dfrac{\partial w}{\partial z}\right)=f(w).$

1°. Functional separable solution for $n\neq 2$, $m\neq 2$, and $\lambda\neq 0$:

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{e^{-\lambda z}}{c\lambda^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}+\frac{A}{r}w'_r=f(w),\qquad A=2\left(\frac{1}{2-n}+\frac{1}{2-m}\right)-1.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(x,\xi), \quad \xi^2 = 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\lambda z}}{c\lambda^2}\right],$$
$$w = V(y,\eta), \quad \eta^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda z}}{c\lambda^2}\right],$$
$$w = W(z,\zeta), \quad \zeta^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2}\right].$$

6. $\displaystyle \frac{\partial}{\partial x}\left(ax^n\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(be^{\beta y}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial z}\left(ce^{\lambda z}\frac{\partial w}{\partial z}\right) = f(w).$

1°. Functional separable solution for $n \neq 2$, $\beta \neq 0$, and $\lambda \neq 0$:

$$w = w(r), \qquad r^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\beta y}}{b\beta^2} + \frac{e^{-\lambda z}}{c\lambda^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{n}{2-n}\frac{1}{r}w'_r = f(w). \tag{1}$$

Example 1. For $n = 0$ and any $f = f(w)$, equation (1) can be solved by quadrature to obtain

$$\int\left[C_1 + 2\int f(w)\,dw\right]^{-1/2} dw = C_2 \pm r,$$

where C_1 and C_2 are arbitrary constants.

Example 2. For $n = 1$ and $f(w) = Ae^{\beta w}$, equation (1) has the one-parameter solution

$$w(r) = \frac{1}{\beta}\ln\left(-\frac{8C}{\beta A}\right) - \frac{2}{\beta}\ln(r^2 + C),$$

where C is an arbitrary constant.

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(x,\xi), \quad \xi^2 = 4\left[\frac{e^{-\beta y}}{b\beta^2} + \frac{e^{-\lambda z}}{c\lambda^2}\right],$$
$$w = V(y,\eta), \quad \eta^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\lambda z}}{c\lambda^2}\right],$$
$$w = W(z,\zeta), \quad \zeta^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\beta y}}{b\beta^2}\right].$$

⊙ *Reference*: A. D. Polyanin and A. I. Zhurov (1998).

7. $\displaystyle \frac{\partial}{\partial x}\left[f_1(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[f_2(y)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[f_3(z)\frac{\partial w}{\partial z}\right] = aw\ln w + bw.$

This is a special case of equation 6.3.3.6 with $g_1(x) = b$ and $g_2(y) = g_3(z) = 0$.

6.3.2. Heat and Mass Transfer Equations with Complicating Factors

1. $\displaystyle (a_1x + b_1y + c_1z + d_1)\frac{\partial w}{\partial x} + (a_2x + b_2y + c_2z + d_2)\frac{\partial w}{\partial y}$
$\displaystyle \qquad + (a_3x + b_3y + c_3z + d_3)\frac{\partial w}{\partial z} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} - f(w).$

This equation describes steady-state mass transfer with a volume chemical reaction in a three-dimensional translational-shear fluid flow.

Let k be a root of the cubic equation

$$\begin{vmatrix} a_1 - k & a_2 & a_3 \\ b_1 & b_2 - k & b_3 \\ c_1 & c_2 & c_3 - k \end{vmatrix} = 0,$$

and the constants A, B, and C solve the degenerate system of linear algebraic equations

$$(a_1 - k)A + a_2 B + a_3 C = 0,$$
$$b_1 A + (b_2 - k)B + b_3 C = 0,$$
$$c_1 A + c_2 B + (c_3 - k)C = 0.$$

One of the equations follows from the other two and, hence, can be omitted.

Solution:

$$w = w(\zeta), \qquad \zeta = Ax + By + Cz,$$

where the function $w(\zeta)$ is determined by the ordinary differential equation

$$(k\zeta + Ad_1 + Bd_2 + Cd_3)w'_\zeta = (A^2 + B^2 + C^2)w''_{\zeta\zeta} - f(w).$$

Remark. In the case of an incompressible fluid, some of the equation coefficients must satisfy the condition $a_1 + b_2 + c_3 = 0$.

2. $\displaystyle \frac{\partial}{\partial x}\left[(a_1 x + b_1 y + c_1 z + d_1)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[(a_2 x + b_2 y + c_2 z + d_2)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[(a_3 x + b_3 y + c_3 z + d_3)\frac{\partial w}{\partial z}\right] = f(w).$

Solutions are sought in the form

$$w = w(\zeta), \qquad \zeta = Ax + By + Cz + D,$$

where the constants A, B, C, and D are determined by solving the algebraic system of equations

$$a_1 A^2 + a_2 B^2 + a_3 C^2 = A,$$
$$b_1 A^2 + b_2 B^2 + b_3 C^2 = B,$$
$$c_1 A^2 + c_2 B^2 + c_3 C^2 = C,$$
$$d_1 A^2 + d_2 B^2 + d_3 C^2 = D.$$

The first three equations are first solved for A, B, and C. The resulting expressions are then substituted into the last equation to evaluate D. The desired function $w(\zeta)$ is determined by the ordinary differential equation

$$\zeta w''_{\zeta\zeta} + (a_1 A + b_2 B + c_3 C)w'_\zeta = f(w).$$

3. $\displaystyle \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(w)\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial z}\left[h(w)\frac{\partial w}{\partial z}\right] = 0.$

This equation describes steady-state heat/mass transfer or combustion processes in inhomogeneous anisotropic media. Here, $f(w)$, $g(w)$, and $h(w)$ are the principal thermal diffusivities (diffusion coefficients) dependent on the temperature w.

1°. Suppose $w(x, y, z)$ is a solution of the equation in question. Then the functions

$$w_1 = w(\pm C_1 x + C_2, \pm C_1 y + C_3, \pm C_1 z + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation (the plus or minus signs in front of C_1 are chosen arbitrarily).

2°. Traveling-wave solution in implicit form:

$$\int \left[k_1^2 f(w) + k_2^2 g(w) + k_3^2 h(w)\right] dw = C_1(k_1 x + k_2 y + k_3 z) + C_2,$$

where C_1, C_2, k_1, k_2, k_3, and λ are arbitrary constants.

3°. Solution:

$$w = w(\theta), \quad \theta = \frac{C_1 y + C_2 z + C_3}{x + C_4}, \tag{1}$$

where $C_1, \ldots, C_4$ are arbitrary constants, and the function $w(\theta)$ is determined by the ordinary differential equation

$$[\theta^2 f(w) w'_\theta]'_\theta + C_1^2 [g(w) w'_\theta]'_\theta + C_2^2 [h(w) w'_\theta]'_\theta = 0,$$

which admits the first integral

$$[\theta^2 f(w) + C_1^2 g(w) + C_2^2 h(w)] w'_\theta = C_5.$$

For $C_5 \neq 0$, treating w as the independent variable, one obtains a Riccati equation for $\theta = \theta(w)$:

$$C_5 \theta'_w = \theta^2 f(w) + C_1^2 g(w) + C_2^2 h(w). \tag{2}$$

For exact solutions of this equation, which can be reduced to a second-order linear equation, see Polyanin and Zaitsev (2003).

Relations (1) and equation (2) can be used to obtain two other "one-dimensional" solutions by means of the following cyclic permutations of variables and determining functions:

$$\begin{array}{ccc} & (x, f) & \\ \nearrow & & \searrow \\ (z, h) & \longleftarrow & (y, g) \end{array} \tag{3}$$

4°. "Two-dimensional" solution (a and b are arbitrary constants):

$$w(x, y, z) = U(x, \zeta), \quad \zeta = ay + bz, \tag{4}$$

where the function $U = U(x, \zeta)$ is determined by a differential equation of the form 5.4.4.8:

$$\frac{\partial}{\partial x}\left[f(U)\frac{\partial U}{\partial x}\right] + \frac{\partial}{\partial \zeta}\left[\psi(U)\frac{\partial U}{\partial \zeta}\right] = 0, \quad \psi(U) = a^2 g(U) + b^2 h(U), \tag{5}$$

which can be reduced to a linear equation.

Relations (4) and equation (5) can be used to obtain two other "two-dimensional" solutions by means of the cyclic permutations of variables and determining functions; see (3).

5°. There are "two-dimensional" solutions of the following forms:

$$w(x, y, z) = V(z_1, z_2), \quad z_1 = a_1 x + a_2 y + a_3 z, \quad z_2 = b_1 x + b_2 y + b_3 z;$$
$$w(x, y, z) = W(\xi, \eta), \quad \xi = y/x, \quad \eta = z/x,$$

where the a_n and b_n are arbitrary constants (the first solution generalizes the one of Item 3°).

6°. Let $g(w) = af(w)$. Then, there is a "two-dimensional" solution of the form

$$w(x, y, z) = u(r, z), \quad r = ax^2 + y^2.$$

7°. Let $g(w) = af(w)$ and $h(w) = bf(w)$. Then, the transformation

$$v = \int f(w)\, dw, \quad y = \sqrt{a}\,\bar{y}, \quad z = \sqrt{b}\,\bar{z}$$

leads to the Laplace equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial \bar{y}^2} + \frac{\partial^2 v}{\partial \bar{z}^2} = 0.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

4. $\dfrac{\partial}{\partial x}\left[f_1(w)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[f_2(w)\dfrac{\partial w}{\partial y}\right]+\dfrac{\partial}{\partial z}\left[f_3(w)\dfrac{\partial w}{\partial z}\right]$
$=(a_1x+b_1y+c_1z+d_1)\dfrac{\partial w}{\partial x}+(a_2x+b_2y+c_2z+d_2)\dfrac{\partial w}{\partial y}+(a_3x+b_3y+c_3z+d_3)\dfrac{\partial w}{\partial z}.$

This equation describes steady-state anisotropic heat/mass transfer with a volume chemical reaction in a three-dimensional translational-shear fluid flow.

Let k be a root of the cubic equation

$$\begin{vmatrix} a_1-k & a_2 & a_3 \\ b_1 & b_2-k & b_3 \\ c_1 & c_2 & c_3-k \end{vmatrix}=0,$$

and the constants A, B, and C solve the degenerate system of linear algebraic equations

$$\begin{gathered}(a_1-k)A+a_2B+a_3C=0,\\ b_1A+(b_2-k)B+b_3C=0,\\ c_1A+c_2B+(c_3-k)C=0.\end{gathered}$$

One of the equations follows from the other two and, hence, can be omitted.

Solution:

$$w=w(\zeta),\qquad \zeta=Ax+By+Cz, \tag{1}$$

where the function $w(\zeta)$ is determined by the ordinary differential equation

$$\begin{gathered}[\varphi(w)w'_\zeta]'_\zeta=(k\zeta+Ad_1+Bd_2+Cd_3)w'_\zeta,\\ \varphi(w)=A^2f_1(w)+B^2f_2(w)+C^2f_3(w).\end{gathered}$$

Remark 1. A more general equation, with an additional term $g(w)$ on the right-hand side, where g is an arbitrary function, also has a solution of the form (1).

Remark 2. In the case of an incompressible fluid, some of the equation coefficients must satisfy the condition $a_1+b_2+c_3=0$.

6.3.3. Other Equations

1. $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}=f(w)\left[\left(\dfrac{\partial w}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial y}\right)^2+\left(\dfrac{\partial w}{\partial z}\right)^2\right].$

The substitution

$$U=\int\frac{dw}{F(w)},\qquad\text{where}\quad F(w)=\exp\left[\int f(w)\,dw\right],$$

leads to the three-dimensional Laplace equation for $U=U(x,y,z)$:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=0.$$

For solutions of this linear equation, see Tikhonov and Samarskii (1990) and Polyanin (2002).

Remark. For a more complicated equation of the form $(\vec{v}\cdot\nabla)w=\Delta w-f(w)|\nabla w|^2$, with an additional convective term, see 6.4.1.1.

2. $ax^n\dfrac{\partial^2 w}{\partial x^2}+by^m\dfrac{\partial^2 w}{\partial y^2}+cz^k\dfrac{\partial^2 w}{\partial z^2}=f(w).$

1°. Functional separable solution for $n\neq 2$, $m\neq 2$, and $k\neq 2$:

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{y^{2-m}}{b(2-m)^2}+\frac{z^{2-k}}{c(2-k)^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r} w'_r = f(w), \qquad A = 5 - 2\left(\frac{1}{2-n} + \frac{1}{2-m} + \frac{1}{2-k}\right).$$

2°. There is a "two-dimensional" solution of the form

$$w(x, y, z) = U(x, \rho), \qquad \rho^2 = 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-k}}{c(2-k)^2}\right].$$

This solution can be used to obtain other "two-dimensional" solutions by means of the following cyclic permutations of variables and determining parameters:

$$\begin{array}{ccc} & (x, a, n) & \\ \nearrow & & \searrow \\ (z, c, k) & \longleftarrow & (y, b, m) \end{array}$$

3. $\boldsymbol{ae^{\lambda x}\dfrac{\partial^2 w}{\partial x^2} + be^{\mu y}\dfrac{\partial^2 w}{\partial y^2} + ce^{\nu z}\dfrac{\partial^2 w}{\partial z^2} = f(w).}$

1°. Functional separable solution for $\lambda \neq 0$, $\mu \neq 0$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = 4\left(\frac{e^{-\lambda x}}{a\lambda^2} + \frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right),$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{5}{r} w'_r = f(w).$$

2°. There is a "two-dimensional" solution of the form

$$w(x, y, z) = U(x, \xi), \qquad \xi^2 = 4\left(\frac{e^{-\mu y}}{b\mu^2} + \frac{e^{-\nu z}}{c\nu^2}\right).$$

This solution can be used to obtain other "two-dimensional" solutions by means of the following cyclic permutations of the variables and determining parameters:

$$\begin{array}{ccc} & (x, a, \lambda) & \\ \nearrow & & \searrow \\ (z, c, \nu) & \longleftarrow & (y, b, \mu) \end{array}$$

4. $\boldsymbol{ax^n\dfrac{\partial^2 w}{\partial x^2} + by^m\dfrac{\partial^2 w}{\partial y^2} + ce^{\nu z}\dfrac{\partial^2 w}{\partial z^2} = f(w).}$

1°. Functional separable solution for $n \neq 2$, $m \neq 2$, and $\nu \neq 0$:

$$w = w(r), \qquad r^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr} + \frac{A}{r} w'_r = f(w), \qquad A = 2\left(\frac{1-n}{2-n} + \frac{1-m}{2-m}\right) + 1.$$

2°. There are "two-dimensional" solutions of the following forms:

$$w = U(x, \xi), \quad \xi^2 = 4\left[\frac{y^{2-m}}{b(2-m)^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

$$w = V(y, \eta), \quad \eta^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{e^{-\nu z}}{c\nu^2}\right],$$

$$w = W(z, \zeta), \quad \zeta^2 = 4\left[\frac{x^{2-n}}{a(2-n)^2} + \frac{y^{2-m}}{b(2-m)^2}\right].$$

5. $ax^n\dfrac{\partial^2 w}{\partial x^2}+be^{\mu y}\dfrac{\partial^2 w}{\partial y^2}+ce^{\nu z}\dfrac{\partial^2 w}{\partial z^2}=f(w).$

1°. Functional separable solution for $n\neq 2$, $\mu\neq 0$, and $\nu\neq 0$:

$$w=w(r),\qquad r^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\mu y}}{b\mu^2}+\frac{e^{-\nu z}}{c\nu^2}\right],$$

where the function $w(r)$ is determined by the ordinary differential equation

$$w''_{rr}+\frac{8-5n}{2-n}\frac{1}{r}w'_r=f(w).$$

2°. There are "two-dimensional" solutions of the following forms:

$$w=U(x,\xi),\quad \xi^2=4\left[\frac{e^{-\mu y}}{b\mu^2}+\frac{e^{-\nu z}}{c\nu^2}\right],$$
$$w=V(y,\eta),\quad \eta^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\nu z}}{c\nu^2}\right],$$
$$w=W(z,\zeta),\quad \zeta^2=4\left[\frac{x^{2-n}}{a(2-n)^2}+\frac{e^{-\mu y}}{b\mu^2}\right].$$

6. $\dfrac{\partial}{\partial x}\left[f_1(x)\dfrac{\partial w}{\partial x}\right]+\dfrac{\partial}{\partial y}\left[f_2(y)\dfrac{\partial w}{\partial y}\right]+\dfrac{\partial}{\partial z}\left[f_3(z)\dfrac{\partial w}{\partial z}\right]=aw\ln w+\big[g_1(x)+g_2(y)+g_3(z)\big]w.$

Multiplicative separable solution:

$$w(x,y,z)=\varphi(x)\psi(y)\chi(z),$$

where the functions $\varphi=\varphi(x)$, $\psi=\psi(y)$, and $\chi=\chi(z)$ are determined by the ordinary differential equations (C_1 and C_2 are arbitrary constants)

$$[f_1(x)\varphi'_x]'_x-a\varphi\ln\varphi-[g_1(x)+C_1]\varphi=0,$$
$$[f_2(y)\psi'_y]'_y-a\psi\ln\psi-[g_2(y)+C_2]\psi=0,$$
$$[f_3(z)\chi'_z]'_z-a\chi\ln\chi-[g_3(z)-C_1-C_2]\chi=0.$$

6.4. Equations with n Independent Variables

6.4.1. Equations of the Form $\dfrac{\partial}{\partial x_1}\left[f_1(x_1)\dfrac{\partial w}{\partial x_1}\right]+\cdots+\dfrac{\partial}{\partial x_n}\left[f_n(x_n)\dfrac{\partial w}{\partial x_n}\right]=g(x_1,\ldots,x_n,w)$

1. $\displaystyle\sum_{k=1}^{n}\frac{\partial^2 w}{\partial x_k^2}=f(w)\sum_{k=1}^{n}\left(\frac{\partial w}{\partial x_k}\right)^2+\sum_{k=1}^{n}g_k(x_1,\ldots,x_n)\frac{\partial w}{\partial x_k}.$

The substitution

$$U=\int\frac{dw}{F(w)},\qquad\text{where}\quad F(w)=\exp\left[\int f(w)\,dw\right],$$

leads to the linear equation

$$\sum_{k=1}^{n}\frac{\partial^2 U}{\partial x_k^2}=\sum_{k=1}^{n}g_k(x_1,\ldots,x_n)\frac{\partial U}{\partial x_k}.$$

2. $\displaystyle\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(a_k x_k^{m_k}\frac{\partial w}{\partial x_k}\right)=f(w).$

Functional separable solution:

$$w=w(r),\qquad r^2=A\sum_{k=1}^{n}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2},$$

where the function $w(r)$ is determined by the ordinary differential equation

$$\frac{d^2w}{dr^2}+\frac{B}{r}\frac{dw}{dr}=\frac{4}{A}f(w),\qquad B=\sum_{k=1}^{n}\frac{2}{2-m_k}-1.$$

Example 1. For $f(w)=bw^p$, there is an exact solution of the form

$$w=\left[\frac{1}{b(1-p)}\left(\frac{p}{1-p}+\sum_{k=1}^{n}\frac{1}{2-m_k}\right)\right]^{\frac{1}{p-1}}\left[\sum_{k=1}^{n}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2}\right]^{\frac{1}{1-p}}.$$

Example 2. For $f(w)=be^{\lambda w}$, there is an exact solution of the form

$$w=-\frac{1}{\lambda}\ln\left[\sum_{k=1}^{n}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2}\right]+\frac{1}{\lambda}\ln\frac{1-B}{2b\lambda},\qquad B=\sum_{k=1}^{n}\frac{2}{2-m_k}-1.$$

⊙ *Reference*: A. D. Polyanin and A. I. Zhurov (1998).

3. $\displaystyle\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(a_k e^{\lambda_k x_k}\frac{\partial w}{\partial x_k}\right)=f(w).$

Functional separable solution:

$$w=w(r),\qquad r^2=A\sum_{k=1}^{n}\frac{e^{-\lambda_k x_k}}{a_k\lambda_k^2},$$

where the function $w(r)$ is determined by the ordinary differential equation

$$\frac{d^2w}{dr^2}-\frac{1}{r}\frac{dw}{dr}=\frac{4}{A}f(w).$$

Example 1. For $f(w)=bw^p$, there is an exact solution of the form

$$w=\left[\frac{p}{b(1-p)^2}\right]^{\frac{1}{p-1}}\left(\sum_{k=1}^{n}\frac{e^{-\lambda_k x_k}}{a_k\lambda_k^2}\right)^{\frac{1}{1-p}}.$$

Example 2. For $f(w)=be^{\beta w}$, there is an exact solution of the form

$$w=-\frac{1}{\beta}\ln\left(b\beta\sum_{k=1}^{n}\frac{e^{-\lambda_k x_k}}{a_k\lambda_k^2}\right).$$

⊙ *Reference*: A. D. Polyanin and A. I. Zhurov (1998).

4. $\displaystyle\sum_{k=1}^{s}\frac{\partial}{\partial x_k}\left(a_k x_k^{m_k}\frac{\partial w}{\partial x_k}\right)+\sum_{k=s+1}^{n}\frac{\partial}{\partial x_k}\left(b_k e^{\lambda_k x_k}\frac{\partial w}{\partial x_k}\right)=f(w).$

1°. Functional separable solution:

$$w=w(r),\qquad r^2=A\sum_{k=1}^{s}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2}+A\sum_{k=s+1}^{n}\frac{e^{-\lambda_k x_k}}{b_k\lambda_k^2},$$

where the function $w(r)$ is determined by the ordinary differential equation

$$\frac{d^2w}{dr^2}+\frac{B}{r}\frac{dw}{dr}=\frac{4}{A}f(w),\qquad B=\sum_{k=1}^{s}\frac{2}{2-m_k}-1.$$

Example 1. For $f(w)=cw^p$, there is an exact solution of the form

$$w=\left[\frac{1}{c(1-p)}\left(\frac{p}{1-p}+\sum_{k=1}^{s}\frac{1}{2-m_k}\right)\right]^{\frac{1}{p-1}}\left[\sum_{k=1}^{s}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2}+\sum_{k=s+1}^{n}\frac{e^{-\lambda_k x_k}}{b_k\lambda_k^2}\right]^{\frac{1}{1-p}}.$$

Example 2. For $f(w)=ce^{\beta w}$, there is an exact solution of the form

$$w=-\frac{1}{\beta}\ln\left[\sum_{k=1}^{s}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2}+\sum_{k=s+1}^{n}\frac{e^{-\lambda_k x_k}}{b_k\lambda_k^2}\right]+\frac{1}{\beta}\ln\frac{1-B}{2c\beta},\qquad B=\sum_{k=1}^{s}\frac{2}{2-m_k}-1.$$

2°. We divide the equation variables into two groups (responsible for both power-law and exponential terms) and look for exact solutions in the form

$$w=w(y,z),$$

where

$$y^2=A_1\sum_{k=1}^{q}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2}+A_1\sum_{k=s+1}^{p}\frac{e^{-\lambda_k x_k}}{b_k\lambda_k^2},\qquad 0\le q\le s\le p\le n;$$

$$z^2=A_2\sum_{k=q+1}^{s}\frac{x_k^{2-m_k}}{a_k(2-m_k)^2}+A_2\sum_{k=p+1}^{n}\frac{e^{-\lambda_k x_k}}{b_k\lambda_k^2}.$$

Then we obtain the following equation for w:

$$A_1\left(\frac{\partial^2 w}{\partial y^2}+\frac{B_1}{y}\frac{\partial w}{\partial y}\right)+A_2\left(\frac{\partial^2 w}{\partial z^2}+\frac{B_1}{z}\frac{\partial w}{\partial z}\right)=4f(w),$$

$$B_1=\sum_{k=1}^{q}\frac{2}{2-m_k}-1,\quad B_2=\sum_{k=q+1}^{s}\frac{2}{2-m_k}-1.$$

For $B_1=B_2=0$ and $A_1=A_2=1$, this equation arises in plane problems of heat and mass transfer (see equations 5.1.1.1, 5.2.1.1, 5.3.1.1, 5.3.2.1, 5.3.3.1, and 5.4.1.1).

⊙ *Reference*: A. D. Polyanin and A. I. Zhurov (1998).

5. $\displaystyle \sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left[f_k(x_k)\frac{\partial w}{\partial x_k}\right]=aw\ln w+w\sum_{k=1}^{n}g_k(x_k).$

Multiplicative separable solution:

$$w(x_1,x_2,\dots,x_n)=\varphi_1(x_1)\varphi_2(x_2)\dots\varphi_n(x_n),$$

where the functions $\varphi_1=\varphi_1(x_1)$, $\varphi_2=\varphi_2(x_2)$, $\dots$, $\varphi_n=\varphi_n(x_n)$ are determined by the ordinary differential equations

$$\frac{d}{dx_k}\left[f_k(x_k)\frac{d\varphi_k}{dx_k}\right]-a\varphi_k\ln\varphi_k-\left[g_k(x_k)+C_k\right]\varphi_k=0;\qquad k=1,\,2,\,\dots,\,n.$$

The arbitrary constants $C_1,\dots,C_n$ are related by a single constraint, $C_1+\dots+C_n=0$.

6.4.2. Other Equations

1. $\displaystyle \sum_{k=1}^{n}f_k(x_k)\frac{\partial^2 w}{\partial x_k^2}+\sum_{k=1}^{n}g_k(x_k)\frac{\partial w}{\partial x_k}=aw\ln w+w\sum_{k=1}^{n}h_k(x_k).$

Multiplicative separable solution:

$$w(x_1,x_2,\dots,x_n)=\varphi_1(x_1)\varphi_2(x_2)\dots\varphi_n(x_n),$$

where the functions $\varphi_1=\varphi_1(x_1)$, $\varphi_2=\varphi_2(x_2)$, $\dots$, $\varphi_n=\varphi_n(x_n)$ are determined by the ordinary differential equations

$$f_k(x_k)\frac{d^2\varphi_k}{dx_k^2}+g_k(x_k)\frac{d\varphi_k}{dx_k}-a\varphi_k\ln\varphi_k-\left[h_k(x_k)+C_k\right]\varphi_k=0;\qquad k=1,\,2,\,\dots,\,n.$$

The arbitrary constants $C_1,\dots,C_n$ are related by a single constraint, $C_1+\dots+C_n=0$.

2. $\displaystyle\sum_{k=1}^{n} a_k x_k^{m_k} \frac{\partial^2 w}{\partial x_k^2} + \sum_{k=1}^{n} b_k x_k^{m_k-1} \frac{\partial w}{\partial x_k} = f(w).$

Functional separable solution:

$$w = w(r), \qquad r^2 = A \sum_{k=1}^{n} \frac{x_k^{2-m_k}}{a_k(2-m_k)^2}$$

where the function $w(r)$ is determined by the ordinary differential equation

$$\frac{d^2w}{dr^2} + \frac{B}{r}\frac{dw}{dr} = \frac{4}{A} f(w), \qquad B = 2\sum_{k=1}^{n} \frac{a_k(1-m_k)+b_k}{a_k(2-m_k)} - 1.$$

Example 1. For $f(w) = cw^p$, there is an exact solution of the form

$$w = \left[\frac{1}{2c(1-p)}\left(\frac{1+p}{1-p} + B\right)\right]^{\frac{1}{p-1}} \left[\sum_{k=1}^{n} \frac{x_k^{2-m_k}}{a_k(2-m_k)^2}\right]^{\frac{1}{1-p}}.$$

Example 2. For $f(w) = ce^{\beta w}$, there is an exact solution of the form

$$w = -\frac{1}{\beta} \ln\left[\sum_{k=1}^{n} \frac{x_k^{2-m_k}}{a_k(2-m_k)^2}\right] + \frac{1}{\beta} \ln \frac{1-B}{2c\beta}.$$

3. $\displaystyle\sum_{k=1}^{n} a_k e^{\lambda_k x_k} \frac{\partial^2 w}{\partial x_k^2} + \sum_{k=1}^{n} b_k e^{\lambda_k x_k} \frac{\partial w}{\partial x_k} = f(w).$

Functional separable solution:

$$w = w(r), \qquad r^2 = A \sum_{k=1}^{n} \frac{e^{-\lambda_k x_k}}{a_k \lambda_k^2},$$

where the function $w(r)$ is determined by the ordinary differential equation

$$\frac{d^2w}{dr^2} + \frac{B}{r}\frac{dw}{dr} = \frac{4}{A} f(w), \qquad B = 2n - 1 - 2\sum_{k=1}^{n} \frac{b_k}{a_k \lambda_k}.$$

Example 1. For $f(w) = cw^p$, there is an exact solution of the form

$$w = \left[\frac{1}{2c(1-p)}\left(\frac{1+p}{1-p} + B\right)\right]^{\frac{1}{p-1}} \left(\sum_{k=1}^{n} \frac{e^{-\lambda_k x_k}}{a_k \lambda_k^2}\right)^{\frac{1}{1-p}}.$$

Example 2. For $f(w) = ce^{\beta w}$, there is an exact solution of the form

$$w = -\frac{1}{\beta} \ln\left(\sum_{k=1}^{n} \frac{e^{-\lambda_k x_k}}{a_k \lambda_k^2}\right) + \frac{1}{\beta} \ln \frac{1-B}{2c\beta}.$$

4. $\displaystyle\sum_{k=1}^{m_1}\left(a_k x_k^{n_k} \frac{\partial^2 w}{\partial x_k^2} + a_k p_k x_k^{n_k-1} \frac{\partial w}{\partial x_k}\right) + \sum_{k=1}^{m_2}\left(b_k e^{\lambda_k x_k} \frac{\partial^2 w}{\partial x_k^2} + b_k q_k e^{\lambda_k x_k} \frac{\partial w}{\partial x_k}\right) = f(w).$

Functional separable solution:

$$w = w(r), \qquad r^2 = A \sum_{k=1}^{m_1} \frac{x_k^{2-n_k}}{a_k(2-n_k)^2} + A \sum_{k=1}^{m_2} \frac{e^{-\lambda_k x_k}}{b_k \lambda_k^2},$$

where the function $w(r)$ is determined by the ordinary differential equation

$$\frac{d^2w}{dr^2} + \frac{B}{r}\frac{dw}{dr} = \frac{4}{A} f(w), \qquad B = 2\sum_{k=1}^{m_1} \frac{1-n_k+p_k}{2-n_k} - 2\sum_{k=1}^{m_2} \frac{q_k}{\lambda_k} + 2m_2 - 1.$$

Chapter 7

Equations Involving Mixed Derivatives and Some Other Equations

Preliminary remarks. Semilinear equations, which can be reduced to the canonical form by standard transformations, are not considered in this chapter. See Section S.1 for information about semilinear equations. For hyperbolic equations that contain mixed derivatives, see Section 3.5.

7.1. Equations Linear in the Mixed Derivative

7.1.1. Calogero Equation

1. $\dfrac{\partial^2 w}{\partial x \partial t} = w\dfrac{\partial^2 w}{\partial x^2} + a.$

This is a special case of equation 7.1.1.3 with $f(u) = a$.

2. $\dfrac{\partial^2 w}{\partial x \partial t} = w\dfrac{\partial^2 w}{\partial x^2} + a\left(\dfrac{\partial w}{\partial x}\right)^2.$

This is a special case of equation 7.1.1.3 with $f(u) = au^2$.

1°. Suppose $w(x, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_2 x + C_2\varphi(t), C_1 C_2 t + C_3) + \varphi'_t(t),$$

where C_1, C_2, and C_3 are arbitrary constants, and $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. General solution in parametric form:

$$w = f'_t(t) + \int \bigl[g(z) - at\bigr]^{\frac{1-a}{a}}\,dz,$$
$$x = -f(t) + \int \bigl[g(z) - at\bigr]^{\frac{1}{a}}\,dz,$$

where $f(t)$ and $g(z)$ are arbitrary functions and z is the parameter.

3°. Conservation laws:

$$D_t\bigl[(w_x)^{1/a}\bigr] + D_x\bigl[-w(w_x)^{1/a}\bigr] = 0,$$
$$D_t\Bigl[(w_{xx})^{\frac{1}{2a+1}}\Bigr] + D_x\Bigl[-w(w_{xx})^{\frac{1}{2a+1}}\Bigr] = 0.$$

⊙ *References*: F. Calogero (1984), J. K. Hunter and R. Saxton (1991), M. V. Pavlov (2001).

3. $\dfrac{\partial^2 w}{\partial x \partial t} = w\dfrac{\partial^2 w}{\partial x^2} + f\left(\dfrac{\partial w}{\partial x}\right).$

Calogero equation.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(C_1 x + C_1\varphi(t), t + C_2) + \varphi_t'(t),$$

where C_1 and C_2 are arbitrary constants and $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solution linear in x:

$$w(x,t) = \varphi(t)x + \psi(t),$$

where $\psi(t)$ is an arbitrary function and the function $\varphi(t)$ is defined implicitly by (C is an arbitrary constant)

$$\int \frac{d\varphi}{f(\varphi)} = t + C.$$

3°. Introduce the notation

$$u = \frac{\partial w}{\partial x}, \quad v = \Phi(u) = \exp\left[\int \frac{u\,du}{f(u)}\right]. \tag{1}$$

The transformation

$$dz = v\,dx + vw\,dt, \quad dy = dt \qquad \left(dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial t}\,dt\right) \tag{2}$$

defines the passage from x, t to the new independent variables z, y in accordance with the rule

$$\frac{\partial}{\partial x} = v\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial y} + vw\frac{\partial}{\partial z}. \tag{3}$$

This results in the first-order equation

$$\frac{\partial u}{\partial y} = f(u),$$

which is independent of z and can be treated as an ordinary differential equation. Integrating yields its solution in implicit form:

$$\int \frac{du}{f(u)} = y + \varphi(z), \tag{4}$$

where $\varphi(z)$ is an arbitrary function. With the first relations of (1) and (3), we obtain the equation

$$\frac{\partial w}{\partial z} = \frac{u}{v} \quad \Longrightarrow \quad \frac{\partial w}{\partial z} = \frac{u}{\Phi(u)},$$

whose general solution is given by

$$w = \int \frac{u\,dz}{\Phi(u)} + \psi(y), \tag{5}$$

where $\psi(y)$ is an arbitrary function, and $u = u(z,y)$ is defined implicitly by (4). The inverse of transformation (2) has the form

$$dx = \frac{1}{\Phi(u)}\,dz - w\,dy, \quad dt = dy. \tag{6}$$

Integrating the first relation in (6) yields

$$x = \int_{z_0}^{z} \frac{d\xi}{\Phi(u(\xi,y))} - \int_{y_0}^{y} w(z_0,\tau)\,d\tau, \tag{7}$$

where $w = w(z,y)$ is defined by (5), and x_0 and y_0 are any numbers.

Formulas (4), (5), and (7) with $y = t$ define the general solution of the equation in question in parametric form (z is the parameter).

⊙ *References*: F. Calogero (1984), M. V. Pavlov (2001).

4°. Conservation law:

$$D_t[\Phi(w_x)] + D_x[-w\Phi(w_x)] = 0,$$

where $D_t = \frac{\partial}{\partial t}$, $D_x = \frac{\partial}{\partial x}$, and the function $\Phi(u)$ is defined in (1).

⦿ *References*: F. Calogero (1984), M. V. Pavlov (2001).

7.1.2. Khokhlov–Zabolotskaya Equation

1. $\displaystyle \frac{\partial^2 w}{\partial x \partial t} - w\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial w}{\partial x}\right)^2 - \frac{\partial^2 w}{\partial y^2} = 0.$

Two-dimensional Khokhlov–Zabolotskaya equation. It describes the propagation of a sound beam in a nonlinear medium; t and y play the role of the space coordinates and x is a linear combination of time and a coordinate.

The equation of unsteady transonic gas flows (see 7.1.3.1 with $a = b = 1/2$)

$$2u_{x\tau} + u_x u_{xx} - u_{yy} = 0$$

can be reduced to the Khokhlov–Zabolotskaya equation; see Lin, Reissner, and Tsien (1948). To this end, one should pass to the new variable $\tau = 2t$, differentiate the equation with respect to x, and then substitute $w = -\partial u/\partial x$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2}C_2^2 w(C_1 x + C_3, C_2 y + C_4, C_1^{-1}C_2^2 t + C_5),$$
$$w_2 = w(x + \lambda y + \varphi(t), y + 2\lambda t, t) + \varphi_t'(t) - \lambda^2,$$

where $C_1, \ldots, C_5$ and λ are arbitrary constants and $\varphi = \varphi(t)$ is an arbitrary function, are also solutions of the equation.

2°. Solutions:

$$w(x, y, t) = -\frac{x}{t + C_1} + \varphi y + \psi,$$
$$w(x, y, t) = 2\varphi x + (\varphi_t' - 2\varphi^2)y^2 + \psi y + \chi,$$
$$w(x, y, t) = (\varphi y + \psi)x - \frac{1}{12\varphi^2}(\varphi y + \psi)^4 + \frac{1}{6}\varphi_t' y^3 + \frac{1}{2}\psi_t' y^2 + \chi y + \theta,$$
$$w(x, y, t) = C_1\sqrt{x + C_2 y + \varphi} + \varphi_t' - C_2^2,$$
$$w(x, y, t) = \frac{C_1}{t}\sqrt{4t(x + \varphi) - (y + C_2)^2} + \varphi_t',$$

where $\varphi = \varphi(t)$, $\psi = \psi(t)$, $\chi = \chi(t)$, and $\theta = \theta(t)$ are arbitrary functions, the prime stands for the differentiation, and C_1 and C_2 are arbitrary constants.

3°. Solution in implicit form:

$$tz + x + \lambda y + \lambda^2 t + \varphi(t) = F(z), \qquad z = w - \varphi_t'(t),$$

where $\varphi(t)$ and $F(z)$ are arbitrary functions. With $\lambda = 0$, this relation determines the general y-independent solution of the original equation.

4°. "Two-dimensional" generalized separable solution quadratic in x:

$$w = f(y, t)x^2 + g(y, t)x + h(y, t),$$

where the functions $f = f(y, t)$, $g = g(y, t)$, and $h = h(y, t)$ are determined by the system of differential equations

$$f_{yy} = -6f^2,$$
$$g_{yy} = -6fg + 2f_t,$$
$$h_{yy} = -2fh + g_t - g^2.$$

The subscripts y and t denote the corresponding partial derivatives. A particular solution of this system is given by

$$f=-\frac{1}{R^2},\quad g=\frac{C_1(t)}{R^2}+C_2(t)R^3-\frac{\varphi'_t(t)}{2R},$$
$$h=\frac{C_3(t)}{R}+C_4(t)R^2+\frac{R^2}{3}\int\frac{1}{R}(g_t-g^2)\,dy-\frac{1}{3R}\int R^2(g_t-g^2)\,dy,\quad R=y+\varphi(t),$$

where $\varphi(t)$, $C_1(t)$, …, $C_4(t)$ are arbitrary functions.

5°. "Two-dimensional" solution:

$$w=xu(\xi,t),\quad \xi=yx^{-1/2},$$

where the function $u=u(\xi,t)$ is determined by the differential equation

$$2\xi\frac{\partial^2 u}{\partial\xi\partial t}+(\xi^2u+4)\frac{\partial^2 u}{\partial\xi^2}+\xi^2\left(\frac{\partial u}{\partial\xi}\right)^2-5\xi u\frac{\partial u}{\partial\xi}-4\frac{\partial u}{\partial t}+4u^2=0.$$

6°. "Two-dimensional" solution:

$$w=v(\zeta,t)+\frac{\alpha'_t+4}{\alpha}x,\quad \zeta=y^2+\alpha x,$$

where $\alpha=\alpha(t)$ is an arbitrary function and the function $v=v(\zeta,t)$ is determined by the differential equation

$$\alpha\frac{\partial^2 v}{\partial\zeta\partial t}-(\alpha^2v+4\zeta)\frac{\partial^2 v}{\partial\zeta^2}-\alpha^2\left(\frac{\partial v}{\partial\zeta}\right)^2-(\alpha'_t+10)\frac{\partial v}{\partial\zeta}+\beta'_t-\beta^2=0,\quad \beta=\frac{\alpha'_t+4}{\alpha}.$$

The last equation has a particular solution of the form $v=\zeta\varphi(t)$, where the function $\varphi=\varphi(t)$ is determined by the Riccati equation $\alpha\varphi'_t-\alpha^2\varphi^2-(\alpha'_t+10)\varphi+\beta'_t-\beta^2=0$.

7°. "Two-dimensional" solution:

$$w=U(r,z),\quad z=x+\beta y+\lambda t,\quad r=y+\mu t,$$

where β, λ, and μ are arbitrary constants, and the function $U=U(r,z)$ is determined by the differential equation

$$(\lambda-\beta^2)\frac{\partial^2 U}{\partial z^2}+(\mu-2\beta)\frac{\partial^2 U}{\partial r\partial z}-\frac{\partial^2 U}{\partial r^2}-U\frac{\partial^2 U}{\partial z^2}-\left(\frac{\partial U}{\partial z}\right)^2=0.$$

With $\lambda=\beta^2$ and $\mu=2\beta$, we obtain an equation of the form 5.1.5.1.

8°. "Two-dimensional" solution:

$$w=x^{-2}V(p,q),\quad p=tx^{-3},\quad q=yx^{-2},$$

where the function $V=V(p,q)$ is determined by the differential equation

$$3p(3Vp+1)\frac{\partial^2 V}{\partial p^2}+(4q^2V+1)\frac{\partial^2 V}{\partial q^2}+2q(6pV+1)\frac{\partial^2 V}{\partial p\partial q}$$
$$+\left(3p\frac{\partial V}{\partial p}+2q\frac{\partial V}{\partial q}\right)^2+(36pV+5)\frac{\partial V}{\partial p}+22qV\frac{\partial V}{\partial q}+10V^2=0.$$

9°. Solution:

$$w=u(r)x^2y^{-2},\quad r=(At+B)^{-1}x^{-1}y^2,$$

where A and B are arbitrary constants, and the function $u=u(r)$ is determined by the ordinary differential equation

$$r^2(u-Ar+4)u''_{rr}+r^2(u'_r)^2-r(6u-Ar+6)u'_r+6(u+1)u=0.$$

⊙ *References for equation* 7.1.2.1: Y. Kodama (1988), Y. Kodama and J. Gibbons (1989), N. H. Ibragimov (1994, pp. 299–300; 1995, pp. 447–450), A. M. Vinogradov and I. S. Krasil'shchik (1997), A. D. Polyanin and V. F. Zaitsev (2002).

2. $\dfrac{\partial^2 w}{\partial x \partial t} + a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + b\dfrac{\partial^2 w}{\partial y^2} = 0.$

The transformation

$$w(x,y,t) = \frac{b}{a}u(x,y,\tau), \qquad \tau = -bt$$

leads to an equation of the form 7.1.2.1:

$$\frac{\partial^2 u}{\partial x \partial \tau} - \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) - \frac{\partial^2 u}{\partial y^2} = 0.$$

3. $\dfrac{\partial^2 w}{\partial x \partial t} - f(t)\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) - g(t)\dfrac{\partial^2 w}{\partial y^2} = 0.$

Generalized Khokhlov–Zabolotskaya equation.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2}w(C_1^2x + C_2, C_1y + C_3, t),$$
$$w_2 = w(\xi,\eta,t) + \varphi(t), \quad \xi = x + \lambda y + \int [f(t)\varphi(t) + \lambda^2 g(t)]\,dt, \quad \eta = y + 2\lambda \int g(t)\,dt,$$

where C_1, C_2, C_3, and λ are arbitrary constants and $\varphi = \varphi(t)$ is an arbitrary function, are also solutions of the equation.

2°. Solutions:

$$w(x,y,t) = -x\left(\int f\,dt + C\right)^{-1} + \varphi y + \psi,$$
$$w(x,y,t) = 2\varphi x + \frac{\varphi_t' - 2f\varphi^2}{g}y^2 + \psi y + \chi,$$
$$w(x,y,t) = (\varphi y + \psi)x - \frac{f}{12g\varphi^2}(\varphi y + \psi)^4 + \frac{\varphi_t'}{6g}y^3 + \frac{\psi_t'}{2g}y^2 + \chi y + \theta,$$

where $\varphi = \varphi(t)$, $\psi = \psi(t)$, $\chi = \chi(t)$, and $\theta = \theta(t)$ are arbitrary functions; C is an arbitrary constant; $f = f(t)$ and $g = g(t)$; the prime denotes a derivative with respect to t.

3°. “Two-dimensional” solution:

$$w(x,y,t) = U(z,t) + \varphi(t), \qquad z = x + \lambda y,$$

where the function $\varphi(t)$ is an arbitrary function, λ is an arbitrary constant, and the function $U = U(z,t)$ is determined by the first-order partial differential equation [$\psi(t)$ is an arbitrary function]

$$\frac{\partial U}{\partial t} - f(t)U\frac{\partial U}{\partial z} - [f(t)\varphi(t) + \lambda^2 g(t)]\frac{\partial U}{\partial z} = \psi(t).$$

A complete integral of this equation is sought in the form $U = A(t)z + B(t)$, which allows obtaining the general solution (see Polyanin, Zaitsev, and Moussiaux, 2002).

4°. “Two-dimensional” generalized separable solution quadratic in x:

$$w = \varphi(y,t)x^2 + \psi(y,t)x + \chi(y,t),$$

where the function $\varphi = \varphi(y,t)$, $\psi = \psi(y,t)$, and $\chi = \chi(y,t)$ are determined by the system of differential equations

$$g\varphi_{yy} = -6f\varphi^2,$$
$$g\psi_{yy} = -6f\varphi\psi + 2\varphi_t,$$
$$g\chi_{yy} = -f(2\varphi\chi + \psi^2) + \psi_t.$$

The subscripts y and t denote the corresponding partial derivatives, $f = f(t)$ and $g = g(t)$.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

4. $\displaystyle \frac{\partial^2 w}{\partial x \partial t} - \left(\frac{\partial w}{\partial x}\right)^2 - w\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial z^2} = 0.$

Three-dimensional Khokhlov–Zabolotskaya equation.

1°. Suppose $w(x, y, z, t)$ is a solution of the equation in question. Then the functions

$$\begin{aligned}
w_1 &= C_1^{-2} C_2^2 w(C_1 x + C_3, C_2 y + C_4, C_2 z + C_5, C_1^{-1} C_2^2 t + C_6),\\
w_2 &= w(x + \lambda y + \mu z + \varphi(t), y + 2\lambda t, z + 2\mu t, t) + \varphi_t'(t) - \lambda^2 - \mu^2,\\
w_3 &= w(x, y\cos\beta + z\sin\beta, -y\sin\beta + z\cos\beta, t),
\end{aligned}$$

where $C_1, \ldots, C_6$, λ, μ, and β are arbitrary constants, and $\varphi = \varphi(t)$ is an arbitrary function, are also solutions of the equation.

2°. Solutions:

$$\begin{aligned}
w(x, y, z, t) &= 2\alpha_1 x + (\alpha_1' - 2\alpha_1^2 - \alpha_2)y^2 + \alpha_3 y + \alpha_2 z^2 + \beta z + \gamma,\\
w(x, y, z, t) &= \frac{C\sqrt{4tx - y^2 - z^2}}{t^{3/2}},
\end{aligned}$$

where α_1, α_2, α_3, β, γ are arbitrary functions of t, and C is an arbitrary constant.

3°. "Three-dimensional" solution:

$$w = u(x, \xi, t), \qquad \xi = y\sin\beta + z\cos\beta,$$

where β is an arbitrary constant and the function $u = u(x, \xi, t)$ is determined by the Khokhlov–Zabolotskaya equation of the form 7.1.2.1:

$$\frac{\partial^2 u}{\partial x \partial t} - \left(\frac{\partial u}{\partial x}\right)^2 - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial \xi^2} = 0.$$

4°. "Three-dimensional" generalized separable solution linear in x:

$$w = f(y, z, t)x + g(y, z, t),$$

where the functions $f = f(y, z, t)$ and $g = g(y, z, t)$ are determined by the differential equations

$$\begin{aligned}
f_{yy} + f_{zz} &= 0,\\
g_{yy} + g_{zz} &= f_t - f^2.
\end{aligned}$$

The subscripts y, z, and t denote the corresponding partial derivatives. The first equation represents the Laplace equation and the second one is a Poisson equation (for g). For solutions of these linear equations, see, for example, Tikhonov and Samarskii (1990) and Polyanin (2002).

5°. "Three-dimensional" generalized separable solution quadratic in x:

$$w = f(y, z, t)x^2 + g(y, z, t)x + h(y, z, t),$$

where the functions $f = f(y, z, t)$, $g = g(y, z, t)$, and $h = h(y, z, t)$ are determined by the system of differential equations

$$\begin{aligned}
f_{yy} + f_{zz} &= -6f^2,\\
g_{yy} + g_{zz} &= -6fg + 2f_t,\\
h_{yy} + h_{zz} &= -2fh + g_t - g^2.
\end{aligned}$$

6°. Solution:

$$w(x,y,z,t)=u(\xi)t^{-\lambda},\quad \xi=t^{\lambda-2}(4xt-y^2-z^2),$$

where λ is an arbitrary constant, and the function $u=u(\xi)$ is determined by the ordinary differential equation

$$[4u+(1-\lambda)\xi]u''_{\xi\xi}+4(u'_\xi)^2=0.$$

For $\lambda\neq 1$, the passage to the inverse $\xi=\xi(u)$, the change of variable $\xi(u)=p(u)-\frac{4}{1-\lambda}u$, and the reduction of order with $p'_u=\frac{4}{1-\lambda}\eta(p)$ result in the first-order equation $p\eta\eta'_p-\eta+1=0$. Integrating yields $(\eta-1)e^\eta=C_1p$.

For $\lambda=1$, we have $u(\xi)=\pm\sqrt{C_1\xi+C_2}$.

7°. Solution:

$$w(x,y,z,t)=\frac{y^2+z^2}{t^2}U(\zeta),\quad \zeta=\frac{y^2+z^2}{xt},$$

where the function $U=U(\zeta)$ is determined by the ordinary differential equation

$$\zeta^2(\zeta^2U-\zeta+4)U''_{\zeta\zeta}+\zeta^4(U'_\zeta)^2+\zeta(2\zeta^2U-3\zeta+12)U'_\zeta+4U=0.$$

8°. Solution:

$$w(x,y,z,t)=\frac{z^2}{t^2}V(q),\quad q=\frac{4tx-y^2}{z^2},$$

where the function $V=V(q)$ is determined by the ordinary differential equation

$$2(4V+q^2-q)V''_{qq}+8(V'_q)^2+(1-q)V'_q+V=0.$$

⊙ *References for equation* 7.1.2.4: A. M. Vinogradov, I. S. Krasil'shchik, and V. V. Lychagin (1986), N. H. Ibragimov (1994, 1995).

5. $\boldsymbol{\dfrac{\partial^2 w}{\partial t\partial x}+a\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right)+b\dfrac{\partial^2 w}{\partial y^2}+c\dfrac{\partial^2 w}{\partial z^2}=0.}$

1°. For $a<0$, $b<0$, and $c<0$, the passage to the new independent variables according to

$$x=\bar{x}\sqrt{-a},\quad y=\bar{y}\sqrt{-b},\quad z=\bar{z}\sqrt{-a},\quad t=\bar{t}/\sqrt{-a}$$

leads to the three-dimensional Khokhlov–Zabolotskaya equation 7.1.2.4.

2°. Suppose $w(x,y,z,t)$ is a solution of the equation in question. Then the functions

$$w_1=C_1^{-2}C_2^2w(C_1x+C_3,C_2y+C_4,C_2z+C_5,C_1^{-1}C_2^2t+C_6),$$
$$w_2=w(x+\lambda y+\mu z+\varphi(t),y-2b\lambda t,z-2c\mu t,t)-\frac{1}{a}\varphi'_t(t)-\frac{b\lambda^2+c\mu^2}{a},$$

where $C_1,\dots,C_6$, λ, μ, and β are arbitrary constants and $\varphi=\varphi(t)$ is an arbitrary function, are also solutions of the equation.

3°. Solutions:

$$w(x,y,z,t)=\alpha y+\beta z+\frac{1}{at+C}x+\gamma,$$
$$w(x,y,z,t)=\alpha\ln(cy^2+bz^2)-(\beta'_t+4abc\beta^2)(cy^2+bz^2)+4bc\beta x+\gamma,$$

where $\alpha=\alpha(t)$, $\beta=\beta(t)$, and $\gamma=\gamma(t)$ are arbitrary functions and C is an arbitrary constant.

4°. "Three-dimensional" generalized separable solution linear in x:

$$w=f(y,z,t)x+g(y,z,t),$$

where the functions $f=f(y,z,t)$ and $g=g(y,z,t)$ are determined by the differential equations

$$bf_{yy}+cf_{zz}=0,\tag{1}$$
$$bg_{yy}+cg_{zz}=-f_t-af^2.\tag{2}$$

The subscripts y, z, t denote the corresponding partial derivatives. If $bc > 0$, then by the scaling $y = \bar{y}\sqrt{|b|}$, $z = \bar{z}\sqrt{|c|}$, equation (1) can be reduced to the Laplace equation, and if $bc < 0$, to the wave equation. Likewise, equation (2) can be reduced to a Poisson equation and a nonhomogeneous wave equation, respectively. For solutions of these linear equations, see, for example, Tikhonov and Samarskii (1990) and Polyanin (2002).

Remark. The above remains true if the coefficients a, b, and c are functions of y, z, and t.

5°. "Three-dimensional" generalized separable solution quadratic in x:

$$w = f(y,z,t)x^2 + g(y,z,t)x + h(y,z,t),$$

where the functions $f = f(y,z,t)$, $g = g(y,z,t)$, and $h = h(y,z,t)$ are determined by the system of differential equations

$$\begin{aligned} bf_{yy} + cf_{zz} &= -6af^2, \\ bg_{yy} + cg_{zz} &= -6afg - 2f_t, \\ bh_{yy} + ch_{zz} &= -2afh - g_t - ag^2. \end{aligned}$$

Remark. This remains true if the coefficients a, b, and c are functions of y, z, and t.

6°. There are "three-dimensional" solutions of the following forms:

$$\begin{aligned} &w(x,y,z,t) = u(x,t,\xi), \quad \xi = cy^2 + bz^2; \\ &w(x,y,z,t) = v(p,q,r)x^{k+2}, \quad p = tx^{k+1}, \quad q = yx^{k/2}, \quad r = zx^{k/2}, \end{aligned}$$

where k is an arbitrary constant.

7°. "Two-dimensional" solution:

$$w(x,y,z,t) = xU(\eta,t), \quad \eta = (cy^2 + bz^2)x^{-1},$$

where the function $U = U(\eta,t)$ is determined by the differential equation

$$\eta(a\eta U + 4bc)\frac{\partial^2 U}{\partial \eta^2} - \eta\frac{\partial^2 U}{\partial t \partial \eta} + a\eta^2\left(\frac{\partial U}{\partial \eta}\right)^2 - 2(a\eta U - 2bc)\frac{\partial U}{\partial \eta} + \frac{\partial U}{\partial t} + aU^2 = 0.$$

8°. "Two-dimensional" solution:

$$w(x,y,z,t) = V(\zeta,t) - \frac{\varphi'_t - 4bc}{a\varphi}x, \quad \zeta = cy^2 + bz^2 + \varphi x,$$

where $\varphi = \varphi(t)$ is an arbitrary function, and $V = V(\zeta,t)$ is determined by the differential equation

$$\begin{aligned} a\varphi^2(a\varphi^2 V + 4bc\zeta)\frac{\partial^2 V}{\partial \zeta^2} + a\varphi^3\frac{\partial^2 V}{\partial t \partial \zeta} + a\varphi^2\left(a\varphi^2\frac{\partial V}{\partial \zeta} - \varphi'_t + 12bc\right)\frac{\partial V}{\partial \zeta} \\ - \varphi''_{tt}\varphi + 2(\varphi'_t)^2 - 12bc\varphi'_t + 16b^2c^2 = 0. \end{aligned}$$

⊙ *References*: P. Kucharczyk (1967), S. V. Sukhinin (1978), N. H. Ibragimov (1994).

7.1.3. Equation of Unsteady Transonic Gas Flows

1. $\displaystyle \frac{\partial^2 w}{\partial x \partial t} + a\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} - b\frac{\partial^2 w}{\partial y^2} = 0.$

This is an equation of an unsteady transonic gas flow; see Lin, Reissner, and Tsien (1948). This is a special case of equation 7.1.3.2 with $f(t) = a$ and $g(t) = -b$.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-3} C_2^2 w(C_1 x + C_3, C_2 y + C_4, C_1^{-1} C_2^2 t + C_5) + C_6 yt + C_7 y + C_8 t + C_9,$$

$$w_2 = w(\xi, \eta, t) + \varphi''_{tt}(t) y^2 + 2b\varphi'_t(t) x + \psi(t) y + \chi(t),$$

$$\xi = x + \lambda y + b\lambda^2 t - 2ab\varphi(t), \quad \eta = y + 2b\lambda t,$$

where the C_n and λ are arbitrary constants and $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are arbitrary functions, are also solutions of the equation.

2°. Solution:

$$w(x,y,t) = \frac{1}{12b^2}(\gamma''_{tt} + 6a\gamma\gamma'_t + 4a^2\gamma^3) y^4 + \frac{1}{6b}(\alpha'_t + 2a\alpha\gamma) y^3 + \frac{1}{2b}[2(\gamma'_t + 2a\gamma^2) x + \beta'_t + 2a\beta\gamma] y^2 + (\alpha x + \delta) y + \gamma x^2 + \beta x + \mu,$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$, $\gamma = \gamma(t)$, $\mu = \mu(t)$, and $\delta = \delta(t)$ are arbitrary functions.

3°. “Two-dimensional” solution:

$$w(x,y,t) = U(z,t) + \varphi(t) y + \psi(t), \quad z = x + \lambda y,$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are arbitrary functions, λ is an arbitrary constant, and the function $U = U(z,t)$ is determined by the first-order partial differential equation

$$\frac{\partial U}{\partial t} + \frac{a}{2}\left(\frac{\partial U}{\partial z}\right)^2 - b\lambda^2 \frac{\partial U}{\partial z} = 0. \tag{1}$$

A complete integral of this equation is given by

$$U = C_1 z + \bigl(b\lambda^2 C_1 - \tfrac{1}{2} a C_1^2\bigr) t + C_2,$$

where C_1 and C_2 are arbitrary constants. The general solution of equation (1) can be written out in parametric form (Polyanin, Zaitsev, and Moussiaux, 2002):

$$U = sz + \bigl(b\lambda^2 s - \tfrac{1}{2} a s^2\bigr) t + f(s),$$

$$z + \bigl(b\lambda^2 - as\bigr) t + f'_s(s) = 0,$$

where $f = f(s)$ is an arbitrary function and s is the parameter.

4°. “Two-dimensional” solution of a more general form:

$$w(x,y,t) = U(z,t) + \varphi(t) y^2 + \psi(t) y + \chi(t) x + \theta(t), \quad z = x + \lambda y,$$

where $\varphi = \varphi(t)$, $\psi = \psi(t)$, $\chi = \chi(t)$, and $\theta = \theta(t)$ are arbitrary functions, λ is an arbitrary constant, and the function $U = U(z,t)$ is determined by the first-order partial differential equation [$\sigma(t)$ is an arbitrary function]:

$$\frac{\partial U}{\partial t} + \frac{a}{2}\left(\frac{\partial U}{\partial z}\right)^2 + \bigl[a\chi(t) - b\lambda^2\bigr]\frac{\partial U}{\partial z} = \bigl[2b\varphi(t) - \chi'_t(t)\bigr] z + \sigma(t).$$

This equation can be fully integrated—a complete integral is sought in the form $U = f(t) z + g(t)$.

5°. “Two-dimensional” generalized separable solution cubic in x:

$$w(x,y,t) = f(y,t) x^3 + g(y,t) x^2 + h(y,t) x + r(y,t),$$

where the functions $f = f(y,t)$, $g = g(y,t)$, $h = h(y,t)$, and $r = r(y,t)$ are determined by the differential equations

$$bf_{yy} = 18af^2,$$

$$bg_{yy} = 18afg + 3f_t,$$

$$bh_{yy} = 6afh + 4ag^2 + 2g_t,$$

$$br_{yy} = 2agh + h_t.$$

The subscripts y and t denote the corresponding partial derivatives. Setting $f = 0$ and $g = \varphi(t) y + \psi(t)$, where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are arbitrary functions, one can integrate the system with respect to y to obtain a solution dependent on six arbitrary functions.

6°. "Two-dimensional" solution:

$$w(x,y,t) = v(x,r)t^{-1}, \quad r = yt^{-1/2},$$

where the function $v = v(x,r)$ is determined by the differential equation

$$r\frac{\partial^2 v}{\partial x \partial r} - 2a\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} + 2b\frac{\partial^2 v}{\partial r^2} + 2\frac{\partial v}{\partial r} = 0.$$

7°. "Two-dimensional" solution:

$$w(x,y,t) = v(p,t) + \frac{\gamma\gamma''_{tt} - 2(\gamma'_t)^2 - 18b\gamma'_t - 40b^2}{12ab\gamma^3}y^4 + \left(\frac{4b+\gamma'_t}{a\gamma^3}p - \delta\right)y^2 + \mu y + \lambda, \quad p = y^2 + \gamma x,$$

where $\gamma = \gamma(t)$, $\mu = \mu(t)$, $\lambda = \lambda(t)$, and $\delta = \delta(t)$ are arbitrary functions, and the function $v = v(p,t)$ is determined by the differential equation

$$\left(\gamma'_t p + a\gamma^3\frac{\partial v}{\partial p}\right)\frac{\partial^2 v}{\partial p^2} + \gamma\frac{\partial^2 v}{\partial t \partial p} + (\gamma'_t - 2b)\frac{\partial v}{\partial p} - \frac{2b[p(\gamma'_t + 4b) - a\gamma^3\delta]}{a\gamma^3} = 0.$$

⊙ *References for equation* 7.1.3.1: E. V. Mamontov (1969), E. M. Vorob'ev, N. V. Ignatovich, and E. O. Semenova (1989), A. D. Polyanin and V. F. Zaitsev (2002).

2. $\displaystyle \frac{\partial^2 w}{\partial x \partial t} + f(t)\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} + g(t)\frac{\partial^2 w}{\partial y^2} = 0.$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-4} w(C_1^2 x + C_2, C_1 y + C_3, t) + C_4 yt + C_5 y + C_6 t + C_7,$$
$$w_2 = w(\xi,\eta,t) - \frac{\varphi'_t(t)}{2g(t)}y^2 + \psi(t)y + \varphi(t)x + \chi(t),$$
$$\xi = x + \lambda y - \int \left[\lambda^2 g(t) + f(t)\varphi(t)\right] dt, \quad \eta = y - 2\lambda\int g(t)\,dt,$$

where $C_1, \ldots, C_7$ and λ are arbitrary constants and $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are arbitrary functions, are also solutions of the equation.

2°. Generalized separable solution in the form of a polynomial of degree 4 in y:

$$w(x,y,t) = a(t)y^4 + b(t)y^3 + [c(t)x + d(t)]y^2 + [\alpha(t)x + \beta(t)]y + \gamma(t)x^2 + \mu(t)x + \delta(t),$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$, $\gamma = \gamma(t)$, $\mu = \mu(t)$, and $\delta = \delta(t)$ are arbitrary functions, and the functions $a = a(t)$, $b = b(t)$, $c = c(t)$, and $d = d(t)$ are given by

$$a = -\frac{c'_t + 2f\gamma c}{12g}, \quad b = -\frac{\alpha'_t + 2f\alpha\gamma}{6g}, \quad c = -\frac{\gamma'_t + 2f\gamma^2}{g}, \quad d = -\frac{\mu'_t + 2f\gamma\mu}{2g}.$$

3°. "Two-dimensional" solution:

$$w(x,y,t) = U(z,t) + \varphi(t)y^2 + \psi(t)y + \chi(t)x + \theta(t), \quad z = x + \lambda y,$$

where $\varphi = \varphi(t)$, $\psi = \psi(t)$, $\chi = \chi(t)$, and $\theta = \theta(t)$ are arbitrary functions, λ is an arbitrary constant, and the function $U = U(z,t)$ is determined by the first-order partial differential equation [$\sigma(t)$ is an arbitrary function]:

$$\frac{\partial U}{\partial t} + \frac{1}{2}f(t)\left(\frac{\partial U}{\partial z}\right)^2 + [f(t)\chi(t) + \lambda^2 g(t)]\frac{\partial U}{\partial z} = -[2g(t)\varphi(t) + \chi'_t(t)]z + \sigma(t).$$

This equation can be fully integrated; a complete integral is sought in the form $U = f(t)z + g(t)$.

4°. "Two-dimensional" generalized separable solution cubic in x:

$$w(x,y,t) = \varphi(y,t)x^3 + \psi(y,t)x^2 + \chi(y,t)x + \theta(y,t),$$

where the functions $\varphi = \varphi(y,t)$, $\psi = \psi(y,t)$, $\chi = \chi(y,t)$, and $\theta = \theta(y,t)$ are determined by the differential equations

$$\begin{aligned}
&g\varphi_{yy} + 18f\varphi^2 = 0,\\
&g\psi_{yy} + 18f\varphi\psi + 3\varphi_t = 0,\\
&g\chi_{yy} + 6f\varphi\chi + 4f\psi^2 + 2\psi_t = 0,\\
&g\theta_{yy} + 2f\psi\chi + \chi_t = 0.
\end{aligned}$$

The subscripts y and t denote the corresponding partial derivatives, $f = f(t)$ and $g = g(t)$. These equations can be treated as ordinary differential equations for y with parameter t; the constants of integration will be functions of t. The first equation has the following particular solutions: $\varphi = 0$ and $\varphi = -\dfrac{g}{3f(y+h)^2}$, where $h = h(t)$ is an arbitrary function.

5°. "Two-dimensional" solution:

$$w(x,y,t) = u(p,t) + a(t)y^4 + [b(t)p + c(t)]y^2 + \mu(t)y + \lambda(t), \quad p = y^2 + \gamma(t)x.$$

Here, $c = c(t)$, $\gamma = \gamma(t)$, $\mu = \mu(t)$, and $\lambda = \lambda(t)$ are arbitrary functions; and the function $u = u(p,t)$ is determined by the differential equation

$$\gamma\frac{\partial^2 u}{\partial p\,\partial t} + \left(\gamma'_t p + f\gamma^3\frac{\partial u}{\partial p}\right)\frac{\partial^2 u}{\partial p^2} + (\gamma'_t + 2g)\frac{\partial u}{\partial p} + 2g(bp + c) = 0,$$

where the functions $a = a(t)$ and $b = b(t)$ are given by

$$a = -\frac{(b\gamma)'_t + 10gb}{12g}, \quad b = \frac{\gamma'_t - 4g}{f\gamma^3}.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

7.1.4. Equations of the Form $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = F\left(x, y, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right)$

1. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = 0.$

General solution:

$$w(x,y) = F\bigl(y + G(x)\bigr),$$

where $F(z)$ and $G(x)$ are arbitrary functions.

⊙ *Reference*: D. Zwillinger (1989, p. 397).

2. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(x).$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{-1} w(x, C_1 y + \varphi(x)) + C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(x)$ is an arbitrary function, are also solutions of the equation.

2°. Generalized separable solutions linear and quadratic in y:

$$w(x,y) = \pm y\left[2\int f(x)\,dx + C_1\right]^{1/2} + \varphi(x),$$
$$w(x,y) = C_1 y^2 + \varphi(x)y + \frac{1}{4C_1}\left[\varphi^2(x) - 2\int f(x)\,dx\right] + C_2,$$

where $\varphi(x)$ is an arbitrary function and C_1 and C_2 are arbitrary constants.

3°. The von Mises transformation

$$\xi = x, \quad \eta = w, \quad U(\xi,\eta) = \frac{\partial w}{\partial y}, \qquad \text{where} \quad w = w(x,y), \tag{1}$$

brings the original equation to the first-order nonlinear equation

$$U\frac{\partial U}{\partial \xi} = f(\xi), \tag{2}$$

which is independent of η. On integrating (2) and taking into account the relations of (1), we obtain the first-order equation

$$\left(\frac{\partial w}{\partial y}\right)^2 = 2\int f(x)\,dx + \psi(w), \tag{3}$$

where $\psi(w)$ is an arbitrary function.

Integrating (3) yields the general solution in implicit form:

$$\int \frac{dw}{\sqrt{2F(x) + \psi(w)}} = \pm y + \varphi(x),$$

where $\varphi(x)$ and $\psi(w)$ are arbitrary functions, $F(x) = \int f(x)\,dx$.

3. $\displaystyle \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = f(x)\frac{\partial w}{\partial y}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-1} w(x, C_1 y + C_2) + C_3,$$
$$w_2 = w\bigl(x, y + \varphi(x)\bigr),$$

where C_1, C_2, and C_3 are arbitrary constants and $\varphi(x)$ is an arbitrary function, are also solutions of the equation.

2°. Generalized separable solutions:

$$w(x,y) = y\left[\int f(x)\,dx + C\right] + \varphi(x),$$
$$w(x,y) = \varphi(x)e^{\lambda y} - \frac{1}{\lambda}\int f(x)\,dx + C,$$

where $\varphi(x)$ is an arbitrary function and C and λ are arbitrary constants.

3°. The equation can be rewritten as the relation that the Jacobian of the functions w and $v = w_y - \int f(x)\,dx$ is equal to zero. It follows that w and v are functionally dependent, which means that v is expressible in terms of w:

$$\frac{\partial w}{\partial y} - \int f(x)\,dx = \varphi(w), \tag{1}$$

where $\varphi(w)$ is an arbitrary function. Any solution of the first-order equation (1) for any $\varphi(w)$ is a solution of the original equation.

Equation (1) can be treated as an ordinary differential equation in the independent variable y with parameter x. Integrating yields its general solution in implicit form:

$$\int \left[\varphi(w) + \int f(x)\,dx\right]^{-1} dw = y + \psi(x),$$

where $\psi(x)$ and $\varphi(w)$ are arbitrary functions.

4. $\displaystyle \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = f(x)\frac{\partial w}{\partial y} + g(y)\frac{\partial w}{\partial x}.$

First integral:

$$\frac{\partial w}{\partial y} = \varphi(w) - \int g(y)\,dy + \int f(x)\,dx,$$

where $\varphi(w)$ is an arbitrary function. This equation can be treated as a first-order ordinary differential equation in the independent variable y with parameter x.

5. $\displaystyle \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = f(x)g\left(\frac{\partial w}{\partial y}\right).$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-1} w(x, C_1 y + C_2) + C_3,$$
$$w_2 = w\bigl(x, y + \varphi(x)\bigr),$$

where C_1, C_2, and C_3 are arbitrary constants and $\varphi(x)$ is an arbitrary function, are also solutions of the equation.

2°. First integral:

$$\int \frac{U\,dU}{g(U)} = \varphi(w) + \int f(x)\,dx, \qquad U = \frac{\partial w}{\partial y},$$

where $\varphi(w)$ is an arbitrary function. This equation can be treated as a first-order ordinary differential equation in the independent variable y with parameter x.

7.1.5. Other Equations with Two Independent Variables

1. $\displaystyle \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} + f(y)\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = g(y)w + h(y)x + s(y).$

Generalized separable solution linear in x:

$$w = \varphi(y)x + \psi(y),$$

where the functions $\varphi(y)$ and $\psi(y)$ are determined by the system of ordinary differential equations

$$f\varphi\varphi''_{yy} + (\varphi'_y)^2 = g\varphi + h,$$
$$f\varphi\psi''_{yy} + \varphi'_y\psi'_y = g\psi + s.$$

2. $\displaystyle \left[1 - \left(\frac{\partial w}{\partial t}\right)^2\right]\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial x \partial t} - \left[1 + \left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial^2 w}{\partial t^2} = 0.$

Born–Infeld equation (see Born and Infeld, 1934). It is used in nonlinear electrodynamics and field theory.

1°. Solutions:

$$w(x,t) = \varphi(x+t),$$
$$w(x,t) = \psi(x-t),$$

where $\varphi(z_1)$ and $\psi(z_2)$ are arbitrary functions.

2°. Cauchy problem with initial conditions:

$$w = f(x) \quad \text{at} \quad t = 0, \qquad \partial_t w = g(x) \quad \text{at} \quad t = 0.$$

The hyperbolicity condition $1 + [f'_x(x)]^2 - g^2(x) > 0$ is assumed to hold.

Solution in parametric form:

$$t = \frac{1}{2}\int_\alpha^\beta \frac{1 + [f'_\zeta(\zeta)]^2}{\sqrt{1 + [f'_\zeta(\zeta)]^2 - g^2(\zeta)}}\, d\zeta,$$
$$x = \frac{\alpha+\beta}{2} - \frac{1}{2}\int_\alpha^\beta \frac{f'_\zeta(\zeta) g(\zeta)\, d\zeta}{\sqrt{1 + [f'_\zeta(\zeta)]^2 - g^2(\zeta)}},$$
$$w = \frac{f(\alpha) + f(\beta)}{2} + \frac{1}{2}\int_\alpha^\beta \frac{g(\zeta)\, d\zeta}{\sqrt{1 + [f'_\zeta(\zeta)]^2 - g^2(\zeta)}}.$$

⊙ *Reference*: B. M. Barbashov and N. A. Chernikov (1966).

3°. By the introduction of the new variables

$$\xi = x - t, \quad \eta = x + t, \quad u = \frac{\partial w}{\partial \xi}, \quad v = \frac{\partial w}{\partial \eta},$$

the Born–Infeld equation can be rewritten as the equivalent system of equations

$$\frac{\partial u}{\partial \eta} - \frac{\partial v}{\partial \xi} = 0,$$
$$v^2 \frac{\partial u}{\partial \xi} - (1 + 2uv)\frac{\partial u}{\partial \eta} + u^2 \frac{\partial v}{\partial \eta} = 0.$$

The hodograph transformation (where u, v are treated as the independent variables and ξ, η as the dependent ones) leads to the linear system

$$\begin{aligned} &\frac{\partial \xi}{\partial v} - \frac{\partial \eta}{\partial u} = 0,\\ &v^2 \frac{\partial \eta}{\partial v} + (1 + 2uv)\frac{\partial \xi}{\partial v} + u^2 \frac{\partial \xi}{\partial u} = 0. \end{aligned} \tag{1}$$

On eliminating η, we can reduce this system to the second-order linear equation

$$u^2 \frac{\partial^2 \xi}{\partial u^2} + (1 + 2uv)\frac{\partial^2 \xi}{\partial u \partial v} + v^2 \frac{\partial^2 \xi}{\partial v^2} + 2u \frac{\partial \xi}{\partial u} + 2v \frac{\partial \xi}{\partial v} = 0.$$

Looking for solutions in the hyperbolic domain, we write out the equation of characteristics

$$u^2\, dv^2 - (1 + 2uv)\, du\, dv + v^2\, du^2 = 0.$$

Integrals of this equation are given by $r = C_1$ and $s = C_2$, where

$$r = \frac{\sqrt{1 + 4uv} - 1}{2v}, \qquad s = \frac{\sqrt{1 + 4uv} - 1}{2u}. \tag{2}$$

Passing in (1) to the new variables of (2), we obtain

$$r^2 \frac{\partial \xi}{\partial r} + \frac{\partial \eta}{\partial r} = 0, \qquad \frac{\partial \xi}{\partial s} + s^2 \frac{\partial \eta}{\partial s} = 0. \tag{3}$$

Eliminating η yields the simple equation

$$\frac{\partial^2 \xi}{\partial r \partial s} = 0,$$

whose solution is the sum of two arbitrary functions with different arguments, $\xi = \varphi(r) + \psi(s)$. The function η is found from system (3).

⊙ *Reference*: G. B. Whitham (1974).

4°. The Legendre transformation

$$w(x,t) + u(\zeta,\tau) = x\zeta + t\tau, \quad \zeta = \frac{\partial w}{\partial x}, \quad \tau = \frac{\partial w}{\partial t}, \quad x = \frac{\partial u}{\partial \zeta}, \quad t = \frac{\partial u}{\partial \tau}$$

leads to the linear equation

$$\left(1 - \tau^2\right) \frac{\partial^2 u}{\partial \tau^2} - 2\zeta\tau \frac{\partial^2 u}{\partial \zeta \partial \tau} - \left(1 + \zeta^2\right) \frac{\partial^2 u}{\partial \zeta^2} = 0.$$

3. $\displaystyle \left[a + \left(\frac{\partial w}{\partial y}\right)^2\right] \frac{\partial^2 w}{\partial x^2} - 2b \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y} + \left[c + \left(\frac{\partial w}{\partial x}\right)^2\right] \frac{\partial^2 w}{\partial y^2} = 0.$

Equation of minimal surfaces (with $a = b = c = 1$). It describes, for example, the shape of a foam film bounded by a given contour.

1°. The Legendre transformation

$$w(x,y) + u(\xi,\eta) = x\xi + y\eta, \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y}, \quad x = \frac{\partial u}{\partial \xi}, \quad y = \frac{\partial u}{\partial \eta}$$

leads to the linear equation

$$\left(a + \eta^2\right) \frac{\partial^2 u}{\partial \eta^2} + 2b\xi\eta \frac{\partial^2 u}{\partial \xi \partial \eta} + \left(c + \xi^2\right) \frac{\partial^2 u}{\partial \xi^2} = 0.$$

2°. General solution in parametric form for $a = b = c = 1$:

$$x = \operatorname{Re} f_1(z), \quad y = \operatorname{Re} f_2(z), \quad w = \operatorname{Re} f_3(z),$$

where the $f_k(z)$ are arbitrary analytic functions of the complex variable $z = \alpha + i\beta$ with derivatives constrained by

$$[f_1'(z)]^2 + [f_2'(z)]^2 + [f_3'(z)]^2 = 0.$$

For example, z can be taken as one of the functions $f_k(z)$.

⊙ *Reference*: R. Courant and D. Hilbert (1989).

4. $\displaystyle \frac{\partial^2 w}{\partial x \partial y} = F\left(y, \frac{\partial w}{\partial x}\right) \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(C_1 x + C_2, y) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. The Euler transformation

$$w(x,y) + u(\xi,\eta) = x\xi, \quad x = \frac{\partial u}{\partial \xi}, \quad y = \eta$$

leads to the linear equation (for details, see Subsection S.2.3)

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = F(\eta,\xi) \frac{\partial u}{\partial \eta}.$$

5. $\left[f^2-\left(\dfrac{\partial w}{\partial x}\right)^2\right]\dfrac{\partial^2 w}{\partial x^2}-2\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\partial y}+\left[f^2-\left(\dfrac{\partial w}{\partial y}\right)^2\right]\dfrac{\partial^2 w}{\partial y^2}=0,\qquad f=f(w_x^2+w_y^2).$

This equation describes a two-dimensional steady-state isentropic flow of a compressible gas; w is the velocity potential and f is the sound speed.

The Legendre transformation

$$w(x,y)+U(\xi,\eta)=x\xi+y\eta,\quad \xi=\frac{\partial w}{\partial x},\quad \eta=\frac{\partial w}{\partial y},\quad x=\frac{\partial U}{\partial \xi},\quad y=\frac{\partial U}{\partial \eta}$$

leads to the linear equation

$$\left(f^2-\xi^2\right)\frac{\partial^2 U}{\partial \eta^2}+2\xi\eta\frac{\partial^2 U}{\partial \xi\partial \eta}+\left(f^2-\eta^2\right)\frac{\partial^2 U}{\partial \xi^2}=0,\qquad f=f(\xi^2+\eta^2).$$

⊙ *Reference*: R. Courant and D. Hilbert (1989).

7.1.6. Other Equations with Three Independent Variables

1. $\dfrac{\partial^2 w}{\partial x\partial t}+f\left(\dfrac{\partial w}{\partial x}\right)\dfrac{\partial^2 w}{\partial x^2}+b\dfrac{\partial^2 w}{\partial y^2}=0.$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the functions

$$w_1=C_1^{-1}w(C_1x+C_2,C_1y+C_3,C_1t+C_4)+C_5yt+C_6y+C_7t+C_8,$$
$$w_2=w(x+\lambda y-b\lambda^2t,y-2b\lambda t,t)+\varphi(t)y+\psi(t),$$

where $C_1,\ldots,C_8$ and λ are arbitrary constants and $\varphi=\varphi(t)$ and $\psi=\psi(t)$ are arbitrary functions, are also solutions of the equation.

2°. “Two-dimensional” solution:

$$w(x,y,t)=U(z,t)+\varphi(t)y^2+\psi(t)y+\chi(t),\quad z=x+\lambda y,$$

where $\varphi=\varphi(t)$, $\psi=\psi(t)$, and $\chi=\chi(t)$ are arbitrary functions, λ is an arbitrary constant, and the function $U=U(z,t)$ is determined by the first-order partial differential equation

$$\frac{\partial U}{\partial t}+F\left(\frac{\partial U}{\partial z}\right)+b\lambda^2\frac{\partial U}{\partial z}+2b\varphi(t)z=\sigma(t),\quad F(u)=\int f(u)\,du,$$

where $\sigma(t)$ is an arbitrary function. A complete integral of this equation has the form

$$U=A(t)z+B(t),$$

where the functions $A(t)$ and $B(t)$ are given by

$$A(t)=-2b\int\varphi(t)\,dt+C_1,\qquad B(t)=\int\Big[\sigma(t)-F(A(t))-b\lambda^2A(t)\Big]\,dt+C_2,$$

and C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

2. $\dfrac{\partial^2 w}{\partial x\partial t}+f(t)\Phi\left(\dfrac{\partial w}{\partial x}\right)\dfrac{\partial^2 w}{\partial x^2}+g(t)\dfrac{\partial^2 w}{\partial y^2}=0.$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the function

$$w_1=w(\xi,\eta,t)+\varphi(t)y+\psi(t),\qquad \xi=x+\lambda y-\lambda^2\int g(t)\,dt+C_1,\quad \eta=y-2\lambda\int g(t)\,dt+C_2,$$

where C_1, C_2, and λ are arbitrary constants and $\varphi=\varphi(t)$ and $\psi=\psi(t)$ are arbitrary functions, is also a solution of the equation.

$2°$. "Two-dimensional" solution:

$$w(x,y,t) = U(z,t) + \varphi(t)y^2 + \psi(t)y + \chi(t), \quad z = x + \lambda y,$$

where $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are arbitrary functions, λ is an arbitrary constant, and the function $U = U(z,t)$ is determined by the first-order partial differential equation

$$\frac{\partial U}{\partial t} + f(t)\Psi\left(\frac{\partial U}{\partial z}\right) + \lambda^2 g(t)\frac{\partial U}{\partial z} + 2g(t)\varphi(t)z = \sigma(t), \quad \Psi(u) = \int \Phi(u)\,du,$$

where $\sigma(t)$ is an arbitrary function. This equation can be fully integrated—a complete integral is sought in the form $U = A(t)z + B(t)$.

7.2. Equations Quadratic in the Highest Derivatives

7.2.1. Equations of the Form $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = F(x,y)$

► *Suppose $w(x,y)$ is a solution of the equation in question. Then the function*

$$w_1 = w(x,y) + C_1 xy + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

1. $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)y^k.$

$1°$. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-k-2} w(x, C_1^2 y) + C_2 xy + C_3 x + C_4 y + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

$2°$. Generalized separable solutions:

$$w(x,y) = (C_1 x + C_2)y^{k+1} + \frac{y}{k(k+1)}\int_0^x \frac{(x-t)f(t)}{(C_1 t + C_2)}\,dt + C_3 xy + C_4 x + C_5 y + C_6,$$

$$w(x,y) = (C_1 x + C_2)y^{k+2} + \frac{1}{(k+1)(k+2)}\int_0^x \frac{(x-t)f(t)}{(C_1 t + C_2)}\,dt + C_3 xy + C_4 x + C_5 y + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants.

$3°$. Generalized separable solution:

$$w(x,y) = \varphi(x)y^{\frac{k+2}{2}} + C_1 xy + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$k(k+2)\varphi\varphi''_{xx} = 4f(x).$$

2. $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)g(y).$

$1°$. Additive separable solution:

$$w(x,y) = C_1\int_0^x (x-t)f(t)\,dt + C_2 x + \frac{1}{C_1}\int_0^y (y-\tau)g(\tau)\,d\tau + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

$2°$. Multiplicative separable solution:

$$w(x,y) = \varphi(x)\psi(y),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined by the ordinary differential equations (C_1 is an arbitrary constant)

$$\varphi\varphi''_{xx} = C_1 f(x),$$
$$\psi\psi''_{yy} = C_1^{-1} g(y).$$

3. $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(ax+by).$

Solutions:

$$w(x,y) = \pm\frac{1}{ab}\int_0^z (z-t)\sqrt{f(t)}\,dt + C_1xy + C_2x + C_3y + C_4, \qquad z = ax+by,$$

where $C_1, \dots, C_4$ are arbitrary constants.

4. $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)y^{2k} + g(x)y^k + h(x)y^{k-1}.$

Generalized separable solution:

$$w(x,y) = \varphi(x)y^{k+1} + \psi(x)y + \chi(x),$$

where the functions $\varphi = \varphi(x)$, $\psi = \psi(x)$, and $\chi = \chi(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned} k(k+1)\varphi\varphi''_{xx} &= f(x),\\ k(k+1)\varphi\psi''_{xx} &= g(x),\\ k(k+1)\varphi\chi''_{xx} &= h(x). \end{aligned}$$

5. $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)e^{\lambda y}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C_1 w\Big(x, y - \frac{2}{\lambda}\ln|C_1|\Big) + C_2xy + C_3x + C_4y + C_5,$$

where $C_1, \dots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution:

$$w(x,y) = (C_1x + C_2)e^{\lambda y} + \frac{1}{\lambda^2}\int_{x_0}^x \frac{(x-t)f(t)}{C_1t + C_2}\,dt + C_3xy + C_4x + C_5y + C_6,$$

where $C_1, \dots, C_6$ are arbitrary constants, and x_0 is any number such that the integrand does not have a singularity at $x = x_0$.

3°. Generalized separable solution:

$$w(x,y) = \varphi(x)e^{\lambda y/2} + C_1xy + C_2x + C_3y + C_4,$$

where $C_1, \dots, C_4$ are arbitrary constants, and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation $\lambda^2\varphi\varphi''_{xx} = 4f(x)$.

6. $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)e^{2\lambda y} + g(x)e^{\lambda y}.$

Generalized separable solution:

$$w(x,y) = \varphi(x)e^{\lambda y} + \frac{1}{\lambda^2}\int_{x_0}^x (x-t)\frac{g(t)}{\varphi(t)}\,dt + C_1xy + C_2x + C_3y + C_4,$$

where $C_1, \dots, C_4$ are arbitrary constants and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation $\lambda^2\varphi\varphi''_{xx} = f(x)$.

7.2.2. Monge–Ampère equation $\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = F(x, y)$

Preliminary remarks.
The Monge–Ampère equation is encountered in differential geometry, gas dynamics, and meteorology.

1°. Suppose $w(x, y)$ is a solution of the Monge–Ampère equation. Then the function

$$w_1 = w(x, y) + C_1 x + C_2 y + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. The transformation

$$\bar{x} = a_1 x + b_1 y + c_1, \quad \bar{y} = a_2 x + b_2 y + c_2, \quad \bar{w} = kw + a_3 x + b_3 y + c_3, \quad \bar{F} = k^2 (a_1 b_2 - a_2 b_1)^{-2} F,$$

where the a_n, b_n, c_n, and k are arbitrary constants, takes the Monge–Ampère equation to an equation of the same form.

3°. The transformation

$$\bar{x} = x(1 + \alpha x + \beta y)^{-1}, \quad \bar{y} = y(1 + \alpha x + \beta y)^{-1}, \quad \bar{w} = w(1 + \alpha x + \beta y)^{-1}, \quad \bar{F} = F(1 + \alpha x + \beta y)^4,$$

where α and β are arbitrary constants, takes the Monge–Ampère equation to an equation of the same form.

⊙ *References*: S. V. Khabirov (1990), N. H. Ibragimov (1994).

4°. In Lagrangian coordinates, the system of equations of the one-dimensional gas dynamics with plane waves is as follows:

$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial \xi} = 0, \qquad \frac{\partial V}{\partial t} - \frac{\partial u}{\partial \xi} = 0,$$

where t is time, u the velocity, p the pressure, ξ the Lagrangian coordinate, and V the specific volume. The equation of state is assumed to have the form $V = V\bigl(p, S(\xi)\bigr)$, where $S = S(\xi)$ is a prescribed entropy profile.

The Martin transformation

$$u(\xi, t) = \frac{\partial w}{\partial x}(x, y), \quad t = \frac{\partial w}{\partial y}(x, y), \quad x = \xi, \quad y = p(\xi, t)$$

reduces the equations of the one-dimensional gas dynamics to the nonhomogeneous Monge–Ampère equation

$$\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = F(x, y),$$

where $F(x, y) = -\frac{\partial V}{\partial p}\bigl(p, S(\xi)\bigr)$.

⊙ *References*: M. N. Martin (1953), B. L. Rozhdestvenskii and N. N. Yanenko (1983).

1. $\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = 0.$

Homogeneous Monge–Ampère equation.

1°. Suppose $w(x, y)$ is a solution of the homogeneous Monge–Ampère equation. Then the functions

$$w_1 = C_1 w(C_2 x + C_3 y + C_4, C_5 x + C_6 y + C_7) + C_8 x + C_9 y + C_{10},$$
$$w_2 = (1 + C_1 x + C_2 y) w\left(\frac{x}{1 + C_1 x + C_2 y}, \frac{y}{1 + C_1 x + C_2 y}\right),$$

where $C_1, \dots, C_{10}$ are arbitrary constants, are also solutions of the equation.

2°. First integrals:

$$\Phi_1\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right) = 0,$$

$$\Phi_2\left(\frac{\partial w}{\partial x}, w - x\frac{\partial w}{\partial x} - y\frac{\partial w}{\partial y}\right) = 0,$$

where $\Phi_1(u, v)$ and $\Phi_2(u, z)$ are arbitrary functions of two arguments.

3°. General solution in parametric form:

$$w = tx + \varphi(t)y + \psi(t),$$
$$x + \varphi'(t)y + \psi'(t) = 0,$$

where t is the parameter, and $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are arbitrary functions.

4°. Solutions involving one arbitrary function:

$$w(x, y) = \varphi(C_1x + C_2y) + C_3x + C_4y + C_5,$$

$$w(x, y) = (C_1x + C_2y)\varphi\left(\frac{y}{x}\right) + C_3x + C_4y + C_5,$$

$$w(x, y) = (C_1x + C_2y + C_3)\varphi\left(\frac{C_4x + C_5y + C_6}{C_1x + C_2y + C_3}\right) + C_7x + C_8y + C_9,$$

where $C_1, \dots, C_9$ are arbitrary constants and $\varphi = \varphi(z)$ is an arbitrary function.

5°. Solutions involving arbitrary constants:

$$w(x, y) = C_1y^2 + C_2xy + \frac{C_2^2}{4C_1}x^2 + C_3y + C_4x + C_5,$$

$$w(x, y) = \frac{1}{x + C_1}\left(C_2y^2 + C_3y + \frac{C_3^2}{4C_2}\right) + C_4y + C_5x + C_6,$$

$$w(x, y) = \pm(C_1x + C_2y + C_3)^k + C_4x + C_5y + C_6,$$

$$w(x, y) = \pm\frac{(C_1x + C_2y + C_3)^{k+1}}{(C_4x + C_5y + C_6)^k} + C_7x + C_8y + C_9,$$

$$w(x, y) = \pm\sqrt{C_1(x + a)^2 + C_2(x + a)(y + b) + C_3(y + b)^2} + C_5x + C_6y + C_7,$$

where the a, b, and the C_n are arbitrary constants.

⊙ *References for equation* 7.2.2.1: E. Goursat (1933), S. V. Khabirov (1990), N. H. Ibragimov (1994).

2. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = A.$

1°. First integrals for $A = a^2 > 0$:

$$\Phi_1\left(\frac{\partial w}{\partial x} + ay, \frac{\partial w}{\partial y} - ax\right) = 0,$$

$$\Phi_2\left(\frac{\partial w}{\partial x} - ay, \frac{\partial w}{\partial y} + ax\right) = 0,$$

where the $\Phi_n(u, v)$ are arbitrary functions of two arguments ($n = 1, 2$).

2°. General solution in parametric form for $A = a^2 > 0$:

$$x = \frac{\beta - \lambda}{2a}, \quad y = \frac{\psi'(\lambda) - \varphi'(\beta)}{2a}, \quad w = \frac{(\beta + \lambda)[\psi'(\lambda) - \varphi'(\beta)] + 2\varphi(\beta) - 2\psi(\lambda)}{4a},$$

where β and λ are the parameters, $\varphi = \varphi(\beta)$ and $\psi = \psi(\lambda)$ are arbitrary functions.

3°. Solutions:

$$w(x,y) = \pm\frac{\sqrt{A}}{C_2}x(C_1x + C_2y) + \varphi(C_1x + C_2y) + C_3x + C_4y,$$

$$w(x,y) = C_1y^2 + C_2xy + \frac{1}{4C_1}(C_2^2 - A)x^2 + C_3y + C_4x + C_5,$$

$$w(x,y) = \frac{1}{x + C_1}\left(C_2y^2 + C_3y + \frac{C_3^2}{4C_2}\right) - \frac{A}{12C_2}(x^3 + 3C_1x^2) + C_4y + C_5x + C_6,$$

$$w(x,y) = \pm\frac{2\sqrt{A}}{3C_1C_2}(C_1x - C_2^2y^2 + C_3)^{3/2} + C_4x + C_5y + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants and $\varphi = \varphi(z)$ is an arbitrary function.

Another five solutions can be obtained:

(a) from the solution of equation 7.2.2.18 with $\alpha = 0$ and $f(u) = A$, where β is an arbitrary constant;
(b) from the solution of equation 7.2.2.20 with $f(u) = A$, where a, b, and c are arbitrary constants;
(c) from the solution of equation 7.2.2.21 with $f(u) = A$, where a, b, c, k, and s are arbitrary constants;
(d) from the solution of equation 7.2.2.22 with $\alpha = 0$ and $f(u) = A$, where β is an arbitrary constant;
(e) from the solution of equation 7.2.2.24 with $\alpha = 0$ and $f(u) = A$, where β is an arbitrary constant.

4°. The Legendre transformation

$$u = x\xi + y\eta - w(x,y), \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y},$$

where $u = u(\xi, \eta)$ is the new independent variable, and ξ and η are the new dependent variables, leads to an equation of the similar form

$$\left(\frac{\partial^2 u}{\partial \xi \partial \eta}\right)^2 - \frac{\partial^2 u}{\partial \xi^2}\frac{\partial^2 u}{\partial \eta^2} = \frac{1}{A}.$$

⊙ *Reference*: E. Goursat (1933).

3. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = f(x).$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-1}w(x, C_2x \pm C_1y + C_3) + C_4x + C_5y + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solutions quadratic in y:

$$w(x,y) = C_1y^2 + C_2xy + \frac{C_2^2}{4C_1}x^2 - \frac{1}{2C_1}\int_0^x (x-t)f(t)\,dt + C_3y + C_4x + C_5,$$

$$w(x,y) = \frac{1}{x + C_1}\left(C_2y^2 + C_3y + \frac{C_3^2}{4C_2}\right) - \frac{1}{2C_2}\int_0^x (x-t)(t + C_1)f(t)\,dt + C_4y + C_5x + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

3°. Generalized separable solutions for $f(x) > 0$:

$$w(x,y) = \pm y\int\sqrt{f(x)}\,dx + \varphi(x) + C_1y,$$

where $\varphi(x)$ is an arbitrary function.

⊙ *References*: M. N. Martin (1953), B. L. Rozhdestvenskii and N. N. Yanenko (1983).

4°. Conservation law:

$$D_x\left[y(w_x w_{yy} - w_y w_{xy} + g_y) - g - w_x w_y\right] + D_y\left[y(w_y w_{xx} - w_x w_{xy} + g_x) + (w_x)^2\right] = 0,$$

where $D_x = \dfrac{\partial}{\partial x}$, $D_y = \dfrac{\partial}{\partial y}$, $g = y\displaystyle\int f(x)\,dx + \varphi(x) + \psi(y)$, and $\varphi(x)$ and $\psi(y)$ are arbitrary functions.

⊙ *Reference*: S. V. Khabirov (1990).

5°. Let us consider some specific functions $f = f(x)$. Solutions that can be obtained by the formulas of Items 1° and 2° are omitted.

5.1. Solutions with $f(x) = Ax^k$ can be obtained:

(a) from the solution of equation 7.2.2.18 with $f(u) = A$ and $\alpha = k/2$, where β is an arbitrary constant;
(b) from the solution of equation 7.2.2.24 with $f(u) = A$ and $\alpha = k/2$, where β is an arbitrary constant.

5.2. Solutions for $f(x) = Ae^{\lambda x}$:

$$w(x,y) = \pm\frac{2\sqrt{A}}{C_2\lambda}e^{\lambda x/2}\sin(C_1x + C_2y + C_3) + C_4x + C_5y + C_6,$$

$$w(x,y) = \pm\frac{2\sqrt{A}}{C_2\lambda}e^{\lambda x/2}\sinh(C_1x + C_2y + C_3) + C_4x + C_5y + C_6,$$

$$w(x,y) = \pm\frac{2\sqrt{-A}}{C_2\lambda}e^{\lambda x/2}\cosh(C_1x + C_2y + C_3) + C_4x + C_5y + C_6.$$

Another solution can be obtained from the solution of equation 7.2.2.22 with $\alpha = \lambda$ and $f(u) = A$, where β is an arbitrary constant.

4. $\displaystyle\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = f(x)y.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{-3}w(x, C_1^2y) + C_2x + C_3y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution quadratic in y:

$$w(x,y) = C_1y^2 - y\int F(x)\,dx + \frac{1}{2C_1}\int_a^x (x-t)F^2(t)\,dt + C_2x + C_3y + C_4,$$

$$F(x) = \frac{1}{2C_1}\int f(x)\,dx + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

3°. Generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where

$$\varphi(x) = \frac{1}{C_1x + C_2}, \quad \psi(x) = C_3\varphi(x) + C_4 + \frac{\varphi(x)}{2C_1}\int\frac{f(x)\,dx}{[\varphi(x)]^3} - \frac{1}{2C_1}\int\frac{f(x)\,dx}{[\varphi(x)]^2},$$

$$\chi(x) = \frac{1}{2}\int_a^x (x-t)\frac{[\psi_t'(t)]^2}{\varphi(t)}\,dt + C_5x + C_6.$$

4°. Generalized separable solutions cubic in y:

$$w(x,y) = C_1y^3 - \frac{1}{6C_1}\int_a^x (x-t)f(t)\,dt + C_2x + C_3y + C_4,$$

$$w(x,y) = \frac{y^3}{(C_1x + C_2)^2} - \frac{1}{6}\int_a^x (x-t)(C_1t + C_2)^2 f(t)\,dt + C_3x + C_4y + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

5°. See solution of equation 7.2.2.7 in Item 3° with $k = 1$.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

5. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)y^2.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = \pm C_1^{-2} w(x, C_1 y) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x)y^2 + \left[C_1 \int \varphi^2(x)\,dx + C_2\right] y + \frac{1}{2}C_1^2 \int_a^x (x-t)\varphi^3(t)\,dt + C_3 x + C_4,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$\varphi\varphi''_{xx} = 2(\varphi'_x)^2 - \tfrac{1}{2} f(x).$$

3°. Generalized separable solutions in the form of polynomials of degree 4 in y:

$$w(x,y) = C_1 y^4 - \frac{1}{12 C_1}\int_a^x (x-t) f(t)\,dt + C_2 x + C_3 y + C_4,$$

$$w(x,y) = \frac{y^4}{(C_1 x + C_2)^3} - \frac{1}{12}\int_a^x (x-t)(C_1 t + C_2)^3 f(t)\,dt + C_3 x + C_4 y + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

4°. See solution of equation 7.2.2.7 in Item 3° with $k = 2$.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

6. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)y^2 + g(x)y + h(x).$

Generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where the functions $\varphi = \varphi(x)$, $\psi = \psi(x)$, and $\chi = \chi(x)$ are determined by the system of ordinary differential equations

$$\varphi\varphi''_{xx} = 2(\varphi'_x)^2 - \tfrac{1}{2} f(x),$$
$$\varphi\psi''_{xx} = 2\varphi'_x\psi'_x - \tfrac{1}{2} g(x),$$
$$\varphi\chi''_{xx} = \tfrac{1}{2}(\psi'_x)^2 - \tfrac{1}{2} h(x).$$

7. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)y^k.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{-k-2} w(x, C_1^2 y) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solutions:

$$w(x,y) = \frac{C_1 y^{k+2}}{(k+1)(k+2)} - \frac{1}{C_1}\int_a^x (x-t) f(t)\,dt + C_2 x + C_3 y + C_4,$$

$$w(x,y) = \frac{y^{k+2}}{(C_1 x + C_2)^{k+1}} - \frac{1}{(k+1)(k+2)}\int_a^x (x-t)(C_1 t + C_2)^{k+1} f(t)\,dt + C_3 x + C_4 y + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

3°. Multiplicative separable solution:

$$w(x,y)=\varphi(x)y^{\frac{k+2}{2}},$$

where the function $\varphi=\varphi(x)$ is determined by the ordinary differential equation

$$k(k+2)\varphi\varphi''_{xx}-(k+2)^2(\varphi'_x)^2+4f(x)=0.$$

4°. Let us consider the case where f is a power-law function of x, $f(x)=Ax^n$, in more detail.

Solutions:

$$w(x,y)=\frac{C_1x^{n+2}}{(n+1)(n+2)}-\frac{Ay^{k+2}}{C_1(k+1)(k+2)}+C_2y+C_3x+C_4,$$
$$w(x,y)=\frac{C_1x^{n+2}}{(n+1)(n+2)y^{n+1}}-\frac{Ay^{k+n+3}}{C_1(k+n+2)(k+n+3)}+C_2y+C_3x+C_4,$$
$$w(x,y)=\frac{C_1y^{k+2}}{(k+1)(k+2)x^{k+1}}-\frac{Ax^{k+n+3}}{C_1(k+n+2)(k+n+3)}+C_2y+C_3x+C_4,$$
$$w(x,y)=(C_1x+C_2)^{-k-1}y^{k+2}-\frac{A}{(k+1)(k+2)}\int_a^x(x-t)t^n(C_1t+C_2)^{k+1}\,dt+C_3y+C_4x,$$
$$w(x,y)=(C_1y+C_2)^{-n-1}x^{n+2}-\frac{A}{(n+1)(n+2)}\int_a^y(y-t)t^k(C_1t+C_2)^{n+1}\,dt+C_3y+C_4x,$$

where $C_1,\,\dots,\,C_4$ are arbitrary constants.

There are also a multiplicative separable solution, see Item 3° with $f(x)=Ax^n$, and a solution of the same type:

$$w(x,y)=\psi(y)x^{\frac{n+2}{2}},$$

where the function $\psi=\psi(y)$ is determined by the ordinary differential equation

$$n(n+2)\psi\psi''_{yy}-(n+2)^2(\psi'_y)^2+4Ay^k=0.$$

The substitution $\psi=U^{-n/2}$ brings it to the Emden–Fowler equation

$$U''_{yy}=\frac{8A}{n^2(n+2)}y^kU^{n+1},$$

whose exact solutions for various values of k and n can be found in the books by Polyanin and Zaitsev (1995, 2003).

Another exact solution for $f(x)=Ax^n$ can be obtained from the solution of equation 7.2.2.18 with $f(u)=Au^k$ and $n=2\alpha+k\beta$, where α and β are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

8. $\displaystyle\left(\frac{\partial^2w}{\partial x\partial y}\right)^2-\frac{\partial^2w}{\partial x^2}\frac{\partial^2w}{\partial y^2}=f(x)y^{2k+2}+g(x)y^k.$

Generalized separable solution:

$$w(x,y)=\varphi(x)y^{k+2}-\frac{1}{(k+1)(k+2)}\int_a^x(x-t)\frac{g(t)}{\varphi(t)}\,dt+C_1x+C_2y+C_3,$$

where the function $\varphi=\varphi(x)$ is determined by the ordinary differential equation

$$(k+1)(k+2)\varphi\varphi''_{xx}-(k+2)^2(\varphi'_x)^2+f(x)=0.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

9. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)e^{\lambda y}.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1 w\Big(x,\ y - \frac{2}{\lambda}\ln|C_1|\Big) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solutions:

$$w(x, y) = C_1 \int_a^x (x - t) f(t)\, dt + C_2 x - \frac{1}{C_1 \lambda^2} e^{\lambda y} + C_3 y + C_4,$$
$$w(x, y) = C_1 e^{\beta x + \lambda y} - \frac{1}{C_1 \lambda^2} \int_a^x (x - t) e^{-\beta t} f(t)\, dt + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ and β are arbitrary constants.

3°. Multiplicative separable solution:

$$w(x, y) = \varphi(x) \exp\big(\tfrac{1}{2}\lambda y\big),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$\varphi\varphi''_{xx} - (\varphi'_x)^2 + 4\lambda^{-2} f(x) = 0.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

10. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)e^{2\lambda y} + g(x)e^{\lambda y}.$

Generalized separable solution:

$$w(x, y) = \varphi(x) e^{\lambda y} - \frac{1}{\lambda^2} \int_a^x (x - t) \frac{g(t)}{\varphi(t)}\, dt + C_1 x + C_2 y + C_3,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$\varphi\varphi''_{xx} - (\varphi'_x)^2 + \lambda^{-2} f(x) = 0.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

11. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)g(y) + A^2.$

Generalized separable solutions:

$$w(x, y) = C_1 \int_a^x (x - t) f(t)\, dt - \frac{1}{C_1} \int_b^y (y - \xi) g(\xi)\, d\xi \pm Axy + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants; a and b are any numbers.

12. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(ax + by).$

1°. Solutions:

$$w(x, y) = \pm\frac{x}{b} \int \sqrt{f(z)}\, dz + \varphi(z) + C_1 x + C_2 y, \qquad z = ax + by,$$

where C_1 and C_2 are arbitrary constants and $\varphi(z)$ is an arbitrary function.

2°. The transformation

$$w = U(x, z), \quad z = ax + by$$

leads to an equation of the form 7.2.2.3:

$$\left(\frac{\partial^2 U}{\partial x \partial z}\right)^2 = \frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial z^2} + b^{-2} f(z).$$

Here, x and z play the role of y and x in 7.2.2.3, respectively.

13. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = x^k f(ax+by).$

The transformation

$$w = U(x,z), \quad z = ax+by$$

leads to an equation of the form 7.2.2.7:

$$\left(\frac{\partial^2 U}{\partial x \partial z}\right)^2 = \frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial z^2} + b^{-2}x^k f(z).$$

Here, x and z play the role of y and x in 7.2.2.7, respectively.

14. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = x^{2k+2} f(ax+by) + x^k g(ax+by).$

The transformation

$$w = U(x,z), \quad z = ax+by$$

leads to an equation of the form 7.2.2.8:

$$\left(\frac{\partial^2 U}{\partial x \partial z}\right)^2 = \frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial z^2} + b^{-2}x^{k+2} f(z) + b^{-2}x^k g(z).$$

Here, x and z play the role of y and x in 7.2.2.8, respectively.

15. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = e^{\lambda x} f(ax+by).$

The transformation

$$w = U(x,z), \quad z = ax+by$$

leads to an equation of the form 7.2.2.9:

$$\left(\frac{\partial^2 U}{\partial x \partial z}\right)^2 = \frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial z^2} + b^{-2}e^{\lambda x} f(z).$$

Here, x and z play the role of y and x in 7.2.2.9, respectively.

16. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = e^{2\lambda x} f(ax+by) + e^{\lambda x} g(ax+by).$

The transformation

$$w = U(x,z), \quad z = ax+by$$

leads to an equation of the form 7.2.2.10:

$$\left(\frac{\partial^2 U}{\partial x \partial z}\right)^2 = \frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial z^2} + b^{-2}e^{2\lambda x} f(z) + b^{-2}e^{\lambda x} g(z).$$

Here, x and z play the role of y and x in 7.2.2.10, respectively.

17. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = \dfrac{1}{x^4} f\left(\dfrac{y}{x}\right).$

This is a special case of equation 7.2.2.18 with $\alpha = -2$ and $\beta = -1$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1 = \pm w(C_1 x,\ C_1 y) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Integral:

$$w - x\frac{\partial w}{\partial x} - y\frac{\partial w}{\partial y} \pm \int \sqrt{f(z)}\,dz = C, \qquad z = \frac{y}{x},$$

where C is an arbitrary constant.

⊙ *References*: M. N. Martin (1953), B. L. Rozhdestvenskii and N. N. Yanenko (1983).

3°. Solutions:

$$w = x\varphi\left(\frac{y}{x}\right) \pm \int \sqrt{f(z)}\,dz + C, \qquad z = \frac{y}{x},$$

where $\varphi(z)$ is an arbitrary function.

4°. Conservation law:

$$D_x\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\right) + D_y\left[-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y} + x^{-3}F\left(\frac{y}{x}\right)\right] = 0,$$

where $D_x = \dfrac{\partial}{\partial x}$, $D_y = \dfrac{\partial}{\partial y}$, and $F(z) = \displaystyle\int f(z)\,dz$.

⊙ *Reference*: S. V. Khabirov (1990).

18. $\displaystyle\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = x^{2\alpha} f(x^\beta y).$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{\beta-\alpha-1} w(C_1 x,\ C_1^{-\beta} y) + C_2 x + C_3 y + C_4,$$

where $C_1, \dots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Self-similar solution:

$$w(x,y) = x^{\alpha-\beta+1} u(z) \quad z = x^\beta y,$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$[\beta(\beta+1)zu'_z + (\alpha-\beta)(\beta-\alpha-1)u]u''_{zz} + (\alpha+1)^2(u'_z)^2 - f(z) = 0.$$

⊙ *Reference*: S. V. Khabirov (1990).

19. $\displaystyle\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = f(ax - by^2).$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = w(x + 2bC_1 y + abC_1, y + aC_1) + C_2 x + C_3 y + C_4,$$

where $C_1, \dots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,y) = \pm \int \sqrt{F(z) + C_1}\,dz + C_2 x + C_3 y + C_4, \qquad F(z) = \frac{1}{a^2 b}\int f(z)\,dz, \quad z = ax - by^2,$$

where $C_1, \dots, C_4$ are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

20. $\displaystyle\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = f(ax^2 + bxy + cy^2).$

Solution for $b^2 \neq 4ac$:

$$w(x,y) = u(z) \quad z = ax^2 + bxy + cy^2,$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$2(4ac - b^2)zu'_z u''_{zz} + (4ac - b^2)(u'_z)^2 + f(z) = 0.$$

Integrating yields

$$u(z) = \pm \int \sqrt{\frac{F(z)}{z}}\, dz + C_1, \quad F(z) = \frac{1}{b^2 - 4ac}\int f(z)\, dz + C_2,$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

21. $\displaystyle\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = f(ax^2 + bxy + cy^2 + kx + sy).$

Solution:

$$w(x,y) = u(z), \quad z = ax^2 + bxy + cy^2 + kx + sy,$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$2\big[(4ac - b^2)z + as^2 + ck^2 - bks\big]u'_z u''_{zz} + (4ac - b^2)(u'_z)^2 + f(z) = 0.$$

The substitution $V(z) = (u'_z)^2$ leads to a first-order linear equation.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

22. $\displaystyle\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = e^{\alpha x} f(e^{\beta x} y).$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{\alpha - 2\beta} w(x - 2\ln C_1,\, C_1^{2\beta} y) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Generalized self-similar solution:

$$w(x,y) = e^{\mu x} U(z), \quad z = e^{\beta x} y, \quad \mu = \tfrac{1}{2}\alpha - \beta,$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$\beta^2 z U'_z U''_{zz} - \mu^2 U U''_{zz} + (\beta + \mu)^2 (U'_z)^2 - f(z) = 0.$$

23. $\displaystyle\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = e^{ky/x} f(x).$

Solution:

$$w(x,y) = \exp\left(\frac{ky}{2x}\right)\varphi(x),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$x^2 \varphi \varphi''_{xx} - x^2 (\varphi'_x)^2 + 2x\varphi\varphi'_x - \varphi^2 + 4k^{-2} x^4 f(x) = 0.$$

24. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = x^{2\alpha} f(x^\beta e^{y/x}).$

Solution:

$$w(x,y) = x^{\alpha+2}u(z), \quad z = x^\beta e^{y/x},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$z^2\left[\beta z u'_z + (\alpha+2)(\alpha+1)u\right]u''_{zz} + z\left\{\left[\beta - (\alpha+1)^2\right]zu'_z + (\alpha+2)(\alpha+1)u\right\}u'_z + f(z) = 0.$$

⊙ *Reference*: S. V. Khabirov (1990).

25. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = y^{-4}\exp(2\alpha y^{-1}) f(xy^{-1} + \beta y^{-2}).$

Solution:

$$w = y\exp(\alpha y^{-1})\varphi(z) + C_1 y + C_2 x + C_3, \quad z = xy^{-1} + \beta y^{-2},$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $\varphi = \varphi(z)$ is determined by the ordinary differential equation

$$(2\beta\varphi'_z + \alpha^2\varphi)\varphi''_{zz} - \alpha^2{\varphi'_z}^2 + f(z) = 0.$$

⊙ *Reference*: S. V. Khabirov (1990).

► *For exact solutions of the nonhomogeneous Monge–Ampère equation for some specific $F = F(x,y)$ (without functional arbitrariness), see Khabirov (1990) and Ibragimov (1994). The Cauchy problem for the Monge–Ampère equation is discussed in Courant and Hilbert (1989).*

7.2.3. Equations of the Form $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = F\left(x, y, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}\right)$

1. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)w + g(x).$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = w(x, y + C_1 x + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution quadratic in y:

$$w = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where $\varphi(x)$, $\psi(x)$, and $\chi(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
&2\varphi\varphi''_{xx} + f(x)\varphi - 4(\varphi'_x)^2 = 0,\\
&2\varphi\psi''_{xx} + f(x)\psi - 4\varphi'_x\psi'_x = 0,\\
&2\varphi\chi''_{xx} + f(x)\chi + g(x) - (\psi'_x)^2 = 0.
\end{aligned}$$

Note that the second equation is linear in ψ and has a particular solution $\psi = \varphi$ (hence, its general solution can be expressed via the particular solution of the first equation).

2. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 - \dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} = f(x)w^2.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, y + C_2 x + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,y) = e^{\lambda y} u(x),$$

where λ is an arbitrary constant and the function $u = u(x)$ is determined by the ordinary differential equation

$$u u''_{xx} - (u'_x)^2 + \lambda^{-2} f(x) u^2 = 0.$$

3. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = f(x) y^n w^k.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C^{n+2} w(x, C^{k-2} y),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Multiplicative separable solution with $n \neq -2$ and $k \neq 2$:

$$w(x,y) = y^{\frac{n+2}{2-k}} U(x),$$

where the function $U(x)$ is determined by the ordinary differential equation

$$(n+2)(n+k) U U''_{xx} - (n+2)^2 (U'_x)^2 + (k-2)^2 f(x) U^k = 0.$$

4. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = f(x) e^{\lambda y} w^k.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C w\Bigl(x, y + \frac{k-2}{\lambda} \ln C\Bigr),$$

where C is an arbitrary constant, is also a solution of the equation.

2°. Multiplicative separable solution with $k \neq 2$ and $\lambda \neq 0$:

$$w(x,y) = \exp\left(\frac{\lambda y}{2-k}\right) U(x),$$

where the function $U(x)$ is determined by the ordinary differential equation

$$U U''_{xx} - (U'_x)^2 + (k-2)^2 \lambda^{-2} f(x) U^k = 0.$$

5. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = f(w).$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = w(A_1 x + B_1 y + C_1, A_2 x + B_2 y + C_2), \qquad |A_2 B_1 - A_1 B_2| = 1,$$

where C_1, C_2, and any three of the four constants A_1, A_2, B_1, and B_2 are arbitrary, is also a solution of the equation.

2°. Functional separable solution:

$$w(x,y) = u(z), \quad z = ax^2 + bxy + cy^2 + kx + sy,$$

where a, b, c, k, and s are arbitrary constants and the function $u = u(z)$ is determined by the ordinary differential equation

$$2\bigl[(4ac - b^2) z + as^2 + ck^2 - bks\bigr] u'_z u''_{zz} + (4ac - b^2)(u'_z)^2 + f(u) = 0.$$

6. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + f(x) \exp\left(\frac{ay}{x}\right) w^k.$

Solution:

$$w(x, y) = \exp\left(\frac{\lambda y}{x}\right) u(x), \qquad \lambda = \frac{a}{2-k},$$

where the function $u = u(x)$ is determined by the ordinary differential equation

$$x^2 u u''_{xx} - (x u'_x - u)^2 + \lambda^{-2} x^4 f(x) u^k = 0.$$

7. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + a\frac{\partial w}{\partial y}.$

This equation is used in meteorology for describing wind fields in near-equatorial regions.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-2} C_2^{-1} w(C_1 x + C_3, C_2 y + C_4 x + C_5) + C_6 x + C_7,$$

where $C_1, \ldots, C_7$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w = \varphi(x),$$
$$w = \tfrac{1}{4}(\sqrt{a}\, x + C)^2 y + \varphi(x),$$

where $\varphi(x)$ is an arbitrary function and C is an arbitrary constant.

3°. Solutions:

$$w = C_1 e^{\lambda y} - \tfrac{1}{2} a \lambda^{-1} x^2 + C_2 x + C_3,$$
$$w = \tfrac{1}{4} a (x + C_1)^2 (y + C_2),$$
$$w = \tfrac{1}{4} a C_2^{-1} (x + C_1)^2 \tanh(C_2 y + C_3),$$
$$w = \tfrac{1}{4} a C_2^{-1} (x + C_1)^2 \coth(C_2 y + C_3),$$
$$w = \tfrac{1}{4} a C_2^{-1} (x + C_1)^2 \tan(C_2 y + C_3),$$

where $\varphi(x)$ is an arbitrary function and C_1, C_2, C_3, and λ are arbitrary constants. The first solution is a solution in additive separable form and the other four are multiplicative separable solutions.

⊙ *Reference*: E. R. Rozendorn (1984).

4°. Generalized separable solution quadratic in y:

$$w = F(x) y^2 + G(x) y + H(x),$$

where

$$F(x) = \frac{1}{C_1 x + C_2}, \quad G(x) = -\frac{a}{6C_1^2}(C_1 x + C_2)^2 + \frac{C_3}{C_1 x + C_2} + C_4,$$
$$H(x) = \frac{1}{2}\int_0^x (x - t) \frac{[G'_t(t)]^2 - aG(t)}{F(t)}\, dt + C_5 x + C_6,$$

and $C_1, \ldots, C_6$ are arbitrary constants.

5°. Generalized separable solution:

$$w = C_1 \exp(C_2 x + C_3 y) - \frac{a}{2C_3} x^2 + C_4 x + C_5.$$

6°. There are exact solutions of the following forms:

$$w(x, y) = |x|^{k+2} U(z), \quad z = y|x|^{-k};$$
$$w(x, y) = e^{kx} V(\xi), \qquad \xi = y e^{-kx};$$
$$w(x, y) = x^2 W(\eta), \qquad \eta = y + k \ln|x|;$$

where k is an arbitrary constant.

8. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + f(x)\frac{\partial w}{\partial y}.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(x, C_1 y + C_2 x + C_3) + C_4 x + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$\begin{aligned} w &= \varphi(x), \\ w &= \frac{1}{4}\left[\int f(x)\,dx + C\right]^2 y + \varphi(x), \\ w &= C_1 \exp(C_2 x + C_3 y) - \frac{1}{C_3}\int_0^x (x-t) f(t)\,dt + C_4 x + C_5, \end{aligned}$$

where $\varphi(x)$ is an arbitrary function and C is an arbitrary constant. For $C_2 = 0$, the last solution is an additive separable solution.

3°. Generalized separable solution quadratic in y:

$$w = \varphi(x) y^2 + \psi(x) y + \chi(x),$$

where

$$\begin{aligned} \varphi(x) &= \frac{1}{C_1 x + C_2}, \quad \psi(x) = -\int\left[\varphi^2(x)\int \frac{f(x)}{\varphi^2(x)}\,dx\right] dx + \frac{C_3}{C_1 x + C_2} + C_4, \\ \chi(x) &= \frac{1}{2}\int_0^x (x-t)\frac{[\psi_t'(t)]^2 - f(t)\psi(t)}{\varphi(t)}\,dt + C_5 x + C_6, \end{aligned}$$

and $C_1, \ldots, C_6$ are arbitrary constants.

9. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + f\left(\frac{\partial w}{\partial x}\right).$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-1} w(C_1 x + C_2 y + C_3, \pm y + C_4) + C_5 y + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution linear in x:

$$w(x, y) = \varphi(y) x + \psi(y),$$

where $\psi(y)$ is an arbitrary function and the function $\varphi(y)$ is defined implicitly by

$$\int \frac{d\varphi}{\sqrt{f(\varphi)}} = \pm y + C,$$

where C is an arbitrary constant.

3°. Additive separable solution:

$$w(x, y) = C_1 y^2 + C_2 y + C_3 + z(x),$$

where the function $z(x)$ is determined by the autonomous ordinary differential equation $2C_1 z''_{xx} + f(z'_x) = 0$. Its general solution can be written out in parametric form as

$$x = -2C_1\int \frac{dt}{f(t)} + C_3, \qquad z = -2C_1\int \frac{t\,dt}{f(t)} + C_4.$$

4°. The Legendre transformation

$$u = x\xi + y\eta - w(x,y), \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y},$$

where $u = u(\xi,\eta)$ is the new dependent variable and ξ and η are the new independent variables, leads to an equation of the form 7.2.2.3:

$$\left(\frac{\partial^2 u}{\partial \xi \partial \eta}\right)^2 - \frac{\partial^2 u}{\partial \xi^2}\frac{\partial^2 u}{\partial \eta^2} = \frac{1}{f(\xi)}.$$

10. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + F\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right).$

The Legendre transformation

$$u = x\xi + y\eta - w(x,y), \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y},$$

where $u = u(\xi,\eta)$ is the new dependent variable and ξ and η are the new independent variables, leads to the simpler equation

$$\left(\frac{\partial^2 u}{\partial \xi \partial \eta}\right)^2 - \frac{\partial^2 u}{\partial \xi^2}\frac{\partial^2 u}{\partial \eta^2} = \frac{1}{F(\xi,\eta)}.$$

For exact solutions of this equation, see Subsection 7.2.2.

7.2.4. Equations of the Form $\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x,y)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + g(x,y)$

1. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x, C_2 y + C_3) + C_4 x + C_5 y + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, is also a solution of the equation.

2°. Degenerate solutions involving arbitrary functions:

$$w(x,y) = \varphi(x) + C_1 y + C_2,$$
$$w(x,y) = \varphi(y) + C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi = \varphi(z)$ is an arbitrary function.

3°. Generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x)y^2 + [C_1\varphi(x) + C_2]y + \frac{C_1^2}{2}\int_0^x (x-t)\frac{[\varphi_t'(t)]^2}{f(t)\varphi(t)}\,dt + C_3 x + C_4,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi\varphi_{xx}'' - 2(\varphi_x')^2 = 0.$$

4°. Generalized separable solution involving an arbitrary power of y:

$$w(x,y) = \varphi(x)y^k + C_1 x + C_2 y + C_3$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$(k-1)f(x)\varphi\varphi_{xx}'' - k(\varphi_x')^2 = 0.$$

5°. Generalized separable solution involving an exponential of y:

$$w(x,y) = \varphi(x)e^{\lambda y} + C_1 x + C_2 y + C_3,$$

where C_1, C_2, C_3, and λ are arbitrary constants and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi\varphi_{xx}'' - (\varphi_x')^2 = 0.$$

2. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x)\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + g(x).$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{-1} w(x, C_1 y + C_2) + C_3 x + C_4 y + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution linear in y:

$$w(x,y) = \pm y \int \sqrt{g(x)}\,dx + \varphi(x) + C_1 y,$$

where $\varphi(x)$ is an arbitrary function.

3°. Generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x) y^2 + [C_1\varphi(x) + C_2] y + \frac{1}{2}\int_0^x (x-t)\frac{C_1^2[\varphi_t'(t)]^2 - g(t)}{f(t)\varphi(t)}\,dt + C_3 x + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi\varphi_{xx}'' - 2(\varphi_x')^2 = 0,$$

which has a particular solution $\varphi = C_6$.

3. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x)\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + g(x)y.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-3} w(x, C_1^2 y) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution cubic in y:

$$w(x,y) = C_1 y^3 + C_2 y - \frac{1}{6C_1}\int_a^x (x-t)\frac{g(t)}{f(t)}\,dt + C_3 x + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

A more general solution is given by

$$w(x,y) = \varphi(x) y^3 + C_1 y - \frac{1}{6}\int_a^x (x-t)\frac{g(t)\,dt}{f(t)\varphi(t)} + C_2 x + C_3,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$2f(x)\varphi\varphi_{xx}'' - 3(\varphi_x')^2 = 0.$$

3°. For an exact solution quadratic in y, see equation 7.2.4.5 with $g_2 = g_0 = 0$.

4°. See the solution of equation 7.2.4.6 in Item 3° with $k = 1$.

4. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x)\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + g(x)y^2.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{-2} w(x, C_1 y) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution involving y to the power of four:

$$w(x,y) = C_1 y^4 + C_2 y - \frac{1}{12C_1}\int_a^x (x-t)\frac{g(t)}{f(t)}\,dt + C_3 x + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

A more general solution is given by

$$w(x,y) = \varphi(x)y^4 + C_1 y - \frac{1}{12}\int_a^x (x-t)\frac{g(t)\,dt}{f(t)\varphi(t)} + C_2 x + C_3,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$3f(x)\varphi\varphi''_{xx} - 4(\varphi'_x)^2 = 0.$$

3°. For an exact solution quadratic in y, see equation 7.2.4.5 with $g_1 = g_0 = 0$.

4°. See the solution of equation 7.2.4.6 in Item 3° with $k = 2$.

5. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + g_2(x)y^2 + g_1(x)y + g_0(x).$

Generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where the functions $\varphi = \varphi(x)$, $\psi = \psi(x)$, and $\chi = \chi(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned} f(x)\varphi\varphi''_{xx} &= 2(\varphi'_x)^2 - \tfrac{1}{2}g_2(x), \\ f(x)\varphi\psi''_{xx} &= 2\varphi'_x\psi'_x - \tfrac{1}{2}g_1(x), \\ f(x)\varphi\chi''_{xx} &= \tfrac{1}{2}(\psi'_x)^2 - \tfrac{1}{2}g_0(x). \end{aligned}$$

6. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + g(x)y^k.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = \pm C_1^{-k-2} w(x, C_1^2 y) + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Additive separable solution:

$$w(x,y) = \frac{C_1 y^{k+2}}{(k+1)(k+2)} + C_2 y - \frac{1}{C_1}\int_a^x (x-t)\frac{g(t)}{f(t)}\,dt + C_3 x + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

3°. Multiplicative separable solution:

$$w(x,y) = \varphi(x)y^{\frac{k+2}{2}},$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$k(k+2)f(x)\varphi\varphi''_{xx} - (k+2)^2(\varphi'_x)^2 + 4g(x) = 0.$$

4°. Generalized separable solution:

$$w(x,y) = \psi(x)y^{k+2} - \frac{1}{(k+1)(k+2)}\int_a^x (x-t)\frac{g(t)}{f(t)\psi(t)}\,dt + C_1 x + C_2 y + C_3,$$

where the function $\psi = \psi(x)$ is determined by the ordinary differential equation

$$(k+1)f(x)\psi\psi''_{xx} - (k+2)(\psi'_x)^2 = 0.$$

7. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x)\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + g(x)y^{2k+2} + h(x)y^k.$

Generalized separable solution:

$$w(x,y) = \varphi(x)y^{k+2} - \frac{1}{(k+1)(k+2)}\int_a^x (x-t)\frac{h(t)}{f(t)\varphi(t)}\,dt + C_1x + C_2y + C_3,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$(k+1)(k+2)f(x)\varphi\varphi''_{xx} - (k+2)^2(\varphi'_x)^2 + g(x) = 0.$$

8. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x)\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + g(x)e^{\lambda y}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = C_1 w\Big(x, y - \frac{2}{\lambda}\ln|C_1|\Big) + C_2x + C_3y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Additive separable solution:

$$w(x,y) = C_1e^{\lambda y} + C_2y - \frac{1}{C_1\lambda^2}\int_a^x (x-t)\frac{g(t)}{f(t)}\,dt + C_3x + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

3°. Multiplicative separable solution:

$$w(x,y) = \varphi(x)\exp\big(\tfrac{1}{2}\lambda y\big),$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi\varphi''_{xx} - (\varphi'_x)^2 + 4\lambda^{-2}g(x) = 0.$$

4°. Generalized separable solution:

$$w(x,y) = \psi(x)e^{\lambda y} - \frac{1}{\lambda^2}\int_a^x (x-t)\frac{g(t)}{f(t)\psi(t)}\,dt + C_1x + C_2y + C_3,$$

where the function $\psi = \psi(x)$ is determined by the ordinary differential equation

$$f(x)\psi\psi''_{xx} - (\psi'_x)^2 = 0.$$

9. $\left(\dfrac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x)\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2} + g(x)e^{2\lambda y} + h(x)e^{\lambda y}.$

Generalized separable solution:

$$w(x,y) = \varphi(x)e^{\lambda y} - \frac{1}{\lambda^2}\int_a^x (x-t)\frac{h(t)}{f(t)\varphi(t)}\,dt + C_1x + C_2y + C_3,$$

where the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$f(x)\varphi\varphi''_{xx} - (\varphi'_x)^2 + \lambda^{-2}g(x) = 0.$$

10. $\displaystyle \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 = f_1(x)g_1(y)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + f_2(x)g_2(y).$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = w(x,y) + C_1 x + C_2 y + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution for $f_1 g_1 \neq 0$:

$$w(x,y) = C_1 \int_a^x (x-t)\frac{f_2(t)}{f_1(t)}\,dt - \frac{1}{C_1}\int_b^y (y-\xi)\frac{g_2(\xi)}{g_1(\xi)}\,d\xi + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

3°. Degenerate solutions for $f_2 g_2 = 0$:

$$w(x,y) = \varphi(x) + C_1 y + C_2,$$
$$w(x,y) = \varphi(y) + C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi = \varphi(z)$ is an arbitrary function.

4°. Generalized separable solution for $f_2 g_2 = 0$:

$$w(x,y) = \varphi(x)\psi(y) + C_1 x + C_2 y + C_3,$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined by the ordinary differential equations

$$f_1(x)\varphi\varphi''_{xx} - C_4(\varphi'_x)^2 = 0,$$
$$C_4 g_1(y)\psi\psi''_{yy} - (\psi'_y)^2 = 0.$$

11. $\displaystyle \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 = f(ax+by)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + g(ax+by).$

Solution:

$$w(x,y) = \varphi(z) + C_1 x^2 + C_2 xy + C_3 y^2 + C_4 x + C_5 y, \qquad z = ax + by,$$

where $C_1, \ldots, C_5$ are arbitrary constants and the function $\varphi(z)$ is determined by the ordinary differential equation

$$(ab\varphi''_{zz} + C_2)^2 = f(z)(a^2\varphi''_{zz} + 2C_1)(b^2\varphi''_{zz} + 2C_3) + g(z),$$

which is easy to integrate; to this end, the equation should first be solved for φ''_{zz}.

7.2.5. Other Equations

1. $\displaystyle \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + f(x)\frac{\partial^2 w}{\partial y^2}.$

The substitution

$$w = U(x,y) - \int_a^x (x-t)f(t)\,dt$$

leads to an equation of the form 7.2.2.1:

$$\left(\frac{\partial^2 U}{\partial x\partial y}\right)^2 = \frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial y^2}.$$

2. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + f(x)\frac{\partial^2 w}{\partial x \partial y}.$

First integral:

$$\frac{\partial w}{\partial y} = \Phi\left(\frac{\partial w}{\partial x}\right) + \int f(x)\,dx,$$

where $\Phi(u)$ is an arbitrary function.

3. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + a_1\frac{\partial^2 w}{\partial x^2} + a_2\frac{\partial^2 w}{\partial y^2} + b.$

The substitution

$$w = U(x,y) - \tfrac{1}{2}a_2 x^2 - \tfrac{1}{2}a_1 y^2$$

leads to an equation of the form 7.2.2.2:

$$\left(\frac{\partial^2 U}{\partial x \partial y}\right)^2 = \frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial y^2} + b - a_1 a_2.$$

4. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + a_1\frac{\partial^2 w}{\partial x^2} + a_2\frac{\partial^2 w}{\partial y^2} + b_1\frac{\partial w}{\partial x} + b_2\frac{\partial w}{\partial y}.$

This equation is used in meteorology for describing horizontal air flows; w is the stream function for the wind velocity, and x and y are coordinates on the earth surface.

⊙ *Reference*: E. R. Rozendorn (1984).

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1, y + C_2) + C_3(b_2 x - b_1 y) + C_4,$$

where $C_1, \dots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w = C_3 \exp\left[-\frac{b_1 C_1 + b_2 C_2}{a_1 C_1^2 + a_2 C_2^2}(C_1 x + C_2 y)\right] + C_4.$$

3°. Generalized separable solution linear in y:

$$w = y(C_1 e^{-\lambda x} + C_2) + \frac{C_1^2}{2a_1}e^{-2\lambda x} + \left(\frac{b_2 C_1}{b_1}x + C_3\right)e^{-\lambda x} - \frac{b_2 C_2}{b_1}x + C_4, \qquad \lambda = \frac{b_1}{a_1}$$

where $C_1, \dots, C_4$ are arbitrary constants.

4°. Generalized separable solution quadratic in y:

$$w = f(x)y^2 + g(x)y + h(x),$$

where the functions $f(x)$, $g(x)$, and $h(x)$ are determined by the system of ordinary differential equations

$$2ff''_{xx} + a_1 f''_{xx} + b_1 f'_x - 4(f'_x)^2 = 0, \qquad (1)$$

$$2fg''_{xx} + a_1 g''_{xx} + b_1 g'_x - 4f'_x g'_x + 2b_2 f = 0, \qquad (2)$$

$$2fh''_{xx} + a_1 h''_{xx} + b_1 h'_x + 2a_2 f + b_2 g - (g'_x)^2 = 0. \qquad (3)$$

This system can be fully integrated. To this end, equation (1) is first reduced, with the change of variable $U(f) = f'_x$, to a first-order linear equation. Equation (2) is linear in g and a fundamental system of solutions of the corresponding homogeneous equation has the form $g_1 = 1$, $g_2 = f(x)$. Equation (3) is finally reduced, with the substitution $V(x) = h'_x$, to first-order linear equation.

Remark. The solutions of Items 3° and 4° can be used to obtain two other solutions by means of the following renaming: $(x, a_1, b_1) \rightleftarrows (y, a_2, b_2)$.

5. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + f_1(x)\frac{\partial^2 w}{\partial x^2} + f_2(x)\frac{\partial^2 w}{\partial y^2} + g_1(x)\frac{\partial w}{\partial x} + g_2(x)\frac{\partial w}{\partial y}.$

There are generalized separable solutions linear and quadratic in y:

$$w(x,y) = \varphi_1(x)y + \varphi_0(x),$$
$$w(x,y) = \psi_2(x)y^2 + \psi_1(x)y + \psi_0(x).$$

6. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = f(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + g(x)w + h_2(x)y^2 + h_1(x)y + h_0(x).$

Generalized separable solution quadratic in y:

$$w(x,y) = \varphi(x)y^2 + \psi(x)y + \chi(x),$$

where the functions $\varphi = \varphi(x)$, $\psi = \psi(x)$, and $\chi = \chi(x)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
&2f(x)\varphi\varphi''_{xx} - 4(\varphi'_x)^2 + g(x)\varphi + h_2(x) = 0,\\
&2f(x)\varphi\psi''_{xx} - 4\varphi'_x\psi'_x + g(x)\psi + h_1(x) = 0,\\
&2f(x)\varphi\chi''_{xx} - (\psi'_x)^2 + g(x)\chi + h_0(x) = 0.
\end{aligned}$$

7. $\displaystyle \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = f_1(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \big[f_2(x)w + f_3(x)y^2 + f_4(x)y + f_5(x)\big]\left(\frac{\partial^2 w}{\partial y^2}\right)^2$
$\displaystyle + g_1(x)\frac{\partial^2 w}{\partial x^2} + \big[g_2(x)y + g_3(x)\big]\frac{\partial^2 w}{\partial x \partial y} + \big[g_4(x)w + g_5(x)y^2 + g_6(x)y + g_7(x)\big]\frac{\partial^2 w}{\partial y^2}$
$\displaystyle + h_1(x)\left(\frac{\partial w}{\partial y}\right)^2 + h_2(x)\frac{\partial w}{\partial x} + \big[h_3(x)y + h_4(x)\big]\frac{\partial w}{\partial y} + s_1(x)w + s_2(x)y^2 + s_3(x)y + s_4(x).$

There is a generalized separable solution of the form

$$w(x,y) = \varphi(x)y^2 + \psi(x)y + \chi(x).$$

8. $\displaystyle \frac{\partial^2 w}{\partial x^2}\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] = \frac{\partial^2 w}{\partial y^2}.$

This equation occurs in plane problems of plasticity; w is the generating function.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1 = \pm C_1^{-2} w(C_1 x + C_2, C_3 y + C_4) + C_5 x + C_6 y + C_7,$$

where $C_1, \ldots, C_7$ are arbitrary constants, are also solutions of the equation.

2°. Introduce the new variable

$$U(x,y) = \frac{\partial w}{\partial x}$$

and apply the Legendre transformation (for details, see Subsection S.2.3)

$$X = \frac{\partial U}{\partial x}, \quad Y = \frac{\partial U}{\partial y}, \quad Z = x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} - U$$

to obtain a second-order linear equation:

$$(1+X^2)^2\frac{\partial^2 Z}{\partial X^2} + 2XY(1+X^2)\frac{\partial^2 Z}{\partial X \partial Y} + Y^2(X^2-1)\frac{\partial^2 Z}{\partial Y^2} = 0. \qquad (1)$$

This equation is hyperbolic. The transformation

$$t = \arctan X, \quad \xi = \tfrac{1}{2}\ln(1+X^2) - \ln Y, \quad F = \frac{Z}{\sqrt{1+X^2}}$$

brings (1) to a constant-coefficient linear equation:

$$\frac{\partial^2 F}{\partial t^2} = \frac{\partial^2 F}{\partial \xi^2} - F. \tag{2}$$

For solutions of equation (2), see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

Remark. The original equation is invariant under the Legendre transformation

$$\overline{x} = \frac{\partial w}{\partial x}, \quad \overline{y} = \frac{\partial w}{\partial y}, \quad \overline{w} = x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} - w.$$

⊙ *References*: Yu. N. Radayev (1988), V. I. Astafiev, Yu. N. Radayev, and L. V. Stepanova (2001).

7.3. Bellman Type Equations and Related Equations

7.3.1. Equations with Quadratic Nonlinearities

1. $\displaystyle \frac{\partial w}{\partial t}\frac{\partial w}{\partial x} - f(t)\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} - g(t)\left(\frac{\partial w}{\partial y}\right)^2 = 0.$

This equation occurs in problems of optimal correction of random disturbances and is a consequence of the Bellman equation; see Chernousko (1971) and Chernousko and Kolmanovskii (1978). The variable $t = T - \tau$ plays the role of "backward" time.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, y + C_3, t) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. "Two-dimensional" solutions:

$$w = U(z, \tau), \quad z = y \pm 2\left[x\int g(t)\,dt + C_1 x\right]^{1/2} + C_2, \quad \tau = \int f(t)\,dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $U = U(z, \tau)$ is determined by the linear heat equation

$$\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial z^2} = 0.$$

⊙ *Reference*: A. S. Bratus' and K. A. Volosov (2002).

3°. "Two-dimensional" solution:

$$w = u(\xi, \tau), \quad \xi = y + C_1 x + \frac{1}{C_1}\int g(t)\,dt + C_2, \quad \tau = \int f(t)\,dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, and the function $u = u(\xi, \eta)$ is determined by the linear heat equation

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial \xi^2} = 0.$$

4°. The solutions of Items 2° and 3° are special cases of a more general solution with the form

$$w = U(z, \tau), \quad z = y + \varphi(x, t), \quad \tau = \int f(t)\,dt,$$

where the function $\varphi = \varphi(x, t)$ satisfies the first-order nonlinear partial differential equation

$$\frac{\partial \varphi}{\partial t}\frac{\partial \varphi}{\partial x} = g(t), \tag{1}$$

and the function $U = U(z, \tau)$ is determined by the linear heat equation

$$\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial z^2} = 0.$$

A complete integral of equation (1) is given by

$$\varphi = C_1 x + \frac{1}{C_1} \int g(t)\, dt + C_2, \tag{2}$$

where C_1 and C_2 are arbitrary constants. The general integral of equation (1) can be represented in parametric form with the complete integral (2) and the two relations (see Kamke, 1965, and Polyanin, Zaitsev, and Moussiaux, 2002)

$$C_2 = \psi(C_1),$$
$$x - \frac{1}{C_1^2} \int g(t)\, dt + \psi'(C_1) = 0,$$

where $\psi = \psi(C_1)$ is an arbitrary function and the prime denotes a derivative with respect to the argument; C_1 and C_2 play the role of parameters.

Remark. To the solution of Item 2° there corresponds $\psi(C_1) = \text{const}$.

⦿ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

5°. "Two-dimensional" solutions:

$$w = \pm \exp\bigl[\lambda y + \zeta(x, t)\bigr],$$

where λ is an arbitrary constant, and the function $\zeta = \zeta(x, t)$ is determined by the first-order partial differential equation

$$\frac{\partial \zeta}{\partial t}\frac{\partial \zeta}{\partial x} - \lambda^2 f(t) \frac{\partial \zeta}{\partial x} - \lambda^2 g(t) = 0. \tag{3}$$

A complete integral of equation (3) is given by

$$\zeta = C_1 x + \lambda^2 \int \left[f(t) + \frac{1}{C_1} g(t) \right] dt + C_2, \tag{4}$$

where C_1 and C_2 are arbitrary constants. The general integral of equation (3) can be represented in parametric form with the complete integral (4) and the two relations

$$C_2 = \varphi(C_1),$$
$$x - \frac{\lambda^2}{C_1^2} \int g(t)\, dt + \varphi'(C_1) = 0,$$

where $\varphi = \varphi(C_1)$ is an arbitrary function; C_1 and C_2 play the role of parameters.

⦿ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

6°. "Two-dimensional" solution:

$$w = e^{\lambda x} \theta(y, t),$$

where λ is an arbitrary constant, and the function $\theta = \theta(y, t)$ is determined by the "two-dimensional" equation

$$\lambda \theta \frac{\partial \theta}{\partial t} - \lambda f(t) \theta \frac{\partial^2 \theta}{\partial y^2} - g(t) \left(\frac{\partial \theta}{\partial y} \right)^2 = 0.$$

7°. Cauchy problems and self-similar solutions of the equation for power-law $f(t)$ and $g(t)$ are discussed in Chernousko (1971) and Chernousko and Kolmanovskii (1978).

2. $\displaystyle \frac{\partial w}{\partial t}\frac{\partial w}{\partial x} - f(t)\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} - g(t)h(x)\left(\frac{\partial w}{\partial y}\right)^2 = 0.$

The substitution $z = \int h(x)\,dx$ leads to an equation of the form 7.3.1.1 for $w = w(z, y, t)$.

3. $\displaystyle \frac{\partial w}{\partial t}\frac{\partial w}{\partial x} - f(t)\frac{\partial w}{\partial x}\left(\frac{\partial^2 w}{\partial y^2} + \frac{n}{y}\frac{\partial w}{\partial y}\right) - g(t)h(x)\left(\frac{\partial w}{\partial y}\right)^2 = 0.$

This equation occurs in problems of optimal correction of random disturbances and is a consequence of the Bellman equation; see Chernousko and Kolmanovskii (1978). The variable $t = T - \tau$ plays the role of "backward" time; $n + 1$ is the dimensionality of the equations of motion of the controllable system (n is a nonnegative integer).

"Two-dimensional" solution:

$$w(x, y, t) = \exp\left[\lambda \int h(x)\,dx\right] U(y, t),$$

where the function $U(y, t)$ is determined by the differential equation (λ is an arbitrary constant)

$$\lambda U \frac{\partial U}{\partial t} - \lambda f(t) U \left(\frac{\partial^2 U}{\partial y^2} + \frac{n}{y}\frac{\partial U}{\partial y}\right) - g(t)\left(\frac{\partial U}{\partial y}\right)^2 = 0.$$

4. $\displaystyle \frac{\partial w}{\partial t}\frac{\partial w}{\partial x} - f(t)\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} - g(x,t)\left(\frac{\partial w}{\partial y}\right)^2 = 0.$

1°. "Two-dimensional" solution:

$$w = U(z, \tau), \quad z = y + \varphi(x, t), \quad \tau = \int f(t)\,dt.$$

Here, the function $\varphi = \varphi(x, t)$ is determined by the first-order nonlinear partial differential equation

$$\frac{\partial \varphi}{\partial t}\frac{\partial \varphi}{\partial x} = g(x, t), \qquad (1)$$

and the function $U = U(z, \tau)$ is determined by the linear heat equation

$$\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial z^2} = 0. \qquad (2)$$

Complete integrals and the general solutions (integrals) of equation (1) for various $g(x, t)$ can be found in Polyanin, Zaitsev, and Moussiaux (2002). For solutions of equation (2), see Tikhonov and Samarskii (1990) and Polyanin (2002).

2°. "Two-dimensional" solutions:

$$w = \pm\exp\bigl[\lambda y + \zeta(x, t)\bigr],$$

where λ is an arbitrary constant and the function $\zeta = \zeta(x, t)$ is determined by the first-order nonlinear partial differential equation

$$\frac{\partial \zeta}{\partial t}\frac{\partial \zeta}{\partial x} - \lambda^2 f(t)\frac{\partial \zeta}{\partial x} - \lambda^2 g(x, t) = 0.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

5. $\displaystyle \frac{\partial w}{\partial t}\frac{\partial w}{\partial x} - f(x,t)\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} - g(x,t)\left(\frac{\partial w}{\partial y}\right)^2 = 0.$

"Two-dimensional" solutions:

$$w = \pm\exp\bigl[\lambda y + \zeta(x, t)\bigr],$$

where λ is an arbitrary constant and the function $\zeta = \zeta(x, t)$ is determined by the first-order nonlinear partial differential equation

$$\frac{\partial \zeta}{\partial t}\frac{\partial \zeta}{\partial x} - \lambda^2 f(x, t)\frac{\partial \zeta}{\partial x} - \lambda^2 g(x, t) = 0.$$

7.3.2. Equations with Power-Law Nonlinearities

1. $\dfrac{\partial w}{\partial t}\left(\dfrac{\partial w}{\partial x}\right)^{k} - f(t)\left(\dfrac{\partial w}{\partial x}\right)^{k}\dfrac{\partial^2 w}{\partial y^2} - g(t)\left(\dfrac{\partial w}{\partial y}\right)^{k+1} = 0.$

This equation occurs in problems of optimal correction of random disturbances and is a consequence of the Bellman equation; see Chernousko (1971) and Chernousko and Kolmanovskii (1978). The variable $t = T - \tau$ plays the role of "backward" time.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, y + C_3, t) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. "Two-dimensional" solution:

$$w = U(z, \tau), \quad \tau = \int f(t)\,dt + C_1,$$

$$z = y + (x + C_2)^{\frac{k}{k+1}}\left[\frac{(k+1)^{k+1}}{k^k}\int g(t)\,dt + C_3\right]^{\frac{1}{k+1}} + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants and the function $U = U(z, \tau)$ is determined by the linear heat equation

$$\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial z^2} = 0.$$

⊙ *Reference*: A. S. Bratus' and K. A. Volosov (2002).

3°. "Two-dimensional" solution:

$$w = u(\xi, \tau), \quad \xi = y + C_1 x + \frac{1}{C_1^k}\int g(t)\,dt + C_2, \quad \tau = \int f(t)\,dt + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants and the function $u = u(\xi, \eta)$ is determined by the linear heat equation

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial \xi^2} = 0.$$

4°. The solutions of Items 2° and 3° are special cases of the more general solution

$$w = U(z, \tau), \quad z = y + \varphi(x, t), \quad \tau = \int f(t)\,dt,$$

where the function $\varphi = \varphi(x, t)$ satisfies the first-order nonlinear partial differential equation

$$\frac{\partial \varphi}{\partial t}\left(\frac{\partial \varphi}{\partial x}\right)^{k} = g(t), \tag{1}$$

and the function $U = U(z, \tau)$ is determined by the linear heat equation

$$\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial z^2} = 0.$$

A complete integral of equation (1) is given by

$$\varphi = C_1 x + \frac{1}{C_1^k}\int g(t)\,dt + C_2, \tag{2}$$

where C_1 and C_2 are arbitrary constants. The general integral of equation (1) can be expressed in parametric form with the complete integral (2) and the two relations (see Kamke, 1965, and Polyanin, Zaitsev, and Moussiaux, 2002)

$$C_2 = \psi(C_1),$$
$$x - \frac{k}{C_1^{k+1}} \int g(t)\,dt + \psi'(C_1) = 0,$$

where $\psi = \psi(C_1)$ is an arbitrary function and the prime denotes a derivative; C_1 and C_2 play the role of parameters.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

5°. "Two-dimensional" solution:

$$w = \exp\bigl[\lambda y + \zeta(x,t)\bigr],$$

where λ is an arbitrary constant and the function $\zeta = \zeta(x,t)$ is determined by the first-order partial differential equation

$$\frac{\partial \zeta}{\partial t}\left(\frac{\partial \zeta}{\partial x}\right)^k - \lambda^2 f(t)\left(\frac{\partial \zeta}{\partial x}\right)^k - \lambda^{k+1} g(t) = 0. \tag{3}$$

A complete integral of this equation has the form (see Polyanin, Zaitsev, and Moussiaux, 2002)

$$\zeta = C_1 x + \int \left[\lambda^2 f(t) + \frac{\lambda^{k+1}}{C_1^k} g(t)\right] dt + C_2, \tag{4}$$

where C_1 and C_2 are arbitrary constants. The general integral of equation (3) can be expressed in parametric form with the complete integral (4) and the two relations

$$C_2 = \varphi(C_1),$$
$$x - k\frac{\lambda^{k+1}}{C_1^{k+1}} \int g(t)\,dt + \varphi'(C_1) = 0,$$

where $\varphi = \varphi(C_1)$ is an arbitrary function; C_1 and C_2 play the role of parameters.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

6°. "Two-dimensional" solution:

$$w = e^{\lambda x}\theta(y,t),$$

where λ is an arbitrary constant and the function $\theta = \theta(y,t)$ is determined by the "two-dimensional" equation

$$\frac{\partial \theta}{\partial t} - f(t)\frac{\partial^2 \theta}{\partial y^2} - \frac{g(t)}{(\lambda\theta)^k}\left(\frac{\partial \theta}{\partial y}\right)^{k+1} = 0.$$

7°. Cauchy problems and self-similar solutions of the equation for power-law $f(t)$ and $g(t)$ are discussed in Chernousko (1971) and Chernousko and Kolmanovskii (1978).

2. $\displaystyle \frac{\partial w}{\partial t}\left(\frac{\partial w}{\partial x}\right)^k - f(t)\left(\frac{\partial w}{\partial x}\right)^k \frac{\partial^2 w}{\partial y^2} - g(t)h(x)\left(\frac{\partial w}{\partial y}\right)^{k+1} = 0.$

The substitution $z = \int [h(x)]^{1/k}\,dx$ leads to an equation of the form 7.3.2.1 for $w = w(z,y,t)$.

3. $\displaystyle \frac{\partial w}{\partial t}\left(\frac{\partial w}{\partial x}\right)^k - f(t)\left(\frac{\partial w}{\partial x}\right)^k \frac{\partial^2 w}{\partial y^2} - g(x,t)\left(\frac{\partial w}{\partial y}\right)^{k+1} = 0.$

1°. "Two-dimensional" solution:

$$w = U(z,\tau), \quad z = y + \varphi(x,t), \quad \tau = \int f(t)\,dt.$$

Here, the function $\varphi = \varphi(x,t)$ is determined by the first-order nonlinear partial differential equation

$$\frac{\partial \varphi}{\partial t}\left(\frac{\partial \varphi}{\partial x}\right)^k = g(x,t), \tag{1}$$

and the function $U = U(z,\tau)$ is determined by the linear heat equation

$$\frac{\partial U}{\partial \tau} - \frac{\partial^2 U}{\partial z^2} = 0. \tag{2}$$

Complete integrals and the general solutions (integrals) of equation (1) for various $g(x,t)$ can be found in Polyanin, Zaitsev, and Moussiaux (2002). For solutions of equation (2), see Tikhonov and Samarskii (1990) and Polyanin (2002).

2°. "Two-dimensional" solution:

$$w = \exp\bigl[\lambda y + \zeta(x,t)\bigr],$$

where λ is an arbitrary constant, and the function $\zeta = \zeta(x,t)$ is determined by the first-order nonlinear partial differential equation

$$\frac{\partial \zeta}{\partial t}\left(\frac{\partial \zeta}{\partial x}\right)^k - \lambda^2 f(t)\left(\frac{\partial \zeta}{\partial x}\right)^k - \lambda^{k+1} g(x,t) = 0.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

4. $\displaystyle \frac{\partial w}{\partial t}\left(\frac{\partial w}{\partial x}\right)^k - f(x,t)\left(\frac{\partial w}{\partial x}\right)^k \frac{\partial^2 w}{\partial y^2} - g(x,t)\left(\frac{\partial w}{\partial y}\right)^{k+1} = 0.$

"Two-dimensional" solution:

$$w = \exp\bigl[\lambda y + \zeta(x,t)\bigr],$$

where λ is an arbitrary constant, and the function $\zeta = \zeta(x,t)$ is determined by the first-order nonlinear partial differential equation

$$\frac{\partial \zeta}{\partial t}\left(\frac{\partial \zeta}{\partial x}\right)^k - \lambda^2 f(x,t)\left(\frac{\partial \zeta}{\partial x}\right)^k - \lambda^{k+1} g(x,t) = 0.$$

Chapter 8

Second Order Equations of General Form

8.1. Equations Involving the First Derivative in t

8.1.1. Equations of the Form $\dfrac{\partial w}{\partial t} = F\Big(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\Big)$

Preliminary remarks. Consider the equation

$$\frac{\partial w}{\partial t} = F\left(w,\ \frac{\partial w}{\partial x},\ \frac{\partial^2 w}{\partial x^2}\right). \tag{1}$$

1°. Suppose $w(x, t)$ is a solution of equation (1). Then the function $w(x + C_1,\ t + C_2)$, where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. In the general case, equation (1) admits traveling-wave solution

$$w = w(\xi), \qquad \xi = kx + \lambda t, \tag{2}$$

where k and λ are arbitrary constants and the function $w(\xi)$ is determined by the ordinary differential equation

$$F\big(w,\ kw'_\xi,\ k^2 w''_{\xi\xi}\big) - \lambda w'_\xi = 0.$$

This subsection presents special cases where equation (1) admits exact solutions other than traveling wave (2).

1. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial^2 w}{\partial x^2}\right)$**.**

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-2} w(C_1 x + C_2, C_1^2 t + C_3) + C_4 x + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x, t) = F(A)t + \tfrac{1}{2}Ax^2 + Bx + C,$$

where A, B, and C are arbitrary constants.

3°. Generalized separable solution:

$$w(x, t) = (Ax + B)t + C + \varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\big(\varphi''_{xx}\big) = Ax + B.$$

4°. Solution:

$$w(x, t) = At + B + \psi(\xi), \qquad \xi = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants and the function $\psi(\xi)$ is determined by the autonomous ordinary differential equation

$$F\big(k^2 \psi''_{\xi\xi}\big) = \lambda \psi'_\xi + A.$$

5°. Solution:

$$w(x,t) = \tfrac{1}{2}Ax^2 + Bx + C + U(\xi), \qquad \xi = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants and the function $U(\xi)$ is determined by the autonomous ordinary differential equation

$$F\big(k^2U''_{\xi\xi} + A\big) = \lambda U'_\xi.$$

6°. Self-similar solution:

$$w(x,t) = t\,\Theta(\zeta), \qquad \zeta = \frac{x}{\sqrt{t}},$$

where the function $\Theta(\zeta)$ is determined by the ordinary differential equation

$$F\big(\Theta''_{\zeta\zeta}\big) + \tfrac{1}{2}\zeta\Theta'_\zeta - \Theta = 0.$$

7°. The substitution $u(x,t) = \dfrac{\partial w}{\partial x}$ brings the original equation to an equation of the form 1.6.18.3:

$$\frac{\partial u}{\partial t} = f\left(\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2}, \qquad f(z) = F'_z(z).$$

8°. The transformation

$$\begin{aligned}
\bar t &= \alpha t + \gamma_1, \quad \bar x = \beta_1 x + \beta_2 w + \gamma_2,\\
\bar w &= \beta_1\big(\beta_4 w + \tfrac{1}{2}\beta_3 x^2 + \gamma_3 x\big) + \gamma_4 t + \gamma_5 + \beta_2\big[\beta_3(xw_x - w) + \gamma_3 w_x + \tfrac{1}{2}\beta_4 w_x^2\big],\\
\bar w_{\bar x} &= \beta_3 x + \beta_4 w_x + \gamma_3,
\end{aligned}$$

where α, the β_i, and the γ_i are arbitrary constants ($\alpha \neq 0$, $\beta_1\beta_4 - \beta_2\beta_3 \neq 0$) and the subscripts x and $\bar x$ denote the corresponding partial derivatives, takes the equation in question to an equation of the same form. The right-hand side of the equation becomes

$$\bar F(\bar w_{\bar x\bar x}) = \frac{\beta_1\beta_4 - \beta_2\beta_3}{\alpha}F(w_{xx}) + \frac{\gamma_4}{\alpha}.$$

⊙ *References*: I. Sh. Akhatov, R. K. Gazizov, and N. H. Ibragimov (1989), N. H. Ibragimov (1994).

Special case 1. Equation:

$$\frac{\partial w}{\partial t} = a\left(\frac{\partial^2 w}{\partial x^2}\right)^k, \qquad k > 0, \quad k \neq 1.$$

1°. Additive separable solution:

$$w(x,t) = \tfrac{1}{2}C_1x^2 + C_2x + aC_1^k t + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

2°. Solution:

$$w(x,t) = \big[a(1-k)t + C_1\big]^{\frac{1}{1-k}}u(x) + C_2,$$

where the function $u(x)$ is determined by the autonomous ordinary differential equation $(u''_{xx})^k - u = 0$, whose general solution can be written out in implicit form:

$$\int\left(\frac{2k}{1+k}u^{\frac{1+k}{k}} + C_3\right)^{-1/2} du = \pm x + C_4.$$

Special case 2. Equation:

$$\frac{\partial w}{\partial t} = a\exp\left(\lambda\frac{\partial^2 w}{\partial x^2}\right).$$

Generalized separable solution:

$$w(x,t) = U(x) - \frac{1}{2\lambda}(x^2 + A_1x + A_2)\ln(B_1t + B_2) + C_1x + C_2,$$

where A_1, A_2, B_1, B_2, C_1, and C_2 are arbitrary constants, and the function $U(x)$ is determined by the ordinary differential equation

$$2a\lambda\exp(\lambda U''_{xx}) + B_1(x^2 + A_1x + A_2) = 0,$$

which is easy to integrate; to this end, the equation should first be solved for U''_{xx}.

Special case 3. Equation:

$$\frac{\partial w}{\partial t} = a \ln\left|\frac{\partial^2 w}{\partial x^2}\right|.$$

Generalized separable solution:

$$w(x,t) = (at + C)\left[\ln \frac{2A^2(at + C)}{\cos^2(Ax + B)} - 1\right] + D,$$

where A, B, C, and D are arbitrary constants.

2. $\displaystyle \frac{\partial w}{\partial t} = F\left(\frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right)$.

Apart from a traveling-wave solution, this equation has a more complicated exact solution of the form

$$w(x,t) = At + B + \varphi(\xi), \qquad \xi = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants, and the function $\varphi(\xi)$ is determined by the autonomous ordinary differential equation

$$F\big(k\varphi'_\xi,\, k^2\varphi''_{\xi\xi}\big) - \lambda\varphi'_\xi - A = 0.$$

Special case. Equation:

$$\frac{\partial w}{\partial t} = a\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}.$$

1°. Generalized separable solution:

$$w(x,t) = \varphi_1(t) + \varphi_2(t)x^{3/2} + \varphi_3(t)x^3,$$

where the functions $\varphi_k = \varphi_k(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}\varphi_1' &= \tfrac{9}{8}a\varphi_2^2,\\ \varphi_2' &= \tfrac{45}{4}a\varphi_2\varphi_3,\\ \varphi_3' &= 18a\varphi_3^2.\end{aligned}$$

The prime denotes a derivative with respect to t.

2°. Generalized separable solution cubic in x:

$$w(x,t) = \psi_1(t) + \psi_2(t)x + \psi_3(t)x^2 + \psi_4(t)x^3,$$

where the functions $\psi_k = \psi_k(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}\psi_1' &= 2a\psi_2\psi_3,\\ \psi_2' &= 2a(2\psi_3^2 + 3\psi_2\psi_4),\\ \psi_3' &= 18a\psi_3\psi_4,\\ \psi_4' &= 18a\psi_4^2.\end{aligned}$$

3°. Generalized separable solution:

$$w(x,t) = \frac{\theta(x) + C_3}{C_1 t + C_2} + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants and the function $\theta = \theta(x)$ is determined by the autonomous ordinary differential equation

$$a\theta'_x\theta''_{xx} + C_1\theta + C_1C_3 = 0,$$

whose solution can be written out in implicit form.

3. $\displaystyle \frac{\partial w}{\partial t} = F\left(\frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right) + aw$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1, t + C_2) + C_3e^{at},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = (C_1x + C_2)e^{at} + e^{at}\int e^{-at}F(C_1e^{at},0)\,dt.$$

3°. Traveling-wave solution:

$$w = w(z), \quad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$F\bigl(w'_z, w''_{zz}\bigr) - \lambda w'_z + aw = 0.$$

4. $\displaystyle \frac{\partial w}{\partial t} = aw\frac{\partial w}{\partial x} + F\left(\frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + aC_1t + C_2, t + C_3) + C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = -\frac{x + C_1}{a\tau} + \frac{1}{\tau}\int \tau F\left(-\frac{1}{a\tau}, 0\right)d\tau, \qquad \tau = t + C_2.$$

3°. Solution:

$$w(x,t) = U(\zeta) + 2C_1t, \qquad \zeta = x + aC_1t^2 + C_2t,$$

where C_1 and C_2 are arbitrary constants and the function $U(\zeta)$ is determined by the autonomous ordinary differential equation

$$F\bigl(U'_\zeta, U''_{\zeta\zeta}\bigr) + aUU'_\zeta = C_2U'_\zeta + 2C_1.$$

In the special case $C_1 = 0$, the above solution converts to a traveling-wave solution.

5. $\displaystyle \frac{\partial w}{\partial t} = aw\frac{\partial w}{\partial x} + F\left(\frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right) + bw.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + aC_1e^{bt} + C_2, t + C_3) + C_1be^{bt},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = \varphi(t)x + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}\varphi'_t &= a\varphi^2 + b\varphi,\\ \psi'_t &= a\varphi\psi + b\psi + F(\varphi,0),\end{aligned}$$

which is easy to integrate (the first equation is a Bernoulli equation and the second one is linear in ψ).

3°. Traveling-wave solution:

$$w = w(z), \quad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$F\bigl(w'_z, w''_{zz}\bigr) + aww'_z - \lambda w'_z + bw = 0.$$

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial}{\partial x}\big[F\big(w, w_x\big)\big], \qquad w_x = \dfrac{\partial w}{\partial x}.$

1°. The transformation

$$\bar{t} = t - t_0, \quad \bar{x} = -\int_{x_0}^{x} w(y,t)\,dy - \int_{t_0}^{t} F\big(w(x_0,\tau), w_x(x_0,\tau)\big)\,d\tau, \quad \bar{w}(\bar{x},\bar{t}) = \frac{1}{w(x,t)}$$

converts a (nonzero) solution $w(x,t)$ of the original equation to a solution $\bar{w}(\bar{x},\bar{t})$ of a similar equation:

$$\frac{\partial \bar{w}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}}\Big[\bar{F}\big(\bar{w},\, \bar{w}_{\bar{x}}\big)\Big],$$

where

$$\bar{F}\big(w, w_x\big) = wF\big(w^{-1},\, w^{-3}w_x\big). \tag{1}$$

2°. In the special case

$$F\big(w, w_x\big) = g(w)(w_x)^k,$$

it follows from (1) that

$$\bar{F}\big(w, w_x\big) = \bar{g}(w)(w_x)^k, \qquad \bar{g}(w) = w^{1-3k} g(w^{-1}).$$

⊙ *References*: W. Strampp (1982), J. R. Burgan, A. Munier, M. R. Feix, and E. Fijalkow (1984), N. H. Ibragimov (1994).

7. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = t\varphi(z), \quad z = kx + \lambda \ln|t|,$$

where k and λ are arbitrary constants and the function $\varphi(z)$ is determined by the autonomous ordinary differential equation

$$F\left(k\frac{\varphi'_z}{\varphi},\, k^2\frac{\varphi''_{zz}}{\varphi}\right) = \lambda\varphi'_z + \varphi.$$

8. $\dfrac{\partial w}{\partial t} = wF\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x + C_2, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = Ce^{\lambda t}\varphi(x),$$

where C and λ are arbitrary constants and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$F\left(\frac{\varphi'_x}{\varphi},\, \frac{\varphi''_{xx}}{\varphi}\right) = \lambda.$$

This equation has particular solutions of the form $\varphi(x) = e^{\alpha x}$, where α is a root of the algebraic (or transcendental) equation $F(\alpha, \alpha^2) - \lambda = 0$.

3°. Solution:

$$w(x,t) = Ce^{\lambda t}\psi(\xi), \quad \xi = kx + \beta t,$$

where C, k, λ, and β are arbitrary constants, and the function $\psi(\xi)$ is determined by the autonomous ordinary differential equation

$$\psi F\left(k\frac{\psi'_\xi}{\psi},\, k^2\frac{\psi''_{\xi\xi}}{\psi}\right) = \beta\psi'_\xi + \lambda\psi.$$

This equation has particular solutions of the form $\psi(\xi) = e^{\mu\xi}$.

9. $\displaystyle \frac{\partial w}{\partial t} = w^\beta F\left(\frac{1}{w}\frac{\partial w}{\partial x},\, \frac{1}{w}\frac{\partial^2 w}{\partial x^2}\right).$

For the cases $\beta = 0$ and $\beta = 1$, see equations 8.1.1.7 and 8.1.1.8, respectively.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2,\, C_1^{\beta-1}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \bigl[(1-\beta)At + B\bigr]^{\frac{1}{1-\beta}}\varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$\varphi^{\beta-1}F\left(\frac{\varphi'_x}{\varphi},\, \frac{\varphi''_{xx}}{\varphi}\right) = A.$$

3°. Solution:

$$w(z,t) = (t+C)^{\frac{1}{1-\beta}}\Theta(z), \quad z = kx + \lambda\ln(t+C),$$

where C, k, and λ are arbitrary constants, and the function $\Theta(z)$ is determined by the autonomous ordinary differential equation

$$\Theta^\beta F\left(k\frac{\Theta'_z}{\Theta},\, k^2\frac{\Theta''_{zz}}{\Theta}\right) = \lambda\Theta'_z + \frac{1}{1-\beta}\Theta.$$

10. $\displaystyle \frac{\partial w}{\partial t} = e^{\beta w} F\left(\frac{\partial w}{\partial x},\, \frac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1,\, C_2 t + C_3) + \frac{1}{\beta}\ln C_2,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = -\frac{1}{\beta}\ln(A\beta t + B) + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$e^{\beta\varphi}F\bigl(\varphi'_x,\, \varphi''_{xx}\bigr) + A = 0.$$

3°. Solution:

$$w(x,t) = -\frac{1}{\beta}\ln(t+C) + \Theta(\xi), \quad \xi = kx + \lambda\ln(t+C),$$

where C, k, and λ are arbitrary constants, and the function $\Theta(\xi)$ is determined by the autonomous ordinary differential equation

$$e^{\beta\Theta}F\bigl(k\Theta'_\xi,\, k^2\Theta''_{\xi\xi}\bigr) = \lambda\Theta'_\xi - \frac{1}{\beta}.$$

11. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial^2 w}{\partial x^2} \Big/ \dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 8.1.1.2.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-1} w(x + C_2, C_1 t + C_3) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = At + B + \varphi(\xi), \qquad \xi = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants, and the function $\varphi(\xi)$ is determined by the autonomous ordinary differential equation

$$F\left(k\varphi''_{\xi\xi}/\varphi'_\xi\right) = \lambda\varphi'_\xi + A.$$

If $A = 0$, the equation has a traveling-wave solution.

3°. Solution:

$$w(x,t) = t\Theta(z) + C, \qquad z = kx + \lambda \ln|t|$$

where C, k, β, and λ are arbitrary constants, and the function $\Theta(z)$ is determined by the autonomous ordinary differential equation

$$F\left(k\Theta''_{zz}/\Theta'_z\right) = \lambda\Theta'_z + \Theta.$$

12. $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x} F\left(\dfrac{\partial^2 w}{\partial x^2} \Big/ \dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 8.1.1.2.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, t + C_3) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = At + B + \varphi(z), \qquad z = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants, and the function $\varphi(z)$ is determined by the autonomous ordinary differential equation

$$k\varphi'_z F\left(k\varphi''_{zz}/\varphi'_z\right) = \lambda\varphi'_z + A.$$

3°. Solution:

$$w(x,t) = Ae^{\beta t}\Theta(\xi) + B, \qquad \xi = kx + \lambda t,$$

where A, B, k, β, and λ are arbitrary constants, and the function $\Theta(\xi)$ is determined by the autonomous ordinary differential equation

$$k\Theta'_\xi F\left(k\Theta''_{\xi\xi}/\Theta'_\xi\right) = \lambda\Theta'_\xi + \beta\Theta.$$

13. $\dfrac{\partial w}{\partial t} = \left(\dfrac{\partial w}{\partial x}\right)^{\beta} F\left(\dfrac{\partial^2 w}{\partial x^2} \Big/ \dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 8.1.1.2. For the cases $\beta = 0$ and $\beta = 1$, see equations 8.1.1.11 and 8.1.1.12, respectively.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, C_1^{\beta-1} t + C_3) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution:

$$w(x,t) = \bigl[A(1-\beta)t + B\bigr]^{\frac{1}{1-\beta}}\varphi(x) + C,$$

where A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$(\varphi'_x)^\beta F\bigl(\varphi''_{xx}/\varphi'_x\bigr) = A\varphi.$$

3°. Solution:

$$w(x,t) = (t+A)^{\frac{1}{1-\beta}}\Theta(z) + B, \qquad z = kx + \lambda\ln(t+A),$$

where A, B, k, and λ are arbitrary constants, and the function $\Theta(z)$ is determined by the autonomous ordinary differential equation

$$k^\beta(\Theta'_z)^\beta F\bigl(k\Theta''_{zz}/\Theta'_z\bigr) = \lambda\Theta'_z + \frac{1}{1-\beta}\Theta.$$

14. $\displaystyle \frac{\partial w}{\partial t} = wF\left(\left(\frac{\partial w}{\partial x}\right)^2 + aw^2,\ \frac{1}{w}\frac{\partial^2 w}{\partial x^2}\right).$

This is a special case of equation 8.1.2.11.

15. $\displaystyle \frac{\partial w}{\partial t} = wF\left(\frac{1}{w}\frac{\partial^2 w}{\partial x^2},\ w\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial w}{\partial x}\right)^2\right).$

This is a special case of equation 8.1.2.12.

16. $\displaystyle \frac{\partial w}{\partial t} = wF\left(\frac{\partial^2 w}{\partial x^2},\ 2w\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial w}{\partial x}\right)^2\right) + G\left(\frac{\partial^2 w}{\partial x^2},\ 2w\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial w}{\partial x}\right)^2\right).$

This is a special case of equation 8.1.2.13.

8.1.2. Equations of the Form $\dfrac{\partial w}{\partial t} = F\left(t, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right)$

1. $\displaystyle \frac{\partial w}{\partial t} = F\left(t, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right) + aw.$

Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{at},$$

where C are arbitrary constants, is also a solution of the equation.

2. $\displaystyle \frac{\partial w}{\partial t} + f(t)w\frac{\partial w}{\partial x} = F\left(t, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right) + g(t)w.$

Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\bigl(x+\psi(t),\ t\bigr) + \varphi(t), \qquad \varphi(t) = C\exp\left[\int g(t)\,dt\right], \qquad \psi(t) = -\int f(t)\varphi(t)\,dt,$$

where C is an arbitrary constant, is also a solution of the equation.

3. $\dfrac{\partial w}{\partial t} = wF\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

Multiplicative separable solutions:

$$w(x,t) = A\exp\left[\lambda x + \int F(t,\lambda^2)\,dt\right],$$

$$w(x,t) = \left[A\cosh(\lambda x) + B\sinh(\lambda x)\right]\exp\left[\int F(t,\lambda^2)\,dt\right],$$

$$w(x,t) = \left[A\cos(\lambda x) + B\sin(\lambda x)\right]\exp\left[\int F(t,-\lambda^2)\,dt\right],$$

where A, B, and λ are arbitrary constants.

4. $\dfrac{\partial w}{\partial t} = wF\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

Multiplicative separable solution:

$$w(x,t) = A\exp\left[\lambda x + \int F(t,\lambda,\lambda^2)\,dt\right],$$

where A and λ are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = wF\left(t,\ w^k\dfrac{\partial w}{\partial x},\ w^{2k+1}\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Multiplicative separable solution for $k \neq -1$:

$$w(x,t) = \left[C_1(k+1)x + C_2\right]^{\frac{1}{k+1}}\varphi(t),$$

where the function $\varphi = \varphi(t)$ is determined by the first-order ordinary differential equation

$$\varphi'_t = \varphi F\left(t, C_1\varphi^{k+1}, -kC_1^2\varphi^{2k+2}\right).$$

2°. For $k = -1$, see equation 8.1.2.4.

6. $\dfrac{\partial w}{\partial t} = f(t)w^\beta\Phi\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + g(t)w.$

The transformation

$$w(x,t) = G(t)u(x,\tau), \quad \tau = \int f(t)G^{\beta-1}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 8.1.1.9:

$$\frac{\partial u}{\partial \tau} = u^\beta\Phi\left(\frac{1}{u}\frac{\partial u}{\partial x},\ \frac{1}{u}\frac{\partial^2 u}{\partial x^2}\right),$$

which has a traveling-wave solution $u = u(Ax + B\tau)$ and a solution in the multiplicative form $u = \varphi(x)\psi(\tau)$.

7. $\dfrac{\partial w}{\partial t} = f(t)\left(\dfrac{\partial w}{\partial x}\right)^k\Phi\left(\dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x}\right) + g(t)w + h(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi'_t = Af(t)\varphi^k + g(t)\varphi, \tag{1}$$
$$\psi'_t = g(t)\psi + Bf(t)\varphi^k + h(t), \tag{2}$$

C is an arbitrary constant, and the function $\Theta(x)$ is determined by the second-order ordinary differential equation

$$\left(\Theta'_x\right)^k \Phi\left(\Theta''_{xx}/\Theta'_x\right) = A\Theta + B. \tag{3}$$

The general solution of system (1), (2) is expressed as

$$\varphi(t) = G(t)\left[C - kA\int f(t)G^{k-1}(t)\,dt\right]^{\frac{1}{1-k}}, \quad G(t) = \exp\left[\int g(t)\,dt\right],$$
$$\psi(t) = DG(t) + G(t)\int\left[Bf(t)\varphi^k(t) + h(t)\right]\frac{dt}{G(t)},$$

where A, B, C, and D are arbitrary constants.

For $k = 1$ and $\Phi(x, y) = \Phi(y)$, a solution to equation (3) is given by

$$\Theta(x) = \alpha e^{\lambda x} - B/A,$$

where α is an arbitrary constant and λ is determined from the algebraic (or transcendental) equation $\lambda\Phi(\lambda) = A$.

8. $\displaystyle \frac{\partial w}{\partial t} = f(t)e^{\beta w}\Phi\left(\frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right) + g(t).$

The transformation

$$w(x,t) = u(x,\tau) + G(t), \quad \tau = \int f(t)\exp\left[\beta G(t)\right]dt, \qquad G(t) = \int g(t)\,dt,$$

leads to a simpler equation of the form 8.1.1.10:

$$\frac{\partial u}{\partial \tau} = e^{\beta u}\Phi\left(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right),$$

which has a traveling-wave solution $u = u(Ax + B\tau)$ and an additive separable solution $u = \varphi(x) + \psi(\tau)$.

9. $\displaystyle \frac{\partial w}{\partial t} = f(t)\Phi\left(w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right) + g(t)\frac{\partial w}{\partial x}.$

With the transformation

$$w = U(z,\tau), \quad z = x + \int g(t)\,dt, \quad \tau = \int f(t)\,dt$$

one arrives at the simpler equation

$$\frac{\partial U}{\partial \tau} = \Phi\left(U, \frac{\partial U}{\partial z}, \frac{\partial^2 U}{\partial z^2}\right),$$

which has a traveling-wave solution $U = U(kz + \lambda\tau)$.

10. $\displaystyle \frac{\partial w}{\partial t} = wF\left(t, w\frac{\partial^2 w}{\partial x^2} + aw^2\right).$

Multiplicative separable solutions:

$$w(x,t) = \left[C_1\sin\left(x\sqrt{a}\right) + C_2\cos\left(x\sqrt{a}\right)\right]\exp\left[\int F(t,0)\,dt\right] \qquad \text{if } a > 0,$$
$$w(x,t) = \left[C_1\sinh\left(x\sqrt{|a|}\right) + C_2\cosh\left(x\sqrt{|a|}\right)\right]\exp\left[\int F(t,0)\,dt\right] \quad \text{if } a < 0,$$

where C_1 and C_2 are arbitrary constants.

11. $\dfrac{\partial w}{\partial t} = wF\left(t,\ \left(\dfrac{\partial w}{\partial x}\right)^2 + aw^2,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Multiplicative separable solution for $a > 0$:

$$w(x,t) = \bigl[C_1 \sin\bigl(x\sqrt{a}\,\bigr) + C_2 \cos\bigl(x\sqrt{a}\,\bigr)\bigr]\varphi(t),$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi'_t = \varphi F\bigl(t,\ a(C_1^2 + C_2^2)\varphi^2,\ -a\bigr)$.

2°. Multiplicative separable solution for $a < 0$:

$$w(x,t) = \bigl(C_1 e^{\sqrt{|a|}\,x} + C_2 e^{-\sqrt{|a|}\,x}\bigr)\varphi(t),$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi'_t = \varphi F\bigl(t,\ 4C_1C_2 a\varphi^2,\ -a\bigr)$.

Example. For $C_1C_2 = 0$, a solution is given by

$$w(x,t) = C\exp\left[\pm\sqrt{|a|}\,x + \int F(t,0,-a)\,dt\right],$$

where C is an arbitrary constant.

12. $\dfrac{\partial w}{\partial t} = wF\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right).$

1°. Multiplicative separable solution:

$$w(x,t) = C\exp\left[\lambda x + \int F(t,\lambda^2,0)\,dt\right],$$

where C and λ are arbitrary constants.

2°. Multiplicative separable solution:

$$w(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi'_t = \varphi F\bigl(t, \lambda^2, 4AB\lambda^2\varphi^2\bigr)$.

3°. Multiplicative separable solution:

$$w(x,t) = [A\sin(\lambda x) + B\cos(\lambda x)]\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi'_t = \varphi F\bigl(t, -\lambda^2, -\lambda^2(A^2 + B^2)\varphi^2\bigr)$.

⊙ *Reference*: Ph. W. Doyle (1996), the case $\partial_t F \equiv 0$ was treated.

13. $\dfrac{\partial w}{\partial t} = wF\left(t,\ \dfrac{\partial^2 w}{\partial x^2},\ 2w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right) + G\left(t,\ \dfrac{\partial^2 w}{\partial x^2},\ 2w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right).$

Generalized separable solution quadratic in x:

$$w = \varphi_1(t)x^2 + \varphi_2(t)x + \varphi_3(t),$$

where the functions φ_1, φ_2, and φ_3 are determined by the system of first-order ordinary differential equations

$$\begin{aligned}
\varphi'_1 &= \varphi_1 F\bigl(t,\ 2\varphi_1,\ 4\varphi_1\varphi_3 - \varphi_2^2\bigr),\\
\varphi'_2 &= \varphi_2 F\bigl(t,\ 2\varphi_1,\ 4\varphi_1\varphi_3 - \varphi_2^2\bigr),\\
\varphi'_3 &= \varphi_3 F\bigl(t,\ 2\varphi_1,\ 4\varphi_1\varphi_3 - \varphi_2^2\bigr) + G\bigl(t,\ 2\varphi_1,\ 4\varphi_1\varphi_3 - \varphi_2^2\bigr).
\end{aligned}$$

It follows from the first two equations that $\varphi_2 = C\varphi_1$, where C is an arbitrary constant.

8.1.3. Equations of the Form $\dfrac{\partial w}{\partial t} = F\Big(x, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\Big)$

Preliminary remarks. Consider the equation

$$\frac{\partial w}{\partial t} = F\left(x,\, w,\, \frac{\partial w}{\partial x},\, \frac{\partial^2 w}{\partial x^2}\right). \qquad (1)$$

Suppose that the auxiliary ordinary differential equation

$$w = F(x,\, w,\, w'_x,\, w''_{xx})$$

is reduced, by a linear transformation

$$x = \varphi(z), \quad w = \psi(z)u + \chi(z)$$

and the subsequent division of the resulting equation by $\psi(z)$, to the autonomous form

$$u = \mathcal{F}(u,\, u'_z,\, u''_{zz}),$$

where $\mathcal{F} = F/\psi$. Then, the original equation (1) can be reduced, by the same transformation

$$x = \varphi(z), \quad w(x,t) = \psi(z)u(z,t) + \chi(z),$$

to the equation

$$\frac{\partial u}{\partial t} = \mathcal{F}\left(u,\, \frac{\partial u}{\partial z},\, \frac{\partial^2 u}{\partial z^2}\right),$$

which has a traveling-wave solution $u = u(kz + \lambda t)$.

The above allows using various known transformations of ordinary differential equations (see Kamke, 1977, and Polyanin and Zaitsev, 2003) for constructing exact solutions to partial differential equations. If the original equation is linear, then such transformations will result in linear constant-coefficient equations.

1. $\dfrac{\partial w}{\partial t} = F\left(x,\, \dfrac{\partial^2 w}{\partial x^2}\right)$.

Generalized separable solution:

$$w(x,t) = Axt + Bt + C + \varphi(x),$$

where A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\big(x,\, \varphi''_{xx}\big) = Ax + B.$$

2. $\dfrac{\partial w}{\partial t} = F\left(x,\, \dfrac{\partial w}{\partial x},\, \dfrac{\partial^2 w}{\partial x^2}\right)$.

Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\big(x,\, \varphi'_x,\, \varphi''_{xx}\big) = A.$$

3. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial w}{\partial x} + F\left(w,\, \dfrac{\partial w}{\partial x},\, \dfrac{\partial^2 w}{\partial x^2}\right)$.

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1 e^{-at}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \quad z = x + Ce^{-at},$$

where C is an arbitrary constant and the function $w(z)$ is determined by the ordinary differential equation

$$F\left(w, w'_z, w''_{zz}\right) + azw'_z = 0.$$

4. $\displaystyle \frac{\partial w}{\partial t} = F\left(\frac{\partial w}{\partial x},\, x\frac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-1} w(C_1 x, C_1 t + C_2) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\left(\varphi'_x,\, x\varphi''_{xx}\right) = A.$$

3°. Solution:

$$w(x,t) = t\Theta(\xi) + C, \quad \xi = x/t,$$

where C is an arbitrary constant, and the function $\Theta(\xi)$ is determined by the ordinary differential equation

$$F\left(\Theta'_\xi,\, \xi\Theta''_{\xi\xi}\right) + \xi\Theta'_\xi - \Theta = 0.$$

5. $\displaystyle \frac{\partial w}{\partial t} = F\left(w,\, x\frac{\partial w}{\partial x},\, x^2\frac{\partial^2 w}{\partial x^2}\right).$

The substitution $x = \pm e^z$ leads to the equation

$$\frac{\partial w}{\partial t} = F\left(w,\, \frac{\partial w}{\partial z},\, \frac{\partial^2 w}{\partial z^2} - \frac{\partial w}{\partial z}\right),$$

which has a traveling-wave solution $w = w(kz + \lambda t)$.

6. $\displaystyle \frac{\partial w}{\partial t} = x^k F\left(w,\, x\frac{\partial w}{\partial x},\, x^2\frac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(C_1 x, C_1^{-k} t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Self-similar solution:

$$w(x,t) = w(z), \quad z = xt^{1/k},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$kz^{k-1} F\left(w,\, zw'_z,\, z^2 w''_{zz}\right) - w'_z = 0.$$

7. $\dfrac{\partial w}{\partial t} = x^k F\left(w,\ x\dfrac{\partial w}{\partial x},\ x^2\dfrac{\partial^2 w}{\partial x^2}\right) + ax\dfrac{\partial w}{\partial x}.$

Passing to the new independent variables

$$z = xe^{at}, \quad \tau = \frac{1}{ak}\left(1 - e^{-akt}\right),$$

we obtain an equation of the form 8.1.3.6:

$$\frac{\partial w}{\partial \tau} = z^k F\left(w,\ z\frac{\partial w}{\partial z},\ z^2\frac{\partial^2 w}{\partial z^2}\right).$$

8. $\dfrac{\partial w}{\partial t} = e^{\lambda x} F\left(w,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1,\ e^{-\lambda C_1}t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = w(z), \quad z = \lambda x + \ln t,$$

where the function $w(z)$ is determined by the ordinary differential equation

$$e^z F\left(w,\ \lambda w'_z,\ \lambda^2 w''_{zz}\right) - w'_z = 0.$$

9. $\dfrac{\partial w}{\partial t} = wF\left(x,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x,\ t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = e^{\lambda t}\varphi(x),$$

where λ is an arbitrary constant and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\left(x,\ \varphi'_x/\varphi,\ \varphi''_{xx}/\varphi\right) = \lambda.$$

10. $\dfrac{\partial w}{\partial t} = w^\beta F\left(x,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

For $\beta = 1$, see equation 8.1.3.9.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x,\ C_1^{\beta-1}t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \left[(1-\beta)At + B\right]^{\frac{1}{1-\beta}}\varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$\varphi^{\beta-1} F\left(x,\ \varphi'_x/\varphi,\ \varphi''_{xx}/\varphi\right) = A.$$

11. $\dfrac{\partial w}{\partial t} = e^{\beta w} F\left(x, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,\ C_1 t + C_2) + \frac{1}{\beta} \ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = -\frac{1}{\beta} \ln(A\beta t + B) + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$e^{\beta\varphi} F\bigl(x,\ \varphi'_x,\ \varphi''_{xx}\bigr) + A = 0.$$

12. $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x} F\left(x, \dfrac{\partial^2 w}{\partial x^2} \Big/ \dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(x,\ t + C_2) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$\varphi'_x F\bigl(x,\ \varphi''_{xx}/\varphi'_x\bigr) = A.$$

3°. Generalized separable solution:

$$w(x,t) = Ae^{\mu t}\Theta(x) + B$$

where A, B, and μ are arbitrary constants, and the function $\Theta(x)$ is determined by the ordinary differential equation

$$\Theta'_x F\bigl(x,\ \Theta''_{xx}/\Theta'_x\bigr) = \mu\Theta.$$

13. $\dfrac{\partial w}{\partial t} = \left(\dfrac{\partial w}{\partial x}\right)^{\beta} F\left(x, \dfrac{\partial^2 w}{\partial x^2} \Big/ \dfrac{\partial w}{\partial x}\right).$

For $\beta = 1$, see equation 8.1.3.12.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x,\ C_1^{\beta-1} t + C_2) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$\bigl(\varphi'_x\bigr)^{\beta} F\bigl(x,\ \varphi''_{xx}/\varphi'_x\bigr) = A.$$

3°. Generalized separable solution:

$$w(x,t) = \bigl[A(1-\beta)t + C_1\bigr]^{\frac{1}{1-\beta}} \bigl[\Theta(x) + B\bigr] + C_2,$$

where A, B, C_1, and C_2 are arbitrary constants, and the function $\Theta(x)$ is determined by the ordinary differential equation

$$\bigl(\Theta'_x\bigr)^{\beta} F\bigl(x,\ \Theta''_{xx}/\Theta'_x\bigr) = A\Theta + AB.$$

8.1.4. Equations of the Form $\frac{\partial w}{\partial t} = F\left(x, t, w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right)$

1. $\displaystyle \frac{\partial w}{\partial t} = a\left(\frac{\partial w}{\partial x}\right)^m \left(\frac{\partial^2 w}{\partial x^2}\right)^n + \left[f(t)x + g(t)\right]\frac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = u(z,\tau)H(t), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^{m+2n}(t)H^{m+n-1}(t)\,dt,$$

where the functions $F(t)$ and $H(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

leads to the simpler equation

$$\frac{\partial u}{\partial \tau} = a\left(\frac{\partial u}{\partial x}\right)^m \left(\frac{\partial^2 u}{\partial z^2}\right)^n.$$

The last equation admits a traveling-wave solution, a self-similar solution, and a multiplicative separable solution.

2. $\displaystyle \frac{\partial w}{\partial t} = f(w)\left(\frac{\partial^2 w}{\partial x^2}\right)^k + \left[xg(t) + h(t)\right]\frac{\partial w}{\partial x}.$

The transformation

$$z = xG(t) + \int h(t)G(t)\,dt, \quad \tau = \int G^{2k}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to the simpler equation

$$\frac{\partial w}{\partial \tau} = f(w)\left(\frac{\partial^2 w}{\partial z^2}\right)^k.$$

The last equation admits a traveling-wave solution and a self-similar solution.

3. $\displaystyle \frac{\partial w}{\partial t} = F\left(x, t, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right) + aw.$

Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x,t) + Ce^{at},$$

where C are arbitrary constants, is also a solution of the equation.

4. $\displaystyle \frac{\partial w}{\partial t} = F\left(ax + bt,\ w,\ \frac{\partial w}{\partial x},\ \frac{\partial^2 w}{\partial x^2}\right).$

Solution:

$$w = w(\xi), \quad \xi = ax + bt,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$F\left(\xi,\ w,\ aw'_\xi,\ a^2 w''_{\xi\xi}\right) - bw'_\xi = 0.$$

5. $\displaystyle \frac{\partial w}{\partial t} = f(t)x^k \Phi\left(w,\ x\frac{\partial w}{\partial x},\ x^2\frac{\partial^2 w}{\partial x^2}\right) + xg(t)\frac{\partial w}{\partial x}.$

Passing to the new independent variables

$$z = xG(t), \quad \tau = \int f(t)G^{-k}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

we obtain a simpler equation of the form 8.1.3.6:

$$\frac{\partial w}{\partial \tau} = z^k \Phi\left(w,\ z\frac{\partial w}{\partial z},\ z^2\frac{\partial^2 w}{\partial z^2}\right).$$

6. $\dfrac{\partial w}{\partial t} = wF\left(t,\ \dfrac{f(x)}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

Multiplicative separable solution:

$$w(x,t) = \varphi(x)\exp\left[\int F(t,\lambda)\,dt\right],$$

where the function $\varphi = \varphi(x)$ satisfies the linear ordinary differential equation $f(x)\varphi''_{xx} = \lambda\varphi$.

7. $\dfrac{\partial w}{\partial t} = w\Phi\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + f(t)e^{\lambda x}.$

Generalized separable solution:

$$w(x,t) = e^{\lambda x}E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + Be^{-\lambda x}E(t), \quad E(t) = \exp\left[\int \Phi(t,\lambda^2)\,dt\right],$$

where A, B, and λ are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = w\Phi\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + f(t)e^{\lambda x} + g(t)e^{-\lambda x}.$

Generalized separable solution:

$$w(x,t) = e^{\lambda x}E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + e^{-\lambda x}E(t)\left[B + \int \frac{g(t)}{E(t)}\,dt\right],$$
$$E(t) = \exp\left[\int \Phi(t,\lambda^2)\,dt\right],$$

where A, B, and λ are arbitrary constants.

9. $\dfrac{\partial w}{\partial t} = wF_1\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + e^{\lambda x}F_2\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + e^{-\lambda x}F_3\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

There is a generalized separable solution of the form

$$w(x,t) = e^{\lambda x}\varphi(t) + e^{-\lambda x}\psi(t).$$

10. $\dfrac{\partial w}{\partial t} = w\Phi\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + f(t)\cosh(\lambda x) + g(t)\sinh(\lambda x).$

Generalized separable solution:

$$w(x,t) = \cosh(\lambda x)E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + \sinh(\lambda x)E(t)\left[B + \int \frac{g(t)}{E(t)}\,dt\right],$$
$$E(t) = \exp\left[\int \Phi(t,\lambda^2)\,dt\right],$$

where A, B, and λ are arbitrary constants.

11. $\dfrac{\partial w}{\partial t} = w\Phi\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + f(t)\cos(\lambda x).$

Generalized separable solution:

$$w(x,t) = \cos(\lambda x)E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + B\sin(\lambda x)E(t),$$
$$E(t) = \exp\left[\int \Phi(t,-\lambda^2)\,dt\right],$$

where A, B, and λ are arbitrary constants.

12. $\dfrac{\partial w}{\partial t} = w\Phi\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + f(t)\cos(\lambda x) + g(t)\sin(\lambda x).$

Generalized separable solution:

$$w(x,t) = \cos(\lambda x)E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + \sin(\lambda x)E(t)\left[B + \int \frac{g(t)}{E(t)}\,dt\right],$$
$$E(t) = \exp\left[\int \Phi(t, -\lambda^2)\,dt\right],$$

where A, B, and λ are arbitrary constants.

13. $\dfrac{\partial w}{\partial t} = wF_1\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + \cos(\lambda x)F_2\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + \sin(\lambda x)F_3\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right).$

There is a generalized separable solution of the form

$$w(x,t) = \cos(\lambda x)\varphi(t) + \sin(\lambda x)\psi(t).$$

14. $\dfrac{\partial w}{\partial t} = w\Phi\left(t,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + f(t)e^{\lambda x}.$

Multiplicative separable solution:

$$w(x,t) = e^{\lambda x}E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right], \quad E(t) = \exp\left[\int \Phi(t, \lambda, \lambda^2)\,dt\right],$$

where A, B, and λ are arbitrary constants.

15. $\dfrac{\partial w}{\partial t} = f(t)w^\beta\Phi\left(x,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + g(t)w.$

The transformation

$$w(x,t) = G(t)u(x,\tau), \quad \tau = \int f(t)G^{\beta-1}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 8.1.3.10:

$$\frac{\partial u}{\partial \tau} = u^\beta \Phi\left(x,\ \frac{1}{u}\frac{\partial u}{\partial x},\ \frac{1}{u}\frac{\partial^2 u}{\partial x^2}\right),$$

which has a multiplicative separable solution $u = \varphi(x)\psi(\tau)$.

16. $\dfrac{\partial w}{\partial t} = f(t)\left(\dfrac{\partial w}{\partial x}\right)^k\Phi\left(x,\ \dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x}\right) + g(t)w + h(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi'_t = Af(t)\varphi^k + g(t)\varphi, \tag{1}$$
$$\psi'_t = g(t)\psi + Bf(t)\varphi^k + h(t), \tag{2}$$

C is an arbitrary constant, and the function $\Theta(x)$ is determined by the second-order ordinary differential equation

$$\left(\Theta'_x\right)^k\Phi\left(x,\ \Theta''_{xx}/\Theta'_x\right) = A\Theta + B. \tag{3}$$

The general solution of system (1), (2) is expressed as

$$\varphi(t) = G(t)\left[C - kA\int f(t)G^{k-1}(t)\,dt\right]^{\frac{1}{1-k}}, \quad G(t) = \exp\left[\int g(t)\,dt\right],$$
$$\psi(t) = DG(t) + G(t)\int \bigl[Bf(t)\varphi^k(t) + h(t)\bigr]\frac{dt}{G(t)},$$

where A, B, C, and D are arbitrary constants.

For $k = 1$ and $\Phi(x, y) = \Phi(y)$, a solution of equation (3) is given by

$$\Theta(x) = \alpha e^{\lambda x} - B/A,$$

where α is an arbitrary constant, and λ is found from the algebraic (or transcendental) equation $\lambda\Phi(\lambda) = A$.

17. $\displaystyle \frac{\partial w}{\partial t} = \bigl[f_1(t)w + f_0(t)\bigr]\left(\frac{\partial w}{\partial x}\right)^k \Phi\left(x,\ \frac{\partial^2 w}{\partial x^2}\Big/\frac{\partial w}{\partial x}\right) + g_1(t)w + g_0(t).$

Generalized separable solution:

$$w(x, t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations (C is an arbitrary constant):

$$\varphi'_t = Cf_1(t)\varphi^{k+1} + g_1(t)\varphi, \tag{1}$$
$$\psi'_t = \bigl[Cf_1(t)\varphi^k + g_1(t)\bigr]\psi + Cf_0(t)\varphi^k + g_0(t), \tag{2}$$

and the function $\Theta(x)$ is determined by the second-order ordinary differential equation

$$\bigl(\Theta'_x\bigr)^k \Phi\bigl(x,\ \Theta''_{xx}/\Theta'_x\bigr) = C. \tag{3}$$

The general solution of system (1), (2) is expressed as

$$\varphi(t) = G(t)\left[A - kC\int f_1(t)G^k(t)\,dt\right]^{-1/k}, \quad G(t) = \exp\left[\int g_1(t)\,dt\right],$$
$$\psi(t) = B\varphi(t) + \varphi(t)\int \bigl[Cf_0(t)\varphi^k(t) + g_0(t)\bigr]\frac{dt}{\varphi(t)},$$

where A, B, and C are arbitrary constants.

Further, we assume that Φ is independent of x explicitly, i.e., $\Phi(x, y) = \Phi(y)$. For $\Phi(0) \neq 0$ and $\Phi(0) \neq \infty$, particular solution to equation (3) has the form $\Theta(x) = \alpha x + \beta$, where $\alpha^k\Phi(0) = C$ and β is an arbitrary constant.

For $k = 0$, the general solution of equation (3) is expressed as

$$\Theta(x) = \alpha e^{\lambda x} + \beta,$$

where α and β are arbitrary constants, and λ is determined from the algebraic (transcendental) equation $\Phi(\lambda) = C$.

18. $\displaystyle \frac{\partial w}{\partial t} = f(t)e^{\beta w}\Phi\left(x,\ \frac{\partial w}{\partial x},\ \frac{\partial^2 w}{\partial x^2}\right) + g(t).$

The transformation

$$w(x, t) = u(x, \tau) + G(t), \quad \tau = \int f(t)\exp\bigl[\beta G(t)\bigr]\,dt, \qquad G(t) = \int g(t)\,dt$$

leads to a simpler equation of the form 8.1.3.11:

$$\frac{\partial u}{\partial \tau} = e^{\beta u}\Phi\left(x,\ \frac{\partial u}{\partial x},\ \frac{\partial^2 u}{\partial x^2}\right),$$

which has an additive separable solution $u = \varphi(x) + \psi(\tau)$.

19. $\dfrac{\partial w}{\partial t} = wF\left(t,\ \dfrac{\partial^2 w}{\partial x^2},\ \dfrac{\partial w}{\partial x} - x\dfrac{\partial^2 w}{\partial x^2},\ 2w - 2x\dfrac{\partial w}{\partial x} + x^2\dfrac{\partial^2 w}{\partial x^2}\right).$

Multiplicative separable solution:

$$w(x,t) = (C_2x^2 + C_1x + C_0)\varphi(t),$$

where C_0, C_1, and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi'_t = \varphi F\big(t, 2C_2\varphi, C_1\varphi, 2C_0\varphi\big)$.

⊙ *Reference*: Ph. W. Doyle (1996), the case $\partial_t F \equiv 0$ was treated.

20. $\dfrac{\partial w}{\partial t} = F\left(x,\ t,\ w,\ \dfrac{\partial w}{\partial x}\right)G\left(x,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial^2 w}{\partial x^2}\right) + h(t).$

Additive separable solution:

$$w(x,t) = \varphi(x) + \int h(t)\,dt,$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$G\big(x, \varphi'_x, \varphi''_{xx}\big) = 0.$$

21. $\dfrac{\partial w}{\partial t} = F\left(x,\ t,\ w,\ \dfrac{\partial w}{\partial x}\right)G\left(x,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + h(t)w.$

Multiplicative separable solution:

$$w(x,t) = C\exp\left[\int h(t)\,dt\right]\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$G\big(x,\ \varphi'_x/\varphi,\ \varphi''_{xx}/\varphi\big) = 0.$$

22. $\dfrac{\partial w}{\partial t} = g_0(t)F_0\left(\dfrac{\partial^2 w}{\partial x^2}\right) + xg_1(t)F_1\left(\dfrac{\partial^2 w}{\partial x^2}\right) + x^2g_2(t)F_2\left(\dfrac{\partial^2 w}{\partial x^2}\right)$
$\qquad + h(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + \big[p_0(t) + xp_1(t)\big]\dfrac{\partial w}{\partial x} + q(t)w + s_0(t) + xs_1(t) + x^2s_2(t).$

There is a generalized separable solution of the form

$$w(x,t) = x^2\varphi(t) + x\psi(t) + \chi(t).$$

23. $\dfrac{\partial w}{\partial t} = x^2f_2\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + xf_1\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + f_0\left(t, \dfrac{\partial^2 w}{\partial x^2}\right).$

Generalized separable solution quadratic in x:

$$w(x,t) = x^2\varphi(t) + x\int f_1(t,2\varphi)\,dt + \int f_0(t,2\varphi)\,dt + C_1x + C_2,$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the first-order ordinary differential equation

$$\varphi'_t = f_2(t,2\varphi).$$

24. $\dfrac{\partial w}{\partial t} = x^2f_2\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + xf_1\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + f_0\left(t, \dfrac{\partial^2 w}{\partial x^2}\right) + g(t)w.$

There is a generalized separable solution of the form

$$w(x,t) = x^2\varphi(t) + x\psi(t) + \chi(t).$$

8.1.5. Equations of the Form $F\left(x, t, w, \dfrac{\partial w}{\partial t}, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right) = 0$

1. $F\left(at + bx,\, w,\, \dfrac{\partial w}{\partial t},\, \dfrac{\partial w}{\partial x},\, \dfrac{\partial^2 w}{\partial x^2}\right) = 0.$

Solution:

$$w = w(\xi), \quad \xi = at + bx,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$F\left(\xi,\, w,\, aw'_\xi,\, bw'_\xi,\, b^2 w''_{\xi\xi}\right) = 0.$$

2. $F\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial t}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) = 0.$

1°. Multiplicative separable solution:

$$w(x,t) = \left[A\cosh(\lambda x) + B\sinh(\lambda x)\right]\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi(t)$ is determined by the first-order ordinary differential equation

$$F\left(t, \varphi'_t/\varphi, \lambda^2\right) = 0.$$

2°. Multiplicative separable solution:

$$w(x,t) = \left[A\cos(\lambda x) + B\sin(\lambda x)\right]\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi(t)$ is determined by the first-order ordinary differential equation

$$F\left(t, \varphi'_t/\varphi, -\lambda^2\right) = 0.$$

3. $F\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial t}, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) = 0.$

Multiplicative separable solution:

$$w(x,t) = Ae^{\lambda x}\varphi(t),$$

where A and λ are arbitrary constants, and the function $\varphi(t)$ is determined by the first-order ordinary differential equation

$$F\left(t, \varphi'_t/\varphi, \lambda, \lambda^2\right) = 0.$$

4. $F\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial t}, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) = 0.$

Multiplicative separable solution:

$$w(x,t) = Ae^{\lambda t}\varphi(x),$$

where A and λ are arbitrary constants, and the function $\varphi(x)$ is determined by the second-order ordinary differential equation

$$F\left(x, \lambda, \varphi'_x/\varphi, \varphi''_{xx}/\varphi\right) = 0.$$

5. $F_1\left(t, \dfrac{\partial w}{\partial t}\right) + F_2\left(x, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right) = kw.$

Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x),$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the first- and second-order ordinary differential equations

$$F_1\left(t, \varphi'_t\right) - k\varphi = C,$$
$$F_2\left(x, \psi'_x, \psi''_{xx}\right) - k\psi = -C,$$

and C is an arbitrary constant.

6. $\boldsymbol{F_1\left(t, \frac{1}{w}\frac{\partial w}{\partial t}\right) + w^k F_2\left(x, \frac{1}{w}\frac{\partial w}{\partial x}, \frac{1}{w}\frac{\partial^2 w}{\partial x^2}\right) = 0.}$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the first- and second-order ordinary differential equations

$$\varphi^{-k} F_1\bigl(t, \varphi'_t/\varphi\bigr) = C,$$
$$\psi^k F_2\bigl(x, \psi'_x/\psi, \psi''_{xx}/\psi\bigr) = -C,$$

and C is an arbitrary constant.

7. $\boldsymbol{F_1\left(t, \frac{\partial w}{\partial t}\right) + e^{\lambda w} F_2\left(x, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right) = 0.}$

Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x),$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the first- and second-order ordinary differential equations

$$e^{-\lambda\varphi} F_1\bigl(t, \varphi'_t\bigr) = C,$$
$$e^{\lambda\psi} F_2\bigl(x, \psi'_x, \psi''_{xx}\bigr) = -C,$$

and C is an arbitrary constant.

8. $\boldsymbol{F_1\left(t, \frac{1}{w}\frac{\partial w}{\partial t}\right) + F_2\left(x, \frac{1}{w}\frac{\partial w}{\partial x}, \frac{1}{w}\frac{\partial^2 w}{\partial x^2}\right) = k \ln w.}$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(x)$ and $\psi(x)$ are determined by the first- and second-order ordinary differential equations

$$F_1\bigl(t, \varphi'_t/\varphi\bigr) - k\ln\varphi = C,$$
$$F_2\bigl(x, \psi'_x/\psi, \psi''_{xx}/\psi\bigr) - k\ln\psi = -C,$$

and C is an arbitrary constant.

8.1.6. Equations with Three Independent Variables

1. $\boldsymbol{\frac{\partial w}{\partial t} = aw\frac{\partial w}{\partial x} + F\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}\right).}$

1°. Suppose $w(x,y,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + aC_1t + C_2, y + C_3, t + C_4) + C_1,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w = w(z), \quad z = C_1x + C_2y + \lambda t,$$

where C_1, C_2, and λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$F\bigl(C_1w'_z, C_2w'_z, C_1^2w''_{zz}, C_2^2w''_{zz}\bigr) + aC_1ww'_z = \lambda w'_z.$$

3°. Solution:

$$w = u(\xi) + 2C_1 t, \quad \xi = x + C_2 y + aC_1 t^2 + C_3 t,$$

where C_1, C_2, and C_3 is an arbitrary constant and the function $u(\xi)$ is determined by the autonomous ordinary differential equation

$$F\big(u'_\xi, C_2 u'_\xi, u''_{\xi\xi}, C_2^2 u''_{\xi\xi}\big) + auu'_\xi = C_3 u'_\xi + 2C_1.$$

4°. "Two-dimensional" solution:

$$w(x, y, t) = U(y, \eta) + 2C_1 t, \quad \eta = x + aC_1 t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(y, \eta)$ is determined by the differential equation

$$2C_1 + C_2 \frac{\partial U}{\partial \eta} = aU \frac{\partial U}{\partial \eta} + F\left(\frac{\partial U}{\partial \eta}, \frac{\partial U}{\partial y}, \frac{\partial^2 U}{\partial \eta^2}, \frac{\partial^2 U}{\partial y^2}\right).$$

5°. There is a "two-dimensional" solution of the form

$$w(x, y, t) = V(\zeta_1, \zeta_2), \quad \zeta_1 = a_1 x + b_1 y + c_1 t, \quad \zeta_2 = a_2 x + b_2 y + c_2 t.$$

2. $\displaystyle \frac{\partial w}{\partial t} + (a_1 x + b_1 y)\frac{\partial w}{\partial x} + (a_2 x + b_2 y)\frac{\partial w}{\partial y} = F\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right).$

1°. Suppose $w(x, y, t)$ is a solution of this equation. Then the functions

$$w_1 = w\big(x + Cb_1 e^{\lambda t}, y + C(\lambda - a_1)e^{\lambda t}, t\big),$$

where C is an arbitrary constant, and $\lambda = \lambda_{1,2}$ are roots of the quadratic equation

$$\lambda^2 - (a_1 + b_2)\lambda + a_1 b_2 - a_2 b_1 = 0, \tag{1}$$

are also solutions of the equation.

2°. Solutions:

$$w = w(z), \quad z = a_2 x + (\lambda - a_1)y + Ce^{\lambda t},$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation (1), and the function $w(z)$ is determined by the ordinary differential equation

$$\big[\lambda z + a_2 c_1 + (\lambda - a_1)c_2\big] w'_z = F\big(w,\ a_2 w'_z,\ (\lambda - a_1)w'_z,\ a_2^2 w''_{zz},\ a_2(\lambda - a_1)w''_{zz},\ (\lambda - a_1)^2 w''_{zz}\big).$$

3°. "Two-dimensional" solutions:

$$w = u(\zeta, t), \quad \zeta = a_2 x + (\lambda - a_1)y,$$

where $\lambda = \lambda_{1,2}$ are roots of the quadratic equation (1), and the function $u(\zeta, t)$ is determined by the differential equation

$$\frac{\partial u}{\partial t} + \big[\lambda\zeta + a_2 c_1 + (\lambda - a_1)c_2\big]\frac{\partial u}{\partial \zeta} = F\left(u,\ a_2\frac{\partial u}{\partial \zeta},\ (\lambda - a_1)\frac{\partial u}{\partial \zeta},\ a_2^2\frac{\partial^2 u}{\partial \zeta^2},\ a_2(\lambda - a_1)\frac{\partial^2 u}{\partial \zeta^2},\ (\lambda - a_1)^2\frac{\partial^2 u}{\partial \zeta^2}\right).$$

8.2. Equations Involving Two or More Second Derivatives

8.2.1. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = F\left(w, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right)$

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} = F\left(\frac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-2} w(C_1 x + C_2, C_1 t + C_3) + C_4 xt + C_5 x + C_6 t + C_7,$$

where the C_n are arbitrary constants, is also a solution of the equation.

2°. Solution quadratic in x and t:

$$w(x,t) = \tfrac{1}{2}Ax^2 + Bxt + \tfrac{1}{2}F(A)t^2 + C_1x + C_2t + C_3,$$

where A, B, C_1, C_2, and C_3 are arbitrary constants.

3°. Generalized separable solution quadratic in x:

$$w(x,t) = \frac{1}{2}(C_1t + C_2)x^2 + (C_3t + C_4)x + \int_0^t (t-\xi)F(C_1\xi + C_2)\,d\xi + C_5t + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants.

4°. Generalized separable solution quadratic in t:

$$w(x,t) = \frac{1}{2}(C_1x + C_2)t^2 + (C_3x + C_4)t + \int_0^x (x-\xi)\Phi(C_1\xi + C_2)\,d\xi + C_5x + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, and the function $\Phi(u)$ is the inverse of $F(u)$.

5°. Self-similar solution:

$$w = t^2U(z), \qquad z = x/t,$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$2U - 2zU'_z + z^2U''_{zz} = F(U''_{zz}).$$

6°. The substitution $u(x,t) = \dfrac{\partial w}{\partial x}$ leads to an equation of the form 3.4.7.7:

$$\frac{\partial^2 u}{\partial t^2} = f\left(\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2}, \qquad f(\xi) = F'_\xi(\xi).$$

⊙ *Reference*: N. H. Ibragimov (1994).

Special case 1. Let $F(\xi) = a\xi^n$.

1°. Multiplicative separable solution:

$$w(x,t) = \varphi(x)\psi(t),$$

where the functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are determined by the second-order autonomous ordinary differential equations (C is an arbitrary constant)

$$\varphi''_{xx} = C\varphi^{1/n},$$
$$\psi''_{tt} = aC^n\psi^n,$$

whose general solutions can be written out in implicit form.

2°. Self-similar solution:

$$w(x,t) = t^\sigma U(z), \qquad z = t^\beta x, \qquad \sigma = \frac{2(1+n\beta)}{1-n},$$

where β is an arbitrary constant, and the function $U = U(z)$ is determined by the ordinary differential equation

$$\sigma(\sigma-1)U + \beta(2\sigma+\beta-1)zU'_z + \beta^2z^2U''_{zz} = a(U''_{zz})^n.$$

3°. Generalized separable solution quadratic in x:

$$w(x,t) = \frac{1}{2}(C_1t + C_2)x^2 + (C_3t + C_4)x + \frac{a}{C_1^2(n+1)(n+2)}(C_1t + C_2)^{n+2} + C_5t + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants.

4°. Generalized separable solution quadratic in t:

$$w(x,t) = \frac{1}{2}(C_1x + C_2)t^2 + (C_3x + C_4)t + \frac{4a^{1/n}}{C_1^2(4n^2-1)}(C_1x + C_2)^{(2n+1)/2} + C_5x + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants.

Special case 2. Let $F(\xi) = a\exp(\lambda\xi)$. Generalized separable solution:

$$w = (A_2x^2 + A_1x + A_0)\varphi(t) + \psi(x),$$

where A_2, A_1, and A_0 are arbitrary constants, and the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations (B_2 is an arbitrary constant)

$$\varphi''_{tt} = aB_2 \exp(2A_2\lambda\varphi), \tag{1}$$

$$\exp(\lambda\psi''_{xx}) = B_2(A_2x^2 + A_1x + A_0). \tag{2}$$

The general solution of equation (1) is expressed as

$$\begin{aligned}
\varphi(t) &= -\frac{1}{2A_2\lambda} \ln\left[\frac{A_2B_2a\lambda}{C_1^2}\cos^2(C_1t + C_2)\right] && \text{if} \quad A_2B_2a\lambda > 0,\\
\varphi(t) &= -\frac{1}{2A_2\lambda} \ln\left[\frac{A_2B_2a\lambda}{C_1^2}\sinh^2(C_1t + C_2)\right] && \text{if} \quad A_2B_2a\lambda > 0,\\
\varphi(t) &= -\frac{1}{2A_2\lambda} \ln\left[-\frac{A_2B_2a\lambda}{C_1^2}\cosh^2(C_1t + C_2)\right] && \text{if} \quad A_2B_2a\lambda < 0,
\end{aligned}$$

where C_1 and C_2 are arbitrary constants. The general solution of equation (2) is given by

$$\psi(x) = \frac{1}{\lambda}\int_{t_0}^{t}(t-\xi)\ln(A_2B_2\xi^2 + A_1B_2\xi + A_0B_2)\,d\xi + B_1t + B_0,$$

where B_1 and B_0 are arbitrary constants.

Special case 3. Let $F(\xi) = a\ln\xi + b$. Generalized separable solution:

$$w = (A_2t^2 + A_1t + A_0)\varphi(x) + \psi(t),$$

where A_2, A_1, and A_0 are arbitrary constants, and the functions $\varphi(x)$ and $\psi(t)$ are determined by the ordinary differential equations (B_2 is an arbitrary constant)

$$a\ln\varphi''_{xx} - 2A_2\varphi = B_2, \tag{3}$$

$$\psi''_{tt} - a\ln(A_2t^2 + A_1t + A_0) - b = B_2. \tag{4}$$

The general solution of equation (3) is given by

$$\begin{aligned}
\varphi(x) &= -\frac{a}{2A_2}\ln\left[\frac{A_2}{aC_1^2}\cos^2(C_1x + C_2)\right] - \frac{B_2}{2A_2} && \text{if } A_2a > 0,\\
\varphi(x) &= -\frac{a}{2A_2}\ln\left[\frac{A_2}{aC_1^2}\sinh^2(C_1x + C_2)\right] - \frac{B_2}{2A_2} && \text{if } A_2a > 0,\\
\varphi(x) &= -\frac{a}{2A_2}\ln\left[-\frac{A_2}{aC_1^2}\cosh^2(C_1x + C_2)\right] - \frac{B_2}{2A_2} && \text{if } A_2a < 0,\\
\varphi(x) &= \tfrac{1}{2}e^{B_2/a}x^2 + C_1x + C_2 && \text{if } A_2 = 0,
\end{aligned}$$

where C_1 and C_2 are arbitrary constants. The general solution of equation (1) is expressed as

$$\varphi(t) = a\int_{t_0}^{t}(t-\xi)\ln(A_2\xi^2 + A_1\xi + A_0)\,d\xi + \frac{1}{2}(B_2 + b)t^2 + B_1t + B_0,$$

where B_1 and B_0 are arbitrary constants.

2. $\dfrac{\partial^2 w}{\partial t^2} = F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1, t + C_2) + C_3t + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w = w(z), \qquad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation

$$F(kw'_z, k^2w''_{zz}) - \lambda^2 w''_{zz} = 0.$$

3°. Additive separable solution:

$$w(x,t) = At^2 + Bt + C + \varphi(x),$$

where A, B, and C are arbitrary constants, and the function $\varphi = \varphi(x)$ is determined by the autonomous ordinary differential equation

$$F(\varphi'_x, \varphi''_{xx}) - 2A = 0.$$

4°. Solution (generalizes the solutions of Items 2° and 3°):

$$w(x,t) = At^2 + Bt + C + \varphi(z), \qquad z = kx + \lambda t,$$

where A, B, C, k, and λ are arbitrary constants, and the function $\varphi = \varphi(z)$ is determined by the autonomous ordinary differential equation

$$F(k\varphi'_z, k^2\varphi''_{zz}) - \lambda^2\varphi''_{zz} - 2A = 0.$$

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = aw\frac{\partial^2 w}{\partial x^2} + F\left(\frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right).$

1°. Degenerate solution linear in x:

$$w = (C_1 t + C_2)x + C_3 t + C_4 + \int_0^t (t-\tau)F(C_1\tau + C_2, 0)\, d\tau.$$

2°. Traveling-wave solution:

$$w(x,t) = w(\xi), \qquad \xi = \beta x + \lambda t,$$

where β and λ are arbitrary constants, and the function $w = w(\xi)$ is determined by the autonomous ordinary differential equation

$$(a\beta^2 w - \lambda^2)w''_{\xi\xi} + F(\beta w'_\xi, \beta^2 w''_{\xi\xi}) = 0.$$

3°. Solution:

$$w = U(z) + 4aC_1^2 t^2 + 4aC_1C_2 t, \qquad z = x + aC_1t^2 + aC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$(aU - a^2C_2^2)U''_{zz} - 2aC_1U'_z + F(U'_z, U''_{zz}) = 8aC_1^2.$$

Special case 1. Let $F(w_x, w_{xx}) = F(w_x)$. Self-similar solution:

$$w(x,t) = t^2 u(\zeta), \qquad \zeta = xt^{-2},$$

where the function $u = u(\zeta)$ is determined by the ordinary differential equation

$$2u - 2\zeta u'_\zeta + 4\zeta^2 u''_{\zeta\zeta} = auu''_{\zeta\zeta} + F(u'_\zeta).$$

Special case 2. Let $F(w_x, w_{xx}) = F(w_{xx})$. Generalized separable solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = 6a\varphi^2,$$
$$\psi''_{tt} = 6a\varphi\psi,$$
$$\chi''_{tt} = 2a\varphi\chi + F(2\varphi).$$

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = wF\left(\left(\frac{\partial w}{\partial x}\right)^2 + aw^2,\ \frac{1}{w}\frac{\partial^2 w}{\partial x^2}\right).$

This is a special case of equation 8.2.2.7.

5. $\dfrac{\partial^2 w}{\partial t^2} = wF\left(\dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ w\dfrac{\partial^2 w}{\partial x^2} + aw^2\right).$

This is a special case of equation 8.2.2.8.

6. $\dfrac{\partial^2 w}{\partial t^2} = wF\left(\dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right).$

This is a special case of equation 8.2.2.9.

7. $\dfrac{\partial^2 w}{\partial t^2} = wF\left(\dfrac{\partial^2 w}{\partial x^2},\ 2w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right) + G\left(\dfrac{\partial^2 w}{\partial x^2},\ 2w\dfrac{\partial^2 w}{\partial x^2} - \left(\dfrac{\partial w}{\partial x}\right)^2\right).$

This is a special case of equation 8.2.2.11.

8.2.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = F\left(x, t, w, \dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial t}, \dfrac{\partial^2 w}{\partial x^2}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = F\left(t, \dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1, t) + C_2 xt + C_3 x + C_4 t + C_5,$$

where the C_n are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution quadratic in x:

$$w(x,t) = \frac{1}{2}(C_1 t + C_2)x^2 + (C_3 t + C_4)x + \int_0^t (t-\xi)F(\xi, C_1\xi + C_2)\,d\xi + C_5 t + C_6,$$

where $C_1, \dots, C_6$ are arbitrary constants.

3°. The substitution $u(x,t) = \frac{\partial w}{\partial x}$ leads to a simpler equation which is linear in the highest derivatives:

$$\frac{\partial^2 u}{\partial t^2} = f\left(t, \frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2}, \qquad f(t,\xi) = \frac{\partial}{\partial \xi}F(t,\xi).$$

2. $\dfrac{\partial^2 w}{\partial t^2} = F\left(x, \dfrac{\partial^2 w}{\partial x^2}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x, t + C_1) + C_2 xt + C_3 x + C_4 t + C_5,$$

where C_n are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution quadratic in t:

$$w(x,t) = \frac{1}{2}(C_1 x + C_2)t^2 + (C_3 x + C_4)t + \varphi(x) + C_5 x + C_6,$$

where $C_1, \dots, C_6$ are arbitrary constants, and the function $\varphi = \varphi(x)$ is determined by the ordinary differential equation

$$C_1 x + C_2 = F(x, \varphi''_{xx}).$$

3. $\dfrac{\partial^2 w}{\partial t^2} = F\left(x,\, w,\, \dfrac{\partial w}{\partial x},\, \dfrac{\partial^2 w}{\partial x^2}\right).$

Suppose the auxiliary ordinary differential equation

$$w = F(x,\, w,\, w'_x,\, w''_{xx})$$

is reduced, with the linear transformation

$$x = \varphi(z), \quad w = \psi(z)u + \chi(z)$$

followed by the division of the resulting equation by $\psi(z)$, to the autonomous form

$$u = \mathcal{F}(u,\, u'_z,\, u''_{zz}),$$

where $\mathcal{F} = F/\psi$. Then the original partial differential equation can be reduced, with the same transformation,

$$x = \varphi(z), \quad w(x,t) = \psi(z)u(z,t) + \chi(z),$$

to

$$\frac{\partial^2 u}{\partial t^2} = \mathcal{F}\left(u,\, \frac{\partial u}{\partial z},\, \frac{\partial^2 u}{\partial z^2}\right),$$

which has a traveling-wave solution $u = u(z + \lambda t)$.

The above allows using various known transformations of ordinary differential equations (see Kamke, 1977; Polyanin and Zaitsev, 2003) for constructing exact solutions to partial differential equations. If the original equation is linear, then such transformations will result in linear constant-coefficient equations.

4. $\dfrac{\partial^2 w}{\partial t^2} = (aw + bx)\dfrac{\partial^2 w}{\partial x^2} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}\right).$

The substitution $w = u - (b/a)x$ leads to an equation of the form 8.2.1.3:

$$\frac{\partial^2 u}{\partial t^2} = au\frac{\partial^2 u}{\partial x^2} + F\left(\frac{\partial u}{\partial x} - \frac{b}{a}, \frac{\partial^2 u}{\partial x^2}\right).$$

5. $\dfrac{\partial^2 w}{\partial t^2} = F\left(x,\, \dfrac{\partial w}{\partial x},\, \dfrac{\partial^2 w}{\partial x^2}\right) + G\left(t,\, \dfrac{\partial w}{\partial t}\right) + bw.$

Additive separable solution:

$$w(x,t) = \varphi(x) + \psi(t),$$

where the functions $\varphi(x)$ and $\psi(t)$ are determined by the ordinary differential equations (C is an arbitrary constant)

$$F\big(x, \varphi'_x, \varphi''_{xx}\big) + b\varphi = C,$$
$$\psi''_{tt} - G\big(t, \psi'_t\big) - b\psi = C.$$

6. $\dfrac{\partial^2 w}{\partial t^2} = F\left(x,\, t,\, \dfrac{\partial w}{\partial x},\, \dfrac{\partial^2 w}{\partial x^2}\right) + aw.$

Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = w(x,t) + C_1\cosh(kt) + C_2\sinh(kt) \quad \text{if } a = k^2 > 0,$$
$$w_2 = w(x,t) + C_1\cos(kt) + C_2\sin(kt) \qquad \text{if } a = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

7. $\displaystyle \frac{\partial^2 w}{\partial t^2} = wF\left(t,\ \frac{1}{w}\frac{\partial w}{\partial t},\ \left(\frac{\partial w}{\partial x}\right)^2 + aw^2,\ \frac{1}{w}\frac{\partial^2 w}{\partial x^2}\right).$

1°. Multiplicative separable solution for $a > 0$:

$$w(x,t) = \left[C_1 \sin\left(x\sqrt{a}\,\right) + C_2 \cos\left(x\sqrt{a}\,\right)\right]\varphi(t),$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi''_{tt} = \varphi F\left(t,\ \varphi'_t/\varphi,\ a(C_1^2 + C_2^2)\varphi^2,\ -a\right).$$

2°. Multiplicative separable solution for $a < 0$:

$$w(x,t) = \left(C_1 e^{\sqrt{|a|}\,x} + C_2 e^{-\sqrt{|a|}\,x}\right)\varphi(t),$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi''_{tt} = \varphi F\left(t,\ \varphi'_t/\varphi,\ 4aC_1C_2\varphi^2,\ -a\right).$$

8. $\displaystyle \frac{\partial^2 w}{\partial t^2} = wF\left(t,\ \frac{1}{w}\frac{\partial w}{\partial t},\ \frac{1}{w}\frac{\partial^2 w}{\partial x^2},\ w\frac{\partial^2 w}{\partial x^2} + aw^2\right).$

Multiplicative separable solutions:

$$\begin{aligned} w(x,t) &= \left[C_1 \sin\left(x\sqrt{a}\,\right) + C_2 \cos\left(x\sqrt{a}\,\right)\right]\varphi(t) && \text{if } a > 0,\\ w(x,t) &= \left(C_1 e^{\sqrt{|a|}\,x} + C_2 e^{-\sqrt{|a|}\,x}\right)\varphi(t) && \text{if } a < 0, \end{aligned}$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi''_{tt} = \varphi F\left(t, \varphi'_t/\varphi, -a, 0\right).$$

9. $\displaystyle \frac{\partial^2 w}{\partial t^2} = wF\left(t,\ \frac{1}{w}\frac{\partial w}{\partial t},\ \frac{1}{w}\frac{\partial^2 w}{\partial x^2},\ w\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial w}{\partial x}\right)^2\right).$

1°. Multiplicative separable solution:

$$w(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi''_{tt} = \varphi F\left(t,\ \varphi'_t/\varphi,\ \lambda^2, 4AB\lambda^2\varphi^2\right)$.

2°. Multiplicative separable solution:

$$w(x,t) = [A\sin(\lambda x) + B\cos(\lambda x)]\varphi(t),$$

where A and B are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi''_{tt} = \varphi F\left(t,\ \varphi'_t/\varphi, -\lambda^2, -\lambda^2(A^2+B^2)\varphi^2\right)$.

10. $\displaystyle \frac{\partial^2 w}{\partial t^2} = wF\left(t,\ \frac{\partial^2 w}{\partial x^2},\ \frac{\partial w}{\partial x} - x\frac{\partial^2 w}{\partial x^2},\ 2w - 2x\frac{\partial w}{\partial x} + x^2\frac{\partial^2 w}{\partial x^2}\right).$

Multiplicative separable solution:

$$w(x,t) = (C_2x^2 + C_1x + C_0)\varphi(t),$$

where C_0, C_1, and C_2 are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation $\varphi''_{tt} = \varphi F\left(t, 2C_2\varphi, C_1\varphi, 2C_0\varphi\right)$.

11. $\displaystyle \frac{\partial^2 w}{\partial t^2} = wF\left(t,\, \frac{\partial^2 w}{\partial x^2},\, 2w\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial w}{\partial x}\right)^2\right) + G\left(t,\, \frac{\partial^2 w}{\partial x^2},\, 2w\frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial w}{\partial x}\right)^2\right).$

Generalized separable solution quadratic in x:

$$w = \varphi_1(t)x^2 + \varphi_2(t)x + \varphi_3(t),$$

where the functions φ_1, φ_2, and φ_3 are determined by the solution of the ordinary differential equations

$$\begin{aligned}
\varphi_1'' &= \varphi_1 F\bigl(t,\, 2\varphi_1,\, 4\varphi_1\varphi_3 - \varphi_2^2\bigr),\\
\varphi_2'' &= \varphi_2 F\bigl(t,\, 2\varphi_1,\, 4\varphi_1\varphi_3 - \varphi_2^2\bigr),\\
\varphi_3'' &= \varphi_3 F\bigl(t,\, 2\varphi_1,\, 4\varphi_1\varphi_3 - \varphi_2^2\bigr) + G\bigl(t,\, 2\varphi_1,\, 4\varphi_1\varphi_3 - \varphi_2^2\bigr).
\end{aligned}$$

It follows from the first two equations that

$$\varphi_2 = C_1\varphi_1 + C_2\varphi_1 \int \frac{dt}{\varphi_1^2},$$

where C_1 and C_2 are arbitrary constants.

12. $\displaystyle \frac{\partial^2 w}{\partial t^2} = wF_1\left(t,\, \frac{1}{w}\frac{\partial^2 w}{\partial x^2}\right) + e^{\lambda x}F_2\left(t,\, \frac{1}{w}\frac{\partial^2 w}{\partial x^2}\right) + e^{-\lambda x}F_3\left(t,\, \frac{1}{w}\frac{\partial^2 w}{\partial x^2}\right).$

There is a generalized separable solution of the form

$$w(x,t) = e^{\lambda x}\varphi(t) + e^{-\lambda x}\psi(t).$$

13. $\displaystyle \frac{\partial^2 w}{\partial t^2} = wF_1\left(t, \frac{1}{w}\frac{\partial^2 w}{\partial x^2}\right) + \cos(\lambda x)F_2\left(t, \frac{1}{w}\frac{\partial^2 w}{\partial x^2}\right) + \sin(\lambda x)F_3\left(t, \frac{1}{w}\frac{\partial^2 w}{\partial x^2}\right).$

There is a generalized separable solution of the form

$$w(x,t) = \cos(\lambda x)\varphi(t) + \sin(\lambda x)\psi(t).$$

14. $\displaystyle \frac{\partial^2 w}{\partial t^2} = x^2 f_2\left(t, \frac{\partial^2 w}{\partial x^2}\right) + xf_1\left(t, \frac{\partial^2 w}{\partial x^2}\right) + f_0\left(t, \frac{\partial^2 w}{\partial x^2}\right).$

Generalized separable solution quadratic in x:

$$w(x,t) = x^2\varphi(t) + x\int_0^t (t-\xi) f_1\bigl(\xi, 2\varphi(\xi)\bigr)\,d\xi + \int_0^t (t-\xi) f_0\bigl(\xi, 2\varphi(\xi)\bigr)\,d\xi + C_1xt + C_2x + C_3t + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi_{tt}'' = f_2(t, 2\varphi).$$

15. $\displaystyle \frac{\partial^2 w}{\partial t^2} = x^2 f_2\left(t, \frac{\partial^2 w}{\partial x^2}\right) + xf_1\left(t, \frac{\partial^2 w}{\partial x^2}\right) + f_0\left(t, \frac{\partial^2 w}{\partial x^2}\right) + g(t)w.$

There is a generalized separable solution of the form

$$w(x,t) = x^2\varphi(t) + x\psi(t) + \chi(t).$$

8.2.3. Equations Linear in the Mixed Derivative

1. $\displaystyle \frac{\partial^2 w}{\partial x \partial t} = F\left(t, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right) + g(t)w\frac{\partial^2 w}{\partial x^2}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + \varphi(t), t) + \frac{\varphi_t'(t)}{g(t)},$$

where $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solution linear in x:

$$w(x,t) = \varphi(t)x + \psi(t),$$

where $\psi(t)$ is an arbitrary function and $\varphi(t)$ is determined by the first-order ordinary differential equation $\varphi'_t = F(t,\varphi,0)$.

3°. For $g(t) = a$ and $F = F(w_x, w_{xx})$, the equation has a traveling-wave solution

$$w = U(z), \quad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$k\lambda U''_{zz} = ak^2 U U''_{zz} + F(kU'_z, k^2 U''_{zz}).$$

2. $\displaystyle f\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right)\frac{\partial^2 w}{\partial x^2} + g\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right)\frac{\partial^2 w}{\partial x \partial y} + h\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right)\frac{\partial^2 w}{\partial y^2} = 0.$

The Legendre transformation

$$w(x,y) + u(\xi,\eta) = x\xi + y\eta, \quad \xi = \frac{\partial w}{\partial x}, \quad \eta = \frac{\partial w}{\partial y}$$

leads to the linear equation (for details, see Subsection S.2.3)

$$f(\xi,\eta)\frac{\partial^2 u}{\partial \eta^2} - g(\xi,\eta)\frac{\partial^2 u}{\partial \xi \partial \eta} + h(\xi,\eta)\frac{\partial^2 u}{\partial \xi^2} = 0.$$

3. $\displaystyle \frac{\partial^2 w}{\partial x \partial t} = F\left(t, w, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}\right) + g(t)\frac{\partial^2 w}{\partial y^2}.$

1°. "Two-dimensional" solution:

$$w(x,y,t) = u(z,t), \quad z = x + C_1 y + C_1^2 \int g(t)\,dt + C_2,$$

where C_1 and C_2 are arbitrary constants and the function $u(z,t)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial z \partial t} = F\left(t, u, \frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial z^2}\right).$$

2°. "Two-dimensional" solution:

$$w(x,y,t) = U(\xi,t), \quad \xi = x + \varphi(t)(y + C_1)^2, \quad \varphi(t) = -\left[4\int g(t)\,dt + C_2\right]^{-1},$$

where the function $U(\xi,t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial \xi \partial t} = F\left(t, U, \frac{\partial U}{\partial \xi}, \frac{\partial^2 U}{\partial \xi^2}\right) + 2g(t)\varphi(t)\frac{\partial U}{\partial \xi}.$$

8.2.4. Equations with Two Independent Variables, Nonlinear in Two or More Highest Derivatives

1. $\displaystyle f_1\left(\frac{\partial^2 w}{\partial x^2}\right) f_2\left(\frac{\partial^2 w}{\partial y^2}\right) = g_1(x)g_2(y).$

Generalized separable solution:

$$w(x,y) = \varphi(x) + \psi(y) + C_1 xy + C_2 x + C_3 y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, and the functions $\varphi = \varphi(x)$ and $\psi = \psi(y)$ are determined by the ordinary differential equations (a is any)

$$f_1(\varphi''_{xx}) = a g_1(x),$$
$$a f_2(\psi''_{yy}) = g_2(y).$$

2. $F\left(x, y, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \dfrac{\partial^2 w}{\partial x \partial y}\right) = 0.$

The substitution $u = \frac{\partial w}{\partial x}$ leads to the first-order partial differential equation

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0.$$

For details about integration methods and exact solutions for such equations (with various F), see Kamke (1965) and Polyanin, Zaitsev, and Moussiaux (2002).

3. $\dfrac{\partial^2 w}{\partial y^2} = F\left(\dfrac{\partial^2 w}{\partial x^2}, \dfrac{\partial^2 w}{\partial x \partial y}\right).$

1°. Solution quadratic in both variables:

$$w(x, y) = \tfrac{1}{2} C_1 x^2 + C_2 xy + \tfrac{1}{2} F(C_1, C_2) y^2 + C_3 x + C_4 y + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants.

2°. We differentiate the equation with respect to x, introduce the new variable

$$U(x, y) = \frac{\partial w}{\partial x},$$

and then apply the Legendre transformation (for details, see Subsection S.2.3)

$$X = \frac{\partial U}{\partial x}, \quad Y = \frac{\partial U}{\partial y}, \quad Z = x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} - U$$

to obtain the second-order linear equation

$$\frac{\partial^2 Z}{\partial X^2} = F_X(X, Y)\frac{\partial^2 Z}{\partial Y^2} - F_Y(X, Y)\frac{\partial^2 Z}{\partial X \partial Y},$$

where the subscripts X and Y denote the corresponding partial derivatives.

Special case. Let $F(X, Y) = aX + f(Y)$, or

$$\frac{\partial^2 w}{\partial y^2} = a\frac{\partial^2 w}{\partial x^2} + f\left(\frac{\partial^2 w}{\partial x \partial y}\right).$$

Solution:

$$w = \varphi(z) + \tfrac{1}{6}(A_2 A_3 - A_1 A_4)x^3 + \tfrac{1}{2} a A_1 A_3 x^2 y + \tfrac{1}{2} a A_2 A_3 x y^2 + \tfrac{1}{6}(a^2 A_1 A_3 + a A_2 A_4) y^3$$
$$+ \tfrac{1}{2} B_1 x^2 + B_2 xy + \tfrac{1}{2} B_3 y^2 + B_4 x + B_5 y + B_6, \qquad z = A_1 x + A_2 y,$$

where the A_n and B_m are arbitrary constants and the function $\varphi(z)$ is determined by the ordinary differential equation

$$(A_2^2 - a A_1^2)\varphi''_{zz} + a A_4 z + B_3 - a B_1 = f(A_1 A_2 \varphi''_{zz} + a A_3 z + B_2).$$

4. $F\left(\dfrac{\partial^2 w}{\partial x^2}, \dfrac{\partial^2 w}{\partial x \partial y}, \dfrac{\partial^2 w}{\partial y^2}\right) = 0.$

1°. Solution quadratic in both variables:

$$w(x, y) = A_{11} x^2 + A_{12} xy + A_{22} y^2 + B_1 x + B_2 y + C,$$

where A_{11}, A_{12}, A_{22}, B_1, B_2, and C are arbitrary constants constrained by $F(2A_{11}, A_{12}, 2A_{22}) = 0$.

2°. Solving the equation for w_{yy} (or w_{xx}), one arrives at an equation of the form 8.2.4.2.

5. $\boldsymbol{F_1\left(x, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \dfrac{\partial^2 w}{\partial x \partial y}\right) + F_2\left(y, \dfrac{\partial w}{\partial y}, \dfrac{\partial^2 w}{\partial x \partial y}, \dfrac{\partial^2 w}{\partial y^2}\right) = kw.}$

Additive separable solution:

$$w(x,y) = \varphi(x) + \psi(y).$$

Here, $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$F_1\bigl(x, \varphi'_x, \varphi''_{xx}, 0\bigr) - k\varphi = C,$$
$$F_2\bigl(y, \psi'_y, 0, \psi''_{yy}\bigr) - k\psi = -C,$$

where C is an arbitrary constant.

6. $\boldsymbol{F_1\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + F_2\left(y, \dfrac{1}{w}\dfrac{\partial w}{\partial y}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial y^2}\right) = \ln w.}$

Multiplicative separable solution:

$$w(x,y) = \varphi(x)\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$F_1\bigl(x, \varphi'_x/\varphi, \varphi''_{xx}/\varphi\bigr) - \ln\varphi = C,$$
$$F_2\bigl(y, \psi'_y/\psi, \psi''_{yy}/\psi\bigr) - \ln\psi = -C,$$

where C is an arbitrary constant.

7. $\boldsymbol{F_1\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}\right) + w^k F_2\left(y, \dfrac{1}{w}\dfrac{\partial w}{\partial y}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial y^2}\right) = 0.}$

Multiplicative separable solution:

$$w(x,y) = \varphi(x)\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$\varphi^{-k} F_1\bigl(x, \varphi'_x/\varphi, \varphi''_{xx}/\varphi\bigr) = C,$$
$$\psi^k F_2\bigl(y, \psi'_y/\psi, \psi''_{yy}/\psi\bigr) = -C,$$

where C is an arbitrary constant.

8. $\boldsymbol{F\left(ax+by,\ w,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial w}{\partial y},\ \dfrac{\partial^2 w}{\partial x^2},\ \dfrac{\partial^2 w}{\partial y^2},\ \dfrac{\partial^2 w}{\partial x \partial y}\right) = 0.}$

Solution:

$$w = w(\xi), \qquad \xi = ax + by,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$F\bigl(\xi,\ w,\ aw'_\xi,\ bw'_\xi,\ a^2 w''_{\xi\xi},\ b^2 w''_{\xi\xi},\ abw''_{\xi\xi}\bigr) = 0.$$

9. $\boldsymbol{F\left(ax+by,\ w+kx+sy,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial w}{\partial y},\ \dfrac{\partial^2 w}{\partial x^2},\ \dfrac{\partial^2 w}{\partial y^2},\ \dfrac{\partial^2 w}{\partial x \partial y}\right) = 0.}$

The substitution $u(x,y) = w(x,y) + kx + sy$ leads to an equation of the form 8.2.4.8:

$$F\left(ax+by,\ u,\ \frac{\partial u}{\partial x} - k,\ \frac{\partial u}{\partial y} - s,\ \frac{\partial^2 u}{\partial x^2},\ \frac{\partial^2 u}{\partial y^2},\ \frac{\partial^2 u}{\partial x \partial y}\right) = 0.$$

10. $(a_1x+b_1y)\dfrac{\partial w}{\partial x}+(a_2x+b_2y)\dfrac{\partial w}{\partial y}=F\left(w,\ \dfrac{\partial w}{\partial x},\dfrac{\partial w}{\partial y},\dfrac{\partial^2 w}{\partial x^2},\dfrac{\partial^2 w}{\partial x\partial y},\dfrac{\partial^2 w}{\partial y^2}\right).$

Traveling-wave solutions:

$$w=w(z),\qquad z=a_2x+(k-a_1)y,$$

where k is a root of the quadratic equation

$$k^2-(a_1+b_2)k+a_1b_2-a_2b_1=0,$$

and the function $w(z)$ is determined by the ordinary differential equation

$$kzw'_z=F\bigl(w,\ a_2w'_z,\ (k-a_1)w'_z,\ a_2^2w''_{zz},\ a_2(k-a_1)w''_{zz},\ (k-a_1)^2w''_{zz}\bigr).$$

11. $(a_1x+b_1y+c_1)\left(\dfrac{\partial w}{\partial x}\right)^k+(a_2x+b_2y+c_2)\left(\dfrac{\partial w}{\partial y}\right)^k$

$$=F\left(w,\ \frac{\partial w}{\partial x},\ \frac{\partial w}{\partial y},\ \frac{\partial^2 w}{\partial x^2},\ \frac{\partial^2 w}{\partial x\partial y},\ \frac{\partial^2 w}{\partial y^2}\right).$$

Exact solutions are sought in the traveling-wave form

$$w=w(z),\qquad z=Ax+By+C,$$

where the constants A, B, and C are determined by solving the algebraic system

$$a_1A^k+a_2B^k=A,\qquad (1)$$

$$b_1A^k+b_2B^k=B,\qquad (2)$$

$$c_1A^k+c_2B^k=C.\qquad (3)$$

Equations (1) and (2) are first solved for A and B, and then C is evaluated from (3).

The desired function $w(z)$ is determined by the ordinary differential equation

$$z(w'_z)^k=F\bigl(w,\ Aw'_z,\ Bw'_z,\ A^2w''_{zz},\ ABw''_{zz},\ B^2w''_{zz}\bigr).$$

12. $(a_1x+b_1y)\dfrac{\partial^2 w}{\partial x^2}+(a_2x+b_2y)\dfrac{\partial^2 w}{\partial x\partial y}+(a_3x+b_3y)\dfrac{\partial^2 w}{\partial y^2}$

$$=F\left(w,\ \frac{\partial w}{\partial x},\frac{\partial w}{\partial y},\frac{\partial^2 w}{\partial x^2},\frac{\partial^2 w}{\partial x\partial y},\frac{\partial^2 w}{\partial y^2}\right).$$

Traveling-wave solutions:

$$w=w(z),\qquad z=Ax+By,$$

where the constants A and B are determined by solving the algebraic system of equation

$$a_1A^2+a_2AB+a_3B^2=A,$$

$$b_1A^2+b_2AB+b_3B^2=B,$$

and the desired function $w(z)$ is determined by the ordinary differential equation

$$zw''_{zz}=F\bigl(w,\ Aw'_z,\ Bw'_z,\ A^2w''_{zz},\ ABw''_{zz},\ B^2w''_{zz}\bigr).$$

8.2.5. Equations with n Independent Variables

1. $\displaystyle\sum_{k=1}^{n} f_k\left(x_k,\dfrac{\partial w}{\partial x_k},\dfrac{\partial^2 w}{\partial x_k^2}\right)=aw.$

Additive separable solution:

$$w(x_1,\dots,x_n)=\sum_{k=1}^{n}\varphi_k(x_k),$$

where the functions $\varphi_k = \varphi_k(x_k)$ are determined by the second-order ordinary differential equations

$$\sum_{k=1}^{n} f_k\left(x_k, \frac{d\varphi_k}{dx_k}, \frac{d^2\varphi_k}{dx_k^2}\right) - a\varphi_k = C_k, \qquad k = 1,\,2,\,\ldots,\,n.$$

The arbitrary constants $C_1, \ldots, C_n$ are related by the constraint $C_1 + \cdots + C_n = 0$.

Remark. The functions f_k in the original equation can also depend on any number of mixed derivatives $\partial_{x_i x_j} w$. In this case, the arguments corresponding to $\partial_{x_i x_j} w$ in the second-order ordinary differential equations obtained will be replaced by zeros.

2. $\displaystyle \sum_{k=1}^{n} f_k\left(x_k, \frac{1}{w}\frac{\partial w}{\partial x_k}, \frac{1}{w}\frac{\partial^2 w}{\partial x_k^2}\right) = a\ln w.$

Multiplicative separable solution:

$$w(x_1, \ldots, x_n) = \prod_{k=1}^{n} \varphi_k(x_k),$$

where the functions $\varphi_k = \varphi_k(x_k)$ are determined by the second-order ordinary differential equations

$$f_k\left(x_k, \frac{1}{\varphi_k}\frac{d\varphi_k}{dx_k}, \frac{1}{\varphi_k}\frac{d^2\varphi_k}{dx_k^2}\right) - a\ln\varphi_k = C_k; \qquad k = 1, \ldots, n.$$

The arbitrary constants $C_1, \ldots, C_n$ are related by a single constraint, $C_1 + \cdots + C_n = 0$.

3. $\displaystyle F\left(x_1, \ldots, x_k; \frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_k}; \frac{\partial^2 w}{\partial x_1^2}, \ldots, \frac{\partial^2 w}{\partial x_k^2}\right)$

$$+ G\left(x_{k+1}, \ldots, x_n; \frac{\partial w}{\partial x_{k+1}}, \ldots, \frac{\partial w}{\partial x_n}; \frac{\partial^2 w}{\partial x_{k+1}^2}, \ldots, \frac{\partial^2 w}{\partial x_n^2}\right) = aw.$$

Additive separable solution:

$$w(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = \varphi(x_1, \ldots, x_k) + \psi(x_{k+1}, \ldots, x_n).$$

Here, the functions $\varphi = \varphi(x_1, \ldots, x_k)$ and $\psi = \psi(x_{k+1}, \ldots, x_n)$ are determined by solving the two simpler partial differential equations

$$F\left(x_1, \ldots, x_k; \frac{\partial\varphi}{\partial x_1}, \ldots, \frac{\partial\varphi}{\partial x_k}; \frac{\partial^2\varphi}{\partial x_1^2}, \ldots, \frac{\partial^2\varphi}{\partial x_k^2}\right) = a\varphi + C,$$

$$G\left(x_{k+1}, \ldots, x_n; \frac{\partial\psi}{\partial x_{k+1}}, \ldots, \frac{\partial\psi}{\partial x_n}; \frac{\partial^2\psi}{\partial x_{k+1}^2}, \ldots, \frac{\partial^2\psi}{\partial x_n^2}\right) = a\psi - C,$$

where C is an arbitrary constant.

4. $\displaystyle F\left(x_1, \ldots, x_k; \frac{1}{w}\frac{\partial w}{\partial x_1}, \ldots, \frac{1}{w}\frac{\partial w}{\partial x_k}; \frac{1}{w}\frac{\partial^2 w}{\partial x_1^2}, \ldots, \frac{1}{w}\frac{\partial^2 w}{\partial x_k^2}\right)$

$$+ G\left(x_{k+1}, \ldots, x_n; \frac{1}{w}\frac{\partial w}{\partial x_{k+1}}, \ldots, \frac{1}{w}\frac{\partial w}{\partial x_n}; \frac{1}{w}\frac{\partial^2 w}{\partial x_{k+1}^2}, \ldots, \frac{1}{w}\frac{\partial^2 w}{\partial x_n^2}\right) = a\ln w.$$

Multiplicative separable solution:

$$w(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = \varphi(x_1, \ldots, x_k)\psi(x_{k+1}, \ldots, x_n).$$

Here, the functions $\varphi = \varphi(x_1, \ldots, x_k)$ and $\psi = \psi(x_{k+1}, \ldots, x_n)$ are determined by solving the two simpler partial differential equations

$$F\left(x_1, \ldots, x_k; \frac{1}{\varphi}\frac{\partial\varphi}{\partial x_1}, \ldots, \frac{1}{\varphi}\frac{\partial\varphi}{\partial x_k}; \frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x_1^2}, \ldots, \frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x_k^2}\right) = a\ln\varphi + C,$$

$$G\left(x_{k+1}, \ldots, x_n; \frac{1}{\psi}\frac{\partial\psi}{\partial x_{k+1}}, \ldots, \frac{1}{\psi}\frac{\partial\psi}{\partial x_n}; \frac{1}{\psi}\frac{\partial^2\psi}{\partial x_{k+1}^2}, \ldots, \frac{1}{\psi}\frac{\partial^2\psi}{\partial x_n^2}\right) = a\ln\psi - C,$$

where C is an arbitrary constant.

$$\textbf{5.}\quad \boldsymbol{F\left(x_1,\dots,x_k;\,\frac{1}{w}\frac{\partial w}{\partial x_1},\dots,\frac{1}{w}\frac{\partial w}{\partial x_k};\,\frac{1}{w}\frac{\partial^2 w}{\partial x_1^2},\dots,\frac{1}{w}\frac{\partial^2 w}{\partial x_k^2}\right)}$$
$$\boldsymbol{{}+w^\beta G\left(x_{k+1},\dots,x_n;\,\frac{1}{w}\frac{\partial w}{\partial x_{k+1}},\dots,\frac{1}{w}\frac{\partial w}{\partial x_n};\,\frac{1}{w}\frac{\partial^2 w}{\partial x_{k+1}^2},\dots,\frac{1}{w}\frac{\partial^2 w}{\partial x_n^2}\right)=0.}$$

Multiplicative separable solution:

$$w(x_1,\dots,x_k,x_{k+1},\dots,x_n)=\varphi(x_1,\dots,x_k)\psi(x_{k+1},\dots,x_n).$$

Here, the functions $\varphi=\varphi(x_1,\dots,x_k)$ and $\psi=\psi(x_{k+1},\dots,x_n)$ are determined by solving the two simpler partial differential equations

$$\varphi^{-\beta}F\left(x_1,\dots,x_k;\,\frac{1}{\varphi}\frac{\partial\varphi}{\partial x_1},\dots,\frac{1}{\varphi}\frac{\partial\varphi}{\partial x_k};\,\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x_1^2},\dots,\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial x_k^2}\right)=C,$$
$$\psi^{\beta}G\left(x_{k+1},\dots,x_n;\,\frac{1}{\psi}\frac{\partial\psi}{\partial x_{k+1}},\dots,\frac{1}{\psi}\frac{\partial\psi}{\partial x_n};\,\frac{1}{\psi}\frac{\partial^2\psi}{\partial x_{k+1}^2},\dots,\frac{1}{\psi}\frac{\partial^2\psi}{\partial x_n^2}\right)=-C,$$

where C is an arbitrary constant.

Chapter 9

Third Order Equations

9.1. Equations Involving the First Derivative in t

9.1.1. Korteweg–de Vries Equation $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + bw\dfrac{\partial w}{\partial x} = 0$

1. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} - 6w\dfrac{\partial w}{\partial x} = 0.$

Korteweg–de Vries equation in canonical form. It is used in many sections of nonlinear mechanics and theoretical physics for describing one-dimensional nonlinear dispersive nondissipative waves (in which the dissipation law for linear waves has the form $\omega = a_1 k + a_3 k^3$, where k is the wavenumber). In particular, the mathematical modeling of moderate-amplitude shallow-water surface waves is based on this equation (see Korteweg and de Vries, 1895).

The Korteweg–de Vries equation is solved by the inverse scattering method; see Items 9 and 10 and the references at the end of this equation.

1. The similarity formula.

Suppose $w(x,t)$ is a solution of the Korteweg–de Vries equation. Then the function

$$w_1 = C_1^2 w(C_1 x + 6C_1 C_2 t + C_3,\, C_1^3 t + C_4) + C_2,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

⊙ *Reference*: P. J. Olver (1986).

2. Traveling-wave solutions. Soliton. Periodic solutions.

2.1. Traveling-wave solution:

$$w = w(z), \quad z = x - vt,$$

where the function $w(z)$ defined in implicit form as

$$\int \frac{dw}{\sqrt{2w^3 + vw^2 + C_1 w + C_2}} = \pm z + C_3. \tag{1}$$

Here, v, C_1, C_2, and C_3 are arbitrary constants; to $v = 0$ there corresponds a stationary solution.

Below are important cases where solution (1) can be written out in explicit form.

2.2. *Soliton.* The unique solution regular for all real values of z and vanishing as $z \to \pm\infty$ is expressed as

$$w(z) = -\frac{v}{2\cosh^2\left[\frac{1}{2}\sqrt{v}\,(z - z_0)\right]}, \tag{2}$$

where z_0 is an arbitrary constant.

2.3. *Cnoidal waves.* There are periodic solutions that are real and regular for any real z:

$$w(z) = A\,\mathrm{cn}^2\left[p(z - z_0), k\right], \quad A = -2p^2k^2, \quad v = 4p^2(2k^2 - 1), \tag{3}$$

They depend on an arbitrary positive constant $k^2 < 1$. Here, $\operatorname{cn}(y,k)$ is the Jacobian elliptic cosine. Solution (2) can be obtained from (3) by letting $k^2 \to 1$. The periods of solution (3) are $\omega_1 = 4K$ and $\omega_2 = 2K + 2iK_*$, where K and K_* are complete elliptic integrals of the first kind:

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad K_* = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k_*^2t^2)}}, \quad k^2 + k_*^2 = 1.$$

2.4. *Rational solution.* It has the form

$$w(z) = \frac{2}{(z-z_0)^2} - \frac{v}{6},$$

where z_0 is an arbitrary constant.

3. Two- and N-soliton solutions.

3.1. Two-soliton solution:

$$w(x,t) = -2\frac{\partial^2}{\partial x^2} \ln\bigl(1 + B_1e^{\theta_1} + B_2e^{\theta_2} + AB_1B_2e^{\theta_1+\theta_2}\bigr),$$
$$\theta_1 = a_1x - a_1^3t, \quad \theta_2 = a_2x - a_2^3t, \quad A = \left(\frac{a_1-a_2}{a_1+a_2}\right)^2,$$

where B_1, B_2, a_1, and a_2 are arbitrary constants.

⊙ *Reference*: R. Hirota (1971, 1972).

3.2. N-soliton solutions:

$$w(x,t) = -2\frac{\partial^2}{\partial x^2}\Bigl\{\ln\det\bigl[\mathbf{I} + \mathbf{C}(\mathbf{x},\mathbf{t})\bigr]\Bigr\}.$$

Here, $\mathbf{I}$ is the $N\times N$ identity matrix and $\mathbf{C}(x,t)$ the $N\times N$ symmetric matrix with entries

$$C_{mn}(x,t) = \frac{\sqrt{\rho_m(t)\rho_n(t)}}{p_m+p_n}\exp\bigl[-(p_m+p_n)x\bigr],$$

where the normalizing factors $\rho_n(t)$ are given by

$$\rho_n(t) = \rho_n(0)\exp\bigl(8p_n^3t\bigr), \quad n = 1, 2, \ldots, N.$$

The solution involves $2N$ arbitrary constants p_n and $\rho_n(0)$.

The following asymptotic formula holds:

$$w(x,t) \approx -2\sum_{n=1}^{N} \frac{p_n^2}{\cosh^2\bigl[p_n\bigl(x-\xi_n^\pm - v_nt\bigr)\bigr]} \quad \text{as} \quad t\to\pm\infty,$$

where $v_n = 4p_n^2$ is the speed of the nth soliton and the real constants $\xi_n^\pm$ are constrained by

$$\xi_n^+ - \xi_n^- = \sum_{m=1}^{n-1} p_n^{-1}\ln\frac{p_n+p_m}{p_n-p_m} - \sum_{m=n+1}^{N} p_n^{-1}\ln\frac{p_n+p_m}{p_n-p_m}.$$

⊙ *Reference*: F. Calogero and A. Degasperis (1982).

4. "Soliton + pole" solutions.

4.1. "One soliton + one pole" solution:

$$w(x,t) = -2p^2\bigl[\cosh^{-2}(pz) - (1+px)^{-2}\tanh^2(pz)\bigr]\bigl[1-(1+px)^{-1}\tanh(pz)\bigr]^{-2}, \qquad z = x - 4p^3t - c,$$

where p and c are arbitrary constants.

4.2. "N solitons + one pole" solution:

$$w(x,t) = -2\frac{\partial^2}{\partial x^2}\Big\{x \ln\det\big[\mathbf{I}+\mathbf{D}(\mathbf{x},\mathbf{t})\big]\Big\}.$$

Here, $\mathbf{I}$ is the $N\times N$ identity matrix and $\mathbf{D}(x,t)$ the $N\times N$ symmetric matrix with entries

$$D_{mn}(x,t) = c_m(t)c_n(t)\big[(p_m+p_n)^{-1} + (p_m p_n x)^{-1}\big]\exp\big[-(p_m+p_n)x\big],$$

where the normalizing factors $c_n(t)$ are given by

$$c_n(t) = c_n(0)\exp\big(4p_n^3 t\big), \quad n = 1, 2, \ldots, N.$$

The solution involves $2N$ arbitrary constants p_n and $c_n(0)$.

⊙ *Reference*: F. Calogero and A. Degasperis (1982).

5. Rational solutions.

5.1. The simplest rational solution is as follows:

$$w(x,t) = 2(x-\xi)^{-2},$$

where ξ is an arbitrary constant that can be complex (if it is real, the solution is singular for real values of x).

5.2. General form of a rational solution:

$$w(x,t) = 2\sum_{j=1}^{N}\big[x-\xi_j(t)\big]^{-2}. \tag{4}$$

The functions $\xi_j(t)$ must meet the conditions

$$\begin{aligned} &\sum_{\substack{k=1\\ j\neq k}}^{N}\big[\xi_j(t)-\xi_k(t)\big]^{-3} = 0, && j = 1, 2, \ldots, N,\\ &\xi_j(t) = -12\sum_{\substack{k=1\\ j\neq k}}^{N}\big[\xi_j(t)-\xi_k(t)\big]^{-2}, && j = 1, 2, \ldots, N. \end{aligned} \tag{5}$$

A solution exists if $N = \frac{1}{2}m(m+1)$, $m = 1, 2, 3, \ldots$; if $m > 1$, there are no real solutions. In particular, if $m = 2$, there are three poles $\xi_j(t) = -e^{2\pi ij/3}(12t)^{1/3}$ ($j = 1, 2, 3$) and solution (4) can be written out as follows:

$$w(x,t) = \frac{6x(x^3-24t)}{(x^3+12t)^2} \qquad (\text{for } N = 3).$$

A solution for $m = 3$ (corresponds to $N = 6$) is given by

$$w(x,t) = -2\frac{\partial^2}{\partial x^2}\big(x^6 + 60x^3 t - 720t^2\big).$$

Note that (4) can be rewritten in the equivalent form

$$w(x,t) = -2\frac{\partial^2}{\partial x^2}\big[\ln P_N(x,t)\big], \quad \text{where} \quad P_N(x,t) = \prod_{j=1}^{N}\big[x-\xi_j(t)\big].$$

⊙ *References*: M. J. Ablowitz and H. Segur (1981), F. Calogero and A. Degasperis (1982).

6. Self-similar solutions.

6.1. Simplest self-similar solution (degenerate solution):

$$w(x,t) = -\frac{1}{6}\frac{x-x_0}{t-t_0}.$$

where x_0 and t_0 are arbitrary constants.

6.2. Self-similar solution:

$$w(x,t) = \bigl[3(t-t_0)\bigr]^{-2/3} f(y), \quad y = \bigl[3(t-t_0)\bigr]^{-1/3}(x-x_0),$$

where the function $f(y)$ is determined by the third-order ordinary differential equation

$$f'''_{yyy} - yf'_y - 2f - 6ff'_y = 0. \tag{6}$$

Equation (6) has a first integral

$$(y+2f)[f''_{yy} - (y+2f)f] - (1+f'_y)f'_y = C,$$

where C is the constant of integration. A solution of equation (6) can be represented as

$$f(y) = g'_y(y) + g^2(y),$$

where the function $g(y)$ is any solution of the second Painlevé equation

$$g''_{yy} - 2g^3 - yg = A, \qquad A \text{ is an arbitrary constant.} \tag{7}$$

For $A = 2^{-2/3}$, equation (7) has a solution

$$g(y) = \frac{d}{dy}\Bigl[\ln F\bigl(-2^{-1/3}y\bigr)\Bigr],$$

where the function $F = F(z)$ satisfies the Airy equation $F''_{zz} = zF$.

⊙ *Reference*: F. Calogero and A. Degasperis (1982).

7. General similarity solutions.

7.1. Solution:

$$w(x,t) = 2\varphi(z) + 2C_1 t, \quad z = x + 6C_1t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants, and the function $\varphi(z)$ is determined by the second-order ordinary differential equation

$$\varphi''_{zz} = 6\varphi^2 - C_2\varphi - C_1 z + C_3,$$

where C_3 is an arbitrary constant. To the case $C_1 = -1$, $C_2 = C_3 = 0$ there corresponds the first Painlevé equation (if all C_n are nonzero, the equation for φ can also be reduced to the first Painlevé equation).

7.2. Solution:

$$w = \varphi^2 F(z) + \frac{1}{6\varphi}(\varphi'_t x + \psi'_t), \quad z = \varphi(t)x + \psi(t).$$

Here, the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are given by

$$\varphi(t) = (3At + C_1)^{-1/3}, \quad \psi(t) = C_2(3At + C_1)^{2/3} + C_3(3At + C_1)^{-1/3},$$

where A, C_1, C_2, and C_3 are arbitrary constants, and the function $F(z)$ is determined by the ordinary differential equation

$$F'''_{zzz} - 6FF'_z - AF + \tfrac{2}{3}A^2 z = 0.$$

⊙ *Reference*: P. A. Clarkson and M. D. Kruskal (1989).

8. Miura transformation and Bäcklund transformations.

8.1. The Korteweg–de Vries equation can be reduced, with the differential change of variable (Miura transformation)

$$w = \frac{\partial u}{\partial x} + u^2, \tag{8}$$

to the form

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6w\frac{\partial w}{\partial x} = \left(\frac{\partial}{\partial x} + 2u\right)\left(\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u^2\frac{\partial u}{\partial x}\right) = 0.$$

It follows that any solution $u = u(x,t)$ of the modified Korteweg–de Vries equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u^2\frac{\partial u}{\partial x} = 0 \tag{9}$$

generates a solution (8) of the Korteweg–de Vries equation.

⊙ *References*: R. M. Miura (1968), F. Calogero and A. Degasperis (1982).

8.2. The Bäcklund transformations

$$\begin{aligned} \frac{\partial u}{\partial x} &= \varepsilon(u^2 - w), \qquad \varepsilon = \pm 1, \\ \frac{\partial u}{\partial t} &= -\varepsilon\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial}{\partial x}(uw) \end{aligned} \tag{10}$$

link solutions of the Korteweg–de Vries equation with those of the modified Korteweg–de Vries equation (9). With $\varepsilon = 1$, the first relation in (10) turns into the Miura transformation (8).

⊙ *References*: G. L. Lamb (1974), N. H. Ibragimov (1985).

8.3. The auto-Bäcklund transformation expressed via the potential functions $\frac{\partial\varphi}{\partial x} = -\frac{1}{2}w$ and $\frac{\partial\widetilde{\varphi}}{\partial x} = -\frac{1}{2}\widetilde{w}$ has the form

$$\begin{aligned} &\frac{\partial}{\partial x}(\widetilde{\varphi} - \varphi) = k^2 - (\widetilde{\varphi} - \varphi)^2, \\ &\frac{\partial}{\partial t}(\widetilde{\varphi} - \varphi) = 6(\widetilde{\varphi} - \varphi)^2\frac{\partial}{\partial x}(\widetilde{\varphi} - \varphi) - 6k^2\frac{\partial}{\partial x}(\widetilde{\varphi} - \varphi) - \frac{\partial^3}{\partial x^3}(\widetilde{\varphi} - \varphi), \end{aligned}$$

where k is an arbitrary constant.

⊙ *Reference*: R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris (1982).

9. Gel'fand–Levitan–Marchenko integral equation.

Any rapidly decreasing function $F = F(x,y;t)$ as $x \to +\infty$ that satisfies simultaneously the two linear equations

$$\begin{gathered} \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} = 0, \\ \frac{\partial F}{\partial t} + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^3 F = 0 \end{gathered}$$

generates a solution of the Korteweg–de Vries equation in the form

$$w = -2\frac{d}{dx}K(x,x;t), \tag{11}$$

where $K(x,y;t)$ is a solution of the linear Gel'fand–Levitan–Marchenko integral equation

$$K(x,y;t) + F(x,y;t) + \int_x^\infty K(x,z;t)F(z,y;t)\,dz = 0. \tag{12}$$

Time t appears in this equation as a parameter.

10. Cauchy problem.

Consider the Cauchy problem for the Korteweg–de Vries equation subject to the initial conditions

$$w = f(x) \quad \text{at} \quad t = 0 \qquad (-\infty < x < \infty), \tag{13}$$

where the function $f(x)$ is quite rapidly vanishing as $|x| \to \infty$. The solution of the Cauchy problem falls into several stages.

First stage. Initially, a linear eigenvalue problem is solved for the auxiliary ordinary differential equation

$$\psi''_{xx} - \bigl[f(x) - \lambda\bigr]\psi = 0. \tag{14}$$

The eigenvalues fall into two types:

$$\begin{aligned} \lambda_n &= -\varkappa_n^2, \quad n = 1,\, 2,\, \dots,\, N \quad &&\text{(discrete spectrum)},\\ \lambda &= k^2, \qquad -\infty < k < \infty &&\text{(continuous spectrum)}. \end{aligned} \tag{15}$$

Let the $\lambda_n = -\varkappa_n^2$ be discrete eigenvalues and let the $\psi_n = \psi_n(x)$ be the corresponding normalized eigenfunctions, which vanish at infinity and are square summable, so that

$$\int_{-\infty}^{\infty} \psi_n^2(x)\, dx = 1.$$

The leading asymptotic term in the expansion of ψ_n for large x is given by

$$\psi_n \to c_n \exp(-\varkappa x) \quad \text{as} \quad x \to \infty. \tag{16}$$

For continuous spectrum, $\lambda = k^2$, the wave function ψ at infinity is determined by a linear combination of the exponentials $\exp(\pm ikx)$ (since $f \to 0$ as $|x| \to \infty$). The conditions

$$\begin{aligned} \psi &\to e^{-ikx} + b(k)e^{ikx} \quad &&\text{as} \quad x \to \infty,\\ \psi &\to a(k)e^{-ikx} &&\text{as} \quad x \to -\infty, \end{aligned} \tag{17}$$

and equation (14) enable us to uniquely determine the transmission and reflection coefficients $a(k)$ and $b(k)$; note that $|a|^2 + |b|^2 = 1$.

Second stage. At the next stage, one considers the linear Gel'fand–Levitan–Marchenko integral equation (12), where

$$F(x, y; t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)e^{i[8k^3 t - k(x+y)]}\, dk + \sum_{n=1}^{N} c_n^2 e^{8\varkappa_n^3 t - \varkappa_n (x+y)}. \tag{18}$$

It involves the constants $\varkappa_n$ and c_n and the function $b(k)$ determined at the first state; see (15)–(17). It is apparent from (18) that $F(x, y; t) = F(x + y; t)$.

Third stage. The solution of the integral equation (12), (18) is finally substituted into (11) to give a solution of the Cauchy problem for the Korteweg–de Vries equation with the initial condition (13).

Remark. Solving the Cauchy problem for the given nonlinear equation is reduced to solving two linear problems sequentially.

⊙ *References*: R. M. Miura (1977), F. Calogero and A. Degasperis (1982), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984).

11. Conservation laws and motion integrals.

11.1. The Korteweg–de Vries equation has infinitely many conservation laws. The simplest conservation laws are expressed as

$$\begin{aligned} &D_t(w) + D_x\bigl(w_{xx} - 3w^2\bigr) = 0,\\ &D_t\bigl(\tfrac{1}{2}w^2\bigr) + D_x\bigl(ww_{xx} - \tfrac{1}{2}w_x^2 - 2w^3\bigr) = 0,\\ &D_t\bigl(w_x^2 + 2w^3\bigr) + D_x\bigl(2w_x w_{xxx} - w_{xx}^2 + 6w^2 w_{xx} - 12ww_x^2 - 9w^4\bigr) = 0,\\ &D_t\bigl(3tw^2 + xw\bigr) + D_x\bigl[t\bigl(6ww_{xx} - 3w_x^2 - 12w^3\bigr) - w_x + xw_{xx} - 3xw^2\bigr] = 0, \end{aligned}$$

where $D_t = \frac{\partial}{\partial t}$, $D_x = \frac{\partial}{\partial x}$.

⊙ *References*: G. B. Whitham (1965), R. M. Miura, C. S. Gardner, and M. D. Kruskal (1968), N. H. Ibragimov (1994).

11.2. The Korteweg–de Vries equation has infinitely many motion integrals:

$$I_n = \int_{-\infty}^{\infty} P_n(w, w_x, \dots)\,dx = \text{const}, \qquad n = 0,\ 1,\ 2,\ \dots,$$

where P_n is a polynomial in w and its derivatives; w is assumed to decay rapidly as $|x| \to \infty$. In particular, the first four polynomials are as follows:

$$P_0 = w, \quad P_1 = w^2, \quad P_2 = w_x^2 + 2w^3, \quad P_3 = \tfrac{1}{2}\bigl(w_{xx}^2 - 5w^2 w_{xx} + 5w^4\bigr).$$

⊙ *References for equation* 9.1.1.1: C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura (1967, 1974), R. M. Miura (1968), P. D. Lax (1968), R. Hirota (1971, 1972), V. E. Zakharov and L. D. Faddeev (1971), C. P. Novikov (1974), R. K. Bullough and P. J. Caudrey (1980), G. L. Lamb (1980), M. J. Ablowitz and H. Segur (1981), F. Calogero and A. Degasperis (1982), R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris (1982), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), G. W. Bluman and S. Kumei (1989), M. J. Ablowitz and P. A. Clarkson (1991), R. S. Palais (1997).

2. $\displaystyle \frac{\partial w}{\partial t} + a\frac{\partial^3 w}{\partial x^3} + bw\frac{\partial w}{\partial x} = 0.$

Unnormalized Korteweg–de Vries equation.

1°. Suppose $w(x,t)$ is a solution of the Korteweg–de Vries equation. Then the function

$$w_1 = C_1^2 w(C_1 x - bC_1C_2 t + C_3,\, C_1^3 t + C_4) + C_2,$$

where $C_1, \dots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Consider the Cauchy problem with the initial condition

$$w = f(x) \quad \text{at} \quad t = 0 \qquad (-\infty < x < \infty),$$

where the function $f(x)$ decays quite rapidly as $|x| \to \infty$.

The asymptotic solution as $t \to \infty$ (for sufficiently large x) is the sum of solitons

$$w(x,t) = 2\sum_{n=1}^{N} |\lambda_n| \cosh^{-2}\left[\sqrt{\frac{b|\lambda_n|}{6a}}\bigl(x - \tfrac{2}{3}b|\lambda_n|t + c_n\bigr)\right],$$

where the λ_n are discrete eigenvalues of the linear Schrödinger equation

$$\Psi''_{xx} + \frac{b}{6a}\bigl[\lambda + f(x)\bigr]\Psi = 0, \qquad \Psi(\pm\infty) = 0.$$

⊙ *References*: C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura (1974), G. I. Barenblatt (1989).

3°. The transformation $w(x,t) = -\dfrac{6a}{b}u(x,\tau)$, $\tau = at$ leads to the Korteweg–de Vries equation in canonical form 9.1.1.1:

$$\frac{\partial u}{\partial \tau} + \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} = 0.$$

9.1.2. Cylindrical, Spherical, and Modified Korteweg–de Vries Equations

1. $\displaystyle \frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6w\frac{\partial w}{\partial x} + \frac{1}{2t}w = 0.$

Cylindrical Korteweg–de Vries equation. This is a special case of equation 9.1.2.3 for $a = 1$, $b = -6$, and $k = \frac{1}{2}$.

The transformation

$$w(x,t) = -\frac{x}{12t} - \frac{1}{2t}u(z,\tau), \quad x = \frac{z}{\tau}, \quad t = -\frac{1}{2\tau^2}$$

leads to the Korteweg–de Vries equation in canonical form 9.1.1.1:

$$\frac{\partial u}{\partial \tau} + \frac{\partial^3 u}{\partial z^3} - 6u\frac{\partial u}{\partial z} = 0.$$

⊙ *References*: R. S. Johnson (1979), F. Calogero and A. Degasperis (1982), G. W. Bluman and S. Kumei (1989).

2. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} - 6w\dfrac{\partial w}{\partial x} + \dfrac{1}{t}w = 0.$

Spherical Korteweg–de Vries equation. This is a special case of equation 9.1.2.3 for $a = 1$, $b = -6$, and $k = 1$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^2 w\bigl(C_1 x + 6C_1C_2 \ln|t| + C_3,\ C_1^3 t\bigr) + C_2 t^{-1},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution linear in x:

$$w(x,t) = \frac{C_1 - x}{t(C_2 + 6\ln|t|)}.$$

3°. Self-similar solution:

$$w(x,t) = t^{-2/3}u(z), \qquad z = xt^{-1/3},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$au'''_{zzz} + buu'_z - \tfrac{1}{3}zu'_z + \tfrac{1}{3}u = 0.$$

3. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + bw\dfrac{\partial w}{\partial x} + \dfrac{k}{t}w = 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^2 w\left(C_1 x - \frac{bC_1C_2}{1-k}t^{1-k} + C_3,\ C_1^3 t\right) + C_2 t^{-k},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution linear in x:

$$w(x,t) = \frac{(1-k)x + C_1}{C_2 t^k + bt}.$$

3°. Self-similar solution:

$$w(x,t) = t^{-2/3}u(z), \qquad z = xt^{-1/3},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$au'''_{zzz} + buu'_z - \tfrac{1}{3}zu'_z + \bigl(k - \tfrac{2}{3}\bigr)u = 0.$$

4. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} - 6w^2\dfrac{\partial w}{\partial x} = 0.$

Modified Korteweg–de Vries equation.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_1 x + C_2, C_1^3 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Self-similar solution (x_0 and t_0 are arbitrary constants):

$$w(x,t) = \bigl[3(t-t_0)\bigr]^{-1/3} f(y), \qquad y = \bigl[3(t-t_0)\bigr]^{-1/3}(x - x_0),$$

where the function $f(y)$ is determined by the third-order ordinary differential equation

$$f'''_{yyy} - yf'_y - f - 6f^2 f'_y = 0.$$

Integrating yields the second Painlevé equation (a is an arbitrary constant):

$$f''_{yy} - 2f^3 - yf = a.$$

3°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function $u(x,t)$ obtained with the Miura transformation

$$u(x,t) = \frac{\partial w}{\partial x} + w^2 \tag{1}$$

satisfies the Korteweg–de Vries equation 9.1.1.1:

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} = 0. \tag{2}$$

In general, the converse is not true: if $u(x,t)$ is a solution of the Korteweg–de Vries equation (2), the function $w(x,t)$ linked to it with the Miura transformation (1) satisfies the nonlinear integro-differential equation

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6w^2\frac{\partial w}{\partial x} = c(t)\exp\left[-2\int w(x,t)\,dx\right].$$

4°. Solutions of the modified Korteweg–de Vries equation

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6\sigma w^2\frac{\partial w}{\partial x} = 0, \qquad \sigma = \pm 1 \tag{3}$$

may be obtained from solutions of the linear Gel'fand–Levitan–Marchenko integral equation. Any function $F = F(x,y;t)$ rapidly decaying as $x \to +\infty$ and satisfying simultaneously the two linear equations

$$\begin{aligned} \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} &= 0, \\ \frac{\partial F}{\partial t} + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^3 F &= 0, \end{aligned} \tag{4}$$

generates a solution of equation (3) in the form

$$w = K(x,x;t),$$

where $K(x,y;t)$ is a solution of the linear Gel'fand–Levitan–Marchenko integral equation,

$$K(x,y;t) = F(x,y;t) + \frac{\sigma}{4}\int_x^\infty\int_x^\infty K(x,z;t)F(z,u;t)F(u,y;t)\,dz\,du. \tag{5}$$

Time t appears in (5) as a parameter. It follows from the first equation in (4) that $F(x,y;t) = F(x+y;t)$.

⊙ *References*: M. J. Ablowitz and H. Segur (1981), F. Calogero and A. Degasperis (1982).

5°. Conservation laws:

$$\begin{aligned} &D_t(w) + D_x\left(w_{xx} - 2w^3\right) = 0, \\ &D_t\left(\tfrac{1}{2}w^2\right) + D_x\left(ww_{xx} - \tfrac{1}{2}w_x^2 - \tfrac{3}{2}w^4\right) = 0, \end{aligned}$$

where $D_t = \frac{\partial}{\partial t}$ and $D_x = \frac{\partial}{\partial x}$.

⊙ *References*: G. B. Whitham (1965), R. M. Miura, C. S. Gardner, and M. D. Kruskal (1968).

5. $\displaystyle \frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} + 6w^2\frac{\partial w}{\partial x} = 0.$

Modified Korteweg–de Vries equation.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_1 x + C_2, C_1^3 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. One-soliton solution:

$$w(x,t) = a + \frac{k^2}{\sqrt{4a^2+k^2}\,\cosh z + 2a}, \qquad z = kx - (6a^2k + k^3)t + b,$$

where a, b, and k are arbitrary constants.

⦿ *Reference*: M. J. Ablowitz and H. Segur (1981).

3°. Two-soliton solution:

$$w(x,t) = 2\frac{a_1e^{\theta_1} + a_2e^{\theta_2} + Aa_2e^{2\theta_1+\theta_2} + Aa_1e^{\theta_1+2\theta_2}}{1 + e^{2\theta_1} + e^{2\theta_2} + 2(1-A)e^{\theta_1+\theta_2} + Ae^{2(\theta_1+\theta_2)}},$$

$$\theta_1 = a_1 - a_1^3t + b_1, \quad \theta_2 = a_2 - a_2^3t + b_2, \quad A = \left(\frac{a_1-a_2}{a_1+a_2}\right)^2,$$

where a_1, a_2, b_1, and b_2 are arbitrary constants.

⦿ *Reference*: R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris (1982).

4°. Rational solutions (algebraic solitons):

$$w(x,t) = a - \frac{4a}{4a^2z^2+1}, \qquad z = x - 6a^2t,$$

$$w(x,t) = a - \frac{12a\left(z^4 + \frac{3}{2}a^{-2}z^2 - \frac{3}{16}a^{-4} - 24tz\right)}{4a^2\left(z^3 + 12t - \frac{3}{4}a^{-2}z\right)^2 + 3\left(z^2 + \frac{1}{4}a^{-2}\right)^2},$$

where a is an arbitrary constant.

⦿ *References*: H. Ono (1976), M. J. Ablowitz and H. Segur (1981).

9.1.3. Generalized Korteweg–de Vries Equation $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + f(w)\dfrac{\partial w}{\partial x} = 0$

Preliminary remarks. For $f(w) = bw$, see equations 9.1.1.1 and 9.1.1.2; for $f(w) = bw^2$, see equations 9.1.2.4 and 9.1.2.5.

1°. Equations of this form admit traveling-wave solutions

$$w = w(z), \qquad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $w(z)$ is determined by the second-order autonomous ordinary differential equation (C is an arbitrary constant)

$$\alpha k^3 w''_{zz} + k\int f(w)\,dw - \lambda w = C.$$

2°. Conservation laws:

$$D_t\bigl(w\bigr) + D_x\bigl[\alpha w_{xx} + F_0(w)\bigr] = 0,$$

$$D_t\bigl(\tfrac{1}{2}w^2\bigr) + D_x\bigl[\alpha w w_{xx} - \tfrac{1}{2}\alpha w_x^2 + F_1(w)\bigr] = 0,$$

where

$$D_t = \frac{\partial}{\partial t}, \quad D_x = \frac{\partial}{\partial x}, \quad F_0(w) = \int f(w)\,dw, \quad F_1(w) = \int wf(w)\,dw.$$

1. $\dfrac{\partial w}{\partial t}+\dfrac{\partial^3 w}{\partial x^3}+aw^k\dfrac{\partial w}{\partial x}=0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1=C_1^{2/k}w(C_1x+C_2,C_1^3t+C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution (soliton):

$$w(x,t)=\frac{A}{\cosh^{2/k}\bigl[Bk(x-4B^2t-C)\bigr]},$$

where B and C are arbitrary constants and $A=\bigl[2(k+1)(k+2)B^2/a\bigr]^{1/k}$.

3°. Self-similar solution:

$$w(x,t)=t^{-\frac{2}{3k}}U(z),\qquad z=xt^{-\frac{1}{3}},$$

where the function $U=U(z)$ is determined by the ordinary differential equation

$$-\frac{2}{3k}U-\frac{1}{3}zU'_z+U'''_{zzz}+aU^kU'_z=0.$$

4°. Conservation laws:

$$D_tw+D_x\left(w_{xx}+\frac{a}{k+1}w^{k+1}\right)=0,$$

$$D_t\bigl(w^2\bigr)+D_x\left(2ww_{xx}-w_x^2+\frac{2a}{k+2}w^{k+2}\right)=0.$$

⊙ *Reference*: M. J. Ablowitz and H. Segur (1981).

2. $\dfrac{\partial w}{\partial t}+\dfrac{\partial^3 w}{\partial x^3}+ae^w\dfrac{\partial w}{\partial x}=0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1=w(C_1x+C_2,C_1^3t+C_3)+2\ln|C_1|,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w=w(z),\qquad z=x+\lambda t,$$

where the function $w(z)$ is determined by the second-order autonomous ordinary differential equation

$$w''_{zz}+\lambda w+ae^w=C,$$

and λ and C are arbitrary constants.

3°. Solution:

$$w(x,t)=U(\xi)-\tfrac{2}{3}\ln t,\qquad \xi=xt^{-\frac{1}{3}},$$

where the function $U=U(\xi)$ is determined by the ordinary differential equation

$$U'''_{\xi\xi\xi}+\bigl(ae^U-\tfrac{1}{3}\xi\bigr)U'_\xi-\tfrac{2}{3}=0.$$

3. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial^3 w}{\partial x^3}+(b\ln w+c)\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t)=\exp\left[\frac{C_2-x}{bt+C_1}+\frac{a}{b}\frac{1}{(bt+C_1)^2}-\frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

⊙ *References*: W. I. Fushchich, N. I. Serov, and T. K. Akhmerov (1991), V. A. Galaktionov (1999).

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + (b \operatorname{arcsinh} w + c)\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \sinh\left[\frac{C_2 - x}{bt + C_1} + \frac{a}{b}\frac{1}{(bt + C_1)^2} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: W. I. Fushchich, N. I. Serov, and T. K. Akhmerov (1991).

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + (b \operatorname{arccosh} w + c)\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \cosh\left[\frac{C_2 - x}{bt + C_1} + \frac{a}{b}\frac{1}{(bt + C_1)^2} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + (b \arcsin w + c)\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \sin\left[\frac{C_2 - x}{bt + C_1} - \frac{a}{b}\frac{1}{(bt + C_1)^2} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: W. I. Fushchich, N. I. Serov, and T. K. Akhmerov (1991).

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + (b \arccos w + c)\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \cos\left[\frac{C_2 - x}{bt + C_1} - \frac{a}{b}\frac{1}{(bt + C_1)^2} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

9.1.4. Equations Reducible to the Korteweg–de Vries Equation

1. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + bw\dfrac{\partial w}{\partial x} = f(t).$

The transformation

$$w = u(z,t) + \int_{t_0}^{t} f(\tau)\,d\tau, \qquad z = x - b\int_{t_0}^{t} (t-\tau) f(\tau)\,d\tau,$$

where t_0 is any, leads to an equation of the form 9.1.1.2:

$$\frac{\partial u}{\partial t} + a\frac{\partial^3 u}{\partial x^3} + bu\frac{\partial u}{\partial x} = 0.$$

2. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + \big[bw^2 + f(t)\big]\dfrac{\partial w}{\partial x} = 0.$

The transformation

$$w = ku(z,t), \qquad z = a^{-1/3}x - a^{-1/3}\int f(t)\,dt, \qquad k = \sqrt{|6a^{1/3}b^{-1}|},$$

leads to an equation of the form 9.1.2.4 or 9.1.2.5:

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial z^3} + 6\,\mathrm{sign}(ab)\,u^2\frac{\partial u}{\partial z} = 0.$$

3. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} - a\left(\dfrac{\partial w}{\partial x}\right)^2 = 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(C_1 x + 2aC_1C_2 t + C_3,\, C_1^3 t + C_4) + C_2 x + aC_2^2 t + C_5,$$

where $C_1, \dots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. The Bäcklund transformation

$$\frac{\partial w}{\partial x} = \frac{3}{a}u, \qquad \frac{\partial w}{\partial t} = -\frac{3}{a}\frac{\partial^2 u}{\partial x^2} + \frac{9}{a}u^2 \tag{1}$$

links the equation in question with the Korteweg–de Vries equation 9.1.1.1:

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} = 0. \tag{2}$$

Let $u = u(x,t)$ be a solution of equation (2). Then the linear system of first-order equations (1) enables us to find the corresponding solution $w = w(x,t)$ of the original equation.

⦿ *Reference*: N. H. Ibragimov (1985).

4. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} - a\left(\dfrac{\partial w}{\partial x}\right)^2 = f(t).$

The substitution $w = u(x,t) + \displaystyle\int f(t)\,dt$ leads to an equation of the form 9.1.4.3:

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - a\left(\frac{\partial u}{\partial x}\right)^2 = 0.$$

5. $\dfrac{\partial w}{\partial t} + \dfrac{\partial^3 w}{\partial x^3} - a\left(\dfrac{\partial w}{\partial x}\right)^3 = 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the functions

$$w_1 = \pm w(C_1 x + C_2,\, C_1^3 t + C_3) + C_4,$$

where $C_1, \dots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. The Bäcklund transformation

$$\frac{\partial w}{\partial x} = bu, \qquad \frac{\partial w}{\partial t} = -b\frac{\partial^2 u}{\partial x^2} + 2bu^3, \qquad \text{where} \quad b = \pm\sqrt{2/a}, \tag{1}$$

links the equation in question with the modified Korteweg–de Vries equation 9.1.2.4:

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} - 6u^2\frac{\partial u}{\partial x} = 0. \tag{2}$$

Let $u = u(x,t)$ be a solution of equation (2). Then the linear system of first-order equations (1) enables us to find the corresponding solution $w = w(x,t)$ of the original equation.

⦿ *Reference*: N. H. Ibragimov (1985).

6. $\dfrac{\partial w}{\partial t}+\dfrac{\partial^3 w}{\partial x^3}-a\left(\dfrac{\partial w}{\partial x}\right)^3=f(t).$

The substitution $w=u(x,t)+\int f(t)\,dt$ leads to an equation of the form 9.1.4.5:

$$\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}-a\left(\frac{\partial u}{\partial x}\right)^3=0.$$

7. $\dfrac{\partial w}{\partial t}=\dfrac{\partial^3 w}{\partial x^3}-\dfrac{1}{8}\left(\dfrac{\partial w}{\partial x}\right)^3-\left(ae^{w}+be^{-w}\right)\dfrac{\partial w}{\partial x}.$

Solutions can be found from the first-order equation

$$\frac{\partial w}{\partial x}-\frac{4}{\sqrt{6}}\left(\sqrt{a}\,e^{w/2}+\sqrt{b}\,e^{-w/2}\right)=4u, \tag{1}$$

where the function $u=u(x,t)$ satisfies

$$\frac{\partial u}{\partial t}=\frac{\partial^3 u}{\partial x^3}+\left(\lambda-6u^2\right)\frac{\partial u}{\partial x},\qquad \lambda=-2\sqrt{ab}. \tag{2}$$

Equation (1) can be treated as an ordinary differential equation for x with parameter t. In the special cases $a=0$ or $b=0$, equation (2) coincides with the modified Korteweg–de Vries equation 9.1.2.4.

⊙ *Reference*: N. H. Ibragimov (1985).

8. $\dfrac{\partial w}{\partial t}=w^3\dfrac{\partial^3 w}{\partial x^3}.$

Harry Dym equation.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1=C_1w(C_2x+C_3,\,C_1^3C_2^3t+C_4),$$

where $C_1,\,\dots,\,C_4$ are arbitrary constants, is also a solution of the equation.

2°. There are solutions of the following forms:

$$\begin{aligned}
&w=U(z), && z=x+\lambda t && \Longrightarrow\quad U''_{zz}+\tfrac{1}{2}\lambda U^{-2}=C;\\
&w=t^{-\lambda-1/3}U(z), && z=xt^{\lambda} && \Longrightarrow\quad U^3U'''_{zzz}-\lambda zU'_z+\left(\lambda+\tfrac{1}{3}\right)U=0;\\
&w=e^{-\lambda t}U(z) && z=xe^{\lambda t} && \Longrightarrow\quad U^3U'''_{zzz}-\lambda zU'_z+\lambda U=0;\\
&w=t^{-1/3}U(z), && z=x+\lambda\ln|t| && \Longrightarrow\quad U^3U'''_{zzz}-\lambda U'_z+\tfrac{1}{3}U=0;
\end{aligned}$$

where λ and C are arbitrary constants. The first solution represents a traveling wave and the second one is a self-similar solution.

3°. We now show that the equation in question is connected with the Korteweg–de Vries equation

$$\frac{\partial u}{\partial t}=\frac{\partial^3 u}{\partial y^3}+u\frac{\partial u}{\partial y}. \tag{1}$$

The substitution

$$u=3\left(\frac{\partial v}{\partial y}\right)^{-1}\frac{\partial^3 v}{\partial y^3}-\frac{3}{2}\left(\frac{\partial v}{\partial y}\right)^{-2}\left(\frac{\partial^2 v}{\partial y^2}\right)^2$$

brings (1) to the form

$$\frac{\partial v}{\partial t}=\frac{\partial^3 v}{\partial y^3}-\frac{3}{2}\left(\frac{\partial v}{\partial y}\right)^{-1}\left(\frac{\partial^2 v}{\partial y^2}\right)^2. \tag{2}$$

Differentiating (2) with respect to y yields

$$\frac{\partial^2 v}{\partial y\partial t}=\frac{\partial^4 v}{\partial y^4}-3\left(\frac{\partial v}{\partial y}\right)^{-1}\frac{\partial^2 v}{\partial y^2}\frac{\partial^3 v}{\partial y^3}+\frac{3}{2}\left(\frac{\partial v}{\partial y}\right)^{-2}\left(\frac{\partial^2 v}{\partial y^2}\right)^3.$$

The transformation $x=v$, $w=\frac{\partial v}{\partial y}$ leads to the original equation.

⊙ *Reference*: N. H. Ibragimov (1985).

9.1.5. Equations of the Form $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + f\Big(w, \dfrac{\partial w}{\partial x}\Big) = 0$

► *For $f(w,u) = bu^2$ and $f(w,u) = bu^3$, see equations 9.1.4.3 and 9.1.4.5, respectively. Equations of this form admit traveling-wave solutions, $w = w(kx + \lambda t)$.*

1. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + bw\dfrac{\partial w}{\partial x} + cw = 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w\big(x + bC_1 e^{-ct} + C_2,\ t + C_3\big) + cC_1 e^{-ct},$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = U(z) + C_1 e^{-ct}, \qquad z = x + bC_1 e^{-ct} + C_2 t,$$

where C_1 and C_2 are arbitrary constants, and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$aU'''_{zzz} + (bU + C_2)U'_z + cU = 0.$$

To the special case $C_1 = 0$ there corresponds a traveling-wave solution.

2. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^3 w}{\partial x^3} + a\Big(\dfrac{\partial w}{\partial x}\Big)^2 + bw^2.$

1°. Generalized separable solutions for $ab < 0$:

$$w(x,t) = \frac{C_2}{(t+C_1)^2}\exp(\lambda x + \lambda^3 t) - \frac{1}{b(t+C_1)}, \qquad \lambda = \pm\sqrt{-\frac{b}{a}},$$

where C_1 and C_2 are arbitrary constants.

2°. Generalized separable solution for $ab < 0$:

$$w(x,t) = \frac{1}{2}\left(\frac{1}{bt+C_1} - \frac{1}{bt+C_2}\right)\cosh(\lambda x + \lambda^3 t + C_3) - \frac{1}{2}\left(\frac{1}{bt+C_1} + \frac{1}{bt+C_2}\right), \qquad \lambda = \sqrt{-\frac{b}{a}},$$

where C_1, C_2, and C_3 are arbitrary constants.

3°. Generalized separable solution for $ab > 0$:

$$w(x,t) = \frac{1}{2}\left(\frac{1}{bt+C_1} - \frac{1}{bt+C_2}\right)\sin(\lambda x - \lambda^3 t + C_3) - \frac{1}{2}\left(\frac{1}{bt+C_1} + \frac{1}{bt+C_2}\right), \qquad \lambda = \sqrt{\frac{b}{a}},$$

where C_1, C_2, and C_3 are arbitrary constants.

⊙ *References*: V. A. Galaktionov and S. A. Posashkov (1989), A. D. Polyanin and V. F. Zaitsev (2002).

3. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^3 w}{\partial x^3} + bw\Big(\dfrac{\partial w}{\partial x}\Big)^k = 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{3-k} w(C_1^k x + C_2, C_1^{3k} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution linear in x:

$$w = x(kbt)^{-1/k} + Ct^{-1/k}.$$

3°. Self-similar solution:

$$w = t^{\frac{k-3}{3k}} U(z), \qquad z = xt^{-\frac{1}{3}},$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$\frac{k-3}{3k}U - \frac{1}{3}zU'_z + bU(U'_z)^k + aU'''_{zzz} = 0.$$

4. $\dfrac{\partial w}{\partial t}+a\dfrac{\partial^3 w}{\partial x^3}+\left(b_1 w^{\frac{2-k}{2}}+b_2 w^{\frac{1-k}{2}}\right)\left(\dfrac{\partial w}{\partial x}\right)^k=0.$

Degenerate solution quadratic in x:

$$w(x,t)=\left[\frac{x}{2}\left(\frac{kb_1 t}{2}\right)^{-1/k}+Ct^{-1/k}-\frac{b_2}{b_1}\right]^2.$$

⊙ *Reference*: W. I. Fushchich, N. I. Serov, and T. K. Akhmerov (1991).

5. $\dfrac{\partial w}{\partial t}+a\dfrac{\partial^3 w}{\partial x^3}+(b_1\ln w+b_2)w^{1-k}\left(\dfrac{\partial w}{\partial x}\right)^k=0.$

Generalized traveling-wave solutions:

$$w=\begin{cases}\exp\left[-a\dfrac{k(kb_1)^{-3/k}}{k-2}t^{(k-3)/k}-\dfrac{b_2}{b_1}+Ct^{-1/k}+(kb_1t)^{-1/k}x\right] & \text{if } k\neq 2,\\[2mm] \exp\left[-a(2b_1)^{-3/2}t^{-1/2}\ln t-\dfrac{b_2}{b_1}+Ct^{-1/2}+(2b_1t)^{-1/2}x\right] & \text{if } k=2,\end{cases}$$

where C is an arbitrary constant.

⊙ *Reference*: W. I. Fushchich, N. I. Serov, and T. K. Akhmerov (1991).

6. $\dfrac{\partial w}{\partial t}-\dfrac{\partial^3 w}{\partial x^3}+(b_1\arcsin w+b_2)(1-w^2)^{\frac{1-k}{2}}\left(\dfrac{\partial w}{\partial x}\right)^k=0.$

Generalized traveling-wave solutions:

$$w=\begin{cases}\sin\left[-\dfrac{k(kb_1)^{-3/k}}{k-2}t^{(k-3)/k}-\dfrac{b_2}{b_1}+Ct^{-1/k}+(kb_1t)^{-1/k}x\right] & \text{if } k\neq 2,\\[2mm] \sin\left[-(2b_1)^{-3/2}t^{-1/2}\ln t-\dfrac{b_2}{b_1}+Ct^{-1/2}+(2b_1t)^{-1/2}x\right] & \text{if } k=2,\end{cases}$$

where C is an arbitrary constant.

⊙ *Reference*: W. I. Fushchich, N. I. Serov, and T. K. Akhmerov (1991).

7. $\dfrac{\partial w}{\partial t}-\dfrac{\partial^3 w}{\partial x^3}+(b_1\,\mathrm{arcsinh}\, w+b_2)(1+w^2)^{\frac{1-k}{2}}\left(\dfrac{\partial w}{\partial x}\right)^k=0.$

Generalized traveling-wave solutions:

$$w=\begin{cases}\sinh\left[\dfrac{k(kb_1)^{-3/k}}{k-2}t^{(k-3)/k}-\dfrac{b_2}{b_1}+Ct^{-1/k}+(kb_1t)^{-1/k}x\right] & \text{if } k\neq 2,\\[2mm] \sinh\left[(2b_1)^{-3/2}t^{-1/2}\ln t-\dfrac{b_2}{b_1}+Ct^{-1/2}+(2b_1t)^{-1/2}x\right] & \text{if } k=2,\end{cases}$$

where C is an arbitrary constant.

⊙ *Reference*: W. I. Fushchich, N. I. Serov, and T. K. Akhmerov (1991).

9.1.6. Equations of the Form $\dfrac{\partial w}{\partial t}+a\dfrac{\partial^3 w}{\partial x^3}+F\left(x,t,w,\dfrac{\partial w}{\partial x}\right)=0$

1. $\dfrac{\partial w}{\partial t}=a\dfrac{\partial^3 w}{\partial x^3}+(bx+c)\dfrac{\partial w}{\partial x}+f(w).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1=w(x+C_1e^{-bt},t+C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \quad z = x + Ce^{-bt},$$

where C is an arbitrary constant and the function $w(z)$ is determined by the ordinary differential equation

$$aw'''_{zzz} + (bz + c)w'_z + f(w) = 0.$$

2. $\displaystyle \frac{\partial w}{\partial t} + a\frac{\partial^3 w}{\partial x^3} + f(t)w\frac{\partial w}{\partial x} + g(t)w = 0.$

Suppose $w(x, t)$ is a solution of this equation. Then the function

$$w_1 = w\bigl(x + C_1\psi(t) + C_2,\ t\bigr) - C_1\varphi(t),$$

where

$$\varphi(t) = \exp\left[-\int g(t)\,dt\right], \quad \psi(t) = \int f(t)\varphi(t)\,dt,$$

is also a solution of the equation (C_1 and C_2 are arbitrary constants).

3. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^3 w}{\partial x^3} + [f(t)\ln w + g(t)]\frac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x, t) = \exp[\varphi(t)x + \psi(t)],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \quad \psi(t) = \varphi(t)\int [g(t) + a\varphi^2(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.*

4. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^3 w}{\partial x^3} + [f(t)\operatorname{arcsinh}(kw) + g(t)]\frac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x, t) = \frac{1}{k}\sinh\bigl[\varphi(t)x + \psi(t)\bigr],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \quad \psi(t) = \varphi(t)\int [g(t) + a\varphi^2(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

5. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^3 w}{\partial x^3} + [f(t)\operatorname{arccosh}(kw) + g(t)]\frac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x, t) = \frac{1}{k}\cosh\bigl[\varphi(t)x + \psi(t)\bigr],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \quad \psi(t) = \varphi(t)\int [g(t) + a\varphi^2(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

* The constant a in equations 9.1.6.2 to 9.1.6.6 and their solutions can be replaced by an arbitrary function of time, $a = a(t)$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + [f(t)\arcsin(kw) + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\sin\bigl[\varphi(t)x + \psi(t)\bigr],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) - a\varphi^2(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + [f(t)\arccos(kw) + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\cos\bigl[\varphi(t)x + \psi(t)\bigr],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) - a\varphi^2(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^3 w}{\partial x^3} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + f(t).$

1°. Degenerate solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi_k = \varphi_k(t)$ satisfy an appropriate system of ordinary differential equations.

2°. Solution:

$$w(x,t) = Ae^{ct} + e^{ct}\int e^{-ct}f(t)\,dt + \theta(z), \qquad z = x + \lambda t,$$

where A and λ are arbitrary constants, and the function $\theta(z)$ is determined by the autonomous ordinary differential equation

$$a\theta'''_{zzz} + b\bigl(\theta'_z\bigr)^2 - \lambda\theta'_z + c\theta = 0.$$

3°. The substitution

$$w = U(x,t) + e^{ct}\int e^{-ct}f(t)\,dt$$

leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^3 U}{\partial x^3} + b\left(\frac{\partial U}{\partial x}\right)^2 + cU.$$

9.1.7. Burgers–Korteweg–de Vries Equation and Other Equations

1. $\dfrac{\partial w}{\partial t} + w\dfrac{\partial w}{\partial x} + a\dfrac{\partial^3 w}{\partial x^3} = b\dfrac{\partial^2 w}{\partial x^2}.$

Burgers–Korteweg–de Vries equation. It describes nonlinear waves in dispersive-dissipative media with instabilities, waves arising in thin films flowing down an inclined surface, changes of the concentration of substances in chemical reactions, etc.

⊙ *References*: Y. Kuramoto and T. Tsuzuki (1976), B. J. Cohen, J. A. Krommes, W. M. Tang, and M. N. Rosenbluth (1976), V. Ya. Shkadov (1977), J. Topper and T. Kawahara (1978), G. I. Sivashinsky (1983).

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x - C_1 t + C_2, t + C_3) + C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = \frac{x + C_1}{t + C_2}.$$

3°. Traveling-wave solutions:

$$w(x,t) = C_1 - \frac{12b^2}{25a(1 + C_2 e^y)^2}, \qquad y = -\frac{b}{5a}x + \left(\frac{b}{5a}C_1 - \frac{6b^3}{125a^2}\right)t;$$

$$w(x,t) = C_1 - \frac{12b^2}{25a(1 - C_2 e^y)^2}, \qquad y = -\frac{b}{5a}x + \left(\frac{b}{5a}C_1 - \frac{6b^3}{125a^2}\right)t;$$

$$w(x,t) = C_1 + \frac{12b^2}{25a}\frac{1 + 2C_2 e^z}{(1 + C_2 e^z)^2}, \quad z = \frac{b}{5a}x - \left(\frac{b}{5a}C_1 + \frac{6b^3}{125a^2}\right)t;$$

where C_1 and C_2 are arbitrary constants.

⊙ *Reference*: N. A. Kudryashov (1990 *a*).

4°. Traveling-wave solutions:

$$w(x,t) = C_1 \mp \frac{12b^2}{25a}\xi^2\varphi(\xi), \qquad \xi = C_2 \exp\left[\frac{b}{5a}x + \left(\frac{6b^3}{125a^2} - \frac{b}{5a}C_1\right)t\right],$$

where the function $\varphi(\xi)$ is defined implicitly by

$$\xi = \int \frac{d\varphi}{\sqrt{\pm(4\varphi^3 - 1)}} - C_3,$$

and C_1, C_2, and C_3 are arbitrary constants. For the upper sign, the inversion of this relation leads to the classical Weierstrass elliptic function, $\varphi(\xi) = \wp(\xi + C_3,\ 0,\ 1)$.

⊙ *Reference*: N. A. Kudryashov (1990 *a*).

5°. Solution:

$$w(x,t) = U(\zeta) + 2C_1 t, \qquad \zeta = x - C_1 t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(\zeta)$ is determined by the second-order ordinary differential equation (C_3 is an arbitrary constant)

$$aU''_{\zeta\zeta} - bU'_\zeta + \tfrac{1}{2}U^2 + C_2 U = -2C_1\zeta + C_3.$$

To the special case $C_1 = 0$ there corresponds a traveling-wave solution.

2. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^2}{\partial x^2}\left(w^{-3/2}\frac{\partial w}{\partial x}\right).$

Modified Harry Dym equation.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^2 C_2^{-2} w(C_1 x + C_3, C_2^3 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. The transformation $u = w^{-1/2}$, $\tau = at$ leads to an equation of the form 9.1.4.8:

$$\frac{\partial u}{\partial \tau} = u^3\frac{\partial^3 u}{\partial x^3}.$$

3°. The equation is invariant under the transformation

$$d\bar{x} = w\,dx + [a(w^{-3/2}w_x)_x]\,dt, \quad d\bar{t} = dt, \quad \bar{w} = 1/w.$$

TABLE 3
Some integrable nonlinear third-order equations of the form 9.1.7.4

Type of generated equation	Form of generated equation	Solvable equation of the form (3)
Linear equation	$\frac{\partial w}{\partial t}=a\frac{\partial^3 w}{\partial x^3}$	$\frac{\partial u}{\partial t}=\frac{\partial^2}{\partial z^2}\left(\frac{a}{u^3}\frac{\partial u}{\partial z}\right)$
Korteweg–de Vries equation 9.1.1.2	$\frac{\partial w}{\partial t}=a\frac{\partial^3 w}{\partial x^3}-bw\frac{\partial w}{\partial x}$	$\frac{\partial u}{\partial t}=\frac{\partial^2}{\partial z^2}\left(\frac{a}{u^3}\frac{\partial u}{\partial z}\right)-\frac{b}{2u^2}\frac{\partial u}{\partial z}$
Modified Korteweg–de Vries equation 9.1.2.4	$\frac{\partial w}{\partial t}=a\frac{\partial^3 w}{\partial x^3}-bw^2\frac{\partial w}{\partial x}$	$\frac{\partial u}{\partial t}=\frac{\partial^2}{\partial z^2}\left(\frac{a}{u^3}\frac{\partial u}{\partial z}\right)-\frac{2b}{3u^3}\frac{\partial u}{\partial z}$

3. $\displaystyle \frac{\partial w}{\partial t}=\frac{\partial^2}{\partial x^2}\left[f(w)\frac{\partial w}{\partial x}\right]+\frac{a}{f(w)}+b.$

Functional separable solution in implicit form:

$$\int f(w)\,dw=at-\frac{1}{6}bx^3+C_1x^2+C_2x+C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

4. $\displaystyle \frac{\partial w}{\partial t}=\frac{\partial^2}{\partial x^2}\left[f(w)\frac{\partial w}{\partial x}\right]+g(w)\frac{\partial w}{\partial x}.$

1°. Traveling-wave solution:

$$w=w(z),\qquad z=kx+\lambda t,$$

where k and λ are arbitrary constants, and the function $w(z)$ is determined by the autonomous ordinary differential equation (C is an arbitrary constant)

$$k^3[f(w)w'_z]'_z+kG(w)-\lambda w+C=0,\qquad G(w)=\int g(w)\,dw.$$

The substitution $U(w)=f(w)w'_z$ leads to a first-order separable equation.

2°. The transformation

$$dz=w\,dx+\bigl\{[f(w)w_x]_x+G(w)\bigr\}\,dt,\quad d\tau=dt,\quad u=1/w\qquad \left(dz=\frac{\partial z}{\partial x}\,dx+\frac{\partial z}{\partial t}\,dt\right)\tag{1}$$

leads to an equation of the similar form

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2}{\partial z^2}\left[\Phi(u)\frac{\partial u}{\partial z}\right]+\Psi(u)\frac{\partial u}{\partial z},\tag{2}$$

where

$$\Phi(u)=\frac{1}{u^3}f\left(\frac{1}{u}\right),\quad \Psi(u)=\frac{1}{u}g\left(\frac{1}{u}\right)-G\left(\frac{1}{u}\right),\quad G(w)=\int g(w)\,dw.$$

The inverse of transformation (1) is written out as

$$dx=\frac{1}{w}\,dz-\frac{1}{w}\bigl\{w[wf(w)w_z]_z+G(w)\bigr\}\,d\tau,\quad dt=d\tau,\quad w=1/u.$$

Table 3 lists some solvable equations of the form (2) generated by known solvable third-order equations.

Equation (2) can be reduced to the form (see equation 9.1.7.5)

$$\frac{\partial v}{\partial \tau}=\varphi(v)\frac{\partial^3 v}{\partial z^3}+\psi(v)\frac{\partial v}{\partial z},\tag{3}$$

where

$$v=\int wf(w)\,dw,\quad \varphi(v)=w^3f(w),\quad \psi(v)=wg(w)-G(w).$$

TABLE 4

Some integrable nonlinear third-order equations of the form 9.1.7.5; $k = (8/a)^{1/2}$

Type of generated equation	Form of generated equation	Solvable equation of the form (3)
Linear equation	$\frac{\partial w}{\partial t} = a\frac{\partial^3 w}{\partial x^3}$	$\frac{\partial U}{\partial t} = kU^{3/2}\frac{\partial^3 U}{\partial z^3}$
Korteweg–de Vries equation 9.1.1.2	$\frac{\partial w}{\partial t} = a\frac{\partial^3 w}{\partial x^3} - bw\frac{\partial w}{\partial x}$	$\frac{\partial U}{\partial t} = kU^{3/2}\frac{\partial^3 U}{\partial z^3} - bU\frac{\partial U}{\partial z}$
Modified Korteweg–de Vries equation 9.1.2.4	$\frac{\partial w}{\partial t} = a\frac{\partial^3 w}{\partial x^3} - bw^2\frac{\partial w}{\partial x}$	$\frac{\partial U}{\partial t} = kU^{3/2}\frac{\partial^3 U}{\partial z^3} - \frac{2bk}{3a}U^{3/2}\frac{\partial U}{\partial z}$

3°. Conservation laws:

$$D_t(w) + D_x\bigl\{-[f(w)w_x]_x - G(w)\bigr\} = 0,$$

$$D_t\bigl[\Phi(w)\bigr] + D_x\bigl\{-F(w)[f(w)w_x]_x + \tfrac{1}{2}[f(w)w_x]^2 - \Psi(w)\bigr\} = 0,$$

where

$$D_t = \frac{\partial}{\partial t},\quad D_x = \frac{\partial}{\partial x},\quad G(w) = \int g(w)\,dw,\quad F(w) = \int f(w)\,dw,$$

$$\Phi(w) = \int F(w)\,dw,\qquad \Psi(w) = \int F(w)g(w)\,dw.$$

5. $\displaystyle \frac{\partial w}{\partial t} = f(w)\frac{\partial^3 w}{\partial x^3} + g(w)\frac{\partial w}{\partial x}.$

1°. Traveling-wave solution:

$$w = w(z),\qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation (C is an arbitrary constant)

$$w''_{zz} = \int \frac{\lambda - g(w)}{f(w)}\,dw + C,$$

which is easy to integrate.

2°. Conservation law:

$$D_t\bigl[\varphi(w)\bigr] + D_x\bigl[-w_{xx} - \psi(w)\bigr] = 0,$$

where

$$D_t = \frac{\partial}{\partial t},\quad D_x = \frac{\partial}{\partial x},\quad \varphi(w) = \int \frac{dw}{f(w)},\quad \psi(w) = \int \frac{g(w)}{f(w)}\,dw. \tag{1}$$

3°. The transformation

$$dz = \varphi(w)\,dx + \bigl[w_{xx} + \psi(w)\bigr]\,dt,\quad d\tau = dt,\quad U = \int \varphi(w)\,dw \qquad \left(dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial t}\,dt\right) \tag{2}$$

leads to an equation of the similar form

$$\frac{\partial U}{\partial \tau} = F(U)\frac{\partial^3 U}{\partial z^3} + G(U)\frac{\partial U}{\partial z}. \tag{3}$$

The functions $F(U)$ and $G(U)$ in (3) are defined parametrically by

$$F(U) = f(w)\varphi^3(w),\quad G(U) = g(w)\varphi(w) - \psi(w),\quad U = \int \varphi(w)\,dw,$$

where $\varphi(w)$ and $\psi(w)$ are defined in (1).

Table 4 presents some solvable equations of the form (3) generated by known solvable third-order equations.

4°. The substitution $\varphi = \int \frac{dw}{f(w)}$ leads to an equation of the form 9.1.7.4:

$$\frac{\partial \varphi}{\partial t} = \frac{\partial^2}{\partial x^2}\left[\mathcal{F}(\varphi)\frac{\partial \varphi}{\partial z}\right] + \mathcal{G}(\varphi)\frac{\partial \varphi}{\partial z},$$

where the functions $\mathcal{F}$ and $\mathcal{G}$ are given by

$$\mathcal{F}(\varphi) = f(w), \quad \mathcal{G}(\varphi) = g(w), \quad \varphi = \int \frac{dw}{f(w)}.$$

6. $\displaystyle \frac{\partial w}{\partial t} = f(w)\frac{\partial^3 w}{\partial x^3} + \big[g(w) + ax\big]\frac{\partial w}{\partial x} + h(w).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1 e^{-at}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + Ce^{-at},$$

where C is an arbitrary constant and the function $w(z)$ is determined by the ordinary differential equation

$$f(w)w'''_{zzz} + [g(w) + az]w'_z + h(w) = 0.$$

7. $\displaystyle w\frac{\partial w}{\partial t} + a\frac{\partial w}{\partial x} + bw\frac{\partial^3 w}{\partial x^3} = 0.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^{-2} w(C_1 x + C_2, C_1^3 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w = U(\xi), \qquad \xi = x + \lambda t,$$

where λ is an arbitrary constant and the function $U = U(\xi)$ is determined by the second-order autonomous ordinary differential equation

$$bU''_{\xi\xi} + a\ln|U| + \lambda U = C_1.$$

3°. Self-similar solution:

$$w = t^{2/3}u(z), \qquad z = xt^{-1/3},$$

where the function $u = u(z)$ is determined by the ordinary differential equation

$$buu'''_{zzz} - \tfrac{1}{3}zuu'_z + au'_z + \tfrac{2}{3}u^2 = 0.$$

9.2. Equations Involving the Second Derivative in t

9.2.1. Equations with Quadratic Nonlinearities

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^3 w}{\partial x^3} + bw\frac{\partial^2 w}{\partial x^2} + c.$

This is a special case of equation 11.3.5.3 with $n = 3$.

1°. Traveling-wave solution:

$$w(x,t) = u(\xi), \qquad \xi = \beta x + \lambda t,$$

where β and λ are arbitrary constants, and the function $u = u(\xi)$ is determined by the autonomous ordinary differential equation

$$a\beta^3 u'''_{\xi\xi\xi} + (b\beta^2 u - \lambda^2)u''_{\xi\xi} + c = 0.$$

2°. Solution:

$$w = U(z) + 4bC_1^2 t^2 + 4bC_1 C_2 t, \qquad z = x + bC_1 t^2 + bC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$aU'''_{zzz} + bUU''_{zz} - b^2 C_2^2 U''_{zz} - 2bC_1 U'_z + c - 8bC_1^2 = 0.$$

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^3 w}{\partial x^3} + b\frac{\partial}{\partial x}\left(w\frac{\partial w}{\partial x}\right) + c.$

This is a special case of equation 11.3.5.3 with $n = 3$.

1°. Traveling-wave solution:

$$w(x,t) = u(\xi), \qquad \xi = \beta x + \lambda t,$$

where β and λ are arbitrary constants, and the function $u = u(\xi)$ is determined by the autonomous ordinary differential equation

$$a\beta^3 u'''_{\xi\xi\xi} + b\beta^2 (uu'_\xi)'_\xi - \lambda^2 u''_{\xi\xi} + c = 0. \tag{1}$$

2°. Solution:

$$w = U(z) + 4bC_1^2 t^2 + 4bC_1 C_2 t, \qquad z = x + bC_1 t^2 + bC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$aU'''_{zzz} + b(UU'_z)'_z - b^2 C_2^2 U''_{zz} - 2bC_1 U'_z + c - 8bC_1^2 = 0. \tag{2}$$

Remark. Equations (1) and (2) can each be integrated once with respect to the independent variable.

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^3 w}{\partial x^3} + b\left(\frac{\partial w}{\partial x}\right)^2 + f(t).$

This is a special case of equation 11.3.3.4 with $n = 3$.

1°. Additive separable solution:

$$w(x,t) = \tfrac{1}{2}At^2 + Bt + C + \int_0^t (t-\tau)f(\tau)\,d\tau + \varphi(x).$$

Here, A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi'''_{xxx} + b(\varphi'_x)^2 - A = 0,$$

whose order can be reduced with the change of variable $U(x) = \varphi'_x$.

2°. The substitution

$$w = u(x,t) + \int_0^t (t-\tau)f(\tau)\,d\tau$$

leads to the simpler equation

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^3 u}{\partial x^3} + b\left(\frac{\partial u}{\partial x}\right)^2.$$

This equation admits a traveling-wave solution, $u = u(kx + \lambda t)$, and a self-similar solution, $u = t^{-2/3}\phi(z)$, where $z = xt^{-2/3}$.

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^3 w}{\partial x^3} + a\left(\dfrac{\partial w}{\partial x}\right)^2 + bw + f(t).$

1°. Solution:

$$w(x,t) = \varphi(t) + \psi(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the functions $\varphi(t)$ and $\psi(z)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - f(t) = 0,$$
$$\psi'''_{zzz} - \lambda^2 \psi''_{zz} + a(\psi'_z)^2 + b\psi = 0.$$

The solution of the first equation is given by

$$\varphi(t) = C_1 \cosh(kt) + C_2 \sinh(kt) + \frac{1}{k}\int_0^t f(\tau)\sinh\bigl[k(t-\tau)\bigr]\,d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1 \cos(kt) + C_2 \sin(kt) + \frac{1}{k}\int_0^t f(\tau)\sin\bigl[k(t-\tau)\bigr]\,d\tau \qquad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution $w = u(x,t) + \varphi(t)$, where the function $\varphi(t)$ is defined in Item 1°, leads to the simpler equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^3 u}{\partial x^3} + a\left(\frac{\partial u}{\partial x}\right)^2 + bu.$$

5. $\dfrac{\partial^2 w}{\partial t^2} + k(t)\dfrac{\partial w}{\partial t} = f(t)w\dfrac{\partial^3 w}{\partial x^3} + g(t)\dfrac{\partial^2 w}{\partial x^2} + h(t)\dfrac{\partial w}{\partial x} + p(t)w + q(t).$

Generalized separable solution cubic in x:

$$w(x,t) = \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t),$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_3'' + k(t)\varphi_3' &= [6f(t)\varphi_3 + p(t)]\varphi_3,\\
\varphi_2'' + k(t)\varphi_2' &= [6f(t)\varphi_3 + p(t)]\varphi_2 + 3h(t)\varphi_3,\\
\varphi_1'' + k(t)\varphi_1' &= [6f(t)\varphi_3 + p(t)]\varphi_1 + 6g(t)\varphi_3 + 2h(t)\varphi_2,\\
\varphi_0'' + k(t)\varphi_0' &= [6f(t)\varphi_3 + p(t)]\varphi_0 + 2g(t)\varphi_2 + h(t)\varphi_1 + q(t).
\end{aligned}$$

6. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^3 w}{\partial x^3} + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)(A_3 x^3 + A_2 x^2 + A_1 x) + \psi(t),$$

where A_1, A_2, and A_3 are arbitrary constants, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = 6A_3 a\varphi^2 + f(t)\varphi,$$
$$\psi''_{tt} = 6A_3 a\varphi\psi + f(t)\psi + g(t).$$

7. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^3 w}{\partial x^3} + bw^2 + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of second-order ordinary differential equations (C is an arbitrary constant)

$$\varphi''_{tt} = C\varphi^2 + b\varphi\psi + f(t)\varphi,$$
$$\psi''_{tt} = C\varphi\psi + b\psi^2 + f(t)\psi + g(t),$$

and the function $\Theta(x)$ satisfies the third-order constant-coefficient linear nonhomogeneous ordinary differential equation

$$a\Theta'''_{xxx} + b\Theta = C.$$

9.2.2. Other Equations

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^3 w}{\partial x^3} + bw\ln w + \bigl[f(x) + g(t)\bigr]w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - \bigl[b\ln\varphi + g(t) + C\bigr]\varphi = 0,$$
$$a\psi'''_{xxx} + \bigl[b\ln\psi + f(x) - C\bigr]\psi = 0,$$

where C is an arbitrary constant.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^3 w}{\partial x^3} + f(x)w\ln w + \bigl[bf(x)t + g(x)\bigr]w.$

Multiplicative separable solution:

$$w(x,t) = e^{-bt}\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi'''_{xxx} + f(x)\varphi\ln\varphi + \bigl[g(x) - b^2\bigr]\varphi = 0.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^3 w}{\partial x^3} + F\left(x,\,\dfrac{\partial w}{\partial x}\right) + g(t).$

1°. Additive separable solution:

$$w(x,t) = C_1t^2 + C_2t + \int_{t_0}^{t}(t-\tau)g(\tau)\,d\tau + \varphi(x),$$

where C_1, C_2, and t_0 are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi'''_{xxx} + F\bigl(x,\varphi'_x\bigr) - 2C_1 = 0,$$

whose order can be reduced with the change of variable $u(x) = \varphi'_x$.

2°. The substitution

$$w = U(x,t) + \int_0^t (t-\tau)g(\tau)\,d\tau$$

leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^3 U}{\partial x^3} + F\left(x,\,\frac{\partial U}{\partial x}\right).$$

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^3 w}{\partial x^3} + F\left(x, \dfrac{\partial w}{\partial x}\right) + bw + g(t).$

1°. Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - g(t) = 0,$$
$$a\psi'''_{xxx} + F\big(x, \psi'_x\big) + b\psi = 0.$$

The solution of the first equation is given by

$$\varphi(t) = C_1 \cosh(kt) + C_2 \sinh(kt) + \frac{1}{k}\int_0^t g(\tau)\sinh\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1 \cos(kt) + C_2 \sin(kt) + \frac{1}{k}\int_0^t g(\tau)\sin\big[k(t-\tau)\big]\,d\tau \qquad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution $w = U(x,t) + \varphi(t)$, where the function $\varphi(t)$ is defined in Item 1°, leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^3 U}{\partial x^3} + F\left(x, \frac{\partial U}{\partial x}\right) + bU.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2}{\partial x^2}\left[f(w)\dfrac{\partial w}{\partial x}\right] - a^2\dfrac{f'(w)}{f^3(w)} + b.$

Functional separable solution in implicit form:

$$\int f(w)\,dw = at - \frac{1}{6}bx^3 + C_1x^2 + C_2x + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants.

6. $\dfrac{\partial^2 w}{\partial t^2} = F\left(\dfrac{\partial^2 w}{\partial x^2}, \dfrac{\partial^3 w}{\partial x^3}\right).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1, y + C_2) + C_3xt + C_4x + C_5t + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w = u(z) + C_3x^2 + C_4t^2, \qquad z = C_1x + C_2t,$$

where the function $u(z)$ is determined by the autonomous ordinary differential equation

$$C_2^2u''_{zz} + 2C_4 = F(C_1^2u''_{zz} + 2C_3,\ C_1^3u'''_{zzz}),$$

whose order can be reduced by two with the change of variable $\theta(z) = u''_{zz}$.

9.3. Hydrodynamic Boundary Layer Equations

9.3.1. Steady Hydrodynamic Boundary Layer Equations for a Newtonian Fluid

1. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = \nu\dfrac{\partial^3 w}{\partial y^3}.$

This is an equation of a steady laminar hydrodynamic boundary layer on a flat plate; w is the stream function, x and y are the longitudinal and normal coordinates, respectively, and ν is the kinematic viscosity of the fluid. A similar equation governs the steady-state flow of a plane laminar jet out of a slot.

Preliminary remarks. The system of hydrodynamic boundary layer equations

$$u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}=\nu\frac{\partial^2 u_1}{\partial y^2},$$
$$\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}=0,$$

where u_1 and u_2 are the longitudinal and normal components of the fluid velocity, respectively, is reduced to the equation in question by the introduction of a stream function w such that $u_1=\frac{\partial w}{\partial y}$ and $u_2=-\frac{\partial w}{\partial x}$.

⊙ *References*: H. Schlichting (1981), L. G. Loitsyanskiy (1996).

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1=w(x,y+\varphi(x)),$$
$$w_2=C_1w(C_2x+C_3,C_1C_2y+C_4)+C_5,$$

where $\varphi(x)$ is an arbitrary function and $C_1,\,\dots,\,C_5$ are arbitrary constants, are also solutions of the equation.

⊙ *References*: Yu. N. Pavlovskii (1961), L. V. Ovsiannikov (1982).

2°. Degenerate solutions (linear and quadratic in y):

$$w(x,y)=C_1y+\varphi(x),$$
$$w(x,y)=C_1y^2+\varphi(x)y+\frac{1}{4C_1}\varphi^2(x)+C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(x)$ is an arbitrary function. These solutions are independent of ν and correspond to inviscid fluid flows.

⊙ *Reference*: D. Zwillinger (1989, pp. 396–397).

3°. Solutions involving arbitrary functions:

$$w(x,y)=\frac{6\nu x+C_1}{y+\varphi(x)}+\frac{C_2}{[y+\varphi(x)]^2}+C_3,$$
$$w(x,y)=\varphi(x)\exp(-C_1y)+\nu C_1x+C_2,$$
$$w(x,y)=C_1\exp\bigl[-C_2y-C_2\varphi(x)\bigr]+C_3y+C_3\varphi(x)+\nu C_2x+C_4,$$
$$w(x,y)=6\nu C_1x^{1/3}\tanh\xi+C_2,\qquad \xi=C_1\frac{y}{x^{2/3}}+\varphi(x),$$
$$w(x,y)=-6\nu C_1x^{1/3}\tan\xi+C_2,\qquad \xi=C_1\frac{y}{x^{2/3}}+\varphi(x),$$

where $C_1,\,\dots,\,C_4$ are arbitrary constants and $\varphi(x)$ is an arbitrary function.

Special case 1. For $C_1=\sqrt{k/\nu}$ and $\varphi(x)=-\sqrt{k\nu}\,x$, the second solution becomes

$$w=\sqrt{k\nu}\,x\bigl[1-\exp\bigl(-\sqrt{k/\nu}\,y\bigr)\bigr]+\text{const}.$$

It describes a fluid flow induced by the motion of surface particles at $y=0$ with a velocity of $u_1|_{y=0}=kx$. The fluid velocity components in this case meet the boundary conditions

$$u_1=0\quad\text{at}\quad x=0,\qquad u_1=kx\quad\text{at}\quad y=0,\qquad u_2=0\quad\text{at}\quad y=0,\qquad u_1\to 0\quad\text{as}\quad y\to\infty.$$

⊙ *References*: N. V. Ignatovich (1993), A. D. Polyanin (2001 *a*).

4°. Table 5 lists invariant solutions to the hydrodynamic boundary layer equation that are obtained with the classical group-theoretic methods. Solution 1 is expressed in additive separable form, solution 2 is in multiplicative separable form, solution 3 is self-similar, and solution 4 is generalized self-similar. Solution 5 degenerates at $a=0$ into a self-similar solution (see solution 3 with $\lambda=-1$).

TABLE 5

Invariant solutions to the hydrodynamic boundary layer equation (the additive constant is omitted)

No.	Solution structure	Function F or equation for F	Remarks
1	$w = F(y) + \nu\lambda x$	$F(y) = \begin{cases} C_1 \exp(-\lambda y) + C_2 y & \text{if } \lambda \neq 0, \\ C_1 y^2 + C_2 y & \text{if } \lambda = 0 \end{cases}$	λ is any
2	$w = F(x)y^{-1}$	$F(x) = 6\nu x + C_1$	—
3	$w = x^{\lambda+1}F(z),\ z = x^\lambda y$	$(2\lambda+1)(F'_z)^2 - (\lambda+1)FF''_{zz} = \nu F'''_{zzz}$	λ is any
4	$w = e^{\lambda x}F(z),\ z = e^{\lambda x}y$	$2\lambda(F'_z)^2 - \lambda FF''_{zz} = \nu F'''_{zzz}$	λ is any
5	$w = F(z) + a\ln\lvert x\rvert,\ z = y/x$	$-(F'_z)^2 - aF''_{zz} = \nu F'''_{zzz}$	a is any

Equations 3–5 for F are autonomous and generalized homogeneous; hence, their order can be reduced by two.

⊙ *References*: Yu. N. Pavlovskii (1961), H. Schlichting (1981), L. G. Loitsyanskiy (1996), G. I. Burde (1996).

Special case 2. The Blasius problem on a translational fluid flow with a velocity U_i past a flat plate is characterized by the boundary conditions

$$\partial_x w = \partial_y w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w \to U_\mathrm{i} \quad \text{as} \quad y \to \infty, \qquad \partial_y w = U_\mathrm{i} \quad \text{at} \quad x = 0.$$

The form of the solution to this problem (in the domain $x \ge 0$, $y \ge 0$) is given in the third row of Table 5 with $\lambda = -1/2$. The boundary conditions for $F(z)$ are as follows:

$$F = F'_z = 0 \quad \text{at} \quad z = 0, \qquad F'_z \to U_i \quad \text{as} \quad z \to \infty.$$

For details, see Blasius (1908), Schlichting (1981), and Loitsyanskiy (1996).

Special case 3. The Schlichting problem on the axisymmetric flow of a plane laminar jet out of a thin slit is characterized by the boundary conditions

$$\partial_x w = \partial_{yy} w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w \to 0 \quad \text{as} \quad y \to \infty,$$

which are supplemented with the integral condition of conservation of momentum

$$\int_0^\infty (\partial_y w)^2\, dy = A \qquad (A = \text{const}).$$

The form of the solution to this problem (in the domain $x \ge 0$, $y \ge 0$) is given in the third row of Table 5 with $\lambda = -2/3$. On integrating the ordinary differential equation for F with appropriate boundary conditions,

$$F = F''_{zz} = 0 \quad \text{at} \quad z = 0, \qquad F'_z \to 0 \quad \text{as} \quad z \to \infty,$$

and the integral condition

$$\int_0^\infty (F'_z)^2 = A,$$

we finally obtain

$$w(x,y) = k(A\nu x)^{1/3} \tanh \xi, \qquad \xi = \tfrac{1}{6}k(A/\nu^2)^{1/3} y x^{-2/3}, \qquad k = 3^{2/3}.$$

For details, see the book by Schlichting (1981) and Loitsyanskiy (1996).

Special case 4. Note two cases where the equation specified in row 3 of Table 5 can be integrated.

For $\lambda = -1$, the solution can be obtained in parametric form:

$$F = -\frac{\nu}{2C_1} \int \frac{\tau\, d\tau}{\sqrt{1+\tau^3}} + C_2, \qquad z = 3C_1 \int \frac{d\tau}{\sqrt{1+\tau^3}} + C_3.$$

There is a solution $F = 6\nu z^{-1}$.

For $\lambda = -\frac{2}{3}$, the twofold integration yields the Riccati equation

$$\nu F'_z + \tfrac{1}{6}F^2 + C_1 z + C_2 = 0.$$

If $C_1 = 0$, it can be readily integrated (since the variables separate); if $C_1 \neq 0$, the solution can be expressed in terms of the Bessel functions or order $1/3$.

5°. Generalized separable solution linear in x:

$$w(x,y) = xf(y) + g(y), \tag{1}$$

where the functions $f = f(y)$ and $g = g(y)$ are determined by the autonomous system of ordinary differential equations

$$(f'_y)^2 - ff''_{yy} = \nu f'''_{yyy}, \tag{2}$$
$$f'_y g'_y - fg''_{yy} = \nu g'''_{yyy}. \tag{3}$$

The order of equation (2) can be reduced by two. Suppose a solution of equation (2) is known. Equation (3) is linear in g and has two linearly independent solutions:

$$g_1 = 1, \quad g_2 = f(y).$$

The second particular solution follows from the comparison of (2) and (3). The general solution of equation (2) can be written out in the form

$$\begin{aligned} g(y) &= C_1 + C_2 f + C_3\left(f\int \psi\,dy - \int f\psi\,dy\right),\\ f &= f(y), \quad \psi = \frac{1}{(f'_y)^2}\exp\left(-\frac{1}{\nu}\int f\,dy\right); \end{aligned} \tag{4}$$

see Polyanin and Zaitsev (2003).

It is not difficult to verify that equation (2) has the following particular solutions:

$$\begin{aligned} f(y) &= 6\nu(y+C)^{-1},\\ f(y) &= Ce^{\lambda y} - \lambda\nu, \end{aligned} \tag{5}$$

where C and λ are arbitrary constants. The first solution in (5) leads, taking into account (1) and (4), to the first solution of Item 3° with $\varphi(x) = \text{const}$. Substituting the second expression of (5) into (1) and (4), one may obtain another solution.

⊙ *Reference*: A. D. Polyanin (2001 *a*).

6°. Generalized separable solution (special case of solution 3 in Item 3°):

$$w(x,y) = (a + be^{-\lambda y})z(x) + cy,$$

where a, b, c, and λ are arbitrary constants, and the function $z = z(x)$ is defined implicitly by

$$c\ln|z| + a\lambda z = \nu\lambda^2 x.$$

⊙ *Reference*: N. V. Ignatovich (1993), B. I. Burde (1996).

7°. Below are two transformations that reduce the order of the boundary layer equation.

7.1. The von Mises transformation

$$\xi = x, \quad \eta = w, \quad U(\xi,\eta) = \frac{\partial w}{\partial y}, \qquad \text{where} \quad w = w(x,y),$$

leads to a nonlinear heat equation of the form 1.10.1.1:

$$\frac{\partial U}{\partial \xi} = \nu\frac{\partial}{\partial \eta}\left(U\frac{\partial U}{\partial \eta}\right).$$

7.2. The Crocco transformation

$$\xi = x, \quad \zeta = \frac{\partial w}{\partial y}, \quad \Psi(\xi,\zeta) = \frac{\partial^2 w}{\partial y^2}, \qquad \text{where} \quad w = w(x,y),$$

leads to the second-order nonlinear equation

$$\zeta\frac{\partial}{\partial \xi}\left(\frac{1}{\Psi}\right) + \nu\frac{\partial^2 \Psi}{\partial \zeta^2} = 0.$$

⊙ *Reference*: L. G. Loitsyanskiy (1996).

TABLE 6
Invariant solutions to the hydrodynamic boundary layer equation with pressure gradient (a, k, m, and β are arbitrary constants)

No.	Function $f(x)$	Form of solution $w = w(x, y)$	Function u or equation for u
1	$f(x) = 0$	See equation 9.3.1.1	See equation 9.3.1.1
2	$f(x) = ax^m$	$w = x^{\frac{m+3}{4}} u(z),\ z = x^{\frac{m-1}{4}} y$	$\frac{m+1}{2}(u'_z)^2 - \frac{m+3}{4} u u''_{zz} = \nu u'''_{zzz} + a$
3	$f(x) = ae^{\beta x}$	$w = e^{\frac{1}{4}\beta x} u(z),\ z = e^{\frac{1}{4}\beta x} y$	$\frac{1}{2}\beta(u'_z)^2 - \frac{1}{4}\beta u u''_{zz} = \nu u'''_{zzz} + a$
4	$f(x) = a$	$w = kx + u(y)$	$u(y) = \begin{cases} C_1 \exp\bigl(-\frac{k}{\nu} y\bigr) - \frac{a}{2k} y^2 + C_2 y & \text{if } k \neq 0, \\ -\frac{a}{6\nu} y^3 + C_2 y^2 + C_1 y & \text{if } k = 0 \end{cases}$
5	$f(x) = ax^{-3}$	$w = k \ln\lvert x\rvert + u(z),\ z = y/x$	$-(u'_z)^2 - k u''_{zz} = \nu u'''_{zzz} + a$

8°. Conservation law:

$$D_x\bigl(w_y^2\bigr) + D_y\bigl(-w_x w_y - \nu w_{yy}\bigr) = 0,$$

where $D_x = \frac{\partial}{\partial x}$ and $D_y = \frac{\partial}{\partial y}$.

2. $$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = \nu \frac{\partial^3 w}{\partial y^3} + f(x).$$

This is a hydrodynamic boundary layer equation with pressure gradient. The formula $f(x) = UU'_x$ holds true; $U = U(x)$ is the fluid velocity in the stream core* at the interface between the core and the boundary layer.

1°. Suppose $w(x, y)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x, y + \varphi(x)) + C,$$
$$w_2 = -w(x, -y + \varphi(x)) + C,$$

where $\varphi(x)$ is an arbitrary function and C is an arbitrary constant, are also solutions of the equation.

⊙ *References*: Yu. N. Pavlovskii (1961), L. V. Ovsiannikov (1982).

2°. Degenerate solutions (linear and quadratic in y) for arbitrary $f(x)$:

$$w(x, y) = \pm y\left[2\int f(x)\,dx + C_1\right]^{1/2} + \varphi(x),$$
$$w(x, y) = C_1 y^2 + \varphi(x) y + \frac{1}{4C_1}\left[\varphi^2(x) - 2\int f(x)\,dx\right] + C_2,$$

where $\varphi(x)$ is an arbitrary function, and C_1 and C_2 are arbitrary constants. These solutions are independent of ν and correspond to inviscid fluid flows.

⊙ *Reference*: A. D. Polyanin (2001 *a*).

3°. Table 6 lists invariant solutions to the hydrodynamic boundary layer equation with pressure gradient that are obtained with the classical group-theoretic methods.

Note that the Falkner–Skan problem (see Falkner and Skan, 1931) on a symmetric fluid flow past a wedge is described by the equation specified in the second row of Table 6. The case $m = 1$ corresponds to a fluid flow near a stagnation point, and the case $m = 0$ corresponds to a symmetric flow past a wedge with an angle of $\alpha = \frac{2}{3}\pi$.

⊙ *References*: Yu. N. Pavlovskii (1961), H. Schlichting (1981), L. G. Loitsyanskiy (1996), G. I. Burde (1996).

* The hydrodynamic problem on the flow of an ideal (inviscid) fluid about the body is solved in the stream core.

4°. Generalized separable solution (linear in x) for $f(x) = ax + b$:

$$w(x, y) = xF(y) + G(y),$$

where the functions $F = F(y)$ and $G = G(y)$ are determined by the system of ordinary differential equations

$$(F'_y)^2 - FF''_{yy} = \nu F'''_{yyy} + a, \tag{1}$$
$$F'_y G'_y - FG''_{yy} = \nu G'''_{yyy} + b. \tag{2}$$

The order of the autonomous equation (1) can be reduced by one. Given a particular solution of equation (1), the corresponding equation (2) can be reduced with the substitution $H(y) = G'_y$ to a second-order equation. For $F(y) = \pm\sqrt{a}\,y + C$, equation (2) is integrable by quadrature (since, for $b = 0$, we know two of its particular solutions: $G_1 = 1$ and $G_2 = \pm\frac{1}{2}\sqrt{a}\,y^2 + Cy$).

⦿ *Reference*: A. D. Polyanin (2001 *a*).

5°. Solutions for $f(x) = -ax^{-5/3}$:

$$w(x, y) = \frac{6\nu x}{y + \varphi(x)} \pm \frac{\sqrt{3a}}{x^{1/3}}[y + \varphi(x)],$$

where $\varphi(x)$ is an arbitrary function.

⦿ *Reference*: B. I. Burde (1996).

6°. Solutions for $f(x) = ax^{-1/3} - bx^{-5/3}$:

$$w(x, y) = \pm\sqrt{3b}\,z + x^{2/3}\theta(z), \qquad z = yx^{-1/3},$$

where the function $\theta = \theta(z)$ is determined by the ordinary differential equation

$$\tfrac{1}{3}(\theta'_z)^2 - \tfrac{2}{3}\theta\theta''_{zz} = \nu\theta'''_{zzz} + a.$$

⦿ *Reference*: B. I. Burde (1996).

7°. Generalized separable solution for $f(x) = ae^{\beta x}$:

$$w(x, y) = \varphi(x)e^{\lambda y} - \frac{a}{2\beta\lambda^2\varphi(x)}e^{\beta x - \lambda y} - \nu\lambda x + \frac{2\nu\lambda^2}{\beta}y + \frac{2\nu\lambda}{\beta}\ln|\varphi(x)|,$$

where $\varphi(x)$ is an arbitrary function and λ is an arbitrary constant.

⦿ *References*: A. D. Polyanin (2001 *a*, 2002).

8°. For

$$f(x) = a^2\nu^2 x^{-3}(xgg'_x - g^2), \qquad g = -\tfrac{1}{4}a \pm \left(\tfrac{1}{16}a^2 + bx^{2/3}\right)^{1/2},$$

there are exact solutions of the form

$$w(x, y) = a\nu z + 6\nu g \tanh z, \qquad z = \frac{yg}{x}.$$

⦿ *Reference*: B. I. Burde (1996).

9°. Below are two transformations that reduce the order of the boundary layer equation.

9.1. The von Mises transformation

$$\xi = x, \quad \eta = w, \quad U(\xi, \eta) = \frac{\partial w}{\partial y}, \qquad \text{where} \quad w = w(x, y),$$

leads to the nonlinear heat equation

$$U\frac{\partial U}{\partial \xi} = \nu U \frac{\partial}{\partial \eta}\left(U\frac{\partial U}{\partial \eta}\right) + f(\xi).$$

9.2. The Crocco transformation

$$\xi = x, \quad \zeta = \frac{\partial w}{\partial y}, \quad \Psi(\xi,\zeta) = \frac{\partial^2 w}{\partial y^2}, \qquad \text{where} \quad w = w(x,y),$$

leads to the second-order nonlinear equation

$$\zeta \frac{\partial}{\partial \xi}\left(\frac{1}{\Psi}\right) + \nu \frac{\partial^2 \Psi}{\partial \zeta^2} - f(\xi)\frac{\partial}{\partial \zeta}\left(\frac{1}{\Psi}\right) = 0.$$

⦿ *Reference*: L. G. Loitsyanskiy (1996).

10°. Conservation law:

$$D_x\left[w_y^2 - F(x)\right] + D_y\left(-w_x w_y - \nu w_{yy}\right) = 0,$$

where $D_x = \dfrac{\partial}{\partial x}$, $D_y = \dfrac{\partial}{\partial y}$, and $F(x) = \displaystyle\int f(x)\,dx$.

3. $\displaystyle \frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial x \partial z} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial z^2} = \nu \frac{\partial}{\partial z}\left(z\frac{\partial^2 w}{\partial z^2}\right) + f(x).$

Preliminary remarks. The system of axisymmetric steady laminar hydrodynamic boundary layer equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + f(x), \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0, \tag{2}$$

where u and v are the axial and radial fluid velocity components, respectively, and x and r are cylindrical coordinates, can be reduced to the equation in question by the introduction of a stream function w and a new variable z such that

$$u = \frac{2}{r}\frac{\partial w}{\partial r}, \quad v = -\frac{2}{r}\frac{\partial w}{\partial x}, \quad z = \frac{1}{4}r^2.$$

System (1), (2) is used for describing an axisymmetric jet and a boundary layer on an extensive body of revolution. The function $f(x)$ is expressed via the longitudinal fluid velocity $U = U(x)$ in the inviscid flow core as $f = UU'_x$.

⦿ *References*: F. L. Crabtree, D. Küchemann, and L. Sowerby (1963), H. Schlichting (1981), L. G. Loitsyanskiy (1996).

1°. Self-similar solution for $f(x) = Ax^k$:

$$w(x,z) = xU(\zeta), \qquad \zeta = zx^{\frac{k-1}{2}},$$

where the function $U = U(\zeta)$ is determined by the ordinary differential equation

$$-\tfrac{1}{2}(k+1)(U'_\zeta)^2 + UU''_{\zeta\zeta} + A + \nu(\zeta U''_{\zeta\zeta})'_\zeta = 0.$$

Special case. An axisymmetric jet is characterized by the values $A = 0$ and $k = -3$. In this case, the solution of the equation just obtained with appropriate boundary conditions is given by

$$U(\zeta) = \frac{2\nu\zeta}{\zeta + C},$$

where the constant of integration C can be expressed via the jet momentum.

⦿ *References*: H. Schlichting (1981), L. G. Loitsyanskiy (1996).

2°. Generalized separable solutions (linear and quadratic in z) for arbitrary $f(x)$:

$$w(x,z) = \pm z\left[2\int f(x)\,dx + C_1\right]^{1/2} + \varphi(x),$$

$$w(x,z) = C_1 z^2 + \varphi(x)z + \frac{1}{4C_1}\varphi^2(x) - \frac{1}{2C_1}\int f(x)\,dx - \nu x + C_2,$$

where $\varphi(x)$ is an arbitrary function and C_1 and C_2 are arbitrary constants. The first solution is "inviscid" (independent of ν).

⦿ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

3°. Functional separable solution for arbitrary $f(x)$:

$$w(x,z) = 2\nu x + \nu F(x)\left(\frac{2C_1}{\xi} + C_2\xi\right), \quad \xi = \frac{z}{F^2(x)} - C_1 F'_x(x),$$
$$F(x) = \pm\nu C_2\left[2\int f(x)\,dx + C_3\right]^{-1/2},$$

where C_1, C_2, and C_3 are arbitrary constants.

⊙ *Reference*: G. I. Burde (1994).

4°. Functional separable solution for $f(x) = ax + b$:

$$w(x,z) = \nu\lambda\varphi(x) + \frac{\nu}{a}(ax+b)\big(Ce^{-\lambda\xi} + \lambda\xi - 3\big), \qquad \xi = z - \varphi'_x(x), \quad \lambda = \pm\frac{\sqrt{a}}{\nu},$$

where C and λ are arbitrary constants and $\varphi(x)$ is an arbitrary function.

⊙ *Reference*: G. I. Burde (1994).

5°. Generalized separable solution (linear in x) for $f(x) = ax + b$:

$$w(x,z) = x\varphi(z) + \psi(z),$$

where the functions $\varphi = \varphi(z)$ and $\psi = \psi(z)$ are determined by the system of ordinary differential equations

$$(\varphi'_z)^2 - \varphi\varphi''_{zz} = \nu(z\varphi''_{zz})'_z + a,$$
$$\varphi'_z\psi'_z - \varphi\psi''_{zz} = \nu(z\psi''_{zz})'_z + b.$$

The first equation has particular solutions $\varphi = \pm\sqrt{a}\,z + C$.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

6°. Additive separable solutions for $f(x) = a$:

$$w(x,z) = \nu(1-k)x + C_1 z^k + \frac{a}{2\nu(k-2)}z^2 + C_2 z + C_3,$$
$$w(x,z) = -\nu x - \frac{a}{2\nu}z^2\ln z + C_1 z^2 + C_2 z + C_3,$$

where $C_1, \ldots, k$ are arbitrary constants.

7°. Conservation law:

$$D_x\big[w_z^2 - F(x)\big] + D_z\big(-w_x w_z - \nu z w_{zz}\big) = 0,$$

where $D_x = \dfrac{\partial}{\partial x}$, $D_z = \dfrac{\partial}{\partial z}$, and $F(x) = \displaystyle\int f(x)\,dx$.

9.3.2. Steady Boundary Layer Equations for Non-Newtonian Fluids

1. $\displaystyle \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = k\left(\frac{\partial^2 w}{\partial y^2}\right)^{n-1}\frac{\partial^3 w}{\partial y^3}.$

This equation describes a boundary layer on a flat plane in the flow of a power-law non-Newtonian fluid; w is the stream function, x and y are the longitudinal and normal coordinates, and n and k are rheological parameters ($n > 0$, $k > 0$).

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(C_1^{2-n}C_2^{2n-1}x + C_3, C_2 y + C_4) + C_5,$$
$$w_2 = w(x, y + \varphi(x)),$$

where $C_1, \ldots, C_5$ are arbitrary constants and $\varphi(x)$ is an arbitrary function, are also solutions of the equation.

2°. Additive separable solutions:

$$w(x,y) = \frac{1}{C_1^2 n(2n-1)} \big[C_1(n-1)y + C_2\big]^{\frac{2n-1}{n-1}} + C_3 y + C_4 - kC_1 x \quad \text{if} \quad n \neq 1/2,$$
$$w(x,y) = -\frac{1}{C_1^2} \ln(C_1 y + C_2) + C_3 y + C_4 + 2kC_1 x \qquad \text{if} \quad n = 1/2.$$

3°. Multiplicative separable solutions:

$$w(x,y) = \big[\lambda(2-n)x + C_1\big]^{\frac{1}{2-n}} F(y) \quad \text{if} \quad n \neq 2,$$
$$w(x,y) = C_1 e^{\lambda x} F(y) \qquad \text{if} \quad n = 2,$$

where $F = F(y)$ is determined by the autonomous ordinary differential equation

$$\lambda(F_y')^2 - \lambda F F_{yy}'' = k(F_{yy}'')^{n-1} F_{yyy}''',$$

whose order can be reduced by two. The equation for F has a particular solution in the form of a power-law function, $F = A_n(y+C)^{\beta_n}$, where $\beta_n = \frac{2n-1}{n-2}$.

4°. Self-similar solution ($n \neq 2$ and λ is any):

$$w(x,y) = x^{\frac{2\lambda n - \lambda + 1}{2-n}} \psi(z), \qquad z = x^\lambda y, \tag{1}$$

where the function $\psi = \psi(z)$ is determined by the autonomous ordinary differential equation

$$\frac{\lambda n + \lambda + 1}{2-n}(\psi_z')^2 - \frac{2\lambda n - \lambda + 1}{2-n}\psi\psi_{zz}'' = k(\psi_{zz}'')^{n-1}\psi_{zzz}''', \tag{2}$$

whose order can be reduced by two.

⊙ *Reference*: Z. P. Shulman and B. M. Berkovskii (1966).

Special case 1. The generalized Blasius problem on a translational flow with an incident velocity U_i past a flat plate is characterized by the boundary conditions

$$\partial_x w = \partial_y w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w \to U_\mathrm{i} \quad \text{as} \quad y \to \infty, \qquad \partial_y w = U_\mathrm{i} \quad \text{at} \quad x = 0.$$

A solution to this problem (in the domain $x \geq 0$, $y \geq 0$) is sought in the form (1) with $\lambda = -\frac{1}{n+1}$. The boundary conditions for $\psi(z)$ are the following:

$$\psi = \psi_z' = 0 \quad \text{at} \quad z = 0, \qquad \psi_z' \to U_i \quad \text{as} \quad z \to \infty. \tag{3}$$

In Zaitsev and Polyanin (1989, 1994), exact solutions to problem (2)–(3) are specified for $\lambda = -\frac{1}{n+1}$ with $n = \frac{1}{5}, \frac{1}{4}, \frac{1}{2}, \frac{3}{5}, \frac{5}{7}, 2$.

Special case 2. The generalized Schlichting problem on the symmetric flow of a plane laminar power-law fluid jet out of a thin slit is characterized by the boundary conditions

$$\partial_x w = \partial_{yy} w = 0 \quad \text{at} \quad y = 0, \qquad \partial_y w \to 0 \quad \text{as} \quad y \to \infty,$$

which are supplemented with the integral condition of conservation of momentum

$$\int_0^\infty (\partial_y w)^2\, dy = A \qquad (A = \text{const}).$$

A solution to this problem (in the domain $x \geq 0$, $y \geq 0$) is sought in the form (1) with $\lambda = -\frac{2}{3n}$. A solution to equation (2) for $\psi(z)$ with appropriate boundary conditions and integral condition (see the conditions in Special case 3, Subsection 9.3.1, where F should be replaced by ψ) can be found in the books by Shulman and Berkovskii (1966) and Polyanin, Kutepov, et al. (2002).

5°. Self-similar solution for $n = 2$ (λ is any):

$$w(x,y) = x^\lambda U(z), \qquad z = yx^{-1/3},$$

where the function $U = U(z)$ is determined by the autonomous ordinary differential equation

$$(\lambda - \tfrac{1}{3})(U_z')^2 + \lambda U U_{zz}'' = kU_{zz}'' U_{zzz}''',$$

whose order can be reduced by two.

6°. Generalized self-similar solution (λ is any):

$$w(x,y) = e^{\lambda(2n-1)x}\Phi(\tau), \qquad \tau = e^{\lambda(2-n)x}y,$$

where the function $\Phi = \Phi(\tau)$ is determined by the autonomous ordinary differential equation

$$\lambda(n+1)(\Phi'_\tau)^2 - \lambda(2n-1)\Phi\Phi''_{\tau\tau} = k(\Phi''_{\tau\tau})^{n-1}\Phi'''_{\tau\tau\tau},$$

whose order can be reduced by two.

⊙ *Reference*: Z. P. Shulman and B. M. Berkovskii (1966).

7°. Solution for $n \neq 1/2$:

$$w(x,y) = C_1 \ln|x| + C_2 + g(\xi), \qquad \xi = x^{\frac{1}{1-2n}}y,$$

where the function $g = g(\xi)$ is determined by the autonomous ordinary differential equation

$$\frac{1}{1-2n}(g'_\xi)^2 - C_1 g''_{\xi\xi} = k(g''_{\xi\xi})^{n-1}g'''_{\xi\xi\xi},$$

whose order can be reduced by two.

8°. Solution for $n = 1/2$:

$$w(x,y) = C_1 x + C_2 + h(\zeta), \qquad \zeta = e^{\lambda x}y,$$

where the function $h = h(\zeta)$ is determined by the autonomous ordinary differential equation

$$\lambda(h'_\zeta)^2 - C_1 h''_{\zeta\zeta} = k(h''_{\zeta\zeta})^{-1/2}h'''_{\zeta\zeta\zeta},$$

whose order can be reduced by two.

9°. Conservation law:

$$D_x\left(nw_y^2\right) + D_y\left(-nw_x w_y - kw_{yy}^n\right) = 0,$$

where $D_x = \frac{\partial}{\partial x}$ and $D_y = \frac{\partial}{\partial y}$.

2. $\displaystyle \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = k\left(\frac{\partial^2 w}{\partial y^2}\right)^{n-1}\frac{\partial^3 w}{\partial y^3} + f(x).$

This is a steady boundary layer equation for a power-law fluid with pressure gradient.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = w(x, y + \varphi(x)) + C,$$

where $\varphi(x)$ is an arbitrary function and C is an arbitrary constant, is also a solution of the equation.

⊙ *Reference*: A. D. Polyanin (2001 *a*).

2°. Degenerate solutions (linear and quadratic in y) for any $f(x)$:

$$w(x,y) = \pm y\left[2\int f(x)\,dx + C_1\right]^{1/2} + \varphi(x),$$
$$w(x,y) = C_1 y^2 + \varphi(x)y + \frac{1}{4C_1}\left[\varphi^2(x) - 2\int f(x)\,dx\right] + C_2,$$

where $\varphi(x)$ is an arbitrary function and C_1 and C_2 are arbitrary constants. These solutions are independent of k and correspond to inviscid fluid flows.

3°. Self-similar solution for $f(x) = ax^m$:

$$w(x,y) = x^{\frac{2nm+2n-m+1}{2(n+1)}}\psi(z), \qquad z = x^{\frac{2m-n-nm}{2(n+1)}}y,$$

where the function $\psi = \psi(z)$ is determined by the autonomous ordinary differential equation

$$\frac{nm+n+m+1}{2(n+1)}(\psi'_z)^2 - \frac{2nm+2n-m+1}{2(n+1)}\psi\psi''_{zz} = k(\psi''_{zz})^{n-1}\psi'''_{zzz} + a.$$

Note that solving the generalized Falkner–Skan problem on a symmetric power-law fluid flow past a wedge is reduced to solving the equation just obtained.

⦿ *Reference*: Z. P. Shulman and B. M. Berkovskii (1966).

4°. Generalized self-similar solution for $f(x) = ae^{\beta x}$:

$$w(x,y) = \exp\left(\beta\frac{2n-1}{2n+2}x\right)\Phi(\tau), \qquad \tau = \exp\left(\beta\frac{2-n}{2n+2}x\right)y,$$

where the function $\Phi = \Phi(\tau)$ is determined by the autonomous ordinary differential equation

$$\frac{1}{2}\beta(\Phi'_\tau)^2 - \beta\frac{2n-1}{2n+2}\Phi\Phi''_{\tau\tau} = k(\Phi''_{\tau\tau})^{n-1}\Phi'''_{\tau\tau\tau} + a.$$

⦿ *Reference*: Z. P. Shulman and B. M. Berkovskii (1966).

5°. Additive separable solution for $f(x) = a$:

$$w(x,y) = C_1x + h(y),$$

where the function $h = h(y)$ is determined by the autonomous ordinary differential equation

$$k(h''_{yy})^{n-1}h'''_{yyy} + C_1h''_{yy} + a = 0.$$

Its general solution can be written out in parametric form:

$$y = -k\int_{C_2}^{t}\frac{u^{n-1}\,du}{C_1u+a}, \qquad h = k^2\int_{C_3}^{t}\frac{u^{n-1}\varphi(u)\,du}{C_1u+a}, \qquad \text{where} \quad \varphi(u) = \int_{C_4}^{u}\frac{v^n\,dv}{C_1v+a}.$$

6°. Multiplicative separable solution for $f(x) = ax^{\frac{n}{2-n}}$, $n \neq 2$:

$$w(x,y) = x^{\frac{1}{2-n}}F(y),$$

where the function $F = F(y)$ is determined by the autonomous ordinary differential equation

$$\frac{1}{2-n}(F'_y)^2 - \frac{1}{2-n}FF''_{yy} = k(F''_{yy})^{n-1}F'''_{yyy} + a.$$

7°. Self-similar solution for $f(x) = ax^m$, $n = 2$:

$$w(x,y) = x^{\frac{1}{2}m+\frac{5}{6}}U(z), \qquad z = yx^{-1/3},$$

where the function $U = U(z)$ is determined by the autonomous ordinary differential equation

$$\tfrac{1}{2}(m+1)(U'_z)^2 + \tfrac{1}{6}(3m+5)UU''_{zz} = kU''_{zz}U'''_{zzz} + a.$$

8°. Multiplicative separable solution for $f(x) = ae^{\lambda x}$, $n = 2$:

$$w(x,y) = e^{\frac{1}{2}\lambda x}G(y),$$

where the function $G = G(y)$ is determined by the autonomous ordinary differential equation

$$\tfrac{1}{2}\lambda(G'_y)^2 - \tfrac{1}{2}\lambda GG''_{yy} = k(G''_{yy})^{n-1}G'''_{yyy} + a.$$

9°. Solution $f(x) = ax^{\frac{2n+1}{1-2n}}$, $n \neq 1/2$:

$$w(x,y) = C_1 \ln|x| + C_2 + g(\xi), \qquad \xi = x^{\frac{1}{1-2n}} y,$$

where the function $g = g(\xi)$ is determined by the autonomous ordinary differential equation

$$k(g''_{\xi\xi})^{n-1} g'''_{\xi\xi\xi} + C_1 g''_{\xi\xi} - \frac{1}{1-2n}(g'_\xi)^2 + a = 0,$$

10°. Solution $f(x) = ae^{\lambda x}$, $n = 1/2$:

$$w(x,y) = C_1 x + C_2 + h(\zeta), \qquad \zeta = e^{\frac{1}{2}\lambda x} y,$$

where the function $h = h(\zeta)$ is determined by the autonomous ordinary differential equation

$$k(h''_{\zeta\zeta})^{-1/2} h'''_{\zeta\zeta\zeta} + C_1 h''_{\zeta\zeta} - \tfrac{1}{2}\lambda(h'_\zeta)^2 + a = 0.$$

11°. Conservation law:

$$D_x\left[nw_y^2 - nF(x)\right] + D_y\left(-nw_x w_y - kw_{yy}^n\right) = 0,$$

where $D_x = \dfrac{\partial}{\partial x}$, $D_y = \dfrac{\partial}{\partial y}$, and $F(x) = \displaystyle\int f(x)\,dx$.

3. $\displaystyle \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left[f\left(\frac{\partial^2 w}{\partial y^2}\right)\right].$

This is an equation of a steady boundary layer on a flat plate in the flow of a non-Newtonian fluid of general form; w is the steam function, and x and y are the coordinates along and normal to the plate.

Preliminary remarks. The system of non-Newtonian fluid boundary layer equations

$$u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} = \frac{\partial}{\partial y}\left[f\left(\frac{\partial u_1}{\partial y}\right)\right],$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0,$$

where u_1 and u_2 are the longitudinal and normal fluid velocity components, can be reduced to the equation in question by the introduction of a stream function w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^{-2} w(C_1^3 x + C_2, C_1 y + C_3) + C_4,$$
$$w_2 = w(x, y + \varphi(x)),$$

where $C_1, \ldots, C_4$ are arbitrary constants and $\varphi(x)$ is an arbitrary function, are also solutions of the equation.

⊙ *Reference*: A. D. Polyanin (2001 *a*).

2°. Solutions involving arbitrary functions:

$$w(x,y) = C_1 y^2 + \varphi(x) y + \frac{1}{4C_1}\varphi^2(x) + C_2,$$
$$w(x,y) = g(z) + C_1 x + C_2, \qquad z = y + \varphi(x),$$

where C_1 and C_2 are arbitrary constants, and $\varphi(x)$ is an arbitrary function. The function $g = g(z)$ in the second formula is determined by the autonomous ordinary differential equation

$$f(g''_{zz}) + C_1 g'_z = C_3,$$

whose general solution can be written out in parametric form:

$$g = \frac{1}{C_1^2}\int \frac{f_t'(t)}{t}\big[f(t) - C_2\big]\,dt + C_3, \qquad z = C_4 - \frac{1}{C_1}\int \frac{f_t'(t)}{t}\,dt.$$

3°. Self-similar solution:

$$w(x,y) = x^{2/3}\psi(\xi), \qquad \xi = yx^{-1/3},$$

where the function $\psi = \psi(\xi)$ is determined by the autonomous ordinary differential equation

$$(\psi_\xi')^2 - 2\psi\psi_{\xi\xi}'' = 3[f(\psi_{\xi\xi}'')]_\xi'.$$

4°. The von Mises transformation

$$\xi = x, \quad \eta = w, \quad U(\xi,\eta) = \frac{\partial w}{\partial y}, \qquad \text{where} \quad w = w(x,y),$$

leads to the second-order nonlinear equation

$$\frac{\partial U}{\partial \xi} = \frac{\partial}{\partial \eta}\left[f\left(U\frac{\partial U}{\partial \eta}\right)\right].$$

It admits, for example, a traveling-wave solution $U = U(a\xi + b\eta)$.

5°. Conservation law:

$$D_x\big(w_y^2\big) + D_y\big[-w_x w_y - f(w_{yy})\big] = 0,$$

where $D_x = \frac{\partial}{\partial x}$ and $D_y = \frac{\partial}{\partial y}$.

4. $\displaystyle \boldsymbol{\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left[f\left(\frac{\partial^2 w}{\partial y^2}\right)\right] + g(x).}$

This is a steady boundary layer equation for a non-Newtonian fluid of general form with pressure gradient.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = w(x, y + \varphi(x)) + C,$$

where $\varphi(x)$ is an arbitrary function and C is an arbitrary constant, is also a solution of the equation.

2°. There are degenerate solutions; see Item 2° in 9.3.2.2, where $f(x)$ should be replaced by $g(x)$.

3°. Solution for $g(x) = a$:

$$w(x,y) = \zeta(z) + C_1 x + C_2, \qquad z = y + \varphi(x),$$

where $\varphi(x)$ is an arbitrary function and C_1 and C_2 are arbitrary constants. The function $\zeta = \zeta(z)$ is determined by the ordinary differential equation

$$f(\zeta_{zz}'') + C_1\zeta_z' + a\zeta = C_3.$$

4°. Self-similar solution for $g(x) = a(x+b)^{-1/3}$:

$$w(x,y) = (x+b)^{2/3}\psi(\xi), \qquad \xi = y(x+b)^{-1/3},$$

where the function $\psi = \psi(\xi)$ is determined by the autonomous ordinary differential equation

$$(\psi_\xi')^2 - 2\psi\psi_{\xi\xi}'' = 3[f(\psi_{\xi\xi}'')]_\xi' + 3a.$$

5°. Conservation law:

$$D_x\big[w_y^2 - G(x)\big] + D_y\big[-w_x w_y - f(w_{yy})\big] = 0,$$

where $D_x = \dfrac{\partial}{\partial x}$, $D_y = \dfrac{\partial}{\partial y}$, and $G(x) = \displaystyle\int g(x)\,dx$.

9.3.3. Unsteady Boundary Layer Equations for a Newtonian Fluid

1. $\dfrac{\partial^2 w}{\partial t \partial y} + \dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = \nu \dfrac{\partial^3 w}{\partial y^3}.$

This equation describes an unsteady hydrodynamic boundary layer on a flat plate; w is the stream function, x and y are the coordinates along and normal to the plate, respectively, and ν is the kinematic viscosity of the fluid. A similar equation describes an unsteady flow of a plane laminar jet out of a thin slit.

Preliminary remarks. The system of unsteady hydrodynamic boundary layer equations

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} = \nu\frac{\partial^2 u_1}{\partial y^2},$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0,$$

where u_1 and u_2 are the longitudinal and normal fluid velocity components, can be reduced to the equation in question by the introduction of a stream function w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x, y + \varphi(x,t), t) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx + \chi(t),$$
$$w_2 = C_1 w(C_2 x + C_2 C_3 t + C_4,\ C_1 C_2 y + C_1 C_2 C_5 t + C_6,\ C_1^2 C_2^2 t + C_7) + C_5 x - C_3 y + C_8,$$

where $\varphi(x,t)$ and $\chi(t)$ are arbitrary functions and the C_n are arbitrary constants, are also solutions of the equation.

⊙ *References*: L. I. Vereshchagina (1973), L. V. Ovsiannikov (1982).

2°. Degenerate solutions linear and quadratic in y:

$$w = C_1 y + \varphi(x,t),$$
$$w = C_1 y^2 + \varphi(x,t) y + \frac{1}{4C_1}\varphi^2(x,t) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx,$$

where $\varphi(x,t)$ is an arbitrary function of two variables and C_1 is an arbitrary constant. Here and henceforth, the additive arbitrary function of time, $\chi = \chi(t)$, in exact solutions for the stream function is omitted. These solutions are independent of ν and correspond to inviscid fluid flows.

3°. Solutions involving arbitrary functions:

$$w = \frac{6\nu x + C_1}{y + \varphi(x,t)} + \frac{C_2}{[y + \varphi(x,t)]^2} + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx,$$
$$w = C_1 \exp\bigl[-C_2 y - C_2\varphi(x,t)\bigr] + C_3 y + C_3\varphi(x,t) + \nu C_2 x + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx,$$
$$w = 6\nu C_1 x^{1/3}\tanh\xi + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx, \qquad \xi = C_1\frac{y + \varphi(x,t)}{x^{2/3}},$$
$$w = -6\nu C_1 x^{1/3}\tan\xi + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx, \qquad \xi = C_1\frac{y + \varphi(x,t)}{x^{2/3}},$$

where $\varphi(x,t)$ is an arbitrary function of two variables, and C_1, C_2, and C_3 are arbitrary constants. The construction of these solutions was based on the simpler, stationary solutions specified in 9.3.1.1.

Note also the solution

$$w = f(x)\exp\bigl[-\lambda y - \lambda g(t)\bigr] + \bigl[\nu\lambda + g_t'(t)\bigr]x,$$

where $f(x)$ and $g(t)$ are arbitrary functions and λ is an arbitrary constant. It can be obtained from the second of the solutions specified above with $\varphi(x,t) = -\frac{1}{\lambda}\ln f(x) + g(t)$, $C_2 = \lambda$, and $C_3 = 0$.

⊙ *References*: G. I. Burde (1995), A. D. Polyanin (2001 *b*, 2002), A. D. Polyanin and V. F. Zaitsev (2001).

TABLE 7
Exact solutions of equation (2) in 9.3.3.1

No.	Function $F = F(y,t)$ (or general form of solution)	Remarks (or determining equations)
1	$F = \psi(t)$	$\psi(t)$ is an arbitrary function
2	$F = \frac{y}{t+C_1} + \psi(t)$	$\psi(t)$ is an arbitrary function, C_1 is any
3	$F = \frac{6\nu}{y+\psi(t)} + \psi'_t(t)$	$\psi(t)$ is an arbitrary function
4	$F = C_1 \exp\bigl[-\lambda y + \lambda\psi(t)\bigr] - \psi'_t(t) + \nu\lambda$	$\psi(t)$ is an arbitrary function, C_1, λ are any
5	$F = F(\xi),\ \xi = y + \lambda t$	$\lambda F''_{\xi\xi} + (F'_\xi)^2 - FF''_{\xi\xi} = \nu F'''_{\xi\xi\xi}$
6	$F = t^{-1/2}\bigl[H(\xi) - \frac{1}{2}\xi\bigr],\ \xi = yt^{-1/2}$	$\frac{3}{4} - 2H'_\xi + (H'_\xi)^2 - HH''_{\xi\xi} = \nu H'''_{\xi\xi\xi}$

4°. Generalized separable solution linear in x:

$$w(x,y,t) = xF(y,t) + G(y,t), \tag{1}$$

where the functions $F = F(y,t)$ and $G = G(y,t)$ are determined from the simpler equations in two variables

$$\frac{\partial^2 F}{\partial t \partial y} + \left(\frac{\partial F}{\partial y}\right)^2 - F\frac{\partial^2 F}{\partial y^2} = \nu\frac{\partial^3 F}{\partial y^3}, \tag{2}$$

$$\frac{\partial^2 G}{\partial t \partial y} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} - F\frac{\partial^2 G}{\partial y^2} = \nu\frac{\partial^3 G}{\partial y^3}. \tag{3}$$

Equation (2) is solved independently of (3). If $F = F(y,t)$ is a solution of equation (2), then the functions

$$F_1 = F(y + \psi(t), t) + \psi'_t(t),$$
$$F_2 = C_1 F(C_1 y + C_1 C_2 t + C_3, C_1^2 t + C_4) + C_2,$$

where $\psi(t)$ is an arbitrary function and $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

Given a particular solution $F = F(y,t)$ of equation (2), the corresponding equation (3) can be reduced, with the substitution $U = \frac{\partial G}{\partial y}$, to the second-order linear equation

$$\frac{\partial U}{\partial t} - F\frac{\partial U}{\partial y} = \nu\frac{\partial^2 U}{\partial y^2} - \frac{\partial F}{\partial y}U. \tag{4}$$

Table 7 lists exact solutions of equation (2). The ordinary differential equations in the last two rows, determining a traveling-wave solution and a self-similar one, are both autonomous and, hence, their order can be reduced.

Table 8 presents transformations that simplify equation (4) corresponding to respective solutions of equation (2) in Table 7. It is apparent that in the first three cases, solutions of equation (4) are expressed via solutions of a linear constant-coefficient heat equation. In the other three cases, equation (4) is reduced to linear equations, which can be solved by the method of separation of variables.

The fourth equation in Table 8 has the following particular solutions (A and B are any):

$$Z(\eta) = A + B\int \Phi(\eta)\,d\eta, \quad \Phi(\eta) = \exp\left(\frac{C_1}{\nu\lambda}e^\eta - \eta\right);$$

$$Z(\eta,t) = A\nu\lambda^2 t + A\int \Phi(\eta)\left[\int \frac{d\eta}{\Phi(\eta)}\right]d\eta.$$

TABLE 8

Transformations of equation (4) for the respective exact solutions of equation (2); the number in the first column corresponds to the number of the exact solution $F = F(y,t)$ in Table 7

No.	Transformation of equation (4)	Resulting equation
1	$U = u(\zeta,t),\ \zeta = y + \int \psi(t)\,dt$	$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial \zeta^2}$
2	$U = \frac{1}{t+C_1} u(z,\tau),\ \tau = \frac{1}{3}(t+C_1)^3 + C_2,$ $z = (t+C_1)y + \int \psi(t)(t+C_1)\,dt + C_3$	$\frac{\partial u}{\partial \tau} = \nu \frac{\partial^2 u}{\partial z^2}$
3	$U = \zeta^{-3} u(\zeta,t),\ \zeta = y + \psi(t)$	$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial \zeta^2}$
4	$U = e^{\eta} Z(\eta,t),\ \eta = -\lambda y + \lambda\psi(t)$	$\frac{\partial Z}{\partial t} = \nu\lambda^2 \frac{\partial^2 Z}{\partial \eta^2} + (\nu\lambda^2 - C_1\lambda e^{\eta})\frac{\partial Z}{\partial \eta}$
5	$U = u(\xi,t),\ \ \xi = y + \lambda t$	$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial \xi^2} + \big[F(\xi) - \lambda\big]\frac{\partial u}{\partial \xi} - F'_\xi(\xi)u$
6	$U = t^{-1/2} u(\xi,\tau),\ \xi = yt^{-1/2},\ \tau = \ln t$	$\frac{\partial u}{\partial \tau} = \nu \frac{\partial^2 u}{\partial \xi^2} + H(\xi)\frac{\partial u}{\partial \xi} + \big[1 - H'_\xi(\xi)\big]u$

For other exact solutions of this equation, see the book by Polyanin (2002), where a more general equation, $\partial_t w = f(x)\partial_{xx} w + g(x)\partial_x w$, was considered.

Equation 5 in Table 8 has a stationary particular solution $u_0 = F'_\xi(\xi)$ (cf. equation 5 in Table 7). Consequently, other particular solutions of this equation are given by

$$u(\xi) = C_1 F'_\xi(\xi) + C_2 F'_\xi(\xi) \int \frac{\Psi(\xi)\,d\xi}{[F'_\xi(\xi)]^2}, \quad \Psi(\xi) = \exp\left[\frac{\lambda}{\nu}\xi - \frac{1}{\nu}\int F(\xi)\,d\xi\right];$$

$$u(\xi,t) = C_1 \nu t F'_\xi(\xi) + C_1 F'_\xi(\xi) \int \frac{\Psi(\xi)\Phi(\xi)}{[F'_\xi(\xi)]^2}\,d\xi, \quad \Phi(\xi) = \int \frac{[F'_\xi(\xi)]^2}{\Psi(\xi)}\,d\xi;$$

see Polyanin (2002).

⊙ *References*: D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (2000), A. D. Polyanin (2001 *b*, 2002), A. D. Polyanin and V. F. Zaitsev (2001, 2002).

Example 1. Solution exponentially dependent on time:

$$w(x,y,t) = f(y)x + e^{-\lambda t} \int g(y)\,dy,$$

where the functions $f = f(y)$ and $g = g(y)$ are determined by the system of ordinary differential equations

$$(f'_y)^2 - f f''_{yy} = \nu f'''_{yyy},$$
$$-\lambda g + g f'_y - f g'_y = \nu g''_{yy}.$$

Example 2. Periodic solution:

$$w(x,y,t) = f(y)x + \sin(\lambda t) \int g(y)\,dy + \cos(\lambda t) \int h(y)\,dy,$$

where the functions $f = f(y)$, $g = g(y)$, and $h = h(y)$ are determined by the system of ordinary differential equations

$$(f'_y)^2 - f f''_{yy} = \nu f'''_{yyy},$$
$$-\lambda h + f'_y g - f g'_y = \nu g''_{yy},$$
$$\lambda g + f'_y h - f h'_y = \nu h''_{yy}.$$

5°. Generalized separable solution:

$$w(x,y,t) = \bigl[A(t)e^{k_1 x} + B(t)e^{k_2 x}\bigr]e^{\lambda y} + \varphi(t)x + ay,$$
$$A(t) = C_1 \exp\left[(\nu\lambda^2 - ak_1)t + \lambda \int \varphi(t)\,dt\right],$$
$$B(t) = C_2 \exp\left[(\nu\lambda^2 - ak_2)t + \lambda \int \varphi(t)\,dt\right],$$

where $\varphi(t)$ is an arbitrary function and C_1, C_2, a, k_1, k_2, and λ are arbitrary constants.

6°. Generalized separable solution:

$$w(x,y,t) = A(t)\exp(kx + \lambda y) + B(t)\exp(\beta kx + \beta\lambda y) + \varphi(t)x + ay,$$
$$A(t) = C_1 \exp\left[(\nu\lambda^2 - ak)t + \lambda \int \varphi(t)\,dt\right],$$
$$B(t) = C_2 \exp\left[(\nu\beta^2\lambda^2 - ak\beta)t + \beta\lambda \int \varphi(t)\,dt\right],$$

where $\varphi(t)$ is an arbitrary function and C_1, C_2, a, k, β, and λ are arbitrary constants.

⊙ *References*: A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2001).

7°. "Two-dimensional" solution:

$$w(x,y,t) = \int u(z,t)\,dz + \varphi(t)y + \psi(t)x, \qquad z = kx + \lambda y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k and λ are arbitrary constants, and the function $u(z,t)$ is determined by the second-order linear differential equation

$$\frac{\partial u}{\partial t} + \bigl[k\varphi(t) - \lambda\psi(t)\bigr]\frac{\partial u}{\partial z} = \nu\lambda^2 \frac{\partial^2 u}{\partial z^2} - \frac{1}{\lambda}\varphi'_t(t).$$

The transformation

$$u = U(\xi,t) - \frac{1}{\lambda}\varphi(t), \qquad \xi = z - \int \bigl[k\varphi(t) - \lambda\psi(t)\bigr]\,dt$$

brings it to the linear heat equation

$$\frac{\partial U}{\partial t} = \nu\lambda^2 \frac{\partial^2 U}{\partial \xi^2}.$$

⊙ *References*: A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2001).

8°. Solutions:

$$w = e^{\nu\lambda^2 t}(C_1 e^{\lambda z} + C_2 e^{-\lambda z}) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx, \qquad z = y + \varphi(x,t);$$
$$w = e^{-\nu\lambda^2 t}\bigl[C_1 \sin(\lambda z) + C_2 \cos(\lambda z)\bigr] + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx, \qquad z = y + \varphi(x,t);$$
$$w = C_1 e^{-\nu\lambda^2 z}\sin(\lambda z - 2\nu\lambda^2 t + C_2) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx, \qquad z = y + \varphi(x,t),$$

where $\varphi(x,t)$ is an arbitrary function of two arguments; C_1, C_2, and λ are arbitrary constants. For periodic function $\varphi(x,t) = \varphi(x,t+T)$, the last solution is also periodic, $w(x,y,t) = w(x,y,t+T)$, if $\lambda = \sqrt{\pi/(\nu T)}$.

9°. "Two-dimensional" solution:

$$w = W(\xi, \eta) + a_1 x + a_2 y, \qquad \xi = k_1 x + \lambda_1 t, \qquad \eta = k_2 y + \lambda_2 t,$$

where the function W is determined by the differential equation

$$(\lambda_1 + a_2 k_1)\frac{\partial^2 W}{\partial\xi\partial\eta} + (\lambda_2 - a_1 k_2)\frac{\partial^2 W}{\partial\eta^2} + k_1 k_2\left(\frac{\partial W}{\partial\eta}\frac{\partial^2 W}{\partial\xi\partial\eta} - \frac{\partial W}{\partial\xi}\frac{\partial^2 W}{\partial\eta^2}\right) = \nu k_2^2 \frac{\partial^3 W}{\partial\eta^3}.$$

In the special case

$$a_1 = \lambda_2/k_2, \qquad a_2 = -\lambda_1/k_1,$$

we have the steady boundary layer equation 9.3.1.1:

$$\frac{\partial W}{\partial\eta}\frac{\partial^2 W}{\partial\xi\partial\eta} - \frac{\partial W}{\partial\xi}\frac{\partial^2 W}{\partial\eta^2} = \beta\frac{\partial^3 W}{\partial\eta^3}, \qquad \beta = \nu\frac{k_2}{k_1}.$$

10°. "Two-dimensional" solution:

$$w = V(\xi, \eta), \qquad \xi = \frac{x}{\sqrt{t}}, \qquad \eta = \frac{y}{\sqrt{t}},$$

where the function V is determined by the differential equation

$$-\frac{1}{2}\frac{\partial V}{\partial\eta} - \frac{1}{2}\xi\frac{\partial^2 V}{\partial\xi\partial\eta} - \frac{1}{2}\eta\frac{\partial^2 V}{\partial\eta^2} + \frac{\partial V}{\partial\eta}\frac{\partial^2 V}{\partial\xi\partial\eta} - \frac{\partial V}{\partial\xi}\frac{\partial^2 V}{\partial\eta^2} = \nu\frac{\partial^3 V}{\partial\eta^3}.$$

For example, this equation has solutions of the form $V = F(\eta)\xi + G(\eta)$.

⊙ *Reference*: L. V. Ovsiannikov (1982).

2. $\displaystyle \frac{\partial^2 w}{\partial t\partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = \nu\frac{\partial^3 w}{\partial y^3} + f(x,t).$

This equation describes an unsteady hydrodynamic boundary layer with pressure gradient.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x, y + \varphi(x,t), t) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx,$$
$$w_2 = -w(x, -y, t) + \psi(t),$$

where $\varphi(x,t)$ and $\psi(t)$ are arbitrary functions, are also solutions of the equation.

⊙ *References*: L. I. Vereshchagina (1973), L. V. Ovsiannikov (1982).

2°. For $f(x,t) = g(t)$, the transformation

$$w = u(\xi, y, t) - h'_t(t)y, \qquad \xi = x + h(t), \qquad \text{where} \quad h(t) = -\int_{t_0}^{t}(t-\tau)g(\tau)\,d\tau, \tag{1}$$

leads to a simpler equation of the form 9.3.3.1:

$$\frac{\partial^2 u}{\partial t\partial y} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial\xi\partial y} - \frac{\partial u}{\partial\xi}\frac{\partial^2 u}{\partial y^2} = \nu\frac{\partial^3 u}{\partial y^3}.$$

Note that $f = g(t)$ and $h = h(t)$ are related by the simple constraint $h''_{tt} = -g$.

In the general case, transformation (1) brings the equation in question to a similar equation with the function $f(x,t)$ modified according to

$$f(x,t) \quad \xrightarrow{\text{transformation (1)}} \quad f(x,t) - g(t).$$

⊙ *Reference*: L. V. Ovsiannikov (1982).

3°. Degenerate solution (quadratic in y) for any $f(x,t)$:

$$w(x,y,t)=Cy^2+\varphi(x,t)y+\frac{1}{4C}\varphi^2(x,t)+\frac{1}{2C}\int\left[\frac{\partial\varphi}{\partial t}-f(x,t)\right]dx,$$

where $\varphi(x,t)$ is an arbitrary function of two arguments and C is an arbitrary constant. From now on, the arbitrary additive function of time $\psi=\psi(t)$ is omitted in exact solutions for the stream function. These solutions are independent of ν and correspond to inviscid fluid flows.

Degenerate solution (linear in y) for any $f(x,t)$:

$$w(x,y,t)=\psi(x,t)y+\varphi(x,t),$$

where $\varphi(x,t)$ is an arbitrary function, and $\psi=\psi(x,t)$ is determined by the first-order partial differential equation

$$\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi}{\partial x}=f(x,t).$$

For information about the methods of integration and exact solutions of such equations (for various f), see the books by Kamke (1965) and Polyanin, Zaitsev, and Moussiaux (2002).

Degenerate solutions for $f(x,t)=f(x)$:

$$w(x,y,t)=\pm y\left[2\int f(x)\,dx+C_1\right]^{1/2}+\varphi(x,t),$$

where $\varphi(x,t)$ is an arbitrary function.

4°. Generalized separable solution (linear in x) for $f(x,t)=f_1(t)x+f_2(t)$:

$$w(x,y,t)=xF(y,t)+G(y,t), \tag{2}$$

where the functions $F=F(y,t)$ and $G=G(y,t)$ are determined by the simpler equations in two variables

$$\frac{\partial^2 F}{\partial t\partial y}+\left(\frac{\partial F}{\partial y}\right)^2-F\frac{\partial^2 F}{\partial y^2}=\nu\frac{\partial^3 F}{\partial y^3}+f_1(t), \tag{3}$$

$$\frac{\partial^2 G}{\partial t\partial y}+\frac{\partial F}{\partial y}\frac{\partial G}{\partial y}-F\frac{\partial^2 G}{\partial y^2}=\nu\frac{\partial^3 G}{\partial y^3}+f_2(t). \tag{4}$$

Equation (3) is solved independently of equation (4).

If $F=F(y,t)$ is a solution to equation (3), then the function

$$F_1=F(y+\psi(t),t)+\psi'_t(t),$$

where $\psi(t)$ is an arbitrary function, is also a solution of the equation.

Table 9 lists exact solutions of equation (3) for various $f_1=f_1(t)$; two more complicated solutions of this equation are given at the end of Item 4°. Note that, for $G\equiv 0$, solutions (2) specified in the first and the last rows of Table 9 were treated in the book by Ovsiannikov (1982).

The substitution $U=\frac{\partial G}{\partial y}$ brings equation (4) to the second-order linear equation

$$\frac{\partial U}{\partial t}-F\frac{\partial U}{\partial y}=\nu\frac{\partial^2 U}{\partial y^2}-\frac{\partial F}{\partial y}U+f_2(t). \tag{5}$$

Let us dwell on the first solution to (3) specified in Table 9:

$$F(y,t)=a(t)y+\psi(t),\qquad\text{where}\qquad a'_t+a^2=f_1(t). \tag{6}$$

The Riccati equation for $a=a(t)$ is reduced by the substitution $a=h'_t/h$ to the second-order linear equation $h''_{tt}-f_1(t)h=0$. Exact solutions of this equation for various $f_1(t)$ can be found in Kamke (1977) and Polyanin and Zaitsev (2003). In particular, for $f_1(t)=\text{const}$ we have

$$a(t)=k\frac{C_1\cos(kt)-C_2\sin(kt)}{C_1\sin(kt)+C_2\cos(kt)}\quad\text{if}\quad f_1=-k^2<0,$$

$$a(t)=k\frac{C_1\cosh(kt)+C_2\sinh(kt)}{C_1\sinh(kt)+C_2\cosh(kt)}\quad\text{if}\quad f_1=k^2>0.$$

TABLE 9
Exact solutions of equation (3) in 9.3.3.2 for various $f_1(t)$; $\psi(t)$ is an arbitrary function

Function $f_1 = f_1(t)$	Function $F = F(y,t)$ (or general form of solution)	Determining equation (or determining coefficients)
Any	$F = a(t)y + \psi(t)$	$a'_t + a^2 = f_1(t)$
$f_1(t) = Ae^{-\beta t}$, $A > 0$, $\beta > 0$	$F = Be^{-\frac{1}{2}\beta t}\sin[\lambda y + \lambda\psi(t)] + \psi'_t(t)$, $F = Be^{-\frac{1}{2}\beta t}\cos[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$B = \pm\sqrt{\frac{2A\nu}{\beta}}$, $\lambda = \sqrt{\frac{\beta}{2\nu}}$
$f_1(t) = Ae^{\beta t}$, $A > 0$, $\beta > 0$	$F = Be^{\frac{1}{2}\beta t}\sinh[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$B = \pm\sqrt{\frac{2A\nu}{\beta}}$, $\lambda = \sqrt{\frac{\beta}{2\nu}}$
$f_1(t) = Ae^{\beta t}$, $A < 0$, $\beta > 0$	$F = Be^{\frac{1}{2}\beta t}\cosh[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$B = \pm\sqrt{\frac{2\lvert A\rvert\nu}{\beta}}$, $\lambda = \sqrt{\frac{\beta}{2\nu}}$
$f_1(t) = Ae^{\beta t}$, A is any, $\beta > 0$	$F = \psi(t)e^{\lambda y} - \frac{Ae^{\beta t - \lambda y}}{4\lambda^2\psi(t)} + \frac{\psi'_t(t)}{\lambda\psi(t)} - \nu\lambda$	$\lambda = \pm\sqrt{\dfrac{\beta}{2\nu}}$
$f_1(t) = At^{-2}$	$F = t^{-1/2}\bigl[H(\xi) - \frac{1}{2}\xi\bigr]$, $\xi = yt^{-1/2}$	$\frac{3}{4} - A - 2H'_\xi + (H'_\xi)^2 - HH''_{\xi\xi} = \nu H'''_{\xi\xi\xi}$
$f_1(t) = A$	$F = F(\xi)$, $\xi = y + \lambda t$	$-A + \lambda F''_{\xi\xi} + (F'_\xi)^2 - FF''_{\xi\xi} = \nu F'''_{\xi\xi\xi}$

On substituting solution (6), with arbitrary $f_1(t)$, into equation (5), one obtains

$$\frac{\partial U}{\partial t} = \nu\frac{\partial^2 U}{\partial y^2} + \bigl[a(t)y + \psi(t)\bigr]\frac{\partial U}{\partial y} - a(t)U + f_2(t).$$

The transformation (Polyanin, 2002)

$$U = \frac{1}{\Phi(t)}\left[u(z,\tau) + \int f_2(t)\Phi(t)\,dt\right], \quad \tau = \int \Phi^2(t)\,dt + C_1,$$
$$z = y\Phi(t) + \int \psi(t)\Phi(t)\,dt + C_2, \quad \Phi(t) = \exp\left[\int a(t)\,dt\right],$$

leads to the linear heat equation

$$\frac{\partial u}{\partial \tau} = \nu\frac{\partial^2 u}{\partial z^2}.$$

⊙ *References*: D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (2000), A. D. Polyanin (2001 *b*, 2002), A. D. Polyanin and V. F. Zaitsev (2001, 2002).

Remark 1. The ordinary differential equations in the last two rows of Table 9 (see the last column), which determine a self-similar and a traveling-wave solution, are both autonomous and, hence, their order can be reduced.

Remark 2. Suppose $w(x,y,t)$ is a solution of the unsteady hydrodynamic boundary layer equation with $f(x,t) = f_1(t)x + f_2(t)$. Then the function

$$w_1 = w(x + h(t), y, t) - h'_t(t)y, \qquad \text{where} \quad h''_{tt} - f_1(t)h = 0,$$

is also a solution of the equation.

⊙ *Reference*: L. V. Ovsiannikov (1982).

Remark 3. In the special case $f_2(t) = 0$, equation (4) admits a particular solution $G = G(t)$, where $G(t)$ is an arbitrary function.

Example 1+. Solution with $f(x,t) = Ax + Be^{-\lambda t}$:

$$w(x,y,t) = xg(y) + e^{-\lambda t} \int h(y)\,dy,$$

where the functions $g = g(y)$ and $h = h(y)$ are determined by the system of ordinary differential equations

$$(g')^2 - gg'' = \nu g''' + A,$$
$$-\lambda h + hg' - gh' = \nu h'' + B.$$

The prime denotes a derivative with respect to y.

Example 2+. Periodic solution with $f(x,t) = Ax + B_1 \sin(\lambda t) + B_2 \cos(\lambda t)$:

$$w(x,y,t) = xg(y) + \sin(\lambda t) \int h_1(y)\,dy + \cos(\lambda t) \int h_2(y)\,dy,$$

where the functions $g = g(y)$, $h_1 = h_1(y)$, and $h_2 = h_2(y)$ are determined by the system of ordinary differential equations

$$(g')^2 - gg'' = \nu g''' + A,$$
$$-\lambda h_2 + g' h_1 - g h_1' = \nu h_1'' + B_1,$$
$$\lambda h_1 + g' h_2 - g h_2' = \nu h_2'' + B_2.$$

Below are two more complex solutions of equation (3).

The solution

$$F(y,t) = -\frac{\gamma_t'}{\gamma} y + \gamma^3 \exp\left(\nu \int \frac{dt}{\gamma^2}\right)\left(A \cosh \frac{y}{\gamma} + B \sinh \frac{y}{\gamma}\right),$$

where A and B are arbitrary constants and $\gamma = \gamma(t)$ is an arbitrary function, corresponds to the right-hand side of equation (3) in the form

$$f_1(t) = -\frac{\gamma_{tt}''}{\gamma} + 2\left(\frac{\gamma_t'}{\gamma}\right)^2 + (B^2 - A^2)\gamma^4 \exp\left(2\nu \int \frac{dt}{\gamma^2}\right).$$

The solution

$$F(y,t) = -\frac{\gamma_t'}{\gamma} y + \gamma^3 \exp\left(-\nu \int \frac{dt}{\gamma^2}\right)\left(A \cos \frac{y}{\gamma} + B \sin \frac{y}{\gamma}\right),$$

where A and B are arbitrary constants and $\gamma = \gamma(t)$ is an arbitrary function, corresponds to the right-hand side of equation (3) in the form

$$f_1(t) = -\frac{\gamma_{tt}''}{\gamma} + 2\left(\frac{\gamma_t'}{\gamma}\right)^2 + (A^2 + B^2)\gamma^4 \exp\left(-2\nu \int \frac{dt}{\gamma^2}\right).$$

This solution was obtained in Burde (1995) for the case $A = 0$.

5°. Generalized separable solution for $f(x,t) = g(x)e^{\beta t}$, $\beta > 0$:

$$w(x,y,t) = \varphi(x,t)e^{\lambda y} + \psi(x,t)e^{-\lambda y} + \frac{1}{\lambda}\frac{\partial}{\partial t} \int \ln|\varphi(x,t)|\,dx - \nu\lambda x,$$

$$\psi(x,t) = -\frac{e^{\beta t}}{2\lambda^2 \varphi(x,t)} \int g(x)\,dx, \qquad \lambda = \pm\sqrt{\frac{\beta}{2\nu}},$$

where $\varphi(x,t)$ is an arbitrary function of two arguments.

⊙ *References*: A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2002).

6°. Generalized separable solutions for $f(x,t) = g(x)e^{\beta t}$, $\beta > 0$:

$$w(x,y,t) = \pm\frac{1}{\lambda} \exp\left(\tfrac{1}{2}\beta t\right) \sqrt{\psi(x)}\, \sinh\left[\lambda y + \varphi(x,t)\right] + \frac{\partial}{\partial t} \int \varphi(x,t)\,dx,$$

$$w(x,y,t) = \pm\frac{1}{\lambda} \exp\left(\tfrac{1}{2}\beta t\right) \sqrt{\psi(x)}\, \cosh\left[\lambda y + \varphi(x,t)\right] + \frac{\partial}{\partial t} \int \varphi(x,t)\,dx,$$

$$\psi(x) = 2 \int g(x)\,dx + C_1, \qquad \lambda = \sqrt{\frac{\beta}{2\nu}},$$

where $\varphi(x,t)$ is an arbitrary function of two arguments.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

7°. Generalized separable solutions for $f(x,t) = g(x)e^{-\beta t}$, $\beta > 0$:

$$w(x,y,t) = \pm\frac{1}{\lambda}\exp\bigl(-\tfrac{1}{2}\beta t\bigr)\sqrt{\psi(x)}\,\sin\bigl[\lambda y + \varphi(x,t)\bigr] + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx,$$
$$w(x,y,t) = \pm\frac{1}{\lambda}\exp\bigl(-\tfrac{1}{2}\beta t\bigr)\sqrt{\psi(x)}\,\cos\bigl[\lambda y + \varphi(x,t)\bigr] + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx,$$
$$\psi(x) = 2\int g(x)\,dx + C_1, \qquad \lambda = \sqrt{\frac{\beta}{2\nu}},$$

where $\varphi(x,t)$ is an arbitrary function of two arguments.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

8°. Solution for $f(x,t) = xg(t)$:

$$w(x,y,t) = \frac{\psi'_t}{\psi}xy + \left(\frac{2\psi'_t}{\psi^2} - \nu\psi\right)x + \varphi(z)\exp(\psi y), \qquad z = \frac{x}{\psi}, \qquad \psi = \psi(t),$$

where $\varphi(z)$ is an arbitrary function and the function $\psi = \psi(t)$ is determined by the second-order linear ordinary differential equation

$$\psi''_{tt} = g(t)\psi.$$

⊙ *Reference*: D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (2000).

9°. Generalized separable solution for $f(x,t) = ae^{\beta x - \gamma t}$:

$$w(x,y,t) = \varphi(x,t)e^{\lambda y} - \frac{a}{2\beta\lambda^2\varphi(x,t)}e^{\beta x - \lambda y - \gamma t}$$
$$+\frac{1}{\lambda}\frac{\partial}{\partial t}\int \ln|\varphi(x,t)|\,dx - \nu\lambda x + \frac{2\nu\lambda^2 + \gamma}{\beta}\left(y + \frac{1}{\lambda}\ln|\varphi(x,t)|\right),$$

where $\varphi(x,t)$ is an arbitrary function of two arguments and λ is an arbitrary constant.

⊙ *References*: A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2002).

10°. Generalized separable solution for $f(x,t) = f(t)$:

$$w(x,y,t) = \int u(z,t)\,dz + \varphi(t)y + \psi(t)x, \qquad z = kx + \lambda y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k and λ are arbitrary constants, and the function $u(z,t)$ is determined by the second-order linear equation

$$\frac{\partial u}{\partial t} + \bigl[k\varphi(t) - \lambda\psi(t)\bigr]\frac{\partial u}{\partial z} = \nu\lambda^2\frac{\partial^2 u}{\partial z^2} - \frac{1}{\lambda}\varphi'_t(t) + \frac{1}{\lambda}f(t).$$

The transformation

$$u = U(\xi,t) - \frac{1}{\lambda}\varphi(t) + \frac{1}{\lambda}\int f(t)\,dt, \qquad \xi = z - \int\bigl[k\varphi(t) - \lambda\psi(t)\bigr]\,dt$$

brings it to the linear heat equation

$$\frac{\partial U}{\partial t} = \nu\lambda^2\frac{\partial^2 U}{\partial \xi^2}.$$

⊙ *References*: A. D. Polyanin (2001 *b*), A. D. Polyanin and V. F. Zaitsev (2002).

11°. Generalized separable solution for $f(x,t) = f(t)$:

$$w(x,y,t) = Ce^{-\lambda y + \lambda\varphi(x,t)} - a(t)\varphi(x,t) - \frac{\partial}{\partial t}\int \varphi(x,t)\,dx + a(t)y + \nu\lambda x, \quad a(t) = \int f(t)\,dt,$$

where $\varphi(x,t)$ is an arbitrary function of two arguments; C and λ are arbitrary constants.

12°. Generalized separable solution for $f(x,t) = f(t)$:

$$w(x,y,t) = \varphi(x,t)e^{\lambda y} + \psi(x,t)e^{-\lambda y} + \chi(x,t) + a(t)y,$$

where λ is any, $\varphi(x,t)$ is an arbitrary function of two arguments, and the remaining functions are given by

$$\psi(x,t) = \frac{C\nu e^{2\nu\lambda^2 t}}{\varphi(x,t)}\left[x - \int a(t)\,dt\right], \quad a(t) = \int f(t)\,dt + Ce^{2\nu\lambda^2 t},$$
$$\chi(x,t) = \frac{1}{\lambda}a(t)\ln|\varphi(x,t)| + \frac{1}{\lambda}\frac{\partial}{\partial t}\int \ln|\varphi(x,t)|\,dx - \nu\lambda x.$$

13°. Solutions for $f(x,t) = f(t)$:

$$w = e^{\nu\lambda^2 t}(C_1 e^{\lambda z} + C_2 e^{-\lambda z}) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx + z\int f(t)\,dt, \qquad z = y + \varphi(x,t);$$
$$w = e^{-\nu\lambda^2 t}\big[C_1\sin(\lambda z) + C_2\cos(\lambda z)\big] + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx + z\int f(t)\,dt, \quad z = y + \varphi(x,t);$$
$$w = C_1 e^{-\lambda z}\sin(\lambda z - 2\nu\lambda^2 t + C_2) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx + z\int f(t)\,dt, \qquad z = y + \varphi(x,t),$$

where $\varphi(x,t)$ is an arbitrary function of two arguments; C_1, C_2, and λ are arbitrary constants. For periodic function $f(t) = f(t+T)$ satisfying the condition $\int_0^T f(t)\,dt = 0$; the last solution is also periodic, $w(x,y,t) = w(x,y,t+T)$, if $\varphi(x,t) = \varphi(x)$ and $\lambda = \sqrt{\pi/(\nu T)}$.

14°. Solutions for $f(x,t) = A$:

$$w = -\frac{A}{6\nu}z^3 + C_2 z^2 + C_1 z + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx, \qquad z = y + \varphi(x,t);$$
$$w = kx + C_1\exp\left(-\frac{k}{\nu}z\right) - \frac{A}{2k}z^2 + C_2 z + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx, \qquad z = y + \varphi(x,t),$$

where $\varphi(x,t)$ is an arbitrary function of two arguments; C_1, C_2, and k are arbitrary constants.

15°. Table 10 presents solutions of the unsteady hydrodynamic boundary layer equation with pressure gradient that depends on two generalized variables (used results of group-theoretic analyses in Ovsiannikov, 1982).

For $f(x,t) = f(k_1 x + \lambda_1 t)$, there is also a wide class of "two-dimensional" solutions with the form

$$w = z(\xi,\eta) + a_1 x + a_2 y, \qquad \xi = k_1 x + \lambda_1 t, \qquad \eta = k_2 y + \lambda_2 t,$$

where the function z is determined by the differential equation

$$(\lambda_1 + a_2 k_1)\frac{\partial^2 z}{\partial\xi\partial\eta} + (\lambda_2 - a_1 k_2)\frac{\partial^2 z}{\partial\eta^2} + k_1 k_2\left(\frac{\partial z}{\partial\eta}\frac{\partial^2 z}{\partial\xi\partial\eta} - \frac{\partial z}{\partial\xi}\frac{\partial^2 z}{\partial\eta^2}\right) = \nu k_2^2\frac{\partial^3 z}{\partial\eta^3} + f(\xi).$$

16°. For

$$f(x,t) = a'(t)X^{-1/3} - \tfrac{1}{3}a^2(t)X^{-5/3} - b''(t), \qquad X = x + b(t),$$

where $a(t)$ and $b(t)$ are some functions, a solution is given by

$$w = [a(t)X^{-1/3} - b'(t)]y + 6\nu X y^{-1}.$$

⊙ *Reference*: Burde (1995).

TABLE 10
Solutions of the unsteady hydrodynamic boundary layer equation that depends on two generalized variables. Notation: $\mathcal{R}[z]=\nu z_{\eta\eta\eta}+z_\xi z_{\eta\eta}-z_\eta z_{\xi\eta}$ and $g=g(u)$ is an arbitrary function.

Function $f=f(x,t)$	General form of solution	Equation for $z=z(\xi,\eta)$
$f=f(x+\lambda t)$	$w=z(\xi,y)-\lambda y$, $\xi=x+\lambda t$	$\nu z_{yyy}+z_\xi z_{yy}-z_y z_{\xi y}+f(\xi)=0$
$f=g(x)t^{-2}$	$w=z(x,\eta)t^{-1/2}$, $\eta=yt^{-1/2}$	$\nu z_{\eta\eta\eta}+z_x z_{\eta\eta}-z_\eta z_{x\eta}+\frac{1}{2}\eta z_{\eta\eta}+z_\eta+g(x)=0$
$f=e^{\lambda t}g(xe^{-\lambda t})$	$w=e^{\lambda t}z(\xi,y)$, $\xi=xe^{-\lambda t}$	$\nu z_{yyy}+z_\xi z_{yy}-z_y z_{\xi y}+\lambda\xi z_{\xi y}-\lambda z_y+g(\xi)=0$
$f=t^{-n-2}g(xt^n)$	$w=z(\xi,\eta)t^{-(2n+1)/2}$, $\xi=xt^n$, $\eta=yt^{-1/2}$	$\mathcal{R}[z]+\frac{1}{2}\eta z_{\eta\eta}-n\xi z_{\xi\eta}+(1+n)z_\eta+g(\xi)=0$
$f=ax^n$	$w=z(\xi,\eta)t^{-(n+3)/(2n-2)}$, $\xi=xt^{2/(n-1)}$, $\eta=yt^{-1/2}$	$\mathcal{R}[z]+\frac{1}{2}\eta z_{\eta\eta}-\frac{2\xi}{n-1}z_{\xi\eta}+\frac{n+1}{n-1}z_\eta+a\xi^n=0$
$f=ae^{\lambda x}$	$w=z(\xi,\eta)t^{-1/2}$, $\xi=x+\frac{2}{\lambda}\ln t$, $\eta=yt^{-1/2}$	$\mathcal{R}[z]+\frac{1}{2}\eta z_{\eta\eta}-\frac{2}{\lambda}z_{\xi\eta}+z_\eta+ae^{\lambda\xi}=0$

3. $\displaystyle \frac{\partial^2 w}{\partial z\partial t}+\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial x\partial z}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial z^2}=\nu\frac{\partial}{\partial z}\left(z\frac{\partial^2 w}{\partial z^2}\right)+f(x,t).$

Preliminary remarks. The system of axisymmetric unsteady laminar boundary layer equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r}=\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)+f(x,t), \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r}=0 \tag{2}$$

where u and v are the axial and radial components of the fluid velocity, respectively, and x and r the axial and radial coordinates, is reduced to the equation in question by the introduction of a stream function w and a new variable z such that

$$u=\frac{2}{r}\frac{\partial w}{\partial r},\quad v=-\frac{2}{r}\frac{\partial w}{\partial x},\quad z=\frac{1}{4}r^2.$$

System (1), (2) describes an axisymmetric jet ($f\equiv 0$) and a boundary layer on an extensive body of revolution ($f\not\equiv 0$).

1°. The equation remains the same under the replacement of w by $w+\varphi(t)$, where $\varphi(t)$ is an arbitrary function.

2°. Generalized separable solution (quadratic in z) for arbitrary $f(x,t)$:

$$w(x,z,t)=Cz^2+\varphi(x,t)z+\frac{1}{4C}\varphi^2(x,t)+\frac{1}{2C}\frac{\partial}{\partial t}\int\varphi(x,t)\,dx-\frac{1}{2C}\int f(x,t)\,dx-\nu x+\psi(t),$$

where $\varphi(x,t)$ and $\psi(t)$ are arbitrary functions and C is an arbitrary constant.

The equation also has an "inviscid" solution of the form $w=\varphi(x,t)z+\psi(x,t)$, where $\psi(x,t)$ is an arbitrary function, and the function $\varphi=\varphi(x,t)$ is described by the first-order partial differential equation $\partial_t\varphi+\varphi\partial_x\varphi=f(x,t)$.

3°. Generalized separable solution (linear in x) for $f(x,t)=a(t)x+b(t)$:

$$w(x,z,t)=x\varphi(z,t)+\psi(z,t),$$

where the functions $\varphi=\varphi(z,t)$ and $\psi=\psi(z,t)$ are described by the system of partial differential equations

$$\frac{\partial^2\varphi}{\partial z\partial t}+\left(\frac{\partial\varphi}{\partial z}\right)^2-\varphi\frac{\partial^2\varphi}{\partial z^2}=\nu\frac{\partial}{\partial z}\left(z\frac{\partial^2\varphi}{\partial z^2}\right)+a(t),$$

$$\frac{\partial^2\psi}{\partial z\partial t}+\frac{\partial\varphi}{\partial z}\frac{\partial\psi}{\partial z}-\varphi\frac{\partial^2\psi}{\partial z^2}=\nu\frac{\partial}{\partial z}\left(z\frac{\partial^2\psi}{\partial z^2}\right)+b(t).$$

The first equation has an exact solution $\varphi = C(t)z$, where the function $C = C(t)$ is determined by the Riccati equation $C'_t + C^2 = a(t)$. The second equation is reduced by the change of variable $V = \frac{\partial \psi}{\partial z}$ to a second-order linear equation.

4°. "Two-dimensional" solution for $f(x,t) = f(x + \lambda t)$:

$$w(x,z,t) = U(\xi, z) - \lambda z, \quad \xi = x + \lambda t,$$

where the function $U = U(\xi, z)$ is determined by the differential equation

$$\frac{\partial U}{\partial z}\frac{\partial^2 U}{\partial \xi \partial z} - \frac{\partial U}{\partial \xi}\frac{\partial^2 U}{\partial z^2} = \nu \frac{\partial}{\partial z}\left(z\frac{\partial^2 U}{\partial z^2}\right) + f(\xi),$$

which coincides, up to renaming, with the stationary equation (see equation 9.3.1.3 and its solutions).

5°. Generalized separable solution (linear in x) for $f(x,t) = f(t)$:

$$w(x,z,t) = A(t)x + B(t) + z\int f(t)\,dt + u(z,t),$$

where $A(t)$ and $B(t)$ are arbitrary functions, and the function $u = u(z,t)$ is determined by the second-order linear parabolic differential equation

$$\frac{\partial u}{\partial t} - A(t)\frac{\partial u}{\partial z} = \nu z \frac{\partial^2 u}{\partial z^2}.$$

6°. Suppose $w(x,z,t)$ is a solution of the unsteady axisymmetric boundary layer equation with $f(x,t) = a(t)x + b(t)$. Then the function

$$w_1 = w(\xi, z, t) - \varphi'_t(t)z + \psi(t), \quad \xi = x + \varphi(t),$$

where $\psi(t)$ is an arbitrary function and $\varphi = \varphi(t)$ is a solution of the linear ordinary differential equation $\varphi''_{tt} - a(t)\varphi = 0$, is also a solution of the equation.

9.3.4. Unsteady Boundary Layer Equations for Non-Newtonian Fluids

1. $\displaystyle \frac{\partial^2 w}{\partial t \partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = k\left(\frac{\partial^2 w}{\partial y^2}\right)^{n-1}\frac{\partial^3 w}{\partial y^3}.$

This equation describes an unsteady boundary layer on a flat plate in a power-law fluid flow; w is the steam function, and x and y are coordinates along and normal to the plate.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(C_1^{n-2}C_2^{2n-1}x + C_1^{n-2}C_2^{2n-1}C_3 t,\ C_2 y + C_2 C_5 t,\ C_1^{n-1}C_2^{2n}t) + C_5 x - C_3 y,$$
$$w_2 = w(x + C_6, y + C_7, t + C_8) + C_9,$$
$$w_3 = w\bigl(x, y + \varphi(x,t), t\bigr) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx + \psi(t),$$

where the C_n are arbitrary constants and $\varphi(x,t)$ and $\psi(t)$ are arbitrary functions, are also solutions of the equation.

2°. Generalized separable solution linear in x:

$$w(x,y,t) = \psi(t)x + \int U(z,t)\,dz, \quad z = y + \int \psi(t)\,dt,$$

where $\psi(t)$ is an arbitrary function, and the function $U(z,t)$ is determined by the second-order differential equation

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial U}{\partial z}\right)^{n-1}\frac{\partial^2 U}{\partial z^2}.$$

For details about this equation, see 1.6.18.2 with $f(x) = \text{const}$ and 1.6.18.3 with $f(U) = kU^{n-1}$ (for $n = 2$, see Special case in equation 8.1.1.2).

3°. Generalized separable solution linear in x:

$$w(x,y,t) = \frac{xy}{t+C_1} + \psi(t)x + \int U(y,t)\,dy,$$

where $\psi(t)$ is an arbitrary function, C_1 is an arbitrary constant, and the function $U(y,t)$ is determined by the second-order differential equation

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial U}{\partial y}\right)^{n-1}\frac{\partial^2 U}{\partial y^2} + \left[\frac{y}{t+C_1} + \psi(t)\right]\frac{\partial U}{\partial y} - \frac{1}{t+C}U.$$

With the transformation

$$U = \frac{1}{t+C_1}u(\zeta,\tau), \quad \tau = \tfrac{1}{3}(t+C_1)^3 + C_2, \quad \zeta = (t+C_1)y + \int \psi(t)(t+C_1)\,dt + C_3$$

one arrives at the simpler equation

$$\frac{\partial u}{\partial \tau} = k\left(\frac{\partial u}{\partial \zeta}\right)^{n-1}\frac{\partial^2 u}{\partial \zeta^2}.$$

For details about this equation, see 1.6.18.2 with $f(x) = \text{const}$ and 1.6.18.3 with $f(U) = kU^{n-1}$.

4°. "Two-dimensional" solution:

$$w(x,y,t) = \int v(\eta,t)\,d\eta + \varphi(t)y + \psi(t)x, \qquad \eta = kx + \lambda y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k and λ are arbitrary constants, and the function $v(\eta,t)$ is determined by the second-order differential equation

$$\frac{\partial v}{\partial t} + \left[k\varphi(t) - \lambda\psi(t)\right]\frac{\partial v}{\partial \eta} = k\lambda^{2n}\left(\frac{\partial v}{\partial \eta}\right)^{n-1}\frac{\partial^2 v}{\partial \eta^2} - \frac{1}{\lambda}\varphi'_t(t).$$

With the transformation

$$v = R(\zeta,t) - \frac{1}{\lambda}\varphi(t), \quad \zeta = \eta - \int \left[k\varphi(t) - \lambda\psi(t)\right]dt$$

one arrives at the simpler equation

$$\frac{\partial R}{\partial t} = k\lambda^{2n}\left(\frac{\partial R}{\partial \zeta}\right)^{n-1}\frac{\partial^2 R}{\partial \zeta^2}.$$

⊙ *Reference for equation* 9.3.4.1: A. D. Polyanin and V. F. Zaitsev (2002).

2. $\displaystyle \frac{\partial^2 w}{\partial t\partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left[f\left(\frac{\partial^2 w}{\partial y^2}\right)\right].$

This equation describes an unsteady boundary layer on a flat plate in a non-Newtonian fluid flow; w is the stream function, and x and y are coordinates along and normal to the plate.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x, y+\varphi(x,t), t) + \frac{\partial}{\partial t}\int \varphi(x,t)\,dx + \psi(t),$$
$$w_2 = C_1^{-2}w(C_1^3 x + C_1^3 C_2 t + C_3,\ C_1 y + C_1 C_4 t + C_5,\ C_1^2 t + C_6) + C_4 x - C_2 y + C_7,$$

where $\varphi(x,t)$ and $\psi(t)$ are arbitrary functions and the C_n are arbitrary constants, are also solutions of the equation.

2°. Generalized separable solution linear in x:

$$w(x,y,t) = \psi(t)x + \int U(z,t)\,dz, \quad z = y + \int \psi(t)\,dt,$$

where $\psi(t)$ is an arbitrary function and the function $U(z,t)$ is determined by the second-order differential equation

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial z}\left[f\left(\frac{\partial U}{\partial z}\right)\right].$$

It admits, for any $f = f(v)$, exact solutions of the following forms:

$$\begin{aligned}
&U(z,t) = H(\zeta), && \zeta = kz + \lambda t && \Longrightarrow \quad \text{equation} \quad \lambda H = kf(kH'_\zeta) + C;\\
&U(z,t) = az + H(\zeta), && \zeta = kz + \lambda t && \Longrightarrow \quad \text{equation} \quad \lambda H = kf(kH'_\zeta + a) + C;\\
&U(z,t) = \sqrt{t}\,H(\zeta), && \zeta = z/\sqrt{t} && \Longrightarrow \quad \text{equation} \quad \tfrac{1}{2}H - \tfrac{1}{2}\zeta H'_\zeta = [f(H'_\zeta)]'_\zeta,
\end{aligned}$$

where a, k, C, and λ are arbitrary constants. Solutions of the first two equations with $H = H(\zeta)$ can be obtained in parametric form; see Kamke (1977) and Polyanin and Zaitsev (2003).

3°. Generalized separable solution linear in x:

$$w(x,y,t) = \frac{xy}{t+C} + \psi(t)x + \int U(y,t)\,dy,$$

where $\psi(t)$ is an arbitrary function, C is an arbitrary constant, and the function $U(y,t)$ is determined by the second-order differential equation

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial y}\left[f\left(\frac{\partial U}{\partial y}\right)\right] + \left[\frac{y}{t+C} + \psi(t)\right]\frac{\partial U}{\partial y} - \frac{1}{t+C}U.$$

With the transformation

$$U = \frac{1}{t+C_1}u(\zeta,\tau), \quad \tau = \tfrac{1}{3}(t+C_1)^3 + C_2, \quad \zeta = (t+C_1)y + \int \psi(t)(t+C_1)\,dt + C_3$$

one arrives at the simpler equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial \zeta}\left[f\left(\frac{\partial u}{\partial \zeta}\right)\right].$$

For details about this equation, see Item 2°.

4°. "Two-dimensional" solution:

$$w(x,y,t) = \int v(\eta,t)\,d\eta + \varphi(t)y + \psi(t)x, \qquad \eta = kx + \lambda y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k and λ are arbitrary constants, and the function $v(\eta,t)$ is determined by the second-order differential equation

$$\frac{\partial v}{\partial t} + \left[k\varphi(t) - \lambda\psi(t)\right]\frac{\partial v}{\partial \eta} = \frac{\partial}{\partial \eta}\left[f\left(\lambda^2\frac{\partial v}{\partial \eta}\right)\right] - \frac{1}{\lambda}\varphi'_t(t).$$

With the transformation

$$v = R(\zeta,t) - \frac{1}{\lambda}\varphi(t), \quad \zeta = \eta - \int \left[k\varphi(t) - \lambda\psi(t)\right]dt$$

one arrives at the simpler equation

$$\frac{\partial R}{\partial t} = \frac{\partial}{\partial \zeta}\left[f\left(\lambda^2\frac{\partial R}{\partial \zeta}\right)\right].$$

⊙ *References for equation* 9.3.4.2: A. D. Polyanin (2001 *b*, 2002), A. D. Polyanin and V. F. Zaitsev (2002).

3. $\dfrac{\partial^2 w}{\partial t\,\partial y}+\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x\,\partial y}-\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}=\dfrac{\partial}{\partial y}\left[f\left(\dfrac{\partial^2 w}{\partial y^2}\right)\right]+g(x,t).$

This is an unsteady boundary layer equation for a non-Newtonian fluid with pressure gradient.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the function

$$w_1=w(x,y+\varphi(x,t),t)+\frac{\partial}{\partial t}\int\varphi(x,t)\,dx+\psi(t),$$

where $\varphi(x,t)$ and $\psi(t)$ are arbitrary functions, is also a solution of the equation.

2°. There are degenerate solutions; see Item 3° in 9.3.3.2, where $f(x,t)$ should be substituted by $g(x,t)$.

3°. For $g(x,t)=g(t)$, the transformation

$$w=u(\xi,y,t)-\varphi'_t(t)y,\qquad \xi=x+\varphi(t),\qquad\text{where}\quad \varphi(t)=-\int_{t_0}^{t}(t-\tau)g(\tau)\,d\tau,$$

leads to a simpler equation of the form 9.3.4.2:

$$\frac{\partial^2 u}{\partial t\,\partial y}+\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial\xi\,\partial y}-\frac{\partial u}{\partial\xi}\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left[f\left(\frac{\partial^2 u}{\partial y^2}\right)\right].$$

Note that $g=g(t)$ and $\varphi=\varphi(t)$ are related by the simple equation $\varphi''_{tt}=-g$.

4°. "Two-dimensional" solution (linear in x) for $g(x,t)=g(t)$:

$$w(x,y,t)=a(t)x+\int U(y,t)\,dy,$$

where the function $U=U(y,t)$ is determined by the second-order differential equation

$$\frac{\partial U}{\partial t}-a(t)\frac{\partial U}{\partial y}=\frac{\partial}{\partial y}\left[f\left(\frac{\partial U}{\partial y}\right)\right]+g(t).$$

With the transformation

$$U=u(\xi,t)+\int g(t)\,dt,\qquad \xi=y+\int a(t)\,dt$$

one arrives at the simpler equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial y}\left[f\left(\frac{\partial u}{\partial y}\right)\right].$$

For details about this equation, see 9.3.4.2, Item 2°.

5°. "Two-dimensional" solution (linear in x) for $g(x,t)=s(t)x+h(t)$:

$$w(x,y,t)=\bigl[a(t)y+\psi(t)\bigr]x+\int Q(y,t)\,dy,$$

where $\psi(t)$ is an arbitrary function and $a=a(t)$ is determined by the Riccati equation

$$a'_t+a^2=s(t),$$

and the function $Q=Q(y,t)$ satisfies the second-order equation

$$\frac{\partial Q}{\partial t}=\frac{\partial}{\partial y}\left[f\left(\frac{\partial Q}{\partial y}\right)\right]+\bigl[a(t)y+\psi(t)\bigr]\frac{\partial Q}{\partial y}-a(t)Q+h(t).$$

With the transformation

$$Q=\frac{1}{\Phi(t)}\left[Z(\xi,\tau)+\int h(t)\Phi(t)\,dt\right],\qquad \tau=\int\Phi^2(t)\,dt+A,\qquad \xi=y\Phi(t)+\int\psi(t)\Phi(t)\,dt+B,$$

where $\Phi(t)=\exp\left[\int a(t)\,dt\right]$, one arrives at the simpler equation

$$\frac{\partial Z}{\partial\tau}=\frac{\partial}{\partial\xi}\left[f\left(\frac{\partial Z}{\partial\xi}\right)\right].$$

For details about this equation, see 9.3.4.2, Item 2°.

6°. "Two-dimensional" solution for $g(x,t)=g(t)$:

$$w(x,y,t)=\int v(\eta,t)\,d\eta+\varphi(t)y+\psi(t)x,\qquad \eta=kx+\lambda y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k and λ are arbitrary constants, and the function $v(\eta,t)$ is determined by the second-order differential equation

$$\frac{\partial v}{\partial t}+\bigl[k\varphi(t)-\lambda\psi(t)\bigr]\frac{\partial v}{\partial\eta}=\frac{\partial}{\partial\eta}\left[f\left(\lambda^2\frac{\partial v}{\partial\eta}\right)\right]-\frac{1}{\lambda}\varphi'_t(t)+\frac{1}{\lambda}g(t).$$

With the transformation

$$v=R(\zeta,t)-\frac{1}{\lambda}\varphi(t)+\frac{1}{\lambda}\int g(t)\,dt,\qquad \zeta=\eta-\int\bigl[k\varphi(t)-\lambda\psi(t)\bigr]\,dt$$

one arrives at the simpler equation

$$\frac{\partial R}{\partial t}=\frac{\partial}{\partial\zeta}\left[f\left(\lambda^2\frac{\partial R}{\partial\zeta}\right)\right].$$

⊙ *References for equation* 9.3.4.3: A. D. Polyanin (2001 *b*, 2002), A. D. Polyanin and V. F. Zaitsev (2002).

9.3.5. Related Equations

1. $\displaystyle \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}=f(x)\frac{\partial^3 w}{\partial y^3}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1(x,y)=C_1w(x,C_1y+\varphi(x))+C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(x)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solutions linear and quadratic in y:

$$w(x,y)=C_1y+\varphi(x),$$
$$w(x,y)=C_1y^2+\varphi(x)y+\frac{1}{4C_1}\varphi^2(x)+C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(x)$ is an arbitrary function.

3°. Generalized separable solution:

$$w(x,y)=\varphi(x)e^{\lambda y}-\lambda\int f(x)\,dx+C,$$

where $\varphi(x)$ is an arbitrary function and C and λ are arbitrary constants.

4°. Generalized separable solution:

$$w(x,y)=\varphi(y)\int f(x)\,dx+\psi(y),$$

where the functions $\varphi=\varphi(y)$ and $\psi=\psi(y)$ are determined by the autonomous system of ordinary differential equations

$$(\varphi'_y)^2-\varphi\varphi''_{yy}=\varphi'''_{yyy},$$
$$\varphi'_y\psi'_y-\varphi\psi''_{yy}=\psi'''_{yyy}.$$

For exact solutions of this system, see 9.3.1.1, Item 5° [equations (2)–(3) with $\nu=1$].

5°. Generalized self-similar solution:

$$w(x,y)=\varphi(x)U(z),\qquad z=\psi(x)y$$

where the functions $\varphi=\varphi(x)$, $\psi=\psi(x)$, and $U=U(z)$ are determined by the system of ordinary differential equations

$$(\varphi\psi)'_x=C_1f(x)\psi^2,$$
$$\varphi'_x=C_2f(x)\psi,$$
$$C_1(U'_z)^2-C_2UU''_{zz}=U'''_{zzz}.$$

2. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(y)\dfrac{\partial^3 w}{\partial y^3} + g(y)x + h(y).$

Generalized separable solution linear in x:

$$w = \varphi(y)x + \psi(y),$$

where the functions $\varphi(y)$ and $\psi(y)$ are determined by the system of ordinary differential equations

$$f\varphi'''_{yyy} + \varphi\varphi''_{yy} - (\varphi'_y)^2 + g = 0,$$
$$f\psi'''_{yyy} + \varphi\psi''_{yy} - \varphi'_y\psi'_y + h = 0.$$

3. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = \dfrac{\partial}{\partial y}\left[f(y)\dfrac{\partial^2 w}{\partial y^2}\right] + g(y)x + h(y).$

Generalized separable solution linear in x:

$$w = \varphi(y)x + \psi(y),$$

where the functions $\varphi(y)$ and $\psi(y)$ are determined by the system of ordinary differential equations

$$(f\varphi''_{yy})'_y + \varphi\varphi''_{yy} - (\varphi'_y)^2 + g = 0,$$
$$(f\psi''_{yy})'_y + \varphi\psi''_{yy} - \varphi'_y\psi'_y + h = 0.$$

4. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(x)\left(\dfrac{\partial^2 w}{\partial y^2}\right)^{n-1}\dfrac{\partial^3 w}{\partial y^3}.$

1°. Suppose $w(x, y)$ is a solution of this equation. Then the function

$$w_1(x, y) = C_1^{2n-1} w(x, C_1^{2-n} y + \varphi(x)) + C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(x)$ is an arbitrary function, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x, y) = \left[(2-n)\int f(x)\,dx + C\right]^{\frac{1}{2-n}} \theta(y),$$

where C is an arbitrary constant and the function $\theta = \theta(y)$ is determined by the autonomous ordinary differential equation

$$(\theta'_y)^2 - \theta\theta''_{yy} = (\theta''_{yy})^{n-1}\theta'''_{yyy}.$$

3°. Generalized traveling-wave solution:

$$w = U(z), \qquad z = y\left[\int f(x)\,dx + C\right]^{\frac{1}{1-2n}} + \varphi(x),$$

where $\varphi(x)$ is an arbitrary function and the function $U = U(z)$ is determined by the autonomous ordinary differential equation

$$(U'_z)^2 = (1-2n)(U''_{zz})^{n-1}U'''_{zzz}.$$

This equation can be fully integrated.

5. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = F\left(x, w, \dfrac{\partial w}{\partial y}, \dfrac{\partial^2 w}{\partial y^2}, \dfrac{\partial^3 w}{\partial y^3}\right).$

This is a special case of equation 11.4.1.5 with $n = 3$.

9.4. Equations of Motion of Ideal Fluid (Euler Equations)

9.4.1. Stationary Equations

1. $$\frac{\partial w}{\partial y}\frac{\partial}{\partial x}(\Delta w)-\frac{\partial w}{\partial x}\frac{\partial}{\partial y}(\Delta w)=0,\qquad \Delta w=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}.$$

Preliminary remarks. The stationary two-dimensional equations of motion of an ideal fluid (Euler equations)

$$u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x},$$
$$u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y},$$
$$\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}=0$$

are reduced to this equation by the introduction of a stream function, w, such that $u_1=\frac{\partial w}{\partial y}$ and $u_2=-\frac{\partial w}{\partial x}$ followed by the elimination of the pressure p (with the cross differentiation) from the first two equations; the third equation is then satisfied automatically.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$w_1=C_1w(C_2x+C_3,\,C_2y+C_4)+C_5,$$
$$w_2=w(x\cos\alpha+y\sin\alpha,\,-x\sin\alpha+y\cos\alpha),$$

where $C_1,\,\dots,\,C_5$ and α are arbitrary constants, are also solutions of the equation.

2°. Solutions of general form:

$$w(x,y)=\varphi_1(\xi),\quad \xi=a_1x+b_1y;$$
$$w(x,y)=\varphi_2(r),\quad r=\sqrt{(x-a_2)^2+(y-b_2)^2};$$

where $\varphi_1(\xi)$ and $\varphi_2(r)$ are arbitrary functions; a_1, b_1, a_2, and b_2 are arbitrary constants.

3°. Any solutions of the linear equations

$$\begin{aligned}
&\Delta w=0 &&\text{(Laplace equation)},\\
&\Delta w=C &&\text{(Poisson equation)},\\
&\Delta w=\lambda w &&\text{(Helmholtz equation)},\\
&\Delta w=\lambda w+C &&\text{(nonhomogeneous Helmholtz equation)},
\end{aligned}$$

where C and λ are arbitrary constants, are also solutions of the original equation. For details about the Laplace, Poisson, and Helmholtz equations, see the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

The solutions of the Laplace equation $\Delta w=0$ correspond to irrotational (potential) solutions of the Euler equation. Such solutions are discussed in detail in textbooks on hydrodynamics (e.g., see Sedov, 1980, and Loitsyanskiy, 1996), where the methods of the theory of functions of a complex variable are extensively used.

4°. The Jacobian of the functions w and $v=\Delta w$ appears on the left-hand side of the equation in question. The fact that the Jacobian of two functions is zero means that the two functions are functionally dependent. Hence, v must be a function of w, so that

$$\Delta w=f(w),\tag{1}$$

where $f(w)$ is an arbitrary function. Any solution of the second-order equation (1) for arbitrary $f(w)$ is a solution of the original equation.

The results of Item 3° correspond to special cases of the linear function $f(w)=\lambda w+C$. For solutions of equation (1) with some nonlinear $f=f(w)$, see 5.1.1.1, 5.2.1.1, 5.3.1.1, 5.3.2.1, 5.3.3.1, 5.4.1.1, and Subsection S.5.3 (Example 12).

5°. Additive separable solutions:

$$w(x,y) = A_1x^2 + A_2x + B_1y^2 + B_2y + C,$$
$$w(x,y) = A_1 \exp(\lambda x) + A_2 \exp(-\lambda x) + B_1 \exp(\lambda y) + B_2 \exp(-\lambda y) + C,$$
$$w(x,y) = A_1 \sin(\lambda x) + A_2 \cos(\lambda x) + B_1 \sin(\lambda y) + B_2 \cos(\lambda y) + C,$$

where A_1, A_2, B_1, B_2, C, and λ are arbitrary constants. These solutions are special cases of solutions presented in Item 3°.

6°. Generalized separable solutions:

$$w(x,y) = (Ax + B)e^{-\lambda y} + C,$$
$$w(x,y) = \bigl[A_1 \sin(\beta x) + A_2 \cos(\beta x)\bigr]\bigl[B_1 \sin(\lambda y) + B_2 \cos(\lambda y)\bigr] + C,$$
$$w(x,y) = \bigl[A_1 \sin(\beta x) + A_2 \cos(\beta x)\bigr]\bigl[B_1 \sinh(\lambda y) + B_2 \cosh(\lambda y)\bigr] + C,$$
$$w(x,y) = \bigl[A_1 \sinh(\beta x) + A_2 \cosh(\beta x)\bigr]\bigl[B_1 \sin(\lambda y) + B_2 \cos(\lambda y)\bigr] + C,$$
$$w(x,y) = \bigl[A_1 \sinh(\beta x) + A_2 \cosh(\beta x)\bigr]\bigl[B_1 \sinh(\lambda y) + B_2 \cosh(\lambda y)\bigr] + C,$$
$$w(x,y) = Ae^{\alpha x+\beta y} + Be^{\gamma x+\lambda y} + C, \qquad \alpha^2 + \beta^2 = \gamma^2 + \lambda^2,$$

where A, B, C, D, k, β, and λ are arbitrary constants. These solutions are special cases of solutions presented in Item 3°.

7°. Solution:

$$w(x,y) = F(z)x + G(z), \qquad z = y + kx,$$

where k is an arbitrary constant and the functions $F = F(z)$ and $G = G(z)$ are determined by the autonomous system of third-order ordinary differential equations:

$$F'_z F''_{zz} - FF'''_{zzz} = 0, \tag{2}$$
$$G'_z F''_{zz} - FG'''_{zzz} = \frac{2k}{(k^2+1)} FF''_{zz}. \tag{3}$$

On integrating the system once, we arrive at the following second-order equations:

$$(F'_z)^2 - FF''_{zz} = A_1, \tag{4}$$
$$G'_z F'_z - FG''_{zz} = \frac{2k}{k^2+1} \int FF''_{zz}\,dz + A_2, \tag{5}$$

where A_1 and A_2 are arbitrary constants.

The autonomous equation (4) can be reduced, with the change of variable $Z(F) = (F'_z)^2$, to a first-order linear equation.

The general solution of equation (2), or (4), is given by

$$F(z) = B_1 z + B_2, \qquad A_1 = B_1^2;$$
$$F(z) = B_1 \exp(\lambda z) + B_2 \exp(-\lambda z), \quad A_1 = -4\lambda^2 B_1 B_2;$$
$$F(z) = B_1 \sin(\lambda z) + B_2 \cos(\lambda z), \qquad A_1 = \lambda^2(B_1^2 + B_2^2),$$

where B_1, B_2, and λ are arbitrary constants.

The general solution of equation (3), or (5), is expressed as

$$G = C_1 \int F\,dz - \int F\biggl(\int \frac{\psi\,dz}{F^2}\biggr)dz + C_2,$$
$$F = F(z), \qquad \psi = \frac{2k}{k^2+1}\int FF''_{zz}\,dz + A_2,$$

where C_1 and C_2 are arbitrary constants.

8°. There are exact solutions of the following forms:

$$w(x,y) = x^a U(\zeta), \qquad \zeta = y/x;$$
$$w(x,y) = e^{ax} V(\rho), \qquad \rho = bx + cy;$$
$$w(x,y) = W(\zeta) + a \ln|x|, \quad \zeta = y/x;$$

where a, b, and c are arbitrary constants.

► *For other exact solutions, see equation 9.4.1.2.*

⊙ *References for equation* 9.4.1.1: A. A. Buchnev (1971), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), A. D. Polyanin and V. F. Zaitsev (2002).

2. $\displaystyle \frac{\partial w}{\partial \theta}\frac{\partial}{\partial r}(\Delta w) - \frac{\partial w}{\partial r}\frac{\partial}{\partial \theta}(\Delta w) = 0, \qquad \Delta w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}.$

Preliminary remarks. Equation 9.4.1.1 is reduced to this equation by passing to polar coordinates r, θ with origin at a point (x_0, y_0), where x_0 and y_0 are any, such that

$$x = r\cos\theta + x_0, \qquad y = r\sin\theta + y_0 \qquad \text{(direct transformation)},$$
$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2}, \quad \tan\theta = \frac{y-y_0}{x-x_0} \qquad \text{(inverse transformation)}.$$

The radial and angular components of the fluid velocity are expressed in terms of the stream function w as follows: $u_r = \frac{1}{r}\frac{\partial w}{\partial \theta}$ and $u_\theta = -\frac{\partial w}{\partial r}$.

1°. Multiplicative separable solution:

$$w(r,\theta) = r^\lambda U(\theta),$$

where the function $U = U(\theta)$ is determined by the second-order autonomous ordinary differential equation

$$U''_{\theta\theta} + \lambda^2 U = CU^{\frac{\lambda-2}{\lambda}} \qquad (\lambda \text{ and } C \text{ are any}).$$

Its general solution can be written out in implicit form. In particular, if $C = 0$, we have

$$U = A_1 \sin(\lambda\theta) + A_2\cos(\lambda\theta) \quad \text{if} \quad \lambda \neq 0,$$
$$U = A_1\theta + A_2 \qquad \text{if} \quad \lambda = 0.$$

To $\lambda = 0$ there corresponds a solution dependent on the angle θ only.

2°. Multiplicative separable solution:

$$w(r,\theta) = f(r)g(\theta),$$

where the functions $f = f(r)$ and $g = g(\theta)$ are determined by the linear ordinary differential equations

$$\mathbf{L}(f) = (\beta - \lambda r^{-2})f,$$
$$g''_{\theta\theta} = \lambda g,$$

where β and λ are arbitrary constants; $\mathbf{L}(f) = r^{-1}(rf'_r)'_r$.

3°. Solution:

$$w = b\theta + U(\xi), \qquad \xi = \theta + a\ln r, \tag{1}$$

where the function $U = U(\xi)$ is determined by the ordinary differential equation

$$abU'''_{\xi\xi\xi} = 2bU''_{\xi\xi} + 2U'_\xi U''_{\xi\xi}.$$

The onefold integration yields

$$abU''_{\xi\xi} = (U'_\xi)^2 + 2bU'_\xi + C_1, \tag{2}$$

where C_1 is an arbitrary constant. The further integration results in

$$\xi = ab \int \frac{dz}{z^2 + 2bz + C_1}, \qquad z = U'_\xi.$$

4°. Generalized separable solution linear in θ:

$$w(r, \theta) = f(r)\theta + g(r).$$

Here, the functions $f = f(r)$ and $g = g(r)$ are determined by the system of ordinary differential equations

$$\begin{aligned} -f'_r \mathbf{L}(f) + f[\mathbf{L}(f)]'_r &= 0, \\ -g'_r \mathbf{L}(f) + f[\mathbf{L}(g)]'_r &= 0, \end{aligned} \tag{3}$$

where $\mathbf{L}(f) = r^{-1}(rf'_r)'_r$.

System (3) admits first integrals, which allow us to obtain the following second-order linear ordinary differential equations for f and g:

$$\begin{aligned} \mathbf{L}(f) &= Af, \\ \mathbf{L}(g) &= Ag + B, \end{aligned} \tag{4}$$

where A and B are arbitrary constants. For $A = 0$, the solutions of equations (4) are given by

$$\begin{aligned} f(r) &= C_1 \ln r + C_2, \\ g(r) &= \tfrac{1}{4}Br^2 + C_3 \ln r + C_4. \end{aligned}$$

For $A \neq 0$, the solutions of equations (4) are expressed in terms of Bessel functions.

► *For other exact solutions, see equation 9.4.1.1.*

⊙ *References for equation* 9.4.1.2: A. A. Buchnev (1971), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), A. D. Polyanin and V. F. Zaitsev (2002).

3. $\displaystyle \frac{\partial w}{\partial z}\frac{\partial \mathrm{E}w}{\partial r} - \frac{\partial w}{\partial r}\frac{\partial \mathrm{E}w}{\partial z} - \frac{2}{r}\frac{\partial w}{\partial z}\mathrm{E}w = 0, \qquad \mathrm{E}w = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}.$

Preliminary remarks. The stationary Euler equations written in cylindrical coordinates for the axisymmetric case are reduced to the equation in question by the introduction of a stream function w such that $u_r = \frac{1}{r}\frac{\partial w}{\partial z}$ and $u_z = -\frac{1}{r}\frac{\partial w}{\partial r}$, where $r = \sqrt{x^2 + y^2}$, and u_r and u_z are the radial and axial fluid velocity components.

1°. Any function $w = w(r, z)$ that solves the second-order linear equation $\mathrm{E}w = 0$ will also be a solution of the given equation.

2°. Solutions:

$$\begin{aligned} w &= \varphi(r), \\ w &= (C_1 z^2 + C_2 z + C_3)r^2 + C_4 z + C_5, \end{aligned}$$

where $\varphi(r)$ is an arbitrary function and $C_1, \dots, C_5$ are arbitrary constants.

3°. Generalized separable solution linear in z:

$$w(r, z) = \varphi(r)z + \psi(r).$$

Here, $\varphi = \varphi(r)$ and $\psi = \psi(r)$ are determined by the system of ordinary differential equations

$$\begin{aligned} \varphi[\mathbf{L}(\varphi)]'_r - \varphi'_r \mathbf{L}(\varphi) - 2r^{-1}\varphi\,\mathbf{L}(\varphi) &= 0, \\ \varphi[\mathbf{L}(\psi)]'_r - \psi'_r \mathbf{L}(\varphi) - 2r^{-1}\varphi\,\mathbf{L}(\psi) &= 0, \end{aligned} \tag{1}$$

where $\mathbf{L}(\varphi) = \varphi''_{rr} - r^{-1}\varphi'_r$.

System (1) admits first integrals, which allow us to obtain the following second-order linear ordinary differential equations for φ and ψ:

$$\begin{aligned}\mathbf{L}(\varphi) &= 4C_1 r^2\varphi,\\ \mathbf{L}(\psi) &= 4C_1 r^2\psi + 4C_2 r^2,\end{aligned} \qquad (2)$$

where C_1 and C_2 are arbitrary constants. The substitution $\xi = r^2$ brings (2) to the linear constant-coefficient equations

$$\begin{aligned}\varphi''_{\xi\xi} &= C_1\varphi,\\ \psi''_{\xi\xi} &= C_1\psi + C_2.\end{aligned}$$

Integrating yields

$$\varphi = \begin{cases} A_1\cosh(k\xi) + B_1\sinh(k\xi) & \text{if } C_1 = k^2 > 0,\\ A_1\cos(k\xi) + B_1\sin(k\xi) & \text{if } C_1 = -k^2 < 0,\\ A_1\xi + B_1 & \text{if } C_1 = 0,\end{cases}$$

$$\psi = \begin{cases} A_2\cosh(k\xi) + B_2\sinh(k\xi) - C_2/C_1 & \text{if } C_1 = k^2 > 0,\\ A_2\cos(k\xi) + B_2\sin(k\xi) - C_2/C_1 & \text{if } C_1 = -k^2 < 0,\\ \tfrac{1}{2}C_2\xi^2 + A_2\xi + B_2 & \text{if } C_1 = 0,\end{cases}$$

where A_1, B_1, A_2, and B_2 are arbitrary constants.

⊙ *References for equation* 9.4.1.3: A. A. Buchnev (1971), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), A. D. Polyanin and V. F. Zaitsev (2002).

9.4.2. Nonstationary Equations

1. $\dfrac{\partial}{\partial t}(\Delta w) + \dfrac{\partial w}{\partial y}\dfrac{\partial}{\partial x}(\Delta w) - \dfrac{\partial w}{\partial x}\dfrac{\partial}{\partial y}(\Delta w) = 0, \qquad \Delta w = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}.$

Preliminary remarks. The two-dimensional nonstationary equations of an ideal incompressible fluid (Euler equations)

$$\begin{aligned}\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x},\\ \frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y},\\ \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} &= 0\end{aligned}$$

are reduced to the equation in question by the introduction of a stream function w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$ followed by the elimination of the pressure p (with cross differentiation) from the first two equations.

For stationary equation, see Subsection 9.4.1.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$\begin{aligned} w_1 &= -w(y,\,x,\,t),\\ w_2 &= C_1 w(C_2x + C_3,\; C_2y + C_4,\; C_1C_2^2t + C_5) + C_6,\\ w_3 &= w(x\cos\alpha + y\sin\alpha,\; -x\sin\alpha + y\cos\alpha,\; t),\\ w_4 &= w(x\cos\beta t + y\sin\beta t,\; -x\sin\beta t + y\cos\beta t,\; t) - \tfrac{1}{2}\beta(x^2+y^2),\\ w_5 &= w(x+\varphi(t),\; y+\psi(t),\; t) + \psi'_t(t)x - \varphi'_t(t)y + \chi(t),\end{aligned}$$

where $C_1, \dots, C_6$, α, and β are arbitrary constants and $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are arbitrary functions, are also solutions of the equation.

2°. Any solution of the Poisson equation $\Delta w = C$ is also a solution of the original equation. Solutions of the Laplace equation $\Delta w = 0$ describe irrotational (potential) flows of an ideal incompressible fluid.

3°. Solutions of general form:

$$w(x,y,t)=Q(z)+\psi'_t(t)x-\varphi'_t(t)y,\quad z=C_1[x+\varphi(t)]+C_2[y+\psi(t)];$$
$$w(x,y,t)=Q(z)+\psi'_t(t)x-\varphi'_t(t)y,\quad z=[x+\varphi(t)]^2+[y+\psi(t)]^2;$$

where $Q(z)$, $\varphi(t)$, and $\varphi(t)$ are arbitrary functions; C_1 and C_2 are arbitrary constants.

Likewise, the formulas of Item 1° can be used to construct nonstationary solution based on other, stationary solutions (see Subsection 9.4.1).

4°. Generalized separable solution linear in x:

$$w(x,y,t)=F(y,t)x+G(y,t),\tag{1}$$

where the functions $F(y,t)$ and $G=G(y,t)$ are determined by the system of one-dimensional third-order equations

$$\frac{\partial^3 F}{\partial t\partial y^2}+\frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial y^2}-F\frac{\partial^3 F}{\partial y^3}=0,\tag{2}$$
$$\frac{\partial^3 G}{\partial t\partial y^2}+\frac{\partial G}{\partial y}\frac{\partial^2 F}{\partial y^2}-F\frac{\partial^3 G}{\partial y^3}=0.\tag{3}$$

Equation (2) is solved independently of (3). If $F=F(y,t)$ is a solution of equation (2), then the functions

$$F_1=F(y+\psi(t),t)+\psi'_t(t),$$
$$F_2=C_1F(C_1y+C_1C_2t+C_3,C_1^2t+C_4)+C_2,$$

where $\psi(t)$ is an arbitrary function and $C_1,\,\dots,\,C_4$ are arbitrary constants, are also solutions of the equation.

Integrating (2) and (3) with respect to y yields the system of second-order equations

$$\frac{\partial^2 F}{\partial t\partial y}+\left(\frac{\partial F}{\partial y}\right)^2-F\frac{\partial^2 F}{\partial y^2}=f_1(t),\tag{4}$$
$$\frac{\partial^2 G}{\partial t\partial y}+\frac{\partial F}{\partial y}\frac{\partial G}{\partial y}-F\frac{\partial^2 G}{\partial y^2}=f_2(t),\tag{5}$$

where $f_1(t)$ and $f_2(t)$ are arbitrary functions. Equation (5) is linear in G. Then the substitution

$$G=\int U\,dy-hF+h'_ty,\quad \text{where}\quad U=U(y,t),\quad F=F(y,t),\tag{6}$$

and the function $h=h(t)$ is determined by the second-order linear ordinary differential equation

$$h''_{tt}-f_1(t)h=f_2(t),\tag{7}$$

brings (5) to the first-order linear homogeneous partial differential equation

$$\frac{\partial U}{\partial t}-F\frac{\partial U}{\partial y}=-\frac{\partial F}{\partial y}U.\tag{8}$$

Thus, whenever a particular solution of equation (2) or (4) is known, finding G is reduced to solving the linear equations (7) and (8) followed by integrating by formula (6).

Solutions of equation (2) are listed in Table 11. The ordinary differential equations in the last two rows can be reduced, with the substitution $H'_z=V(H)$, to first-order separable equations. Table 12 presents the general solutions of equation (5) that correspond to exact solutions of equation (2) in Table 11.

TABLE 11

Solutions of equations (2) and (4)

No.	Function $F = F(y,t)$ (or general form of solutions)	Function $f_1(t)$ in equation (4)	Determining functions (of determining equation)
1	$F = \varphi(t)y + \psi(t)$	$f_1(t) = \varphi'_t + \varphi^2$	$\varphi(t)$ and $\psi(t)$ are arbitrary
2	$F = A\exp[-\lambda y - \lambda\psi(t)] + \psi'_t(t)$	$f_1(t) = 0$	$\psi(t)$ is arbitrary; A and λ are any
3	$F = A\sinh[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$f_1(t) = A^2\lambda^2$	$\psi(t)$ is arbitrary; A and λ are any
4	$F = A\cosh[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$f_1(t) = -A^2\lambda^2$	$\psi(t)$ is arbitrary; A and λ are any
5	$F = A\sin[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$f_1(t) = A^2\lambda^2$	$\psi(t)$ is arbitrary; A and λ are any
6	$F = A\cos[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$f_1(t) = A^2\lambda^2$	$\psi(t)$ is arbitrary; A and λ are any
7	$F = t^{-1}H(z) + \psi'_t(t),\ z = y + \psi(t)$	$f_1(t) = At^{-2}$	$-A - H'_z + (H'_z)^2 - HH''_{zz} = 0$
8	$F = t^{-1/2}\big[H(z) - \frac{1}{2}z\big],\ z = yt^{-1/2}$	$f_1(t) = At^{-2}$	$\frac{3}{4} - A - 2H'_z + (H'_z)^2 - HH''_{zz} = 0$

TABLE 12

Solutions of equation (5); $\Theta(\xi)$ is an arbitrary function everywhere; the number in the first column corresponds to the number of an exact solution in Table 11

No.	General solution of equation (5)	Notation
1	$G = \frac{1}{\Phi^2(t)}\Theta(\xi) + \frac{y}{\Phi(t)}\int f_2(t)\Phi(t)\,dt,\ \xi = y\Phi(t) + \int \psi(t)\Phi(t)\,dt$	$\Phi(t) = \exp\big[\int \varphi(t)\,dt\big]$
2	Formula (6), where $U = e^{-\lambda z}\Theta(\xi),\ \xi = t + \frac{1}{A\lambda}e^{\lambda z}$	$z = y + \psi(t)$
3	Formula (6), where $U = \sinh(\lambda z)\Theta(\xi),\ \xi = t + \frac{1}{A\lambda}\ln\big\lvert\tanh\frac{\lambda z}{2}\big\rvert$	$z = y + \psi(t)$
4	Formula (6), where $U = \cosh(\lambda z)\Theta(\xi),\ \xi = t + \frac{2}{A\lambda}\arctan\big(e^{\lambda z}\big)$	$z = y + \psi(t)$
5	Formula (6), where $U = \sin(\lambda z)\Theta(\xi),\ \xi = t + \frac{1}{A\lambda}\ln\big\lvert\tan\frac{\lambda z}{2}\big\rvert$	$z = y + \psi(t)$
6	Formula (6), where $U = \cos(\lambda z)\Theta(\xi),\ \xi = t + \frac{1}{A\lambda}\ln\big\lvert\tan\big(\frac{\lambda z}{2} + \frac{\pi}{4}\big)\big\rvert$	$z = y + \psi(t)$
7	Formula (6), where $U = \Theta(\xi)H(z),\ \xi = t\exp\big[\int \frac{dz}{H(z)}\big]$	$z = y + \psi(t)$
8	Formula (6), where $U = \Theta(\xi)H(z)\exp\big[-\frac{1}{2}\int \frac{dz}{H(z)}\big],\ \xi = t\exp\big[\int \frac{dz}{H(z)}\big]$	$z = \frac{y}{\sqrt{t}}$

The general solution of the linear nonhomogeneous equation (7) can be obtained by the formula

$$h(t) = C_1h_1(t) + C_2h_2(t) + \frac{1}{W_0}\left[h_2(t)\int h_1(t)f_2(t)\,dt - h_1(t)\int h_2(t)f_2(t)\,dt\right], \qquad (9)$$

where $h_1 = h_1(t)$ and $h_2 = h_2(t)$ are fundamental solutions of the corresponding homogeneous equation (with $f_2 \equiv 0$), and $W_0 = h_1(h_2)'_t - h_2(h_1)'_t$ is the Wronskian determinant ($W_0 = \text{const}$).

For exact solutions 2–8 in Table 11, one should set

$$
\begin{array}{llll}
h_1 = 1, & h_2 = t, & W_0 = 1 & \text{for solution 2;} \\
h_1 = e^{-A\lambda t}, & h_2 = e^{A\lambda t}, & W_0 = 2A\lambda & \text{for solutions 3, 5, 6;} \\
h_1 = \cos(A\lambda t), & h_2 = \sin(A\lambda t), & W_0 = A\lambda & \text{for solution 4;} \\
h_1 = |t|^{\frac{1}{2}-\mu}, & h_2 = |t|^{\frac{1}{2}+\mu}, & W_0 = 2\mu = (1+4A)^{\frac{1}{2}} & \text{for solutions 7, 8}
\end{array}
$$

in formula (9).

5°. Solution:

$$w(x,y,t) = F(\zeta,t)x + G(\zeta,t), \qquad \zeta = y + kx,$$

where the functions $F(\zeta,t)$ and $G = G(\zeta,t)$ are determined from the system of one-dimensional third-order equations

$$\frac{\partial^3 F}{\partial t \partial \zeta^2} + \frac{\partial F}{\partial \zeta}\frac{\partial^2 F}{\partial \zeta^2} - F\frac{\partial^3 F}{\partial \zeta^3} = 0, \tag{10}$$

$$\frac{\partial^3 G}{\partial t \partial \zeta^2} + \frac{\partial G}{\partial \zeta}\frac{\partial^2 F}{\partial \zeta^2} - F\frac{\partial^3 G}{\partial \zeta^3} = \frac{2k}{k^2+1}\left(F\frac{\partial^2 F}{\partial \zeta^2} - \frac{\partial^2 F}{\partial t \partial \zeta}\right). \tag{11}$$

Integrating (10) and (11) with respect to ζ yields

$$\frac{\partial^2 F}{\partial t \partial \zeta} + \left(\frac{\partial F}{\partial \zeta}\right)^2 - F\frac{\partial^2 F}{\partial \zeta^2} = f_1(t), \tag{12}$$

$$\frac{\partial^2 G}{\partial t \partial \zeta} + \frac{\partial F}{\partial \zeta}\frac{\partial G}{\partial \zeta} - F\frac{\partial^2 G}{\partial \zeta^2} = Q(\zeta,t), \tag{13}$$

where $f_1(t)$ is an arbitrary function, and the function $Q(\zeta,t)$ is given by

$$Q(\zeta,t) = -\frac{2k}{k^2+1}\frac{\partial F}{\partial t} + \frac{2k}{k^2+1}\int F\frac{\partial^2 F}{\partial \zeta^2}\,d\zeta + f_2(t), \qquad f_2(t) \text{ is any.}$$

Equation (13) is linear in G. Consequently, the substitution $U = \frac{\partial G}{\partial \zeta}$ brings it to the first-order linear equation

$$\frac{\partial U}{\partial t} - F\frac{\partial U}{\partial \zeta} = -\frac{\partial F}{\partial \zeta}U + Q(\zeta,t). \tag{14}$$

Equation (10) coincides, up to renaming, with equation (2), whose exact solutions are listed in Table 11. In these cases, solutions of the corresponding equation (14) can be found by quadrature.

6°. Solution [special case of a solution of the form (1)]:

$$w(x,y,t) = \exp\left[-\lambda y - \lambda\int\varphi(t)\,dt\right]\left[C_1 x + C_2 - C_1\int\psi(t)\,dt\right] + \varphi(t)x + \psi(t)y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions and C_1, C_2, and λ are arbitrary constants.

7°. Generalized separable solution:

$$w(x,y,t) = e^{-\lambda y}\left[A(t)e^{\beta x} + B(t)e^{-\beta x}\right] + \varphi(t)x + \psi(t)y,$$

$$A(t) = C_1\exp\left[-\beta\int\psi(t)\,dt - \lambda\int\varphi(t)\,dt\right],$$

$$B(t) = C_2\exp\left[\beta\int\psi(t)\,dt - \lambda\int\varphi(t)\,dt\right],$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions and C_1, C_2, λ, and β are arbitrary constants.

8°. Generalized separable solution:

$$w(x,y,t) = e^{-\lambda y}\big[A(t)\sin(\beta x) + B(t)\cos(\beta x)\big] + \varphi(t)x + \psi(t)y,$$
$$A(t) = \exp\biggl(-\lambda\int\varphi\,dt\biggr)\biggl[C_1\sin\biggl(\beta\int\psi\,dt\biggr) + C_2\cos\biggl(\beta\int\psi\,dt\biggr)\biggr],$$
$$B(t) = \exp\biggl(-\lambda\int\varphi\,dt\biggr)\biggl[C_1\cos\biggl(\beta\int\psi\,dt\biggr) - C_2\sin\biggl(\beta\int\psi\,dt\biggr)\biggr],$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are arbitrary functions and C_1, C_2, λ, and β are arbitrary constants.

9°. Generalized separable solutions:

$$w(x,y,t) = A(t)\exp(k_1x + \lambda_1y) + B(t)\exp(k_2x + \lambda_2y) + \varphi(t)x + \psi(t)y,$$
$$A(t) = C_1\exp\biggl[\lambda_1\int\varphi(t)\,dt - k_1\int\psi(t)\,dt\biggr],$$
$$B(t) = C_2\exp\biggl[\lambda_2\int\varphi(t)\,dt - k_2\int\psi(t)\,dt\biggr],$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions; C_1 and C_2 are arbitrary constants; and k_1, λ_1, k_2, and λ_2 are arbitrary parameters related by one of the two constraints

$$k_1^2 + \lambda_1^2 = k_2^2 + \lambda_2^2 \quad \text{(first family of solutions)},$$
$$k_1\lambda_2 = k_2\lambda_1 \quad \text{(second family of solutions)}.$$

10°. Generalized separable solution:

$$w(x,y,t) = \big[C_1\sin(\lambda x) + C_2\cos(\lambda x)\big]\big[A(t)\sin(\beta y) + B(t)\cos(\beta y)\big] + \varphi(t)x,$$
$$A(t) = C_3\cos\biggl(\beta\int\varphi\,dt + C_4\biggr), \quad B(t) = C_3\sin\biggl(\beta\int\varphi\,dt + C_4\biggr),$$

where $\varphi = \varphi(t)$ is an arbitrary function and $C_1, \ldots, C_4$, λ, and β are arbitrary constants.

11°. Generalized separable solution:

$$w(x,y,t) = \big[C_1\sinh(\lambda x) + C_2\cosh(\lambda x)\big]\big[A(t)\sin(\beta y) + B(t)\cos(\beta y)\big] + \varphi(t)x,$$
$$A(t) = C_3\cos\biggl(\beta\int\varphi\,dt + C_4\biggr), \quad B(t) = C_3\sin\biggl(\beta\int\varphi\,dt + C_4\biggr),$$

where $\varphi = \varphi(t)$ is an arbitrary function and $C_1, \ldots, C_4$, λ, and β are arbitrary constants.

12°. Solution:

$$w(x,y,t) = f(z) + g(t)z + \varphi(t)x + \psi(t)y, \qquad z = kx + \lambda y + \int\big[\lambda\varphi(t) - k\psi(t)\big]\,dt,$$

where $f(z)$, $g(t)$, $\varphi(t)$, and $\psi(t)$ are arbitrary functions and k and λ are arbitrary constants.

13°+. There is a "two-dimensional" solution of the form

$$w = W(\rho_1,\rho_2) + c_1x + c_2y, \qquad \rho_1 = a_1x + a_2y + a_3t, \qquad \rho_2 = b_1x + b_2y + b_3t.$$

14°+. "Two-dimensional" solution:

$$w = t^{(2-k)/k}\Psi(\xi,\eta), \quad \xi = t^{-1/k}\big[x\cos(\lambda\ln t) - y\sin(\lambda\ln t)\big], \quad \eta = t^{-1/k}\big[x\sin(\lambda\ln t) + y\cos(\lambda\ln t)\big],$$

where k and λ are arbitrary constants and the function $\Psi(\xi,\eta)$ is determined by the differential equation

$$-\widetilde{\Delta}\Psi + \left(\frac{\partial\Psi}{\partial\eta} - \frac{1}{k}\xi - \lambda\eta\right)\frac{\partial}{\partial\xi}\widetilde{\Delta}\Psi - \left(\frac{\partial\Psi}{\partial\xi} + \frac{1}{k}\eta - \lambda\xi\right)\frac{\partial}{\partial\eta}\widetilde{\Delta}\Psi = 0, \qquad \widetilde{\Delta} = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}.$$

15°+. "Two-dimensional" solution:

$$w(x,y,t)=\frac{\varphi_t'(x^2-y^2+2\varphi xy)}{2(1+\varphi^2)}+\frac{y-\varphi x}{1+\varphi^2}F(\zeta,t)-2G(\zeta,t),\qquad \zeta=x+\varphi y,$$

where $\varphi=\varphi(t)$ is an arbitrary function and the functions $F=F(\zeta,t)$ and $G=G(\zeta,t)$ are determined by the differential equations

$$F\frac{\partial^3F}{\partial\zeta^3}-\frac{\partial F}{\partial\zeta}\frac{\partial^2F}{\partial\zeta^2}+\frac{2\varphi\varphi_t'}{1+\varphi^2}\frac{\partial^2F}{\partial\zeta^2}+\frac{\partial^3F}{\partial\zeta^2t}=0,\tag{15}$$

$$F\frac{\partial^3G}{\partial\zeta^3}-\frac{\partial^2F}{\partial\zeta^2}\frac{\partial G}{\partial\zeta}+\frac{2\varphi\varphi_t'}{1+\varphi^2}\frac{\partial^2G}{\partial\zeta^2}+\frac{\partial^3G}{\partial\zeta^2t}=-\frac{\varphi_t'}{(1+\varphi^2)^2}\zeta\frac{\partial^2F}{\partial\zeta^2}.\tag{16}$$

Equation (15) is solved independently of equation (16). If $F=F(\zeta,t)$ is a solution to (15), then the function

$$F_1=F(y+\sigma(t),t)-\sigma_t'(t),$$

where $\sigma(t)$ is an arbitrary function, is also a solution of the equation.

Integrating (15) and (16) with respect to ζ yields

$$F\frac{\partial^2F}{\partial\zeta^2}-\left(\frac{\partial F}{\partial\zeta}\right)^2+\frac{2\varphi\varphi_t'}{1+\varphi^2}\frac{\partial F}{\partial\zeta}+\frac{\partial^2F}{\partial\zeta\partial t}=\psi_1(t),$$

$$F\frac{\partial^2G}{\partial\zeta^2}-\frac{\partial F}{\partial\zeta}\frac{\partial G}{\partial\zeta}+\frac{2\varphi\varphi_t'}{1+\varphi^2}\frac{\partial G}{\partial\zeta}+\frac{\partial^2G}{\partial\zeta\partial t}=\frac{\varphi_t'}{(1+\varphi^2)^2}\left(F-\zeta\frac{\partial F}{\partial\zeta}\right)+\psi_2(t),$$

where $\psi_1(t)$ and $\psi_2(t)$ are arbitrary functions. The change of variable $u=\frac{\partial G}{\partial\zeta}$ brings the last equation to a first-order linear equation (for known F).

Note that equation (15) admits particular solutions of the following forms:

$$F(\zeta,t)=a(t)\zeta+b(t),$$

$$F(\zeta,t)=a(t)e^{-\lambda\zeta}+\frac{a_t'(t)}{\lambda a(t)}+\frac{2\varphi\varphi_t'}{\lambda(1+\varphi^2)},$$

where $a(t)$ and $b(t)$ are arbitrary functions and λ is an arbitrary constant.

⊙ *References for equation* 9.4.2.1: A. A. Buchnev (1971), B. J. Cantwell (1978), P. J. Olver (1986), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (1999), A. D. Polyanin and V. F. Zaitsev (2002).

2. $\displaystyle \frac{\partial Q}{\partial t}+\frac{1}{r}\frac{\partial w}{\partial\theta}\frac{\partial Q}{\partial r}-\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial Q}{\partial\theta}=0,\qquad Q=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2w}{\partial\theta^2}.$

Preliminary remarks. Equation 9.4.4.1 is reduced to the equation in question by passing to the polar coordinate system with center at a point (x_0,y_0), where x_0 and y_0 are any, by the formulas

$$x=r\cos\theta+x_0,\qquad y=r\sin\theta+y_0\qquad\text{(direct transformation)},$$

$$r=\sqrt{(x-x_0)^2+(y-y_0)^2},\quad \tan\theta=\frac{y-y_0}{x-x_0}\qquad\text{(inverse transformation)}.$$

The radial and angular components of the fluid velocity are expressed via the stream function w as follows: $u_r=\frac{1}{r}\frac{\partial w}{\partial\theta}$, $u_\theta=-\frac{\partial w}{\partial r}$.

1°. Generalized separable solution linear in θ:

$$w(r,\theta,t)=f(r,t)\theta+g(r,t),\tag{1}$$

where the functions $f=f(r,t)$ and $g=g(r,t)$ satisfy the system of equations

$$\mathbf{L}(f_t)-r^{-1}f_r\mathbf{L}(f)+r^{-1}f[\mathbf{L}(f)]_r=0,\tag{2}$$

$$\mathbf{L}(g_t)-r^{-1}g_r\mathbf{L}(f)+r^{-1}f[\mathbf{L}(g)]_r=0.\tag{3}$$

Here, the subscripts r and t denote the corresponding partial derivatives, $\mathbf{L}(f)=r^{-1}(rf_r)_r$.

2°. For particular solutions to equation (2) of the form

$$f = \varphi(t)\ln r + \psi(t) \tag{4}$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are arbitrary functions, equation (3) can be reduced, with the change of variable $U = \mathbf{L}(g)$, to the first-order linear equation $U_t + r^{-1}fU_r = 0$. Two families of particular solutions to this equation are given by

$$U = \Theta(\zeta), \quad \zeta = r^2 - 2\int \psi(t)\,dt \qquad \text{(first family of solutions, } \varphi = 0\text{)},$$

$$U = \Theta(\zeta), \quad \zeta = \int \frac{r\,dr}{\ln r} - \int \varphi(t)\,dt \qquad \text{(second family of solutions, } \psi = 0\text{)},$$

where $\Theta(\zeta)$ is an arbitrary function. The second term in solution (1) is expressed via $U = U(r,t)$, provided the first term has the form (4), as follows:

$$g(r,t) = C_1(t)\ln r + C_2(t) + \int \Phi(r,t)\,dr, \qquad \Phi(r,t) = \frac{1}{r}\int rU(r,t)\,dr,$$

where $C_1(t)$ and $C_2(t)$ are arbitrary functions.

Remark. Equation (2) has also a solution $f = -\dfrac{r^2}{2(t+C)}$.

3°. “Two-dimensional” solution:

$$w(r,\theta,t) = Ar^2t + H(\xi,\eta), \qquad \xi = r\cos(\theta + At^2), \qquad \eta = r\sin(\theta + At^2),$$

where A is an arbitrary constant and the function $H(\xi,\eta)$ is determined by the differential equation

$$\frac{\partial H}{\partial \eta}\frac{\partial}{\partial \xi}\widetilde{\Delta} H - \frac{\partial H}{\partial \xi}\frac{\partial}{\partial \eta}\widetilde{\Delta} H + 4A = 0, \qquad \widetilde{\Delta} = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}.$$

► *For other exact solutions, see equation 9.4.2.1.*

⊙ *References for equation* 9.4.2.2: A. A. Buchnev (1971), P. J. Olver (1986), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999), D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (1999), A. D. Polyanin and V. F. Zaitsev (2002).

9.5. Other Third-Order Nonlinear Equations

9.5.1. Equations Involving Second-Order Mixed Derivatives

1. $\dfrac{\partial^2 w}{\partial x \partial t} = aw\dfrac{\partial^2 w}{\partial x^2} + b\dfrac{\partial^3 w}{\partial x^3}.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(C_1 x + aC_1\varphi(t), C_1^2 t + C_2) + \varphi'_t(t),$$

where C_1 and C_3 are arbitrary constants and $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. There are exact solutions of the following forms:

$$w = U(z), \qquad z = x + \lambda t \qquad \text{traveling-wave solution;}$$

$$w = |t|^{-1/2}V(\xi), \quad \xi = x|t|^{-1/2} \qquad \text{self-similar solution.}$$

2. $\dfrac{\partial^2 w}{\partial x \partial t} + \left(\dfrac{\partial w}{\partial x}\right)^2 - w\dfrac{\partial^2 w}{\partial x^2} = \nu \dfrac{\partial^3 w}{\partial x^3}.$

This equation occurs in fluid dynamics; see 9.3.3.1, equation (2) and 10.3.3.1, equation (4) with $f_1(t) = 0$.

1°. Suppose $w = w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x + \psi(t), t) + \psi'_t(t),$$
$$w_2 = C_1 w(C_1 x + C_1 C_2 t + C_3, C_1^2 t + C_4) + C_2,$$

where $\psi(t)$ is an arbitrary function and $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,t) = \frac{C_1 x}{C_1 t + C_2} + \psi(t),$$
$$w(x,t) = \frac{6\nu}{x + \psi(t)} + \psi'_t(t),$$
$$w(x,t) = C_1 \exp\bigl[-\lambda x + \lambda\psi(t)\bigr] - \psi'_t(t) + \nu\lambda,$$

where $\psi(t)$ is an arbitrary function and C_1, C_2, and λ are arbitrary constants. The first solution is "inviscid" (independent of ν).

3°. Traveling-wave solution (λ is an arbitrary constant):

$$w = F(z), \qquad z = x + \lambda t,$$

where the function $F(z)$ is determined by the autonomous ordinary differential equation

$$\lambda F''_{zz} + (F'_z)^2 - F F''_{zz} = \nu F'''_{zzz}.$$

4°. Self-similar solution:

$$w = t^{-1/2}\bigl[G(\xi) - \tfrac{1}{2}\xi\bigr], \qquad \xi = x t^{-1/2},$$

where the function $G = G(z)$ is determined by the autonomous ordinary differential equation

$$\tfrac{3}{4} - 2G'_\xi + (G'_\xi)^2 - G G''_{\xi\xi} = \nu G'''_{\xi\xi\xi}.$$

The solutions of Items 3° and 4° can be generalized using the formulas of Item 1°.

⊙ *References*: A. D. Polyanin (2001 *b*, 2002).

3. $\dfrac{\partial^2 w}{\partial x \partial t} + \left(\dfrac{\partial w}{\partial x}\right)^2 - w\dfrac{\partial^2 w}{\partial x^2} = \nu \dfrac{\partial^3 w}{\partial x^3} + f(t).$

This equation occurs in fluid dynamics; see 9.3.3.2, equation (3) and 10.3.3.1, equation (4).

1°. Suppose $w = w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + \psi(t), t) + \psi'_t(t),$$

where $\psi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solution (linear in x) for any $f(t)$:

$$w(x,t) = \varphi(t) x + \psi(t),$$

where $\psi(t)$ is an arbitrary function, and the function $\varphi = \varphi(t)$ is described by the Riccati equation $\varphi'_t + \varphi^2 = f(t)$. For exact solutions of this equation, see Polyanin and Zaitsev (2003).

3°. Generalized separable solutions for $f(t) = Ae^{-\beta t}$, $A > 0$, $\beta > 0$:

$$w(x,t) = Be^{-\frac{1}{2}\beta t} \sin[\lambda x + \lambda\psi(t)] + \psi'_t(t), \qquad B = \pm\sqrt{\frac{2A\nu}{\beta}}, \qquad \lambda = \sqrt{\frac{\beta}{2\nu}},$$
$$w(x,t) = Be^{-\frac{1}{2}\beta t} \cos[\lambda x + \lambda\psi(t)] + \psi'_t(t),$$

where $\psi(t)$ is an arbitrary function.

4°. Generalized separable solution for $f(t) = Ae^{\beta t}$, $A > 0$, $\beta > 0$:

$$w(x,t) = Be^{\frac{1}{2}\beta t}\sinh[\lambda x + \lambda\psi(t)] + \psi'_t(t), \qquad B = \pm\sqrt{\frac{2A\nu}{\beta}}, \quad \lambda = \sqrt{\frac{\beta}{2\nu}},$$

where $\psi(t)$ is an arbitrary function.

5°. Generalized separable solution for $f(t) = Ae^{\beta t}$, $A < 0$, $\beta > 0$:

$$w(x,t) = Be^{\frac{1}{2}\beta t}\cosh[\lambda x + \lambda\psi(t)] + \psi'_t(t), \qquad B = \pm\sqrt{\frac{2|A|\nu}{\beta}}, \quad \lambda = \sqrt{\frac{\beta}{2\nu}},$$

where $\psi(t)$ is an arbitrary function.

6°. Generalized separable solution for $f(t) = Ae^{\beta t}$, A is any, $\beta > 0$:

$$w(x,t) = \psi(t)e^{\lambda x} - \frac{Ae^{\beta t-\lambda x}}{4\lambda^2\psi(t)} + \frac{\psi'_t(t)}{\lambda\psi(t)} - \nu\lambda, \qquad \lambda = \pm\sqrt{\frac{\beta}{2\nu}},$$

where $\psi(t)$ is an arbitrary function.

7°. Self-similar solution for $f(t) = At^{-2}$:

$$w(x,t) = t^{-1/2}\bigl[u(z) - \tfrac{1}{2}z\bigr], \qquad z = xt^{-1/2},$$

where the function $u = u(z)$ is determined by the autonomous ordinary differential equation

$$\tfrac{3}{4} - A - 2u'_z + (u'_z)^2 - uu''_{zz} = \nu u'''_{zzz},$$

whose order can be reduced by one.

8°. Traveling-wave solution for $f(t) = A$:

$$w = w(\xi), \qquad \xi = x + \lambda t,$$

where the function $w(\xi)$ is determined by the autonomous ordinary differential equation

$$-A + \lambda w''_{\xi\xi} + (w'_\xi)^2 - ww''_{\xi\xi} = \nu w'''_{\xi\xi\xi},$$

whose order can be reduced by one.

⊙ *References*: V. A. Galaktionov (1995), A. D. Polyanin (2001 *b*, 2002).

4. $$\frac{\partial^2 w}{\partial x\partial t} + \left(\frac{\partial w}{\partial x}\right)^2 - w\frac{\partial^2 w}{\partial x^2} = f(t)\frac{\partial^3 w}{\partial x^3}.$$

1°. Suppose $w = w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + \psi(t), t) + \psi'_t(t),$$

where $\psi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Generalized separable solutions:

$$w(x,t) = \frac{C_1 x}{C_1 t + C_2} + \varphi(t),$$

$$w(x,t) = \varphi(t)e^{-\lambda x} - \frac{\varphi'_t(t)}{\lambda\varphi(t)} + \lambda f(t),$$

where $\varphi(t)$ is an arbitrary function and C_1, C_2, and λ are arbitrary constants. The first solution is degenerate.

► *For other equations involving second-order mixed derivatives, see Sections 9.3 and 9.4.*

9.5.2. Equations Involving Third-Order Mixed Derivatives

1. $$\frac{\partial^3 w}{\partial x^2\partial y} = ae^{\lambda w}.$$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = w\bigl(C_1 x + C_2,\ C_3 y + C_4\bigr) + \frac{1}{\lambda}\ln(C_1^2 C_3),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w(x,y) = -\frac{3}{\lambda}\ln z, \qquad z = f(y)x - \frac{1}{6}a\lambda f(y)\int \frac{dy}{f^3(y)},$$

where $f(y)$ is an arbitrary function.

2. $\displaystyle \frac{\partial w}{\partial t} = w\frac{\partial w}{\partial x} + \beta\frac{\partial^3 w}{\partial x^2 \partial t}.$

BBM equation (Benjamin–Bona–Mahony equation). It describes long waves in dispersive systems.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(\pm x + C_2, \pm C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (either plus or minus signs are taken).

2°. Traveling-wave solution:

$$w(x,t) = -a + \wp\left(\frac{x - at}{2\sqrt{3a\beta}} + C_1, C_2, C_3\right),$$

where $\wp(z, C_2, C_3)$ is the Weierstrass elliptic function $\left(\wp'_z = \sqrt{4\wp^3 - C_2\wp - C_3}\,\right)$; a, C_1, C_2, and C_3 are arbitrary constants. See also equation 9.5.2.3, Item 2° with $a = -1$, $b = \beta$, and $k = 1$.

3°. Multiplicative separable solution:

$$w(x,t) = u(x)/t,$$

where the function $u = u(x)$ is determined by the autonomous ordinary differential equation $\beta u''_{xx} - uu'_x - u = 0$. Its solution can be written out in parametric form

$$u = \sqrt{2\beta}(\tau - \ln|\tau| + C_1)^{1/2}, \qquad x = \frac{1}{2}\sqrt{2\beta}\int \frac{d\tau}{\tau(\tau - \ln|\tau| + C_1)^{1/2}} + C_2.$$

4°. Solution:

$$w(x,t) = U(\xi)/t, \qquad \xi = x - a\ln|t|,$$

where the function $U = U(\xi)$ is determined by the autonomous ordinary differential equation $\beta(aU'''_{\xi\xi\xi} + U''_{\xi\xi}) - (U + a)U'_\xi - U = 0$.

5°. Conservation laws for $\beta = 1$:

$$D_t w + D_x\left(-w_{tx} - \tfrac{1}{2}w^2\right) = 0,$$
$$D_t\left(\tfrac{1}{2}w^2 + \tfrac{1}{2}w_x^2\right) + D_x\left(-ww_{tx} - \tfrac{1}{3}w^3\right) = 0,$$
$$D_t\left(\tfrac{1}{3}w^3\right) + D_x\left(w_t^2 - w_{tx}^2 - w^2 w_{tx} - \tfrac{1}{4}w^4\right) = 0,$$

where $D_x = \frac{\partial}{\partial x}$ and $D_t = \frac{\partial}{\partial t}$.

⊙ *References*: D. N. Peregrine (1966), T. B. Benjamin, J. L. Bona, and J. J. Mahony (1972), P. O. Olver (1979), N. H. Ibragimov (1994).

3. $\displaystyle \frac{\partial w}{\partial t} + aw^k\frac{\partial w}{\partial x} - b\frac{\partial^3 w}{\partial x^2 \partial t} = 0.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(\pm x + C_2, \pm C_1^k t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (either plus or minus signs are taken).

2°. Traveling-wave solution (soliton):

$$w(x,t) = \left\{\frac{C_1(k+1)(k+2)}{2a}\cosh^{-2}\left[\frac{k}{2\sqrt{b}}(x - C_1 t + C_2)\right]\right\}^{1/k},$$

where C_1 and C_2 are arbitrary constants.

⊙ *Private communications*: W. E. Schiesser (2003), S. Hamdi, W. H. Enright, W. E. Schiesser, and J. J. Gottlieb (2003).

3°. There is a multiplicative separable solution of the form $w(x,t) = t^{-1/k}\theta(x)$.

4. $\dfrac{\partial w}{\partial t} + aw^k \dfrac{\partial w}{\partial x} + b\dfrac{\partial^3 w}{\partial x \partial t^2} = 0.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w(\pm C_1^{-k} x + C_2, \pm C_1^k t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (either plus or minus signs are taken).

2°. Traveling-wave solution (soliton):

$$w(x,t) = \left\{ \frac{C_1(k+1)(k+2)}{2a} \cosh^{-2} \left[\frac{k}{2\sqrt{bC_1}}(x - C_1 t + C_2) \right] \right\}^{1/k},$$

where C_1 and C_2 are arbitrary constants.

⊙ *Private communication*: W. E. Schiesser (2003).

3°. There is a self-similar solution of the form $w(x,t) = x^{2/k} U(z)$, where $z = xt$.

4°. Generalized separable solution for $k = 1$:

$$w(x,t) = \frac{x + C_2}{at + C_1} + \frac{2ab}{(at + C_1)^2}.$$

5. $\dfrac{\partial^3 w}{\partial x^2 \partial t} = kw \dfrac{\partial^3 w}{\partial x^3}.$

This equation is encountered at the interface between projective geometry and gravitational theory.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w\bigl(C_2 x + C_2 k\varphi(t), C_1 C_2 t + C_3\bigr) + \varphi'_t(t),$$

where C_1, C_2, and C_3 are arbitrary constants and $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = Cx^2 + \varphi(t)x + \psi(t),$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions and C is an arbitrary constant.

3°. Self-similar solution:

$$w(x,t) = t^{-\alpha-1} U(z), \qquad z = t^\alpha x,$$

where α is an arbitrary constant and the function $U(z)$ is determined by the ordinary differential equation $(\alpha - 1)U''_{zz} + \alpha z U'''_{zzz} = kUU'''_{zzz}$.

4°. Multiplicative separable solution:

$$w(x,t) = (Akt + B)^{-1} u(x),$$

where A and B are arbitrary constants, and the function $u(x)$ is determined by the autonomous ordinary differential equation $uu'''_{xxx} + Au''_{xx} = 0$.

5°. There is a first integral:

$$\frac{\partial^2 w}{\partial x \partial t} = kw \frac{\partial^2 w}{\partial x^2} - \frac{k}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \psi(t),$$

where $\psi(t)$ is an arbitrary function. For $\psi = 0$, the substitution $u = kw$ leads to an equation of the form 7.1.1.2 with $a = -\frac{1}{2}$.

⊙ *References*: V. S. Dryuma (2000), M. V. Pavlov (2001).

6. $\dfrac{\partial^3 w}{\partial x^2 \partial t} = f(t) w \dfrac{\partial^3 w}{\partial x^3} + g(x,t).$

There is a first integral:

$$\frac{\partial^2 w}{\partial x \partial t} = f(t) w \frac{\partial^2 w}{\partial x^2} - \frac{1}{2} f(t) \left(\frac{\partial w}{\partial x} \right)^2 + \int g(x,t)\, dx + \varphi(t),$$

where $\varphi(t)$ is an arbitrary function.

7. $\dfrac{\partial w}{\partial y} \dfrac{\partial^3 w}{\partial x^2 \partial y} - \dfrac{\partial w}{\partial x} \dfrac{\partial^3 w}{\partial x \partial y^2} = 0.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1 = C_1 w(C_2 x + C_3, C_4 y + C_5) + C_6,$$

where $C_1, \ldots, C_6$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$\begin{aligned}
&w(x,y) = axy + f(x) + g(y);\\
&w(x,y) = \frac{1}{\lambda}\bigl[f(x) + g(y)\bigr] - \frac{2}{\lambda} \ln \left| b \int \exp\bigl[f(x)\bigr]\, dx + \frac{a\lambda}{2b} \int \exp\bigl[g(y)\bigr]\, dy \right|,\\
&w(x,y) = \varphi(z), \qquad z = ax + by;\\
&w(x,y) = \psi(\xi), \qquad \xi = xy;
\end{aligned}$$

where $f = f(y)$, $g = g(y)$, $\varphi(z)$, and $\psi(\xi)$ are arbitrary functions; a, b, and λ are arbitrary constants.

3°. There are exact solutions of the following forms:

$$\begin{aligned}
&w(x,y) = |x|^a F(r), \qquad r = y|x|^b;\\
&w(x,y) = e^{ax} G(\eta), \qquad \eta = bx + cy;\\
&w(x,y) = e^{ax} H(\zeta), \qquad \zeta = ye^{bx};\\
&w(x,y) = |x|^a U(\rho), \qquad \rho = y + b \ln|x|;\\
&w(x,y) = V(r) + a \ln|x|, \qquad r = y|x|^b;\\
&w(x,y) = W(\rho) + a \ln|x|, \qquad \rho = y + b \ln|x|;
\end{aligned}$$

where a, b, and c are arbitrary constants. Another set of solutions can be obtained by swapping x and y in the above formulas.

4°. The left-hand side of the original equation represents the Jacobian of w and $v = w_{xy}$. The fact that the Jacobian of two quantities is zero means that these are functionally dependent, i.e., v can be treated as a function of w:

$$\frac{\partial^2 w}{\partial x \partial y} = \Phi(w), \tag{1}$$

where $\Phi(w)$ is an arbitrary function. Any solution of the second-order equation (1) with arbitrary $\Phi(w)$ will be a solution of the original equation.

8. $\dfrac{\partial w}{\partial y} \dfrac{\partial^3 w}{\partial x^3} - \dfrac{\partial w}{\partial x} \dfrac{\partial^3 w}{\partial x^2 \partial y} = 0.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$\begin{aligned}
w_1 &= C_1 w(C_2 x + C_3, C_4 y + C_5) + C_6,\\
w_2 &= w\bigl(x + \varphi(y), y\bigr),
\end{aligned}$$

where $C_1, \ldots, C_6$ are arbitrary constants and $\varphi(y)$ is an arbitrary function, are also solutions of the equation.

2°. Solutions:

$$w(x,y) = Ax^2 + fx + g;$$
$$w(x,y) = f\exp(Ax) + g\exp(-Ax);$$
$$w(x,y) = f\sin(Ax) + g\cos(Ax);$$
$$w(x,y) = A\ln\big[(x+f)^2\big] + B;$$
$$w(x,y) = A\ln\big[\sin^2(fx+g)\big] + B;$$
$$w(x,y) = A\ln\big[\sinh^2(fx+g)\big] + B;$$
$$w(x,y) = A\ln\big[\cosh^2(fx+g)\big] + B;$$
$$w(x,y) = \varphi(z), \qquad z = Ax + By;$$

where $f = f(y)$, $g = g(y)$, and $\varphi(z)$ are arbitrary functions; A and B are arbitrary constants.

3°. The left-hand side of the original equation represents the Jacobian of w and $v = w_{xx}$. The fact that the Jacobian of two quantities is zero means that these are functionally dependent, i.e., v can be treated as a function of w:

$$\frac{\partial^2 w}{\partial x^2} = \varphi(w), \tag{1}$$

where $\varphi(w)$ is an arbitrary function. Any solution of the second-order equation (1) with arbitrary $\varphi(w)$ will be a solution of the original equation.

Integrating (1) yields the general solution of the original equation in implicit form:

$$\int\left[f(y) + 2\int\varphi(w)\,dw\right]^{-1/2} dw = g(y) \pm x,$$

where $f = f(y)$, $g = g(y)$, and $\varphi(w)$ are arbitrary functions.

9. $\displaystyle \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x^3} - \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^2 \partial y} = f(y)\frac{\partial w}{\partial x}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the functions

$$w_1 = C_1^{-2} w(C_1 x + C_2, y) + C_3,$$
$$w_2 = w\big(x + \varphi(y), y\big),$$

where C_1, C_2, and C_3 are arbitrary constants and $\varphi(y)$ is an arbitrary function, are also solutions of the equation.

2°. Generalized separable solution quadratic in x:

$$w(x,y) = -\frac{1}{2}x^2\left[\int f(y)\,dy + C\right] + x\varphi(y) + \psi(y),$$

where $\varphi(y)$ and $\psi(y)$ are arbitrary functions and C is an arbitrary constant.

3°. Additive separable solutions:

$$w(x,y) = C_1 e^{kx} + C_2 e^{-kx} + C_3 + \frac{1}{k^2}\int f(y)\,dy,$$
$$w(x,y) = C_1\cos(kx) + C_2\sin(kx) + C_3 - \frac{1}{k^2}\int f(y)\,dy,$$

where C_1, C_2, C_3, and k are arbitrary constants.

4°. The original equation can be rewritten as the relation where the Jacobian of two functions, w and $v = w_{xx} + \int f(y)\,dy$, is equal to zero. It follows that w and v are functionally dependent, i.e., v can be treated as a function of w:

$$\frac{\partial^2 w}{\partial x^2} + \int f(y)\,dy = \varphi(w), \tag{1}$$

where $\varphi(w)$ is an arbitrary function. Any solution of the second-order equation (1) with arbitrary $\varphi(w)$ will be a solution of the original equation.

Equation (1) may be treated as an ordinary differential equation with independent variable x and parameter y. Integrating yields the general solution of (1) in implicit form:

$$\int \left[\psi_1(y) - 2w \int f(y)\,dy + 2\int \varphi(w)\,dw\right]^{-1/2} dw = \psi_2(y) \pm x,$$

where $\psi_1(y)$, $\psi_2(y)$, and $\varphi(w)$ are arbitrary functions.

10. $\displaystyle \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x^3} - \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^2 \partial y} = f(y)\frac{\partial w}{\partial x} + g(x)\frac{\partial w}{\partial y}.$

First integral:

$$\frac{\partial^2 w}{\partial x^2} = \varphi(w) + \int g(x)\,dx - \int f(y)\,dy,$$

where $\varphi(w)$ is an arbitrary function. This equation can be treated as a second-order ordinary differential equation with independent variable x and parameter y.

11. $\displaystyle \frac{\partial w}{\partial y}\left(\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial y^3} - \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x \partial y^2}\right) = 2\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y}\right).$

General solution:

$$w = f\big(\varphi(x)y + \psi(x)\big),$$

where $\varphi(x)$, $\psi(x)$, and $f(z)$ are arbitrary functions.

Remark. The equation in question can be represented as the equality of the Jacobian of two functions, w and v, to zero:

$$w_x v_y - w_y v_x = 0, \qquad \text{where} \quad v = w_{yy}/w_y^2.$$

12. $\displaystyle \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\left(\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x \partial y^2} - \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x^2 \partial y}\right) = \frac{\partial^2 w}{\partial x \partial y}\left[\left(\frac{\partial w}{\partial x}\right)^2 \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial w}{\partial y}\right)^2 \frac{\partial^2 w}{\partial x^2}\right].$

Two forms of representation of the general solution:

$$w = f\big(\varphi(x) + \psi(y)\big),$$
$$w = \bar{f}\big(\bar{\varphi}(x)\bar{\psi}(y)\big),$$

where $\varphi(x)$, $\psi(y)$, $\bar{\varphi}(x)$, $\bar{\psi}(y)$, $f(z_1)$, and $\bar{f}(z_2)$ are arbitrary functions.

Remark. The equation in question can be represented as the equality of the Jacobian of two functions to zero:

$$w_x v_y - w_y v_x = 0, \qquad \text{where} \quad v = w_{xy}/(w_x w_y).$$

9.5.3. Equations Involving $\dfrac{\partial^3 w}{\partial x^3}$ and $\dfrac{\partial^3 w}{\partial y^3}$

1. $\displaystyle a\frac{\partial^3 w}{\partial x^3} + b\frac{\partial^3 w}{\partial y^3} = (ay^3 + bx^3)f(w).$

Solution:

$$w = w(z), \qquad z = xy,$$

where the function $w(w)$ is determined by the autonomous ordinary differential equation

$$w'''_{zzz} = f(w).$$

Remark. The above remains true if the constants a and b in the original equation are replaced by arbitrary functions $a = a(x, y, w, w_x, w_y, \dots)$ and $b = b(x, y, w, w_x, w_y, \dots)$.

2. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x^2}+a\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2}=b\dfrac{\partial^3 w}{\partial x^3}+c\dfrac{\partial^3 w}{\partial y^3}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1=w(C_1x+C_2,C_1y+C_3)+C_4,$$

where $C_1,\dots,C_4$ are arbitrary constants, is also a solution of the equation.

2°. Additive separable solutions:

$$w(x,y)=c\lambda x+C_1e^{a\lambda y}+C_2y+C_3,$$
$$w(x,y)=C_1e^{\lambda x}+C_2x+b\lambda y+C_3,$$
$$w(x,y)=C_1e^{-a\lambda x}+\frac{c\lambda}{a}x+C_2e^{\lambda y}-ab\lambda y+C_3,$$

where C_1, C_2, C_3, and λ are arbitrary constants.

3°. Solution:

$$w=u(z)+C_3x,\qquad z=C_1x+C_2y,$$

where C_1, C_2 and C_3 are arbitrary constants. The function $u(z)$ is determined by the second-order autonomous ordinary differential equation (C_4 is an arbitrary constant)

$$C_1C_2(C_1+aC_2)(u'_z)^2+2aC_2^2C_3u'_z=2(bC_1^3+cC_2^3)u''_{zz}+C_4.$$

To $C_3=0$ there corresponds a traveling-wave solution. In this case, the substitution $F(u)=(u'_z)^2$ leads to a first-order linear equation.

4°. There is a self-similar solution of the form $w=w(y/x)$.

3. $\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial^2 w}{\partial y^2}=a\dfrac{\partial^3 w}{\partial x^3}+b\dfrac{\partial^3 w}{\partial y^3}.$

1°. Suppose $w(x,y)$ is a solution of this equation. Then the function

$$w_1=C_1^{-1}w(C_1x+C_2,C_1y+C_3)+C_4xy+C_5x+C_6y+C_7,$$

where $C_1,\dots,C_7$ are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w(x,y)=-\frac{aC_1^3+bC_2^3}{C_1^2C_2^2}z(\ln|z|-1),\qquad z=C_1x+C_2y+C_3.$$

3°. Additive separable solutions:

$$w(x,y)=\tfrac{1}{2}bC_1x^2+C_2x+C_3\exp(C_1y)+C_4y+C_5,$$
$$w(x,y)=\tfrac{1}{2}aC_1y^2+C_2y+C_3\exp(C_1x)+C_4x+C_5,$$

where $C_1,\dots,C_5$ are arbitrary constants.

4°. Solution:

$$w=U(\zeta)+C_3x^2+C_4y^2,\qquad \zeta=C_1x+C_2y,$$

where the function $U(\zeta)$ is determined by the autonomous ordinary differential equation

$$(C_1^2U''_{\zeta\zeta}+2C_3)(C_2^2U''_{\zeta\zeta}+2C_4)=(aC_1^3+bC_2^3)U'''_{\zeta\zeta\zeta},$$

which can be integrated with the substitution $F(\zeta)=U''_{\zeta\zeta}$.

5°. There is a self-similar solution of the form $w=xu(y/x)$.

Chapter 10

Fourth Order Equations

10.1. Equations Involving the First Derivative in t

10.1.1. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + F\Big(x, t, w, \dfrac{\partial w}{\partial x}\Big)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + bw\ln w + f(t)w.$

1°. Generalized traveling-wave solution:

$$w(x,t) = \exp\left[Ae^{bt}x + Be^{bt} + \frac{aA^4}{3b}e^{4bt} + e^{bt}\int e^{-bt}f(t)\,dt\right],$$

where A and B are arbitrary constants.

2°. Solution:

$$w(x,t) = \exp\left[Ae^{bt} + e^{bt}\int e^{-bt}f(t)\,dt\right]\varphi(z), \qquad z = x + \lambda t,$$

where A and λ are arbitrary constants, and the function $\varphi = \varphi(z)$ is determined by the autonomous ordinary differential equation

$$a\varphi''''_{zzzz} - \lambda\varphi'_z + b\varphi\ln\varphi = 0,$$

whose order can be reduced by one.

3°. The substitution

$$w(x,t) = \exp\left[e^{bt}\int e^{-bt}f(t)\,dt\right]u(x,t)$$

leads to the simpler equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^4 u}{\partial x^4} + bu\ln u.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + f(t)w\ln w + [g(t)x + h(t)]w.$

This is a special case of equation 11.1.2.5 with $n = 4$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + (bx + c)\dfrac{\partial w}{\partial x} + f(w).$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + C_1e^{-bt}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + C_1e^{-bt},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$aw''''_{zzzz} + (bz + c)w'_z + f(w) = 0.$$

4. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial w}{\partial x} - b\dfrac{\partial^4 w}{\partial x^4}.$

This equation describes the evolution of nonlinear waves in a dispersive medium; see Rudenko and Robsman (2002).

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^3 w(C_1 x + aC_1C_2 t + C_3, C_1^4 t + C_4) + C_2,$$

where $C_1, \dots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t) = -\frac{x + C_1}{at + C_2},$$
$$w(x,t) = -\frac{120b}{a(x + aC_1 t + C_2)^3} + C_1.$$

The first solution is degenerate and the second one is a traveling-wave solution.

3°. Traveling-wave solution in implicit form:

$$\left(\frac{40b}{9a}\right)^{1/3} \int_0^{C_1^3 w + C_2} \frac{d\eta}{(1-\eta^2)^{2/3}} = C_1 x + aC_1C_2 t + C_3.$$

With $C_1 = 1$ and $C_2 = C_3 = 0$, we have the stationary solution obtained in Rudenko and Robsman (2002).

4°. Traveling-wave solution (generalizes the second solution of Item 2° and the solution of Item 3°):

$$w = w(\xi), \quad \xi = x - \lambda t,$$

where the function $w(\xi)$ is determined by the third-order autonomous ordinary differential equation

$$bw'''_{\xi\xi\xi} = \tfrac{1}{2}aw^2 + \lambda w + C.$$

Here, C and λ are arbitrary constants.

5°. Self-similar solution:

$$w(x,t) = t^{-3/4} u(\eta), \qquad \eta = xt^{-1/4},$$

where the function $u(\eta)$ is determined by the ordinary differential equation

$$bu''''_{\eta\eta\eta\eta} = auu'_\eta + \tfrac{1}{4}\eta u'_\eta + \tfrac{3}{4}u.$$

6°. Solution:

$$w(x,t) = U(\zeta) + 2C_1 t, \qquad \zeta = x + aC_1 t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(\zeta)$ is determined by the third-order ordinary differential equation

$$bU'''_{\zeta\zeta\zeta} - \tfrac{1}{2}aU^2 + C_2 U = -2C_1\zeta + C_3.$$

7°. Solution:

$$w = \varphi^3 F(z) + \frac{1}{a\varphi}(\varphi'_t x + \psi'_t), \qquad z = \varphi(t)x + \psi(t).$$

Here, the functions $\varphi(t)$ and $\psi(t)$ are defined by

$$\varphi(t) = (4At + C_1)^{-1/4},$$
$$\psi(t) = C_2(4At + C_1)^{3/4} + C_3(4At + C_1)^{-1/4},$$

where A, C_1, C_2, and C_3 are arbitrary constants, and the function $F(z)$ is determined by the ordinary differential equation

$$bF''''_{zzzz} - aFF'_z - 2AF + 3\frac{A^2}{a}z = 0.$$

8°. Let us set $a = b = -1$ (the original equation can be reduced to this case by appropriate scaling of the independent variables).

The equation admits a formal series solution of the form

$$w(x,t) = \frac{1}{[x-\varphi(t)]^3} \sum_{n=0}^{\infty} w_n(t)[x-\varphi(t)]^n.$$

The series coefficients $w_n = w_n(t)$ are expressed as

$$w_0 = -120, \quad w_1 = w_2 = 0, \quad w_3 = -\varphi'(t), \quad w_4 = w_5 = 0, \quad w_6 = \psi(t),$$
$$(n+1)(n-6)(n^2 - 13n + 60)w_n = \sum_{m=6}^{n-6} (m-3)w_{n-m}w_m + w'_{n-4},$$

where $\varphi(t)$ and $\psi(t)$ is an arbitrary function. This solution has a singularity at $x = \varphi(t)$.

9°. If $a = b = -1$, the equation also admits the formal series solution

$$w(x,t) = \frac{x}{t} + \frac{1}{x} \sum_{n=1}^{\infty} \left(\frac{t}{x^4}\right)^{n-1} \sum_{k=0}^{n-1} A_k^n x^{2k},$$

where A_0^1 is an arbitrary constant and the other coefficients can be expressed in terms of A_0^1 with recurrence relations. This solution can be generalized with the help of translations in the independent variables.

⊙ The solutions of Items 8° and 9° were obtained by V. G. Baydulov (private communication, 2002).

5. $\displaystyle \frac{\partial w}{\partial t} = aw\frac{\partial w}{\partial x} - b\frac{\partial^4 w}{\partial x^4} + f(t).$

The transformation

$$w = u(z,t) + \int_{t_0}^{t} f(\tau)\,d\tau, \qquad z = x + a\int_{t_0}^{t} (t-\tau)f(\tau)\,d\tau,$$

where t_0 is any, leads to an equation of the form 10.1.1.4:

$$\frac{\partial u}{\partial t} = au\frac{\partial u}{\partial x} - b\frac{\partial^4 u}{\partial x^4}.$$

6. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^4 w}{\partial x^4} + bw\frac{\partial w}{\partial x} + cw.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = w(x + bC_1e^{ct} + C_2,\ t + C_3) + C_1ce^{ct},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w = U(z) + C_1ce^{ct}, \qquad z = x + bC_1e^{ct} + C_2t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$aU''''_{zzzz} + bUU'_z - C_2U'_z + cU = 0.$$

If $C_1 = 0$, we have a traveling-wave solution.

3°. There is a degenerate solution linear in x:

$$w(x,t) = \varphi(t)x + \psi(t).$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + [f(t)\ln w + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \exp[\varphi(t)x + \psi(t)],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) + a\varphi^3(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + b\left(\dfrac{\partial w}{\partial x}\right)^2.$

1°. Suppose $w(x,t)$ is a solution of this equation. Then the function

$$w_1 = C_1^2 w(C_1 x + 2bC_1C_2 t + C_3, C_1^4 t + C_4) + C_2 x + bC_2^2 t + C_5,$$

where $C_1, \ldots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = C_1 t + C_2 + \int \theta(z)\,dz, \qquad z = x + \lambda t,$$

where C_1, C_2, and λ are arbitrary constants, and the function $\theta(z)$ is determined by the third-order autonomous ordinary differential equation

$$a\theta'''_{zzz} + b\theta^2 - \lambda\theta - C_1 = 0.$$

To $C_1 = 0$ there corresponds a traveling-wave solution.

3°. Self-similar solution:

$$w(x,t) = t^{-1/2}u(\zeta), \qquad \zeta = xt^{-1/4},$$

where the function $u(\zeta)$ is determined by the ordinary differential equation

$$au''''_{\zeta\zeta\zeta\zeta} + b(u'_\zeta)^2 + \tfrac{1}{4}\zeta u'_\zeta + \tfrac{1}{2}u = 0.$$

4°. There is a degenerate solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(t).$

The substitution $w = U(x,t) + \int f(t)\,dt$ leads to a simpler equation of the form 10.1.1.8:

$$\frac{\partial U}{\partial t} = a\frac{\partial^4 U}{\partial x^4} + b\left(\frac{\partial U}{\partial x}\right)^2.$$

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^4 w}{\partial x^4} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw + f(t).$

1°. Solution:

$$w(x,t) = Ae^{ct} + e^{ct}\int e^{-ct}f(t)\,dt + \theta(z), \qquad z = x + \lambda t,$$

where A and λ are arbitrary constants, and the function $\theta(z)$ is determined by the autonomous ordinary differential equation

$$a\theta''''_{zzzz} + b\big(\theta'_z\big)^2 - \lambda\theta'_z + c\theta = 0.$$

2°. There is a degenerate solution of the form

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

3°. The substitution

$$w = U(x,t) + e^{ct} \int e^{-ct} f(t)\,dt$$

leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^4 U}{\partial x^4} + b\left(\frac{\partial U}{\partial x}\right)^2 + cU.$$

10.1.2. Other Equations

1. $\displaystyle \frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} + \alpha\frac{\partial^2 w}{\partial x^2} + \beta\frac{\partial^3 w}{\partial x^3} + \gamma\frac{\partial^4 w}{\partial x^4} = 0.$

Kuramoto–Sivashinsky equation. It describes nonlinear waves in dispersive-dissipative media with an instability, waves arising in a fluid flowing down an inclined plane, the evolution of the concentration of a substance in chemical reactions, and others.

⊙ *References*: Y. Kuramoto and T. Tsuzuki (1976), B. J. Cohen, J. A. Krommes, W. M. Tang, and M. N. Rosenbluth (1976), V. Ya. Shkadov (1977), J. Topper and T. Kawahara (1978), G. I. Sivashinsky (1983).

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x - C_1 t + C_2,\, t + C_3) + C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = \frac{x + C_1}{t + C_2},$$

3°. Traveling-wave solutions:

$$w(x,t) = C_1 + \left[\tfrac{15}{76}\left(16\alpha - \beta^2\gamma^{-1}\right) + 15\beta k + 60\gamma k^2\right]F - \left(15\beta + 180\gamma k\right)F^2 + 60\gamma F^3,$$
$$F = k\left[1 + C_2 \exp(-kx - \lambda t)\right]^{-1},$$

where C_1 and C_2 are arbitrary constants, and the coefficients β, k, and λ are defined by

$$\beta = 0, \qquad k = \pm\sqrt{\tfrac{11}{19}\alpha\gamma^{-1}}, \quad \lambda = -C_1 k - \tfrac{30}{19}\alpha k^2 \quad \text{(first set of solutions)};$$
$$\beta = \pm 4\sqrt{\alpha\gamma}, \qquad k = \pm\sqrt{\alpha\gamma^{-1}}, \quad \lambda = -C_1 k - \tfrac{3}{2}\beta k^3 \quad \text{(second set of solutions)};$$
$$\beta = \pm\tfrac{12}{\sqrt{47}}\sqrt{\alpha\gamma}, \quad k = \pm\sqrt{\tfrac{1}{47}\alpha\gamma^{-1}}, \quad \lambda = -C_1 k - \tfrac{60}{47}\alpha k^2 \quad \text{(third set of solutions)};$$
$$\beta = \pm\tfrac{16}{\sqrt{73}}\sqrt{\alpha\gamma}, \quad k = \pm\sqrt{\tfrac{1}{73}\alpha\gamma^{-1}}, \quad \lambda = -C_1 k - \tfrac{90}{73}\alpha k^2 \quad \text{(fourth set of solutions)}.$$

⊙ *Reference*: N. A. Kudryashov (1989, 1990 *b*).

Special case. For $\beta = 0$, $\alpha = \gamma = 1$, $C_1 = 0$, and $C_2 = 1$, we have a solution

$$w(x,t) = \tfrac{15}{19}k\left(11H^3 - 9H + 2\right), \quad H = \tanh\left(\tfrac{1}{2}kx - \tfrac{15}{19}k^2 t\right), \quad k = \pm\sqrt{\tfrac{11}{19}},$$

which describes concentration waves in chemical reactions.

⊙ *Reference*: J. Kuramoto and T. Tsuzuki (1976).

4°. Solution:

$$w(x,t) = U(\zeta) + 2C_1 t, \qquad \zeta = x - C_1 t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(\zeta)$ is determined by the third-order ordinary differential equation (C_3 is an arbitrary constant)

$$\gamma U'''_{\zeta\zeta\zeta} + \beta U''_{\zeta\zeta} + \alpha U'_\zeta + \tfrac{1}{2}U^2 + C_2 U = -2C_1\zeta + C_3.$$

The special case $C_1 = 0$ corresponds to a traveling-wave solution.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial}{\partial x}\left(w^k \dfrac{\partial^3 w}{\partial x^3}\right).$

With $k = 3$, this equation occurs in problems on the motion of long bubbles in tubes and on the spread of drops over a rigid surface; see Bretherton (1962) and Starov (1983).

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_2 x + C_3, C_1^k C_2^4 t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t) = A(x + C_1 t + C_2)^{3/k}, \qquad A = \left[\frac{C_1 k^3}{3a(k-3)(2k-3)}\right]^{1/k};$$

$$w(x,t) = (Bt + C_1)^{-1/k}(x + C_2)^{4/k}, \qquad B = 8ak^{-3}(k+4)(k-4)(2-k).$$

3°. Traveling-wave solution (generalizes the first solution of Item 2°):

$$w = w(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w = w(z)$ is determined by the third-order autonomous ordinary differential equation $aw^k w'''_{zzz} - \lambda w = C_1$. The substitution $U(z) = (w'_z)^2$ leads to the second-order equation

$$aU''_{ww} = \pm 2(\lambda w^{1-k} + C_1 w^{-k})U^{-1/2}.$$

For its solutions at some values of k and C_1, see Polyanin and Zaitsev (2003).

4°. Self-similar solution:

$$w(x,t) = t^{-\frac{4\beta+1}{k}} u(\xi), \qquad \xi = xt^\beta,$$

where β is an arbitrary constant and the function $u = u(\xi)$ is determined by the ordinary differential equation

$$-(4\beta + 1)u + k\beta\xi u'_\xi = ak(u^k u'''_{\xi\xi\xi})'_\xi.$$

⊙ *Reference*: V. M. Starov (1983, the case $k = 3$ was considered).

5°. Solution:

$$w(x,t) = (C_1 t + C_2)^{-1/k} V(\zeta), \qquad \zeta = x + C_3 \ln|C_1 t + C_2|,$$

where the function $V = V(\zeta)$ is determined by the autonomous ordinary differential equation

$$ak(V^k V'''_{\zeta\zeta\zeta})'_\zeta - kC_1 C_3 V'_\zeta + C_1 V = 0.$$

Remark. For a special case $C_3 = 0$, we have a solution in multiplicative separable form.

6°. Generalized self-similar solution:

$$w(x,t) = e^{-4\beta t}\varphi(\eta), \qquad \eta = xe^{k\beta t},$$

where β is an arbitrary constant and the function $\varphi = \varphi(\eta)$ is determined by the ordinary differential equation

$$-4\beta\varphi + k\beta\eta\varphi'_\eta = a(\varphi^k \varphi'''_{\eta\eta\eta})'_\eta.$$

3. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^3}{\partial x^3}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{a}{f(w)} + b.$

Functional separable solution in implicit form:

$$\int f(w)\,dw = at - \frac{b}{24}x^4 + C_1 x^3 + C_2 x^2 + C_3 x + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

10.2. Equations Involving the Second Derivative in t

10.2.1. Boussinesq Equation and Its Modifications

1. $\dfrac{\partial^2 w}{\partial t^2} + \dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + \dfrac{\partial^4 w}{\partial x^4} = 0.$

Boussinesq equation in canonical form. This equation arises in several physical applications: propagation of long waves in shallow water, one-dimensional nonlinear lattice-waves, vibrations in a nonlinear string, and ion sound waves in a plasma.

⊙ *References*: Boussinesq (1872), M. Toda (1975), A. C. Scott (1975).

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w(C_1 x + C_2, \pm C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$\begin{aligned}
w(x,t) &= 2C_1 x - 2C_1^2 t^2 + C_2 t + C_3,\\
w(x,t) &= (C_1 t + C_2)x - \frac{1}{12C_1^2}(C_1 t + C_2)^4 + C_3 t + C_4,\\
w(x,t) &= -\frac{(x+C_1)^2}{(t+C_2)^2} + \frac{C_3}{t+C_2} + C_4(t+C_2)^2,\\
w(x,t) &= -\frac{x^2}{t^2} + C_1 t^3 x - \frac{C_1^2}{54}t^8 + C_2 t^2 + \frac{C_4}{t},\\
w(x,t) &= -\frac{(x+C_1)^2}{(t+C_2)^2} - \frac{12}{(x+C_1)^2},\\
w(x,t) &= -3\lambda^2 \cos^{-2}\left[\tfrac{1}{2}\lambda(x \pm \lambda t) + C_1\right],
\end{aligned}$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants.

3°. Traveling-wave solution (generalizes the last solution of Item 2°):

$$w = w(\zeta), \qquad \zeta = x + \lambda t,$$

where the function $w(\zeta)$ is determined by the second-order ordinary differential equation (C_1 and C_2 are arbitrary constants)

$$w''_{\zeta\zeta} + w^2 + 2\lambda^2 w + C_1\zeta + C_2 = 0.$$

For $C_1 = 0$, this equation is integrable by quadrature.

⊙ *References*: T. Nishitani and M. Tajiri (1982), G. R. W. Quispel, F. W. Nijhoff, and H. W. Capel (1982).

4°. Self-similar solution:

$$w = \frac{1}{t}U(z), \qquad z = \frac{x}{\sqrt{t}},$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$U''''_{zzzz} + (UU'_z)'_z + \tfrac{1}{4}z^2 U''_{zz} + \tfrac{7}{4}zU'_z + 2U = 0.$$

⊙ *Reference*: T. Nishitani and M. Tajiri (1982).

5°. Degenerate solution (generalizes the first four solutions of Item 2°):

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned}
\varphi''_{tt} &= -6\varphi^2,\\
\psi''_{tt} &= -6\varphi\psi,\\
\chi''_{tt} &= -2\varphi\chi - \psi^2.
\end{aligned}$$

6°. Solution:

$$w = f(\xi) - 4C_1^2 t^2 - 4C_1 C_2 t, \qquad \xi = x - C_1 t^2 - C_2 t,$$

where the function $f(\xi)$ is determined by the third-order ordinary differential equation

$$f'''_{\xi\xi\xi} + f f'_\xi + C_2^2 f'_\xi - 2C_1 f = 8C_1^2 \xi + C_3, \tag{1}$$

and C_1, C_2, and C_3 are arbitrary constants. Equation (1) is reduced to the second Painlevé equation.

⊙ *References*: T. Nishitani and M. Tajiri (1982), G. R. W. Quispel, F. W. Nijhoff, and H. W. Capel (1982), P. A. Clarkson and M. D. Kruskal (1989).

7°. Generalized separable solution (generalizes the penultimate solution of Item 2°):

$$w = (x + C_1)^2 u(t) - \frac{12}{(x + C_1)^2},$$

where the function $u = u(t)$ is determined by the second-order autonomous ordinary differential equation

$$u''_{tt} = -6u^2.$$

The function $u(t)$ is representable in terms of the Weierstrass elliptic function.

⊙ *Reference*: P. A. Clarkson and M. D. Kruskal (1989).

8°. Solution:

$$w = \frac{1}{t} F(z) - \frac{1}{4}\left(\frac{x}{t} + Ct\right)^2, \qquad z = \frac{x}{\sqrt{t}} - \frac{1}{3} C t^{3/2},$$

where C is an arbitrary constant and the function $F = F(z)$ is determined by the fourth-order ordinary differential equation

$$F''''_{zzzz} + (F F'_z)'_z + \tfrac{3}{4} z F'_z + \tfrac{3}{2} F - \tfrac{9}{8} z^2 = 0.$$

Its solutions are expressed via solutions of the fourth Painlevé equation.

⊙ *Reference*: P. A. Clarkson and M. D. Kruskal (1989).

9°. Solution:

$$w(x,t) = (a_1 t + a_0)^2 U(z) - \left(\frac{a_1 x + b_1}{a_1 t + a_0}\right)^2, \qquad z = x(a_1 t + a_0) + b_1 t + b_0.$$

Here, a_1, a_0, b_1, and b_0 are arbitrary constants, and the function $U = U(z)$ is determined by the second-order ordinary differential equation

$$U''_{zz} + \tfrac{1}{2} U^2 = c_1 z + c_2, \tag{2}$$

where c_1 and c_2 are arbitrary constants. For $c_1 = 0$, the general solution of equation (2) can be written out in implicit form. If $c_1 \neq 0$, the equation is reduced to the first Painlevé equation.

⊙ *Reference*: P. A. Clarkson and M. D. Kruskal (1989).

10°. Solution:

$$w(x,t) = (a_1 t + a_0)^2 U(z) - \left[\frac{a_1^2 x + \lambda (a_1 t + a_0)^5 + a_1 b_1}{a_1 (a_1 t + a_0)}\right]^2,$$

$$z = x(a_1 t + a_0) + \frac{\lambda}{6a_1^2}(a_1 t + a_0)^6 + b_1 t + b_0.$$

Here, a_1, a_0, b_1, and b_0 are arbitrary constants, and the function $U = U(z)$ is determined by the third-order ordinary differential equation

$$U'''_{zzz} + U U'_z + 5\lambda U = 50\lambda^2 z + c, \tag{3}$$

where c is an arbitrary constant. Equation (3) is reduced to the second Painlevé equation.

⊙ *Reference*: P. A. Clarkson and M. D. Kruskal (1989).

11°. Solution:

$$w(x,t) = \varphi^2(t)U(z) - \frac{1}{\varphi^2(t)}\left[x\varphi'_t(t) + \psi'_t(t)\right]^2, \quad z = \varphi(t)x + \psi(t).$$

Here, the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the autonomous system of second-order ordinary differential equations

$$\varphi''_{tt} = A\varphi^5, \tag{4}$$
$$\psi''_{tt} = A\varphi^4\psi, \tag{5}$$

where A is an arbitrary constant and the function $U = U(z)$ is determined by the fourth-order ordinary differential equation

$$U''''_{zzzz} + UU''_{zz} + (U'_z)^2 + AzU'_z + 2AU = 2A^2z^2.$$

A first integral of equation (4) is given by

$$(\varphi'_t)^2 = \tfrac{1}{3}A\varphi^6 + B,$$

where B is an arbitrary constant. The general solution of this equation can be expressed in terms of Jacobi elliptic functions. The general solution of equation (5) can be expressed in terms of $\varphi = \varphi(t)$ by

$$\psi = C_1\varphi(t) + C_2\varphi(t)\int\frac{dt}{\varphi^2(t)},$$

where C_1 and C_2 are arbitrary constants.

⊙ *References*: P. A. Clarkson and M. D. Kruskal (1989), P. A. Clarkson, D. K. Ludlow, and T. J. Priestley (1997).

12°. The Boussinesq equation is solved by the inverse scattering method. Any rapidly decaying function $F = F(x,y;t)$ as $x \to +\infty$ and satisfying simultaneously the two linear equations

$$\frac{1}{\sqrt{3}}\frac{\partial F}{\partial t} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} = 0,$$
$$\frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial y^3} = 0$$

generates a solution of the Boussinesq equation in the form

$$w = 12\frac{d}{dx}K(x,x;t),$$

where $K(x,y;t)$ is a solution of the linear Gel'fand–Levitan–Marchenko integral equation

$$K(x,y;t) + F(x,y;t) + \int_x^\infty K(x,s;t)F(s,y;t)\,ds = 0.$$

Time t appears here as a parameter.

⊙ *References*: V. E. Zakharov (1973), M. J. Ablowitz and H. Segur (1981), J. Weiss (1984).

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(w\frac{\partial w}{\partial x}\right) + b\frac{\partial^4 w}{\partial x^4}.$

Unnormalized Boussinesq equation.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1^2 w(C_1x + C_2, \pm C_1^2 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solutions:

$$w(x,t) = 2C_1x + 2aC_1^2t^2 + C_2t + C_3,$$
$$w(x,t) = (C_1t + C_2)x + \frac{a}{12C_1^2}(C_1t + C_2)^4 + C_3t + C_4,$$
$$w(x,t) = \frac{(x + C_1)^2}{a(t + C_2)^2} + \frac{C_3}{t + C_2} + C_4(t + C_2)^2,$$
$$w(x,t) = \frac{x^2}{at^2} + C_1t^3x + \frac{aC_1^2}{54}t^8 + C_2t^2 + \frac{C_4}{t},$$
$$w(x,t) = \frac{(x + C_1)^2}{a(t + C_2)^2} - \frac{12b}{a(x + C_1)^2},$$
$$w(x,t) = \frac{3\lambda^2}{a}\cosh^{-2}\left[\frac{\lambda}{2\sqrt{b}}(x \pm \lambda t) + C_1\right],$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants.

3°. Traveling-wave solution (generalizes the last solution of Item 2°):

$$w = u(\zeta), \qquad \zeta = x + \lambda t,$$

where the function $u = u(\zeta)$ is determined by the second-order ordinary differential equation (C_1 and C_2 are arbitrary constants)

$$bu''_{\zeta\zeta} + au^2 - 2\lambda^2 u + C_1\zeta + C_2 = 0.$$

For $C_1 = 0$, this equation is integrable by quadrature.

4°. Self-similar solution:

$$w = \frac{1}{t}U(z), \qquad z = \frac{x}{\sqrt{t}},$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$2U + \tfrac{7}{4}zU'_z + \tfrac{1}{4}z^2U''_{zz} = a(UU'_z)'_z + bU''''_{zzzz}.$$

5°. Degenerate solution (generalizes the first four solutions of Item 2°):

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t),$$

where the functions $\varphi = \varphi(t)$, $\psi = \psi(t)$, and $\chi = \chi(t)$ are determined by the autonomous system of ordinary differential equations

$$\varphi''_{tt} = 6a\varphi^2,$$
$$\psi''_{tt} = 6a\varphi\psi,$$
$$\chi''_{tt} = 2a\varphi\chi + a\psi^2.$$

6°. Solution:

$$w = f(\xi) + 4aC_1^2t^2 + 4aC_1C_2t, \qquad \xi = x + aC_1t^2 + aC_2t,$$

where the function $f(\xi)$ is determined by the third-order ordinary differential equation

$$bf'''_{\xi\xi\xi} + aff'_\xi - a^2C_2^2f'_\xi - 2aC_1f = 8aC_1^2\xi + C_3,$$

and C_1, C_2, and C_3 are arbitrary constants.

7°. Solution (generalizes the penultimate solution of Item 2°):

$$w = (x + C_1)^2u(t) - \frac{12b}{a(x + C_1)^2},$$

where the function $u = u(t)$ is determined by the second-order autonomous ordinary differential equation

$$u''_{tt} = 6au^2.$$

The function $u(t)$ is expressible in terms of the Weierstrass elliptic function.

8°. Solution:

$$w = \frac{1}{t}F(z) + \frac{1}{4a}\left(\frac{x}{t} + Ct\right)^2, \qquad z = \frac{x}{\sqrt{t}} - \frac{1}{3}Ct^{3/2},$$

where C is an arbitrary constant and the function $F = F(z)$ is determined by the ordinary differential equation

$$a(FF'_z)'_z + bF''''_{zzzz} = \frac{3}{4}zF'_z + \frac{3}{2}F + \frac{9}{8a}z^2.$$

9°. See also equation 10.2.1.3, Item 6°.

⊙ *References for equation* 10.2.1.2: T. Nishitani and M. Tajiri (1982), G. R. W. Quispel, F. W. Nijhoff, and H. W. Capel (1982), P. A. Clarkson and M. D. Kruskal (1989).

3. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + 6\frac{\partial}{\partial x}\left(w\frac{\partial w}{\partial x}\right) + \frac{\partial^4 w}{\partial x^4}.$

Solutions of this equation can be represented in the form

$$w(x,t) = 2\frac{\partial^2}{\partial x^2}(\ln u), \tag{1}$$

where the function $u = u(x,t)$ is determined by the bilinear equation

$$u\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial u}{\partial t}\right)^2 - u\frac{\partial^4 u}{\partial x^4} + 4\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3} - 3\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 = 0. \tag{2}$$

1°. One- or two-soliton solutions of the original equation are generated by the following solutions of equation (2):

$$u = 1 + A\exp\left(kx \pm kt\sqrt{1+k^2}\,\right),$$
$$u = 1 + A_1\exp(k_1x + m_1t) + A_2\exp(k_2x + m_2t) + A_1A_2p_{12}\exp\left[(k_1+k_2)x + (m_1+m_2)t\right],$$

where A, A_1, A_2, k, k_1, and k_2 are arbitrary constants, and

$$m_i = \pm k_i\sqrt{1+k_i^2}, \quad p_{12} = \frac{3(k_1-k_2)^2 + (n_1-n_2)^2}{3(k_1+k_2)^2 + (n_1-n_2)^2}, \quad n_i = \frac{m_i}{k_i}.$$

⊙ *References*: R. Hirota (1973), M. J. Ablowitz and H. Segur (1981).

2°. Rational solutions are generated by the following solutions of equation (2):

$$u = x \pm t,$$
$$u = x^2 - t^2 - 3,$$
$$u = (x \pm t)^3 + x \mp 5t.$$

⊙ *Reference*: M. J. Ablowitz and H. Segur (1981).

3°. Solution of equation (2):

$$u = \exp(2kx - 2mt) + (Cx - At)\exp(kx - mt) - B,$$
$$A = \frac{C(2k^2+1)}{\sqrt{1+k^2}}, \quad B = \frac{C^2(4k^2+3)}{12k^2(1+k^2)}, \quad m = \sqrt{k^2+k^4},$$

where k and C are arbitrary constants.

⊙ *Reference*: O. V. Kaptsov (1998).

4°. Solutions of equation (2):

$$u = \sin(kx - mt) + Ax + Bt,$$
$$u = \sin(kx) + C\sin(mt) + E\cos(mt),$$

where k and C are arbitrary constants,

$$m = \sqrt{k^2 - k^4}, \quad A = \sqrt{\frac{3m^2}{3 - 4k^2}}, \quad B = \frac{A(2k^2 - 1)}{\sqrt{1 - k^2}}, \quad E = \sqrt{\frac{1 - C^2 + k^2C^2 - 4k^2}{1 - k^2}}.$$

⊙ *Reference*: O. V. Kaptsov (1998).

5°. Solution (C is an arbitrary constant):

$$u = \sin(kx) + C\exp\bigl(t\sqrt{k^4 - k^2}\,\bigr) + \frac{4k^2 - 1}{4C(k^2 - 1)}\exp\bigl(-t\sqrt{k^4 - k^2}\,\bigr).$$

⊙ *Reference*: O. V. Kaptsov (1998).

6°. The substitution $w = \frac{1}{6}(U - 1)$ leads to an equation of the form 10.2.1.2:

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial x}\left(U\frac{\partial U}{\partial x}\right) + \frac{\partial^4 U}{\partial x^4}.$$

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + b\frac{\partial}{\partial x}\left(w\frac{\partial w}{\partial x}\right) + c\frac{\partial^4 w}{\partial x^4}.$

The substitution $w = U - (a/b)$ leads to an equation of the form 10.2.1.2:

$$\frac{\partial^2 U}{\partial t^2} = b\frac{\partial}{\partial x}\left(U\frac{\partial U}{\partial x}\right) + c\frac{\partial^4 U}{\partial x^4}.$$

10.2.2. Equations with Quadratic Nonlinearities

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^4 w}{\partial x^4} + bw\frac{\partial^2 w}{\partial x^2} + c.$

1°. Traveling-wave solution:

$$w(x, t) = u(\xi), \qquad \xi = \beta x + \lambda t,$$

where β and λ are arbitrary constants, and the function $u = u(\xi)$ is determined by the autonomous ordinary differential equation

$$a\beta^4 u''''_{\xi\xi\xi\xi} + (b\beta^2 u - \lambda^2)u''_{\xi\xi} + c = 0.$$

2°. Solution:

$$w(x, t) = U(z) + bC_1^2 t^2 + 2bC_1C_2 t, \qquad z = x - \tfrac{1}{2}bC_1t^2 - bC_2 t,$$

where C_1 and C_2 are arbitrary constants, and the function $U = U(z)$ is determined by the autonomous ordinary differential equation

$$aU''''_{zzzz} + bUU''_{zz} - b^2C_2^2U''_{zz} + bC_1U'_z + c - 2bC_1^2 = 0.$$

3°. There is a degenerate solution quadratic in x:

$$w(x, t) = f_2(t)x^2 + f_1(t)x + f_0(t).$$

2. $\displaystyle \frac{\partial^2 w}{\partial t^2} = \frac{\partial^4 w}{\partial x^4} + aw\frac{\partial^2 w}{\partial x^2} + b\frac{\partial w}{\partial x} + c.$

This is a special case of equation 11.3.5.3 with $n = 4$.

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^4 w}{\partial x^4} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + f(t).$

This is a special case of equation 11.3.2.2 with $n = 4$.

1°. Additive separable solution:

$$w(x,t) = \tfrac{1}{2}At^2 + Bt + C + \int_0^t (t-\tau)f(\tau)\,d\tau + \varphi(x).$$

Here, A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''''_{xxxx} + b(\varphi'_x)^2 - A = 0,$$

whose order can be reduced with the change of variable $U(x) = \varphi'_x$.

2°. The substitution

$$w = u(x,t) + \int_0^t (t-\tau)f(\tau)\,d\tau$$

leads to the simpler equation

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^4 u}{\partial x^4} + b\left(\frac{\partial u}{\partial x}\right)^2.$$

This equation admits a traveling-wave solution $u = u(kx+\lambda t)$ and a self-similar solution $u = t^{-1}\phi(z)$, where $z = xt^{-1/2}$.

4. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^4 w}{\partial x^4} + a\left(\dfrac{\partial w}{\partial x}\right)^2 + bw + f(t).$

1°. Solution:

$$w(x,t) = \varphi(t) + \psi(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the functions $\varphi(t)$ and $\psi(z)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - f(t) = 0,$$
$$\psi''''_{zzzz} - \lambda^2\psi''_{zz} + a(\psi'_z)^2 + b\psi = 0.$$

The general solution of the first equation is given by

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t f(\tau)\sinh\bigl[k(t-\tau)\bigr]\,d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t f(\tau)\sin\bigl[k(t-\tau)\bigr]\,d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution $w = u(x,t) + \varphi(t)$, where the function $\varphi(t)$ is defined in Item 1°, leads to the simpler equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^4 u}{\partial x^4} + a\left(\frac{\partial u}{\partial x}\right)^2 + bu.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^4 w}{\partial x^4} + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)(A_4x^4 + A_3x^3 + A_2x^2 + A_1x) + \psi(t),$$

where A_1, A_2, A_3, and A_4 are arbitrary constants, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = 24A_4a\varphi^2 + f(t)\varphi,$$
$$\psi''_{tt} = 24A_4a\varphi\psi + f(t)\psi + g(t).$$

6. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^4 w}{\partial x^4} + bw^2 + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the following system of second-order ordinary differential equations (C is an arbitrary constant):

$$\varphi''_{tt} = C\varphi^2 + b\varphi\psi + f(t)\varphi,$$
$$\psi''_{tt} = C\varphi\psi + b\psi^2 + f(t)\psi + g(t),$$

and the function $\Theta(x)$ satisfies the fourth-order constant-coefficient linear nonhomogeneous ordinary differential equation

$$a\Theta''''_{xxxx} + b\Theta = C.$$

7. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^2 w}{\partial x^2} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw^2 - f(t)\dfrac{\partial^2 w}{\partial x^2} - g(t)\dfrac{\partial^4 w}{\partial x^4} - h(t)w - p(t).$

1°. Generalized separable solution for $c/(a+b) = k^2 > 0$:

$$w(x,t) = \varphi_1(t) + \varphi_2(t)\cos(kx) + \varphi_3(t)\sin(kx),$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\varphi_1'' = c\varphi_1^2 + bk^2(\varphi_2^2 + \varphi_3^2) - h(t)\varphi_1 - p(t),$$
$$\varphi_2'' = (2c - ak^2)\varphi_1\varphi_2 + [k^2 f(t) - k^4 g(t) - h(t)]\varphi_2,$$
$$\varphi_3'' = (2c - ak^2)\varphi_1\varphi_3 + [k^2 f(t) - k^4 g(t) - h(t)]\varphi_3.$$

The prime denotes a derivative with respect to t. From the last two equations we have $\varphi_2''/\varphi_2 = \varphi_3''/\varphi_3$. It follows that

$$\varphi_3 = C_1\varphi_2 + C_2\varphi_2\int\frac{dt}{\varphi_2^2}, \tag{1}$$

where C_1 and C_2 are arbitrary constants.

2°. Generalized separable solution for $c/(a+b) = -k^2 < 0$:

$$w(x,t) = \varphi_1(t) + \varphi_2(t)\cosh(kx) + \varphi_3(t)\sinh(kx),$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\varphi_1'' = c\varphi_1^2 + bk^2(\varphi_3^2 - \varphi_2^2) - h(t)\varphi_1 - p(t),$$
$$\varphi_2'' = (2c + ak^2)\varphi_1\varphi_2 - [k^2 f(t) + k^4 g(t) + h(t)]\varphi_2,$$
$$\varphi_3'' = (2c + ak^2)\varphi_1\varphi_3 - [k^2 f(t) + k^4 g(t) + h(t)]\varphi_3.$$

The function φ_3 can be expressed in terms of φ_2 by formula (1).

3°. *Special case*: $a/b = -\frac{4}{3}$ and $bc < 0$.

Generalized separable solution:

$$w(x,t) = \psi_1(t) + \psi_2(t)\cos(kx) + \psi_3(t)\cos\left(\tfrac{1}{2}kx\right), \qquad k = \sqrt{-3c/b}.$$

Here, the functions $\psi_n = \psi_n(t)$ are determined by the system of ordinary differential equations

$$\psi_1'' = c\psi_1^2 + bk^2\psi_2^2 + \left(A + \tfrac{1}{4}bk^2\right)\psi_3^2 - h(t)\psi_1 - p(t),$$
$$\psi_2'' = (2c - ak^2)\psi_1\psi_2 + A\psi_3^2 + \left[k^2 f(t) - k^4 g(t) - h(t)\right]\psi_2,$$
$$\psi_3'' = \left(2c - \tfrac{1}{4}ak^2\right)\psi_1\psi_3 + bk^2\psi_2\psi_3 + \left[\tfrac{1}{4}k^2 f(t) - \tfrac{1}{16}k^4 g(t) - h(t)\right]\psi_3,$$

where $A = \frac{1}{8}\left[4c - (a+b)k^2\right]$.

There is a more general solution of the form

$$w(x,t) = \psi_1(t) + \psi_2(t)\cos(kx) + \psi_3(t)\sin(kx) + \psi_4(t)\cos(\tfrac{1}{2}kx) + \psi_5(t)\sin(\tfrac{1}{2}kx),$$

where $k = \sqrt{-3c/b}$.

4°. *Special case*: $a/b = -\frac{4}{3}$ and $bc > 0$.

Generalized separable solution:

$$w(x,t) = \psi_1(t) + \psi_2(t)\cosh(kx) + \psi_3(t)\cosh(\tfrac{1}{2}kx), \qquad k = \sqrt{3c/b}.$$

Here, the functions $\psi_n = \psi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\psi_1'' &= c\psi_1^2 - bk^2\psi_2^2 + \bigl(A - \tfrac{1}{4}bk^2\bigr)\psi_3^2 - h(t)\psi_1 - p(t),\\
\psi_2'' &= (2c + ak^2)\psi_1\psi_2 + A\psi_3^2 - \bigl[k^2 f(t) + k^4 g(t) + h(t)\bigr]\psi_2,\\
\psi_3'' &= \bigl(2c + \tfrac{1}{4}ak^2\bigr)\psi_1\psi_3 - k^2\psi_2\psi_3 - \bigl[\tfrac{1}{4}k^2 f(t) + \tfrac{1}{16}k^4 g(t) + h(t)\bigr]\psi_3,
\end{aligned}$$

where $A = \frac{1}{8}\bigl[4c + (a+b)k^2\bigr]$.

There is a more general solution of the form

$$w(x,t) = \psi_1(t) + \psi_2(t)\cosh(kx) + \psi_3(t)\sinh(kx) + \psi_4(t)\cosh(\tfrac{1}{2}kx) + \psi_5(t)\sinh(\tfrac{1}{2}kx),$$

where $k = \sqrt{3c/b}$.

⊙ *Reference for equation* 10.2.2.5: V. A. Galaktionov (1995).

8. $\displaystyle \frac{\partial^2 w}{\partial t^2} = w\frac{\partial^2 w}{\partial x^2} - \frac{3}{4}\left(\frac{\partial w}{\partial x}\right)^2 - a(t)\frac{\partial^4 w}{\partial x^4} - b(t)\frac{\partial^3 w}{\partial x^3} - c(t)\frac{\partial^2 w}{\partial x^2} - d(t)\frac{\partial w}{\partial x} - e(t)w - f(t).$

There is a generalized separable solution in the form of a fourth-degree polynomial in x:

$$w(x,t) = \varphi_4(t)x^4 + \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t).$$

⊙ *Reference*: V. A. Galaktionov (1995).

9. $\displaystyle \frac{\partial^2 w}{\partial t^2} = f(t)w\frac{\partial^4 w}{\partial x^4} + g(t)\frac{\partial^2 w}{\partial x^2} + h(t)\frac{\partial w}{\partial x} + p(t)w + q(t).$

Generalized separable solution in the form of a fourth-degree polynomial in x:

$$w(x,t) = \varphi_4(t)x^4 + \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t),$$

where the functions $\varphi_n = \varphi_n(t)$ are determined by the system of ordinary differential equations

$$\begin{aligned}
\varphi_4'' &= (24f\varphi_4 + p)\varphi_4,\\
\varphi_3'' &= (24f\varphi_4 + p)\varphi_3 + 4h\varphi_4,\\
\varphi_2'' &= (24f\varphi_4 + p)\varphi_2 + 12g\varphi_4 + 3h\varphi_3,\\
\varphi_1'' &= (24f\varphi_4 + p)\varphi_1 + 6g\varphi_3 + 2h\varphi_2,\\
\varphi_0'' &= (24f\varphi_4 + p)\varphi_2 + 2g\varphi_2 + h\varphi_1 + q.
\end{aligned}$$

► *For other equations with quadratic nonlinearities, see Subsection 10.2.1.*

10.2.3. Other Equations

1. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^4 w}{\partial x^4} + bw\ln w + \bigl[f(x) + g(t)\bigr]w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\begin{aligned}
\varphi_{tt}'' - \bigl[b\ln\varphi + g(t) + C\bigr]\varphi &= 0,\\
a\psi_{xxxx}'''' + \bigl[b\ln\psi + f(x) - C\bigr]\psi &= 0,
\end{aligned}$$

where C is an arbitrary constant.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^4 w}{\partial x^4} + f(x)w \ln w + \big[bf(x)t + g(x)\big]w.$

Multiplicative separable solution:

$$w(x,t) = e^{-bt}\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''''_{xxxx} + f(x)\varphi \ln \varphi + \big[g(x) - b^2\big]\varphi = 0.$$

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^4 w}{\partial x^4} + f\left(x, \dfrac{\partial w}{\partial x}\right) + g(t).$

1°. Additive separable solution:

$$w(x,t) = C_1 t^2 + C_2 t + \int_{t_0}^{t} (t - \tau) g(\tau)\, d\tau + \varphi(x),$$

where C_1, C_2, and t_0 are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi''''_{xxxx} + f\big(x, \varphi'_x\big) - 2C_1 = 0,$$

whose order can be reduced with the change of variable $u(x) = \varphi'_x$.

2°. The substitution $w = U(x,t) + \displaystyle\int_0^t (t - \tau) g(\tau)\, d\tau$ leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^4 U}{\partial x^4} + f\left(x, \frac{\partial U}{\partial x}\right).$$

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^4 w}{\partial x^4} + f\left(x, \dfrac{\partial w}{\partial x}\right) + bw + g(t).$

1°. Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - g(t) = 0,$$
$$a\psi''''_{xxxx} + f\big(x, \psi'_x\big) + b\psi = 0.$$

The general solution of the first equation is given by

$$\varphi(t) = C_1 \cosh(kt) + C_2 \sinh(kt) + \frac{1}{k}\int_0^t g(\tau) \sinh\big[k(t - \tau)\big]\, d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1 \cos(kt) + C_2 \sin(kt) + \frac{1}{k}\int_0^t g(\tau) \sin\big[k(t - \tau)\big]\, d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution $w = U(x,t) + \varphi(t)$, where the function $\varphi(t)$ is given in Item 1°, leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^4 U}{\partial x^4} + f\left(x, \frac{\partial U}{\partial x}\right) + bU.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^3}{\partial x^3}\left[f(w)\dfrac{\partial w}{\partial x}\right] - a^2\dfrac{f'(w)}{f^3(w)} + b.$

Functional separable solution in implicit form:

$$\int f(w)\, dw = at - \frac{1}{24}bx^4 + C_1 x^3 + C_2 x^2 + C_3 x + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

10.3. Equations Involving Mixed Derivatives

10.3.1. Kadomtsev–Petviashvili Equation

1. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial w}{\partial t}+\dfrac{\partial^3 w}{\partial x^3}-6w\dfrac{\partial w}{\partial x}\right)+a\dfrac{\partial^2 w}{\partial y^2}=0.$

Kadomtsev–Petviashvili equation in canonical form (Kadomtsev and Petviashvili, 1970). It arises in the theory of long, weakly nonlinear surface waves propagating in the x-direction, with the variation in y being sufficiently slow.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1=C_1^2w\bigl(C_1x+6C_1\varphi(t),\ \pm C_1^2y+C_2,\ C_1^3t+C_3\bigr)+\varphi'_t(t),$$

where C_1, C_2, and C_3 are arbitrary constants and $\varphi(t)$ is an arbitrary function, are also solutions of the equation.

2°. The time-invariant solutions satisfy the Boussinesq equation 10.2.1.2 (see also 10.2.1.1). The y-independent solutions satisfy the Korteweg–de Vries equation 9.1.1.1.

3°. One-soliton solution:

$$w(x,y,t)=-2\frac{\partial^2}{\partial x^2}\ln\bigl[1+Ae^{kx+kpy-k(k^2+ap^2)t}\bigr],$$

where A, k, and p are arbitrary constants.

4°. Two-soliton solution:

$$w(x,y,t)=-2\frac{\partial^2}{\partial x^2}\ln\bigl(1+A_1e^{\eta_1}+A_2e^{\eta_2}+A_1A_2Be^{\eta_1+\eta_2}\bigr),$$

$$\eta_i=k_ix+k_ip_iy-k_i(k_i^2+ap_i^2)t,\qquad B=\frac{3(k_1-k_2)^2-a(p_1-p_2)^2}{3(k_1+k_2)^2-a(p_1-p_2)^2},$$

where A_1, A_2, k_1, k_2, p_1, and p_2 are arbitrary constants.

5°. N-soliton solution:

$$w(x,y,t)=-2\frac{\partial^2}{\partial x^2}\ln\det\mathbf{A},$$

where $\mathbf{A}$ is an $N\times N$ matrix with entries

$$A_{nm}=\delta_{nm}+f_n(y,t)\frac{\exp[(p_n+q_m)x]}{p_n+q_m},\qquad \delta_{nm}=\begin{cases}1 & \text{if } n=m,\\ 0 & \text{if } n\neq m,\end{cases}$$

$$f_n(y,t)=C_n\exp\bigl[\sqrt{3/a}\,(q_n^2-p_n^2)y+4(p_n^3+q_n^3)t\bigr],\qquad n,m=1,\,2,\,\dots,\,N,$$

and the p_n, q_m, and C_n are arbitrary constants ($C_n>0$).

6°. Rational solutions:

$$w(x,y,t)=-2\frac{\partial^2}{\partial x^2}\ln(x+py-ap^2t),$$

$$w(x,y,t)=-2\frac{\partial^2}{\partial x^2}\ln\left[(x+p_1y-ap_1^2t)(x+p_2y-ap_2^2t)+\frac{12}{a(p_1-p_2)^2}\right],$$

where p, p_1, and p_2 are arbitrary constants.

7°. Two-dimensional power-law decaying solution ($a=-1$):

$$w(x,y,t)=4\frac{(\widetilde{x}+\beta\widetilde{y})^2-\gamma^2(\widetilde{y})^2-3/\gamma^2}{\bigl[(\widetilde{x}+\beta\widetilde{y})^2+\gamma^2(\widetilde{y})^2+3/\gamma^2\bigr]^2},\qquad \widetilde{x}=x-(\beta^2+\gamma^2)t,\quad \widetilde{y}=y+2\beta t,$$

where β and γ are arbitrary constants.

⊙ *Reference*: M. J. Ablowitz and H. Segur (1981).

8°. "Two-dimensional" solution:

$$w = U(z,t) + \tfrac{1}{6}a\lambda^2, \qquad z = x + \lambda y,$$

where λ is an arbitrary constant and the function $U = U(z,t)$ is determined by a third-order differential equation of the form 9.1.4.1:

$$\frac{\partial U}{\partial t} + \frac{\partial^3 U}{\partial z^3} - 6U\frac{\partial U}{\partial z} = \varphi(t),$$

with $\varphi(t)$ being an arbitrary function. For $\varphi = 0$ we have the Korteweg–de Vries equation 9.1.1.1.

9°. "Two-dimensional" solution:

$$w = V(\xi,t), \qquad \xi = x + C_1 y - aC_1^2 t,$$

where C_1 and C_2 are arbitrary constants and the function $V = V(\xi,t)$ is determined by a third-order differential equation of the form 9.1.4.1:

$$\frac{\partial V}{\partial t} + \frac{\partial^3 V}{\partial \xi^3} - 6V\frac{\partial V}{\partial \xi} = \varphi(t),$$

with $\varphi(t)$ being an arbitrary function. For $\varphi = 0$ we have the Korteweg–de Vries equation 9.1.1.1.

10°+. "Two-dimensional" solution:

$$w(x,y,t) = u(\eta,t), \qquad \eta = x + \frac{y^2}{4at},$$

where the function $u(\eta,t)$ is determined by the third-order differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial \eta^3} - 6u\frac{\partial u}{\partial \eta} + \frac{1}{2t}u = \psi(t),$$

with $\psi(t)$ being an arbitrary function. For $\psi = 0$ we have the cylindrical Korteweg–de Vries equation 9.1.2.1.

⊙ *References*: R. S. Johnson (1979), F. Calogero and A. Degasperis (1982).

11°+. There is a degenerate solution quadratic in x:

$$w = x^2\varphi(y,t) + x\psi(y,t) + \chi(y,t).$$

12°. The Kadomtsev–Petviashvili equation is solved by the inverse scattering method. Any rapidly decaying function $F = F(x,z;y,t)$ as $x \to +\infty$ and satisfying simultaneously the two linear equations

$$\sqrt{\frac{a}{3}}\,\frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial z^2} = 0,$$
$$\frac{\partial F}{\partial t} + 4\left(\frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial z^3}\right) = 0$$

generates a solution of the Kadomtsev–Petviashvili equation in the form

$$w = -2\frac{d}{dx}K(x,x;y,t),$$

where $K = K(x,z;y,t)$ is a solution to the linear Gel'fand–Levitan–Marchenko integral equation

$$K(x,z;y,t) + F(x,z;y,t) + \int_x^\infty K(x,s;y,t)F(s,z;y,t)\,ds = 0.$$

The quantities y and t appear here as parameters.

⊙ *References*: V. S. Dryuma (1974), V. E. Zakharov and A. B. Shabat (1974), I. M. Krichever and S. P. Novikov (1978), M. J. Ablowitz and H. Segur (1981), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), V. E. Adler, A. B. Shabat, and R. I. Yamilov (2000).

2. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial w}{\partial t}+a\dfrac{\partial^3 w}{\partial x^3}+bw\dfrac{\partial w}{\partial x}\right)+c\dfrac{\partial^2 w}{\partial y^2}=0.$

Unnormalized Kadomtsev–Petviashvili equation. The transformation $w=-\dfrac{6a}{b}U(x,y,\tau)$, $\tau=at$ leads to an equation of the form 10.3.1.1:

$$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial t}+\frac{\partial^3 U}{\partial x^3}-6U\frac{\partial U}{\partial x}\right)+\frac{c}{a}\frac{\partial^2 U}{\partial y^2}=0.$$

10.3.2. Stationary Hydrodynamic Equations (Navier–Stokes Equations)

1. $\dfrac{\partial w}{\partial y}\dfrac{\partial}{\partial x}(\Delta w)-\dfrac{\partial w}{\partial x}\dfrac{\partial}{\partial y}(\Delta w)=\nu\Delta\Delta w,\qquad \Delta w=\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}.$

Preliminary remarks. The two-dimensional stationary equations of a viscous incompressible fluid

$$\begin{aligned}
u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}&=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\Delta u_1,\\
u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}&=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\Delta u_2,\\
\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}&=0
\end{aligned}$$

are reduced to the equation in question by the introduction of a stream function w such that $u_1=\frac{\partial w}{\partial y}$ and $u_2=-\frac{\partial w}{\partial x}$ followed by the elimination of the pressure p (with cross differentiation) from the first two equations.

⊙ *Reference*: L. G. Loitsyanskiy (1996).

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the functions

$$\begin{aligned}
w_1&=-w(y,\,x),\\
w_2&=w(C_1x+C_2,\,C_1y+C_3)+C_4,\\
w_3&=w(x\cos\alpha+y\sin\alpha,\,-x\sin\alpha+y\cos\alpha),
\end{aligned}$$

where $C_1,\,\dots,\,C_4$ and α are arbitrary constants, are also solutions of the equation.

⊙ *Reference*: V. V. Pukhnachov (1960).

2°. Any solution of the Poisson equation $\Delta w=C$ is also a solution of the original equation (these are "inviscid" solutions). On the utilization of these solutions in the hydrodynamics of ideal fluids, see Lamb (1945), Batchelor (1970), Lavrent'ev and Shabat (1973), Sedov (1980), and Loitsyanskiy (1996).

3°. Solutions in the form of a one-variable function or the sum of functions with different arguments:

$$\begin{aligned}
w(y)&=C_1y^3+C_2y^2+C_3y+C_4,\\
w(x,y)&=C_1x^2+C_2x+C_3y^2+C_4y+C_5,\\
w(x,y)&=C_1\exp(-\lambda y)+C_2y^2+C_3y+C_4+\nu\lambda x,\\
w(x,y)&=C_1\exp(\lambda x)-\nu\lambda x+C_2\exp(\lambda y)+\nu\lambda y+C_3,\\
w(x,y)&=C_1\exp(\lambda x)+\nu\lambda x+C_2\exp(-\lambda y)+\nu\lambda y+C_3,
\end{aligned}$$

where $C_1,\,\dots,\,C_5$ and λ are arbitrary constants.

⊙ *References*: V. V. Pukhnachov (1960), L. G. Loitsyanskiy (1996), A. D. Polyanin (2001 *c*).

4°. Generalized separable solutions:

$$w(x,y) = A(kx+\lambda y)^3 + B(kx+\lambda y)^2 + C(kx+\lambda y) + D,$$
$$w(x,y) = Ae^{-\lambda(y+kx)} + B(y+kx)^2 + C(y+kx) + \nu\lambda(k^2+1)x + D,$$

where A, B, C, D, k, β, and λ are arbitrary constants.

⊙ *Reference*: V. V. Pukhnachov (1960).

5°. Generalized separable solutions:

$$w(x,y) = 6\nu x(y+\lambda)^{-1} + A(y+\lambda)^3 + B(y+\lambda)^{-1} + C(y+\lambda)^{-2} + D \qquad (\nu \neq 0),$$
$$w(x,y) = (Ax+B)e^{-\lambda y} + \nu\lambda x + C,$$
$$w(x,y) = \bigl[A\sinh(\beta x) + B\cosh(\beta x)\bigr]e^{-\lambda y} + \frac{\nu}{\lambda}(\beta^2+\lambda^2)x + C,$$
$$w(x,y) = \bigl[A\sin(\beta x) + B\cos(\beta x)\bigr]e^{-\lambda y} + \frac{\nu}{\lambda}(\lambda^2-\beta^2)x + C,$$
$$w(x,y) = Ae^{\lambda y+\beta x} + Be^{\gamma x} + \nu\gamma y + \frac{\nu}{\lambda}\gamma(\beta-\gamma)x + C, \qquad \gamma = \pm\sqrt{\lambda^2+\beta^2},$$

where A, B, C, D, k, β, and λ are arbitrary constants.

⊙ *Reference*: A. D. Polyanin (2001 *c*).

Special case. Setting $A=-\nu\lambda$, $B=C=0$, $\lambda=\sqrt{k/\nu}$ in the second solution, we obtain $w=\sqrt{k\nu}\,x\bigl[1-\exp\bigl(-\sqrt{k/\nu}\,y\bigr)\bigr]$. This solution describes the steady-state motion of a fluid due to the motion of the surface particles at $y=0$ with a velocity $u_1|_{y=0} = kx$.

6°. Generalized separable solution linear in x:

$$w(x,y) = F(y)x + G(y), \tag{1}$$

where the functions $F=F(y)$ and $G=G(y)$ are determined by the autonomous system of fourth-order ordinary differential equations

$$F'_y F''_{yy} - FF'''_{yyy} = \nu F''''_{yyyy}, \tag{2}$$
$$G'_y F''_{yy} - FG'''_{yyy} = \nu G''''_{yyyy}. \tag{3}$$

On integrating the equations once, we obtain the system of third-order equations

$$(F'_y)^2 - FF''_{yy} = \nu F'''_{yyy} + A, \tag{4}$$
$$G'_y F'_y - FG''_{yy} = \nu G'''_{yyy} + B, \tag{5}$$

where A and B are arbitrary constants. The order of the autonomous equation (4) can be reduced by one.

Equation (2) has the following particular solutions:

$$F(y) = ay + b, \tag{6}$$
$$F(y) = 6\nu(y+a)^{-1}, \tag{7}$$
$$F(y) = ae^{-\lambda y} + \lambda\nu, \tag{8}$$

where a, b, and λ are arbitrary constants.

In the general case, equation (5) is reduced, with the substitution $U = G'_y$, to the second-order linear nonhomogeneous equation

$$\nu U''_{yy} + FU'_y - F'_y U + B = 0, \qquad \text{where} \quad U = G'_y. \tag{9}$$

The corresponding homogeneous equation (with $B = 0$) has two linearly independent particular solutions:

$$U_1 = \begin{cases} F''_{yy} & \text{if } F''_{yy} \not\equiv 0, \\ F & \text{if } F''_{yy} \equiv 0, \end{cases} \qquad U_2 = U_1 \int \frac{\Phi\,dy}{U_1^2}, \qquad \text{where} \quad \Phi = \exp\biggl(-\frac{1}{\nu}\int F\,dy\biggr); \tag{10}$$

the first solution follows from the comparison of (2) and (9) with $B=0$. Therefore the general solutions of equations (9) and (3) are given by

$$U = C_1U_1 + C_2U_2 + C_3\left(U_2\int\frac{U_1}{\Phi}\,dy - U_1\int\frac{U_2}{\Phi}\,dy\right), \quad G = \int U\,dy + C_4, \quad C_3 = -\frac{B}{\nu}; \qquad (11)$$

see Polyanin and Zaitsev (2003).

The general solution of equation (3) corresponding to the particular solution (7) is expressed as

$$G(y) = \widetilde{C}_1(y+a)^3 + \widetilde{C}_2 + \widetilde{C}_3(y+a)^{-1} + \widetilde{C}_4(y+a)^{-2},$$

where $\widetilde{C}_1$, $\widetilde{C}_2$, $\widetilde{C}_3$, and $\widetilde{C}_4$ are arbitrary constants (these are expressed in terms of $C_1, \ldots, C_4$).

The general solutions of equation (3) corresponding to the particular solutions (6) and (8) are given by (10) and (11), respectively.

⊙ *References*: R. Berker (1963), A. D. Polyanin (2001 *c*).

Special case. A solution of the form (1) with $G(y) = kF(y)$ describes a laminar fluid flow in a plane channel with porous walls. In this case, equation (3) is satisfied by virtue of (2).

⊙ *Reference*: A. S. Berman (1953).

7°. Solution (generalizes the solution of Item 6°):

$$w(x,y) = F(z)x + G(z), \qquad z = y + kx,$$

where the functions $F=F(z)$ and $G=G(z)$ are determined by the autonomous system of fourth-order ordinary differential equations

$$F'_zF''_{zz} - FF'''_{zzz} = \nu(k^2+1)F''''_{zzzz}, \qquad (12)$$

$$G'_zF''_{zz} - FG'''_{zzz} = \nu(k^2+1)G''''_{zzzz} + 4k\nu F'''_{zzz} + \frac{2k}{k^2+1}FF''_{zz}. \qquad (13)$$

On integrating the equations once, we obtain the system of third-order equations

$$(F'_z)^2 - FF''_{zz} = \nu(k^2+1)F'''_{zzz} + A, \qquad (14)$$

$$G'_zF'_z - FG''_{zz} = \nu(k^2+1)G'''_{zzz} + \psi(z) + B, \qquad (15)$$

where A and B are arbitrary constants, and the function $\psi(z)$ is defined by

$$\psi(z) = 4k\nu F''_{zz} + \frac{2k}{k^2+1}\int FF''_{zz}\,dz.$$

The order of the autonomous equation (14) can be reduced by one.

Equation (12) has the following particular solutions:

$$F(z) = az + b, \qquad z = y + kx,$$
$$F(z) = 6\nu(k^2+1)(z+a)^{-1},$$
$$F(z) = ae^{-\lambda z} + \lambda\nu(k^2+1),$$

where a, b, and λ are arbitrary constants.

In the general case, equation (15) is reduced, with the substitution $U = G'_z$, to a second-order linear nonhomogeneous equation, a particular solution of which, in the homogeneous case $\psi = B = 0$, is given by

$$U = \begin{cases} F''_{zz} & \text{if } F''_{zz} \not\equiv 0, \\ F & \text{if } F''_{zz} \equiv 0. \end{cases}$$

Consequently, the general solution to (15) can be expressed by quadrature; see Polyanin and Zaitsev (2003).

⊙ *Reference*: A. D. Polyanin (2001 *c*).

8°. Self-similar solution:

$$w = \int F(z)\,dz + C_1, \quad z = \arctan\left(\frac{x}{y}\right),$$

where the function F is determined by the first-order autonomous ordinary differential equation

$$3\nu(F'_z)^2 - 2F^3 + 12\nu F^2 + C_2F + C_3 = 0, \tag{16}$$

and C_1, C_2, and C_3 are arbitrary constants. The general solution of equation (16) can be written out in implicit form and also can be expressed in terms of the Weierstrass elliptic function.

⊙ *Reference*: L. G. Loitsyanskiy (1996).

9°. There is an exact solution of the form

$$w = a\ln|x| + \int V(z)\,dz + C_1, \quad z = \arctan\left(\frac{x}{y}\right).$$

To $a = 0$ there corresponds a self-similar solution of (16).

► *For other exact solutions, see equation 10.3.2.4.*

2. $\displaystyle \frac{\partial w}{\partial y}\frac{\partial}{\partial x}(\Delta w) - \frac{\partial w}{\partial x}\frac{\partial}{\partial y}(\Delta w) = \nu\Delta\Delta w + f(y), \qquad \Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}.$

Preliminary remarks. The system

$$u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u_1 + F(y),$$

$$u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\Delta u_2,$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0$$

can be reduced to the equation in question by the introduction of a stream function w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$. The above system of equations describes the plane flow of a viscous incompressible fluid under the action of a transverse force. Here, $f(y) = F'_y(y)$.

The case $F(y) = a\sin(\lambda y)$ corresponds to A. N. Kolmogorov's model, which is used for describing subcritical and transitional (laminar-to-turbulent) flow modes.

⊙ *Reference*: O. M. Belotserkovskii and A. M. Oparin (2000).

1°. Solution in the form of a one-argument function:

$$w(y) = -\frac{1}{6\nu}\int_0^y (y-z)^3 f(z)\,dz + C_1y^3 + C_2y^2 + C_3y + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants.

2°. Additive separable solution for arbitrary $f(y)$:

$$w(x,y) = -\frac{1}{2\nu}\int_0^y (y-z)^2\Phi(z)\,dz + C_1e^{-\lambda y} + C_2y^2 + C_3y + C_4 + \nu\lambda x,$$

$$\Phi(z) = e^{-\lambda z}\int e^{\lambda z} f(z)\,dz,$$

where $C_1, \ldots, C_4$ and λ are arbitrary constants.

Special case. If $f(y) = a\beta\cos(\beta y)$, which corresponds to $F(y) = a\sin(\beta y)$, it follows from the preceding formula with $C_1 = C_2 = C_4 = 0$ and $B = -\nu\lambda$ that

$$w(x,y) = -\frac{a}{\beta^2(B^2+\nu^2\beta^2)}\bigl[B\sin(\beta y) + \nu\beta\cos(\beta y)\bigr] + Cy - Bx,$$

where B and C are arbitrary constants. This solution is specified in the book by Belotserkovskii and Oparin (2000); it describes a flow with a periodic structure.

3°. Additive separable solution for $f(y) = Ae^{\lambda y} + Be^{-\lambda y}$:

$$w(x,y) = C_1e^{-\lambda x} + C_2x - \frac{A}{\lambda^3(C_2+\nu\lambda)}e^{\lambda y} + \frac{B}{\lambda^3(C_2-\nu\lambda)}e^{-\lambda y} - \nu\lambda y,$$

where C_1 and C_2 are arbitrary constants.

4°. Generalized separable solution linear in x:

$$w(x,y)=\varphi(y)x+\psi(y),$$

where the functions $\varphi=\varphi(y)$ and $\psi=\psi(y)$ are determined by the system of fourth-order ordinary differential equations

$$\varphi'_y\varphi''_{yy}-\varphi\varphi'''_{yyy}=\nu\varphi''''_{yyyy}, \tag{1}$$
$$\psi'_y\varphi''_{yy}-\varphi\psi'''_{yyy}=\nu\psi''''_{yyyy}+f(y). \tag{2}$$

On integrating once, we obtain the system of third-order equations

$$(\varphi'_y)^2-\varphi\varphi''_{yy}=\nu\varphi'''_{yyy}+A, \tag{3}$$
$$\psi'_y\varphi'_y-\varphi\psi''_{yy}=\nu\psi'''_{yyy}+\int f(y)\,dy+B, \tag{4}$$

where A and B are arbitrary constants. The order of the autonomous equation (3) can be reduced by one.

Equation (1) has the following particular solutions:

$$\varphi(y)=ay+b,$$
$$\varphi(y)=6\nu(y+a)^{-1},$$
$$\varphi(y)=ae^{-\lambda y}+\lambda\nu,$$

where a, b, and λ are arbitrary constants.

In the general case, equation (4) is reduced, with the substitution $U=\psi'_y$, to the second-order linear nonhomogeneous equation

$$\nu U''_{yy}+\varphi U'_y-\varphi'_yU+F=0, \qquad \text{where} \quad U=\psi'_y, \quad F=\int f(y)\,dy+B. \tag{5}$$

The corresponding homogeneous equation (with $F=0$) has two linearly independent particular solutions:

$$U_1=\begin{cases}\varphi''_{yy} & \text{if } \varphi\neq ay+b,\\ \varphi & \text{if } \varphi=ay+b,\end{cases} \qquad U_2=U_1\int\frac{\Phi\,dy}{U_1^2}, \qquad \text{where} \quad \Phi=\exp\left(-\frac{1}{\nu}\int\varphi\,dy\right);$$

the first solution follows from the comparison of (1) and (5) with $F=0$. Consequently, the general solutions of equations (5) and (2) are given by

$$U=C_1U_1+C_2U_2+\frac{1}{\nu}U_1\int U_2\frac{F}{\Phi}\,dy-\frac{1}{\nu}U_2\int U_1\frac{F}{\Phi}\,dy, \quad \psi=\int U\,dy+C_4;$$

see Polyanin and Zaitsev (2003).

3. $\displaystyle \left(\frac{\partial w}{\partial y}+ax\right)\frac{\partial}{\partial x}(\Delta w)-\left(\frac{\partial w}{\partial x}-ay\right)\frac{\partial}{\partial y}(\Delta w)+2a\Delta w=\nu\Delta\Delta w.$

This equation is used for describing the motion of a viscous incompressible fluid induced by two parallel disks, moving towards each other; see Craik (1989) and equation 10.3.3.2 in the stationary case.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$w_1=-w(y,\,x),$$
$$w_2=w(x+C_1,\,y+C_2)-aC_2x+aC_1y+C_3,$$
$$w_3=w(x\cos\beta+y\sin\beta,\,-x\sin\beta+y\cos\beta),$$

where C_1, C_2, C_3, and β are arbitrary constants, are also solutions of the equation.

2°. Any solution of the Poisson equation $\Delta w = C$ is also a solution of the original equation (these are "inviscid" solutions). For details about the Poisson, see, for example, the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

3°. Solution dependent on a single coordinate x:

$$w(x) = \int_0^x (x-\xi)U(\xi)\,d\xi + C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants and the function $U(x)$ is determined by the second-order linear ordinary differential equation

$$axU'_x + 2aU = \nu U''_{xx}.$$

The general solution to this equation can be found in Polyanin and Zaitsev (2003).

Likewise, we can obtain solutions of the form $w = w(y)$.

4°. Generalized separable solution linear in x:

$$w(x,y) = F(y)x + G(y), \tag{1}$$

where the functions $F = F(y)$ and $G = G(y)$ are determined by the fourth-order ordinary differential equations

$$F'_y F''_{yy} - FF'''_{yyy} + a(3F''_{yy} + yF'''_{yyy}) = \nu F''''_{yyyy}, \tag{2}$$
$$F''_{yy} G'_y - FG'''_{yyy} + a(2G''_{yy} + yG'''_{yyy}) = \nu G''''_{yyyy}. \tag{3}$$

Equation (2) is solved independently of equation (3). If $F = F(y)$ is a solution to (2), then the function

$$F_1 = F(y + C) - aC,$$

where C is an arbitrary constant, is also a solution of the equation.

Integrating (2) and (3) with respect to y yields

$$(F'_y)^2 - FF''_{yy} + a(2F'_y + yF''_{yy}) = \nu F'''_{yyy} + C_1, \tag{4}$$
$$F'_y G'_y - FG''_{yy} + a(G'_y + yG''_{yy}) = \nu G'''_{yyy} + C_2, \tag{5}$$

where C_1 and C_2 are arbitrary constants.

Equation (2) has a particular solution

$$F(y) = Ay + B, \tag{6}$$

where A and B are arbitrary constants. On substituting (6) into (5) and performing the change of variable $Q = G''_{yy}$, we obtain the second-order linear ordinary differential equation

$$-\bigl[(A-a)y + B\bigr]Q'_y + 2aQ = \nu Q''_{yy},$$

whose general solution can be found in Polyanin and Zaitsev (2003).

Solutions of the form $w(x,y) = f(x)y + g(x)$ can be obtained likewise.

⊙ *Reference*: S. N. Aristov and I. M. Gitman (2002).

5°. Note that equation (2) has the following particular solutions:

$$F = ay + C_1 \exp\bigl(-2\sqrt{a/\nu}\,y\bigr) + C_2 \exp\bigl(2\sqrt{a/\nu}\,y\bigr) \quad \text{if} \quad a > 0,$$
$$F = ay + C_1 \cos\bigl(2\sqrt{-a/\nu}\,y\bigr) + C_2 \sin\bigl(2\sqrt{-a/\nu}\,y\bigr) \quad \text{if} \quad a < 0,$$

where C_1 and C_2 are arbitrary constants.

4. $\dfrac{1}{r}\dfrac{\partial w}{\partial \theta}\dfrac{\partial}{\partial r}(\Delta w)-\dfrac{1}{r}\dfrac{\partial w}{\partial r}\dfrac{\partial}{\partial \theta}(\Delta w)=\nu\Delta\Delta w, \qquad \Delta w=\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial w}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial \theta^2}.$

Preliminary remarks. Equation 10.3.2.1 is reduced to the equation in question by passing to the polar coordinate system with origin at (x_0, y_0), where x_0 and y_0 are any numbers, according to

$$x = r\cos\theta + x_0, \qquad y = r\sin\theta + y_0 \qquad \text{(direct transformation)},$$
$$r=\sqrt{(x-x_0)^2+(y-y_0)^2}, \quad \tan\theta=\frac{y-y_0}{x-x_0} \qquad \text{(inverse transformation)}.$$

The radial and angular fluid velocity components are expressed via the stream function w as follows: $u_r=\frac{1}{r}\frac{\partial w}{\partial \theta}$, $u_\theta=-\frac{\partial w}{\partial r}$.

1°. Any solution of the Poisson equation $\Delta w = C$ is also a solution of the original equation (these are "inviscid" solutions).

2°. Solutions in the form of a one-variable function and the sum of functions with different arguments:

$$w(r)=C_1r^2\ln r+C_2r^2+C_3\ln r+C_4,$$
$$w(r,\theta)=A\nu\theta+C_1r^{A+2}+C_2r^2+C_3\ln r+C_4,$$

where $A, C_1, \ldots, C_4$ are arbitrary constants.

⊙ *References*: G. B. Jeffery (1915), V. V. Pukhnachov (1960).

3°. Solution:

$$w=b\theta+U(\xi), \qquad \xi=\theta+a\ln r, \tag{1}$$

where the function $U(\xi)$ is determined by the autonomous ordinary differential equation

$$\nu(a^2+1)U^{(4)}_\xi-a(b+4\nu)U'''_{\xi\xi\xi}+2(b+2\nu)U''_{\xi\xi}+2U'_\xi U''_{\xi\xi}=0.$$

The onefold integration yields

$$\nu(a^2+1)U'''_{\xi\xi\xi}-a(b+4\nu)U''_{\xi\xi}+2(b+2\nu)U'_\xi+(U'_\xi)^2=C_1, \tag{2}$$

where C_1 is an arbitrary constant. Equation (2) is autonomous and independent of U explicitly. The transformation

$$z=U'_\xi, \qquad u(z)=U''_{\xi\xi}$$

brings it to the Abel equation of the second kind

$$\nu(a^2+1)uu'_z-a(b+4\nu)u+2(b+2\nu)z+z^2=C_1, \tag{3}$$

which is integrable by quadrature in some cases; for example, in the cases $a = 0$ and $b = -4\nu$, we have

$$\nu u^2+\tfrac{2}{3}z^3+2(b+2\nu)z^2=2C_1z+C_2 \quad \text{if} \quad a=0,$$
$$\nu(a^2+1)u^2+\tfrac{2}{3}z^3-4\nu z^2=2C_1z+C_2 \quad \text{if} \quad b=-4\nu.$$

Four other solvable cases for equation (3) are presented in the book by Polyanin and Zaitsev (2003); (3) is first reduced to a canonical form with the change of variable $u = k\bar{u}$, where $k = \text{const}$.

Note that to $a = b = 0$ in (1)–(3) there corresponds a solution dependent on the angle θ alone; this solution can be written out in implicit form, see equation 10.3.2.1, Item 8°.

⊙ *Reference*: L. G. Loitsyanskiy (1996).

4°. Generalized separable solution linear in θ:

$$w(r,\theta)=f(r)\theta+g(r).$$

Here, $f = f(r)$ and $g = g(r)$ are determined by the system of ordinary differential equations

$$-f'_r\mathbf{L}(f)+f[\mathbf{L}(f)]'_r=\nu r\mathbf{L}^2(f), \tag{4}$$
$$-g'_r\mathbf{L}(f)+f[\mathbf{L}(g)]'_r=\nu r\mathbf{L}^2(g), \tag{5}$$

where $\mathbf{L}(f)=r^{-1}(rf'_r)'_r$.

A particular solution to (4) is given by $f(r) = C_1 \ln r + C_2$. The corresponding equation (5) is reduced, with the substitution $Q = \mathbf{L}(g)$, to a second-order linear equation, which is easy to integrate (since it has a particular solution $Q = 1$). Consequently, we obtain an exact solution of system (4)–(5) in the form

$$f(r) = C_1 \ln r + C_2, \qquad g(r) = C_3 r^2 + C_4 \ln r + C_5 \int \left[\int r Q(r)\, dr \right] \frac{dr}{r} + C_6,$$

$$Q(r) = \int r^{(C_2/\nu)-1} \exp\left(\frac{C_1}{2\nu} \ln^2 r \right) dr,$$

where $C_1, \dots, C_6$ are arbitrary constants.

⊙ *References*: R. Berker (1963), A. D. Polyanin (2001 *c*).

5. $\displaystyle \frac{1}{r}\left(\frac{\partial w}{\partial z} \frac{\partial \mathbf{E}w}{\partial r} - \frac{\partial w}{\partial r} \frac{\partial \mathbf{E}w}{\partial z} \right) - \frac{2}{r^2} \frac{\partial w}{\partial z} \mathbf{E}w = \nu \mathbf{E}^2 w,$

where $\displaystyle \mathbf{E}w = r \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial w}{\partial r} \right) + \frac{\partial^2 w}{\partial z^2}, \quad \mathbf{E}^2 w = \mathbf{E}(\mathbf{E}w).$

Preliminary remarks. The stationary Navier–Stokes equations written in cylindrical coordinates for the axisymmetric case can be reduced to the equation in question by the introduction of a stream function w such that $u_r = \frac{1}{r}\frac{\partial w}{\partial z}$ and $u_z = -\frac{1}{r}\frac{\partial w}{\partial r}$, where $r = \sqrt{x^2 + y^2}$, and u_r and u_z are the radial and axial fluid velocity components.

⊙ *Reference*: J. Happel and H. Brenner (1965).

1°. Any function $w = w(r, z)$ that solves the second-order linear equation $\mathrm{E}w = 0$ is also a solution of the original equation.

2°. Solutions in the form of a one-argument function and the sum of functions with different arguments:

$$w(r) = C_1 r^4 + C_2 r^2 \ln r + C_3 r^2 + C_4,$$
$$w(r, z) = A\nu z + C_1 r^{A+2} + C_2 r^4 + C_3 r^2 + C_4,$$

where A, $C_1, \dots, C_4$ are arbitrary constants.

3°. Multiplicative separable solution:

$$w(r, z) = r^2 f(z),$$

where the function $f = f(z)$ is determined by the ordinary differential equation (C is an arbitrary constant):

$$\nu f'''_{zzz} + 2 f f''_{zz} - (f'_z)^2 = C. \tag{1}$$

This solution describes an axisymmetric fluid flow towards a plane (flow near a stagnation point).

⊙ *Reference*: H. Schlichting (1981).

4°. Generalized separable solution quadratic in r (generalizes the solution of Item 3°):

$$w(r, z) = r^2 f(z) + Az + B,$$

where A and B are arbitrary constants, and the function $f = f(z)$ is determined by the ordinary differential equation (1).

5°. Generalized separable solution linear in z:

$$w(r, z) = \varphi(r) z + \psi(r).$$

Here, $\varphi = \varphi(r)$ and $\psi = \psi(r)$ are determined by the system of ordinary differential equations

$$\varphi[\mathbf{L}(\varphi)]'_r - \varphi'_r \mathbf{L}(\varphi) - 2r^{-1} \varphi\, \mathbf{L}(\varphi) = \nu r \mathbf{L}^2(\varphi), \tag{2}$$
$$\varphi[\mathbf{L}(\psi)]'_r - \psi'_r \mathbf{L}(\varphi) - 2r^{-1} \varphi\, \mathbf{L}(\psi) = \nu r \mathbf{L}^2(\psi), \tag{3}$$

where $\mathbf{L}(\varphi) = \varphi''_{rr} - r^{-1} \varphi'_r$.

Particular solution of equation (2):

$$\varphi(r) = C_1 r^2 + C_2,$$

where C_1 and C_2 are arbitrary constants. In this case, the change of variable $U = \mathbf{L}(\psi)$ brings (3) to a second-order linear equation.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

6. $$\frac{1}{r^2\sin\theta}\left(\frac{\partial w}{\partial\theta}\frac{\partial \mathrm{E}w}{\partial r} - \frac{\partial w}{\partial r}\frac{\partial \mathrm{E}w}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\left(2\cot\theta\frac{\partial w}{\partial r} - \frac{2}{r}\frac{\partial w}{\partial\theta}\right)\mathrm{E}w = \nu\mathrm{E}^2 w,$$

$$\text{where } \mathrm{E}w = \frac{\partial^2 w}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial w}{\partial\theta}\right), \qquad \mathrm{E}^2 w = \mathrm{E}(\mathrm{E}w).$$

Preliminary remarks. The stationary Navier–Stokes equations written in spherical coordinates for the axisymmetric case are reduced to the given equation through the introduction of a stream function w such that $u_r = \frac{1}{r^2\sin\theta}\frac{\partial w}{\partial\theta}$ and $u_\theta = -\frac{1}{r\sin\theta}\frac{\partial w}{\partial r}$, where $r = \sqrt{x^2+y^2+z^2}$, and u_r and u_θ are the radial and angular fluid velocity components.

⊙ *References*: A. Nayfeh (1973), M. D. Van Dyke (1975).

1°. Any function $w = w(r,\theta)$ that solves the second-order linear equation $\mathrm{E}w = 0$ is also a solution of the equation in question.

Example. Solution:

$$w(r,\theta) = (C_1 r^2 + C_2 r^{-1})\sin^2\theta,$$

where C_1 and C_2 are arbitrary constants.

2°. Self-similar solution:

$$w(r,\theta) = \nu r f(\xi), \qquad \xi = \cos\theta,$$

where the function $f = f(\xi)$ is determined by the first-order ordinary differential equation

$$2(1-\xi^2)f'_\xi - f^2 + 4\xi f + C_1\xi^2 + C_2\xi + C_3 = 0, \tag{1}$$

and C_1, C_2, and C_3 are arbitrary constants.

The Riccati equation (1) is reduced, with the change of variable $f = -2(1-\xi^2)g'_\xi/g$, to the hypergeometric equation

$$(1-\xi^2)^2 g''_{\xi\xi} + (C_1\xi^2 + C_2\xi + C_3)g = 0,$$

which, in the case $C_1\xi^2 + C_2\xi + C_3 = A(1-\xi^2)$, has power-law solutions:

$$g = (1+\xi)^k, \qquad k = \tfrac{1}{2}\bigl(1 \pm \sqrt{1+A}\bigr).$$

Special case. In the Landau problem on the outflow of an axisymmetric submerged jet source, the solution of equation (1) is given by

$$f(\xi) = \frac{2(1-\xi^2)}{B-\xi} \qquad (C_1 = C_2 = C_3 = 0),$$

where the constant of integration B can be expressed via the jet momentum.

⊙ *References*: N. A. Slezkin (1934), L. D. Landau and E. M. Lifshitz (1987), L. G. Loitsyanskiy (1996).

3°. The homogeneous translational fluid flow with a velocity U_0 about a rigid spherical particle of radius a is characterized by the boundary conditions

$$w = \frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = a, \qquad w \to \tfrac{1}{2}U_0 r^2\sin^2\theta \quad \text{as} \quad r \to \infty. \tag{2}$$

The asymptotic solution of the equation in question subject to the boundary conditions (2) for low Reynolds numbers, $\mathrm{Re} = aU_0/\nu \to 0$, in the domain $r/a \le O(\mathrm{Re}^{-1})$ is given by

$$\frac{w}{U_0} = \frac{1}{4}(r-a)^2\left(2+\frac{a}{r}\right)\sin^2\theta + \frac{3}{32}\mathrm{Re}\,(r-a)^2\left[2+\frac{a}{r} - \left(2+\frac{a}{r}+\frac{a^2}{r^2}\right)\cos\theta\right]\sin^2\theta + O(\mathrm{Re}^2).$$

For the case $\mathrm{Re} = aU_0/\nu \to 0$ in the domain $r/a \ge O(\mathrm{Re}^{-1})$, Oseen asymptotic solution holds true; specifically,

$$\frac{w}{U_0} = \frac{1}{2}r^2\sin^2\theta - \frac{3}{2\,\mathrm{Re}}(1+\cos\theta)\bigl[1 - e^{-\frac{1}{2}\mathrm{Re}\,r(1-\cos\theta)}\bigr] + O(1).$$

⊙ *References*: I. Proudman and J. R. A. Pearson (1957), A. Nayfeh (1973), M. D. Van Dyke (1975).

10.3.3. Nonstationary Hydrodynamic Equations (Navier–Stokes equations)

1. $\dfrac{\partial}{\partial t}(\Delta w) + \dfrac{\partial w}{\partial y}\dfrac{\partial}{\partial x}(\Delta w) - \dfrac{\partial w}{\partial x}\dfrac{\partial}{\partial y}(\Delta w) = \nu\Delta\Delta w, \qquad \Delta w = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}.$

Preliminary remarks. The two-dimensional nonstationary equations of a viscous incompressible fluid,

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u_1,$$
$$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\Delta u_2,$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0,$$

are reduced to the equation in question through the introduction of a stream function w such that $u_1 = \frac{\partial w}{\partial y}$ and $u_2 = -\frac{\partial w}{\partial x}$ followed by the elimination of the pressure p (with cross differentiation) from the first two equations.

⊙ *Reference*: L. G. Loitsyanskiy (1996).

For stationary solutions, see equation 10.3.2.1.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$\begin{aligned}
w_1 &= -w(y,\, x,\, t),\\
w_2 &= w(C_1x + C_2,\ C_1y + C_3,\ C_1^2t + C_4) + C_5,\\
w_3 &= w(x\cos\alpha + y\sin\alpha,\ -x\sin\alpha + y\cos\alpha,\ t),\\
w_4 &= w(x\cos\beta t + y\sin\beta t,\ -x\sin\beta t + y\cos\beta t,\ t) - \tfrac{1}{2}\beta(x^2 + y^2),\\
w_5 &= w(x + \varphi(t),\ y + \psi(t),\ t) + \psi'_t(t)x - \varphi'_t(t)y + \chi(t),
\end{aligned}$$

where $C_1, \dots, C_4$, α, and β are arbitrary constants and $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are arbitrary functions, are also solutions of the equation.

⊙ *References*: V. V. Pukhnachov (1960), B. J. Cantwell (1978), S. P. Lloyd (1981), L. V. Ovsiannikov (1982).

2°. Any solution of the Poisson equation $\Delta w = C$ is also a solution of the original equation (these are "inviscid" solutions). For details about the Poisson equation, see, for example, the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

Example of an inviscid solution involving five arbitrary functions:

$$w = \varphi(t)x^2 + \psi(t)xy + [C - \varphi(t)]y^2 + a(t)x + b(t)y + c(t).$$

3°. Solution dependent on a single space variable:

$$w = W(x, t),$$

where the function W satisfies the linear nonhomogeneous heat equation

$$\frac{\partial W}{\partial t} - \nu\frac{\partial^2 W}{\partial x^2} = f_1(t)x + f_0(t),$$

and $f_1(t)$ and $f_0(t)$ are arbitrary functions. Solutions of the form $w = V(y, t)$ are determined by a similar equation.

4°. Generalized separable solution linear in x:

$$w(x, y, t) = F(y, t)x + G(y, t), \tag{1}$$

where the functions $F(y, t)$ and $G = G(y, t)$ are determined by the system of fourth-order one-dimensional equations

$$\frac{\partial^3 F}{\partial t\partial y^2} + \frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial y^2} - F\frac{\partial^3 F}{\partial y^3} = \nu\frac{\partial^4 F}{\partial y^4}, \tag{2}$$
$$\frac{\partial^3 G}{\partial t\partial y^2} + \frac{\partial G}{\partial y}\frac{\partial^2 F}{\partial y^2} - F\frac{\partial^3 G}{\partial y^3} = \nu\frac{\partial^4 G}{\partial y^4}. \tag{3}$$

Equation (2) is solved independently of (3). If $F = F(y,t)$ is a solution of equation (2), then the functions

$$F_1 = F(y+\psi(t),t) + \psi'_t(t),$$
$$F_2 = C_1 F(C_1 y + C_1 C_2 t + C_3, C_1^2 t + C_4) + C_2,$$

where $\psi(t)$ is an arbitrary function and $C_1, \ldots, C_4$ are arbitrary constants, are also solutions of the equation.

Integrating (2) and (3) with respect to y yields

$$\frac{\partial^2 F}{\partial t \partial y} + \left(\frac{\partial F}{\partial y}\right)^2 - F\frac{\partial^2 F}{\partial y^2} = \nu \frac{\partial^3 F}{\partial y^3} + f_1(t), \tag{4}$$

$$\frac{\partial^2 G}{\partial t \partial y} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} - F\frac{\partial^2 G}{\partial y^2} = \nu \frac{\partial^3 G}{\partial y^3} + f_2(t), \tag{5}$$

where $f_1(t)$ and $f_2(t)$ are arbitrary functions. Equation (5) is linear in G. The substitution

$$G = \int U\,dy - hF + h'_t y, \qquad \text{where} \quad U = U(y,t), \quad F = F(y,t), \tag{6}$$

and the function $h = h(t)$ satisfies the linear ordinary differential equation

$$h''_{tt} - f_1(t)h = f_2(t), \tag{7}$$

brings (5) to the linear homogeneous parabolic second-order equation

$$\frac{\partial U}{\partial t} = \nu\frac{\partial^2 U}{\partial y^2} + F\frac{\partial U}{\partial y} - \frac{\partial F}{\partial y}U. \tag{8}$$

Thus, whenever a particular solution of equation (2) or (4) is known, determining the function G is reduced to solving the linear equations (7)–(8) followed by computing integrals by formula (6).

Exact solutions of equation (2) are listed in Table 13 (two more complicated solutions are specified at the end of Item 4°). The ordinary differential equations in the last two rows, which determine a traveling-wave solution and a self-similar solution, are autonomous and, therefore, its order can be reduced. Note that solutions of the form (1) with $F(y,t) = Cy/t$ were treated in Pukhnachov (1960); these solutions correspond to $\varphi(t) = C/t$ in the first row.

The general solution of the linear nonhomogeneous equation (7) is expressed as

$$h(t) = C_1 h_1(t) + C_2 h_2(t) + \frac{1}{W_0}\left[h_2(t)\int h_1(t) f_2(t)\,dt - h_1(t)\int h_2(t) f_2(t)\,dt\right], \tag{9}$$

where $h_1 = h_1(t)$ and $h_2 = h_2(t)$ are fundamental solutions of the corresponding homogeneous equation (with $f_2 \equiv 0$) and $W_0 = h_1(h_2)'_t - h_2(h_1)'_t$ is the Wronskian determinant (in this case, $W_0 = \text{const}$). Table 14 lists fundamental solutions of the homogeneous equation (7) corresponding to the exact solutions of (2) specified in Table 13.

Equation (8) with any $F = F(y,t)$ has the trivial solution. The expressions in Tables 13–14 together with formulas (6) and (9) with $U = 0$ describe some exact solutions of the form (1). Nontrivial solutions of equation (8) generate a wider class of exact solutions.

Table 15 presents transformations that simplify equation (8) for some of the solutions to (2) or (4) listed in Table 13. It is apparent that solutions to (8) are expressed via solutions to the linear constant-coefficient heat equation in the first two cases. Equation (8) admits the application of the method of separation of variables in three other cases.

The third equation in Table 15 has the following particular solutions (B_1 and B_2 are arbitrary constants):

$$Z(\eta) = B_1 + B_2\int \Phi(\eta)\,d\eta, \qquad \Phi(\eta) = \exp\left(\frac{A}{\nu\lambda}e^{\eta} - \eta\right),$$

$$Z(\eta,t) = B_1\nu\lambda^2 t + B_1\int \Phi(\eta)\left[\int\frac{d\eta}{\Phi(\eta)}\right]d\eta.$$

TABLE 13
Solutions of equations (2) and (4); $\varphi(t)$ and $\psi(t)$ are arbitrary functions, and A and λ are arbitrary constants

No.	Function $F = F(y,t)$ (or general form of solution)	Function $f_1(t)$ in equation (4)	Determining coefficients (or determining equation)
1	$F = \varphi(t)y + \psi(t)$	$f_1(t) = \varphi'_t + \varphi^2$	N/A
2	$F = \frac{6\nu}{y+\psi(t)} + \psi'_t(t)$	$f_1(t) = 0$	N/A
3	$F = A\exp\bigl[-\lambda y - \lambda\psi(t)\bigr] + \psi'_t(t) + \nu\lambda$	$f_1(t) = 0$	N/A
4	$F = Ae^{-\beta t}\sin[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$f_1(t) = Be^{-2\beta t}$	$\beta = \nu\lambda^2$, $B = A^2\lambda^2 > 0$
5	$F = Ae^{-\beta t}\cos[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$f_1(t) = Be^{-2\beta t}$	$\beta = \nu\lambda^2$, $B = A^2\lambda^2 > 0$
6	$F = Ae^{\beta t}\sinh[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$f_1(t) = Be^{2\beta t}$	$\beta = \nu\lambda^2$, $B = A^2\lambda^2 > 0$
7	$F = Ae^{\beta t}\cosh[\lambda y + \lambda\psi(t)] + \psi'_t(t)$	$f_1(t) = Be^{2\beta t}$	$\beta = \nu\lambda^2$, $B = -A^2\lambda^2 < 0$
8	$F = \psi(t)e^{\lambda y} - \frac{Ae^{\beta t - \lambda y}}{4\lambda^2\psi(t)} + \frac{\psi'_t(t)}{\lambda\psi(t)} - \nu\lambda$	$f_1(t) = Ae^{\beta t}$	$\beta = 2\nu\lambda^2$
9	$F = F(\xi),\ \xi = y + \lambda t$	$f_1(t) = A$	$-A + \lambda F''_{\xi\xi} + (F'_\xi)^2 - FF''_{\xi\xi} = \nu F'''_{\xi\xi\xi}$
10	$F = t^{-1/2}\bigl[U(\xi) - \frac{1}{2}\xi\bigr],\ \xi = yt^{-1/2}$	$f_1(t) = At^{-2}$	$\frac{3}{4} - A - 2U'_\xi + (U'_\xi)^2 - UU''_{\xi\xi} = \nu U'''_{\xi\xi\xi}$

For other exact solutions of this equation, see the book by Polyanin (2002), where a more general solution of the form $\partial_t w = f(x)\partial_{xx} w + g(x)\partial_x w$ was considered.

⊙ *References*: R. Berker (1963), A. D. Polyanin (2001 *c*, 2002), A. D. Polyanin and V. F. Zaitsev (2002).

Special case 1. Solution exponentially dependent on time:

$$w(x,y,t) = f(y)x + e^{-\lambda t}\int g(y)\,dy,$$

where the functions $f = f(y)$ and $g = g(y)$ are determined by the system of ordinary differential equations

$$(f'_y)^2 - ff''_{yy} = \nu f'''_{yyy} + C_1,$$
$$-\lambda g + gf'_y - fg'_y = \nu g''_{yy} + C_2,$$

and C_1 and C_2 are arbitrary constants.

⊙ *Reference*: N. Rott (1956).

Special case 2. Periodic solution:

$$w(x,y,t) = f(y)x + \sin(\lambda t)\int g(y)\,dy + \cos(\lambda t)\int h(y)\,dy,$$

where the functions $f = f(y)$, $g = g(y)$, and $h = h(y)$ are determined by the solution of ordinary differential equations

$$(f'_y)^2 - ff''_{yy} = \nu f'''_{yyy} + C_1,$$
$$-\lambda h + f'_y g - fg'_y = \nu g''_{yy} + C_2,$$
$$\lambda g + f'_y h - fh'_y = \nu h''_{yy} + C_3.$$

Below are another two exact solutions of equation (2):

$$F(y,t) = -\frac{\gamma'_t}{\gamma}y + \gamma^3\exp\biggl(\nu\int\frac{dt}{\gamma^2}\biggr)\biggl(A\cosh\frac{y}{\gamma} + B\sinh\frac{y}{\gamma}\biggr),$$
$$F(y,t) = -\frac{\gamma'_t}{\gamma}y + \gamma^3\exp\biggl(-\nu\int\frac{dt}{\gamma^2}\biggr)\biggl(A\cos\frac{y}{\gamma} + B\sin\frac{y}{\gamma}\biggr),$$

TABLE 14

Fundamental system of solutions determining the general solution (9) of the nonhomogeneous equation (7); the number in the first column corresponds to the respective number of an exact solution in Table 13

No.	Fundamental system of solutions	Wronskian W_0	Notation and remarks
1	$h_1=\Phi(t)$, $h_2=\Phi(t)\int\frac{dt}{\Phi^2(t)}$	$W_0=1$	$\Phi(t)=\exp\bigl[\int\varphi(t)\,dt\bigr]$
2	$h_1=1$, $h_2=t$	$W_0=1$	N/A
3	$h_1=1$, $h_2=t$	$W_0=1$	N/A
4	$h_1=I_0\bigl(\frac{A\lambda}{\beta}e^{-\beta t}\bigr)$, $h_2=K_0\bigl(\frac{A\lambda}{\beta}e^{-\beta t}\bigr)$	$W_0=\beta$	$I_0(z)$, $K_0(z)$ are modified Bessel functions; $\beta=\nu\lambda^2$
5	$h_1=I_0\bigl(\frac{A\lambda}{\beta}e^{-\beta t}\bigr)$, $h_2=K_0\bigl(\frac{A\lambda}{\beta}e^{-\beta t}\bigr)$	$W_0=\beta$	$I_0(z)$, $K_0(z)$ are modified Bessel functions; $\beta=\nu\lambda^2$
6	$h_1=I_0\bigl(\frac{A\lambda}{\beta}e^{\beta t}\bigr)$, $h_2=K_0\bigl(\frac{A\lambda}{\beta}e^{\beta t}\bigr)$	$W_0=-\beta$	$I_0(z)$, $K_0(z)$ are modified Bessel functions; $\beta=\nu\lambda^2$
7	$h_1=J_0\bigl(\frac{A\lambda}{\beta}e^{\beta t}\bigr)$, $h_2=Y_0\bigl(\frac{A\lambda}{\beta}e^{\beta t}\bigr)$	$W_0=\frac{2\beta}{\pi}$	$J_0(z)$, $Y_0(z)$ are Bessel functions; $\beta=\nu\lambda^2$
8	$h_1=I_0\bigl(\frac{2\sqrt{A}}{\beta}e^{\beta t/2}\bigr)$, $h_2=K_0\bigl(\frac{2\sqrt{A}}{\beta}e^{\beta t/2}\bigr)$	$W_0=-\frac{\beta}{2}$	$I_0(z)$, $K_0(z)$ are modified Bessel functions; $\beta=2\nu\lambda^2$
9	$h_1=\cosh(kt)$, $h_2=\sinh(kt)$ $h_1=\cos(kt)$, $h_2=\sin(kt)$	$W_0=k$ $W_0=k$	if $A=k^2>0$ if $A=-k^2<0$
10	$h_1=\lvert t\rvert^{\frac12-\mu}$, $h_2=\lvert t\rvert^{\frac12+\mu}$ $h_1=\lvert t\rvert^{\frac12}$, $h_2=\lvert t\rvert^{\frac12}\ln\lvert t\rvert$ $h_1=\lvert t\rvert^{\frac12}\cos(\mu\ln\lvert t\rvert)$, $h_2=\lvert t\rvert^{\frac12}\sin(\mu\ln\lvert t\rvert)$	$W_0=2\mu$ $W_0=1$ $W_0=\mu$	if $A>-\frac14$; $\mu=\frac12\lvert 1+4A\rvert^{\frac12}$ if $A=-\frac14$ if $A<-\frac14$; $\mu=\frac12\lvert 1+4A\rvert^{\frac12}$

where A and B are arbitrary constants, and $\gamma=\gamma(t)$ is an arbitrary function. The first formula of the two displayed after (3) allows us to generalize the above expressions to obtain solutions involving two arbitrary functions.

5°. Solution (generalizes the solution of Item 4°):

$$w(x,y,t)=F(\xi,t)x+G(\xi,t),\qquad \xi=y+kx,$$

where k is an arbitrary constant and the functions $F(\xi,t)$ and $G=G(\xi,t)$ are determined from the system of one-dimensional fourth-order equations

$$\frac{\partial^3 F}{\partial t\partial\xi^2}+\frac{\partial F}{\partial\xi}\frac{\partial^2 F}{\partial\xi^2}-F\frac{\partial^3 F}{\partial\xi^3}=\nu(k^2+1)\frac{\partial^4 F}{\partial\xi^4},\tag{10}$$

$$\frac{\partial^3 G}{\partial t\partial\xi^2}+\frac{\partial G}{\partial\xi}\frac{\partial^2 F}{\partial\xi^2}-F\frac{\partial^3 G}{\partial\xi^3}=\nu(k^2+1)\frac{\partial^4 G}{\partial\xi^4}+4\nu k\frac{\partial^3 F}{\partial\xi^3}+\frac{2k}{k^2+1}\left(F\frac{\partial^2 F}{\partial\xi^2}-\frac{\partial^2 F}{\partial t\partial\xi}\right).\tag{11}$$

Integrating (10) and (11) with respect to ξ yields

$$\frac{\partial^2 F}{\partial t\partial\xi}+\left(\frac{\partial F}{\partial\xi}\right)^2-F\frac{\partial^2 F}{\partial\xi^2}=\nu(k^2+1)\frac{\partial^3 F}{\partial\xi^3}+f_1(t),\tag{12}$$

TABLE 15

Transformations of equation (8) for the corresponding exact solutions of equation (4); the number in the first column corresponds to the respective number of an exact solution $F = F(y, t)$ in Table 13

No.	Transformations of equation (8)	Resulting equation
1	$U = \frac{1}{\Phi(t)} u(z,\tau)$, $\tau = \int \Phi^2(t)\,dt + C_1$, $z = y\Phi(t) + \int \psi(t)\Phi(t)\,dt + C_2$, $\Phi(t) = \exp\bigl[\int \varphi(t)\,dt\bigr]$	$\frac{\partial u}{\partial \tau} = \nu \frac{\partial^2 u}{\partial z^2}$
2	$U = \zeta^{-3} u(\zeta, t)$, $\zeta = y + \psi(t)$	$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial \zeta^2}$
3	$U = e^{\eta} Z(\eta, t)$, $\eta = -\lambda y - \lambda\psi(t)$	$\frac{\partial Z}{\partial t} = \nu\lambda^2 \frac{\partial^2 Z}{\partial \eta^2} + (\nu\lambda^2 - A\lambda e^{\eta}) \frac{\partial Z}{\partial \eta}$
9	$U = u(\xi, t)$, $\xi = y + \lambda t$	$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial \xi^2} + \bigl[F(\xi) - \lambda\bigr] \frac{\partial u}{\partial \xi} - F'_{\xi}(\xi) u$
10	$U = t^{-1/2} u(\xi, \tau)$, $\xi = y t^{-1/2}$, $\tau = \ln t$	$\frac{\partial u}{\partial \tau} = \nu \frac{\partial^2 u}{\partial \xi^2} + H(\xi) \frac{\partial u}{\partial \xi} + \bigl[1 - H'_{\xi}(\xi)\bigr] u$

$$\frac{\partial^2 G}{\partial t \partial \xi} + \frac{\partial F}{\partial \xi}\frac{\partial G}{\partial \xi} - F\frac{\partial^2 G}{\partial \xi^2} = \nu(k^2+1)\frac{\partial^3 G}{\partial \xi^3} + Q(\xi, t), \tag{13}$$

where $f_1(t)$ is an arbitrary function, and the function $Q(\xi, t)$ is given by

$$Q(\xi, t) = 4\nu k \frac{\partial^2 F}{\partial \xi^2} - \frac{2k}{k^2+1}\frac{\partial F}{\partial t} + \frac{2k}{k^2+1}\int F \frac{\partial^2 F}{\partial \xi^2}\,d\xi + f_2(t), \qquad f_2(t) \text{ is any.}$$

Equation (13) is linear in G. The substitution $U = \frac{\partial G}{\partial \xi}$ brings (13) to the second-order linear equation

$$\frac{\partial U}{\partial t} = \nu(k^2+1)\frac{\partial^2 U}{\partial \xi^2} + F\frac{\partial U}{\partial \xi} - \frac{\partial F}{\partial \xi}U + Q(\xi, t). \tag{14}$$

Thus, whenever a particular solution of equation (10) or (12) is known, determining the function G is reduced to solving the second-order linear equation (14). Equation (10) is reduced, by scaling the independent variables so that $\xi = (k^2+1)\zeta$ and $t = (k^2+1)\tau$, to equation (2) in which y and t should be replaced by ζ and τ; exact solutions of equation (2) are listed in Table 13.

⊙ *Reference*: A. D. Polyanin (2001 *c*).

6°. Solutions:

$$w(x, y, t) = Az^3 + Bz^2 + Cz + \psi'_t(t)x, \qquad z = y + kx + \psi(t);$$
$$w(x, y, t) = Ae^{-\lambda z} + Bz^2 + Cz + \nu\lambda(k^2+1)x + \psi'_t(t)x,$$

where A, B, C, k, and λ are arbitrary constants and $\psi(t)$ is an arbitrary function.

7°. Generalized separable solution [special case of a solution of the form (1)]:

$$w(x, y, t) = e^{-\lambda y}\bigl[f(t)x + g(t)\bigr] + \varphi(t)x + \psi(t)y,$$
$$f(t) = C_1 E(t), \qquad E(t) = \exp\left[\nu\lambda^2 t - \lambda\int \varphi(t)\,dt\right],$$
$$g(t) = C_2 E(t) - C_1 E(t)\int \psi(t)\,dt,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions and C_1, C_2, and λ are arbitrary constants.

8°. Generalized separable solution:

$$w(x,y,t) = e^{-\lambda y}\bigl[A(t)e^{\beta x} + B(t)e^{-\beta x}\bigr] + \varphi(t)x + \psi(t)y,$$
$$A(t) = C_1 \exp\left[\nu(\lambda^2+\beta^2)t - \beta\int\psi(t)\,dt - \lambda\int\varphi(t)\,dt\right],$$
$$B(t) = C_2 \exp\left[\nu(\lambda^2+\beta^2)t + \beta\int\psi(t)\,dt - \lambda\int\varphi(t)\,dt\right],$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions and C_1, C_2, λ, and β are arbitrary constants.

9°. Generalized separable solution:

$$w(x,y,t) = e^{-\lambda y}\bigl[A(t)\sin(\beta x) + B(t)\cos(\beta x)\bigr] + \varphi(t)x + \psi(t)y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, λ and β are arbitrary constants, and the functions $A(t)$ and $B(t)$ satisfy the linear nonautonomous system of ordinary differential equations

$$\begin{aligned} A'_t &= \bigl[\nu(\lambda^2-\beta^2) - \lambda\varphi(t)\bigr]A + \beta\psi(t)B,\\ B'_t &= \bigl[\nu(\lambda^2-\beta^2) - \lambda\varphi(t)\bigr]B - \beta\psi(t)A. \end{aligned} \tag{15}$$

The general solution of system (15) is expressed as

$$A(t) = \exp\left[\nu(\lambda^2-\beta^2)t - \lambda\int\varphi\,dt\right]\left[C_1\sin\left(\beta\int\psi\,dt\right) + C_2\cos\left(\beta\int\psi\,dt\right)\right],$$
$$B(t) = \exp\left[\nu(\lambda^2-\beta^2)t - \lambda\int\varphi\,dt\right]\left[C_1\cos\left(\beta\int\psi\,dt\right) - C_2\sin\left(\beta\int\psi\,dt\right)\right],$$

where $\varphi = \varphi(t)$ and $\psi = \psi(t)$; C_1 and C_2 are arbitrary constants. In particular, for $\varphi = \dfrac{\nu}{\lambda}(\lambda^2-\beta^2)$ and $\psi = a$, we obtain the periodic solution

$$A(t) = C_1\sin(a\beta t) + C_2\cos(a\beta t),$$
$$B(t) = C_1\cos(a\beta t) - C_2\sin(a\beta t).$$

⦿ *Reference*: A. D. Polyanin (2001 *c*).

10°. Generalized separable solution:

$$w(x,y,t) = A(t)\exp(k_1x + \lambda_1y) + B(t)\exp(k_2x + \lambda_2y) + \varphi(t)x + \psi(t)y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k_1, λ_1, k_2, and λ_2 are arbitrary constants, constrained by one of the two relations

$$k_1^2 + \lambda_1^2 = k_2^2 + \lambda_2^2 \quad \text{(first family of solutions)},$$
$$k_1\lambda_2 = k_2\lambda_1 \quad \text{(second family of solutions)},$$

and the functions $A(t)$ and $B(t)$ satisfy the linear ordinary differential equations

$$A'_t = \bigl[\nu(k_1^2+\lambda_1^2) + \lambda_1\varphi(t) - k_1\psi(t)\bigr]A,$$
$$B'_t = \bigl[\nu(k_2^2+\lambda_2^2) + \lambda_2\varphi(t) - k_2\psi(t)\bigr]B.$$

These equations can be readily integrated to obtain

$$A(t) = C_1\exp\left[\nu(k_1^2+\lambda_1^2)t + \lambda_1\int\varphi(t)\,dt - k_1\int\psi(t)\,dt\right],$$
$$B(t) = C_2\exp\left[\nu(k_2^2+\lambda_2^2)t + \lambda_2\int\varphi(t)\,dt - k_2\int\psi(t)\,dt\right].$$

⦿ *Reference*: A. D. Polyanin (2001 *c*).

11°. Generalized separable solution:

$$w(x,y,t) = \big[C_1 \sin(\lambda x) + C_2 \cos(\lambda x)\big]\big[A(t)\sin(\beta y) + B(t)\cos(\beta y)\big] + \varphi(t)x,$$

where $\varphi(t)$ is an arbitrary function, C_1, C_2, λ, and β are arbitrary constants, and the functions $A(t)$ and $B(t)$ satisfy the linear nonautonomous system of ordinary differential equations

$$\begin{aligned} A'_t &= -\nu(\lambda^2+\beta^2)A - \beta\varphi(t)B, \\ B'_t &= -\nu(\lambda^2+\beta^2)B + \beta\varphi(t)A. \end{aligned} \tag{16}$$

The general solution of system (16) is expressed as

$$\begin{aligned} A(t) &= \exp\big[-\nu(\lambda^2+\beta^2)t\big]\left[C_3\sin\left(\beta\int\varphi\,dt\right) + C_4\cos\left(\beta\int\varphi\,dt\right)\right], \qquad \varphi=\varphi(t), \\ B(t) &= \exp\big[-\nu(\lambda^2+\beta^2)t\big]\left[-C_3\cos\left(\beta\int\varphi\,dt\right) + C_4\sin\left(\beta\int\varphi\,dt\right)\right], \end{aligned}$$

where C_3 and C_4 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin (2001 *c*).

12°. Generalized separable solution:

$$w(x,y,t) = \big[C_1 \sinh(\lambda x) + C_2 \cosh(\lambda x)\big]\big[A(t)\sin(\beta y) + B(t)\cos(\beta y)\big] + \varphi(t)x,$$

where $\varphi(t)$ is an arbitrary function, C_1, C_2, λ, and β are arbitrary constants, and the functions $A(t)$ and $B(t)$ satisfy the linear nonautonomous system of ordinary differential equations

$$\begin{aligned} A'_t &= \nu(\lambda^2-\beta^2)A - \beta\varphi(t)B, \\ B'_t &= \nu(\lambda^2-\beta^2)B + \beta\varphi(t)A. \end{aligned} \tag{17}$$

The general solution of system (17) is expressed as

$$\begin{aligned} A(t) &= \exp\big[\nu(\lambda^2-\beta^2)t\big]\left[C_3\sin\left(\beta\int\varphi\,dt\right) + C_4\cos\left(\beta\int\varphi\,dt\right)\right], \qquad \varphi=\varphi(t), \\ B(t) &= \exp\big[\nu(\lambda^2-\beta^2)t\big]\left[-C_3\cos\left(\beta\int\varphi\,dt\right) + C_4\sin\left(\beta\int\varphi\,dt\right)\right], \end{aligned}$$

where C_3 and C_4 are arbitrary constants.

⊙ *Reference*: A. D. Polyanin (2001 *c*).

13°. "Two-dimensional" solution:

$$w(x,y,t) = u(z,t) + \varphi(t)x + \psi(t)y, \qquad z = kx + \lambda y,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, k and λ are arbitrary constants, and the function $u(z,t)$ is determined by the fourth-order linear equation

$$\frac{\partial^3 u}{\partial t\partial z^2} + \big[k\psi(t) - \lambda\varphi(t)\big]\frac{\partial^3 u}{\partial z^3} = \nu(k^2+\lambda^2)\frac{\partial^4 u}{\partial z^4}.$$

The transformation

$$U(\xi,t) = \frac{\partial^2 u}{\partial z^2}, \qquad \xi = z - \int\big[k\psi(t) - \lambda\varphi(t)\big]\,dt$$

brings it to the linear heat equation

$$\frac{\partial U}{\partial t} = \nu(k^2+\lambda^2)\frac{\partial^2 U}{\partial \xi^2}.$$

⊙ *Reference*: A. D. Polyanin (2001 *c*).

14°. There are "two-dimensional" solutions of the form

$$w(x,y,t)=W(\rho_1,\rho_2)+c_1x+c_2y, \qquad \rho_1=a_1x+a_2y+a_3t, \qquad \rho_2=b_1x+b_2y+b_3t.$$

15°+. "Two-dimensional" solution (a, b, and c are arbitrary constants):

$$w(x,y,t)=Z(X,Y), \qquad X=\frac{x+a}{\sqrt{t+c}}, \qquad Y=\frac{y+b}{\sqrt{t+c}},$$

where the function $Z=Z(X,Y)$ is determined by the differential equation

$$-\bar{\Delta}Z+\left(\frac{\partial Z}{\partial Y}-\frac{1}{2}X\right)\frac{\partial}{\partial X}(\bar{\Delta}Z)-\left(\frac{\partial Z}{\partial X}+\frac{1}{2}Y\right)\frac{\partial}{\partial Y}(\bar{\Delta}Z)=\nu\bar{\Delta}\bar{\Delta}Z, \qquad \bar{\Delta}=\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}.$$

⊙ *Reference*: V. V. Pukhnachov (1960).

16°+. "Two-dimensional" solution:

$$w(x,y,t)=\Psi(\xi,\eta), \quad \xi=t^{-1/2}\big[x\cos(k\ln t)-y\sin(k\ln t)\big], \quad \eta=t^{-1/2}\big[x\sin(k\ln t)+y\cos(k\ln t)\big],$$

where k is an arbitrary constant and the function $\Psi(\xi,\eta)$ is determined by the differential equation

$$-\widetilde{\Delta}\Psi+\left(\frac{\partial\Psi}{\partial\eta}-\frac{1}{2}\xi-k\eta\right)\frac{\partial}{\partial\xi}\widetilde{\Delta}\Psi-\left(\frac{\partial\Psi}{\partial\xi}+\frac{1}{2}\eta-k\xi\right)\frac{\partial}{\partial\eta}\widetilde{\Delta}\Psi=\nu\widetilde{\Delta}\widetilde{\Delta}\Psi, \qquad \widetilde{\Delta}=\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}.$$

⊙ *Reference*: B. J. Cantwell (1978).

17°+. "Two-dimensional" solution:

$$w(x,y,t)=\frac{\varphi'_t(x^2-y^2+2\varphi xy)}{2(1+\varphi^2)}+\frac{y-\varphi x}{1+\varphi^2}F(\zeta,t)-2G(\zeta,t), \qquad \zeta=x+\varphi y,$$

where $\varphi=\varphi(t)$ is an arbitrary function and the functions $F=F(\zeta,t)$ and $G=G(\zeta,t)$ are determined by the differential equations

$$\nu(1+\varphi^2)\frac{\partial^4 F}{\partial\zeta^4}-F\frac{\partial^3 F}{\partial\zeta^3}+\frac{\partial F}{\partial\zeta}\frac{\partial^2 F}{\partial\zeta^2}-\frac{2\varphi\varphi'_t}{1+\varphi^2}\frac{\partial^2 F}{\partial\zeta^2}-\frac{\partial^3 F}{\partial\zeta^2\partial t}=0, \tag{18}$$

$$\nu(1+\varphi^2)\frac{\partial^4 G}{\partial\zeta^4}-F\frac{\partial^3 G}{\partial\zeta^3}+\frac{\partial^2 F}{\partial\zeta^2}\frac{\partial G}{\partial\zeta}-\frac{2\varphi\varphi'_t}{1+\varphi^2}\frac{\partial^2 G}{\partial\zeta^2}-\frac{\partial^3 G}{\partial\zeta^2\partial t}=\frac{\varphi'_t}{(1+\varphi^2)^2}\zeta\frac{\partial^2 F}{\partial\zeta^2}. \tag{19}$$

⊙ *Reference*: D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (1999).

Equation (18) is solved independently of equation (19). If $F=F(\zeta,t)$ is a solution to (18), the function

$$F_1=F(y+\sigma(t),t)-\sigma'_t(t),$$

where $\sigma(t)$ is an arbitrary function, is also a solution of the equation.

Integrating (18) and (19) with respect to ζ yields

$$\nu(1+\varphi^2)\frac{\partial^3 F}{\partial\zeta^3}-F\frac{\partial^2 F}{\partial\zeta^2}+\left(\frac{\partial F}{\partial\zeta}\right)^2-\frac{2\varphi\varphi'_t}{1+\varphi^2}\frac{\partial F}{\partial\zeta}-\frac{\partial^2 F}{\partial\zeta\partial t}=\psi_1(t),$$

$$\nu(1+\varphi^2)\frac{\partial^3 G}{\partial\zeta^3}-F\frac{\partial^2 G}{\partial\zeta^2}+\frac{\partial F}{\partial\zeta}\frac{\partial G}{\partial\zeta}-\frac{2\varphi\varphi'_t}{1+\varphi^2}\frac{\partial G}{\partial\zeta}-\frac{\partial^2 G}{\partial\zeta\partial t}=\frac{\varphi'_t}{(1+\varphi^2)^2}\left(\zeta\frac{\partial F}{\partial\zeta}-F\right)+\psi_2(t),$$

where $\psi_1(t)$ and $\psi_2(t)$ are arbitrary functions. The change of variable $u=\frac{\partial G}{\partial\zeta}$ brings the last equation to a second-order parabolic linear equation (with known F).

Note that equation (18) admits particular solutions of the forms

$$F(\zeta,t)=a(t)\zeta+b(t),$$

$$F(\zeta,t)=a(t)e^{-\lambda\zeta}+\frac{a'_t(t)}{\lambda a(t)}+\frac{2\varphi\varphi'_t}{\lambda(1+\varphi^2)}-\nu\lambda(1+\varphi^2),$$

where $a(t)$ and $b(t)$ are arbitrary functions and λ is an arbitrary constant.

► *For other exact solutions, see equation 10.3.3.3.*

2. $\dfrac{\partial}{\partial t}(\Delta w)+\left(\dfrac{\partial w}{\partial y}+ax\right)\dfrac{\partial}{\partial x}(\Delta w)-\left(\dfrac{\partial w}{\partial x}-ay\right)\dfrac{\partial}{\partial y}(\Delta w)+2a\Delta w=\nu\Delta\Delta w.$

Preliminary remarks. The system

$$\begin{aligned}\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}&=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\Delta u_1,\\ \frac{\partial u_2}{\partial t}+u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}&=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\Delta u_2,\\ \frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}&=2a,\end{aligned}$$

describing the motion of a viscous incompressible fluid induced by two parallel disks moving towards each other is reduced to the given equation. Here, a is the relative velocity of the disks, u_1 and u_2 are the horizontal velocity components, and $u_3=-2az$ is the vertical velocity component. The introduction of a stream function w such that $u_1=ax+\frac{\partial w}{\partial y}$ and $u_2=ay-\frac{\partial w}{\partial x}$ followed by the elimination of the pressure p (with the help of cross differentiation) leads to the equation in question. For $a=0$, see equation 10.3.3.1.

⦿ *Reference*: A. Craik (1989).

For stationary solutions, see equation 10.3.2.3.

1°. Suppose $w(x,y,t)$ is a solution of the equation in question. Then the functions

$$\begin{aligned}w_1&=-w(y,\,x,\,t),\\ w_2&=w(x\cos\beta+y\sin\beta,\,-x\sin\beta+y\cos\beta,\,t),\\ w_3&=w\bigl(x+\varphi(t),\,y+\psi(t),\,t+C\bigr)+\bigl[\psi'_t(t)-a\psi(t)\bigr]x+\bigl[a\varphi(t)-\varphi'_t(t)\bigr]y+\chi(t),\end{aligned}$$

where $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are arbitrary functions and C and β are arbitrary constants, are also solutions of the equation.

2°. Any solution of the Poisson equation $\Delta w=C$ is also a solution of the original equation (these are "inviscid" solutions). For details about the Poisson equation, see, for example, the books by Tikhonov and Samarskii (1990) and Polyanin (2002).

3°. Solution dependent on a single coordinate x:

$$w(x,t)=\int_0^x(x-\xi)U(\xi,t)\,d\xi+f_1(t)x+f_0(t),$$

where $f_1(t)$ and $f_0(t)$ are arbitrary functions and the function $U(x,t)$ satisfies the linear nonhomogeneous parabolic equation

$$\frac{\partial U}{\partial t}+ax\frac{\partial U}{\partial x}+2aU=\nu\frac{\partial^2 U}{\partial x^2},$$

which can be reduced to a linear constant-coefficient heat equation; see Polyanin (2002, page 93).

Solutions of the form $w=w(y,t)$ can be obtained likewise.

4°. Generalized separable solution linear in x:

$$w(x,y,t)=F(y,t)x+G(y,t),\tag{1}$$

where the functions $F(y,t)$ and $G=G(y,t)$ are determined by the system of one-dimensional fourth-order equations

$$\frac{\partial^3 F}{\partial t\partial y^2}+\frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial y^2}-F\frac{\partial^3 F}{\partial y^3}+a\left(3\frac{\partial^2 F}{\partial y^2}+y\frac{\partial^3 F}{\partial y^3}\right)=\nu\frac{\partial^4 F}{\partial y^4},\tag{2}$$

$$\frac{\partial^3 G}{\partial t\partial y^2}+\frac{\partial G}{\partial y}\frac{\partial^2 F}{\partial y^2}-F\frac{\partial^3 G}{\partial y^3}+a\left(2\frac{\partial^2 G}{\partial y^2}+y\frac{\partial^3 G}{\partial y^3}\right)=\nu\frac{\partial^4 G}{\partial y^4}.\tag{3}$$

Equation (2) is solved independently of equation (3). If $F=F(y,t)$ is a solution to (2), then the function

$$F_1=F\bigl(y+\psi(t),t\bigr)+\psi'_t(t)-a\psi(t),$$

where $\psi(t)$ is an arbitrary function, is also a solution of the equation.

Integrating (2) and (3) with respect to y yields

$$\frac{\partial^2 F}{\partial t \partial y} + \left(\frac{\partial F}{\partial y}\right)^2 - F\frac{\partial^2 F}{\partial y^2} + a\left(2\frac{\partial F}{\partial y} + y\frac{\partial^2 F}{\partial y^2}\right) = \nu\frac{\partial^3 F}{\partial y^3} + f_1(t), \tag{4}$$

$$\frac{\partial^2 G}{\partial t \partial y} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} - F\frac{\partial^2 G}{\partial y^2} + a\left(\frac{\partial G}{\partial y} + y\frac{\partial^2 G}{\partial y^2}\right) = \nu\frac{\partial^3 G}{\partial y^3} + f_2(t), \tag{5}$$

where $f_1(t)$ and $f_2(t)$ are arbitrary functions.

Equation (2) has a particular solution

$$F(y,t) = f_1(t)y + f_0(t), \tag{6}$$

where $f_1 = f_1(t)$ and $f_0 = f_0(t)$ are arbitrary functions. On substituting (6) into (5), we arrive at a linear equation whose order can be reduced by two:

$$\frac{\partial Q}{\partial t} - \big[(f_1 - a)y + f_0\big]\frac{\partial Q}{\partial y} + 2aQ = \nu\frac{\partial^2 Q}{\partial y^2}, \qquad Q = \frac{\partial^2 G}{\partial y^2}.$$

The equation for Q can be reduced to a linear constant-coefficient heat equation; see Polyanin (2002, page 135).

Note that equation (2) has the following particular solutions:

$$\begin{aligned} F(y,t) &= ay + \big[C_1\exp(-\lambda y) + C_2\exp(\lambda y)\big]\exp\big[(\nu\lambda^2 - 4a)t\big],\\ F(y,t) &= ay + \big[C_1\cos(\lambda y) + C_2\sin(\lambda y)\big]\exp\big[-(\nu\lambda^2 + 4a)t\big], \end{aligned} \tag{7}$$

where C_1, C_2, and λ are arbitrary constants.

Solutions of the form $w(x,y,t) = f(x,t)y + g(x,t)$ can be obtained likewise.

Remark. The results of Items 1°–4° exclusive of formula (7) remain true if $a = a(t)$ is an arbitrary function in the original equation (in this case, one should set $C = 0$ in Item 1°).

► *For other exact solutions, see equation 10.3.3.4.*

3. $$\frac{\partial Q}{\partial t} + \frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial Q}{\partial r} - \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial Q}{\partial \theta} = \nu\Delta Q, \quad Q = \Delta w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}.$$

Preliminary remarks. Equation 10.3.3.1 is reduced to the given equation by passing to polar coordinates with origin at a point (x_0, y_0), where x_0 and y_0 are any numbers, according to

$$\begin{aligned} &x = r\cos\theta + x_0, && y = r\sin\theta + y_0 && \text{(direct transformation)},\\ &r = \sqrt{(x-x_0)^2 + (y-y_0)^2}, && \tan\theta = \frac{y-y_0}{x-x_0} && \text{(inverse transformation)}. \end{aligned}$$

The radial and angular fluid velocity components are expressed in terms of the stream function w as follows: $u_r = \frac{1}{r}\frac{\partial w}{\partial \theta}$ and $u_\theta = -\frac{\partial w}{\partial r}$.

1°. Solutions with axial symmetry

$$w = W(r,t)$$

are described by the linear nonhomogeneous heat equation

$$\frac{\partial W}{\partial t} - \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right) = \varphi(t)\ln r + \psi(t), \tag{1}$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions. For particular solutions of equation (1) that occur in fluid dynamics, see Pukhnachov (1960) and Loitsyanskiy (1996).

2°. Generalized separable solution linear in θ:

$$w(r,\theta,t) = f(r,t)\theta + g(r,t), \tag{2}$$

where the functions $f = f(r,t)$ and $g = g(r,t)$ are determined by the differential equations

$$\mathbf{L}(f_t) - r^{-1} f_r \mathbf{L}(f) + r^{-1} f[\mathbf{L}(f)]_r = \nu \mathbf{L}^2(f), \tag{3}$$

$$\mathbf{L}(g_t) - r^{-1} g_r \mathbf{L}(f) + r^{-1} f[\mathbf{L}(g)]_r = \nu \mathbf{L}^2(g). \tag{4}$$

Here, the subscripts r and t denote partial derivatives with respect to r and t, $\mathbf{L}(f) = r^{-1}(r f_r)_r$, and $\mathbf{L}^2(f) = \mathbf{L}\mathbf{L}(f)$.

Equation (3) has a particular solution of the form

$$f = \varphi(t) \ln r + \psi(t),$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions. In this case, equation (4) is reduced by the change of variable $U = \mathbf{L}(g)$ to a second-order linear equation.

Remark. Equation (3) has also a particular solution $f = -\dfrac{r^2}{2(t+C)}$.

⊙ *References*: R. Berker (1963), D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (1999).

3°. Let us consider the case $f = \psi(t)$ in Item 2° in more detail. This case corresponds to $w = \psi(t)\theta + g(r,t)$; the existence of such an exact solution was established by Pukhnachov (1960). For g, we have the equation

$$\frac{\partial U}{\partial t} + \frac{\psi(t)}{r}\frac{\partial U}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right), \qquad \text{where} \qquad U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right). \tag{5}$$

Below are some exact solutions of equation (5):

$$U = \frac{a}{t}\exp\left[-\frac{r^2}{4\nu t} + \frac{1}{2\nu}\int \frac{\psi(t)}{t}\,dt\right] + b,$$

$$U = r^2 + 4\nu t - 2\int \psi(t)\,dt + a,$$

$$U = r^4 + p(t)r^2 + q(t), \quad p(t) = 16\nu t - 4\int \psi(t)\,dt + a, \quad q(t) = 2\int \bigl[2\nu - \psi(t)\bigr]p(t)\,dt + b,$$

where a and b are arbitrary constants. The second and the third solutions are special cases of solutions having the form

$$U = r^{2n} + A_{2n-2}(t)r^{2n-2} + \cdots + A_2(t)r^2 + A_0(t)$$

with n arbitrary constants.

The function $g(r,t)$ can be expressed in terms of $U(r,t)$ by

$$g(r,t) = C_1(t)\ln r + C_2(t) + \int \Phi(r,t)\,dr, \qquad \Phi(r,t) = \frac{1}{r}\int rU(r,t)\,dr,$$

where $C_1(t)$ and $C_2(t)$ are arbitrary functions.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

4°. "Two-dimensional" solution:

$$w(r,\theta,t) = Ar^2 t + \nu H(\xi,\eta), \qquad \xi = r\cos(\theta + At^2), \qquad \eta = r\sin(\theta + At^2),$$

where A is an arbitrary constant and the function $H(\xi,\eta)$ is determined by the differential equation

$$\widetilde{\Delta}\widetilde{\Delta}H - \frac{\partial H}{\partial \eta}\frac{\partial}{\partial \xi}\widetilde{\Delta}H + \frac{\partial H}{\partial \xi}\frac{\partial}{\partial \eta}\widetilde{\Delta}H - \frac{4A}{\nu^2} = 0, \qquad \widetilde{\Delta} = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}.$$

⊙ *Reference*: D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (1999).

4. $\displaystyle \frac{\partial Q}{\partial t}+ar\frac{\partial Q}{\partial r}+2aQ+\frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial Q}{\partial r}-\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial Q}{\partial \theta}=\nu\Delta Q,$

where $\displaystyle Q=\Delta w=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}.$

Equation 10.3.3.2 is reduced to the given equation by passing to polar coordinates r,θ: $x=r\cos\theta$, $y=r\sin\theta$.

1°. Solutions with axial symmetry,

$$w=W(r,t),$$

are described by the linear parabolic equation

$$\frac{\partial Q}{\partial t}+ar\frac{\partial Q}{\partial r}+2aQ=\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial Q}{\partial r}\right),\qquad Q=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right).$$

2°. Generalized separable solution linear in θ:

$$w(r,\theta,t)=f(r,t)\theta+g(r,t), \tag{1}$$

where the functions $f=f(r,t)$ and $g=g(r,t)$ are determined by the differential equations

$$\mathbf{L}(f_t)+ar[\mathbf{L}(f)]_r+2a\mathbf{L}(f)-r^{-1}f_r\mathbf{L}(f)+r^{-1}f[\mathbf{L}(f)]_r=\nu\mathbf{L}^2(f), \tag{2}$$
$$\mathbf{L}(g_t)+ar[\mathbf{L}(g)]_r+2a\mathbf{L}(g)-r^{-1}g_r\mathbf{L}(f)+r^{-1}f[\mathbf{L}(g)]_r=\nu\mathbf{L}^2(g). \tag{3}$$

Here, the subscripts r and t denote partial derivatives with respect to r and t, $\mathbf{L}(f)=r^{-1}(rf_r)_r$, and $\mathbf{L}^2(f)=\mathbf{LL}(f)$.

Equation (2) has particular solutions of the form

$$f=\varphi(t)\ln r+\psi(t),$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions. In this case, equation (3) is reduced by the change of variable $U=\mathbf{L}(g)$ to a second-order linear equation.

5. $\displaystyle \frac{\partial \mathrm{E}w}{\partial t}+\frac{1}{r}\left(\frac{\partial w}{\partial z}\frac{\partial \mathrm{E}w}{\partial r}-\frac{\partial w}{\partial r}\frac{\partial \mathrm{E}w}{\partial z}\right)-\frac{2}{r^2}\frac{\partial w}{\partial z}\mathrm{E}w=\nu\mathrm{E}^2w,$

where $\displaystyle \mathrm{E}w=r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)+\frac{\partial^2 w}{\partial z^2},\quad \mathrm{E}^2w=\mathrm{E}(\mathrm{E}w).$

Preliminary remarks. The nonstationary Navier–Stokes equations written in cylindrical coordinates for the axisymmetric case are reduced to the equation in question by the introduction of a stream function w such that $u_r=\frac{1}{r}\frac{\partial w}{\partial z}$ and $u_z=-\frac{1}{r}\frac{\partial w}{\partial r}$, where $r=\sqrt{x^2+y^2}$, and u_r and u_z are the radial and axial fluid velocity components.

⊙ *Reference*: J. Happel and H. Brenner (1965).

1°. Any function $w=w(r,z,t)$ that solves the second-order linear stationary equation $\mathrm{E}w=0$ is also a solution of the original equation.

2°. Solution with axial symmetry:

$$w=U(r,t)+\varphi(t)r^2+\psi(t),$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions and the function $U=U(r,t)$ is determined by the linear parabolic equation

$$\frac{\partial U}{\partial t}-\nu r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial U}{\partial r}\right)=0.$$

3°. Generalized separable solution linear in z:

$$w(r,z,t) = f(r,t)z + g(r,t).$$

Here, $f = f(r,t)$ and $g = g(r,t)$ satisfy the system

$$\mathbf{L}(f_t) + r^{-1} f[\mathbf{L}(f)]_r - r^{-1} f_r \mathbf{L}(f) - 2r^{-2} f\,\mathbf{L}(f) = \nu \mathbf{L}^2(f), \tag{1}$$
$$\mathbf{L}(g_t) + r^{-1} f[\mathbf{L}(g)]_r - r^{-1} g_r \mathbf{L}(f) - 2r^{-2} f\,\mathbf{L}(g) = \nu \mathbf{L}^2(g), \tag{2}$$

where $\mathbf{L}(f) = f_{rr} - r^{-1} f_r$; the subscripts denote the corresponding partial derivatives.

Particular solution of equation (1):

$$f(r,t) = C_1(t) r^2 + C_2(t),$$

where $C_1(t)$ and $C_2(t)$ are arbitrary functions. In this case, the change of variable $U = \mathbf{L}(g)$ brings (2) to a second-order linear equation.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

10.3.4. Other Equations

1. $\displaystyle \frac{\partial^3 w}{\partial t \partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} - w\frac{\partial^3 w}{\partial x^3} = f(t)\frac{\partial^4 w}{\partial x^4}.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w\big(x + \varphi(t), t\big) + \varphi'_t(t),$$

where $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Multiplicative separable solutions:

$$w = (Ae^{\lambda x} + Be^{-\lambda x}) \exp\left[\lambda^2 \int f(t)\,dt\right],$$
$$w = A\sin(\lambda x + B) \exp\left[-\lambda^2 \int f(t)\,dt\right],$$

where A, B, and C are arbitrary constants.

3°. On integrating once with respect to x, we obtain the third-order equation

$$\frac{\partial^2 w}{\partial t \partial x} + \left(\frac{\partial w}{\partial x}\right)^2 - w\frac{\partial^2 w}{\partial x^2} = f(t)\frac{\partial^3 w}{\partial x^3} + \varphi(t),$$

where $\varphi(t)$ is an arbitrary function.

2. $\displaystyle \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = f(x)\frac{\partial^4 w}{\partial y^4}.$

This is a special case of equation 11.4.1.2 with $n = 4$.

3. $\displaystyle \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = f(x)\frac{\partial^4 w}{\partial y^4} + g(x).$

This is a special case of equation 11.4.1.3 with $n = 2$.

4. $\dfrac{\partial^2 w}{\partial x \partial t} + f(t)\dfrac{\partial^4 w}{\partial x^4} + g(t)\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + h(t)\dfrac{\partial^2 w}{\partial y^2} = 0.$

Generalized Kadomtsev–Petviashvili equation. This is a special case of equation 11.4.1.9.

1°. Suppose $w(x, y, t)$ is a solution of the equation in question. Then the functions

$$w_1 = w\bigl(x + \varphi(t), \pm y + C, t\bigr) - \frac{\varphi_t'(t)}{g(t)},$$

where C is an arbitrary constant and $\varphi(t)$ is an arbitrary function, are also solutions of the equation.

2°. "Two-dimensional" solution:

$$w(x, y, t) = u(z, t), \qquad z = x + C_1 y - C_1^2 \int h(t)\,dt + C_2,$$

where C_1 and C_2 are arbitrary constants and the function $u(z, t)$ is determined by the third-order differential equation

$$\frac{\partial u}{\partial t} + f(t)\frac{\partial^3 u}{\partial z^3} + g(t)u\frac{\partial u}{\partial z} = \varphi(t),$$

with $\varphi(t)$ being an arbitrary function.

3°. "Two-dimensional" solution:

$$w(x, y, t) = U(\xi, t), \qquad \xi = x + \theta(t)(y + C_1)^2, \qquad \theta(t) = \left[4\int h(t)\,dt + C_2\right]^{-1},$$

where the function $U(\xi, t)$ is determined by the third-order differential equation

$$\frac{\partial U}{\partial t} + f(t)\frac{\partial^3 U}{\partial \xi^3} + g(t)U\frac{\partial U}{\partial \xi} + 2h(t)\theta(t)U = \psi(t),$$

with $\psi(t)$ being an arbitrary function.

Chapter 11

Equations of Higher Orders

11.1. Equations Involving the First Derivative in t and Linear in the Highest Derivative

11.1.1. Fifth-Order Equations

1. $\dfrac{\partial w}{\partial t} + w\dfrac{\partial w}{\partial x} = a\dfrac{\partial^5 w}{\partial x^5}.$

This is a special case of equation 11.1.3.1 with $n = 5$ and $b = -1$.

2. $\dfrac{\partial w}{\partial t} - bw^k\dfrac{\partial w}{\partial x} = a\dfrac{\partial^5 w}{\partial x^5}.$

This is a special case of equation 11.1.3.2 with $n = 5$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^5 w}{\partial x^5} + be^{\lambda w}\dfrac{\partial w}{\partial x}.$

This is a special case of equation 11.1.3.3 with $n = 5$.

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^5 w}{\partial x^5} + (b\ln w + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 11.1.3.4 with $n = 5$.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^5 w}{\partial x^5} + (b\,\mathrm{arcsinh}\,w + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 11.1.3.5 with $n = 2$ and $k = 1$.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^5 w}{\partial x^5} + (b\,\mathrm{arccosh}\,w + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 11.1.3.6 with $n = 2$ and $k = 1$.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^5 w}{\partial x^5} + (b\arcsin w + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 11.1.3.7 with $n = 2$ and $k = 1$.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^5 w}{\partial x^5} + (b\arccos w + c)\dfrac{\partial w}{\partial x}.$

This is a special case of equation 11.1.3.8 with $n = 2$ and $k = 1$.

9. $\dfrac{\partial w}{\partial t}+w\dfrac{\partial w}{\partial x}+a\dfrac{\partial^3 w}{\partial x^3}=b\dfrac{\partial^5 w}{\partial x^5}.$

Kawahara's equation. It describes magnetoacoustic waves in plasma and long water waves under ice cover.

⦿ *References*: T. Kawahara (1972), A. V. Marchenko (1988).

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1=w(\pm x+C_1,\pm t+C_2),$$
$$w_2=w(x-C_3t,t)+C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation (either plus or minus signs are taken in the first formula).

2°. Degenerate solution:

$$w(x,t)=\frac{x+C_1}{t+C_2}.$$

3°. Traveling-wave solutions:

$$w(x,t)=\frac{105a^2}{169b\cosh^4 z}+2C_1,\quad z=\tfrac{1}{2}kx-(18bk^5+C_1k)t+C_2,\quad k=\sqrt{\frac{a}{13b}}\quad \text{if } ab>0;$$
$$w(x,t)=\frac{105a^2}{169b\sinh^4 z}+2C_1,\quad z=\tfrac{1}{2}kx-(18bk^5+C_1k)t+C_2,\quad k=\sqrt{\frac{a}{13b}}\quad \text{if } ab>0;$$
$$w(x,t)=\frac{105a^2}{169b\cos^4 z}+2C_1,\quad z=\tfrac{1}{2}kx-(18bk^5+C_1k)t+C_2,\quad k=\sqrt{-\frac{a}{13b}}\quad \text{if } ab<0,$$

where C_1 and C_2 are arbitrary constants.

⦿ *Reference*: N. A. Kudryashov (1990 a, the first solution was obtained).

4°. Traveling-wave solution for $a=0$:

$$w(x,t)=\frac{1680b}{(x+C_1t+C_2)^4}-C_1.$$

5°. Solution:

$$w(x,t)=U(\zeta)+2C_1t,\qquad \zeta=x-C_1t^2+C_2t,$$

where C_1 and C_2 are arbitrary constants and the function $U(\zeta)$ is determined by the fourth-order ordinary differential equation (C_3 is an arbitrary constant)

$$bU''''_{\zeta\zeta\zeta\zeta}-aU''_{\zeta\zeta}-\tfrac{1}{2}U^2-C_2U=2C_1\zeta+C_3.$$

The special case $C_1=0$ corresponds to a traveling-wave solution.

10. $\dfrac{\partial w}{\partial t}+aw\dfrac{\partial w}{\partial x}+b\dfrac{\partial^3 w}{\partial x^3}=c\dfrac{\partial^5 w}{\partial x^5}+kw.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1=w(x-aC_1e^{kt}+C_2,t+C_3)+C_1ke^{kt},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w=U(z)+C_1ke^{kt},\qquad z=x-aC_1e^{kt}+C_2t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$cU^{(5)}_z-bU'''_{zzz}-aUU'_z-C_2U'_z+kU=0.$$

If $C_1=0$, we have a traveling-wave solution.

3°. There is a degenerate solution linear in x:

$$w(x,t)=\varphi(t)x+\psi(t).$$

11. $\dfrac{\partial w}{\partial t} + a_1\dfrac{\partial w}{\partial x} + a_2 w\dfrac{\partial w}{\partial x} + a_3\dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial x^2} + a_4\dfrac{\partial^3 w}{\partial x^3} + a_5 w\dfrac{\partial^3 w}{\partial x^3} + a_6\dfrac{\partial^5 w}{\partial x^5} = 0.$

This equation describes long water waves with surface tension (Olver, 1984).

1°. Traveling-wave solutions:

$$w(x,t) = A + C_1 \exp(kx + C_2 t), \qquad k = \pm\sqrt{-\frac{a_2}{a_3 + a_5}}, \quad A = -\frac{a_6 k^5 + a_4 k^3 + a_1 k + C_2}{a_5 k^3 + a_2 k};$$

$$w(x,t) = A + C_1 \sinh(kx + C_2 t + C_3), \quad k = \pm\sqrt{-\frac{a_2}{a_3 + a_5}}, \quad A = -\frac{a_6 k^5 + a_4 k^3 + a_1 k + C_2}{a_5 k^3 + a_2 k};$$

$$w(x,t) = A + C_1 \cosh(kx + C_2 t + C_3), \quad k = \pm\sqrt{-\frac{a_2}{a_3 + a_5}}, \quad A = -\frac{a_6 k^5 + a_4 k^3 + a_1 k + C_2}{a_5 k^3 + a_2 k};$$

$$w(x,t) = A + C_1 \sin(kx + C_2 t + C_3), \quad k = \pm\sqrt{\frac{a_2}{a_3 + a_5}}, \quad A = \frac{a_6 k^5 - a_4 k^3 + a_1 k + C_2}{a_5 k^3 - a_2 k},$$

where C_1, C_2, and C_3 are arbitrary constants.

2°. There are traveling-wave solutions of the following forms:

$$w(x,t) = A + \frac{B}{\cosh z} + \frac{C}{\cosh^2 z},$$
$$w(x,t) = A + \frac{B}{\sinh z} + \frac{C}{\sinh^2 z},$$
$$w(x,t) = A + B\frac{\sinh z}{\cosh z} + \frac{C}{\cosh^2 z},$$
$$w(x,t) = A + \frac{B + C\sinh z + D\cosh z}{(E + \cosh z)^2},$$

where $z = kx + \lambda t + \text{const}$, and the constants A, B, C, D, E, k, and λ are identified by substituting these solutions into the original equation.

⊙ *References*: N. A. Kudryashov and M. B. Sukharev (2001), P. Saucez, A. Vande Wouwer, W. E. Schiesser, and P. Zegeling (2003).

11.1.2. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(x + bt, w).$

Solution:

$$w = w(\xi), \qquad \xi = x + bt,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$aw_\xi^{(n)} - bw'_\xi + f(\xi, w) = 0.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + bw\ln w + f(t)w.$

1°. Generalized traveling-wave solution:

$$w(x,t) = \exp\left[Ae^{bt}x + Be^{bt} + \frac{aA^n}{b(n-1)}e^{nbt} + e^{bt}\int e^{-bt} f(t)\,dt\right],$$

where A and B are arbitrary constants.

2°. Solution:

$$w(x,t) = \exp\left[Ae^{bt} + e^{bt}\int e^{-bt} f(t)\,dt\right]\varphi(z), \quad z = x + \lambda t,$$

where A and λ are arbitrary constants, and the function $\varphi = \varphi(z)$ is determined by the autonomous ordinary differential equation

$$a\varphi_z^{(n)} - \lambda\varphi_z' + b\varphi\ln\varphi = 0,$$

whose order can be reduced by one.

3°. The substitution

$$w(x,t) = \exp\left[e^{bt}\int e^{-bt} f(t)\,dt\right]u(x,t)$$

leads to the simpler equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial x^n} + bu\ln u.$$

3. $\displaystyle\frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + bw\ln w + \big[f(x) + g(t)\big]w.$

1°. Multiplicative separable solution:

$$w(x,t) = \exp\left[Ce^{bt} + e^{bt}\int e^{-bt} g(t)\,dt\right]\varphi(x),$$

where C is an arbitrary constant and the function $\varphi(t)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + b\varphi\ln\varphi + f(x)\varphi = 0.$$

2°. The substitution

$$w(x,t) = \exp\left[e^{bt}\int e^{-bt} g(t)\,dt\right]u(x,t)$$

leads to the simpler equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial x^n} + bu\ln u + f(x)u.$$

4. $\displaystyle\frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + f(t)w\ln w + g(t)w.$

Generalized traveling-wave solution:

$$w(x,t) = \exp\big[\varphi(t)x + \psi(t)\big].$$

Here, the functions $\varphi(t)$ and $\psi(t)$ are given by

$$\varphi(t) = Ae^F, \quad \psi(t) = Be^F + e^F\int e^{-F}(aA^n e^{nF} + g)\,dt, \qquad F = \int f\,dt,$$

where A and B are arbitrary constants.

5. $\displaystyle\frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + f(t)w\ln w + \big[g(t)x + h(t)\big]w.$

Generalized traveling-wave solution:

$$w(x,t) = \exp\big[\varphi(t)x + \psi(t)\big].$$

Here, the functions $\varphi(t)$ and $\psi(t)$ are given by

$$\varphi(t) = Ae^F + e^F\int e^{-F} g\,dt, \quad F = \int f\,dt,$$

$$\psi(t) = Be^F + e^F\int e^{-F}(a\varphi^n + h)\,dt,$$

where A and B are arbitrary constants.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(x)w\ln w + \big[bf(x)t + g(x)\big]w.$

Multiplicative separable solution:

$$w(x,t) = e^{-bt}\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f(x)\varphi\ln\varphi + \big[g(x) + b\big]\varphi = 0.$$

11.1.3. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(w)\dfrac{\partial w}{\partial x}$

Preliminary remarks. Equations of this form admit traveling-wave solutions:

$$w = w(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the $(n-1)$st-order autonomous ordinary differential equation (C is an arbitrary constant)

$$aw_z^{(n-1)} + \int f(w)\,dw - \lambda w = C.$$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + bw\dfrac{\partial w}{\partial x}.$

Generalized Burgers–Korteweg–de Vries equation.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{n-1}w(C_1x + bC_1C_2t + C_3,\, C_1^n t + C_4) + C_2,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solutions:

$$w(x,t) = -\frac{x + C_1}{b(t + C_2)},$$
$$w(x,t) = (-1)^n\frac{a(2n-2)!}{b(n-1)!}\,\frac{1}{(x + bC_1t + C_2)^{n-1}} + C_1.$$

The first solution is degenerate and the second one is a traveling-wave solution (a special case of the solution of Item 3°).

3°. Traveling-wave solution:

$$w = w(\xi), \qquad \xi = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(\xi)$ is determined by the $(n-1)$st-order autonomous ordinary differential equation

$$aw_\xi^{(n-1)} + \tfrac{1}{2}bw^2 = \lambda w + C.$$

4°. Self-similar solution:

$$w(x,t) = t^{\frac{1-n}{n}}u(\eta), \qquad \eta = xt^{-\frac{1}{n}},$$

where the function $u(\eta)$ is determined by the ordinary differential equation

$$au_\eta^{(n)} + buu'_\eta + \frac{1}{n}\eta u'_\eta + \frac{n-1}{n}u = 0.$$

5°. Solution:

$$w(x,t) = U(\zeta) + 2C_1t, \qquad \zeta = x + bC_1t^2 + C_2t,$$

where C_1 and C_2 are arbitrary constants and the function $U(\zeta)$ is determined by the $(n-1)$st-order ordinary differential equation

$$aU_\zeta^{(n-1)} + \tfrac{1}{2}bU^2 - C_2U = 2C_1\zeta + C_3.$$

6°. Solution:

$$w = \varphi^{n-1} F(z) + \frac{1}{b\varphi}(\varphi'_t x + \psi'_t), \qquad z = \varphi(t)x + \psi(t).$$

Here, the functions $\varphi(t)$ and $\psi(t)$ are defined by

$$\varphi(t) = (Ant + C_1)^{-\frac{1}{n}},$$
$$\psi(t) = C_2(Ant + C_1)^{\frac{n-1}{n}} + C_3(Ant + C_1)^{-\frac{1}{n}} + \frac{B}{A^2(n-1)},$$

where A, B, C_1, C_2, and C_3 are arbitrary constants, and the function $F(z)$ is determined by the ordinary differential equation

$$aF_z^{(n)} + bFF'_z + A(n-2)F + \frac{A^2}{b}(1-n)z + \frac{B}{b} = 0.$$

2. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + bw^k \frac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{n-1} w(C_1^k x + C_2, C_1^{nk} t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Self-similar solution:

$$w(x,t) = t^{\frac{1-n}{nk}} U(z), \qquad z = xt^{-\frac{1}{n}},$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$aU_z^{(n)} + bU^k U'_z + \frac{1}{n} zU'_z + \frac{n-1}{nk} U = 0.$$

3. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + be^{\lambda w} \frac{\partial w}{\partial x}.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1 x + C_2, C_1^n t + C_3) + \frac{n-1}{\lambda} \ln C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = U(z) + \frac{1-n}{n\lambda} \ln t, \qquad z = xt^{-\frac{1}{n}},$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$aU_z^{(n)} + be^{\lambda U} U'_z + \frac{1}{n} zU'_z + \frac{n-1}{n\lambda} = 0.$$

4. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + (b\ln w + c)\frac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \exp\left[\frac{x + C_2}{C_1 - bt} + \frac{a}{b(n-2)} \frac{1}{(C_1 - bt)^{n-1}} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [b\,\mathrm{arcsinh}(kw) + c]\dfrac{\partial w}{\partial x}$.

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\sinh\left[\frac{x+C_2}{C_1-bt} + \frac{a}{b(2n-1)}\frac{1}{(C_1-bt)^{2n}} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [b\,\mathrm{arccosh}(kw) + c]\dfrac{\partial w}{\partial x}$.

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\cosh\left[\frac{x+C_2}{C_1-bt} + \frac{a}{b(2n-1)}\frac{1}{(C_1-bt)^{2n}} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [b\arcsin(kw) + c]\dfrac{\partial w}{\partial x}$.

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\sin\left[\frac{x+C_2}{C_1-bt} + \frac{a(-1)^n}{b(2n-1)}\frac{1}{(C_1-bt)^{2n}} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

8. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [b\arccos(kw) + c]\dfrac{\partial w}{\partial x}$.

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\cos\left[\frac{x+C_2}{C_1-bt} + \frac{a(-1)^n}{b(2n-1)}\frac{1}{(C_1-bt)^{2n}} - \frac{c}{b}\right],$$

where C_1 and C_2 are arbitrary constants.

11.1.4. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(x,t,w)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + (bx + c)\dfrac{\partial w}{\partial x} + f(w)$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1e^{-bt}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + C_1e^{-bt},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$aw_z^{(n)} + (bz + c)w'_z + f(w) = 0.$$

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(t)\dfrac{\partial w}{\partial x} + g(w).$

The transformation $w = u(z,t)$, $z = x + \int f(t)\,dt$ leads to the simpler equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial z^n} + g(u),$$

which has a traveling-wave solution $u = u(kz + \lambda t)$.

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + \bigl[bx + f(t)\bigr]\dfrac{\partial w}{\partial x} + g(w).$

Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + Ce^{-bt} + e^{-bt}\int e^{bt} f(t)\,dt,$$

where C is an arbitrary constant and the function $w(z)$ is determined by the ordinary differential equation

$$aw_z^{(n)} + bzw'_z + g(w) = 0.$$

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(x)\dfrac{\partial w}{\partial x} + bw\ln w + \bigl[g(x) + h(t)\bigr]w.$

Multiplicative separable solution:

$$w(x,t) = \exp\left[Ce^{bt} + e^{bt}\int e^{-bt}h(t)\,dt\right]\varphi(x),$$

where C is an arbitrary constant and the function $\varphi(t)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f(x)\varphi'_x + b\varphi\ln\varphi + g(x)\varphi = 0.$$

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + bw\dfrac{\partial w}{\partial x} + f(t).$

The transformation

$$w = u(z,t) + \int_{t_0}^{t} f(\tau)\,d\tau, \qquad z = x + b\int_{t_0}^{t}(t-\tau)f(\tau)\,d\tau,$$

where t_0 is any, leads to an equation of the form 11.1.3.1:

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial x^n} + bu\frac{\partial u}{\partial x}.$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + bw\dfrac{\partial w}{\partial x} + cw.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + bC_1e^{ct} + C_2,\ t + C_3) + C_1ce^{ct},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w = U(z) + C_1ce^{ct}, \qquad z = x + bC_1e^{ct} + C_2t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$aU_z^{(n)} + bUU'_z - C_2U'_z + cU = 0.$$

For $C_1 = 0$, we have a traveling-wave solution.

3°. There is a degenerate solution linear in x:

$$w(x,t) = \varphi(t)x + \psi(t).$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + \big[bw + f(t)\big]\dfrac{\partial w}{\partial x} + g(t).$

The transformation

$$w = u(z,t) + \int_{t_0}^{t} g(\tau)\,d\tau, \qquad z = x + \int_{t_0}^{t} f(\tau)\,d\tau + b\int_{t_0}^{t}(t-\tau)g(\tau)\,d\tau,$$

where t_0 is any, leads to an equation of the form 11.1.3.1:

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial x^n} + bu\frac{\partial u}{\partial x}.$$

8. $\dfrac{\partial w}{\partial t} + a\dfrac{\partial^n w}{\partial x^n} + f(t)w\dfrac{\partial w}{\partial x} + g(t)w = 0.$

Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w\big(x + C_1\psi(t) + C_2,\ t\big) - C_1\varphi(t),$$

where

$$\varphi(t) = \exp\biggl[-\int g(t)\,dt\biggr], \qquad \psi(t) = \int f(t)\varphi(t)\,dt,$$

is also a solution of the equation (C_1 and C_2 are arbitrary constants).

Remark. This also remains true if a in the equation is an arbitrary function of time, $a = a(t)$.

9. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + [f(t)\ln w + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \exp[\varphi(t)x + \psi(t)],$$

where

$$\varphi(t) = -\biggl[\int f(t)\,dt + C_1\biggr]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) + a\varphi^{n-1}(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.*

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [f(t)\,\mathrm{arcsinh}(kw) + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\sinh\big[\varphi(t)x + \psi(t)\big],$$

where

$$\varphi(t) = -\biggl[\int f(t)\,dt + C_1\biggr]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) + a\varphi^{2n}(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [f(t)\,\mathrm{arccosh}(kw) + g(t)]\dfrac{\partial w}{\partial x}.$

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\cosh\big[\varphi(t)x + \psi(t)\big],$$

where

$$\varphi(t) = -\biggl[\int f(t)\,dt + C_1\biggr]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) + a\varphi^{2n}(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

* In equations 11.1.4.9 to 11.1.4.13 and their solutions, a can be an arbitrary function of time, $a = a(t)$.

12. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [f(t)\arcsin(kw) + g(t)]\dfrac{\partial w}{\partial x}$.

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\sin\bigl[\varphi(t)x + \psi(t)\bigr],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) + a(-1)^n\varphi^{2n}(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

13. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n+1} w}{\partial x^{2n+1}} + [f(t)\arccos(kw) + g(t)]\dfrac{\partial w}{\partial x}$.

Generalized traveling-wave solution:

$$w(x,t) = \frac{1}{k}\cos\bigl[\varphi(t)x + \psi(t)\bigr],$$

where

$$\varphi(t) = -\left[\int f(t)\,dt + C_1\right]^{-1}, \qquad \psi(t) = \varphi(t)\int [g(t) + a(-1)^n\varphi^{2n}(t)]\,dt + C_2\varphi(t),$$

and C_1 and C_2 are arbitrary constants.

11.1.5. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + F\left(x, t, w, \dfrac{\partial w}{\partial x}\right)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + b\left(\dfrac{\partial w}{\partial x}\right)^2$.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{n-2} w(C_1 x + 2bC_1C_2 t + C_3, C_1^n t + C_4) + C_2 x + bC_2^2 t + C_5,$$

where $C_1, \dots, C_5$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = C_1 t + C_2 + \int \theta(z)\,dz, \qquad z = x + \lambda t,$$

where C_1, C_2, and λ are arbitrary constants, and the function $\theta(z)$ is determined by the $(n-1)$st-order autonomous ordinary differential equation

$$a\theta_z^{(n-1)} + b\theta^2 - \lambda\theta - C_1 = 0.$$

To $C_1 = 0$ there corresponds a traveling-wave solution.

3°. Self-similar solution:

$$w(x,t) = t^{\frac{2-n}{n}} u(\zeta), \qquad \zeta = xt^{-\frac{1}{n}},$$

where the function $u(\zeta)$ is determined by the ordinary differential equation

$$au_\zeta^{(n)} + b(u'_\zeta)^2 + \frac{1}{n}\zeta u'_\zeta + \frac{n-2}{n}u = 0.$$

4°. There is a degenerate solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

5°. The Bäcklund transformation

$$\frac{\partial w}{\partial x}=\frac{u}{2}, \qquad \frac{\partial w}{\partial t}=\frac{a}{2}\frac{\partial^{n-1}u}{\partial x^{n-1}}+\frac{b}{4}u^2 \tag{1}$$

connects the original equation with the generalized Burgers–Korteweg–de Vries equation 11.1.3.1:

$$\frac{\partial u}{\partial t}=a\frac{\partial^n u}{\partial x^n}+bu\frac{\partial u}{\partial x}. \tag{2}$$

If $u=u(x,t)$ is a solution of equation (2), then the corresponding solution $w=w(x,t)$ of the original equation can be found from the linear system of first-order equations (1).

2. $\displaystyle \frac{\partial w}{\partial t}=a\frac{\partial^n w}{\partial x^n}+b\left(\frac{\partial w}{\partial x}\right)^2+f(t).$

1°. Solution:

$$w(x,t)=C_1t+C_2+\int f(t)\,dt+\Theta(z), \qquad z=x+\lambda t,$$

where C_1, C_2, and λ are arbitrary constants, and the function $\Theta(z)$ is determined by the autonomous ordinary differential equation

$$a\Theta_z^{(n)}+b\left(\Theta_z'\right)^2-\lambda\Theta_z'-C_1=0.$$

2°. The substitution $w=U(x,t)+\int f(t)\,dt$ leads to a simpler equation of the form 11.1.5.1:

$$\frac{\partial U}{\partial t}=a\frac{\partial^n U}{\partial x^n}+b\left(\frac{\partial U}{\partial x}\right)^2.$$

3. $\displaystyle \frac{\partial w}{\partial t}=a\frac{\partial^n w}{\partial x^n}+b\left(\frac{\partial w}{\partial x}\right)^2+cw+f(t).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1=w(x+C_1,t)+C_2e^{ct},$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t)=Ae^{ct}+e^{ct}\int e^{-ct}f(t)\,dt+\theta(z), \qquad z=x+\lambda t,$$

where A and λ are arbitrary constants, and the function $\theta(z)$ is determined by the autonomous ordinary differential equation

$$a\theta_z^{(n)}+b\left(\theta_z'\right)^2-\lambda\theta_z'+c\theta=0.$$

3°. There is a degenerate solution quadratic in x:

$$w(x,t)=\varphi(t)x^2+\psi(t)x+\chi(t).$$

4°. The substitution $w=U(x,t)+e^{ct}\int e^{-ct}f(t)\,dt$ leads to the simpler equation

$$\frac{\partial U}{\partial t}=a\frac{\partial^n U}{\partial x^n}+b\left(\frac{\partial U}{\partial x}\right)^2+cU.$$

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + b\left(\dfrac{\partial w}{\partial x}\right)^2 + cw\dfrac{\partial w}{\partial x} + kw^2 + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t) + \psi(t)\exp(\lambda x),$$

where λ is a root of the quadratic equation $b\lambda^2 + c\lambda + k = 0$, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi_t' = k\varphi^2 + f(t)\varphi + g(t), \tag{1}$$

$$\psi_t' = \bigl[(c\lambda + 2k)\varphi + f(t) + a\lambda^n\bigr]\psi. \tag{2}$$

The Riccati equation (1) is integrable by quadrature in some special cases, for example,

$$\text{(a)}\ \ k = 0, \qquad \text{(b)}\ \ g(t) \equiv 0, \qquad \text{(c)}\ \ f(t) = \text{const}, \quad g(t) = \text{const}.$$

See also Kamke (1977) and Polyanin and Zaitsev (2003). Whenever a solution of equation (1) is found, one can obtain the corresponding solution of the linear equation (2).

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + g(x) + h(t).$

1°. Additive separable solution:

$$w(x,t) = At + B + \int h(t)\,dt + \varphi(x).$$

Here, A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f(x)\bigl(\varphi_x'\bigr)^2 + g(x) - A = 0.$$

2°. The substitution $w = U(x,t) + \int h(t)\,dt$ leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^n U}{\partial x^n} + f(x)\left(\frac{\partial U}{\partial x}\right)^2 + g(x).$$

6. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + bw + g(x) + h(t).$

1°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ae^{bt} + e^{bt}\int e^{-bt}h(t)\,dt.$$

Here, A is an arbitrary constant and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f(x)(\varphi_x')^2 + b\varphi + g(x) = 0.$$

2°. The substitution $w = U(x,t) + e^{bt}\int e^{-bt}h(t)\,dt$ leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^n U}{\partial x^n} + f(x)\left(\frac{\partial U}{\partial x}\right)^2 + bU + g(x).$$

7. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + bf(t)w^2 + g(t)w + h(t).$

1°. Generalized separable solutions involving exponentials of x:

$$w(x,t) = \varphi(t) + \psi(t)\exp\bigl(\pm x\sqrt{-b}\,\bigr), \qquad b < 0, \tag{1}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = bf\varphi^2 + g\varphi + h, \tag{2}$$

$$\psi'_t = \bigl[2bf\varphi + g + a(\pm\sqrt{-b}\,)^n\bigr]\psi. \tag{3}$$

The arguments of the functions f, g, and h are not specified.

Equation (2) is a Riccati equation for $\varphi = \varphi(t)$ and, hence, can be reduced to a second-order linear equation. The books by Kamke (1977) and Polyanin and Zaitsev (2003) present a large number of solutions to this equation for various f, g, and h.

Whenever a solution of equation (2) is known, the corresponding solution of equation (3) is computed by the formula

$$\psi(t) = C\exp\left[a(\pm\sqrt{-b}\,)^n t + \int(2bf\varphi + g)\,dt\right], \tag{4}$$

where C is an arbitrary constant.

Note two special integrable cases of equation (2).

Solution of equation (2) for $h \equiv 0$:

$$\varphi(t) = e^G\left(C_1 - b\int fe^G\,dt\right)^{-1}, \quad G = \int g\,dt,$$

where C_1 is an arbitrary constant.

If the functions f, g, and h are proportional,

$$g = \alpha f, \quad h = \beta f \qquad (\alpha,\ \beta = \text{const}),$$

the solution of equation (2) is expressed as

$$\int \frac{d\varphi}{b\varphi^2 + \alpha\varphi + \beta} = \int f\,dt + C_2, \tag{5}$$

where C_2 is an arbitrary constant. On integrating the left-hand side of (5), one may obtain $\varphi = \varphi(t)$ in explicit form.

2°. Generalized separable solution (generalizes the solutions of Item 1°):

$$w(x,t) = \varphi(t) + \psi(t)\exp\bigl(x\sqrt{-b}\,\bigr) + \chi(t)\exp\bigl(-x\sqrt{-b}\,\bigr), \qquad b < 0, \tag{6}$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi'_t = bf\varphi^2 + g\varphi + h + 4bf\psi\chi, \tag{7}$$

$$\psi'_t = \bigl[2bf\varphi + g + a(\sqrt{-b}\,)^n\bigr]\psi, \tag{8}$$

$$\chi'_t = \bigl[2bf\varphi + g + a(-\sqrt{-b}\,)^n\bigr]\chi. \tag{9}$$

For equations of even order, with $n = 2m$, $m = 1,\ 2,\ \ldots$, it follows from (8) and (9) that $\psi(t)$ and $\chi(t)$ are proportional. Then, by setting $\psi(t) = A\theta(t)$ and $\chi(t) = B\theta(t)$, we can rewrite solution (6) in the form

$$w(x,t) = \varphi(t) + \theta(t)\bigl[A\exp\bigl(x\sqrt{-b}\,\bigr) + B\exp\bigl(-x\sqrt{-b}\,\bigr)\bigr], \qquad b < 0, \tag{10}$$

where the functions $\varphi(t)$ and $\theta(t)$ are determined by the system of ordinary differential equations

$$\varphi_t' = bf\left(\varphi^2 + 4AB\theta^2\right) + g\varphi + h, \tag{11}$$

$$\theta_t' = \left[2bf\varphi + g + (-1)^m ab^m\right]\theta. \tag{12}$$

The function φ can be expressed from (12) via θ and then substituted into (11) to obtain a second-order nonlinear equation for θ. For $f,\ g,\ h = \text{const}$, this equation is autonomous and its order can be reduced.

Note two special cases where solution (10) is expressed in terms of hyperbolic functions:

$$w(x,t) = \varphi(t) + \theta(t)\cosh\left(x\sqrt{-b}\right) \qquad \text{if} \quad A = \tfrac{1}{2},\ B = \tfrac{1}{2};$$

$$w(x,t) = \varphi(t) + \theta(t)\sinh\left(x\sqrt{-b}\right) \qquad \text{if} \quad A = \tfrac{1}{2},\ B = -\tfrac{1}{2}.$$

3°. Generalized separable solution involving trigonometric functions of x:

$$w(x,t) = \varphi(t) + \psi(t)\cos\left(x\sqrt{b}\right) + \chi(t)\sin\left(x\sqrt{b}\right), \qquad b > 0, \tag{13}$$

where the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ are determined by a system of ordinary differential equations (which is not written out here).

For equations of even order, with $n = 2m$, $m = 1,\ 2,\ \ldots$, there are exact solutions of the form

$$w(x,t) = \varphi(t) + \theta(t)\cos\left(x\sqrt{b} + c\right), \qquad b > 0, \tag{14}$$

where c is an arbitrary constant and the functions $\varphi(t)$ and $\theta(t)$ are determined by the system of first-order ordinary differential equations with variable coefficients

$$\varphi_t' = bf\left(\varphi^2 + \theta^2\right) + g\varphi + h, \tag{15}$$

$$\theta_t' = \left[2bf\varphi + g + (-1)^m ab^m\right]\theta. \tag{16}$$

The function φ can be expressed from (16) via θ and then substituted into (15) to obtain a second-order nonlinear equation for θ. For $f,\ g,\ h = \text{const}$, this equation is autonomous and its order can be reduced.

⊙ *References*: V. A. Galaktionov (1995), A. D. Polyanin and V. F. Zaitsev (2002).

8. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + f(w)\left(\frac{\partial w}{\partial x}\right)^n + \left[xg(t) + h(t)\right]\frac{\partial w}{\partial x}.$

Passing to the new independent variables

$$\tau = \int \varphi^n(t)\,dt, \quad z = \varphi(t)x + \int h(t)\varphi(t)\,dt, \quad \varphi(t) = \exp\left[\int g(t)\,dt\right],$$

one arrives to the simpler equation

$$\frac{\partial w}{\partial \tau} = a\frac{\partial^n w}{\partial z^n} + f(w)\left(\frac{\partial w}{\partial z}\right)^n,$$

which has a traveling-wave solution $w = u(kz + \lambda\tau)$ and a self-similar solution $w = v(z\tau^{-1/n})$.

9. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + f\left(x,\ \frac{\partial w}{\partial x}\right) + g(t).$

1°. Additive separable solution:

$$w(x,t) = At + B + \int g(t)\,dt + \varphi(x).$$

Here, A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f\left(x, \varphi_x'\right) - A = 0.$$

2°. The substitution

$$w = U(x,t) + \int g(t)\,dt$$

leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^n U}{\partial x^n} + f\left(x,\ \frac{\partial U}{\partial x}\right).$$

10. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f\left(x,\ \dfrac{\partial w}{\partial x}\right) + bw + g(t).$

1°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ae^{bt} + e^{bt}\int e^{-bt} g(t)\,dt.$$

Here, A is an arbitrary constant and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f\left(x, \varphi'_x\right) + b\varphi = 0.$$

2°. The substitution

$$w = U(x,t) + e^{bt}\int e^{-bt} g(t)\,dt$$

leads to the simpler equation

$$\frac{\partial U}{\partial t} = a\frac{\partial^n U}{\partial x^n} + f\left(x,\ \frac{\partial U}{\partial x}\right) + bU.$$

11. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + wf\left(t,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x}\right).$

Multiplicative separable solution:

$$w(x,t) = A\exp\left[\lambda x + a\lambda^n t + \int f(t,\lambda)\,dt\right],$$

where A and λ are arbitrary constants.

11.1.6. Equations of the Form $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + F\left(x, t, w, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^{n-1} w}{\partial x^{n-1}}\right)$

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + f(t)\displaystyle\sum_{i,j=0}^{i,j<n} b_{ij}\dfrac{\partial^i w}{\partial x^i}\dfrac{\partial^j w}{\partial x^j} + \sum_{k=0}^{n-1} g_k(t)\dfrac{\partial^k w}{\partial x^k} + h(t).$

Here, we adopt the notation: $\frac{\partial^0 w}{\partial x^0} \equiv w$.

1°. In the general case, the equation has generalized separable solutions of the form

$$w(x,t) = \varphi(t) + \psi(t)\exp(\lambda x),$$

where λ is a root of the algebraic equation $\displaystyle\sum_{i,j=0}^{i,j<n} b_{ij}\lambda^{i+j} = 0.$

2°. Let n be an even number and let all coefficients b_{ij} be zero for odd $i+j$. In this case, the original equation has also generalized separable solutions of the form

$$w(x,t) = \varphi_1(t) + \psi_1(t)\left[A\cosh(\lambda x) + B\sinh(\lambda x)\right],$$
$$w(x,t) = \varphi_2(t) + \psi_2(t)\left[A\cos(\lambda x) + B\sin(\lambda x)\right],$$

where A and B are arbitrary constants, the parameter λ is determined by solving algebraic equations, and the functions $\varphi_1(t)$, $\psi_1(t)$ and $\varphi_2(t)$, $\psi_2(t)$ are found from appropriate systems of first-order ordinary differential equations.

2. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + F\left(x, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^{n-1} w}{\partial x^{n-1}}\right) + g(t).$

1°. Additive separable solution:

$$w(x,t) = At + B + \int g(t)\,dt + \varphi(x).$$

Here, A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + F\bigl(x, \varphi_x', \ldots, \varphi_x^{(n-1)}\bigr) - A = 0,$$

whose order can be reduced with the substitution $U(x) = \varphi_x'$.

2°. The substitution $w = u(x,t) + \int g(t)\,dt$ leads to the simpler equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial x^n} + F\left(x, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{n-1} u}{\partial x^{n-1}}\right).$$

3. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + F\left(x, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^{n-1} w}{\partial x^{n-1}}\right) + bw + g(t).$

1°. Additive separable solution:

$$w(x,t) = \varphi(x) + Ae^{bt} + e^{bt}\int e^{-bt} g(t)\,dt.$$

Here, A is an arbitrary constant and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + F\bigl(x, \varphi_x', \ldots, \varphi_x^{(n-1)}\bigr) + b\varphi = 0.$$

2°. The substitution $w = u(x,t) + e^{bt}\int e^{-bt} g(t)\,dt$ leads to the simpler equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial x^n} + F\left(x, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{n-1} u}{\partial x^{n-1}}\right) + bu.$$

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n w}{\partial x^n} + wF\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \ldots, \dfrac{1}{w}\dfrac{\partial^{n-1} w}{\partial x^{n-1}}\right).$

Multiplicative separable solution:

$$w(x,t) = A\exp\left[\lambda x + a\lambda^n t + \int F(t, \lambda, \ldots, \lambda^{n-1})\,dt\right],$$

where A and λ are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^{2n} w}{\partial x^{2n}} + wF\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^{2n-2} w}{\partial x^{2n-2}}\right).$

Multiplicative separable solutions:

$$w(x,t) = \bigl[A\cosh(\lambda x) + B\sinh(\lambda x)\bigr]\exp\left[a\lambda^{2n} t + \int F(t, \lambda^2, \ldots, \lambda^{2n-2})\,dt\right],$$

$$w(x,t) = \bigl[A\cos(\lambda x) + B\sin(\lambda x)\bigr]\exp\bigl[(-1)^n a\lambda^{2n} t + \Phi(t)\bigr],$$

$$\Phi(t) = \int F\bigl(t, -\lambda^2, \ldots, (-1)^{n-1}\lambda^{2n-2}\bigr)\,dt,$$

where A, B, and λ are arbitrary constants.

11.1.7. Equations of the Form $\frac{\partial w}{\partial t} = aw\frac{\partial^n w}{\partial x^n} + f(x,t,w)\frac{\partial w}{\partial x} + g(x,t,w)$

1. $\frac{\partial w}{\partial t} = aw\frac{\partial^n w}{\partial x^n} + f(t)w + g(t).$

1°. Degenerate solution:

$$w(x,t) = F(t)\bigl(A_{n-1}x^{n-1} + \cdots + A_1x + A_0\bigr) + F(t)\int \frac{g(t)}{F(t)}\,dt, \quad F(t) = \exp\biggl[\int f(t)\,dt\biggr],$$

where A_0, A_1, ..., A_{n-1} are arbitrary constants.

2°. Generalized separable solution:

$$w(x,t) = \varphi(t)\bigl(x^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0\bigr) + \varphi(t)\int \frac{g(t)}{\varphi(t)}\,dt,$$
$$\varphi(t) = F(t)\biggl[C - an!\int F(t)\,dt\biggr]^{-1}, \quad F(t) = \exp\biggl[\int f(t)\,dt\biggr],$$

where A_0, A_1, ..., A_{n-1}, and C are arbitrary constants.

2. $\frac{\partial w}{\partial t} = aw\frac{\partial^n w}{\partial x^n} + f(x)w + \sum_{k=0}^{n-1} b_k x^k.$

Generalized separable solution:

$$w(x,t) = t\sum_{k=0}^{n-1} b_k x^k + \sum_{k=0}^{n-1} C_k x^k - \frac{1}{a(n-1)!}\int_{x_0}^{x} (x-\xi)^{n-1} f(\xi)\,d\xi,$$

where C_0, C_1, ..., C_{n-1}, and x_0 are arbitrary constants.

3. $\frac{\partial w}{\partial t} = aw\frac{\partial^n w}{\partial x^n} + bw^2 + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t).$$

Here, the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi'_t = C\varphi^2 + b\varphi\psi + f(t)\varphi,$$
$$\psi'_t = C\varphi\psi + b\psi^2 + f(t)\psi + g(t),$$

where C is an arbitrary constant and the function $\Theta(x)$ satisfies the nth-order linear ordinary differential equation

$$a\Theta_x^{(n)} + b\Theta = C.$$

4. $\frac{\partial w}{\partial t} = aw\frac{\partial^{2n} w}{\partial x^{2n}} - ak^{2n}w^2 + f(x)w + b_1\sinh(kx) + b_2\cosh(kx).$

Generalized separable solution linear in t:

$$w(x,t) = t\bigl[b_1\sinh(kx) + b_2\cosh(kx)\bigr] + \varphi(x).$$

Here, the function $\varphi(x)$ is determined from the constant-coefficient linear nonhomogeneous ordinary differential equation

$$a\varphi_x^{(2n)} - ak^{2n}\varphi + f(x) = 0.$$

5. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^n w}{\partial x^n} + \bigl[xf(t) + g(t)\bigr]\dfrac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = H(t)u(z,\tau), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^n(t)H(t)\,dt,$$

where the functions $F(t)$ and $H(t)$ are given by

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

leads to the simpler equation

$$\frac{\partial u}{\partial \tau} = au\frac{\partial^n u}{\partial z^n},$$

which admits, for example, a traveling-wave solution $u = u(kz + \lambda\tau)$ and a self-similar solution of the form $u = u(\xi)$, $\xi = z\tau^{-1/n}$.

6. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^n w}{\partial x^n} + f(x)w\dfrac{\partial w}{\partial x} + g(t)w + h(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\Theta(x)$ are determined by the ordinary differential equations

$$\begin{aligned}
&\varphi'_t = C\varphi^2 + g(t)\varphi,\\
&\psi'_t = \bigl[C\varphi + g(t)\bigr]\psi + h(t),\\
&a\Theta_x^{(n)} + f(x)\Theta'_x = C,
\end{aligned}$$

where C is an arbitrary constant. On integrating the first two equations successively, one obtains

$$\begin{aligned}
&\varphi(t) = G(t)\left[A - C\int G(t)\,dt\right]^{-1}, \quad G(t) = \exp\left[\int g(t)\,dt\right],\\
&\psi(t) = B\varphi(t) + \varphi(t)\int \frac{h(t)}{\varphi(t)}\,dt,
\end{aligned}$$

where A and B are arbitrary constants.

7. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial^n w}{\partial x^n} + f(x)w\dfrac{\partial w}{\partial x} + g(x)w^2 + h(t)w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(x)H(t)\left[A + B\int H(t)\,dt\right]^{-1}, \quad H(t) = \exp\left[\int h(t)\,dt\right],$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the linear ordinary differential equation

$$a\varphi_x^{(n)} + f(x)\varphi'_x + g(x)\varphi + B = 0.$$

11.1.8. Other Equations

1. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n}{\partial x^n}\left(w^m\dfrac{\partial^k w}{\partial x^k}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(C_2 x + C_3, C_1^m C_2^{n+k} t + C_4),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Traveling-wave solution:

$$w = w(z), \qquad z = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(z)$ is determined by the autonomous ordinary differential equation $a[w^m w_z^{(k)}]_z^{(n)} - \lambda w_z' = 0$.

3°. Self-similar solution:

$$w(x,t) = t^{-\frac{(n+k)\beta+1}{m}} u(\xi), \qquad \xi = xt^\beta,$$

where β is an arbitrary constant and the function $u = u(\xi)$ is determined by the ordinary differential equation

$$-[(n+k)\beta + 1]u + m\beta\xi u_\xi' = am[u^m u_\xi^{(k)}]_\xi^{(n)}.$$

4°. Solution:

$$w(x,t) = (C_1 t + C_2)^{-1/m} V(\zeta), \qquad \zeta = x + C_3 \ln|C_1 t + C_2|,$$

where the function $V = V(\zeta)$ is determined by the autonomous ordinary differential equation

$$am[V^m V_\zeta^{(k)}]_\zeta^{(n)} - mC_1C_3 V_\zeta' + C_1 V = 0.$$

Remark. For a special case $C_3 = 0$, we have a solution in multiplicative separable form.

5°. Generalized self-similar solution:

$$w(x,t) = e^{-(n+k)\beta t}\varphi(\eta), \qquad \eta = xe^{m\beta t},$$

where β is an arbitrary constant and the function $\varphi = \varphi(\eta)$ is determined by the ordinary differential equation

$$-(n+k)\beta\varphi + m\beta\eta\varphi_\eta' = a[\varphi^m \varphi_\eta^{(k)}]_\eta^{(n)}.$$

2. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^n}{\partial x^n}\left(w^m \frac{\partial^k w}{\partial x^k}\right) + \big[xf(t) + g(t)\big]\frac{\partial w}{\partial x} + h(t)w.$

The transformation

$$w(x,t) = u(z,\tau)H(t), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^{n+k}(t)H^m(t)\,dt,$$

where the functions $F(t)$ and $H(t)$ are given by

$$F(t) = \exp\biggl[\int f(t)\,dt\biggr], \quad H(t) = \exp\biggl[\int h(t)\,dt\biggr],$$

leads to a simpler equation of the form 11.1.8.1:

$$\frac{\partial u}{\partial \tau} = a\frac{\partial^n}{\partial z^n}\left(u^m \frac{\partial^k u}{\partial z^k}\right).$$

3. $\displaystyle \frac{\partial w}{\partial t} = a\frac{\partial^n}{\partial x^n}\left(e^{\lambda w} \frac{\partial^k w}{\partial x^k}\right) + f(t).$

The transformation

$$w(x,t) = u(x,\tau) + F(t), \quad \tau = \int \exp\bigl[\lambda F(t)\bigr]\,dt, \qquad F(t) = \int f(t)\,dt,$$

leads to the simpler equation

$$\frac{\partial u}{\partial \tau} = a\frac{\partial^n}{\partial x^n}\left(e^{\lambda u} \frac{\partial^k u}{\partial x^k}\right).$$

It admits, for example, exact solutions of the forms

$$u = U(kx + \lambda\tau) \qquad \text{(traveling-wave solution)},$$
$$u = V\bigl(x\tau^{-1/(n+k)}\bigr) \qquad \text{(self-similar solution)},$$
$$u = \varphi(x) + \psi(\tau) \qquad \text{(additive separable solution)}.$$

4. $\dfrac{\partial w}{\partial t} = a\dfrac{\partial^n}{\partial x^n}\left(e^{\lambda w}\dfrac{\partial^k w}{\partial x^k}\right) + f(x)e^{\lambda w}.$

Additive separable solution:

$$w = -\frac{1}{\lambda}\ln(\lambda t + C) + \varphi(x),$$

where λ and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\frac{d^n}{dx^n}\left(e^{\lambda\varphi}\frac{d^k\varphi}{dx^k}\right) + f(x)e^{\lambda\varphi} + 1 = 0.$$

For $k = 1$, it is reduced with the change of variable $\psi = e^{\lambda\varphi}$ to a linear equation.

5. $\dfrac{\partial w}{\partial t} = \displaystyle\sum_{k=0}^{n}[f_k(t)\ln w + g_k(t)]\dfrac{\partial^k w}{\partial x^k}.$

Generalized traveling-wave solution:

$$w(x,t) = \exp[\varphi(t)x + \psi(t)],$$

where the functions $\varphi = \varphi(t)$ and $\psi = \psi(t)$ are determined by the system of first-order ordinary differential equations

$$\varphi'_t = \sum_{k=0}^{n} f_k(t)\varphi^{k+1},$$

$$\psi'_t = \sum_{k=0}^{n} \varphi^k[f_k(t)\psi + g_k(t)].$$

6. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^{n-1}}{\partial x^{n-1}}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{a}{f(w)} + b.$

Functional separable solution in implicit form:

$$\int f(w)\,dw = at - \frac{b}{n!}x^n + C_{n-1}x^{n-1} + \cdots + C_1x + C_0,$$

where C_0, C_1, …, C_{n-1} are arbitrary constants.

7. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^{n-1}}{\partial x^{n-1}}\left[f(w)\dfrac{\partial w}{\partial x}\right] + \dfrac{g(t)}{f(w)} + h(x).$

Functional separable solution in implicit form:

$$\int f(w)\,dw = \int g(t)\,dt - \frac{1}{(n-1)!}\int_{x_0}^{x}(x-\xi)^{n-1}h(\xi)\,d\xi + C_{n-1}x^{n-1} + \cdots + C_1x + C_0,$$

where C_0, C_1, …, C_{n-1} are arbitrary constants and x_0 is any number.

8. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^n}{\partial x^n}\left[f(w)\dfrac{\partial^k w}{\partial x^k}\right].$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1x + C_2, C_1^{n+k}t + C_3),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. There are solutions of the forms

$$w(x,t) = u(\xi),\quad \xi = kx + \lambda t \quad \text{(traveling-wave solution)},$$

$$w(x,t) = z(\zeta),\quad \zeta = x^{n+k}/t \quad \text{(self-similar solution)}.$$

9. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^n}{\partial x^n}\left[f(w)\dfrac{\partial^k w}{\partial x^k}\right] + \big[xg(t) + h(t)\big]\dfrac{\partial w}{\partial x}.$

The transformation of the independent variables

$$z = xG(t) + \int h(t)G(t)\,dt, \quad \tau = \int G^{n+k}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 11.1.8.8:

$$\frac{\partial w}{\partial \tau} = \frac{\partial^n}{\partial z^n}\left[f(w)\frac{\partial^k w}{\partial z^k}\right].$$

10. $\dfrac{\partial w}{\partial t} = f(w)\Big(\dfrac{\partial^n w}{\partial x^n}\Big)^k + \big[xg(t) + h(t)\big]\dfrac{\partial w}{\partial x}.$

The transformation

$$z = xG(t) + \int h(t)G(t)\,dt, \quad \tau = \int G^{nk}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to the simpler equation

$$\frac{\partial w}{\partial \tau} = f(w)\left(\frac{\partial^n w}{\partial z^n}\right)^k.$$

It admits a traveling-wave solution and a self-similar solution.

11. $\dfrac{\partial w}{\partial t} = \dfrac{\partial^n}{\partial x^n}\big[f(x, w)\big] + \dfrac{g(t)}{f_w(x, w)} + h(x).$

Solution in implicit form:

$$f(x, w) = \int g(t)\,dt - \frac{1}{(n-1)!}\int (x-\xi)^{n-1}h(\xi)\,d\xi + C_{n-1}x^{n-1} + \cdots + C_1 x + C_0,$$

where C_0, C_1, ..., C_{n-1} are arbitrary constants.

11.2. General Form Equations Involving the First Derivative in t

11.2.1. Equations of the Form $\dfrac{\partial w}{\partial t} = F\Big(w, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n}\Big)$

Preliminary remarks. Consider the equation

$$\frac{\partial w}{\partial t} = F\left(w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\right). \tag{1}$$

1°. Suppose $w(x, t)$ is a solution of equation (1). Then the function $w(x + C_1, t + C_2)$, where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. In the general case, equation (1) admits a traveling-wave solution

$$w = w(\xi), \qquad \xi = kx + \lambda t, \tag{2}$$

where k and λ are arbitrary constants, and the function $w(\xi)$ is determined by the ordinary differential equation

$$F\big(w,\, kw'_\xi,\, \ldots,\, k^n w^{(n)}_\xi\big) - \lambda w'_\xi = 0.$$

Special cases of equation (1) that admit, apart from traveling-wave solutions (2), also other types of solution are presented in this subsection.

1. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-n} w(C_1 x + C_2, C_1^n t + C_3) + \sum_{k=0}^{n-1} A_k x^k,$$

where C_1, C_2, C_3, and the A_k are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = F(A)t + \frac{A}{n!}x^n + C_{n-1}x^{n-1} + \cdots + C_1 x + C_0,$$

where A, C_0, C_1, …, C_{n-1} are arbitrary constants.

3°. Solution linear in t:

$$w(x,t) = t\sum_{k=0}^{n-1} A_k x^k + \sum_{k=0}^{n-1} B_k x^k + \int_0^x \frac{(x-\xi)^{n-1}}{(n-1)!}\Phi\left(\sum_{k=0}^{n-1} A_k \xi^k\right) d\xi,$$

where the A_k and B_k are arbitrary constants and $\Phi(u)$ is the inverse of the function $F(u)$.

4°. Solution:

$$w(x,t) = A_1 t + \frac{1}{n!}A_2 x^n + \sum_{m=0}^{n-1} B_m x^m + U(z), \qquad z = kx + \lambda t,$$

where A_1, A_2, the B_m, k, and λ are arbitrary constants, and the function $U = U(z)$ is determined by the autonomous ordinary differential equation

$$A_1 + \lambda U'_z = F\bigl(A_2 + k^n U_z^{(n)}\bigr).$$

5°. Self-similar solution:

$$w(x,t) = t\,\Theta(\zeta), \quad \zeta = xt^{-1/n},$$

where the function $\Theta(\zeta)$ is determined by the ordinary differential equation

$$nF\bigl(\Theta_\zeta^{(n)}\bigr) + \zeta\Theta'_\zeta - n\Theta = 0.$$

2. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial w}{\partial x},\ \dfrac{\partial^2 w}{\partial x^2},\ \dots,\ \dfrac{\partial^n w}{\partial x^n}\right).$

Solution:

$$w(x,t) = At + B + \varphi(\xi), \qquad \xi = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants, and the function $\varphi(\xi)$ is determined by the autonomous ordinary differential equation

$$F\bigl(k\varphi'_\xi,\ k^2\varphi''_{\xi\xi},\ \dots,\ k^n\varphi_\xi^{(n)}\bigr) - \lambda\varphi'_\xi - A = 0.$$

3. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial w}{\partial x},\ \dfrac{\partial^2 w}{\partial x^2},\ \dots,\ \dfrac{\partial^n w}{\partial x^n}\right) + aw.$

This is a special case of equation 11.2.2.1 with $g(t) = a$ and $F_t = 0$.

4. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial w}{\partial x} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + aC_1 t + C_2, t + C_3) + C_1,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,t) = -\frac{x + C_1}{a\tau} + \frac{1}{\tau}\int \tau F\Big(-\frac{1}{a\tau}, 0, \ldots, 0\Big)\,d\tau, \qquad \tau = t + C_2.$$

3°. Solution:

$$w(x,t) = U(\zeta) + 2C_1 t, \qquad \zeta = x + aC_1 t^2 + C_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(\zeta)$ is determined by the autonomous ordinary differential equation

$$F\big(U'_\zeta, U''_{\zeta\zeta}, \ldots, U^{(n)}_\zeta\big) + aUU'_\zeta = C_2 U'_\zeta + 2C_1.$$

In the special case $C_1 = 0$, we have a traveling-wave solution.

5. $\dfrac{\partial w}{\partial t} = aw\dfrac{\partial w}{\partial x} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + bw.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + aC_1 e^{bt} + C_2, t + C_3) + C_1 b e^{bt},$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. There is a degenerate solution linear in x:

$$w(x,t) = \varphi(t)x + \psi(t).$$

3°. Traveling-wave solution:

$$w = w(\xi), \qquad \xi = x + \lambda t,$$

where λ is an arbitrary constant and the function $w(\xi)$ is determined by the autonomous ordinary differential equation

$$F\big(w'_\xi, w''_{\xi\xi}, \ldots, w^{(n)}_\xi\big) + aww'_\xi - \lambda w'_\xi + bw = 0.$$

6. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-1} w(x + C_2, C_1 t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = t\varphi(\xi), \qquad \xi = kx + \lambda \ln|t|,$$

where k and λ are arbitrary constants, and the function $\varphi(\xi)$ is determined by the autonomous ordinary differential equation

$$F\left(\frac{k}{\varphi}\varphi'_\xi,\ \frac{k^2}{\varphi}\varphi''_{\xi\xi},\ \ldots,\ \frac{k^n}{\varphi}\varphi^{(n)}_\xi\right) = \lambda\varphi'_\xi + \varphi.$$

7. $\displaystyle \frac{\partial w}{\partial t} = wF\left(\frac{1}{w}\frac{\partial w}{\partial x}, \frac{1}{w}\frac{\partial^2 w}{\partial x^2}, \ldots, \frac{1}{w}\frac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = Ce^{\lambda t}\varphi(x),$$

where C and λ are arbitrary constants, and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$F\left(\frac{\varphi'_x}{\varphi}, \frac{\varphi''_{xx}}{\varphi}, \ldots, \frac{\varphi^{(n)}_x}{\varphi}\right) = \lambda.$$

This equation has particular solutions of the form $\varphi(x) = e^{\alpha x}$, where α is a root of the algebraic (or transcendental) equation $F\bigl(\alpha, \alpha^2, \ldots, \alpha^n\bigr) - \lambda = 0$.

3°. Solution:

$$w(x,t) = Ce^{\lambda t}\psi(\xi), \quad \xi = kx + \beta t$$

where C, k, λ, and β are arbitrary constants, and the function $\psi(\xi)$ is determined by the autonomous ordinary differential equation

$$\psi F\left(\frac{k}{\psi}\psi'_\xi, \frac{k^2}{\psi}\psi''_{\xi\xi}, \ldots, \frac{k^n}{\psi}\psi^{(n)}_\xi\right) = \beta\psi'_\xi + \lambda\psi.$$

This equation has particular solutions of the form $\psi(\xi) = e^{\mu\xi}$.

8. $\displaystyle \frac{\partial w}{\partial t} = w^\beta F\left(\frac{1}{w}\frac{\partial w}{\partial x}, \frac{1}{w}\frac{\partial^2 w}{\partial x^2}, \ldots, \frac{1}{w}\frac{\partial^n w}{\partial x^n}\right).$

For $\beta = 0$, see equation 11.2.1.6, and for $\beta = 1$, see 11.2.1.7.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, C_1^{\beta-1}t + C_3),$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \bigl[(1-\beta)At + B\bigr]^{\frac{1}{1-\beta}}\varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$\varphi^{\beta-1}F\left(\frac{\varphi'_x}{\varphi}, \frac{\varphi''_{xx}}{\varphi}, \ldots, \frac{\varphi^{(n)}_x}{\varphi}\right) = A.$$

3°. Solution:

$$w(z,t) = (t + C)^{\frac{1}{1-\beta}}\Theta(z), \quad z = kx + \lambda\ln(t + C),$$

where C, k, and λ are arbitrary constants, and the function $\Theta(z)$ is determined by the autonomous ordinary differential equation

$$\Theta^\beta F\left(k\frac{\Theta'_z}{\Theta}, k^2\frac{\Theta''_{zz}}{\Theta}, \ldots, k^n\frac{\Theta^{(n)}_z}{\Theta}\right) = \lambda\Theta'_z + \frac{1}{1-\beta}\Theta.$$

9. $\dfrac{\partial w}{\partial t} = e^{\beta w} F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1, C_2 t + C_3) + \frac{1}{\beta}\ln C_2,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = -\frac{1}{\beta}\ln(A\beta t + B) + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$e^{\beta\varphi} F\left(\varphi'_x, \varphi''_{xx}, \ldots, \varphi^{(n)}_x\right) + A = 0.$$

3°. Solution:

$$w(x,t) = -\frac{1}{\beta}\ln(t + C) + \Theta(\xi), \qquad \xi = kx + \lambda\ln(t + C),$$

where C, k, and λ are arbitrary constants, and the function $\Theta(\xi)$ is determined by the autonomous ordinary differential equation

$$e^{\beta\Theta} F\left(k\Theta'_\xi, k^2\Theta''_{\xi\xi}, \ldots, k^n\Theta^{(n)}_\xi\right) = \lambda\Theta'_\xi - \frac{1}{\beta}.$$

10. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n}\Big/\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 11.2.1.2.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-1} w(x + C_2, C_1 t + C_3) + C_4,$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = At + B + \varphi(\xi), \qquad \xi = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants, and the function $\varphi(\xi)$ is determined by the autonomous ordinary differential equation

$$F\left(k\varphi''_{\xi\xi}/\varphi'_\xi, \ldots, k^{n-1}\varphi^{(n)}_\xi/\varphi'_\xi\right) = \lambda\varphi'_\xi + A.$$

3°. Solution:

$$w(x,t) = (t + C_1)\Theta(z) + C_2, \qquad z = kx + \lambda\ln|t + C_1|,$$

where C_1, C_2, k, and λ are arbitrary constants, and the function $\Theta(z)$ is determined by the autonomous ordinary differential equation

$$F\left(k\Theta''_{zz}/\Theta'_z, \ldots, k^{n-1}\Theta^{(n)}_z/\Theta'_z\right) = \lambda\Theta'_z + \Theta.$$

11. $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x} F\left(\dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n}\Big/\dfrac{\partial w}{\partial x}\right).$

This is a special case of equation 11.2.1.2.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, t + C_3) + C_4,$$

where $C_1, \dots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = At + B + \varphi(z), \qquad z = kx + \lambda t,$$

where A, B, k, and λ are arbitrary constants, and the function $\varphi(z)$ is determined by the autonomous ordinary differential equation

$$k\varphi'_z F\bigl(k\varphi''_{zz}/\varphi'_z, \dots, k^{n-1}\varphi^{(n)}_z/\varphi'_z\bigr) = \lambda\varphi'_z + A.$$

3°. Solution:

$$w(x,t) = Ae^{\beta t}\Theta(\xi) + B, \qquad \xi = kx + \lambda t,$$

where A, B, k, β, and λ are arbitrary constants, and the function $\Theta(\xi)$ is determined by the autonomous ordinary differential equation

$$k\Theta'_\xi F\bigl(k\Theta''_{\xi\xi}/\Theta'_\xi, \dots, k^{n-1}\Theta^{(n)}_\xi/\Theta'_\xi\bigr) = \lambda\Theta'_\xi + \beta\Theta.$$

12. $\displaystyle \frac{\partial w}{\partial t} = \left(\frac{\partial w}{\partial x}\right)^{\beta} F\left(\frac{\partial^2 w}{\partial x^2}\Big/\frac{\partial w}{\partial x}, \dots, \frac{\partial^n w}{\partial x^n}\Big/\frac{\partial w}{\partial x}\right).$

This is a special case of equation 11.2.1.2. For $\beta = 0$ and $\beta = 1$ see equations 11.2.1.10 and 11.2.1.11.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, C_1^{\beta-1}t + C_3) + C_4,$$

where $C_1, \dots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution:

$$w(x,t) = \bigl[A(1-\beta)t + B\bigr]^{\frac{1}{1-\beta}}\varphi(x) + C,$$

where A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the autonomous ordinary differential equation

$$\bigl(\varphi'_x\bigr)^{\beta} F\bigl(\varphi''_{xx}/\varphi'_x, \dots, \varphi^{(n)}_x/\varphi'_x\bigr) = A\varphi.$$

3°. Solution:

$$w(x,t) = (t + A)^{\frac{1}{1-\beta}}\Theta(z) + B, \qquad z = kx + \lambda\ln(t + A),$$

where A, B, k, and λ are arbitrary constants, and the function $\Theta(z)$ is determined by the autonomous ordinary differential equation

$$k^{\beta}\bigl(\Theta'_z\bigr)^{\beta} F\bigl(k\Theta''_{zz}/\Theta'_z, \dots, k^{n-1}\Theta^{(n)}_z/\Theta'_z\bigr) = \lambda\Theta'_z + \frac{1}{1-\beta}\Theta.$$

11.2.2. Equations of the Form $\dfrac{\partial w}{\partial t} = F\left(t, w, \dfrac{\partial w}{\partial x}, \dots, \dfrac{\partial^n w}{\partial x^n}\right)$

1. $\displaystyle \frac{\partial w}{\partial t} = F\left(t, \frac{\partial w}{\partial x}, \frac{\partial^2 w}{\partial x^2}, \dots, \frac{\partial^n w}{\partial x^n}\right) + g(t)w.$

Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x,t) + C\exp\left[\int g(t)\,dt\right],$$

where C is an arbitrary constant, is also a solution of the equation.

2. $\dfrac{\partial w}{\partial t} = F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + aw\dfrac{\partial w}{\partial x} + g(t).$

The transformation

$$w = u(z,t) + \int_{t_0}^{t} g(\tau)\,d\tau, \qquad z = x + a\int_{t_0}^{t} (t-\tau)g(\tau)\,d\tau,$$

where t_0 is any, leads to a simpler equation of the form 11.2.1.4:

$$\frac{\partial u}{\partial t} = au\frac{\partial u}{\partial x} + F\left(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^n u}{\partial x^n}\right).$$

3. $\dfrac{\partial w}{\partial t} = F\left(t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + f(t)w\dfrac{\partial w}{\partial x} + g(t)w.$

Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w\bigl(x + C_1\psi(t) + C_2,\ t\bigr) + C_1\varphi(t),$$

where

$$\varphi(t) = \exp\left[\int g(t)\,dt\right], \quad \psi(t) = \int f(t)\varphi(t)\,dt, \quad C_1 \text{ and } C_2 \text{ are arbitrary constants,}$$

is also a solution of the equation.

4. $\dfrac{\partial w}{\partial t} = wF\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \ldots, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, t),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = A\exp\left[\lambda x + \int F(t, \lambda, \ldots, \lambda^n)\,dt\right],$$

where A and λ are arbitrary constants.

5. $\dfrac{\partial w}{\partial t} = wF\left(t, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, t),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solutions:

$$w(x,t) = A\exp\left[\lambda x + \int F\bigl(t, \lambda^2, \ldots, \lambda^{2n}\bigr)\,dt\right],$$

$$w(x,t) = \bigl[A\cosh(\lambda x) + B\sinh(\lambda x)\bigr]\exp\left[\int F\bigl(t, \lambda^2, \ldots, \lambda^{2n}\bigr)\,dt\right],$$

$$w(x,t) = \bigl[A\cos(\lambda x) + B\sin(\lambda x)\bigr]\exp\left[\int F\bigl(t, -\lambda^2, \ldots, (-1)^n\lambda^{2n}\bigr)\,dt\right],$$

where A, B, and λ are arbitrary constants.

6. $\dfrac{\partial w}{\partial t} = f(t)w^{\beta}\Phi\left(\dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \dots,\ \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right) + g(t)w.$

The transformation

$$w(x,t) = G(t)u(x,\tau), \quad \tau = \int f(t)G^{\beta-1}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 11.2.1.8:

$$\frac{\partial u}{\partial \tau} = u^{\beta}\Phi\left(\frac{1}{u}\frac{\partial u}{\partial x},\ \frac{1}{u}\frac{\partial^2 u}{\partial x^2},\ \dots,\ \frac{1}{u}\frac{\partial^n u}{\partial x^n}\right),$$

which has, for instance, a traveling-wave solution $u = u(ax + b\tau)$ and a multiplicative solution of the form $u = \varphi(x)\psi(\tau)$.

7. $\dfrac{\partial w}{\partial t} = f(t)e^{\beta w}\Phi\left(\dfrac{\partial w}{\partial x},\ \dfrac{\partial^2 w}{\partial x^2},\ \dots,\ \dfrac{\partial^n w}{\partial x^n}\right) + g(t).$

The transformation

$$w(x,t) = u(x,\tau) + G(t), \quad \tau = \int f(t)\exp\bigl[\beta G(t)\bigr]\,dt, \qquad G(t) = \int g(t)\,dt,$$

leads to a simpler equation of the form 11.2.1.9:

$$\frac{\partial u}{\partial \tau} = e^{\beta u}\Phi\left(\frac{\partial u}{\partial x},\ \frac{\partial^2 u}{\partial x^2},\ \dots,\ \frac{\partial^n u}{\partial x^n}\right),$$

which has, for instance, a traveling-wave solution $u = u(ax + b\tau)$ and an additive separable solution of the form $u = \varphi(x) + \psi(\tau)$.

8. $\dfrac{\partial w}{\partial t} = f(t)\Phi\left(w,\ \dfrac{\partial w}{\partial x},\ \dfrac{\partial^2 w}{\partial x^2},\ \dots,\ \dfrac{\partial^n w}{\partial x^n}\right) + g(t)\dfrac{\partial w}{\partial x}.$

The transformation

$$w = u(z,\tau), \qquad z = x + \int g(t)\,dt, \qquad \tau = \int f(t)\,dt,$$

leads to the simpler equation

$$\frac{\partial u}{\partial \tau} = \Phi\left(u,\ \frac{\partial u}{\partial z},\ \frac{\partial^2 u}{\partial z^2},\ \dots,\ \frac{\partial^n u}{\partial z^n}\right),$$

which has a traveling-wave solution $u = u(kz + \lambda\tau)$.

9. $\dfrac{\partial w}{\partial t} = w\Phi_0\left(t,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dots,\ \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right) + \displaystyle\sum_{k=1}^{m}\dfrac{\partial^k w}{\partial x^k}\Phi_k\left(t,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dots,\ \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right).$

The equation has a multiplicative solution of the form

$$w(x,t) = Ae^{\lambda x}\Theta(t),$$

where A and λ are arbitrary constants.

10. $\dfrac{\partial w}{\partial t} = w\Phi_0\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \dots,\ \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right) + \displaystyle\sum_{k=1}^{m}\dfrac{\partial^{2k} w}{\partial x^{2k}}\Phi_k\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \dots,\ \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right).$

The equation has multiplicative solutions of the following forms:

$$w(x,t) = Ae^{\lambda x}\Theta_1(t),$$
$$w(x,t) = \bigl[A\cosh(\lambda x) + B\sinh(\lambda x)\bigr]\Theta_1(t),$$
$$w(x,t) = \bigl[A\cos(\lambda x) + B\sin(\lambda x)\bigr]\Theta_2(t),$$

where A, B, and λ are arbitrary constants.

11.2.3. Equations of the Form $\dfrac{\partial w}{\partial t} = F\Big(x, w, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n}\Big)$

1. $\dfrac{\partial w}{\partial t} = F\Big(x,\, \dfrac{\partial^2 w}{\partial x^2},\, \ldots,\, \dfrac{\partial^n w}{\partial x^n}\Big).$

Generalized separable solution linear in t:

$$w(x,t) = Axt + Bt + C + \varphi(x),$$

where A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\big(x,\, \varphi''_{xx},\, \ldots,\, \varphi^{(n)}_x\big) = Ax + B.$$

2. $\dfrac{\partial w}{\partial t} = F\Big(x,\, \dfrac{\partial w}{\partial x},\, \dfrac{\partial^2 w}{\partial x^2},\, \ldots,\, \dfrac{\partial^n w}{\partial x^n}\Big).$

Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\big(x,\, \varphi'_x,\, \varphi''_{xx},\, \ldots,\, \varphi^{(n)}_x\big) = A.$$

3. $\dfrac{\partial w}{\partial t} = F\Big(\dfrac{\partial w}{\partial x},\, x\dfrac{\partial^2 w}{\partial x^2},\, \ldots,\, x^{n-1}\dfrac{\partial^n w}{\partial x^n}\Big).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-1} w(C_1 x, C_1 t + C_2) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\big(\varphi'_x,\, x\varphi''_{xx},\, \ldots,\, x^{n-1}\varphi^{(n)}_x\big) = A.$$

3°. Solution:

$$w(x,t) = tU(z) + C, \qquad z = x/t,$$

where C is an arbitrary constant and the function $U(z)$ is determined by the ordinary differential equation

$$F\big(U'_z,\, zU''_{zz},\, \ldots,\, z^{n-1}U^{(n)}_z\big) + zU'_z - U = 0.$$

4. $\dfrac{\partial w}{\partial t} = ax\dfrac{\partial w}{\partial x} + F\Big(w,\, \dfrac{\partial w}{\partial x},\, \dfrac{\partial^2 w}{\partial x^2},\, \ldots,\, \dfrac{\partial^n w}{\partial x^n}\Big).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1 e^{-at}, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = w(z), \qquad z = x + Ce^{-at},$$

where C is an arbitrary constant and the function $w(z)$ is determined by the ordinary differential equation

$$F\big(w, w'_z, w''_{zz}, \ldots, w^{(n)}_z\big) + azw'_z = 0.$$

5. $\dfrac{\partial w}{\partial t} = F\left(w,\ x\dfrac{\partial w}{\partial x},\ x^2\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ x^n\dfrac{\partial^n w}{\partial x^n}\right).$

The substitution $x = \pm e^z$ leads to an equation of the form 11.2.1.2.

6. $\dfrac{\partial w}{\partial t} = x^k F\left(w,\ x\dfrac{\partial w}{\partial x},\ \ldots,\ x^n\dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(C_1 x, C_1^{-k} t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Self-similar solution:

$$w(x,t) = w(z), \quad z = x t^{1/k},$$

where the function $w(z)$ is determined by the ordinary differential equation

$$k z^{k-1} F\bigl(w,\ z w'_z,\ \ldots,\ z^n w_z^{(n)}\bigr) - w'_z = 0.$$

7. $\dfrac{\partial w}{\partial t} = x^k F\left(w,\ x\dfrac{\partial w}{\partial x},\ \ldots,\ x^n\dfrac{\partial^n w}{\partial x^n}\right) + ax\dfrac{\partial w}{\partial x}.$

Passing to the new independent variables

$$z = x e^{at}, \quad \tau = \frac{1}{ak}\bigl(1 - e^{-akt}\bigr),$$

we obtain an equation of the form 11.2.3.6:

$$\frac{\partial w}{\partial \tau} = z^k F\left(w,\ z\frac{\partial w}{\partial z},\ \ldots,\ z^n\frac{\partial^n w}{\partial z^n}\right).$$

8. $\dfrac{\partial w}{\partial t} = e^{\lambda x} F\left(w,\ \dfrac{\partial w}{\partial x},\ \ldots,\ \dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1, e^{-\lambda C_1} t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w(x,t) = w(z), \quad z = \lambda x + \ln t,$$

where the function $w(z)$ is determined by the ordinary differential equation

$$e^z F\bigl(w,\ \lambda w'_z,\ \ldots,\ \lambda^n w_z^{(n)}\bigr) - w'_z = 0.$$

9. $\dfrac{\partial w}{\partial t} = wF\left(x,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x, t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = A e^{\mu t} \varphi(x),$$

where A and μ are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$F\bigl(x,\ \varphi'_x/\varphi,\ \varphi''_{xx}/\varphi,\ \ldots,\ \varphi_x^{(n)}/\varphi\bigr) = \mu.$$

10. $\dfrac{\partial w}{\partial t} = w^{\beta} F\left(x,\, \dfrac{1}{w}\dfrac{\partial w}{\partial x},\, \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\, \ldots,\, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right).$

For $\beta = 1$, see equation 11.2.3.9.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x, C_1^{\beta-1} t + C_2),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \bigl[(1-\beta)At + B\bigr]^{\frac{1}{1-\beta}} \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$\varphi^{\beta-1} F\bigl(x,\, \varphi'_x/\varphi,\, \varphi''_{xx}/\varphi,\, \ldots,\, \varphi_x^{(n)}/\varphi\bigr) = A.$$

11. $\dfrac{\partial w}{\partial t} = e^{\beta w} F\left(x,\, \dfrac{\partial w}{\partial x},\, \dfrac{\partial^2 w}{\partial x^2},\, \ldots,\, \dfrac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x, C_1 t + C_2) + \frac{1}{\beta} \ln C_1,$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = -\frac{1}{\beta} \ln(A\beta t + B) + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$e^{\beta\varphi} F\bigl(x,\, \varphi'_x,\, \varphi''_{xx},\, \ldots,\, \varphi_x^{(n)}\bigr) + A = 0.$$

12. $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x} F\left(x,\, \dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x},\, \ldots,\, \dfrac{\partial^n w}{\partial x^n}\Big/\dfrac{\partial w}{\partial x}\right).$

1°. Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$\varphi'_x F\bigl(x,\, \varphi''_{xx}/\varphi'_x,\, \ldots,\, \varphi_x^{(n)}/\varphi'_x\bigr) = A.$$

2°. Generalized separable solution:

$$w(x,t) = Ae^{\mu t}\Theta(x) + B,$$

where A, B, and μ are arbitrary constants, and the function $\Theta(x)$ is determined by the ordinary differential equation

$$\Theta'_x F\bigl(x,\, \Theta''_{xx}/\Theta'_x\, \ldots,\, \Theta_x^{(n)}/\Theta'_x\bigr) = \mu\Theta.$$

13. $\dfrac{\partial w}{\partial t} = \left(\dfrac{\partial w}{\partial x}\right)^{\beta} F\left(x,\ \dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x},\ \ldots,\ \dfrac{\partial^n w}{\partial x^n}\Big/\dfrac{\partial w}{\partial x}\right).$

For $\beta = 1$, see equation 11.2.3.12.

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x, C_1^{\beta-1} t + C_2) + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Additive separable solution:

$$w(x,t) = At + B + \varphi(x),$$

where A and B are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$\left(\varphi'_x\right)^{\beta} F\left(x,\ \varphi''_{xx}/\varphi'_x,\ \ldots,\ \varphi^{(n)}_x/\varphi'_x\right) = A.$$

3°. Generalized separable solution:

$$w(x,t) = \left[A(1-\beta)t + C_1\right]^{\frac{1}{1-\beta}} \left[\Theta(x) + B\right] + C_2,$$

where A, B, C_1, and C_2 are arbitrary constants, and the function $\Theta(x)$ is determined by the ordinary differential equation

$$\left(\Theta'_x\right)^{\beta} F\left(x,\ \Theta''_{xx}/\Theta'_x\ \ldots,\ \Theta^{(n)}_x/\Theta'_x\right) = A\Theta + AB.$$

11.2.4. Equations of the Form $\dfrac{\partial w}{\partial t} = F\left(x, t, w, \dfrac{\partial w}{\partial x}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right)$

1. $\dfrac{\partial w}{\partial t} = F\left(x, t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + g(t)w.$

Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x,t) + C \exp\left[\int g(t)\,dt\right],$$

where C are arbitrary constants, is also a solution of the equation.

2. $\dfrac{\partial w}{\partial t} = F\left(ax + bt,\ w,\ \dfrac{\partial w}{\partial x},\ \ldots,\ \dfrac{\partial^n w}{\partial x^n}\right).$

Solution:

$$w = w(\xi), \quad \xi = ax + bt,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$F\left(\xi,\ w,\ aw'_\xi,\ \ldots,\ a^n w^{(n)}_\xi\right) - bw'_\xi = 0.$$

3. $\dfrac{\partial w}{\partial t} = F\left(ax + bt,\ \dfrac{\partial w}{\partial x},\ \ldots,\ \dfrac{\partial^n w}{\partial x^n}\right).$

Solution:

$$w = \varphi(\xi) + Ct, \quad \xi = ax + bt,$$

where C is an arbitrary constant and the function $\varphi(\xi)$ is determined by the ordinary differential equation

$$F\left(\xi,\ a\varphi'_\xi,\ \ldots,\ a^n \varphi^{(n)}_\xi\right) - b\varphi'_\xi - C = 0,$$

whose order can be reduced with the substitution $U(\xi) = \varphi'_\xi$.

4. $\displaystyle \frac{\partial w}{\partial t} = f(t)x^k \Phi\left(w,\ x\frac{\partial w}{\partial x},\ \ldots,\ x^n\frac{\partial^n w}{\partial x^n}\right) + xg(t)\frac{\partial w}{\partial x}.$

Passing to the new independent variables

$$z = xG(t), \quad \tau = \int f(t)G^{-k}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

one arrives at a simpler equation of the form 11.2.3.6:

$$\frac{\partial w}{\partial \tau} = z^k \Phi\left(w,\ z\frac{\partial w}{\partial z},\ \ldots,\ z^n\frac{\partial^n w}{\partial z^n}\right).$$

5. $\displaystyle \frac{\partial w}{\partial t} = w\Phi\left(t,\ \frac{1}{w}\frac{\partial^2 w}{\partial x^2},\ \ldots,\ \frac{1}{w}\frac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)e^{\lambda x}.$

Generalized separable solution:

$$w(x,t) = e^{\lambda x}E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + Be^{-\lambda x}E(t),$$
$$E(t) = \exp\left[\int \Phi\bigl(t, \lambda^2, \ldots, \lambda^{2n}\bigr)\,dt\right],$$

where A and B are arbitrary constants.

6. $\displaystyle \frac{\partial w}{\partial t} = w\Phi\left(t,\ \frac{1}{w}\frac{\partial^2 w}{\partial x^2},\ \ldots,\ \frac{1}{w}\frac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)e^{\lambda x} + g(t)e^{-\lambda x}.$

Generalized separable solution:

$$w(x,t) = e^{\lambda x}E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + e^{-\lambda x}E(t)\left[B + \int \frac{g(t)}{E(t)}\,dt\right],$$
$$E(t) = \exp\left[\int \Phi\bigl(t, \lambda^2, \ldots, \lambda^{2n}\bigr)\,dt\right],$$

where A and B are arbitrary constants.

7. $\displaystyle \frac{\partial w}{\partial t} = w\Phi\left(t,\ \frac{1}{w}\frac{\partial^2 w}{\partial x^2},\ \ldots,\ \frac{1}{w}\frac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)\cosh(\lambda x) + g(t)\sinh(\lambda x).$

Generalized separable solution:

$$w(x,t) = \cosh(\lambda x)E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + \sinh(\lambda x)E(t)\left[B + \int \frac{g(t)}{E(t)}\,dt\right],$$
$$E(t) = \exp\left[\int \Phi\bigl(t, \lambda^2, \ldots, \lambda^{2n}\bigr)\,dt\right],$$

where A and B are arbitrary constants.

8. $\displaystyle \frac{\partial w}{\partial t} = w\Phi\left(t,\ \frac{1}{w}\frac{\partial^2 w}{\partial x^2},\ \ldots,\ \frac{1}{w}\frac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)\cos(\lambda x).$

Generalized separable solution:

$$w(x,t) = \cos(\lambda x)E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + B\sin(\lambda x)E(t),$$
$$E(t) = \exp\left[\int \Phi\bigl(t, -\lambda^2, \ldots, (-1)^n\lambda^{2n}\bigr)\,dt\right],$$

where A and B are arbitrary constants.

9. $\dfrac{\partial w}{\partial t} = w\Phi\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ \dfrac{1}{w}\dfrac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)\cos(\lambda x) + g(t)\sin(\lambda x).$

Generalized separable solution:

$$w(x,t) = \cos(\lambda x)E(t)\left[A + \int \frac{f(t)}{E(t)}\,dt\right] + \sin(\lambda x)E(t)\left[B + \int \frac{g(t)}{E(t)}\,dt\right],$$
$$E(t) = \exp\left[\int \Phi\bigl(t, -\lambda^2, \ldots, (-1)^n\lambda^{2n}\bigr)\,dt\right],$$

where A and B are arbitrary constants.

10. $\dfrac{\partial w}{\partial t} = f(t)w^\beta\Phi\left(x,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \ldots,\ \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right) + g(t)w.$

The transformation

$$w(x,t) = G(t)u(x,\tau), \quad \tau = \int f(t)G^{\beta-1}(t)\,dt, \qquad G(t) = \exp\left[\int g(t)\,dt\right],$$

leads to a simpler equation of the form 11.2.3.10:

$$\frac{\partial u}{\partial \tau} = u^\beta\Phi\left(x,\ \frac{1}{u}\frac{\partial u}{\partial x},\ \frac{1}{u}\frac{\partial^2 u}{\partial x^2},\ \frac{1}{u}\frac{\partial^n u}{\partial x^n}\right),$$

which has a multiplicative separable solution $u = \varphi(x)\psi(\tau)$.

11. $\dfrac{\partial w}{\partial t} = f(t)\left(\dfrac{\partial w}{\partial x}\right)^k\Phi\left(x,\ \dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x},\ \ldots,\ \dfrac{\partial^n w}{\partial x^n}\Big/\dfrac{\partial w}{\partial x}\right) + g(t)w + h(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations (C is an arbitrary constant)

$$\varphi_t' = Af(t)\varphi^k + g(t)\varphi, \tag{1}$$
$$\psi_t' = g(t)\psi + Bf(t)\varphi^k + h(t), \tag{2}$$

and the function $\Theta(x)$ satisfies the nth-order ordinary differential equation

$$\bigl(\Theta_x'\bigr)^k\Phi\bigl(x,\ \Theta_{xx}''/\Theta_x',\ \ldots,\ \Theta_x^{(n)}/\Theta_x'\bigr) = A\Theta + B.$$

The general solution of system (1), (2) is given by

$$\varphi(t) = G(t)\left[C + A(1-k)\int f(t)G^{k-1}(t)\,dt\right]^{\frac{1}{1-k}}, \quad G(t) = \exp\left[\int g(t)\,dt\right],$$
$$\psi(t) = DG(t) + G(t)\int\bigl[Bf(t)\varphi^k(t) + h(t)\bigr]\frac{dt}{G(t)},$$

where A, B, C, and D are arbitrary constants.

12. $\dfrac{\partial w}{\partial t} = \bigl[f_1(t)w + f_0(t)\bigr]\left(\dfrac{\partial w}{\partial x}\right)^k\Phi\left(x,\ \dfrac{\partial^2 w}{\partial x^2}\Big/\dfrac{\partial w}{\partial x},\ldots,\ \dfrac{\partial^n w}{\partial x^n}\Big/\dfrac{\partial w}{\partial x}\right) + g_1(t)w + g_0(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of first-order ordinary differential equations (C is an arbitrary constant):

$$\varphi_t' = Cf_1(t)\varphi^{k+1} + g_1(t)\varphi, \tag{1}$$
$$\psi_t' = \big[Cf_1(t)\varphi^k + g_1(t)\big]\psi + Cf_0(t)\varphi^k + g_0(t), \tag{2}$$

and the function $\Theta(x)$ satisfies the nth-order ordinary differential equation

$$\big(\Theta_x'\big)^k\Phi\big(x,\,\Theta_{xx}''/\Theta_x',\,\dots,\,\Theta_x^{(n)}/\Theta_x'\big) = C.$$

The general solution of system (1), (2) is given by

$$\varphi(t) = G(t)\left[A - kC\int f_1(t)G^k(t)\,dt\right]^{-1/k}, \quad G(t) = \exp\left[\int g_1(t)\,dt\right],$$
$$\psi(t) = B\varphi(t) + \varphi(t)\int\big[Cf_0(t)\varphi^k(t) + g_0(t)\big]\frac{dt}{\varphi(t)},$$

where A, B, and C are arbitrary constants.

13. $\displaystyle \frac{\partial w}{\partial t} = f(t)e^{\beta w}\Phi\left(x,\,\frac{\partial w}{\partial x},\,\frac{\partial^2 w}{\partial x^2},\,\dots,\,\frac{\partial^n w}{\partial x^n}\right) + g(t).$

The transformation

$$w(x,t) = u(x,\tau) + G(t), \quad \tau = \int f(t)\exp\big[\beta G(t)\big]\,dt, \qquad G(t) = \int g(t)\,dt,$$

leads to a simpler equation of the form 11.2.3.11:

$$\frac{\partial u}{\partial \tau} = e^{\beta u}\Phi\left(x,\,\frac{\partial u}{\partial x},\,\frac{\partial^2 u}{\partial x^2},\,\dots,\,\frac{\partial^n u}{\partial x^n}\right),$$

which has a solution in the additive separable form $u = \varphi(x) + \psi(\tau)$.

14. $\displaystyle \frac{\partial w}{\partial t} = w\Phi_0\left(t,\,\frac{1}{w}\frac{\partial^2 w}{\partial x^2},\,\dots,\,\frac{1}{w}\frac{\partial^{2n} w}{\partial x^{2n}}\right) + \sum_{k=1}^m \frac{\partial^{2k} w}{\partial x^{2k}}\Phi_k\left(t,\,\frac{1}{w}\frac{\partial^2 w}{\partial x^2},\,\dots,\,\frac{1}{w}\frac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)e^{\lambda x} + g(t)e^{-\lambda x}.$

There is a generalized separable solution of the form

$$w(x,t) = e^{\lambda x}\varphi(t) + e^{-\lambda x}\psi(t).$$

15. $\displaystyle \frac{\partial w}{\partial t} = w\Phi_0\left(t,\,\frac{1}{w}\frac{\partial^2 w}{\partial x^2},\,\dots,\,\frac{1}{w}\frac{\partial^{2n} w}{\partial x^{2n}}\right) + \sum_{k=1}^m \frac{\partial^{2k} w}{\partial x^{2k}}\Phi_k\left(t,\,\frac{1}{w}\frac{\partial^2 w}{\partial x^2},\,\dots,\,\frac{1}{w}\frac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)\cosh(\lambda x) + g(t)\sinh(\lambda x).$

There is a generalized separable solution of the form

$$w(x,t) = \cosh(\lambda x)\varphi(t) + \sinh(\lambda x)\psi(t).$$

16. $\displaystyle \frac{\partial w}{\partial t} = w\Phi_0\left(t,\,\frac{1}{w}\frac{\partial^2 w}{\partial x^2},\,\dots,\,\frac{1}{w}\frac{\partial^{2n} w}{\partial x^{2n}}\right) + \sum_{k=1}^m \frac{\partial^{2k} w}{\partial x^{2k}}\Phi_k\left(t,\,\frac{1}{w}\frac{\partial^2 w}{\partial x^2},\,\dots,\,\frac{1}{w}\frac{\partial^{2n} w}{\partial x^{2n}}\right) + f(t)\cos(\lambda x) + g(t)\sin(\lambda x).$

There is a generalized separable solution of the form

$$w(x,t) = \cos(\lambda x)\varphi(t) + \sin(\lambda x)\psi(t).$$

17. $\dfrac{\partial w}{\partial t} = wF(t, \zeta_0, \zeta_1, \ldots, \zeta_n), \qquad \zeta_k = \sum_{i=k}^{n} \dfrac{(-1)^{i+k}}{k!\,(i-k)!} x^{i-k} \dfrac{\partial^i w}{\partial x^i}, \quad k = 0, 1, \ldots, n.$

Multiplicative separable solution:

$$w(x,t) = (C_0 + C_1 x + \cdots + C_n x^n)\varphi(t),$$

where C_0, C_1, …, C_n are arbitrary constants, and the function $\varphi = \varphi(t)$ is determined by the ordinary differential equation

$$\varphi'_t = \varphi F(t, C_0\varphi, C_1\varphi, \ldots, C_n\varphi).$$

⊙ *Reference*: Ph. W. Doyle (1996), the case $\partial_t F \equiv 0$ was treated.

11.3. Equations Involving the Second Derivative in t

11.3.1. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f(x,t,w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f(x + bt, w).$

Solution:

$$w = w(\xi), \qquad \xi = x + bt,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$aw_\xi^{(n)} - b^2 w''_{\xi\xi} + f(\xi, w) = 0.$$

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + bw\ln w + f(t)w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - \bigl[b\ln\varphi + f(t) + C\bigr]\varphi = 0,$$
$$a\psi_x^{(n)} + \bigl(b\ln\psi - C\bigr)\psi = 0,$$

where C is an arbitrary constant.

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + bw\ln w + \bigl[f(x) + g(t)\bigr]w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - \bigl[b\ln\varphi + g(t) + C\bigr]\varphi = 0,$$
$$a\psi_x^{(n)} + \bigl[b\ln\psi + f(x) - C\bigr]\psi = 0,$$

where C is an arbitrary constant.

4. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f(x)w\ln w + \bigl[bf(x)t + g(x)\bigr]w.$

Multiplicative separable solution:

$$w(x,t) = e^{-bt}\varphi(x),$$

where the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f(x)\varphi\ln\varphi + \bigl[g(x) - b^2\bigr]\varphi = 0.$$

11.3.2. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + F\Big(x, t, w, \dfrac{\partial w}{\partial x}\Big)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f(x)\dfrac{\partial w}{\partial x} + bw\ln w + \big[g(x) + h(t)\big]w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(t)\psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - \big[b\ln\varphi + h(t) + C\big]\varphi = 0,$$
$$a\psi_x^{(n)} + f(x)\psi'_x + \big[b\ln\psi + g(x) - C\big]\psi = 0,$$

where C is an arbitrary constant.

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + b\Big(\dfrac{\partial w}{\partial x}\Big)^2 + f(t).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + C_1, t) + C_2 t + C_3,$$

where C_1, C_2, and C_3 are arbitrary constants, is also a solution of the equation.

2°. Solution:

$$w(x,t) = C_1 t^2 + C_2 t + \int_{t_0}^{t}(t-\tau)f(\tau)\,d\tau + \theta(z), \qquad z = x + \lambda t,$$

where C_1, C_2, t_0, and λ are arbitrary constants, and the function $\theta(z)$ is determined by the autonomous ordinary differential equation

$$a\theta_z^{(n)} - \lambda^2\theta''_{zz} + b\big(\theta'_z\big)^2 - 2C_1 = 0.$$

3°. There is a degenerate solution quadratic in x:

$$w(x,t) = \varphi(t)x^2 + \psi(t)x + \chi(t).$$

4°. The substitution $w = U(x,t) + \displaystyle\int_0^t (t-\tau)f(\tau)\,d\tau$ leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^n U}{\partial x^n} + b\Big(\frac{\partial U}{\partial x}\Big)^2,$$

which admits a self-similar solution of the form $U = t^{\frac{2(2-n)}{n}}u(\zeta)$, where $\zeta = xt^{-\frac{2}{n}}$.

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + b\Big(\dfrac{\partial w}{\partial x}\Big)^2 + cw + f(t).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x + C_1, t) + C_2\cosh(kt) + C_3\sinh(kt) \quad \text{if } c = k^2 > 0,$$
$$w_2 = w(x + C_1, t) + C_2\cos(kt) + C_3\sin(kt) \quad \text{if } c = -k^2 < 0,$$

where C_1, C_2, and C_3 are arbitrary constants, are also solutions of the equation.

2°. Solution:

$$w(x,t) = \varphi(t) + \psi(z), \quad z = x + \lambda t,$$

where λ is an arbitrary constant and the functions $\varphi(t)$ and $\psi(z)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - c\varphi - f(t) = 0,$$
$$a\psi^{(n)}_z - \lambda^2\psi''_{zz} + b\big(\psi'_z\big)^2 + c\psi = 0.$$

The general solution of the first equation is expressed as

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t f(\tau)\sinh\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad c = k^2 > 0,$$
$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t f(\tau)\sin\big[k(t-\tau)\big]\,d\tau \quad \text{if} \quad c = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

3°. There is a degenerate solution quadratic in x:

$$w(x,t) = \varphi_2(t)x^2 + \varphi_1(t)x + \varphi_0(t).$$

4°. The substitution $w = U(x,t) + \varphi(t)$, where the function $\varphi(t)$ is defined in Item 2°, leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^n U}{\partial x^n} + b\left(\frac{\partial U}{\partial x}\right)^2 + cU.$$

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^n w}{\partial x^n} + b\left(\frac{\partial w}{\partial x}\right)^2 + cw\frac{\partial w}{\partial x} + kw^2 + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t) + \psi(t)\exp(\lambda x),$$

where λ are roots of the quadratic equation $b\lambda^2 + c\lambda + k = 0$, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = k\varphi^2 + f(t)\varphi + g(t), \qquad (1)$$
$$\psi''_{tt} = \big[(c\lambda + 2k)\varphi + f(t) + a\lambda^n\big]\psi. \qquad (2)$$

In the special case $f(t) = \text{const}$ and $g(t) = \text{const}$, equation (1) has particular solutions of the form $\varphi = \text{const}$ and, due to autonomy, can be integrated by quadrature. Equation (2) is linear in ψ, and, hence, for $\varphi = \text{const}$, its general solution is expressed in terms of exponentials or sine and cosine.

5. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^n w}{\partial x^n} + f(x)\left(\frac{\partial w}{\partial x}\right)^2 + g(x) + h(t).$

1°. Additive separable solution:

$$w(x,t) = C_1t^2 + C_2t + \int_{t_0}^t (t-\tau)h(\tau)\,d\tau + \varphi(x).$$

Here, C_1, C_2, and t_0 are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi^{(n)}_x + f(x)\big(\varphi'_x\big)^2 + g(x) - 2C_1 = 0.$$

2°. The substitution $w = U(x,t) + \displaystyle\int_0^t (t-\tau)h(\tau)\,d\tau$ leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^n U}{\partial x^n} + f(x)\left(\frac{\partial U}{\partial x}\right)^2 + g(x).$$

6. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f(x)\left(\dfrac{\partial w}{\partial x}\right)^2 + bw + g(x) + h(t).$

Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x).$$

Here, the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} - b\varphi - h(t) = 0,$$
$$a\psi^{(n)}_x + f(x)(\psi'_x)^2 + b\psi + g(x) = 0.$$

The general solution of the first equation is given by

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t h(\tau)\sinh\bigl[k(t-\tau)\bigr]\,d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t h(\tau)\sin\bigl[k(t-\tau)\bigr]\,d\tau \qquad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

7. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^{2n} w}{\partial x^{2n}} + f(t)\left(\dfrac{\partial w}{\partial x}\right)^2 + bf(t)w^2 + g(t)w + h(t).$

1°. Generalized separable solutions involving exponentials of x:

$$w(x,t) = \varphi(t) + \psi(t)\exp\bigl(\pm x\sqrt{-b}\,\bigr), \qquad b < 0, \tag{1}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the following system of variable-coefficient second-order ordinary differential equations:

$$\varphi''_{tt} = bf\varphi^2 + g\varphi + h, \tag{2}$$
$$\psi''_{tt} = \bigl[2bf\varphi + g + (-1)^n ab^n\bigr]\psi; \tag{3}$$

the arguments of the functions f, g, and h are not specified.

In the special case of constant f, g, and h, equation (2) has particular solutions of the form $\varphi = \text{const}$. In this case, the general solution of equation (3) is expressed in terms of exponentials or sine and cosine.

2°. Generalized separable solution (generalizes the solutions of Item 1°):

$$w(x,t) = \varphi(t) + \psi(t)\bigl[A\exp\bigl(x\sqrt{-b}\,\bigr) + B\exp\bigl(-x\sqrt{-b}\,\bigr)\bigr], \qquad b < 0, \tag{4}$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the following system of variable-coefficient second-order ordinary differential equations:

$$\varphi''_{tt} = bf\bigl(\varphi^2 + 4AB\psi^2\bigr) + g\varphi + h, \tag{5}$$
$$\psi''_{tt} = \bigl[2bf\varphi + g + (-1)^n ab^n\bigr]\psi. \tag{6}$$

One can express φ via ψ in (6) and substitute the resulting expression into (5) to obtain a fourth-order nonlinear equation for ψ. For $f,\ g,\ h = \text{const}$, this equation is autonomous and its order can be reduced.

Note two special cases where solution (4) is expressed in terms of hyperbolic functions:

$$w(x,t) = \varphi(t) + \psi(t)\cosh\bigl(x\sqrt{-b}\,\bigr) \quad \text{if } A = \tfrac{1}{2},\ B = \tfrac{1}{2};$$
$$w(x,t) = \varphi(t) + \psi(t)\sinh\bigl(x\sqrt{-b}\,\bigr) \quad \text{if } A = \tfrac{1}{2},\ B = -\tfrac{1}{2}.$$

3°. Generalized separable solution involving trigonometric functions of x:

$$w(x,t) = \varphi(t) + \psi(t)\cos\bigl(x\sqrt{b} + c\bigr), \qquad b > 0,$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the following system of variable-coefficient second-order ordinary differential equations:

$$\varphi''_{tt} = bf\bigl(\varphi^2 + \psi^2\bigr) + g\varphi + h,$$
$$\psi''_{tt} = \bigl[2bf\varphi + g + (-1)^n ab^n\bigr]\psi.$$

⊙ *References*: V. A. Galaktionov (1995), A. D. Polyanin and V. F. Zaitsev (2002).

8. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f\left(x, \dfrac{\partial w}{\partial x}\right) + g(t).$

1°. Additive separable solution:

$$w(x,t) = C_1 t^2 + C_2 t + \int_{t_0}^{t} (t-\tau) g(\tau)\, d\tau + \varphi(x),$$

where C_1, C_2, and t_0 are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + f\left(x, \varphi_x'\right) - 2C_1 = 0,$$

whose order can be reduced with the substitution $u(x) = \varphi_x'$.

2°. The substitution $w = U(x,t) + \int_0^t (t-\tau) g(\tau)\, d\tau$ leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^n U}{\partial x^n} + f\left(x, \frac{\partial U}{\partial x}\right).$$

9. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f\left(x, \dfrac{\partial w}{\partial x}\right) + bw + g(t).$

1°. Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x),$$

where the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi_{tt}'' - b\varphi - g(t) = 0,$$
$$a\psi_x^{(n)} + f\left(x, \psi_x'\right) + b\psi = 0.$$

The general solution of the first equation is given by

$$\varphi(t) = C_1 \cosh(kt) + C_2 \sinh(kt) + \frac{1}{k}\int_0^t g(\tau) \sinh\bigl[k(t-\tau)\bigr]\, d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1 \cos(kt) + C_2 \sin(kt) + \frac{1}{k}\int_0^t g(\tau) \sin\bigl[k(t-\tau)\bigr]\, d\tau \quad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution $w = U(x,t) + \varphi(t)$, where the function $\varphi(t)$ is specified Item 1°, leads to the simpler equation

$$\frac{\partial^2 U}{\partial t^2} = a\frac{\partial^n U}{\partial x^n} + f\left(x, \frac{\partial U}{\partial x}\right) + bU.$$

10. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + wf\left(t, \dfrac{1}{w}\dfrac{\partial w}{\partial x}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(x + C_2, t),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = e^{\lambda x}\varphi(t),$$

where λ is an arbitrary constant and the function $\varphi(t)$ is determined by the second-order linear ordinary differential equation

$$\varphi_{tt}'' = \bigl[a\lambda^n + f(t,\lambda)\bigr]\varphi.$$

11.3.3. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + F\left(x, t, w, \dfrac{\partial w}{\partial x}, \dots, \dfrac{\partial^{n-1} w}{\partial x^{n-1}}\right)$

1. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + bw\dfrac{\partial^2 w}{\partial x^2} + c.$

1°. Traveling-wave solution:

$$w(x,t) = u(\xi), \qquad \xi = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $u = u(\xi)$ is determined by the autonomous ordinary differential equation

$$ak^n u_\xi^{(n)} + (bk^2 u - \lambda^2)u''_{\xi\xi} + c = 0.$$

2°. Solution:

$$w = U(z) + 4bC_1^2 t^2 + 4bC_1C_2 t, \qquad z = x + bC_1 t^2 + bC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$aU_z^{(n)} + bUU''_{zz} - b^2C_2^2 U''_{zz} - 2bC_1 U'_z = 8bC_1^2 - c.$$

3°. There is a degenerate solution quadratic in x:

$$w(x,t) = f_2(t)x^2 + f_1(t)x + f_0(t).$$

2. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + b\dfrac{\partial}{\partial x}\left(w\dfrac{\partial w}{\partial x}\right) + c.$

1°. Traveling-wave solution:

$$w(x,t) = u(\xi), \qquad \xi = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $u = u(\xi)$ is determined by the autonomous ordinary differential equation

$$ak^n u_\xi^{(n)} + bk^2(uu'_\xi)'_\xi - \lambda^2 u''_{\xi\xi} + c = 0.$$

2°. Solution:

$$w = U(z) + 4bC_1^2 t^2 + 4bC_1C_2 t, \qquad z = x + bC_1 t^2 + bC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$aU_z^{(n)} + b(UU'_z)'_z - b^2C_2^2 U''_{zz} - 2bC_1 U'_z = 8bC_1^2 - c.$$

3°. There is a degenerate solution quadratic in x:

$$w(x,t) = f_2(t)x^2 + f_1(t)x + f_0(t).$$

3. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + f(t)\sum\limits_{i,j=0}^{i,j<n} b_{ij}\dfrac{\partial^i w}{\partial x^i}\dfrac{\partial^j w}{\partial x^j} + \sum\limits_{k=0}^{n-1} g_k(t)\dfrac{\partial^k w}{\partial x^k} + h(t).$

Here, we adopt the notation $\dfrac{\partial^0 w}{\partial x^0} \equiv w$.

1°. In the general case, the equation has generalized separable solutions of the form

$$w(x,t) = \varphi(t) + \psi(t)\exp(\lambda x),$$

where λ is a root of the algebraic equation $\sum\limits_{i,j=0}^{i,j<n} b_{ij}\lambda^{i+j} = 0.$

2°. Let n be an even number and let all coefficients b_{ij} be zero for odd $i+j$. In this case, the original equation has also generalized separable solutions of the form

$$w(x,t) = \varphi_1(t) + \psi_1(t)\bigl[A\cosh(\lambda x) + B\sinh(\lambda x)\bigr],$$
$$w(x,t) = \varphi_2(t) + \psi_2(t)\bigl[A\cos(\lambda x) + B\sin(\lambda x)\bigr],$$

where A and B are arbitrary constants, the parameter λ is determined by solving algebraic equations, and the functions $\varphi_1(t)$, $\psi_1(t)$ and $\varphi_2(t)$, $\psi_2(t)$ are found from appropriate systems of first-order ordinary differential equations.

4. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^n w}{\partial x^n} + F\left(x,\ \frac{\partial w}{\partial x},\ \ldots,\ \frac{\partial^{n-1} w}{\partial x^{n-1}}\right) + g(t).$

1°. Additive separable solution:

$$w(x,t) = \tfrac{1}{2}At^2 + Bt + C + \int_0^t (t-\tau)g(\tau)\,d\tau + \varphi(x).$$

Here, A, B, and C are arbitrary constants, and the function $\varphi(x)$ is determined by the ordinary differential equation

$$a\varphi_x^{(n)} + F\bigl(x, \varphi_x', \ldots, \varphi_x^{(n-1)}\bigr) - A = 0,$$

whose order can be reduced with the substitution $U(x) = \varphi_x'$.

2°. The substitution

$$w = u(x,t) + \int_0^t (t-\tau)g(\tau)\,d\tau$$

leads to the simpler equation

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^n u}{\partial x^n} + F\left(x,\ \frac{\partial u}{\partial x},\ \ldots,\ \frac{\partial^{n-1} u}{\partial x^{n-1}}\right).$$

5. $\displaystyle \frac{\partial^2 w}{\partial t^2} = a\frac{\partial^n w}{\partial x^n} + F\left(x,\ \frac{\partial w}{\partial x},\ \ldots,\ \frac{\partial^{n-1} w}{\partial x^{n-1}}\right) + bw + g(t).$

1°. Additive separable solution:

$$w(x,t) = \varphi(t) + \psi(x).$$

Here, the functions $\varphi(t)$ and $\psi(x)$ are determined by the ordinary differential equations

$$\varphi_{tt}'' - b\varphi - g(t) = 0,$$
$$a\psi_x^{(n)} + F\bigl(x, \psi_x', \ldots, \psi_x^{(n-1)}\bigr) + b\psi = 0.$$

The general solution of the first equation is expressed as

$$\varphi(t) = C_1\cosh(kt) + C_2\sinh(kt) + \frac{1}{k}\int_0^t g(\tau)\sinh\bigl[k(t-\tau)\bigr]\,d\tau \quad \text{if} \quad b = k^2 > 0,$$
$$\varphi(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k}\int_0^t g(\tau)\sin\bigl[k(t-\tau)\bigr]\,d\tau \qquad \text{if} \quad b = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants.

2°. The substitution $w = u(x,t) + \varphi(t)$, where the function $\varphi(t)$ is specified in Item 1°, leads to the simpler equation

$$\frac{\partial^2 u}{\partial t^2} = a\frac{\partial^n u}{\partial x^n} + F\left(x,\ \frac{\partial u}{\partial x},\ \ldots,\ \frac{\partial^{n-1} u}{\partial x^{n-1}}\right) + bu.$$

6. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^n w}{\partial x^n} + wF\left(t,\ \dfrac{1}{w}\dfrac{\partial w}{\partial x},\ \dots,\ \dfrac{1}{w}\dfrac{\partial^{n-1} w}{\partial x^{n-1}}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = C_1 w(x + C_2, t),$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = e^{\lambda x}\varphi(t),$$

where λ is an arbitrary constant and the function $\varphi(t)$ is determined by the second-order linear ordinary differential equation

$$\varphi''_{tt} = \bigl[a\lambda^n + F(t, \lambda, \dots, \lambda^{n-1})\bigr]\varphi.$$

7. $\dfrac{\partial^2 w}{\partial t^2} = a\dfrac{\partial^{2n} w}{\partial x^{2n}} + wF\left(t,\ \dfrac{1}{w}\dfrac{\partial^2 w}{\partial x^2},\ \dots,\ \dfrac{1}{w}\dfrac{\partial^{2n-2} w}{\partial x^{2n-2}}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1 w(x + C_2, t),$$

where C_1 and C_2 are arbitrary constants, is also a solution of the equation.

2°. Multiplicative separable solution:

$$w(x,t) = \bigl[A\cosh(\lambda x) + B\sinh(\lambda x)\bigr]\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi(t)$ is determined by the second-order linear ordinary differential equation

$$\varphi''_{tt} = \Phi(t)\varphi, \qquad \Phi(t) = a\lambda^{2n} + F\bigl(t, \lambda^2, \dots, \lambda^{2n-2}\bigr).$$

3°. Multiplicative separable solution:

$$w(x,t) = \bigl[A\cos(\lambda x) + B\sin(\lambda x)\bigr]\varphi(t),$$

where A, B, and λ are arbitrary constants, and the function $\varphi(t)$ is determined by the second-order linear ordinary differential equation

$$\varphi''_{tt} = \Phi(t)\varphi, \qquad \Phi(t) = (-1)^n a\lambda^{2n} + F\bigl(t, -\lambda^2, \dots, (-1)^{n-1}\lambda^{2n-2}\bigr).$$

11.3.4. Equations of the Form $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^n w}{\partial x^n} + f(x,t,w)\dfrac{\partial w}{\partial x} + g(x,t,w)$

1. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^n w}{\partial x^n} + f(x)w + \displaystyle\sum_{k=0}^{n-1} b_k x^k.$

Generalized separable solution:

$$w(x,t) = \tfrac{1}{2}t^2\sum_{k=0}^{n-1} b_k x^k + t\sum_{k=0}^{n-1} A_k x^k + \sum_{k=0}^{n-1} B_k x^k - \frac{1}{a(n-1)!}\int_0^x (x-\xi)^{n-1} f(\xi)\,d\xi,$$

where A_0, A_1, $\dots$, A_{n-1} and B_0, B_1, $\dots$, B_{n-1} are arbitrary constants.

2. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^n w}{\partial x^n} + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\big(A_n x^n + \cdots + A_1 x\big) + \psi(t),$$

where $A_1, \dots, A_n$ are arbitrary constants, and the functions $\varphi(t)$ and $\psi(t)$ are determined by the system of ordinary differential equations

$$\varphi''_{tt} = A_n a n!\,\varphi^2 + f(t)\varphi,$$
$$\psi''_{tt} = A_n a n!\,\varphi\psi + f(t)\psi + g(t).$$

3. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^n w}{\partial x^n} + bw^2 + f(t)w + g(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the following system of second-order ordinary differential equations (C is an arbitrary constant):

$$\varphi''_{tt} = C\varphi^2 + b\varphi\psi + f(t)\varphi,$$
$$\psi''_{tt} = C\varphi\psi + b\psi^2 + f(t)\psi + g(t),$$

and the function $\Theta(x)$ satisfies the nth-order constant-coefficient linear ordinary differential equation

$$a\Theta^{(n)}_x + b\Theta = C.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^{2n} w}{\partial x^{2n}} - ak^{2n}w^2 + f(x)w + b_1 \sinh(kx) + b_2 \cosh(kx).$

Generalized separable solution quadratic in t:

$$w(x,t) = \tfrac{1}{2}(t+C)^2\big[b_1\sinh(kx) + b_2\cosh(kx)\big] + \varphi(x).$$

Here, C is an arbitrary constant and the function $\varphi(x)$ is found from the constant-coefficient linear nonhomogeneous ordinary differential equation

$$a\varphi^{(2n)}_x - ak^{2n}\varphi + f(x) = 0.$$

5. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^n w}{\partial x^n} + f(x)w\dfrac{\partial w}{\partial x} + g(t)w + h(t).$

Generalized separable solution:

$$w(x,t) = \varphi(t)\Theta(x) + \psi(t),$$

where the functions $\varphi(t)$, $\psi(t)$, and $\Theta(x)$ are determined by the ordinary differential equations

$$\varphi''_{tt} = C\varphi^2 + g(t)\varphi,$$
$$\psi''_{tt} = \big[C\varphi + g(t)\big]\psi + h(t),$$
$$a\Theta^{(n)}_x + f(x)\Theta'_x = C,$$

and C is an arbitrary constant.

6. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^n w}{\partial x^n} + f(x)w\dfrac{\partial w}{\partial x} + g(x)w^2 + h(t)w.$

Multiplicative separable solution:

$$w(x,t) = \varphi(x)\psi(t),$$

where the functions $\varphi(t)$ and $\psi(t)$ are determined by the ordinary differential equations

$$a\varphi^{(n)}_x + f(x)\varphi'_x + g(x)\varphi - C = 0,$$
$$\psi''_{tt} - C\psi^2 - h(t)\psi = 0,$$

and C is an arbitrary constant.

11.3.5. Equations of the Form $\frac{\partial^2 w}{\partial t^2} = F\left(x, t, w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\right)$

1. $\frac{\partial^2 w}{\partial t^2} = \frac{\partial^{n-1}}{\partial x^{n-1}}\left[f(w)\frac{\partial w}{\partial x}\right] - a^2\frac{f'(w)}{f^3(w)} + b.$

Functional separable solution in implicit form:

$$\int f(w)\,dw = at - \frac{b}{n!}x^n + C_{n-1}x^{n-1} + \cdots + C_1 x + C_0,$$

where C_0, C_1, …, C_{n-1} are arbitrary constants.

2. $\frac{\partial^2 w}{\partial t^2} = F\left(\frac{\partial^n w}{\partial x^n}\right).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = C_1^{-2n} w(C_1^2 x + C_2, C_1^n t + C_3) + \sum_{k=0}^{n-1}(A_k t + B_k)x^k,$$

where C_1, C_2, C_3, the A_k, and the B_k are arbitrary constants, is also a solution of the equation.

2°. Generalized separable solution in the form of an nth-degree polynomial in x:

$$w(x,t) = \frac{1}{n!}(C_1 t + C_2)x^n + \sum_{k=0}^{n-1}(A_k t + B_k)x^k + \int_0^t (t-\xi)F(C_1\xi + C_2)\,d\xi,$$

where C_1, C_2, the A_k, and the B_k are arbitrary constants.

3°. Generalized separable solution quadratic in t:

$$w(x,t) = \frac{1}{2}t^2\sum_{k=0}^{n-1}A_k x^k + t\sum_{k=0}^{n-1}B_k x^k + \sum_{k=0}^{n-1}C_k x^k + \int_0^x \frac{(x-\xi)^{n-1}}{(n-1)!}\Phi\left(\sum_{k=0}^{n-1}A_k\xi^k\right)d\xi,$$

where the A_k, B_k, and C_k are arbitrary constants, and $\Phi(u)$ is the inverse of the function $F(u)$.

4°. Solution:

$$w(x,t) = \frac{1}{2}A_1 t^2 + \frac{1}{n!}A_2 x^n + \sum_{m=0}^{n-1}(B_m t + C_m)x^m + \varphi(\zeta), \qquad \zeta = kx + \lambda t,$$

where A_1, A_2, the B_m, the C_m, k, and λ are arbitrary constants, and the function $\varphi = \varphi(\zeta)$ is determined by the autonomous ordinary differential equation

$$A_1 + \lambda^2\varphi''_{\zeta\zeta} = F\left(A_2 + k^n\varphi^{(n)}_\zeta\right).$$

5°. Self-similar solution:

$$w = t^2 U(z), \qquad z = xt^{-2/n},$$

where the function $U = U(z)$ is determined by the ordinary differential equation

$$2U + \frac{2(2-3n)}{n^2}zU'_z + \frac{4}{n^2}z^2 U''_{zz} = F(U^{(n)}_z).$$

3. $\dfrac{\partial^2 w}{\partial t^2} = aw\dfrac{\partial^2 w}{\partial x^2} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right).$

1°. Degenerate solution linear in x:

$$w = (C_1 t + C_2)x + C_3 t + C_4 + \int_0^t (t-\tau)F(C_1\tau + C_2, 0, \ldots, 0)\,d\tau.$$

2°. Traveling-wave solution:

$$w(x,t) = u(\xi), \qquad \xi = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $u = u(\xi)$ is determined by the autonomous ordinary differential equation

$$(ak^2 u - \lambda^2)u''_{\xi\xi} + F(ku'_\xi, k^2 u''_{\xi\xi}, \ldots, k^n u^{(n)}_\xi) = 0.$$

3°. Solution:

$$w = U(z) + 4aC_1^2 t^2 + 4aC_1C_2 t, \qquad z = x + aC_1 t^2 + aC_2 t,$$

where C_1 and C_2 are arbitrary constants and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$(aU - a^2C_2^2)U''_{zz} - 2aC_1U'_z + F(U'_z, U''_{zz}, \ldots, U^{(n)}_z) = 8aC_1^2.$$

4. $\dfrac{\partial^2 w}{\partial t^2} = (aw + bx)\dfrac{\partial^2 w}{\partial x^2} + F\left(\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right).$

The substitution $w = u - (b/a)x$ leads to an equation of the form 11.3.5.3:

$$\frac{\partial^2 u}{\partial t^2} = au\frac{\partial^2 u}{\partial x^2} + F\left(\frac{\partial u}{\partial x} - \frac{b}{a}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^n u}{\partial x^n}\right).$$

5. $\dfrac{\partial^2 w}{\partial t^2} = F\left(x, t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\right) + aw.$

Suppose $w(x,t)$ is a solution of the equation in question. Then the functions

$$w_1 = w(x,t) + C_1\cosh(kt) + C_2\sinh(kt) \quad \text{if } a = k^2 > 0,$$
$$w_2 = w(x,t) + C_1\cos(kt) + C_2\sin(kt) \quad \text{if } a = -k^2 < 0,$$

where C_1 and C_2 are arbitrary constants, are also solutions of the equation.

11.4. Other Equations

11.4.1. Equations Involving Mixed Derivatives

1. $\dfrac{\partial^2 w}{\partial x \partial t} + \left(\dfrac{\partial w}{\partial x}\right)^2 - w\dfrac{\partial^2 w}{\partial x^2} = f(t)\dfrac{\partial^n w}{\partial x^n}.$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1(x,t) = w\big(x + \varphi(t), t\big) + \varphi'_t(t),$$

where $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Generalized separable solution:

$$w = \varphi(t)e^{\lambda x} + \frac{1}{\lambda}\frac{\varphi'_t(t)}{\varphi(t)} - \lambda^{n-2}f(t),$$

where $\varphi(t)$ is an arbitrary function and λ is an arbitrary constant.

Remark. This equation with $n = 3$ occurs in fluid dynamics; see 9.3.3.1, equation (2) and 10.3.3.1, equation (4) with $f_1(t) = 0$.

2. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(x)\dfrac{\partial^n w}{\partial y^n}.$

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1(x,y) = C_1^{n-2} w(x, C_1 y + \varphi(x)) + C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(x)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solution:

$$w(x,y) = \sum_{k=0}^{n-1} C_k \bigl[y + \varphi(x)\bigr]^k,$$

where $\varphi(x)$ is an arbitrary function and the C_k are arbitrary constants.

3°. Generalized separable solution:

$$w(x,y) = \varphi(x) e^{\lambda y} - \lambda^{n-2} \int f(x)\, dx + C,$$

where $\varphi(x)$ is an arbitrary function and C and λ are arbitrary constants.

4°. Generalized separable solution:

$$w(x,y) = \varphi(y) \int f(x)\, dx + \psi(y),$$

where the functions $\varphi = \varphi(y)$ and $\psi = \psi(y)$ are determined by the autonomous system of ordinary differential equations

$$(\varphi'_y)^2 - \varphi\varphi''_{yy} = \varphi_y^{(n)},$$
$$\varphi'_y \psi'_y - \varphi\psi''_{yy} = \psi_y^{(n)}.$$

5°. Generalized self-similar solution:

$$w(x,y) = \varphi(x) U(z), \qquad z = \psi(x) y$$

where the functions $\varphi = \varphi(x)$, $\psi = \psi(x)$, and $U = U(z)$ are determined by the system of ordinary differential equations

$$(\varphi\psi)'_x = C_1 f(x) \psi^{n-1},$$
$$\varphi'_x = C_2 f(x) \psi^{n-2},$$
$$C_1 (U'_z)^2 - C_2 U U''_{zz} = U_z^{(n)}.$$

6°. See also equation 11.4.1.3 with $g(x) = 0$.

Remark. This equation with $n = 3$ occurs in fluid dynamics; see 9.3.1.1 with $f(x) = \text{const}$.

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

3. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(x)\dfrac{\partial^{2n} w}{\partial y^{2n}} + g(x).$

This is a special case of equation 11.4.1.5.

Generalized separable solution:

$$w(x,y) = \varphi(x) e^{\lambda y} - \frac{1}{2\lambda^2 \varphi(x)} \left[\int g(x)\, dx + C_1\right] e^{-\lambda y} - \lambda^{2n-2} \int f(x)\, dx + C_2,$$

where $\varphi(x)$ is an arbitrary function and C_1, C_2, and λ are arbitrary constants.

4. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = f(x)\Big(\dfrac{\partial^2 w}{\partial y^2}\Big)^{k-1}\dfrac{\partial^n w}{\partial y^n}$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1(x,y) = C_1^{2k+n-4} w\big(x, C_1^{2-k} y + \varphi(x)\big) + C_2,$$

where C_1 and C_2 are arbitrary constants and $\varphi(x)$ is an arbitrary function, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w = U(z), \qquad z = y\left[\int f(x)\,dx + C\right]^{\frac{1}{4-2k-n}} + \varphi(x),$$

where $\varphi(x)$ is an arbitrary function and the function $U = U(z)$ is determined by the autonomous ordinary differential equation

$$(U'_z)^2 = (4-2k-n)(U''_{zz})^{k-1} U^{(n)}_z.$$

3°. Multiplicative separable solution:

$$w(x,y) = \left[(2-k)\int f(x)\,dx + C\right]^{\frac{1}{2-k}} \theta(y),$$

where the function $\theta(y)$ is determined by the autonomous ordinary differential equation

$$(\theta'_y)^2 - \theta\theta''_{yy} = (\theta''_{yy})^{k-1}\theta^{(n)}_y.$$

5. $\dfrac{\partial w}{\partial y}\dfrac{\partial^2 w}{\partial x \partial y} - \dfrac{\partial w}{\partial x}\dfrac{\partial^2 w}{\partial y^2} = F\left(x, w, \dfrac{\partial w}{\partial y}, \ldots, \dfrac{\partial^n w}{\partial y^n}\right)$.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1(x,y) = w\big(x, y + \varphi(x)\big),$$

where $\varphi(x)$ is an arbitrary function, is also a solution of the equation.

2°. Suppose the right-hand side of the equation is independent of x explicitly. Then there is a generalized traveling-wave solution of the form

$$w = w(z), \qquad z = y + \varphi(x),$$

where $\varphi(x)$ is an arbitrary function, and the function $w(z)$ is determined by the autonomous ordinary differential equation $F\big(w, w'_z, \ldots, w^{(n)}_z\big) = 0$.

3°. Suppose the right-hand side of the equation is independent of x and w explicitly. Then there is an exact solution of the form

$$w = Cx + g(z), \qquad z = y + \varphi(x),$$

where $\varphi(x)$ is an arbitrary function, C is an arbitrary constant, and the function $g(z)$ is determined by the autonomous ordinary differential equation $F\big(g'_z, \ldots, g^{(n)}_z\big) + Cg''_{zz} = 0$.

4°. The von Mises transformation

$$\xi = x, \quad \eta = w, \quad u(\xi,\eta) = \frac{\partial w}{\partial y}, \qquad \text{where} \quad w = w(x,y),$$

reduces the order of the equation by one. Formulas for computing derivatives:

$$\frac{\partial w}{\partial y} = u, \ \ \frac{\partial^2 w}{\partial y^2} = u\frac{\partial u}{\partial \eta}, \ \ \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = u\frac{\partial u}{\partial \xi}, \ \ \frac{\partial^3 w}{\partial y^3} = u\frac{\partial}{\partial \eta}\left(u\frac{\partial u}{\partial \eta}\right), \ \ \frac{\partial}{\partial y} = u\frac{\partial}{\partial \eta}.$$

⊙ *Reference*: A. D. Polyanin and V. F. Zaitsev (2002).

6. $\dfrac{\partial^2 w}{\partial x \partial t} = a(t) w \dfrac{\partial^2 w}{\partial x^2} + F\Big(t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\Big).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + \varphi(t), t) + \frac{\varphi'_t(t)}{a(t)},$$

where $\varphi(t)$ is an arbitrary function, is also a solution of the equation.

2°. Degenerate solution linear in x:

$$w(x,t) = \varphi(t)x + \psi(t),$$

where $\psi(t)$ is an arbitrary function, and $\varphi(t)$ is determined by the first-order ordinary differential equation $\varphi'_t = F(t, \varphi, 0, \ldots, 0)$.

3°. For $a = \text{const}$ and $F = F(w_x, w_{xx}, \ldots, w_x^{(n)})$, the equation has a traveling-wave solution

$$w = U(z), \quad z = kx + \lambda t,$$

where k and λ are arbitrary constants, and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$k\lambda U''_{zz} = ak^2 U U''_{zz} + F\big(kU'_z, k^2 U''_{zz}, \ldots, k^n U_z^{(n)}\big).$$

7. $\dfrac{\partial^{n+1} w}{\partial x^n \partial y} = ae^{\lambda w}.$

Generalized Liouville equation.

1°. Suppose $w(x,y)$ is a solution of the equation in question. Then the function

$$w_1 = w\big(C_1 x + C_2,\ C_3 y + C_4\big) + \frac{1}{\lambda}\ln(C_1^n C_3),$$

where $C_1, \ldots, C_4$ are arbitrary constants, is also a solution of the equation.

2°. Generalized traveling-wave solution:

$$w(x,y) = -\frac{n+1}{\lambda}\ln z, \qquad z = \varphi(y)x + \frac{a\lambda(-1)^{n+1}}{(n+1)!}\varphi(y)\int \frac{dy}{[\varphi(y)]^{n+1}},$$

where $\varphi(y)$ is an arbitrary function.

8. $\dfrac{\partial^{k+1} w}{\partial x^k \partial t} = a(t) w \dfrac{\partial^{k+1} w}{\partial x^{k+1}} + F\Big(t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \ldots, \dfrac{\partial^n w}{\partial x^n}\Big).$

1°. Suppose $w(x,t)$ is a solution of the equation in question. Then the function

$$w_1 = w(x + \varphi(t), t) + \frac{\varphi'_t(t)}{a(t)},$$

where $\varphi(t)$ is an arbitrary function, is also a solution of the equation for $k = 1, 2, \ldots$

2°. For $a = \text{const}$ and $F = F(w_x, w_{xx}, \ldots, w_x^{(n)})$, the equation has a traveling-wave solution

$$w = U(z), \quad z = \beta x + \lambda t,$$

where β and λ are arbitrary constants, and the function $U(z)$ is determined by the autonomous ordinary differential equation

$$\lambda\beta^k U_z^{(k+1)} = a\beta^{k+1} U U_z^{(k+1)} + F\big(\beta U'_z, \beta^2 U''_{zz}, \ldots, \beta^n U_z^{(n)}\big).$$

9. $\dfrac{\partial^2 w}{\partial x \partial t} = F\left(t, \dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x^2}, \dots, \dfrac{\partial^n w}{\partial x^n}\right) + g(t)\dfrac{\partial^2 w}{\partial y^2}.$

1°. "Two-dimensional" solution:

$$w(x,y,t) = u(z,t), \qquad z = x + C_1 y + C_1^2 \int g(t)\,dt + C_2,$$

where C_1 and C_2 are arbitrary constants and the function $u(z,t)$ is determined by the differential equation

$$\frac{\partial^2 u}{\partial z \partial t} = F\left(t, u, \frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial z^2}, \dots, \frac{\partial^n u}{\partial z^n}\right).$$

2°. "Two-dimensional" solution:

$$w(x,y,t) = U(\xi,t), \qquad \xi = x + \varphi(t)(y + C_1)^2, \qquad \varphi(t) = -\left[4\int g(t)\,dt + C_2\right]^{-1},$$

where the function $U(\xi,t)$ is determined by the differential equation

$$\frac{\partial^2 U}{\partial \xi \partial t} = F\left(t, U, \frac{\partial U}{\partial \xi}, \frac{\partial^2 U}{\partial \xi^2}, \dots, \frac{\partial^n U}{\partial \xi^n}\right) + 2g(t)\varphi(t)\frac{\partial U}{\partial \xi}.$$

11.4.2. Equations Involving $\dfrac{\partial^n w}{\partial x^n}$ and $\dfrac{\partial^m w}{\partial y^m}$

1. $a\dfrac{\partial^n w}{\partial x^n} + b\dfrac{\partial^n w}{\partial y^n} = (ay^n + bx^n)f(w).$

Solution:

$$w = w(z), \qquad z = xy,$$

where the function $w(z)$ is determined by the autonomous ordinary differential equation

$$w_z^{(n)} = f(w).$$

Remark. This remains true if the constants a and b in the equation are replaced by arbitrary functions $a = a(x, y, w, w_x, w_y, \dots)$ and $b = b(x, y, w, w_x, w_y, \dots)$.

2. $F\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dots, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}; \dfrac{1}{w}\dfrac{\partial w}{\partial y}, \dots, \dfrac{1}{w}\dfrac{\partial^m w}{\partial y^m}\right) = 0.$

Multiplicative separable solution:

$$w(x,y) = Ae^{\lambda y}\varphi(x),$$

where A and λ are arbitrary constants, and the function $\varphi(x)$ is determined by the nth-order ordinary differential equation

$$F\bigl(x, \varphi_x'/\varphi, \dots, \varphi_x^{(n)}/\varphi; \lambda, \dots, \lambda^m\bigr) = 0.$$

3. $F\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dots, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}; \dfrac{1}{w}\dfrac{\partial^2 w}{\partial y^2}, \dots, \dfrac{1}{w}\dfrac{\partial^{2m} w}{\partial y^{2m}}\right) = 0.$

1°. Multiplicative separable solution:

$$w(x,y) = \bigl[A\cosh(\lambda y) + B\sinh(\lambda y)\bigr]\varphi(x),$$

where A, B, and λ are arbitrary constants, and the function $\varphi(x)$ is determined by the nth-order ordinary differential equation

$$F\bigl(x, \varphi_x'/\varphi, \dots, \varphi_x^{(n)}/\varphi; \lambda^2, \dots, \lambda^{2m}\bigr) = 0.$$

2°. Multiplicative separable solution:

$$w(x,y) = \bigl[A\cos(\lambda y) + B\sin(\lambda y)\bigr]\varphi(x),$$

where A, B, and λ are arbitrary constants, and the function $\varphi(x)$ is determined by the nth-order ordinary differential equation

$$F\bigl(x, \varphi_x'/\varphi, \dots, \varphi_x^{(n)}/\varphi; -\lambda^2, \dots, (-1)^m\lambda^{2m}\bigr) = 0.$$

4. $F_1\left(x, \dfrac{\partial w}{\partial x}, \dots, \dfrac{\partial^n w}{\partial x^n}\right) + F_2\left(y, \dfrac{\partial w}{\partial y}, \dots, \dfrac{\partial^m w}{\partial y^m}\right) = kw.$

Additive separable solution:

$$w(x,y) = \varphi(x) + \psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$F_1\bigl(x, \varphi'_x, \dots, \varphi^{(n)}_x\bigr) - k\varphi = C,$$
$$F_2\bigl(y, \psi'_y, \dots, \psi^{(m)}_y\bigr) - k\psi = -C,$$

where C is an arbitrary constant.

5. $F_1\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dots, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right) + w^k F_2\left(y, \dfrac{1}{w}\dfrac{\partial w}{\partial y}, \dots, \dfrac{1}{w}\dfrac{\partial^m w}{\partial y^m}\right) = 0.$

Multiplicative separable solution:

$$w(x,y) = \varphi(x)\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$\varphi^{-k} F_1\bigl(x, \varphi'_x/\varphi, \dots, \varphi^{(n)}_x/\varphi\bigr) = C,$$
$$\psi^k F_2\bigl(y, \psi'_y/\psi, \dots, \psi^{(m)}_y/\psi\bigr) = -C,$$

where C is an arbitrary constant.

6. $F_1\left(x, \dfrac{\partial w}{\partial x}, \dots, \dfrac{\partial^n w}{\partial x^n}\right) + e^{\lambda w} F_2\left(y, \dfrac{\partial w}{\partial y}, \dots, \dfrac{\partial^m w}{\partial y^m}\right) = 0.$

Additive separable solution:

$$w(x,y) = \varphi(x) + \psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$e^{-\lambda\varphi} F_1\bigl(x, \varphi'_x, \dots, \varphi^{(n)}_x\bigr) = C,$$
$$e^{\lambda\psi} F_2\bigl(y, \psi'_y, \dots, \psi^{(m)}_y\bigr) = -C,$$

where C is an arbitrary constant.

7. $F_1\left(x, \dfrac{1}{w}\dfrac{\partial w}{\partial x}, \dots, \dfrac{1}{w}\dfrac{\partial^n w}{\partial x^n}\right) + F_2\left(y, \dfrac{1}{w}\dfrac{\partial w}{\partial y}, \dots, \dfrac{1}{w}\dfrac{\partial^m w}{\partial y^m}\right) = k \ln w.$

Multiplicative separable solution:

$$w(x,y) = \varphi(x)\psi(y).$$

Here, the functions $\varphi(x)$ and $\psi(y)$ are determined by the ordinary differential equations

$$F_1\bigl(x, \varphi'_x/\varphi, \dots, \varphi^{(n)}_x/\varphi\bigr) - k\ln\varphi = C,$$
$$F_2\bigl(y, \psi'_y/\psi, \dots, \psi^{(m)}_y/\psi\bigr) - k\ln\psi = -C,$$

where C is an arbitrary constant.

8. $F\left(ax + by,\ w,\ \dfrac{\partial w}{\partial x}, \dots, \dfrac{\partial^n w}{\partial x^n},\ \dfrac{\partial w}{\partial y}, \dots, \dfrac{\partial^m w}{\partial y^m}\right) = 0.$

Solution:

$$w = w(\xi), \qquad \xi = ax + by,$$

where the function $w(\xi)$ is determined by the ordinary differential equation

$$F\bigl(\xi,\ w,\ aw'_\xi, \dots, a^n w^{(n)}_\xi,\ bw'_\xi, \dots, b^m w^{(m)}_\xi\bigr) = 0.$$

9. $\boldsymbol{F\Big(ax+by,\ \frac{\partial w}{\partial x},\dots,\frac{\partial^n w}{\partial x^n},\ \frac{\partial w}{\partial y},\dots,\frac{\partial^m w}{\partial y^m}\Big)=0.}$

Solution:

$$w=\varphi(\xi)+Cx,\qquad \xi=ax+by,$$

where C is an arbitrary constant and the function $\varphi(\xi)$ is determined by the ordinary differential equation

$$F\big(\xi,\ a\varphi'_\xi+C,\ a^2\varphi''_{\xi\xi},\dots,a^n\varphi^{(n)}_\xi,\ b\varphi'_\xi,\dots,b^m\varphi^{(m)}_\xi\big)=0.$$

10. $\boldsymbol{\frac{\partial^n}{\partial x^n}\Big\{\big[a_1x+b_1y+f(w)\big]\frac{\partial^m w}{\partial x^m}\Big\}+\frac{\partial^n}{\partial y^n}\Big\{\big[a_2x+b_2y+g(w)\big]\frac{\partial^m w}{\partial y^m}\Big\}=0.}$

Solutions are sought in the traveling-wave form

$$w=w(z),\qquad z=Ax+By,$$

where the constants A and B are evaluated from the algebraic system of equations

$$a_1A^{n+m}+a_2B^{n+m}=A,$$
$$b_1A^{n+m}+b_2B^{n+m}=B.$$

The desired function $w(z)$ is determined by the mth-order ordinary differential equation

$$\big[z+A^{n+m}f(w)+B^{n+m}g(w)\big]w^{(m)}_z=C_0+C_1z+\dots+C_{n-1}z^{n-1},$$

where C_0, C_1, …, C_{n-1} are arbitrary constants.

11. $\boldsymbol{(a_1x+b_1y)\frac{\partial^n w}{\partial x^n}+(a_2x+b_2y)\frac{\partial^n w}{\partial y^n}=F\Big(w,\ \frac{\partial w}{\partial x},\dots,\frac{\partial^m w}{\partial x^m},\ \frac{\partial w}{\partial y},\dots,\frac{\partial^k w}{\partial y^k}\Big).}$

Generalized traveling-wave solution:

$$w=w(z),\qquad z=Ax+By,$$

where the constants A and B are evaluated from the algebraic system of equations

$$a_1A^n+a_2B^n=A,$$
$$b_1A^n+b_2B^n=B,$$

and the desired function $w(z)$ is determined by the ordinary differential equation

$$zw^{(n)}_z=F\big(w,\ Aw'_z,\dots,A^mw^{(m)}_z,\ Bw'_z,\dots,B^kw^{(k)}_z\big).$$

Remark. If the right-hand side of the equation is also dependent on mixed derivatives, solutions are constructed likewise.

Supplements

Exact Methods for Solving Nonlinear Partial Differential Equations

S.1. Classification of Second-Order Semilinear Partial Differential Equations in Two Independent Variables

S.1.1. Types of Equations. Characteristic Equation

Consider a second-order semilinear partial differential equation in two independent variables of the form

$$a(x,y)\frac{\partial^2 w}{\partial x^2}+2b(x,y)\frac{\partial^2 w}{\partial x \partial y}+c(x,y)\frac{\partial^2 w}{\partial y^2}=F\left(x,y,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\right), \tag{1}$$

where a, b, and c are some functions of x and y that have continuous derivatives up to the second-order inclusive.

Given a point (x, y), equation (1) is said to be

$$\begin{array}{ll} \text{parabolic} & \text{if } b^2-ac=0, \\ \text{hyperbolic} & \text{if } b^2-ac>0, \\ \text{elliptic} & \text{if } b^2-ac<0 \end{array}$$

at this point.

In order to reduce equation (1) to a canonical form, one should first write out the characteristic equation

$$a\,dy^2-2b\,dx\,dy+c\,dx^2=0,$$

which splits into two equations

$$a\,dy-\left(b+\sqrt{b^2-ac}\right)dx=0, \tag{2}$$

and

$$a\,dy-\left(b-\sqrt{b^2-ac}\right)dx=0, \tag{3}$$

and find their general integrals.

S.1.2. Canonical Form of Parabolic Equations (Case $b^2 - ac = 0$)

In this case, equations (2) and (3) coincide and have a common general integral,

$$\varphi(x,y)=C.$$

By passing from x, y to new independent variables ξ, η in accordance with the relations

$$\xi=\varphi(x,y), \qquad \eta=\eta(x,y),$$

where $\eta = \eta(x, y)$ is any twice differentiable function that satisfies the condition of nondegeneracy of the Jacobian $\frac{D(\xi,\eta)}{D(x,y)}$ in a given domain, we reduce equation (1) to the canonical form

$$\frac{\partial^2 w}{\partial \eta^2} = F_1\left(\xi, \eta, w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}\right). \tag{4}$$

As η, one can take $\eta = x$ or $\eta = y$. Often $\eta = \eta(x, y)$ is selected so as to simplify the right-hand side of equation (4) as much as possible. In the special case $F_1 = \partial_\xi w$, we have the classical linear heat equation.

It is apparent that, the transformed equation (4) has only one highest-derivative term.

$Remark.$ In the degenerate case where the function F_1 is independent of the derivative $\partial_\xi w$, equation (4) is an ordinary differential equation for η, in which ξ serves as a parameter.

S.1.3. Canonical Form of Hyperbolic Equations (Case $b^2 - ac > 0$)

The general integrals

$$\varphi(x, y) = C_1, \qquad \psi(x, y) = C_2$$

of equations (2) and (3) are real and different. These integrals determine two different families of real characteristics.

By passing from x, y to new independent variables ξ, η in accordance with the relations

$$\xi = \varphi(x, y), \qquad \eta = \psi(x, y),$$

we reduce equation (1) to

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = F_2\left(\xi, \eta, w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}\right).$$

This is the so-called first canonical form of a hyperbolic equation.

The transformation

$$\xi = t + z, \qquad \eta = t - z$$

brings this equation to another canonical form,

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial z^2} = F_3\left(t, z, w, \frac{\partial w}{\partial t}, \frac{\partial w}{\partial z}\right), \tag{5}$$

where $F_3 = 4F_2$. This is the so-called second canonical form of a hyperbolic equation.

In the special case $F_3 = 0$, equation (5) is the classical linear wave equation.

S.1.4. Canonical Form of Elliptic Equations (Case $b^2 - ac < 0$)

In this case, the general integrals of equations (2) and (3) are complex conjugate; these determine two families of complex characteristics.

Let the general integral of equation (2) have the form

$$\varphi(x, y) + i\psi(x, y) = C, \qquad i^2 = -1,$$

where $\varphi(x, y)$ and $\psi(x, y)$ are real-valued functions.

By passing from x, y to new independent variables ξ, η in accordance with the relations

$$\xi = \varphi(x, y), \qquad \eta = \psi(x, y),$$

we reduce equation (1) to the canonical form

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} = F_4\left(\xi, \eta, w, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}\right).$$

In the special case $F_4 = 0$, we have the linear Laplace equation.

S.2. Transformations of Equations of Mathematical Physics

S.2.1. Point Transformations

Let $w = w(x, y)$ be a function of independent variables x and y. In general, a point transformation is defined by the formulas

$$x = X(\xi, \eta, u), \qquad y = Y(\xi, \eta, u), \qquad w = W(\xi, \eta, u), \tag{1}$$

where ξ and η are new independent variables, $u = u(\xi, \eta)$ is a new dependent variable, and the functions X, Y, W may be either given or unknown (have to be found).

A point transformation not only preserves the order or the equation to which it is applied but also mostly preserves the structure of the equation, since the highest-order derivatives of the new variables are linearly dependent on the highest-order derivatives of the original variables.

Transformation (1) is invertible if

$$\det \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial X}{\partial w} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial w} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \frac{\partial W}{\partial w} \end{pmatrix} \neq 0.$$

In the general case, a point transformation (1) reduces a second-order equation with two independent variables

$$F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0 \tag{2}$$

to an equation

$$G\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \frac{\partial^2 u}{\partial \xi^2}, \frac{\partial^2 u}{\partial \xi \partial \eta}, \frac{\partial^2 u}{\partial \eta^2}\right) = 0. \tag{3}$$

If $u = u(\xi, \eta)$ is a solution of equation (3), then formulas (1) define the corresponding solution of equation (2) in parametric form.

Point transformations are employed to simplify equations and their reduction to known equations. Sometimes, point transformations can be used for the reduction of nonlinear equations to linear ones.

Example 1. The equation

$$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^m \frac{\partial w}{\partial x}\right) + \left[xf(t) + g(t)\right]\frac{\partial w}{\partial x} + h(t)w$$

can be simplified to obtain

$$\frac{\partial u}{\partial \tau} = \frac{\partial}{\partial z}\left(u^m \frac{\partial u}{\partial z}\right)$$

with the help of the transformation

$$w(x, t) = u(z, \tau)H(t), \quad z = xF(t) + \int g(t)F(t)\,dt, \quad \tau = \int F^2(t)H^m(t)\,dt,$$

where

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad H(t) = \exp\left[\int h(t)\,dt\right].$$

Example 2. The nonlinear equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + a\left(\frac{\partial w}{\partial x}\right)^2 + f(x, t)$$

can be reduced to the linear equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + af(x, t)u$$

for the function $u = u(x, t)$ by means of the transformation $u = \exp(aw)$.

S.2.2. Hodograph Transformation

In some cases, nonlinear equations and systems of partial differential equations can be simplified by means of the hodograph transformation.

1°. For an equation with two independent variables x, t and an unknown function $w = w(x, t)$, the hodograph transformation consists in representing the solution in implicit form

$$x = x(t, w) \tag{4}$$

or $t = t(x, w)$. Thus, t and w are treated as independent variables, while x is taken to be the dependent variable. The hodograph transformation (4) does not change the order of the equation and belongs to the class of point transformations (equivalently, it can be represented as $x = \widetilde{w}$, $t = \widetilde{t}$, $w = \widetilde{x}$).

2°. For a system of two equations with two independent variables x, y and two dependent variables $w = w(x, y)$, $v = v(x, y)$, the hodograph transformation implies that w, v are treated as the independent variables and x, y as the dependent variables. In other words, one looks for a solution in the form

$$x = x(w, v), \qquad y = y(w, v). \tag{5}$$

The hodograph transformation is used in gas dynamics and the theory of jets for the linearization of equations and finding solutions of certain boundary value problems.

Below we consider some applications of the hodograph transformation to solving specific equations of mathematical physics.

Example 3. Consider the nonlinear second-order equation

$$\frac{\partial w}{\partial t}\left(\frac{\partial w}{\partial x}\right)^2 = f(t, w)\frac{\partial^2 w}{\partial x^2}. \tag{6}$$

Let us seek its solution in implicit form. Differentiating relation (4) with respect to both variables as an implicit function and taking into account that $w = w(x, t)$, we get

$$\begin{aligned} 1 &= x_w w_x && \text{(differentiation in } x\text{)},\\ 0 &= x_w w_t + x_t && \text{(differentiation in } t\text{)},\\ 0 &= x_{ww} w_x^2 + x_w w_{xx} && \text{(double differentiation in } x\text{)}, \end{aligned}$$

where the subscripts indicate the corresponding partial derivatives. We solve these relations to express the "old" derivatives through the "new" ones,

$$w_x = \frac{1}{x_w}, \quad w_t = -\frac{x_t}{x_w}, \quad w_{xx} = -\frac{w_x^2 x_{ww}}{x_w} = -\frac{x_{ww}}{x_w^3}.$$

Substituting these expressions into (6), we obtain the following second-order linear equation:

$$\frac{\partial x}{\partial t} = f(t, w)\frac{\partial^2 x}{\partial w^2}.$$

Example 4. Let us represent the equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial y}\left[f(w)\frac{\partial w}{\partial y}\right] = 0 \tag{7}$$

as the following system of equations:

$$\frac{\partial w}{\partial x} = \frac{\partial v}{\partial y}, \quad -f(w)\frac{\partial w}{\partial y} = \frac{\partial v}{\partial x}. \tag{8}$$

We now take advantage of the hodograph transformation (5), which amounts to taking w, v as the independent variables and x, y as dependent variables. Differentiating each relation in (5) with respect to x and y (as composite functions) and eliminating the partial derivatives x_w, x_v, y_w, y_v from the resulting relations, we obtain

$$\frac{\partial x}{\partial w} = \frac{1}{J}\frac{\partial v}{\partial y}, \quad \frac{\partial x}{\partial v} = -\frac{1}{J}\frac{\partial w}{\partial y}, \quad \frac{\partial y}{\partial w} = -\frac{1}{J}\frac{\partial v}{\partial x}, \quad \frac{\partial y}{\partial v} = \frac{1}{J}\frac{\partial w}{\partial x}, \quad \text{where} \quad J = \frac{\partial w}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial w}{\partial y}\frac{\partial v}{\partial x}. \tag{9}$$

Using (9) to eliminate the derivatives w_x, w_y, v_x, v_y from (8), we arrive at the system

$$\frac{\partial y}{\partial v} = \frac{\partial x}{\partial w}, \quad -f(w)\frac{\partial x}{\partial v} = \frac{\partial y}{\partial w}. \tag{10}$$

Let us differentiate the first equation in w and the second in v, and then eliminate the mixed derivative y_{wv}. As a result, we obtain the following linear equation for the function $x = x(w, v)$:

$$\frac{\partial^2 x}{\partial w^2} + f(w)\frac{\partial^2 x}{\partial v^2} = 0. \tag{11}$$

Similarly, from system (10), we obtain another linear equation for the function $y = y(w, v)$,

$$\frac{\partial^2 y}{\partial v^2} + \frac{\partial}{\partial w}\left[\frac{1}{f(w)}\frac{\partial y}{\partial w}\right] = 0. \tag{12}$$

Given a particular solution $x = x(w, v)$ of equation (11), we substitute this solution into system (10) and find $y = y(w, v)$ by straightforward integration. Eliminating v from (5), we obtain an exact solution $w = w(x, y)$ of the nonlinear equation (7).

1°. Equation (11) with an arbitrary $f(w)$ admits a simple particular solution, namely,

$$x = C_1 wv + C_2 w + C_3 v + C_4, \tag{13}$$

where $C_1, \dots, C_4$ are arbitrary constants. Substituting this solution into system (10), we obtain

$$\frac{\partial y}{\partial v} = C_1 v + C_2, \qquad \frac{\partial y}{\partial w} = -(C_1 w + C_3)f(w). \tag{14}$$

Integrating the first equation in (14) yields $y = \frac{1}{2}C_1 v^2 + C_2 v + \varphi(w)$. Substituting this solution into the second equation in (14), we find the function $\varphi(w)$, and consequently

$$y = \tfrac{1}{2}C_1 v^2 + C_2 v - \int (C_1 w + C_3)f(w)\,dw + C_5. \tag{15}$$

Formulas (13) and (15) define an exact solution of equation (7) in parametric form (v is the parameter).

2°. In a similar way, one can construct a more complex solution of equation (7) in parametric form,

$$x = C_1 v^2 + C_2 wv + C_3 v + C_4 w - 2C_1 \int_a^w (x - t)f(t)\,dt + C_5,$$
$$y = \tfrac{1}{2}C_2 v^2 + C_4 v - 2C_1 v \int f(w)\,dw - \int (C_2 w + C_3)f(w)\,dw + C_6.$$

3°. Using a particular solution of equation (12), we obtain another exact solution of equation (7):

$$x = -\tfrac{1}{2}C_1 v^2 - C_2 v + C_1 \int F(w)\,dw + C_3 w + C_4,$$
$$y = (C_1 v + C_2)F(w) + C_3 v + C_5, \qquad F(w) = \int f(w)\,dw.$$

See also 5.4.4.8 for a more general equation and some other solutions.

Example 5. Consider the system of gas dynamic type equations

$$f_1(w,v)\frac{\partial w}{\partial x} + f_2(w,v)\frac{\partial w}{\partial y} + f_3(w,v)\frac{\partial v}{\partial x} + f_4(w,v)\frac{\partial v}{\partial y} = 0,$$
$$g_1(w,v)\frac{\partial w}{\partial x} + g_2(w,v)\frac{\partial w}{\partial y} + g_3(w,v)\frac{\partial v}{\partial x} + g_4(w,v)\frac{\partial v}{\partial y} = 0.$$

Treating w, v as the independent variables and x, y as the dependent ones, we arrive at the following system of linear equations (the calculations are similar to those of Example 4):

$$f_1(w,v)\frac{\partial y}{\partial v} - f_2(w,v)\frac{\partial x}{\partial v} - f_3(w,v)\frac{\partial y}{\partial w} + f_4(w,v)\frac{\partial x}{\partial w} = 0,$$
$$g_1(w,v)\frac{\partial y}{\partial v} - g_2(w,v)\frac{\partial x}{\partial v} - g_3(w,v)\frac{\partial y}{\partial w} + g_4(w,v)\frac{\partial x}{\partial w} = 0.$$

⊙ *References for Subsection* S.2.2: N. E. Kochin, I. A. Kibel', and N. V. Roze (1963), B. L. Rozhdestvenskii and N. N. Yanenko (1983), A. M. Siddiqui, P. N. Kaloni, and O. P. Chandna (1985), G. G. Chernyi (1988), R. Courant and D. Hilbert (1989), P. A. Clarkson, A. S. Fokas, and M. J. Ablowitz (1989), V. F. Zaitsev and A. D. Polyanin (2001 *b*).

S.2.3. Contact Transformations. Legendre and Euler Transformations

S.2.3-1. General form of contact transformations.

Consider functions of two variables $w = w(x, y)$. A common property of contact transformations is the dependence of the original variables on the new variables and their first derivatives:

$$x = X\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right), \quad y = Y\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right), \quad w = W\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \qquad (16)$$

The functions X, Y, and W cannot be arbitrary and are selected so as to ensure that the first derivatives of the original variables depend only on the transformed variables and, possibly, their first derivatives,

$$\frac{\partial w}{\partial x} = U\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right), \quad \frac{\partial w}{\partial y} = V\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}\right). \qquad (17)$$

Contact transformations (16)–(17) do not increase the order of the equations to which they are applied.

We now outline the procedure for finding the functions U and V in (17) and the relations that must hold for the functions X, Y, and W in (16).

Let us differentiate the first and second expressions in (16) with respect to x and y as composite functions taking into account that $u = u(\xi, \eta)$. Thus, we obtain the following four relations:

$$\begin{aligned}
&\left(\frac{\partial X}{\partial \xi} + \frac{\partial X}{\partial u}p + \frac{\partial X}{\partial p}p_\xi + \frac{\partial X}{\partial q}p_\eta\right)\frac{\partial \xi}{\partial x} + \left(\frac{\partial X}{\partial \eta} + \frac{\partial X}{\partial u}q + \frac{\partial X}{\partial p}q_\xi + \frac{\partial X}{\partial q}q_\eta\right)\frac{\partial \eta}{\partial x} = 1,\\
&\left(\frac{\partial Y}{\partial \xi} + \frac{\partial Y}{\partial u}p + \frac{\partial Y}{\partial p}p_\xi + \frac{\partial Y}{\partial q}p_\eta\right)\frac{\partial \xi}{\partial x} + \left(\frac{\partial Y}{\partial \eta} + \frac{\partial Y}{\partial u}q + \frac{\partial Y}{\partial p}q_\xi + \frac{\partial Y}{\partial q}q_\eta\right)\frac{\partial \eta}{\partial x} = 0,\\
&\left(\frac{\partial X}{\partial \xi} + \frac{\partial X}{\partial u}p + \frac{\partial X}{\partial p}p_\xi + \frac{\partial X}{\partial q}p_\eta\right)\frac{\partial \xi}{\partial y} + \left(\frac{\partial X}{\partial \eta} + \frac{\partial X}{\partial u}q + \frac{\partial X}{\partial p}q_\xi + \frac{\partial X}{\partial q}q_\eta\right)\frac{\partial \eta}{\partial y} = 0,\\
&\left(\frac{\partial Y}{\partial \xi} + \frac{\partial Y}{\partial u}p + \frac{\partial Y}{\partial p}p_\xi + \frac{\partial Y}{\partial q}p_\eta\right)\frac{\partial \xi}{\partial y} + \left(\frac{\partial Y}{\partial \eta} + \frac{\partial Y}{\partial u}q + \frac{\partial Y}{\partial p}q_\xi + \frac{\partial Y}{\partial q}q_\eta\right)\frac{\partial \eta}{\partial y} = 1,
\end{aligned} \qquad (18)$$

where $p = \frac{\partial u}{\partial \xi}$, $q = \frac{\partial u}{\partial \eta}$, and $p_\eta = q_\xi$; the subscripts ξ and η denote the corresponding partial derivatives. The first two relations in (18) constitute a system of linear algebraic equations for $\frac{\partial \xi}{\partial x}$ and $\frac{\partial \eta}{\partial x}$, and the other two relations form a system of linear algebraic equations for $\frac{\partial \xi}{\partial y}$ and $\frac{\partial \eta}{\partial y}$. Having solved these systems, we find the derivatives: $\frac{\partial \xi}{\partial x} = A$, $\frac{\partial \eta}{\partial x} = B$, $\frac{\partial \xi}{\partial y} = C$, $\frac{\partial \eta}{\partial y} = D$. Then, differentiating the third relation in (16) with respect to x and y, we express $U = \frac{\partial w}{\partial x}$ and $V = \frac{\partial w}{\partial y}$ in terms of the new variables to obtain

$$\begin{aligned}
U &= A\left(\frac{\partial W}{\partial \xi} + \frac{\partial W}{\partial u}p + \frac{\partial W}{\partial p}p_\xi + \frac{\partial W}{\partial q}p_\eta\right) + B\left(\frac{\partial W}{\partial \eta} + \frac{\partial W}{\partial u}q + \frac{\partial W}{\partial p}q_\xi + \frac{\partial W}{\partial q}q_\eta\right),\\
V &= C\left(\frac{\partial W}{\partial \xi} + \frac{\partial W}{\partial u}p + \frac{\partial W}{\partial p}p_\xi + \frac{\partial W}{\partial q}p_\eta\right) + D\left(\frac{\partial W}{\partial \eta} + \frac{\partial W}{\partial u}q + \frac{\partial W}{\partial p}q_\xi + \frac{\partial W}{\partial q}q_\eta\right).
\end{aligned}$$

Relations (17) require that U and W should be independent of the second derivatives, i.e.,

$$\frac{\partial U}{\partial p_\xi} = \frac{\partial V}{\partial p_\xi} = \frac{\partial U}{\partial p_\eta} = \frac{\partial V}{\partial p_\eta} = \frac{\partial U}{\partial q_\eta} = \frac{\partial V}{\partial q_\eta} = 0 \quad (p_\eta \equiv q_\xi),$$

which results in additional relations for the functions X, Y, W.

In general, a contact transformation (16)–(17) reduces a second-order equation in two independent variables

$$F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0 \qquad (19)$$

to an equation of the form

$$G\left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \frac{\partial^2 u}{\partial \xi^2}, \frac{\partial^2 u}{\partial \xi \partial \eta}, \frac{\partial^2 u}{\partial \eta^2}\right) = 0. \qquad (20)$$

In some cases, equation (20) turns out to be more simple than (19). If $u = u(\xi, \eta)$ is a solution of equation (20), then formulas (16) define the corresponding solution of equation (19) in parametric form.

S.2.3-2. Legendre transformation.

An important special case of contact transformations is the Legendre transformation defined by the relations

$$w(x,y)+u(\xi,\eta)=x\xi+y\eta,\quad x=\frac{\partial u}{\partial \xi},\quad y=\frac{\partial u}{\partial \eta},\tag{21}$$

where u is the new dependent variable and ξ, η are the new independent variables.

Differentiating the first relation in (21) with respect to x and y and taking into account the other two relations, we obtain the first derivatives:

$$\frac{\partial w}{\partial x}=\xi,\quad \frac{\partial w}{\partial y}=\eta.\tag{22}$$

With (21)–(22), we find the second derivatives

$$\frac{\partial^2 w}{\partial x^2}=J\frac{\partial^2 u}{\partial \eta^2},\quad \frac{\partial^2 w}{\partial x\partial y}=\frac{\partial^2 w}{\partial y\partial x}=-J\frac{\partial^2 u}{\partial \xi\partial \eta},\quad \frac{\partial^2 w}{\partial y^2}=J\frac{\partial^2 u}{\partial \xi^2},$$

where

$$J=\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2,\quad \frac{1}{J}=\frac{\partial^2 u}{\partial \xi^2}\frac{\partial^2 u}{\partial \eta^2}-\left(\frac{\partial^2 u}{\partial \xi\partial \eta}\right)^2.$$

The Legendre transformation (21), with $J\neq 0$, allows us to rewrite a general second-order equation with two independent variables

$$F\left(x,y,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y},\frac{\partial^2 w}{\partial x^2},\frac{\partial^2 w}{\partial x\partial y},\frac{\partial^2 w}{\partial y^2}\right)=0\tag{23}$$

in the form

$$F\left(\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial \eta},\xi\frac{\partial u}{\partial \xi}+\eta\frac{\partial u}{\partial \eta}-u,\xi,\eta,J\frac{\partial^2 u}{\partial \eta^2},-J\frac{\partial^2 u}{\partial \xi\partial \eta},J\frac{\partial^2 u}{\partial \xi^2}\right)=0.\tag{24}$$

Sometimes equation (24) may be simpler than (23).

Let $u=u(\xi,\eta)$ be a solution of equation (24). Then the formulas

$$w=\xi\frac{\partial u}{\partial \xi}+\eta\frac{\partial u}{\partial \eta}-u(\xi,\eta),\quad x=\frac{\partial u}{\partial \xi},\quad y=\frac{\partial u}{\partial \eta}$$

define the corresponding solution of equation (23) in parametric form.

Remark. The Legendre transformation may result in the loss of solutions for which $J=0$.

Example 6. The Legendre transformation (21) reduces the nonlinear equation

$$f\left(\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\right)\frac{\partial^2 w}{\partial x^2}+g\left(\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\right)\frac{\partial^2 w}{\partial x\partial y}+h\left(\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\right)\frac{\partial^2 w}{\partial y^2}=0$$

to the following linear equation with variable coefficients:

$$f(\xi,\eta)\frac{\partial^2 u}{\partial \eta^2}-g(\xi,\eta)\frac{\partial^2 u}{\partial \xi\partial \eta}+h(\xi,\eta)\frac{\partial^2 u}{\partial \xi^2}=0.$$

S.2.3-3. Euler transformation.

The Euler transformation belongs to the class of contact transformations and is defined by the relations

$$w(x,y)+u(\xi,\eta)=x\xi,\quad x=\frac{\partial u}{\partial \xi},\quad y=\eta.\tag{25}$$

Differentiating the first relation in (25) with respect to x and y and taking into account the other two relations, we find that

$$\frac{\partial w}{\partial x}=\xi,\quad \frac{\partial w}{\partial y}=-\frac{\partial u}{\partial \eta}.\tag{26}$$

Differentiating these expressions in x and y, we find the second derivatives:

$$w_{xx} = \frac{1}{u_{\xi\xi}}, \quad w_{xy} = -\frac{u_{\xi\eta}}{u_{\xi\xi}}, \quad w_{yy} = \frac{u_{\xi\eta}^2 - u_{\xi\xi}u_{\eta\eta}}{u_{\xi\xi}}. \tag{27}$$

The subscripts indicate the corresponding partial derivatives.

The Euler transformation (25)–(27) is employed in finding solutions and linearization of certain nonlinear partial differential equations.

The Euler transformation (25) allows us to reduce a general second-order equation with two independent variables

$$F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0 \tag{28}$$

to the equation

$$F\left(u_\xi, \eta, \xi u_\xi - u, \xi, -u_\eta, \frac{1}{u_{\xi\xi}}, -\frac{u_{\xi\eta}}{u_{\xi\xi}}, \frac{u_{\xi\eta}^2 - u_{\xi\xi}u_{\eta\eta}}{u_{\xi\xi}}\right) = 0. \tag{29}$$

In some cases, equation (29) may become simpler than equation (28).

Let $u = u(\xi, \eta)$ be a solution of equation (29). Then formulas (25) define the corresponding solution of equation (28) in parametric form.

Example 7. The equation

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2} = f\left(y, \frac{\partial w}{\partial x}\right)$$

can be linearized with the help of the Euler transformation (25)–(27) to obtain

$$\frac{\partial u}{\partial \eta} = -f(\eta, \xi)\frac{\partial^2 u}{\partial \xi^2}.$$

Example 8. The equation

$$\frac{\partial^2 w}{\partial x \partial y} = f\left(y, \frac{\partial w}{\partial x}\right)\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2}$$

can be linearized by the Euler transformation (25)–(27) to obtain

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = f(\eta, \xi)\frac{\partial u}{\partial \eta}.$$

⊙ *References for Subsection* S.2.3: M. G. Kurenskii (1934), N. H. Ibragimov (1985, 1994), H. Stephani (1989), B. J. Cantwell (2002), A. D. Polyanin and V. F. Zaitsev (2002).

S.2.4. Bäcklund Transformations. Differential Substitutions

S.2.4-1. Bäcklund transformations.

1°. Let $w = w(x, y)$ be a solution of the equation

$$F_1\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}\right) = 0, \tag{30}$$

and let $u = u(x, y)$ be a solution of another equation

$$F_2\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\right) = 0. \tag{31}$$

Equations (30) and (31) are said to be related by the Bäcklund transformation

$$\begin{aligned} &\Phi_1\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0, \\ &\Phi_2\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \end{aligned} \tag{32}$$

if the compatibility of the pair (30), (32) implies equation (31), and the compatibility of the pair (31), (32) implies (30). If, for some specific solution $u = u(x, y)$ of equation (31), one succeeds in solving equations (32) for $w = w(x, y)$, then this function $w = w(x, y)$ will be a solution of equation (30). Relations (32) are also called differential constraints.

Bäcklund transformations may preserve the form of equations* (such transformations are used for obtaining new solutions) or establish relations between solutions of different equations (such transformations are used for obtaining solutions of one equation from solutions of another equation).

2°. For two nth-order evolution equations of the forms

$$\frac{\partial w}{\partial t} = F_1\left(x, w, \frac{\partial w}{\partial x}, \dots, \frac{\partial^n w}{\partial x^n}\right),$$
$$\frac{\partial u}{\partial t} = F_2\left(x, u, \frac{\partial u}{\partial x}, \dots, \frac{\partial^n u}{\partial x^n}\right),$$

a Bäcklund transformation is often sought in the form of a differential constraint

$$\Phi\left(x, w, \frac{\partial w}{\partial x}, \dots, \frac{\partial^m w}{\partial x^m}, u, \frac{\partial u}{\partial x}, \dots, \frac{\partial^k u}{\partial x^k}\right) = 0$$

containing derivatives in only one variable x (the second variable, t, is present implicitly through the functions w, u). This constraint can be regarded as an ordinary differential equation in one of the dependent variables.

S.2.4-2. Differential substitutions.

In mathematical physics, apart from the Bäcklund transformations, one often resorts to the so-called differential substitutions. For second-order differential equations, differential substitutions have the form

$$w = \Psi\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right).$$

A differential substitution increases the order of an equation (if it is inserted into an equation for w) and allows us to obtain solutions of one equation from those of another. The relationship between the solutions of the two equations is generally not invertible and is, in a sense, unilateral. A differential substitution may be obtained as a consequence of a Bäcklund transformation (although this is not always the case).

S.2.4-3. Examples of Bäcklund transformations and differential substitutions.

Example 9. The Burgers equation

$$\frac{\partial w}{\partial t} = w\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} \tag{33}$$

is related to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{34}$$

by the Bäcklund transformation

$$\begin{aligned}\frac{\partial u}{\partial x} - \frac{1}{2}uw &= 0,\\ \frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial (uw)}{\partial x} &= 0.\end{aligned} \tag{35}$$

Eliminating w from (35), we obtain equation (34).

* In such cases, these are referred to as auto-Bäcklund transformations.

Conversely, let $u(x,t)$ be a nonzero solution of the heat equation (34). Dividing (34) by u, differentiating the resulting equation with respect to x, and taking into account that $(u_t/u)_x = (u_x/u)_t$, we obtain

$$\left(\frac{u_x}{u}\right)_t = \left(\frac{u_{xx}}{u}\right)_x.$$

Hence, taking into account the relations that follow from the first equation in (35),

$$\frac{u_x}{u} = \frac{w}{2} \quad\Longrightarrow\quad \frac{u_{xx}}{u} - \left(\frac{u_x}{u}\right)^2 = \frac{w_x}{2} \quad\Longrightarrow\quad \frac{u_{xx}}{u} = \frac{w_x}{2} + \frac{1}{4}w^2,$$

we obtain the Burgers equation (34).

Remark. The first relation in (35) can be rewritten as the differential substitution (the Hopf–Cole transformation)

$$w = \frac{2u_x}{u}. \tag{36}$$

Substituting (36) into (33), we obtain the equation

$$\frac{2u_{tx}}{u} - \frac{2u_t u_x}{u^2} = \frac{2u_{xxx}}{u} - \frac{2u_x u_{xx}}{u^2},$$

which can be converted to

$$\frac{\partial}{\partial x}\left[\frac{1}{u}\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right)\right] = 0.$$

Thus, using formula (36), one can transform each solution of the linear heat equation (34) into a solution of the Burgers equation (33). The converse is not generally true. Indeed, a solution of equation (33) generates a solution of the more general equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(t)u,$$

where $f(t)$ is a function of t.

Example 10. The nonlinear Schrödinger equation with a cubic nonlinearity

$$i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + |w|^2 w = 0,$$

where w is a complex-valued function of real variables x and t $(i^2 = -1)$, is invariant under the Bäcklund transformation

$$\begin{aligned}
\frac{\partial w}{\partial x} - \frac{\partial \widetilde{w}}{\partial x} &= iaf_1 - \frac{i}{2}f_2 g_1,\\
\frac{\partial w}{\partial t} - \frac{\partial \widetilde{w}}{\partial t} &= \frac{1}{2}g_1\left(\frac{\partial w}{\partial x} + \frac{\partial \widetilde{w}}{\partial x}\right) - ag_2 + \frac{i}{4}f_1\bigl(|f_1|^2 + |f_2|^2\bigr).
\end{aligned}$$

Here, we have used the notation

$$f_1 = w - \widetilde{w}, \quad f_2 = w + \widetilde{w}, \quad g_1 = i\varepsilon\bigl(b - 2|f_1|^2\bigr)^{1/2}, \quad g_2 = i\bigl(af_1 - \tfrac{1}{2}f_2 g_1\bigr),$$

where a and b are arbitrary real constants, $\varepsilon = \pm 1$.

Example 11. The Korteweg–de Vries equation

$$\frac{\partial w}{\partial t} + 6w\frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} = 0$$

and the modified Korteweg–de Vries equation

$$\frac{\partial u}{\partial t} - 6u^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

are related by the Bäcklund transformation

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \varepsilon(w + u^2), \quad \varepsilon = \pm 1,\\
\frac{\partial u}{\partial t} &= \varepsilon\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial}{\partial x}(uw).
\end{aligned} \tag{37}$$

The first relation in (37) is a Miura transformation which can be rewritten as a differential substitution by solving (37) for w.

S.2.4-4. Bäcklund transformations based on conservation laws.

Consider a differential equation written as a conservation law,

$$\frac{\partial}{\partial x}\left[F\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \dots\right)\right] + \frac{\partial}{\partial y}\left[G\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \dots\right)\right] = 0. \tag{38}$$

The Bäcklund transformation

$$dz = F(w, w_x, w_y, \dots)\,dy - G(w, w_x, w_y, \dots)\,dx, \quad d\eta = dy \tag{39}$$

$$\left(dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy \quad \Longrightarrow \quad \frac{\partial z}{\partial x} = -G, \quad \frac{\partial z}{\partial x} = F\right)$$

determines the passage from the variables x and y to the new independent variables z and η according to the rule

$$\frac{\partial}{\partial x} = -G\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \eta} + F\frac{\partial}{\partial z}.$$

Here, F and G are the same as in (38). The transformation (39) preserves the order of the equation under consideration.

Remark. Often one may encounter transformations (39) that are supplemented with a transformation of the unknown function in the form $u = \varphi(w)$.

Example 12. Consider the third-order nonlinear equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2}{\partial x^2}\left[f(w)\frac{\partial w}{\partial x}\right], \tag{40}$$

which represents a special case of equation (38) for $y = t$, $F = [f(w)w_x]_x$, and $G = -w$.

In this case, transformation (39) has the form

$$dz = w\,dx + [f(w)w_x]_x\,dt, \quad d\eta = dt \tag{41}$$

and determines a transformation from the variables x and y to the new independent variables z and η according to the rule

$$\frac{\partial}{\partial x} = w\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \eta} + [f(w)w_x]_x\frac{\partial}{\partial z}.$$

Applying transformation (41) to equation (40), we obtain

$$\frac{\partial w}{\partial \eta} = w^2\frac{\partial^2}{\partial z^2}\left[wf(w)\frac{\partial w}{\partial z}\right]. \tag{42}$$

The substitution $w = 1/u$ reduces (42) to an equation of the form (40),

$$\frac{\partial u}{\partial \eta} = \frac{\partial^2}{\partial z^2}\left[\frac{1}{u^3}f\left(\frac{1}{u}\right)\frac{\partial u}{\partial z}\right].$$

In the special case of $f(w) = aw^{-3}$, the nonlinear equation (40) is reduced to the linear equation $u_\eta = au_{zzz}$ by the transformation (41).

⊙ *References for Subsection* S.2.4: G. L. Lamb (1974), R. M. Miura (1976), R. L. Anderson and N. H. Ibragimov (1979), A. S. Fokas and R. L. Anderson (1979), A. S. Fokas and B. Fuchssteiner (1981), M. J. Ablowitz and H. Segur (1981), N. H. Ibragimov (1985, 1994), H. Stephani (1989), B. J. Cantwell (2002).

S.3. Traveling-Wave Solutions and Self-Similar Solutions. Similarity Methods

S.3.1. Preliminary Remarks

There are a number of methods for the construction of exact solutions to equations of mathematical physics that are based on the reduction of the original equations to equations in fewer dependent and/or independent variables. The main idea is to find such variables and, by passing to them, to obtain simpler equations. In particular, in this way, finding exact solutions of some partial differential equations in two independent variables may be reduced to finding solutions of appropriate ordinary differential equations (or systems of ordinary differential equations). Naturally, the ordinary differential equations thus obtained do not give all solutions of the original partial differential equation, but provide only a class of solutions with some specific properties.

The simplest classes of exact solutions described by ordinary differential equations involve traveling-wave solutions and self-similar solutions. The existence of such solutions is due to the invariance of the equations in question under translations and scaling transformations.

Traveling-wave solutions and self-similar solutions often occur in various applications. Below we consider some characteristic features of such solutions.

It is assumed that the unknown w depends on two variables, x and t, where t plays the role of time and x is a spatial coordinate.

S.3.2. Traveling Wave Solutions. Invariance of Equations Under Translations

1°. *Traveling-wave solutions*, by definition, are of the form

$$w(x,t) = W(z), \qquad z = x + \lambda t, \tag{1}$$

where λ plays the role of the wave propagation velocity (the sign of λ can be arbitrary and the value $\lambda = 0$ corresponds to a stationary solution). Traveling-wave solutions are characterized by the fact that the profiles of these solutions at different time instants are obtained from one another by appropriate shifts (translations) along the x-axis. Consequently, a Cartesian coordinate system moving with a constant speed can be introduced in which the profile of the desired quantity is stationary.

A traveling-wave solution is found by directly substituting the representation (1) into the original equation and taking into account the relations $w_x = W'$, $w_t = \lambda W'$, etc. (the prime denotes a derivative with respect to z).

Traveling-wave solutions occur for equations that do not explicitly involve independent variables,

$$F\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial t}, \frac{\partial^2 w}{\partial t^2}, \dots\right) = 0. \tag{2}$$

Substituting (1) into (2), we obtain an autonomous ordinary differential equation for the function $W(z)$:

$$F(W, W', \lambda W', W'', \lambda W'', \lambda^2 W'', \dots) = 0,$$

where λ is an arbitrary constant.

2°. It should be observed that equations of the form (2) are invariant (i.e., preserve their form) under translations in both independent variables:

$$x = \bar{x} + C_1, \qquad t = \bar{t} + C_2, \tag{3}$$

where C_1 and C_2 are arbitrary constants. The property of the invariance of specific equations under translation transformations (3) is inseparably linked with the existence of traveling-wave solutions of such equations (the former implies the latter).

Traveling-wave solutions are simplest *invariant solutions*, i.e., solutions whose properties are due to the fact that the equations are invariant under certain transformations (containing arbitrary constants).

Example 1. The nonlinear heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right] \tag{4}$$

admits a traveling-wave solution. Substituting (1) into (4), we arrive at the ordinary differential equation

$$[f(W)W']' - \lambda W' = 0.$$

Integrating this equation twice yields its solution in implicit form:

$$\int \frac{f(W)\,dW}{\lambda W + C_1} = z + C_2,$$

where C_1 and C_2 are arbitrary constants.

Example 2. Consider the homogeneous Monge–Ampère equation

$$\left(\frac{\partial^2 w}{\partial x \partial t}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial t^2} = 0. \tag{5}$$

Inserting (1) into this equation, we obtain an identity. Therefore, equation (5) admits solutions of the form

$$w = W(x + \lambda t),$$

where $W(z)$ is an arbitrary function and λ is an arbitrary constant.

S.3.3. Self Similar Solutions. Invariance of Equations Under Scaling Transformations

By definition, a *self-similar solution* is a solution of the form

$$w(x,t) = t^\alpha U(\zeta), \qquad \zeta = xt^\beta. \tag{6}$$

The profiles of these solutions at different time instants are obtained from one another by a similarity transformation (like scaling).

Self-similar solutions exist if the scaling of the independent and dependent variables,

$$t = C\bar{t}, \qquad x = C^k \bar{x}, \qquad w = C^m \bar{w}, \qquad \text{where } C \neq 0 \text{ is an arbitrary constant,} \tag{7}$$

for some k and m, is equivalent to the identical transformation. This means that the original equation

$$F(x, t, w, w_x, w_t, w_{xx}, w_{xt}, w_{tt}, \ldots) = 0, \tag{8}$$

when subjected to transformation (7), turns into the same equation in the new variables,

$$F(\bar{x}, \bar{t}, \bar{w}, \bar{w}_{\bar{x}}, \bar{w}_{\bar{t}}, \bar{w}_{\bar{x}\bar{x}}, \bar{w}_{\bar{x}\bar{t}}, \bar{w}_{\bar{t}\bar{t}}, \ldots) = 0. \tag{9}$$

In practice, the above existence criterion is checked: if a pair of k and m in (7) has been found such that (9) holds true, there is a self-similar solution of the form (6), where

$$\alpha = m, \qquad \beta = -k. \tag{10}$$

These relations follow from the condition that the scaling transformation (7) must preserve the form of the variables (6):

$$w = t^\alpha U(\zeta),\ \ \zeta = xt^\beta \quad \Longrightarrow \quad \bar{w} = \bar{t}^\alpha U(\bar{\zeta}),\ \ \bar{\zeta} = \bar{x}\bar{t}^\beta.$$

The method of constructing self-similar solutions on the basis of scaling transformations (7) is called the *similarity method*. It is significant that these transformations involve the arbitrary constant C as a parameter.

Example 3. Consider the heat equation with a nonlinear power-law source term

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw^n. \tag{11}$$

The scaling transformation (7) converts equation (11) into

$$C^{m-1}\frac{\partial \bar{w}}{\partial \bar{t}} = aC^{m-2k}\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + bC^{mn}\bar{w}^n.$$

Equating the powers of C yields the following system of linear algebraic equations for the constants k and m:

$$m - 1 = m - 2k = mn.$$

This system admits a unique solution: $k = \frac{1}{2}$, $m = \frac{1}{1-n}$. Using this solution together with relations (6) and (10), we obtain self-similar variables in the form

$$w = t^{1/(1-n)} U(\zeta), \quad \zeta = x t^{-1/2}.$$

Inserting these into (11), we arrive at the following ordinary differential equation for the function $U(\zeta)$:

$$aU''_{\zeta\zeta} + \frac{1}{2}\zeta U'_\zeta + \frac{1}{n-1}U + bU^n = 0.$$

Example 4. Consider the nonlinear equation

$$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(w^n \frac{\partial w}{\partial x}\right), \tag{12}$$

which occurs in problems of wave and gas dynamics. Inserting (7) into (12) yields

$$C^{m-2}\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} = aC^{mn+m-2k}\frac{\partial}{\partial \bar{x}}\left(\bar{w}^n \frac{\partial \bar{w}}{\partial \bar{x}}\right).$$

Equating the powers of C results in a single linear equation, $m-2 = mn+m-2k$. Hence, we obtain $k = \frac{1}{2}mn + 1$, where m is arbitrary. Further, using (6) and (10), we find self-similar variables:

$$w = t^m U(\zeta), \quad \zeta = x t^{-\frac{1}{2}mn-1}, \quad m \text{ is arbitrary}.$$

Substituting these into (12), one obtains an ordinary differential equation for the function $U(\zeta)$.

Remark. Traveling-wave solutions are closely related to self-similar solutions. Indeed, taking

$$w = \ln u, \quad W = \ln F, \quad t = \ln \tau, \quad x = \ln y$$

in (1), we obtain a representation of a traveling wave in self-similar form, $u = F(x+\lambda t) = F\big(\ln(y\tau^\lambda)\big) = F_1(y\tau^\lambda)$.

S.3.4. Exponential Self Similar Solutions. Equations Invariant Under Combined Translation and Scaling

By definition, an *exponential self-similar solution* is a solution of the form

$$w(x,t) = e^{\alpha t} V(\xi), \quad \xi = x e^{\beta t}. \tag{13}$$

An exponential self-similar solution exists if the equation under consideration is invariant under the transformation

$$t = \bar{t} + \ln C, \quad x = C^k \bar{x}, \quad w = C^m \bar{w}, \qquad \text{where } C > 0 \text{ is an arbitrary constant}, \tag{14}$$

for some k and m. Transformation (14) is a combination of a shift in t and scaling in x and w. Observe that these transformations contain an arbitrary constant C as a parameter.

In practice, the above existence criterion is checked: if a pair of k and m in (14) has been found such that the equation remains the same, then there exists an exponential self-similar solution with the new variables having the form (13), where

$$\alpha = m, \quad \beta = -k. \tag{15}$$

These relations follow from the condition that the scaling transformation (14) must preserve the form of the variables of (13):

$$w = e^{\alpha t} V(\xi), \ \ \xi = x e^{\beta t} \quad \Longrightarrow \quad \bar{w} = e^{\alpha \bar{t}} V(\bar{\xi}), \ \ \bar{\xi} = \bar{x} e^{\beta \bar{t}}.$$

Remark. Solutions of the form (13) are sometimes called limit self-similar solutions.

Example 5. Let us show that the nonlinear heat equation

$$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^n \frac{\partial w}{\partial x}\right) \tag{16}$$

TABLE 16
Invariant solutions found by using combining translations and scaling (C, C_1, and C_2 are arbitrary constants)

No.	Form of solution	Invariant transformation	Relations for coefficients
1	$w=U(z),\ z=\alpha x+\beta y$	$t=\bar{t}+C_1,\ x=\bar{x}+C_2$	α and β are arbitrary constants
2	$w=t^{\alpha}U(z),\ z=xt^{\beta}$	$t=C\bar{t},\ x=C^k\bar{x},\ w=C^m\bar{w}$	$\alpha=m,\ \beta=-k$
3	$w=e^{\alpha t}U(z),\ z=xe^{\beta t}$	$t=\bar{t}+\ln C,\ x=C^k\bar{x},\ w=C^m\bar{w}$	$\alpha=m,\ \beta=-k$
4	$w=t^{\alpha}U(z),\ z=x+\beta\ln t$	$t=C\bar{t},\ x=\bar{x}+k\ln C,\ w=C^m\bar{w}$	$\alpha=m,\ \beta=-k$

admits an exponential self-similar solution. Substituting (14) into (16) yields

$$C^m\frac{\partial\bar{w}}{\partial\bar{t}}=aC^{mn+m-2k}\frac{\partial}{\partial\bar{x}}\left(\bar{w}^n\frac{\partial\bar{w}}{\partial\bar{x}}\right).$$

Equating the exponents of C, we obtain one linear equation, $m=mn+m-2k$. Hence, we have $k=\frac{1}{2}mn$, where m is arbitrary. Further, using formulas (13) and (15) and taking (without loss of generality) $m=2$, which is equivalent to scaling of time t, we find the new variables:

$$w=e^{2t}V(\xi),\qquad \zeta=xe^{-nt}. \tag{17}$$

Inserting these into (16), we obtain an ordinary differential equation for the function $V(\xi)$:

$$a(V^nV'_\xi)'_\xi+n\xi V'_\xi-2V=0.$$

Example 6. With this method, it can be shown that equation (12) also admits an exponential self-similar solution of the form (17).

Table 16 lists invariant solutions which can be found by combining translation and scaling of the independent variables and scaling of the dependent variable. Apart from traveling-wave (row 1), self-similar (row 2), and exponential self-similar (row 3) solutions considered above, the last row in the table describes another invariant solution. Below we give an example that illustrates the method for the construction of such a solution.

Example 7. Let us show that the nonlinear heat equation (16) admits a solution having the form specified in the fourth row of Table 16. To that end, we use the transformation

$$t=C\bar{t},\qquad x=\bar{x}+k\ln C,\qquad w=C^m\bar{w}$$

to obtain

$$C^{m-1}\frac{\partial\bar{w}}{\partial\bar{t}}=aC^{mn+m}\frac{\partial}{\partial\bar{x}}\left(\bar{w}^n\frac{\partial\bar{w}}{\partial\bar{x}}\right).$$

Equating the powers of C yields one linear equation, $m-1=mn+m$. Hence, we find that $m=-1/n$ and k may be arbitrary. Therefore (see row 4 in Table 16), equation (16) has a solution of the form

$$w=t^{-1/n}U(z),\qquad z=x+\beta\ln t,\qquad \text{where } \beta \text{ is arbitrary}. \tag{18}$$

Substituting (18) into (16), we arrive at the autonomous differential equation

$$a(U^nU'_z)'_z-\beta U'_z+\frac{1}{n}U=0.$$

The value $\beta=0$ corresponds to an additively separable solution.

The examples considered in Section S.3 show that the construction of exact solutions by means of reducing the dimension of a partial differential equation is possible, provided that the equation in question is invariant under certain transformations (containing one or more arbitrary parameters) or, in other words, the equation possesses a certain symmetry. Below, in Section S.7, we describe a more general approach to the construction of exact solutions. This approach is based on the methods of group-theoretic analysis of differential equations. These methods provide a regular procedure for obtaining invariant solutions of an analogous or more complex structure.

⊙ *References for Section* S.3: P. W. Bridgman (1931), W. F. Ames (1972), G. W. Bluman and J. D. Cole (1974), G. I. Barenblatt and Ya. B. Zel'dovich (1972), W. F. Ames, R. J. Lohner, and E. Adams (1981), L. Dresner (1983), G. I. Barenblatt (1989), L. I. Sedov (1993).

S.4. Method of Generalized Separation of Variables

S.4.1. Introduction

S.4.1-1. Preliminary remarks. Multiplicative and additive separable solutions.

Separation of variables is the most common approach to solve linear equations of mathematical physics. For equations in two independent variables x, y and a dependent variable w, this approach involves searching for exact solutions in the form of the product of functions depending on different arguments:

$$w(x,t) = \varphi(x)\psi(t). \tag{1}$$

The integration of a few classes of first-order nonlinear partial differential equations is based on searching for exact solutions in the form of the sum of functions depending on different arguments:

$$w(x,t) = \varphi(x) + \psi(t). \tag{2}$$

Some second- and higher-order nonlinear equations of mathematical physics also have exact solutions of the form (1) or (2). Such solutions are called *multiplicative separable* and *additive separable*, respectively.

⊙ *References*: D. Zwillinger (1989), A. N. Tikhonov and A. A. Samarskii (1990), A. D. Polyanin (2002), A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

S.4.1-2. Simple cases of variable separation in nonlinear partial differential equations.

In isolated cases, the separation of variables in nonlinear equations is carried out following the same technique as in linear equations. Specifically, an exact solution is sought in the form of the product or sum of functions depending on different arguments. On substituting it into the equation and performing elementary algebraic manipulations, one obtains an equation with the two sides dependent on different variables (for equations with two variables). Then one concludes that the expressions on each side must be equal to the same constant quantity, called a separation constant. Below we consider specific examples.

Example 1. The heat equation with a power nonlinearity

$$\frac{\partial w}{\partial t} = a\frac{\partial}{\partial x}\left(w^k \frac{\partial w}{\partial x}\right) \tag{3}$$

has a multiplicative separable solution. Substituting (1) into (3) yields

$$\varphi\psi'_t = a\psi^{k+1}(\varphi^k\varphi'_x)'_x.$$

Separating the variables by dividing both sides by $\varphi\psi^{k+1}$, we obtain

$$\frac{\psi'_t}{\psi^{k+1}} = \frac{a(\varphi^k\varphi'_x)'_x}{\varphi}.$$

The left-hand side depends on t alone and the right-hand side on x alone. This is possible only if

$$\frac{\psi'_t}{\psi^{k+1}} = C, \qquad \frac{a(\varphi^k\varphi'_x)'_x}{\varphi} = C, \tag{4}$$

where C is an arbitrary constant (separation constant). On solving the ordinary differential equations (4), we obtain a solution of equation (3) with the form (1).

The procedure for constructing a separable solution (1) of the nonlinear equation (3) is identical to that used in solving linear equations [in particular, equation (3) with $k = 0$]. We refer to the cases of similar separation of variables as *simple separable* cases.

Example 2. The wave equation with an exponential nonlinearity

$$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(e^{\lambda w}\frac{\partial w}{\partial x}\right) \tag{5}$$

has an additive separable solution. On substituting (2) into (5) and dividing by $e^{\lambda\psi}$, we arrive at the equation

$$e^{-\lambda\psi}\psi''_{tt} = a(e^{\lambda\varphi}\varphi'_x)'_x,$$

whose left-hand side depends on t alone and the right-hand side on x alone. This is possible only if

$$e^{-\lambda\psi}\psi''_{tt} = C, \quad a(e^{\lambda\varphi}\varphi'_x)'_x = C, \tag{6}$$

where C is an arbitrary constant. Solving the ordinary differential equations (6) yields a solution of equation (5) with the form (2).

Example 3. The steady-state heat equation in an anisotropic medium with a logarithmic source

$$\frac{\partial}{\partial x}\left[f(x)\frac{\partial w}{\partial x}\right] + \frac{\partial}{\partial y}\left[g(y)\frac{\partial w}{\partial y}\right] = aw\ln w \tag{7}$$

has a multiplicative separable solution

$$w = \varphi(x)\psi(y). \tag{8}$$

On substituting (8) into (7), dividing by $\varphi\psi$, and rearranging individual terms of the resulting equation, we obtain

$$\frac{1}{\varphi}[f(x)\varphi'_x]'_x - a\ln\varphi = -\frac{1}{\psi}[g(y)\psi'_y]'_y + a\ln\psi.$$

The left-hand side of this equation depends only on x and the right-hand only on y. By equating both sides to a constant quantity, one obtains ordinary differential equations for $\varphi(x)$ and $\psi(y)$.

⦿ *References*: L. V. Ovsiannikov (1982), A. D. Polyanin (2002, Supplement B).

S.4.1-3. Examples of nontrivial variable separation in nonlinear partial differential equations.

Unlike linear equations, the variables in nonlinear equations often separate differently. We exemplify this below.

Example 4. Consider the equation with a cubic nonlinearity

$$\frac{\partial w}{\partial t} = f(t)\frac{\partial^2 w}{\partial x^2} + w\left(\frac{\partial w}{\partial x}\right)^2 - aw^3, \tag{9}$$

where $f(t)$ is an arbitrary function, $a > 0$. We look for exact solutions in the product form. We substitute (1) into (9) and divide the resulting equation by $f(t)\varphi(x)\psi(t)$ to obtain

$$\frac{\psi'_t}{f\psi} = \frac{\varphi''_{xx}}{\varphi} + \frac{\psi^2}{f}[(\varphi'_x)^2 - a\varphi^2]. \tag{10}$$

In the general case, this expression cannot be represented as the sum of two functions depending on different arguments. This however does not mean that equation (9) has no solutions of the form (1).

1°. One can make sure by direct check that the functional-differential equation (10) has solutions

$$\varphi(x) = C\exp\bigl(\pm x\sqrt{a}\,\bigr), \quad \psi(t) = \exp\left[a\int f(t)\,dt\right], \tag{11}$$

where C is an arbitrary constant. Solutions (11) for φ make the expression in square brackets in (10) vanish, which allows the separation of variables.

2°. There is a more general solution of the functional-differential equation (10):

$$\varphi(x) = C_1\exp\bigl(x\sqrt{a}\,\bigr) + C_2\exp\bigl(-x\sqrt{a}\,\bigr),$$
$$\psi(t) = e^F\left(C_3 + 8aC_1C_2\int e^{2F}\,dt\right)^{-1/2}, \quad F = a\int f(t)\,dt,$$

where C_1, C_2, and C_3 are arbitrary constants. The function $\varphi = \varphi(x)$ is such that both x-dependent expressions in (10) are constant simultaneously:

$$\varphi''_{xx}/\varphi = \text{const}, \quad (\varphi'_x)^2 - a\varphi^2 = \text{const}.$$

It is this circumstance that makes it possible to separate the variables.

Note that the function $\psi = \psi(t)$ satisfies the Bernoulli equation $\psi'_t = af(t)\psi - 4aC_1C_2\psi^3$.

Example 5. Consider the third-order equation with a quadratic nonlinearity

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2} + a\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = b\frac{\partial^3 w}{\partial x^3} + c\frac{\partial^3 w}{\partial y^3}. \tag{12}$$

We look for additive separable solutions

$$w = f(x) + g(y). \tag{13}$$

Substituting (13) into (12) yields

$$g'_y f''_{xx} + af'_x g''_{yy} = bf'''_{xxx} + cg'''_{yyy}. \tag{14}$$

This expression cannot be rewritten as the equality of two functions depending on different arguments.

It is not difficult to see that the functional-differential equation (14) is satisfied

$$\begin{aligned}
&\text{if } g'_y = C_1 \quad\Longrightarrow\quad g(y) = C_1 y + C_2, \quad f(x) = C_3 \exp(C_1 x/b) + C_4 x \quad \text{(case 1)},\\
&\text{if } f'_x = C_1 \quad\Longrightarrow\quad f(x) = C_1 x + C_2, \quad g(y) = C_3 \exp(aC_1 y/c) + C_4 y \quad \text{(case 2)},
\end{aligned}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants. In both cases, two terms of the four in (14) vanish, which makes it possible to separate the variables.

In addition, equation (12) has a more complicated solution of the form (13):

$$w = C_1 e^{-a\lambda x} + \frac{c\lambda}{a} x + C_2 e^{\lambda y} - ab\lambda y + C_3,$$

where C_1, C_2, C_3, and λ are arbitrary constants. The mechanism of separation of variables is different here: both nonlinear terms on the left-hand side in (14) contain terms which cannot be rewritten in additive form but are equal in magnitude and have unlike signs. In adding, the two terms cancel out, thus resulting in separation of variables:

$$\begin{array}{rcl}
g'_y f''_{xx} & = & C_1 C_2 a^2 \lambda^3 e^{\lambda y - a\lambda x} - C_1 b(a\lambda)^3 e^{-a\lambda x} \\
+ \quad a f'_x g''_{yy} & = & -C_1 C_2 a^2 \lambda^3 e^{\lambda y - a\lambda x} + C_2 c\lambda^3 e^{\lambda y} \\
\hline
\multicolumn{3}{c}{g'_y f''_{xx} + a f'_x g''_{yy} = -C_1 b(a\lambda)^3 e^{-a\lambda x} + C_2 c\lambda^3 e^{\lambda y} = b f'''_{xxx} + c g'''_{yyy}}
\end{array}.$$

Example 6. Consider the second-order equation with a cubic nonlinearity

$$(1 + w^2)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - 2w\left(\frac{\partial w}{\partial x}\right)^2 - 2w\left(\frac{\partial w}{\partial y}\right)^2 = aw(1 - w^2). \tag{15}$$

We seek an exact solution of this equation in the product form

$$w = f(x)g(y). \tag{16}$$

Substituting (16) into (15) yields

$$(1 + f^2 g^2)(g f''_{xx} + f g''_{yy}) - 2fg[g^2 (f'_x)^2 + f^2 (g'_y)^2] = afg(1 - f^2 g^2). \tag{17}$$

This expression cannot be rewritten as the equality of two functions with different arguments. Nevertheless, equation (15) has solutions of the form (16). One can make sure by direct check that the functions $f = f(x)$ and $g = g(y)$ satisfying the nonlinear ordinary differential equations

$$\begin{aligned}
(f'_x)^2 &= Af^4 + Bf^2 + C,\\
(g'_y)^2 &= Cg^4 + (a - B)g^2 + A,
\end{aligned} \tag{18}$$

where A, B, and C are arbitrary constants, reduce equation (17) to an identity; to verify this, one should use the relations $f''_{xx} = 2Af^3 + Bf$ and $g''_{yy} = 2Cg^3 + (a - B)g$ that follow from (18).

Remark. By the change of variable $u = 4 \arctan w$ equation (15) can be reduced to a nonlinear heat equation with a sinusoidal source, $\Delta u = a \sin u$.

The examples considered above illustrate some specific features of separable solutions to nonlinear equations. Sections S.4.2–S.4.4 outline fairly general methods for constructing similar and more complicated solutions to nonlinear partial differential equations.

⊙ *References*: R. Steuerwald (1936), A. D. Polyanin (2002, Supplement B).

S.4.2. Structure of Generalized Separable Solutions

S.4.2-1. General form of solutions. The classes of nonlinear equations considered.

To simplify the presentation, we confine ourselves to the case of mathematical physics equations in two independent variables x, y and a dependent variable w (one of the independent variables can play the role of time).

Linear separable equations of mathematical physics admit exact solutions in the form

$$w(x, y) = \varphi_1(x)\psi_1(y) + \varphi_2(x)\psi_2(y) + \cdots + \varphi_n(x)\psi_n(y), \tag{19}$$

where the $w_i = \varphi_i(x)\psi_i(y)$ are particular solutions; the functions $\varphi_i(x)$, as well as the functions $\psi_i(y)$, with different numbers i are not related to one another.

Many nonlinear partial differential equations with quadratic or power nonlinearities,

$$f_1(x)g_1(y)\Pi_1[w] + f_2(x)g_2(y)\Pi_2[w] + \cdots + f_m(x)g_m(y)\Pi_m[w] = 0, \tag{20}$$

also have exact solutions of the form (19). In (20), the $\Pi_i[w]$ are differential forms that are the products of nonnegative integer powers of the function w and its partial derivatives $\partial_x w$, $\partial_y w$, $\partial_{xx} w$, $\partial_{xy} w$, $\partial_{yy} w$, $\partial_{xxx} w$, etc. We will refer to solutions (19) of nonlinear equations (20) as *generalized separable solutions*. Unlike linear equations, in nonlinear equations the functions $\varphi_i(x)$ with different subscripts i are usually related to one another [and to functions $\psi_j(y)$]. In general, the functions $\varphi_i(x)$ and $\psi_j(y)$ in (19) are not known in advance and are to be identified. Subsections S.4.1-2 and S.4.1-3 give examples of exact solutions (19) to nonlinear equations (20) for some simple cases with $n = 1$ or $n = 2$ (for $\psi_1 = \varphi_2 = 1$).

Note that most common of the generalized separable solutions are solutions of the special form

$$w(x, y) = \varphi(x)\psi(y) + \chi(x);$$

the independent variables on the right-hand side can be swapped. In the special case $\chi(x) = 0$, this is a multiplicative separable solution, and if $\varphi(x) = 1$, this is an additive separable solution.

Remark. Expressions of the form (19) are often used in applied and computational mathematics for constructing approximate solutions to differential equations by the Galerkin method (and its modifications).

S.4.2-2. General form of functional-differential equations.

In general, on substituting expression (19) into the differential equation (20), one arrives at a functional-differential equation

$$\Phi_1(X)\Psi_1(Y) + \Phi_2(X)\Psi_2(Y) + \cdots + \Phi_k(X)\Psi_k(Y) = 0 \tag{21}$$

for the $\varphi_i(x)$ and $\psi_i(y)$. The functionals $\Phi_j(X)$ and $\Psi_j(Y)$ depend only on x and y, respectively,

$$\begin{aligned}\Phi_j(X) &\equiv \Phi_j\bigl(x, \varphi_1, \varphi_1', \varphi_1'', \ldots, \varphi_n, \varphi_n', \varphi_n''\bigr),\\ \Psi_j(Y) &\equiv \Psi_j\bigl(y, \psi_1, \psi_1', \psi_1'', \ldots, \psi_n, \psi_n', \psi_n''\bigr).\end{aligned} \tag{22}$$

Here, for simplicity, the formulas are written out for the case of a second-order equation (20); for higher-order equations, the right-hand sides of relations (22) will contain higher-order derivatives of φ_i and ψ_j.

Further, Subsections S.4.3 and S.4.4 outline two different methods for solving functional-differential equations of the form (21), (22).

Remark. Unlike ordinary differential equations, equation (21)–(22) involves several functions (and their derivatives) with different arguments.

⊙ *References for Subsection* S.4.2: S. S. Titov (1988), V. A. Galaktionov and S. A. Posashkov (1989, 1994), V. A. Galaktionov (1995), A. D. Polyanin (2002, Supplement B).

S.4.3. Solution of Functional Differential Equations by Differentiation

S.4.3-1. Description of the method.

Below we describe a procedure for constructing solutions to functional-differential equations. It involves three successive stages.

1°. Assume that $\Psi_k \not\equiv 0$. We divide equation (21) by Ψ_k and differentiate with respect to y. This results in a similar equation but with fewer terms:

$$\widetilde{\Phi}_1(X)\widetilde{\Psi}_1(Y) + \widetilde{\Phi}_2(X)\widetilde{\Psi}_2(Y) + \cdots + \widetilde{\Phi}_{k-1}(X)\widetilde{\Psi}_{k-1}(Y) = 0,$$
$$\widetilde{\Phi}_j(X) = \Phi_j(X), \quad \widetilde{\Psi}_j(Y) = [\Psi_j(Y)/\Psi_k(Y)]'_y.$$

We continue the above procedure until we obtain a separable two-term equation

$$\widehat{\Phi}_1(X)\widehat{\Psi}_1(Y) + \widehat{\Phi}_2(X)\widehat{\Psi}_2(Y) = 0. \tag{23}$$

Three cases must be considered.

Nondegenerate case: $|\widehat{\Phi}_1(X)| + |\widehat{\Phi}_2(X)| \not\equiv 0$ and $|\widehat{\Psi}_1(Y)| + |\widehat{\Psi}_2(Y)| \not\equiv 0$. Then equation (23) is equivalent to the ordinary differential equations

$$\widehat{\Phi}_1(X) + C\widehat{\Phi}_2(X) = 0, \quad C\widehat{\Psi}_1(Y) - \widehat{\Psi}_2(Y) = 0,$$

where C is an arbitrary constant. The equations $\widehat{\Phi}_2 = 0$ and $\widehat{\Psi}_1 = 0$ correspond to the limit case $C = \infty$.

Two degenerate cases:

$$\widehat{\Phi}_1(X) \equiv 0, \quad \widehat{\Phi}_2(X) \equiv 0 \quad \Longrightarrow \quad \widehat{\Psi}_{1,2}(Y) \text{ are any;}$$
$$\widehat{\Psi}_1(Y) \equiv 0, \quad \widehat{\Psi}_2(Y) \equiv 0 \quad \Longrightarrow \quad \widehat{\Phi}_{1,2}(X) \text{ are any.}$$

2°. The solutions of the two-term equation (23) should be substituted into the original functional-differential equation (21) to “remove” redundant constants of integration [these arise because equation (23) is obtained from (21) by differentiation].

3°. The case $\Psi_k \equiv 0$ should be treated separately (since we divided the equation by Ψ_k at the first stage). Likewise, we have to study all other cases where the functionals by which the intermediate functional-differential equations were divided vanish.

Remark 1. The functional-differential equation (21) can happen to have no solutions.

Remark 2. At each subsequent stage, the number of terms in the functional-differential equation can be reduced by differentiation with respect to either y or x. For example, we can assume at the first stage that $\Phi_k \not\equiv 0$. On dividing equation (21) by Φ_k and differentiating with respect to x, we again obtain a similar equation that has fewer terms.

S.4.3-2. Examples of constructing exact generalized separable solutions.

Below we consider specific examples illustrating the application of the above method to constructing exact generalized separable solutions of nonlinear equations.

Example 7. Let us consider the nth-order nonlinear equation

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = f(x)\frac{\partial^n w}{\partial y^n}, \tag{24}$$

where $f(x)$ is an arbitrary function. In the special case $n = 3$ and $f(x) = \text{const}$, it coincides with the equation of a steady boundary layer on a flat plate for the stream function (see Schlichting, 1981, and Loitsyanskiy, 1996).

We look for generalized separable solutions to equation (24) in the form

$$w(x, y) = \varphi(x)\psi(y) + \chi(x). \tag{25}$$

On substituting (25) into (24) and cancelling by φ, we arrive at the functional-differential equation

$$\varphi'_x[(\psi'_y)^2 - \psi\psi''_{yy}] - \chi'_x\psi''_{yy} = f(x)\psi^{(n)}_y. \tag{26}$$

We divide (26) by $f = f(x)$ and then differentiate with respect to x to obtain

$$(\varphi'_x/f)'_x[(\psi'_y)^2 - \psi\psi''_{yy}] - (\chi'_x/f)'_x\psi''_{yy} = 0. \tag{27}$$

Nondegenerate case. On separating the variables in (27), we get

$$(\chi'_x/f)'_x = C_1(\varphi'_x/f)'_x,$$
$$(\psi'_y)^2 - \psi\psi''_{yy} - C_1\psi''_{yy} = 0.$$

Integrating yields

$$\psi(y) = C_4 e^{\lambda y} - C_1, \quad \varphi(x) \text{ is any}, \quad \chi(x) = C_1\varphi(x) + C_2\int f(x)\,dx + C_3, \tag{28}$$

where $C_1, \dots, C_4$, and λ are constants of integration. On substituting (28) into (26), we establish the relationship between constants to obtain $C_2 = -\lambda^{n-2}$. Ultimately, taking into account the aforesaid and formulas (25) and (28), we arrive at a solution of equation (24) of the form (25):

$$w(x,y) = \varphi(x)e^{\lambda y} - \lambda^{n-2}\int f(x)\,dx + C,$$

where $\varphi(x)$ is an arbitrary function, C and λ are arbitrary constants ($C = C_3$, $C_4 = 1$).

Degenerate case. It follows from (27) that

$$(\varphi'_x/f)'_x = 0, \quad (\chi'_x/f)'_x = 0, \quad \psi(y) \text{ is any}. \tag{29}$$

Integrating the first two equations in (29) twice yields

$$\varphi(x) = C_1\int f(x)\,dx + C_2, \quad \chi(x) = C_3\int f(x)\,dx + C_4, \tag{30}$$

where $C_1, \dots, C_4$ are arbitrary constants.

Substituting (25) into (26) and taking into account (30), we arrive at an ordinary differential equation for $\psi = \psi(y)$:

$$C_1(\psi'_y)^2 - (C_1\psi + C_3)\psi''_{yy} = \psi^{(n)}_y. \tag{31}$$

Formulas (25) and (30) together with equation (31) determine an exact solution of equation (24).

Example 8. The two-dimensional stationary equations of motion of a viscous incompressible fluid are reduced to a single fourth-order nonlinear equation for the stream function (see Loitsyanskiy, 1996):

$$\frac{\partial w}{\partial y}\frac{\partial}{\partial x}(\Delta w) - \frac{\partial w}{\partial x}\frac{\partial}{\partial y}(\Delta w) = \nu\Delta\Delta w, \qquad \Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}. \tag{32}$$

We seek exact separable solutions of equation (32) in the form

$$w = f(x) + g(y). \tag{33}$$

Substituting (33) into (32) yields

$$g'_y f'''_{xxx} - f'_x g'''_{yyy} = \nu f''''_{xxxx} + \nu g''''_{yyyy}. \tag{34}$$

Differentiating (34) with respect to x and y, we obtain

$$g''_{yy} f''''_{xxxx} - f''_{xx} g''''_{yyyy} = 0. \tag{35}$$

Nondegenerate case. If $f''_{xx} \not\equiv 0$ and $g''_{yy} \not\equiv 0$, we separate the variables in (35) to obtain the ordinary differential equations

$$f''''_{xxxx} = Cf''_{xx}, \tag{36}$$
$$g''''_{yyyy} = Cg''_{yy}, \tag{37}$$

which have different solutions depending on the value of the integration constant C.

1°. Solutions of equations (36) and (37) for $C = 0$:

$$\begin{aligned} f(x) &= A_1 + A_2 x + A_3 x^2 + A_4 x^3, \\ g(y) &= B_1 + B_2 y + B_3 y^2 + B_4 y^3, \end{aligned} \tag{38}$$

where the A_k and B_k are arbitrary constants ($k = 1, 2, 3, 4$). On substituting (38) into (34), we evaluate the integration constants. Three cases are possible:

$$\begin{array}{lll} A_4 = B_4 = 0, & A_n, B_n \text{ are any numbers} & (n = 1, 2, 3); \\ A_k = 0, & B_k \text{ are any numbers} & (k = 1, 2, 3, 4); \\ B_k = 0, & A_k \text{ are any numbers} & (k = 1, 2, 3, 4). \end{array}$$

The first two sets of constants determine two simple solutions (33) of equation (32):

$$w = C_1 x^2 + C_2 x + C_3 y^2 + C_4 y + C_5,$$
$$w = C_1 y^3 + C_2 y^2 + C_3 y + C_4,$$

where $C_1, \dots, C_5$ are arbitrary constants.

2°. Solutions of equations (36) and (37) for $C = \lambda^2 > 0$:

$$f(x) = A_1 + A_2 x + A_3 e^{\lambda x} + A_4 e^{-\lambda x},\\ g(y) = B_1 + B_2 y + B_3 e^{\lambda y} + B_4 e^{-\lambda y}. \tag{39}$$

Substituting (39) into (34), dividing by λ^3, and collecting terms, we obtain

$$A_3(\nu\lambda - B_2)e^{\lambda x} + A_4(\nu\lambda + B_2)e^{-\lambda x} + B_3(\nu\lambda + A_2)e^{\lambda y} + B_4(\nu\lambda - A_2)e^{-\lambda y} = 0.$$

Equating the coefficients of the exponentials to zero, we find

$$\begin{aligned} &A_3 = A_4 = B_3 = 0, \quad A_2 = \nu\lambda && \text{(case 1)},\\ &A_3 = B_3 = 0, \quad A_2 = \nu\lambda, \quad B_2 = -\nu\lambda && \text{(case 2)},\\ &A_3 = B_4 = 0, \quad A_2 = -\nu\lambda, \quad B_2 = -\nu\lambda && \text{(case 3)}. \end{aligned}$$

(The other constants are arbitrary.) These sets of constants determine three solutions (33) of equation (32):

$$\begin{aligned} w &= C_1 e^{-\lambda y} + C_2 y + C_3 + \nu\lambda x,\\ w &= C_1 e^{-\lambda x} + \nu\lambda x + C_2 e^{-\lambda y} - \nu\lambda y + C_3,\\ w &= C_1 e^{-\lambda x} - \nu\lambda x + C_2 e^{\lambda y} - \nu\lambda y + C_3, \end{aligned}$$

where C_1, C_2, C_3, and λ are arbitrary constants.

3°. Solution of equations (36) and (37) for $C = -\lambda^2 < 0$:

$$f(x) = A_1 + A_2 x + A_3 \cos(\lambda x) + A_4 \sin(\lambda x),\\ g(y) = B_1 + B_2 y + B_3 \cos(\lambda y) + B_4 \sin(\lambda y). \tag{40}$$

Substituting (40) into (34) does not yield new real solutions.

Degenerate cases. If $f''_{xx} \equiv 0$ or $g''_{yy} \equiv 0$, equation (35) becomes an identity for any $g = g(y)$ or $f = f(x)$, respectively. These cases should be treated separately from the nondegenerate case. For example, if $f''_{xx} \equiv 0$, we have $f(x) = Ax + B$, where A and B are arbitrary numbers. Substituting this f into (34), we arrive at the equation $-Ag'''_{yyy} = \nu g''''_{yyyy}$. Its general solution is given by $g(y) = C_1 \exp(-Ay/\nu) + C_2 y^2 + C_3 y + C_4$. Thus, we obtain another solution (33) of equation (32):

$$w = C_1 e^{-\lambda y} + C_2 y^2 + C_3 y + C_4 + \nu\lambda x \qquad (A = \nu\lambda,\ B = 0).$$

Example 9. Consider the second-order nonlinear parabolic equation

$$\frac{\partial w}{\partial t} = aw\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + c. \tag{41}$$

We look for exact separable solutions of equation (41) in the form

$$w = \varphi(t) + \psi(t)\theta(x). \tag{42}$$

Substituting (42) into (41) and collecting terms yields

$$\varphi'_t - c + \psi'_t\theta = a\varphi\psi\theta''_{xx} + \psi^2\left[a\theta\theta''_{xx} + b(\theta'_x)^2\right]. \tag{43}$$

On dividing this relation by ψ^2 and differentiating with respect to t and x, we obtain

$$(\psi'_t/\psi^2)'_t\theta'_x = a(\varphi/\psi)'_t\theta'''_{xxx}.$$

Separating the variables, we arrive at the ordinary differential equations

$$\theta'''_{xxx} = K\theta'_x, \tag{44}$$

$$(\psi'_t/\psi^2)'_t = aK(\varphi/\psi)'_t, \tag{45}$$

where K is an arbitrary constant. The general solution of equation (44) is given by

$$\theta = \begin{cases} A_1 x^2 + A_2 x + A_3 & \text{if } K = 0,\\ A_1 e^{\lambda x} + A_2 e^{-\lambda x} + A_3 & \text{if } K = \lambda^2 > 0,\\ A_1 \sin(\lambda x) + A_2 \cos(\lambda x) + A_3 & \text{if } K = -\lambda^2 < 0, \end{cases} \tag{46}$$

where A_1, A_2, and A_3 are arbitrary constants. Integrating (45) yields

$$\begin{aligned} &\psi = \frac{B}{t + C_1}, \quad \varphi(t) \text{ is any} && \text{if } K = 0,\\ &\varphi = B\psi + \frac{1}{aK}\frac{\psi'_t}{\psi}, \quad \psi(t) \text{ is any} && \text{if } K \neq 0, \end{aligned} \tag{47}$$

where B is an arbitrary constant. On substituting solutions (46) and (47) into (43), one can “remove” the redundant constants and define the functions φ and ψ. Below we summarize the results.

1°. Solution for $a \neq -b$ and $a \neq -2b$:

$$w = \frac{c(a+2b)}{2(a+b)}(t+C_1) + C_2(t+C_1)^{-\frac{a}{a+2b}} - \frac{(x+C_3)^2}{2(a+2b)(t+C_1)} \qquad \text{(corresponds to } K=0\text{)},$$

where C_1, C_2, and C_3 are arbitrary constants.

2°. Solution for $b = -a$:

$$w = \frac{1}{a\lambda^2}\frac{\psi'_t}{\psi} + \psi(A_1 e^{\lambda x} + A_2 e^{-\lambda x}) \qquad \text{(corresponds to } K = \lambda^2 > 0\text{)},$$

where the function $\psi = \psi(t)$ is determined from the autonomous ordinary differential equation

$$Z''_{tt} = ac\lambda^2 + 4a^2\lambda^4 A_1 A_2 e^{2Z}, \qquad \psi = e^Z,$$

whose solution can be found in implicit form. In the special case $A_1 = 0$ or $A_2 = 0$, we have $\psi = C_1 \exp\bigl(\frac{1}{2}ac\lambda^2 t^2 + C_2 t\bigr)$.

3°. Solution for $b = -a$:

$$w = -\frac{1}{a\lambda^2}\frac{\psi'_t}{\psi} + \psi[A_1 \sin(\lambda x) + A_2 \cos(\lambda x)] \qquad \text{(corresponds to } K = -\lambda^2 < 0\text{)},$$

where the function $\psi = \psi(t)$ is determined from the autonomous ordinary differential equation

$$Z''_{tt} = -ac\lambda^2 + a^2\lambda^4 (A_1^2 + A_2^2) e^{2Z}, \qquad \psi = e^Z,$$

whose solution can be found in implicit form.

Remark. The structure of solutions to equation (41) was obtained by Galaktionov (1995) by a different method (see Subsection S.4.6, Example 14).

⊙ *References for Subsection* S.4.3: A. D. Polyanin (2002, Supplement B), A. D. Polyanin and V. F. Zaitsev (2002).

S.4.4. Solution of Functional Differential Equations by Splitting

S.4.4-1. Preliminary remarks. Description of the method.

As one reduces the number of terms in the functional-differential equation (21)–(22) by differentiation, redundant constants of integration arise. These constants must be "removed" at the final stage. Furthermore, the resulting equation can be of a higher-order than the original equation. To avoid these difficulties, it is convenient to reduce the solution of the functional-differential equation to the solution of a bilinear functional equation of a standard form and solution of a system of ordinary differential equations. Thus, the original problem splits into two simpler problems. Below we outline the basic stages of the splitting method.

1°. At the first stage, we treat equation (21) as a purely functional equation that depends on two variables X and Y, where $\Phi_n = \Phi_n(X)$ and $\Psi_n = \Psi_n(Y)$ are unknown quantities ($n = 1, \ldots, k$).

It can be shown that the bilinear functional equation (21) has $k-1$ different solutions:

$$\begin{aligned} \Phi_i(X) &= C_{i,1}\Phi_{m+1}(X) + C_{i,2}\Phi_{m+2}(X) + \cdots + C_{i,k-m}\Phi_k(X), \qquad i = 1, \ldots, m; \\ \Psi_{m+j}(Y) &= -C_{1,j}\Psi_1(Y) - C_{2,j}\Psi_2(Y) - \cdots - C_{m,j}\Psi_m(Y), \qquad j = 1, \ldots, k-m; \\ m &= 1, 2, \ldots, k-1; \end{aligned} \tag{48}$$

where the $C_{i,j}$ are arbitrary constants. The functions $\Phi_{m+1}(X)$, …, $\Phi_k(X)$, $\Psi_1(Y)$, …, $\Psi_m(Y)$ on the right-hand sides of equations (48) are defined arbitrarily. It is apparent that for fixed m, solution (48) contains $m(k-m)$ arbitrary constants.

2°. At the second stage, we successively substitute the $\Phi_i(X)$ and $\Psi_j(Y)$ of (22) into all solutions (48) to obtain systems of ordinary differential equations* for the unknown functions $\varphi_p(x)$ and $\psi_q(y)$. Solving these systems, we get generalized separable solutions of the form (19).

Remark 1. It is important that, for fixed k, the bilinear functional equation (21) used in the splitting method is the same for different classes of original nonlinear mathematical physics equations.

* Such systems are usually overdetermined.

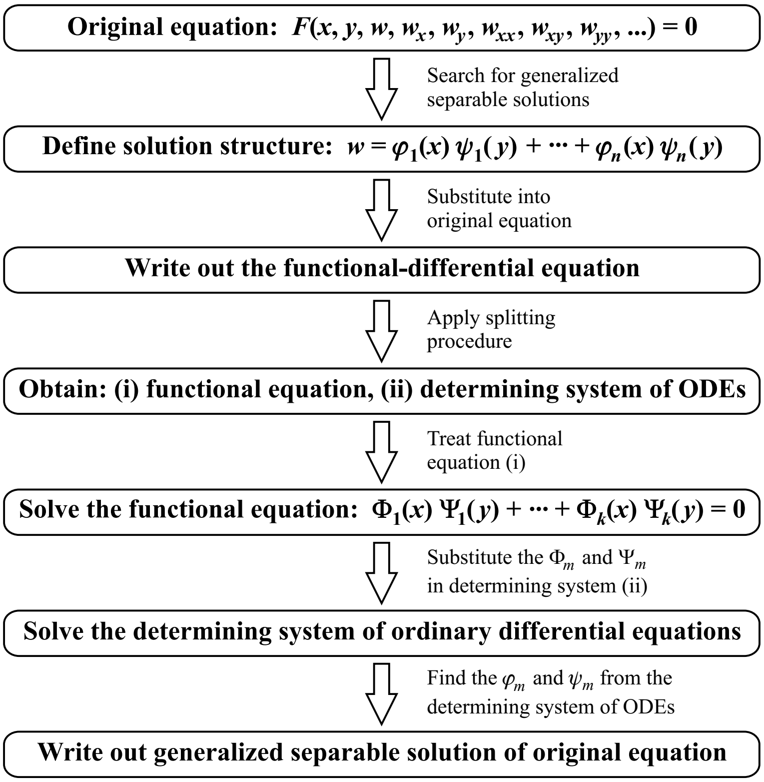

Figure 1. General scheme for constructing generalized separable solutions by the splitting method. Abbreviation: ODE stands for ordinary differential equation.

Remark 2. For fixed m, solution (48) contains $m(k-m)$ arbitrary constants $C_{i,j}$. Given k, the solutions having the maximum number of arbitrary constants are defined by

Solution number	*Number of arbitrary constants*	*Conditions on k*
$m = \frac{1}{2}k$	$\frac{1}{4}k^2$	if k is even,
$m = \frac{1}{2}(k \pm 1)$	$\frac{1}{4}(k^2 - 1)$	if k is odd.

It is these solutions of the bilinear functional equation that most frequently result in nontrivial generalized separable solution in nonlinear partial differential equations.

Remark 3. The bilinear functional equation (21) and its solutions (48) play an important role in the method of functional separation of variables.

For visualization, the main stages of constructing generalized separable solutions by the splitting method are displayed in Fig. 1.

S.4.4-2. Solutions of simple functional equations and their application.

Below we give solutions to two simple bilinear functional equations of the form (21) that will be used subsequently for solving specific nonlinear partial differential equations.

1°. The functional equation

$$\Phi_1\Psi_1 + \Phi_2\Psi_2 + \Phi_3\Psi_3 = 0, \tag{49}$$

where the Φ_i are all functions of the same argument and the Ψ_i are all functions of another argument, has two solutions:

$$\begin{aligned} &\Phi_1 = A_1\Phi_3, \quad \Phi_2 = A_2\Phi_3, \quad \Psi_3 = -A_1\Psi_1 - A_2\Psi_2; \\ &\Psi_1 = A_1\Psi_3, \quad \Psi_2 = A_2\Psi_3, \quad \Phi_3 = -A_1\Phi_1 - A_2\Phi_2. \end{aligned} \tag{50}$$

The arbitrary constants are renamed as follows: $A_1 = C_{1,1}$ and $A_2 = C_{2,1}$ in the first solution, and $A_1 = -1/C_{1,2}$ and $A_2 = C_{1,1}/C_{1,2}$ in the second solution. The functions on the right-hand sides of the equations in (50) are assumed to be arbitrary.

2°. The functional equation

$$\Phi_1\Psi_1 + \Phi_2\Psi_2 + \Phi_3\Psi_3 + \Phi_4\Psi_4 = 0, \tag{51}$$

where the Φ_i are all functions of the same argument and the Ψ_i are all functions of another argument, has a solution

$$\begin{aligned} \Phi_1 &= A_1\Phi_3 + A_2\Phi_4, & \Phi_2 &= A_3\Phi_3 + A_4\Phi_4, \\ \Psi_3 &= -A_1\Psi_1 - A_3\Psi_2, & \Psi_4 &= -A_2\Psi_1 - A_4\Psi_2 \end{aligned} \tag{52}$$

dependent on four arbitrary constants $A_1, \dots, A_4$; see solution (48) with $k = 4$, $m = 2$, $C_{1,1} = A_1$, $C_{1,2} = A_2$, $C_{2,1} = A_3$, and $C_{2,2} = A_4$. The functions on the right-hand sides of the equations in (50) are assumed to be arbitrary.

Equation (51) has also two other solutions

$$\begin{aligned} &\Phi_1 = A_1\Phi_4, \quad \Phi_2 = A_2\Phi_4, \quad \Phi_3 = A_3\Phi_4, \quad \Psi_4 = -A_1\Psi_1 - A_2\Psi_2 - A_3\Psi_3; \\ &\Psi_1 = A_1\Psi_4, \quad \Psi_2 = A_2\Psi_4, \quad \Psi_3 = A_3\Psi_4, \quad \Phi_4 = -A_1\Phi_1 - A_2\Phi_2 - A_3\Phi_3 \end{aligned} \tag{53}$$

involving three arbitrary constants. In the first solution, $A_1 = C_{1,1}$, $A_2 = C_{2,1}$, and $A_3 = C_{3,1}$, and in the second solution, $A_1 = -1/C_{1,3}$, $A_2 = C_{1,1}/C_{1,3}$, and $A_3 = C_{1,2}/C_{1,3}$.

Solutions (53) will sometimes be called *degenerate*, to emphasize the fact that they contain fewer arbitrary constants than solution (52).

Example 10. Consider the nonlinear hyperbolic equation

$$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial}{\partial x}\left(w\frac{\partial w}{\partial x}\right) + f(t)w + g(t), \tag{54}$$

where $f(t)$ and $g(t)$ are arbitrary functions. We look for generalized separable solutions of the form

$$w(x,t) = \varphi(x)\psi(t) + \chi(t). \tag{55}$$

Substituting (55) into (54) and collecting terms yields

$$a\psi^2(\varphi\varphi'_x)'_x + a\psi\chi\varphi''_{xx} + (f\psi - \psi''_{tt})\varphi + f\chi + g - \chi''_{tt} = 0.$$

This equation can be represented as a functional equation (51) in which

$$\begin{aligned} &\Phi_1 = (\varphi\varphi'_x)'_x, \quad \Phi_2 = \varphi''_{xx}, \quad \Phi_3 = \varphi, \qquad\qquad \Phi_4 = 1, \\ &\Psi_1 = a\psi^2, \qquad\quad \Psi_2 = a\psi\chi, \quad \Psi_3 = f\psi - \psi''_{tt}, \quad \Psi_4 = f\chi + g - \chi''_{tt}. \end{aligned} \tag{56}$$

On substituting (56) into (52), we obtain the following overdetermined system of ordinary differential equations for the functions $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $\chi = \chi(t)$:

$$\begin{aligned} (\varphi\varphi'_x)'_x &= A_1\varphi + A_2, & \varphi''_{xx} &= A_3\varphi + A_4, \\ f\psi - \psi''_{tt} &= -A_1 a\psi^2 - A_3 a\psi\chi, & f\chi + g - \chi''_{tt} &= -A_2 a\psi^2 - A_4 a\psi\chi. \end{aligned} \tag{57}$$

The first two equations in (57) are consistent only if

$$A_1 = 6B_2, \quad A_2 = B_1^2 - 4B_0B_2, \quad A_3 = 0, \quad A_4 = 2B_2, \tag{58}$$

where B_0, B_1, and B_2 are arbitrary constants, and the solution is given by

$$\varphi(x) = B_2x^2 + B_1x + B_0. \tag{59}$$

On substituting the expressions (58) into the last two equations in (57), we obtain the following system of equations for $\psi(t)$ and $\chi(t)$:

$$\begin{aligned} \psi''_{tt} &= 6aB_2\psi^2 + f(t)\psi, \\ \chi''_{tt} &= [2aB_2\psi + f(t)]\chi + a(B_1^2 - 4B_0B_2)\psi^2 + g(t). \end{aligned} \tag{60}$$

Relations (55), (59) and system (60) determine a generalized separable solution of equation (54). The first equation in (60) can be solved independently; it is linear if $B_2 = 0$ and is integrable by quadrature for $f(t) = \text{const}$. The second equation in (60) is linear in χ (for ψ known).

Equation (54) does not have other solutions with the form (55) if f and g are arbitrary functions and $\varphi \not\equiv 0$, $\psi \not\equiv 0$, and $\chi \not\equiv 0$.

Remark. It can be shown that equation (54) has a more general solution with the form (Galaktionov, 1995)

$$w(x,y)=\varphi_1(x)\psi_1(t)+\varphi_2(x)\psi_2(t)+\psi_3(t),\qquad \varphi_1(x)=x^2,\quad \varphi_2(x)=x, \tag{61}$$

where the functions $\psi_i=\psi_i(t)$ are determined by the ordinary differential equations

$$\begin{aligned}\psi_1''&=6a\psi_1^2+f(t)\psi_1,\\ \psi_2''&=[6a\psi_1+f(t)]\psi_2,\\ \psi_3''&=[2a\psi_1+f(t)]\psi_3+a\psi_2^2+g(t).\end{aligned} \tag{62}$$

(The prime denotes a derivative with respect to t.) The second equation in (62) has a particular solution $\psi_2=\psi_1$. Hence, its general solution can be represented as (see Polyanin and Zaitsev, 2003)

$$\psi_2=C_1\psi_1+C_2\psi_1\int\frac{dt}{\psi_1^2}.$$

The solution obtained in Example 10 corresponds to the special case $C_2=0$.

Example 11. Consider the nonlinear equation

$$\frac{\partial^2 w}{\partial x\partial t}+\left(\frac{\partial w}{\partial x}\right)^2-w\frac{\partial^2 w}{\partial x^2}=\nu\frac{\partial^3 w}{\partial x^3}, \tag{63}$$

which arises in hydrodynamics [see 9.3.3.1, equation (2) and 10.3.3.1, equation (4) with $f_1(t)=0$].

We look for exact solutions of the form

$$w=\varphi(t)\theta(x)+\psi(t). \tag{64}$$

Substituting (64) into (63) yields

$$\varphi_t'\theta_x'-\varphi\psi\theta_{xx}''+\varphi^2\left[(\theta_x')^2-\theta\theta_{xx}''\right]-\nu\varphi\theta_{xxx}'''=0.$$

This functional-differential equation can be reduced to the functional equation (51) by setting

$$\begin{aligned}&\Phi_1=\varphi_t',\quad \Phi_2=\varphi\psi,\quad \Phi_3=\varphi^2,\quad \Phi_4=\nu\varphi,\\ &\Psi_1=\theta_x',\quad \Psi_2=-\theta_{xx}'',\quad \Psi_3=(\theta_x')^2-\theta\theta_{xx}'',\quad \Psi_4=-\theta_{xxx}'''.\end{aligned} \tag{65}$$

On substituting these expressions into (52), we obtain the system of equations

$$\begin{aligned}&\varphi_t'=A_1\varphi^2+A_2\nu\varphi, && \varphi\psi=A_3\varphi^2+A_4\nu\varphi,\\ &(\theta_x')^2-\theta\theta_{xx}''=-A_1\theta_x'+A_3\theta_{xx}'', && \theta_{xxx}'''=A_2\theta_x'-A_4\theta_{xx}''.\end{aligned} \tag{66}$$

It can be shown that the last two equations in (66) are consistent only if the function θ and its derivative are linearly dependent,

$$\theta_x'=B_1\theta+B_2. \tag{67}$$

The six constants B_1, B_2, A_1, A_2, A_3, and A_4 must satisfy the three conditions

$$\begin{aligned}B_1(A_1+B_2-A_3B_1)&=0,\\ B_2(A_1+B_2-A_3B_1)&=0,\\ B_1^2+A_4B_1-A_2&=0.\end{aligned} \tag{68}$$

Integrating (67) yields

$$\theta=\begin{cases}B_3\exp(B_1x)-\dfrac{B_2}{B_1} & \text{if } B_1\neq 0,\\ B_2x+B_3 & \text{if } B_1=0,\end{cases} \tag{69}$$

where B_3 is an arbitrary constant.

The first two equations in (66) lead to the following expressions for φ and ψ:

$$\varphi=\begin{cases}\dfrac{A_2\nu}{C\exp(-A_2\nu t)-A_1} & \text{if } A_2\neq 0,\\ -\dfrac{1}{A_1t+C} & \text{if } A_2=0,\end{cases}\qquad \psi=A_3\varphi+A_4\nu, \tag{70}$$

where C is an arbitrary constant.

Formulas (69), (70) and relations (68) allow us to find the following solutions of equation (63) with the form (64):

$$\begin{aligned}w&=\frac{x+C_1}{t+C_2}+C_3 && \text{if}\quad A_2=B_1=0,\ B_2=-A_1;\\ w&=\frac{C_1e^{-\lambda x}+1}{\lambda t+C_2}+\nu\lambda && \text{if}\quad A_2=0,\ B_1=-A_4,\ B_2=-A_1-A_3A_4;\\ w&=C_1e^{-\lambda(x+\beta\nu t)}+\nu(\lambda+\beta) && \text{if}\quad A_1=A_3=B_2=0,\ A_2=B_1^2+A_4B_1;\\ w&=\frac{\nu\beta+C_1e^{-\lambda x}}{1+C_2e^{-\nu\lambda\beta t}}+\nu(\lambda-\beta) && \text{if}\quad A_1=A_3B_1-B_2,\ A_2=B_1^2+A_4B_1,\end{aligned}$$

where C_1, C_2, C_3, β, and λ are arbitrary constants (these can be expressed in terms of the A_k and B_k).

The analysis of the second solution (53) of the functional equation (51) leads to the following two more general solutions of the differential equation (63):

$$w = \frac{x}{t + C_1} + \psi(t),$$

$$w = \varphi(t)e^{-\lambda x} - \frac{\varphi_t'(t)}{\lambda\varphi(t)} + \nu\lambda,$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions, and C_1 and λ are arbitrary constants.

⊙ *References for Subsection* S.4.4: E. R. Rozendorn (1984), A. D. Polyanin (2002, Supplement B), A. D. Polyanin and A. I. Zhurov (2002).

S.4.5. Simplified Scheme for Constructing Generalized Separable Solutions

S.4.5-1. Description of the simplified scheme.

To construct exact solutions of equations (20) with quadratic or power nonlinearities that do not depend explicitly on x (all f_i constant), it is reasonable to use the following simplified approach. As before, we seek solutions in the form of finite sums (19). We assume that the system of coordinate functions $\{\varphi_i(x)\}$ is governed by linear differential equations with constant coefficients. The most common solutions of such equations are of the forms

$$\varphi_i(x) = x^i, \quad \varphi_i(x) = e^{\lambda_i x}, \quad \varphi_i(x) = \sin(\alpha_i x), \quad \varphi_i(x) = \cos(\beta_i x). \tag{71}$$

Finite chains of these functions (in various combinations) can be used to search for separable solutions (19), where the quantities λ_i, α_i, and β_i are regarded as free parameters. The other system of functions $\{\psi_i(y)\}$ is determined by solving the nonlinear equations resulting from substituting (19) into the equation under consideration.

This simplified approach lacks the generality of the methods outlined in Subsections S.4.2–S.4.4. However, specifying one of the systems of coordinate functions, $\{\varphi_i(x)\}$, simplifies the procedure of finding exact solutions substantially. The drawback of this approach is that some solutions of the form (19) can be overlooked. It is significant that the overwhelming majority of generalized separable solutions known to date, for partial differential equations with quadratic nonlinearities, are determined by coordinate functions (71) (usually with $n = 2$).

S.4.5-2. Examples of constructing exact solutions of higher-order equations.

Below we consider specific examples that illustrate the application of the above simplified scheme to constructing generalized separable solutions of higher-order nonlinear equations.

Example 12. The equations of a laminar boundary layer on a flat plate are reduced to a single third-order nonlinear equation for the stream function (see Schlichting, 1981, and Loitsyanskiy, 1996):

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = \nu\frac{\partial^3 w}{\partial y^3}. \tag{72}$$

We look for generalized separable solutions with the form

$$w(x, y) = x\psi(y) + \theta(y), \tag{73}$$

which corresponds to the simplest set of functions $\varphi_1(x) = x$, $\varphi_2(x) = 1$ with $n = 2$ in formula (19). On substituting (73) into (72) and collecting terms, we obtain

$$x[(\psi')^2 - \psi\psi'' - \nu\psi'''] + [\psi'\theta' - \psi\theta'' - \nu\theta'''] = 0.$$

(The prime denotes a derivative with respect to y.) To meet this equation for any x, one should equate both expressions in square brackets to zero. This results in a system of ordinary differential equations for $\psi = \psi(y)$ and $\theta = \theta(y)$:

$$(\psi')^2 - \psi\psi'' - \nu\psi''' = 0,$$
$$\psi'\theta' - \psi\theta'' - \nu\theta''' = 0.$$

For example, this system has an exact solution

$$\psi = \frac{6\nu}{y + C_1}, \quad \theta = \frac{C_2}{y + C_1} + \frac{C_3}{(y + C_1)^2} + C_4,$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

Other generalized separable solutions of equation (72) can be found in Subsection 9.3.1; see also Example 7 with $n = 3$ and $f(x) = \nu$.

Example 13. Consider the nth-order nonlinear equation

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = f(x)\frac{\partial^n w}{\partial y^n}, \tag{74}$$

where $f(x)$ is an arbitrary function. In the special case $n = 3$ with $f(x) = \nu = \text{const}$, this equation coincides with the boundary layer equation (72).

We look for generalized separable solutions of the form

$$w(x, y) = \varphi(x)e^{\lambda y} + \theta(x), \tag{75}$$

which correspond to the set of functions $\psi_1(y) = e^{\lambda y}$, $\psi_2(y) = 1$ in (19). On substituting (75) into (74) and rearranging terms, we obtain

$$\lambda^2 e^{\lambda y}\varphi[\theta'_x + \lambda^{n-2} f(x)] = 0.$$

This equation is met if

$$\theta(x) = -\lambda^{n-2}\int f(x)\,dx + C, \quad \varphi(x) \text{ is any}, \tag{76}$$

where C is an arbitrary constant. (The other case, $\varphi = 0$ and θ is any, is of little interest.) Formulas (75) and (76) define an exact solution of equation (74),

$$w(x, y) = \varphi(x)e^{\lambda y} - \lambda^{n-2}\int f(x)\,dx + C, \tag{77}$$

which involves an arbitrary function $\varphi(x)$ and two arbitrary constants C and λ.

Note that solution (77) with $n = 3$ and $f(x) = \text{const}$ was obtained by Ignatovich (1993) by a more complicated approach.

Example 14. Consider the nth-order nonlinear equation

$$\frac{\partial^2 w}{\partial x \partial t} + \left(\frac{\partial w}{\partial x}\right)^2 - w\frac{\partial^2 w}{\partial x^2} = f(t)\frac{\partial^n w}{\partial x^n}, \tag{78}$$

where $f(t)$ is an arbitrary function. In the special case $n = 3$ and $f(t) = \text{const}$, it coincides with equation (63).

We look for exact solutions of the form

$$w = \varphi(t)e^{\lambda x} + \psi(t). \tag{79}$$

On substituting (79) into (78), we have

$$\varphi'_t - \lambda\varphi\psi = \lambda^{n-1} f(t)\varphi.$$

We now solve this equation for ψ and substitute the resulting expression into (79) to obtain a solution of equation (78) in the form

$$w = \varphi(t)e^{\lambda x} + \frac{1}{\lambda}\frac{\varphi'_t(t)}{\varphi(t)} - \lambda^{n-2} f(t),$$

where $\varphi(t)$ is an arbitrary function and λ is an arbitrary constant.

⊙ *References for Subsection* S.4.5: A. D. Polyanin (2002, Supplement B), A. D. Polyanin and V. F. Zaitsev (2002).

S.4.6. Titov–Galaktionov Method

S.4.6-1. Description of the method. Linear subspaces invariant under a nonlinear operator.

Consider the nonlinear evolution equation

$$\frac{\partial w}{\partial t} = F[w], \tag{80}$$

where $F[w]$ is a differential operator of the form

$$F[w] \equiv F\left(w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\right). \tag{81}$$

Definition. A finite-dimensional linear subspace

$$\mathscr{L}_k = \{\varphi_1(x), \dots, \varphi_k(x)\} \tag{82}$$

formed by linear combinations of linearly independent functions $\varphi_1(x), \dots, \varphi_k(x)$ is called invariant under the operator F if $F[\mathscr{L}_k] \subseteq \mathscr{L}_k$. This means that there exist functions $f_1, \dots, f_k$ such that

$$F\left[\sum_{i=1}^{k} C_i\varphi_i(x)\right] = \sum_{i=1}^{k} f_i(C_1, \dots, C_k)\varphi_i(x) \tag{83}$$

for arbitrary constants $C_1, \dots, C_k$.

Let the linear subspace (82) be invariant under the operator F. Then equation (80) possesses generalized separable solutions of the form

$$w(x,t) = \sum_{i=1}^{k} \psi_i(t)\varphi_i(x). \tag{84}$$

Here, the functions $\psi_1(t), \dots, \psi_k(t)$ are described by the autonomous system of ordinary differential equations

$$\psi_i' = f_i(\psi_1, \dots, \psi_k), \qquad i = 1, \dots, k, \tag{85}$$

where the prime denotes a derivative with respect to t.

The following example illustrates the scheme for constructing generalized separable solutions.

Example 15. Consider the nonlinear second-order parabolic equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2 + kw^2 + bw + c. \tag{86}$$

Obviously, the nonlinear differential operator $F[w] = aw_{xx} + (w_x)^2 + kw^2 + bw + c$ for $k > 0$ has a two-dimensional invariant subspace $\mathscr{L}_2 = \{1, \cos(x\sqrt{k})\}$. Indeed, for arbitrary C_1 and C_2 we have

$$F\left[C_1 + C_2\cos(x\sqrt{k})\right] = k(C_1^2 + C_2^2) + bC_1 + c + C_2(2kC_1 - ak + b)\cos(x\sqrt{k}).$$

Therefore, there is a generalized separable solution of the form

$$w(x,t) = \psi_1(t) + \psi_2(t)\cos(x\sqrt{k}), \tag{87}$$

where the functions $\psi_1(t)$ and $\psi_2(t)$ are determined by the autonomous system of ordinary differential equations

$$\begin{aligned} \psi_1' &= k(\psi_1^2 + \psi_2^2) + b\psi_1 + c, \\ \psi_2' &= \psi_2(2k\psi_1 - ak + b). \end{aligned} \tag{88}$$

Remark 1. For $k > 0$, the nonlinear differential operator $F[w]$ has a three-dimensional invariant subspace $\mathscr{L}_3 = \{1, \sin(x\sqrt{k}), \cos(x\sqrt{k})\}$.

Remark 2. For $k < 0$, the nonlinear differential operator $F[w]$ has a three-dimensional invariant subspace $\mathscr{L}_3 = \{1, \sinh(x\sqrt{k}), \cosh(x\sqrt{k})\}$.

Remark 3. A more general equation (86), with $a = a(t)$, $b = b(t)$, and $c = c(t)$ being arbitrary functions, and $k = \text{const} < 0$, also admits a generalized separable solution of the form (87), where the functions $\psi_1(t)$ and $\psi_2(t)$ are determined by the system of ordinary differential equations (88).

S.4.6-2. Some generalizations.

Likewise, one can consider a more general equation of the form

$$L_1[w] = L_2[U], \qquad U = F[w], \tag{89}$$

where $L_1[w]$ and $L_2[U]$ are linear differential operators with respect to t,

$$L_1[w] \equiv \sum_{i=0}^{m_1} a_i(t)\frac{\partial^i w}{\partial t^i}, \qquad L_2[U] \equiv \sum_{j=0}^{m_2} b_j(t)\frac{\partial^j U}{\partial t^j}, \tag{90}$$

and $F[w]$ is a nonlinear differential operator with respect to x,

$$F[w] \equiv F\left(t, w, \frac{\partial w}{\partial x}, \ldots, \frac{\partial^n w}{\partial x^n}\right), \tag{91}$$

and may depend on t as a parameter.

Let the linear subspace (82) be invariant under the operator F, i.e., for arbitrary constants $C_1, \ldots, C_k$ the following relation holds:

$$F\left[\sum_{i=1}^{k} C_i \varphi_i(x)\right] = \sum_{i=1}^{k} f_i(t, C_1, \ldots, C_k)\varphi_i(x). \tag{92}$$

Then equation (89) possesses generalized separable solutions of the form (84), where the functions $\psi_1(t), \ldots, \psi_k(t)$ are described by the system of ordinary differential equations

$$L_1\big[\psi_i(t)\big] = L_2\big[f_i(t, \psi_1, \ldots, \psi_k)\big], \qquad i = 1, \ldots, k. \tag{93}$$

Example 16. Consider the equation

$$a_2(t)\frac{\partial^2 w}{\partial t^2} + a_1(t)\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}, \tag{94}$$

which, in the special case of $a_2(t) = k_2$ and $a_1(t) = k_1/t$, is used for describing transonic gas flows (where t plays the role of a spatial variable).

Equation (94) is a special case of equation (89), where $L_1[w] = a_2(t)w_{tt} + a_1(t)w_t$, $L_2[U] = U$, and $F[w] = w_x w_{xx}$. It can be shown that the nonlinear differential operator $F[w] = w_x w_{xx}$ admits the three-dimensional invariant subspace $\mathscr{L}_3 = \{1, x^{3/2}, x^3\}$. Therefore, equation (94) possesses generalized separable solutions of the form

$$w(x,t) = \psi_1(t) + \psi_2(t)x^{3/2} + \psi_3(t)x^3,$$

where the functions $\psi_1(t)$, $\psi_2(t)$, and $\psi_3(t)$ are described by the system of ordinary differential equations

$$\begin{aligned}
a_2(t)\psi_1'' + a_1(t)\psi_1' &= \tfrac{9}{8}\psi_2^2,\\
a_2(t)\psi_2'' + a_1(t)\psi_2' &= \tfrac{45}{4}\psi_2\psi_3,\\
a_2(t)\psi_3'' + a_1(t)\psi_3' &= 18\psi_3^2.
\end{aligned}$$

Remark. The operator $F[w]$ also has a four-dimensional invariant subspace $\mathscr{L}_4 = \{1, x, x^2, x^3\}$, which corresponds to a generalized separable solution of the form

$$w(x,t) = \psi_1(t) + \psi_2(t)x + \psi_3(t)x^2 + \psi_4(t)x^3.$$

See also Example 17 with $a_0(t) = 0$, $k = 1$, and $n = 2$.

Example 17. Consider the more general nth-order equation

$$a_2(t)\frac{\partial^2 w}{\partial t^2} + a_1(t)\frac{\partial w}{\partial t} + a_0(t)w = \left(\frac{\partial w}{\partial x}\right)^k \frac{\partial^n w}{\partial x^n}. \tag{95}$$

The nonlinear operator $F[w] = (w_x)^k w_x^{(n)}$ has a two-dimensional invariant subspace $\mathscr{L}_2 = \{1, \varphi(x)\}$, where the function $\varphi(x)$ is determined by the autonomous ordinary differential equation $(\varphi_x')^k \varphi_x^{(n)} = \varphi$. Therefore, equation (95) possesses generalized separable solutions of the form

$$w(x,t) = \psi_1(t) + \psi_2(t)\varphi(x),$$

where the functions $\psi_1(t)$ and $\psi_2(t)$ are described by two independent ordinary differential equations

$$\begin{aligned}
a_2(t)\psi_1'' + a_1(t)\psi_1' + a_0(t)\psi_1 &= 0,\\
a_2(t)\psi_2'' + a_1(t)\psi_2' + a_0(t)\psi_2 &= \psi_2^{k+1}.
\end{aligned}$$

Many other examples of this type, as well as some modifications and generalizations of the method described here, can be found in the literature cited below. The basic difficulty of using the Titov–Galaktionov method for the construction of exact solutions of specific equations consists in finding linear subspaces which are invariant under a given nonlinear operator. Moreover, the original equation may be of a different type than the equations considered here (it is not always possible to single out a suitable nonlinear operator $F[w]$).

⊙ *References for Subsection* S.4.6: S. S. Titov (1988), V. A. Galaktionov and S. A. Posashkov (1994), V. A. Galaktionov (1995), V. A. Galaktionov, S. A. Posashkov, and S. R. Svirshchevskii (1995), S. R. Svirshchevskii (1995, 1996).

S.5. Method of Functional Separation of Variables

S.5.1. Structure of Functional Separable Solutions

Suppose a nonlinear equation for $w = w(x, y)$ is obtained from a linear mathematical physics equation for $z = z(x, y)$ by a nonlinear change of variable $w = F(z)$. Then, if the linear equation for z admits separable solutions, the transformed nonlinear equation for w will have exact solutions of the form

$$w(x, y) = F(z), \quad \text{where} \quad z = \sum_{m=1}^{n} \varphi_m(x)\psi_m(y). \tag{1}$$

It is noteworthy that many nonlinear partial differential equations that are not reducible to linear equations have exact solutions of the form (1) as well. We will call such solutions *functional separable solutions*. In general, the functions $\varphi_m(x)$, $\psi_m(y)$, and $F(z)$ in (1) are not known in advance and are to be identified.

Main idea: the functional-differential equation resulting from the substitution of (1) in the original partial differential equation should be reduced to the standard bilinear functional equation (21) of Subsection S.4.2, or to a functional-differential equation of the form (21)–(22) of Subsection S.4.2.

Remark 1. In functional separation of variables, searching for solutions in the forms $w = F\big(\varphi(x) + \psi(y)\big)$ and $w = F\big(\varphi(x)\psi(y)\big)$ leads to equivalent results, because the two forms are functionally equivalent. Indeed, we have $F\big(\varphi(x)\psi(y)\big) = F_1\big(\varphi_1(x) + \psi_1(y)\big)$, where $F_1(z) = F(e^z)$, $\varphi_1(x) = \ln\varphi(x)$, and $\psi_1(y) = \ln\psi(y)$.

Remark 2. In constructing functional separable solutions with the form $w = F\big(\varphi(x) + \psi(y)\big)$, it is assumed that $\varphi \not\equiv \text{const}$ and $\psi \not\equiv \text{const}$.

Remark 3. The function $F(z)$ can be determined by a single ordinary differential equation or by an overdetermined system of equations; both possibilities must be taken into account.

S.5.2. Special Functional Separable Solutions

S.5.2-1. Generalized traveling-wave solutions. Examples.

To simplify the analysis, some of the functions in (1) can be specified a priori and the other functions will be defined in the analysis. We call such solutions *special functional separable solutions*.

Consider functional separable solutions of the form (1) in the special case where the composite argument z is linear in one of the independent variables (e.g., in x). We substitute (1) into the equation under study and eliminate x using the expression of z to obtain a functional-differential equation with two arguments. In many cases, this equation can be solved by the methods outlined in Subsections S.4.2–S.4.4.

Below are the simplest functional separable solutions of special forms (x and y can be swapped):

$$\begin{aligned}
&w = F(z), \quad z = \psi_1(y)x + \psi_2(y) && (z \text{ is linear in } x);\\
&w = F(z), \quad z = \psi_1(y)x^2 + \psi_2(y) && (z \text{ is quadratic in } x);\\
&w = F(z), \quad z = \psi_1(y)e^{\lambda x} + \psi_2(y) && (z \text{ contains an exponential of } x).
\end{aligned}$$

The first solution will be called a generalized traveling-wave solution. In the last formula, $e^{\lambda x}$ can be replaced by $\cosh(ax + b)$, $\sinh(ax + b)$, or $\sin(ax + b)$ to obtain another three modifications.

After substituting any of the above expressions into the original equation, one should eliminate x with the help of the expression for z. This will result in a functional-differential equation with two arguments, y and z. Its solution may be obtained in some cases with the methods outlined in Subsections S.4.2–S.4.4.

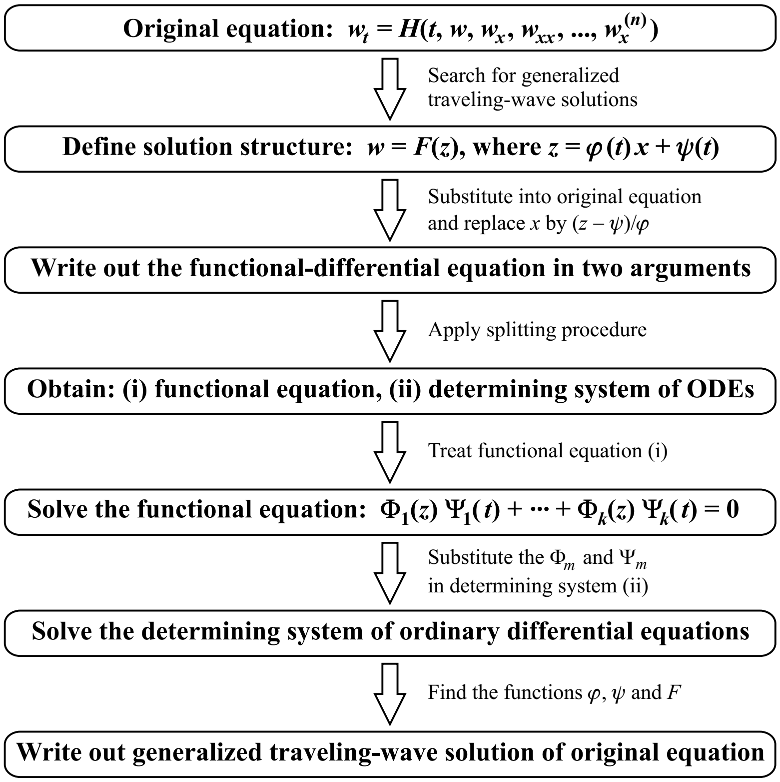

Figure 2. Algorithm for constructing generalized traveling-wave solutions for evolution equations. Abbreviation: ODE stands for ordinary differential equation.

For visualization, the general scheme for constructing generalized traveling-wave solutions for evolution equations is displayed in Fig. 2.

Remark 1. The algorithm presented in Fig. 2 can also be used for finding exact solutions of the more general form $w = \sigma(t)F(z) + \varphi_1(t)x + \psi_2(t)$, where $z = \varphi_1(t)x + \psi_2(t)$. For an example of this sort of solution, see Subsection S.6.3 (Example 6).

Remark 2. A generalized separable solution (see Section S.4) is a functional separable solution of the special form corresponding to $F(z) = z$.

We consider below examples of nonlinear equations that admit functional separable solutions of the special form where the argument z is linear or quadratic in one of the independent variables.

Example 1. Consider the nonstationary heat equation with a nonlinear source

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \mathcal{F}(w). \tag{2}$$

We look for functional separable solutions of the special form

$$w = w(z), \quad z = \varphi(t)x + \psi(t). \tag{3}$$

The functions $w(z)$, $\varphi(t)$, $\psi(t)$, and $\mathcal{F}(w)$ are to be determined.

On substituting (3) into (2) and on dividing by w'_z, we have

$$\varphi'_t x + \psi'_t = \varphi^2 \frac{w''_{zz}}{w'_z} + \frac{\mathcal{F}(w)}{w'_z}. \tag{4}$$

We express x from (3) in terms of z and substitute into (4) to obtain a functional-differential equation with two variables, t and z,

$$-\psi'_t + \frac{\psi}{\varphi}\varphi'_t - \frac{\varphi'_t}{\varphi} z + \varphi^2 \frac{w''_{zz}}{w'_z} + \frac{\mathcal{F}(w)}{w'_z} = 0,$$

which can be treated as the functional equation (51) in Subsection S.4.4 where

$$\Phi_1 = -\psi'_t + \frac{\psi}{\varphi}\varphi'_t, \quad \Phi_2 = -\frac{\varphi'_t}{\varphi}, \quad \Phi_3 = \varphi^2, \qquad \Phi_4 = 1,$$
$$\Psi_1 = 1, \qquad\qquad \Psi_2 = z, \qquad \Psi_3 = \frac{w''_{zz}}{w'_z}, \quad \Psi_4 = \frac{\mathcal{F}(w)}{w'_z}.$$

Substituting these expressions into relations (52) of Subsection S.4.4 yields the system of ordinary differential equations

$$\begin{gathered}
-\psi'_t + \frac{\psi}{\varphi}\varphi'_t = A_1\varphi^2 + A_2, \quad -\frac{\varphi'_t}{\varphi} = A_3\varphi^2 + A_4,\\
\frac{w''_{zz}}{w'_z} = -A_1 - A_3 z, \quad \frac{\mathcal{F}(w)}{w'_z} = -A_2 - A_4 z,
\end{gathered} \tag{5}$$

where $A_1, \dots, A_4$ are arbitrary constants.

Case 1. For $A_4 \neq 0$, the solution of system (5) is given by

$$\begin{aligned}
\varphi(t) &= \pm\left(C_1 e^{2A_4 t} - \frac{A_3}{A_4}\right)^{-1/2},\\
\psi(t) &= -\varphi(t)\left[A_1\int\varphi(t)\,dt + A_2\int\frac{dt}{\varphi(t)} + C_2\right],\\
w(z) &= C_3\int\exp\left(-\tfrac{1}{2}A_3z^2 - A_1z\right)dz + C_4,\\
\mathcal{F}(w) &= -C_3(A_4z + A_2)\exp\left(-\tfrac{1}{2}A_3z^2 - A_1z\right),
\end{aligned} \tag{6}$$

where $C_1, \dots, C_4$ are arbitrary constants. The dependence $\mathcal{F} = \mathcal{F}(w)$ is defined by the last two relations in parametric form (z is considered the parameter). If $A_3 \neq 0$ in (6), the source function is expressed in terms of elementary functions and the inverse of the error function.

In the special case $A_3 = C_4 = 0$, $A_1 = -1$, and $C_3 = 1$, the source function can be represented in explicit form as

$$\mathcal{F}(w) = -w(A_4 \ln w + A_2). \tag{7}$$

Solutions of equation (2) in this case were obtained by Dorodnitsyn (1982) with group-theoretic methods.

Case 2. For $A_4 = 0$, the solution to the first two equations in (5) is given by

$$\varphi(t) = \pm\frac{1}{\sqrt{2A_3t + C_1}}, \qquad \psi(t) = \frac{C_2}{\sqrt{2A_3t + C_1}} - \frac{A_1}{A_3} - \frac{A_2}{3A_3}(2A_3t + C_1),$$

and the solutions to the other equations are determined by the last two formulas in (6) where $A_4 = 0$.

Example 2. Consider the more general equation

$$\frac{\partial w}{\partial t} = a(t)\frac{\partial^2 w}{\partial x^2} + b(t)\frac{\partial w}{\partial x} + c(t)\mathcal{F}(w).$$

We look for solutions in the form (3). In this case, only the first two equations in system (5) will change, and the functions $w(z)$ and $\mathcal{F}(w)$ will be given by (6).

Example 3. The nonlinear heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[\mathcal{G}(w)\frac{\partial w}{\partial x}\right] + \mathcal{F}(w)$$

has also solutions of the form (3). The unknown quantities are governed by system (5) in which w''_{zz} must be replaced by $[\mathcal{G}(w)w'_z]'_z$. The functions $\varphi(t)$ and $\psi(t)$ are determined by the first two formulas in (6). One of the two functions $\mathcal{G}(w)$ and $\mathcal{F}(w)$ can be assumed arbitrary and the other is identified in the course of the solution. The special case $\mathcal{F}(w) = \text{const}$ yields $\mathcal{G}(w) = C_1e^{2ke} + (C_2w + C_3)e^{kw}$.

Functional separable solutions (3) of the given equation are discussed in more detail in 1.6.15.2, Items 3° and 4°; some other solutions are also specified there.

Example 4. We can treat the nth-order nonlinear equation

$$\frac{\partial w}{\partial t} = \frac{\partial^n w}{\partial x^n} + \mathcal{F}(w)$$

likewise. As before, we look for solutions in the form (3). In this case, the quantities φ^2 and w''_{zz} in (5) must be replaced by φ^n and $w_z^{(n)}$, respectively. In particular, for $A_3 = 0$, apart from equations with logarithmic nonlinearities of the form (7), we obtain other equations.

Example 5. For the nth-order nonlinear equation

$$\frac{\partial w}{\partial t}=\frac{\partial^n w}{\partial x^n}+\mathcal{F}(w)\frac{\partial w}{\partial x},$$

the search for exact solutions of the form (3) leads to the following system of equations for $\varphi(t)$, $\psi(t)$, $w(z)$, and $\mathcal{F}(w)$:

$$-\psi'_t+\frac{\psi}{\varphi}\varphi'_t=A_1\varphi^n+A_2\varphi,\qquad -\frac{\varphi'_t}{\varphi}=A_3\varphi^n+A_4\varphi,$$
$$\frac{w^{(n)}_z}{w'_z}=-A_1-A_3z,\qquad \mathcal{F}(w)=-A_2-A_4z,$$

where $A_1,\,\dots,\,A_4$ are arbitrary constants.

In the case $n=3$, we assume $A_3=0$ and $A_1>0$ to find in particular that $\mathcal{F}(w)=-A_2-A_4\arcsin(kw)$.

Some functional separable solutions (3) of the given equation can be found in Subsection 11.1.3.

Example 6. In addition, searching for solutions of equation (2) with z quadratically dependent on x,

$$w=w(z),\qquad z=\varphi(t)x^2+\psi(t),\tag{8}$$

also makes sense here. Indeed, on substituting (8) into (2), we arrive at an equation that contains terms with x^2 and does not contain terms linear in x. Eliminating x^2 from the resulting equation with the aid of (8), we obtain

$$-\psi'_t+\frac{\psi}{\varphi}\varphi'_t+2\varphi-\frac{\varphi'_t}{\varphi}z+4\varphi z\frac{w''_{zz}}{w'_z}-4\varphi\psi\frac{w''_{zz}}{w'_z}+\frac{\mathcal{F}(w)}{w'_z}=0.$$

To solve this functional-differential equation with two arguments, we apply the splitting method outlined in Subsection S.4.4. It can be shown that, for equations (2), this equation has a solution with a logarithmic nonlinearity of the form (7).

Example 7. Consider the mth-order nonlinear equation

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}=f(x)\left(\frac{\partial^2 w}{\partial y^2}\right)^{n-1}\frac{\partial^m w}{\partial y^m},$$

which, in the special case of $f(x)=\text{const}$ and $m=3$, describes a boundary layer of a power-law fluid on a flat plate; w is the stream function, x and y are coordinates along and normal to the plate, and n is a rheological parameter (the value $n=1$ corresponds to a Newtonian fluid). Searching for solutions in the form

$$w=w(z),\qquad z=\varphi(x)y+\psi(x),$$

leads to the equation $\varphi'_x(w'_z)^2=f(x)\varphi^{2n+m-3}(w''_{zz})^{n-1}w^{(m)}_z$, which is independent of ψ. Separating the variables and integrating yields

$$\varphi(x)=\left[\int f(x)\,dx+C\right]^{\frac{1}{4-2n-m}},\qquad \psi(x)\text{ is arbitrary,}$$

and the function $w=w(z)$ is determined by solving the ordinary differential equation $(w'_z)^2=(4-2n-m)(w''_{zz})^{n-1}w^{(m)}_z$.

Example 8. Consider the equation

$$\frac{\partial^{n+1}w}{\partial x^n\partial y}=f(w).\tag{9}$$

We look for functional separable solutions of the special form

$$w=w(z),\qquad z=\varphi(y)x+\psi(y).\tag{10}$$

We substitute (10) in (9), eliminate x with the expression for z, divide the resulting equation by $w^{(n+1)}_z$, and rearrange terms to obtain the functional-differential equation with two arguments

$$\varphi^n\psi'_y-\varphi^{n-1}\psi\varphi'_y+\varphi^{n-1}\varphi'_y\left(z+n\frac{w^{(n)}_z}{w^{(n+1)}_z}\right)-\frac{f(w)}{w^{(n+1)}_z}=0.\tag{11}$$

It is reduced to a three-term bilinear functional equation, which has two solutions (see Subsection S.4.4). Accordingly, we consider two cases.

1°. First, we set the expression in parentheses and the last fraction in (11) equal to constants. On rearranging terms, we obtain

$$(z-C_1)w^{(n+1)}_z+nw^{(n)}_z=0,$$
$$C_2w^{(n+1)}_z-f(w)=0,$$
$$\varphi^n\psi'_y-\varphi^{n-1}\psi\varphi'_y+C_1\varphi^{n-1}\varphi'_y-C_2=0,$$

where C_1 and C_2 are arbitrary constants. Setting $C_1 = 0$, which corresponds to a translation in z and renaming ψ, and integrating yields

$$
\begin{aligned}
w &= A \ln|z| + B_{n-1} z^{n-1} + \cdots + B_1 z + B_0, \\
f(w) &= A C_2 n!\,(-1)^n z^{-n-1}, \\
\psi(y) &= C_2 \varphi(y) \int \frac{dy}{[\varphi(y)]^{n+1}} + C_3 \varphi(y),
\end{aligned}
\tag{12}
$$

where A, the B_m, and C_3 are arbitrary constants and $\varphi(y)$ is an arbitrary function.

The first two formulas in (12) give a parametric representation of $f(w)$. In the special case of $B_{n-1} = \cdots = B_0 = 0$, on eliminating z, we arrive at the exponential dependence

$$
f(w) = \alpha e^{\beta w}, \qquad \alpha = A C_2 n!\,(-1)^n, \quad \beta = -(n+1)/A.
$$

By virtue of (12), the corresponding solution of equation (9) will have functional arbitrariness.

2°. In the second case, (11) splits into three ordinary differential equations:

$$
\begin{aligned}
\varphi^{n-1} \varphi'_y &= C_1, \\
\varphi^n \psi'_y - \varphi^{n-1} \psi \varphi'_y &= C_2, \\
(C_1 z + C_2) w_z^{(n+1)} + C_1 n w_z^{(n)} - f(w) &= 0,
\end{aligned}
\tag{13}
$$

where C_1 and C_2 are arbitrary constants. The solutions of the first two equations are given by

$$
\varphi = (C_1 n t + C_3)^{1/n}, \qquad \psi = C_4 (C_1 n t + C_3)^{1/n} - \frac{C_2}{C_1}.
$$

Together with the last equation in (13), these formulas define a self-similar solution of the form (10).

⊙ *References for Subsection* S.5.2-1: A. D. Polyanin (2002, Supplement B), A. D. Polyanin and A. I. Zhurov (2002), A. D. Polyanin and V. F. Zaitsev (2002).

S.5.2-2. Solution by reduction to equations with quadratic (or power) nonlinearities.

In some cases, solutions of the form (1) can be searched for in two stages. First, one looks for a transformation that would reduce the original equation to an equation with a quadratic (or power) nonlinearity. Then the methods outlined in Subsections S.4.2–S.4.4 are used for finding solutions of the resulting equation.

Sometimes, quadratically nonlinear equations can be obtained using the substitutions

$$
\begin{aligned}
w(z) &= z^\lambda && \text{(for equations with power nonlinearities)}, \\
w(z) &= \lambda \ln z && \text{(for equations with exponential nonlinearities)}, \\
w(z) &= e^{\lambda z} && \text{(for equations with logarithmic nonlinearities)},
\end{aligned}
$$

where λ is a constant to be determined. This approach is equivalent to specifying the form of the function $F(z)$ in (1) a priori.

Galaktionov and Posashkov (1989, 1994) and Galaktionov (1995) describe a large number of nonlinear equations of different type that can be reduced with similar transformations to equations with quadratic nonlinearities.

Example 9. The nonlinear heat equation with a logarithmic source

$$
\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + f(t) w \ln w + g(t) w
$$

can be reduced by the change of variable $w = e^z$ to the quadratically nonlinear equation

$$
\frac{\partial z}{\partial t} = a \frac{\partial^2 z}{\partial x^2} + a \left(\frac{\partial z}{\partial x} \right)^2 + f(t) z + g(t),
$$

which admits separable solutions with the form

$$
z = \varphi_1(x) \psi_1(t) + \varphi_2(x) \psi_2(t) + \psi_3(t),
$$

where $\varphi_1(x) = x^2$ and $\varphi_2(x) = x$, and the functions $\psi_k(t)$ are determined by an appropriate system of ordinary differential equations.

⊙ *References for Subsection* S.5.2-2: V. A. Galaktionov and S. A. Posashkov (1989, 1994), V. A. Galaktionov (1995), A. D. Polyanin (2002, Supplement B), A. D. Polyanin and V. F. Zaitsev (2002).

S.5.3. Differentiation Method

S.5.3-1. Basic ideas of the method. Reduction to a standard equation.

In general, the substitution of expression (1) into the nonlinear partial differential equation under study leads to a functional-differential equation with three arguments—two arguments are usual, x and y, and the third is composite, z. In many cases, the resulting equation can be reduced by differentiation to a standard functional-differential equation with two arguments (either x or y is eliminated). To solve the two-argument equation, one can use the methods outlined in Subsections S.4.2–S.4.4.

S.5.3-2. Examples of constructing functional separable solutions.

Below we consider specific examples illustrating the application of the differentiation method for constructing functional separable solutions of nonlinear equations.

Example 10. Consider the nonlinear heat equation

$$\frac{\partial w}{\partial t}=\frac{\partial}{\partial x}\left[\mathcal{F}(w)\frac{\partial w}{\partial x}\right]. \tag{14}$$

We look for exact solutions with the form

$$w=w(z),\quad z=\varphi(x)+\psi(t). \tag{15}$$

On substituting (15) into (14) and dividing by w'_z, we obtain the functional-differential equation with three variables

$$\psi'_t=\varphi''_{xx}\mathcal{F}(w)+(\varphi'_x)^2H(z), \tag{16}$$

where

$$H(z)=\mathcal{F}(w)\frac{w''_{zz}}{w'_z}+\mathcal{F}'_z(w),\qquad w=w(z). \tag{17}$$

Differentiating (16) with respect to x yields

$$\varphi'''_{xxx}\mathcal{F}(w)+\varphi'_x\varphi''_{xx}[\mathcal{F}'_z(w)+2H(z)]+(\varphi'_x)^3H'_z=0. \tag{18}$$

This functional-differential equation with two variables can be treated as the functional equation (49) of Subsection S.4.4. This three-term functional equation has two different solutions. Accordingly, we consider two cases.

Case 1. The solutions of the functional-differential equation (18) are determined from the system of ordinary differential equations

$$\begin{gathered}\mathcal{F}'_z+2H=2A_1\mathcal{F},\quad H'_z=A_2\mathcal{F},\\ \varphi'''_{xxx}+2A_1\varphi'_x\varphi''_{xx}+A_2(\varphi'_x)^3=0,\end{gathered} \tag{19}$$

where A_1 and A_2 are arbitrary constants.

The first two equations (19) are linear and independent of the third equation. Their general solution is given by

$$\mathcal{F}=\begin{cases}e^{A_1z}(B_1e^{kz}+B_2e^{-kz}) & \text{if } A_1^2>2A_2,\\ e^{A_1z}(B_1+B_2z) & \text{if } A_1^2=2A_2,\\ e^{A_1z}[B_1\sin(kz)+B_2\cos(kz)] & \text{if } A_1^2<2A_2,\end{cases}\qquad H=A_1\mathcal{F}-\tfrac{1}{2}\mathcal{F}'_z,\qquad k=\sqrt{|A_1^2-2A_2|}. \tag{20}$$

Substituting H of (20) into (17) yields an ordinary differential equation for $w=w(z)$. On integrating this equation, we obtain

$$w=C_1\int e^{A_1z}|\mathcal{F}(z)|^{-3/2}dz+C_2, \tag{21}$$

where C_1 and C_2 are arbitrary constants. The expression of $\mathcal{F}$ in (20) together with expression (21) define the function $\mathcal{F}=\mathcal{F}(w)$ in parametric form.

Without full analysis, we will study the case $A_2=0$ ($k=A_1$) and $A_1\neq 0$ in more detail. It follows from (20) and (21) that

$$\mathcal{F}(z)=B_1e^{2A_1z}+B_2,\quad H=A_1B_2,\quad w(z)=C_3(B_1+B_2e^{-2A_1z})^{-1/2}+C_2\quad (C_1=A_1B_2C_3). \tag{22}$$

Eliminating z yields

$$\mathcal{F}(w)=\frac{B_2C_3^2}{C_3^2-B_1w^2}. \tag{23}$$

The last equation in (19) with $A_2 = 0$ has the first integral $\varphi''_{xx} + A_1(\varphi'_x)^2 = \text{const}$. The corresponding general solution is given by

$$
\begin{aligned}
\varphi(x) &= -\frac{1}{2A_1}\ln\left[\frac{D_2}{D_1}\frac{1}{\sinh^2\bigl(A_1\sqrt{D_2}\,x + D_3\bigr)}\right] && \text{for}\quad D_1 > 0 \ \text{and}\ D_2 > 0;\\
\varphi(x) &= -\frac{1}{2A_1}\ln\left[-\frac{D_2}{D_1}\frac{1}{\cos^2\bigl(A_1\sqrt{-D_2}\,x + D_3\bigr)}\right] && \text{for}\quad D_1 > 0 \ \text{and}\ D_2 < 0;\\
\varphi(x) &= -\frac{1}{2A_1}\ln\left[-\frac{D_2}{D_1}\frac{1}{\cosh^2\bigl(A_1\sqrt{D_2}\,x + D_3\bigr)}\right] && \text{for}\quad D_1 < 0 \ \text{and}\ D_2 > 0;
\end{aligned}
\tag{24}
$$

where D_1, D_2, and D_3 are constants of integration. In all three cases, the following relations hold:

$$
(\varphi'_x) = D_1 e^{-2A_1\varphi} + D_2, \qquad \varphi''_{xx} = -A_1 D_1 e^{-2A_1\varphi}. \tag{25}
$$

We substitute (22) and (25) into the original functional-differential equation (16). With reference to the expression of z in (15), we obtain the following equation for $\psi = \psi(t)$:

$$
\psi'_t = -A_1 B_1 D_1 e^{2A_1\psi} + A_1 B_2 D_2.
$$

Its general solution is given by

$$
\psi(t) = \frac{1}{2A_1}\ln\frac{B_2 D_2}{D_4\exp(-2A_1^2 B_2 D_2 t) + B_1 D_1}, \tag{26}
$$

where D_4 is an arbitrary constant.

Formulas (15), (22) for w, (24), and (26) define three solutions of the nonlinear equation (14) with $\mathcal{F}(w)$ of the form (23) [recall that these solutions correspond to the special case $A_2 = 0$ in (20) and (21)].

Case 2. The solutions of the functional-differential equation (18) are determined from the system of ordinary differential equations

$$
\begin{gathered}
\varphi'''_{xxx} = A_1(\varphi'_x)^3, \qquad \varphi'_x\varphi''_{xx} = A_2(\varphi'_x)^3,\\
A_1\mathcal{F} + A_2(\mathcal{F}'_z + 2H) + H'_z = 0.
\end{gathered}
\tag{27}
$$

The first two equations in (27) are consistent in the two cases

$$
\begin{aligned}
A_1 = A_2 = 0 \quad &\Longrightarrow\quad \varphi(x) = B_1 x + B_2,\\
A_1 = 2A_2^2 \quad &\Longrightarrow\quad \varphi(x) = -\frac{1}{A_2}\ln|B_1 x + B_2|.
\end{aligned}
\tag{28}
$$

The first solution in (28) eventually leads to the traveling-wave solution $w = w(B_1 x + B_2 t)$ of equation (14) and the second solution to the self-similar solution of the form $w = \widetilde{w}(x^2/t)$. In both cases, the function $\mathcal{F}(w)$ in (14) is arbitrary.

⦿ *References*: P. W. Doyle and P. J. Vassiliou (1998), A. D. Polyanin (2002, Supplement B), A. D. Polyanin and V. F. Zaitsev (2002).

Remark. The more general nonlinear heat equation

$$
\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[\mathcal{F}(w)\frac{\partial w}{\partial x}\right] + \mathcal{G}(w)
$$

has also solutions of the form (15). For the unknown functions $\varphi(x)$ and $\psi(t)$, we have the functional-differential equation in three variables

$$
\psi'_t = \varphi''_{xx}\mathcal{F}(w) + (\varphi'_x)^2 H(z) + \mathcal{G}(w)/w'_z,
$$

where $w = w(z)$ and $H(z)$ is defined by (17). Differentiating with respect to x yields

$$
\varphi'''_{xxx}\mathcal{F}(w) + \varphi'_x\varphi''_{xx}[\mathcal{F}'_z(w) + 2H(z)] + (\varphi'_x)^3 H'_z + \varphi'_x[\mathcal{G}(w)/w'_z]'_z = 0.
$$

This functional-differential equation in two variables can be treated as the bilinear functional equation (51) of Subsection S.4.4 with $\Phi_1 = \varphi'''_{xxx}$, $\Phi_2 = \varphi'_x\varphi''_{xx}$, $\Phi_3 = (\varphi'_x)^3$, and $\Phi_4 = \varphi'_x$.

See also Estévez, Qu, and Zhang (2002), where a more general equation was considered.

Example 11. Consider the nonlinear Klein–Gordon equation

$$
\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = \mathcal{F}(w). \tag{29}
$$

We look for functional separable solutions in additive form:

$$
w = w(z), \qquad z = \varphi(x) + \psi(t). \tag{30}
$$

Substituting (30) into (29) yields

$$
\psi''_{tt} - \varphi''_{xx} + \bigl[(\psi'_t)^2 - (\varphi'_x)^2\bigr]\, g(z) = h(z), \tag{31}
$$

where

$$
g(z) = w''_{zz}/w'_z, \quad h(z) = \mathcal{F}\bigl(w(z)\bigr)/w'_z. \tag{32}
$$

On differentiating (31) first with respect to t and then with respect to x and on dividing by $\psi'_t\varphi'_x$, we have

$$2(\psi''_{tt}-\varphi''_{xx})\,g'_z+\bigl[(\psi'_t)^2-(\varphi'_x)^2\bigr]\,g''_{zz}=h''_{zz}.$$

Eliminating $\psi''_{tt}-\varphi''_{xx}$ from this equation with the aid of (31), we obtain

$$\bigl[(\psi'_t)^2-(\varphi'_x)^2\bigr](g''_{zz}-2gg'_z)=h''_{zz}-2g'_zh. \tag{33}$$

This relation holds in the following cases:

$$\begin{aligned}&g''_{zz}-2gg'_z=0,\quad h''_{zz}-2g'_zh=0 &&\text{(case 1)},\\&(\psi'_t)^2=A\psi+B,\quad(\varphi'_x)^2=-A\varphi+B-C,\quad h''_{zz}-2g'_zh=(Az+C)(g''_{zz}-2gg'_z)&&\text{(case 2)},\end{aligned}\tag{34}$$

where A, B, and C are arbitrary constants. We consider both cases.

Case 1. The first two equations in (34) enable one to determine $g(z)$ and $h(z)$. Integrating the first equation once yields $g'_z=g^2+\text{const}$. Further, the following cases are possible:

$$g=k, \tag{35a}$$
$$g=-1/(z+C_1), \tag{35b}$$
$$g=-k\tanh(kz+C_1), \tag{35c}$$
$$g=-k\coth(kz+C_1), \tag{35d}$$
$$g=k\tan(kz+C_1), \tag{35e}$$

where C_1 and k are arbitrary constants.

The second equation in (34) has a particular solution $h=g(z)$. Hence, its general solution in expressed by (e.g., see Polyanin and Zaitsev (2003))

$$h=C_2g(z)+C_3g(z)\int\frac{dz}{g^2(z)}, \tag{36}$$

where C_2 and C_3 are arbitrary constants.

The functions $w(z)$ and $\mathcal{F}(w)$ are found from (32) as

$$w(z)=B_1\int G(z)\,dz+B_2,\quad \mathcal{F}(w)=B_1h(z)G(z),\quad\text{where}\quad G(z)=\exp\biggl[\int g(z)\,dz\biggr], \tag{37}$$

and B_1 and B_2 are arbitrary constants ($\mathcal{F}$ is defined parametrically).

Let us dwell on the case (35b). According to (36),

$$h=A_1(z+C_1)^2+\frac{A_2}{z+C_1}, \tag{38}$$

where $A_1=-C_3/3$ and $A_2=-C_2$ are any numbers. Substituting (35b) and (38) into (37) yields

$$w=B_1\ln|z+C_1|+B_2,\quad \mathcal{F}=A_1B_1(z+C_1)+\frac{A_2B_1}{(z+C_1)^2}.$$

Eliminating z, we arrive at the explicit form of the right-hand side of equation (29):

$$\mathcal{F}(w)=A_1B_1e^u+A_2B_1e^{-2u},\quad\text{where}\quad u=\frac{w-B_2}{B_1}. \tag{39}$$

For simplicity, we set $C_1=0$, $B_1=1$, and $B_2=0$ and denote $A_1=a$ and $A_2=b$. Thus, we have

$$w(z)=\ln|z|,\quad \mathcal{F}(w)=ae^w+be^{-2w},\quad g(z)=-1/z,\quad h(z)=az^2+b/z. \tag{40}$$

It remains to determine $\psi(t)$ and $\varphi(x)$. We substitute (40) into the functional-differential equation (31). Taking into account (30), we find

$$[\psi''_{tt}\psi-(\psi'_t)^2-a\psi^3-b]-[\varphi''_{xx}\varphi-(\varphi'_x)^2+a\varphi^3]+(\psi''_{tt}-3a\psi^2)\varphi-\psi(\varphi''_{xx}+3a\varphi^2)=0. \tag{41}$$

Differentiating (41) with respect to t and x yields the separable equation*

$$(\psi'''_{ttt}-6a\psi\psi'_t)\varphi'_x-(\varphi'''_{xxx}+6a\varphi\varphi'_x)\psi'_t=0,$$

whose solution is determined by the ordinary differential equations

$$\psi'''_{ttt}-6a\psi\psi'_t=A\psi'_t,$$
$$\varphi'''_{xxx}+6a\varphi\varphi'_x=A\varphi'_x,$$

where A is the separation constant. Each equation can be integrated twice, thus resulting in

$$\begin{aligned}(\psi'_t)^2&=2a\psi^3+A\psi^2+C_1\psi+C_2,\\(\varphi'_x)^2&=-2a\varphi^3+A\varphi^2+C_3\varphi+C_4,\end{aligned}\tag{42}$$

* To solve equation (41), one can use the solution of equation (51) in Subsection S.4.4 [see (52)].

TABLE 17

Nonlinear Klein–Gordon equations $\partial_{tt}w - \partial_{xx}w = \mathcal{F}(w)$ admitting functional separable solutions of the form $w = w(z)$, $z = \varphi(x) + \psi(t)$. Notation: A, C_1, and C_2 are arbitrary constants; $\sigma = 1$ for $z > 0$ and $\sigma = -1$ for $z < 0$

No.	Right-hand side $\mathcal{F}(w)$	Solution $w(z)$	Equations for $\psi(t)$ and $\varphi(x)$
1	$aw\ln w + bw$	e^z	$(\psi'_t)^2 = C_1e^{-2\psi} + a\psi - \frac{1}{2}a + b + A$, $(\varphi'_x)^2 = C_2e^{-2\varphi} - a\varphi + \frac{1}{2}a + A$
2	$ae^w + be^{-2w}$	$\ln\lvert z\rvert$	$(\psi'_t)^2 = 2a\psi^3 + A\psi^2 + C_1\psi + C_2$, $(\varphi'_x)^2 = -2a\varphi^3 + A\varphi^2 - C_1\varphi + C_2 + b$
3	$a\sin w + b\left(\sin w \ln\tan\dfrac{w}{4} + 2\sin\dfrac{w}{4}\right)$	$4\arctan e^z$	$(\psi'_t)^2 = C_1e^{2\psi} + C_2e^{-2\psi} + b\psi + a + A$, $(\varphi'_x)^2 = -C_2e^{2\varphi} - C_1e^{-2\varphi} - b\varphi + A$
4	$a\sinh w + b\left(\sinh w \ln\tanh\dfrac{w}{4} + 2\sinh\dfrac{w}{2}\right)$	$2\ln\left\lvert\coth\dfrac{z}{2}\right\rvert$	$(\psi'_t)^2 = C_1e^{2\psi} + C_2e^{-2\psi} - \sigma b\psi + a + A$, $(\varphi'_x)^2 = C_2e^{2\varphi} + C_1e^{-2\varphi} + \sigma b\varphi + A$
5	$a\sinh w + 2b\left(\sinh w \arctan e^{w/2} + \cosh\dfrac{w}{2}\right)$	$2\ln\left\lvert\tan\dfrac{z}{2}\right\rvert$	$(\psi'_t)^2 = C_1\sin 2\psi + C_2\cos 2\psi + \sigma b\psi + a + A$, $(\varphi'_x)^2 = -C_1\sin 2\varphi + C_2\cos 2\varphi - \sigma b\varphi + A$

where $C_1, \ldots, C_4$ are arbitrary constants. Eliminating the derivatives from (41) with the aid of (42), we find that the arbitrary constants are related by $C_3 = -C_1$ and $C_4 = C_2 + b$. So, the functions $\psi(t)$ and $\varphi(x)$ are determined by the first-order nonlinear autonomous equations

$$(\psi'_t)^2 = 2a\psi^3 + A\psi^2 + C_1\psi + C_2,$$
$$(\varphi'_x)^2 = -2a\varphi^3 + A\varphi^2 - C_1\varphi + C_2 + b.$$

The solutions of these equations are expressed in terms of elliptic functions.

For the other cases in (35), the analysis is performed in a similar way. Table 17 presents the final results for the cases (35a)–(35e).

Case 2. Integrating the third and fourth equations in (34) yields

$$\begin{aligned} &\psi = \pm\sqrt{B}\,t + D_1, && \varphi = \pm\sqrt{B - C}\,t + D_2 && \text{if}\quad A = 0;\\ &\psi = \frac{1}{4A}(At + D_1)^2 - \frac{B}{A}, && \varphi = -\frac{1}{4A}(Ax + D_2)^2 + \frac{B - C}{A} && \text{if}\quad A \neq 0; \end{aligned} \tag{43}$$

where D_1 and D_2 are arbitrary constants. In both cases, the function $\mathcal{F}(w)$ in equation (29) is arbitrary. The first row in (43) corresponds to the traveling wave solution $w = w(kx + \lambda t)$. The second row leads to a solution of the form $w = w(x^2 - t^2)$.

⊙ *References*: A. M. Grundland and E. Infeld (1992), J. Miller and L. A. Rubel (1993), R. Z. Zhdanov (1994), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

Example 12. The nonlinear stationary heat (diffusion) equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \mathcal{F}(w)$$

is analyzed in much the same way as the nonlinear Klein–Gordon equation considered in Example 11. The final results are listed in Table 18; the traveling wave solutions $w = w(kx + \lambda t)$ and solutions of the form $w = w(x^2 + y^2)$, existing for any $\mathcal{F}(w)$, are omitted.

⊙ *References*: A. M. Grundland and E. Infeld (1992), J. Miller and L. A. Rubel (1993), R. Z. Zhdanov (1994), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

S.5.4. Splitting Method. Reduction to a Functional Equation with Two Variables

S.5.4-1. Splitting method. Reduction to a standard functional equation.

The procedure for constructing functional separable solutions, which is based on the splitting method, involves several stages outlined below.

1°. Substitute expression (1) into the nonlinear partial differential equation under study. This results in a functional-differential equation with three arguments—the first two are usual, x and y, and the third is composite, z.

TABLE 18

Nonlinear equations $\partial_{xx}w + \partial_{yy}w = \mathcal{F}(w)$ admitting functional separable solutions of the form $w = w(z)$, $z = \varphi(x) + \psi(y)$. Notation: A, C_1, and C_2 are arbitrary constants; $\sigma = 1$ for $z > 0$, $\sigma = -1$ for $z < 0$

No.	Right-hand side $\mathcal{F}(w)$	Solution $w(z)$	Equations for $\varphi(x)$ and $\psi(y)$
1	$aw\ln w + bw$	e^z	$(\varphi'_x)^2 = C_1e^{-2\varphi} + a\varphi - \frac{1}{2}a + b + A$, $(\psi'_y)^2 = C_2e^{-2\psi} + a\psi - \frac{1}{2}a - A$
2	$ae^w + be^{-2w}$	$\ln\lvert z\rvert$	$(\varphi'_x)^2 = 2a\varphi^3 + A\varphi^2 + C_1\varphi + C_2$, $(\psi'_y)^2 = 2a\psi^3 - A\psi^2 + C_1\psi - C_2 - b$
3	$a\sin w + b\left(\sin w\ln\tan\dfrac{w}{4} + 2\sin\dfrac{w}{4}\right)$	$4\arctan e^z$	$(\varphi'_x)^2 = C_1e^{2\varphi} + C_2e^{-2\varphi} + b\varphi + a + A$, $(\psi'_y)^2 = C_2e^{2\psi} + C_1e^{-2\psi} + b\psi - A$
4	$a\sinh w + b\left(\sinh w\ln\tanh\dfrac{w}{4} + 2\sinh\dfrac{w}{2}\right)$	$2\ln\left\lvert\coth\dfrac{z}{2}\right\rvert$	$(\varphi'_x)^2 = C_1e^{2\varphi} + C_2e^{-2\varphi} - \sigma b\varphi + a + A$, $(\psi'_y)^2 = -C_2e^{2\psi} - C_1e^{-2\psi} - \sigma b\psi - A$
5	$a\sinh w + 2b\left(\sinh w\arctan e^{w/2} + \cosh\dfrac{w}{2}\right)$	$2\ln\left\lvert\tan\dfrac{z}{2}\right\rvert$	$(\varphi'_x)^2 = C_1\sin 2\varphi + C_2\cos 2\varphi + \sigma b\varphi + a + A$, $(\psi'_y)^2 = C_1\sin 2\psi - C_2\cos 2\psi + \sigma b\psi - A$

2°. Reduce the functional-differential equation to a purely functional equation with three arguments x, y, and z with the aid of elementary differential substitutions (by selecting and renaming terms with derivatives).

3°. Reduce the three-argument functional-differential equation by the differentiation method to the standard functional equation with two arguments (either x or y is eliminated) considered in Subsection S.4.2.

4°. Construct the solutions of the two-argument functional equation using the formulas given in Subsection S.4.4.

5°. Solve the (overdetermined) systems formed by the solutions of Item 4° and the differential substitutions of Item 2°.

6°. Substitute the solutions of Item 5° into the original functional-differential equation of Item 1° to establish the relations for the constants of integration and determine all unknown quantities.

7°. Consider all degenerate cases possibly arising due to the violation of assumptions adopted in the previous analysis.

Remark. Stage 3° is the most difficult here; it may not always be realizable.

The splitting method reduces solving the three-argument functional-differential equation to (i) solving a purely functional equation with three arguments (by reducing it to a standard functional equation with two arguments) and (ii) solving systems of ordinary differential equations. Thus, the initial problem splits into several simpler problems. Examples of constructing functional separable solutions by the splitting method are given in Subsection S.5.5.

S.5.4-2. Three-argument functional equations of special form.

The substitution of expression (1) with $n = 2$ into a nonlinear partial differential equation often leads to functional-differential equations of the form

$$\Phi_1(x)\Psi_1(y,z) + \cdots + \Phi_k(x)\Psi_k(y,z) + \Psi_{k+1}(y,z) + \Psi_{k+2}(y,z) + \cdots + \Psi_n(y,z) = 0, \qquad (44)$$

where the $\Phi_j(x)$ and $\Psi_j(y,z)$ are functionals dependent on the variables x and y, z, respectively,

$$\Phi_j(x) \equiv \Phi_j\big(x, \varphi, \varphi'_x, \varphi''_{xx}\big), \quad \Psi_j(y,z) \equiv \Psi_j\big(y, \psi, \psi'_y, \psi''_{yy}, F, F'_z, F''_{zz}\big). \qquad (45)$$

(These expressions apply to a second-order equation.)

It is reasonable to solve equation (44) by the splitting method. At the first stage, we treat (44) as a purely functional equation, thus disregarding (45). Assuming that $\Psi_1 \not\equiv 0$, we divide (44) by Ψ_1 and differentiate with respect to y to obtain a similar equation but with fewer terms containing Φ_m:

$$\Phi_2(x)\Psi_2^{(2)}(y,z) + \cdots + \Phi_k(x)\Psi_k^{(2)}(y,z) + \Psi_{k+1}^{(2)}(y,z) + \cdots + \Psi_n^{(2)}(y,z) = 0, \tag{46}$$

where $\Psi_m^{(2)} = \frac{\partial}{\partial y}\bigl(\Psi_m/\Psi_1\bigr) + \psi'_y \frac{\partial}{\partial z}\bigl(\Psi_m/\Psi_1\bigr)$. We continue this procedure until an equation independent of x explicitly is obtained:

$$\Psi_{k+1}^{(k+1)}(y,z) + \cdots + \Psi_n^{(k+1)}(y,z) = 0, \tag{47}$$

where $\Psi_m^{(k+1)} = \frac{\partial}{\partial y}\bigl(\Psi_m^{(k)}/\Psi_k^{(k)}\bigr) + \psi'_y \frac{\partial}{\partial z}\bigl(\Psi_m^{(k)}/\Psi_k^{(k)}\bigr)$.

Relation (47) can be regarded as an equation with two independent variables y and z. If $\Psi_m^{(k+1)}(y,z) = Q_m(y)R_m(z)$ for all $m = k+1, \ldots, n$, then equation (47) can be solved using the results of Subsections S.4.2–S.4.4.

S.5.5. Solutions of Some Nonlinear Functional Equations and Their Applications

In this subsection, we discuss several types of three-argument functional equations that arise most frequently in the functional separation of variables in nonlinear equations of mathematical physics. The results are used for constructing exact solutions for some classes of nonlinear heat and wave equations.

S.5.5-1. The functional equation $f(x) + g(y) = Q(z)$, where $z = \varphi(x) + \psi(y)$.

Here, one of the two functions $f(x)$ and $\varphi(x)$ is prescribed and the other is assumed unknown, also one of the functions $g(y)$ and $\psi(y)$ is prescribed and the other is unknown, and the function $Q(z)$ is assumed unknown.*

Differentiating the equation with respect to x and y yields $Q''_{zz} = 0$. Consequently, the solution is given by

$$f(x) = A\varphi(x) + B, \quad g(y) = A\psi(y) - B + C, \quad Q(z) = Az + C, \tag{48}$$

where A, B, and C are arbitrary constants.

S.5.5-2. The functional equation $f(t) + g(x) + h(x)Q(z) + R(z) = 0$, where $z = \varphi(x) + \psi(t)$.

Differentiating the equation with respect to x yields the two-argument equation

$$g'_x + h'_x Q + h\varphi'_x Q'_z + \varphi'_x R'_z = 0. \tag{49}$$

Such equations were discussed in Subsections S.4.2–S.4.4. Hence, the following relations hold [see formulas (51) and (52) in Subsection S.4.4]:

$$\begin{aligned} g'_x &= A_1 h\varphi'_x + A_2\varphi'_x, \\ h'_x &= A_3 h\varphi'_x + A_4\varphi'_x, \\ Q'_z &= -A_1 - A_3 Q, \\ R'_z &= -A_2 - A_4 Q, \end{aligned} \tag{50}$$

where $A_1, \ldots, A_4$ are arbitrary constants. By integrating system (50) and substituting the resulting solutions into the original functional equation, one obtains the results given below.

* In similar equations with a composite argument, it is assumed that $\varphi(x) \not\equiv \text{const}$ and $\psi(y) \not\equiv \text{const}$.

Case 1. If $A_3 = 0$ in (50), the corresponding solution of the functional equation is given by

$$\begin{aligned} f &= -\tfrac{1}{2}A_1A_4\psi^2 + (A_1B_1 + A_2 + A_4B_3)\psi - B_2 - B_1B_3 - B_4, \\ g &= \tfrac{1}{2}A_1A_4\varphi^2 + (A_1B_1 + A_2)\varphi + B_2, \\ h &= A_4\varphi + B_1, \\ Q &= -A_1z + B_3, \\ R &= \tfrac{1}{2}A_1A_4z^2 - (A_2 + A_4B_3)z + B_4, \end{aligned} \tag{51}$$

where the A_k and B_k are arbitrary constants and $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are arbitrary functions.

Case 2. If $A_3 \neq 0$ in (50), the corresponding solution of the functional equation is

$$\begin{aligned} f &= -B_1B_3e^{-A_3\psi} + \left(A_2 - \frac{A_1A_4}{A_3}\right)\psi - B_2 - B_4 - \frac{A_1A_4}{A_3^2}, \\ g &= \frac{A_1B_1}{A_3}e^{A_3\varphi} + \left(A_2 - \frac{A_1A_4}{A_3}\right)\varphi + B_2, \\ h &= B_1e^{A_3\varphi} - \frac{A_4}{A_3}, \\ Q &= B_3e^{-A_3z} - \frac{A_1}{A_3}, \\ R &= \frac{A_4B_3}{A_3}e^{-A_3z} + \left(\frac{A_1A_4}{A_3} - A_2\right)z + B_4, \end{aligned} \tag{52}$$

where the A_k and B_k are arbitrary constants and $\varphi = \varphi(x)$ and $\psi = \psi(t)$ are arbitrary functions.

Case 3. In addition, the functional equation has the two degenerate solutions:

$$f = A_1\psi + B_1, \quad g = A_1\varphi + B_2, \quad h = A_2, \quad R = -A_1z - A_2Q - B_1 - B_2, \tag{53a}$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $Q = Q(z)$ are arbitrary functions, A_1, A_2, B_1, and B_2 are arbitrary constants, and

$$f = A_1\psi + B_1, \quad g = A_1\varphi + A_2h + B_2, \quad Q = -A_2, \quad R = -A_1z - B_1 - B_2, \tag{53b}$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $h = h(x)$ are arbitrary functions, A_1, A_2, B_1, and B_2 are arbitrary constants. The degenerate solutions (53a) and (53b) can be obtained directly from the original equation or its consequence (49) using formulas (53) in Subsection S.4.4.

Example 13. Consider the nonstationary heat equation with a nonlinear source

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \mathcal{F}(w). \tag{54}$$

We look for exact solutions of the form

$$w = w(z), \quad z = \varphi(x) + \psi(t). \tag{55}$$

Substituting (55) into (54) and dividing by w'_z yields the functional-differential equation

$$\psi'_t = \varphi''_{xx} + (\varphi'_x)^2\frac{w''_{zz}}{w'_z} + \frac{\mathcal{F}(w(z))}{w'_z}.$$

We rewrite it as the functional equation S.5.5-2 in which

$$f(t) = -\psi'_t, \quad g(x) = \varphi''_{xx}, \quad h(x) = (\varphi'_x)^2, \quad Q(z) = w''_{zz}/w'_z, \quad R(z) = f(w(z))/w'_z. \tag{56}$$

We now use the solutions of equation S.5.5-2. On substituting the expressions of g and h of (56) into (51)–(53), we arrive at overdetermined systems of equations for $\varphi = \varphi(x)$.

Case 1. The system

$$\begin{aligned} \varphi''_{xx} &= \tfrac{1}{2}A_1A_4\varphi^2 + (A_1B_1 + A_2)\varphi + B_2, \\ (\varphi'_x)^2 &= A_4\varphi + B_1 \end{aligned}$$

following from (51) and corresponding to $A_3 = 0$ in (50) is consistent in the cases

$$\begin{aligned} &\varphi = C_1 x + C_2 &&\text{for}\quad A_2 = -A_1 C_1^2,\ A_4 = B_2 = 0,\ B_1 = C_1^2,\\ &\varphi = \tfrac{1}{4} A_4 x^2 + C_1 x + C_2 &&\text{for}\quad A_1 = A_2 = 0,\ B_1 = C_1^2 - A_4 C_2,\ B_2 = \tfrac{1}{2} A_4, \end{aligned} \tag{57}$$

where C_1 and C_2 are arbitrary constants.

The first solution in (57) with $A_1 \neq 0$ leads to a right-hand side of equation (54) containing the inverse of the error function [the form of the right-hand side is identified from the last two relations in (51) and (56)]. The second solution in (57) corresponds to the right-hand side $\mathcal{F}(w) = k_1 w \ln w + k_2 w$ in (54). In both cases, the first relation in (51) is, taking into account that $f = -\psi'_t$, a first-order linear solution with constant coefficients, whose solution is an exponential plus a constant.

Case 2. The system

$$\varphi''_{xx} = \frac{A_1 B_1}{A_3} e^{A_3\varphi} + \left(A_2 - \frac{A_1 A_4}{A_3}\right)\varphi + B_2,$$
$$(\varphi'_x)^2 = B_1 e^{A_3\varphi} - \frac{A_4}{A_3},$$

following from (52) and corresponding to $A_3 \neq 0$ in (50) is consistent in the following cases:

$$\begin{aligned}
&\varphi = \pm\sqrt{-A_4/A_3}\, x + C_1 &&\text{for}\ \ A_2 = A_1 A_4/A_3,\ B_1 = B_2 = 0,\\
&\varphi = -\frac{2}{A_3} \ln|x| + C_1 &&\text{for}\ \ A_1 = \tfrac{1}{2}A_3^2,\ A_2 = A_4 = B_2 = 0,\ B_1 = 4A_3^{-2} e^{-A_3 C_1},\\
&\varphi = -\frac{2}{A_3} \ln\bigl|\cos\bigl(\tfrac{1}{2}\sqrt{A_3 A_4}\, x + C_1\bigr)\bigr| + C_2 &&\text{for}\ \ A_1 = \tfrac{1}{2}A_3^2,\ A_2 = \tfrac{1}{2}A_3 A_4,\ B_2 = 0,\ A_3 A_4 > 0,\\
&\varphi = -\frac{2}{A_3} \ln\bigl|\sinh\bigl(\tfrac{1}{2}\sqrt{-A_3 A_4}\, x + C_1\bigr)\bigr| + C_2 &&\text{for}\ \ A_1 = \tfrac{1}{2}A_3^2,\ A_2 = \tfrac{1}{2}A_3 A_4,\ B_2 = 0,\ A_3 A_4 < 0,\\
&\varphi = -\frac{2}{A_3} \ln\bigl|\cosh\bigl(\tfrac{1}{2}\sqrt{-A_3 A_4}\, x + C_1\bigr)\bigr| + C_2 &&\text{for}\ \ A_1 = \tfrac{1}{2}A_3^2,\ A_2 = \tfrac{1}{2}A_3 A_4,\ B_2 = 0,\ A_3 A_4 < 0,
\end{aligned}$$

where C_1 and C_2 are arbitrary constants. The right-hand sides of equation (54) corresponding to these solutions are represented in parametric form.

Case 3. Traveling wave solutions of the nonlinear heat equation (54) and solutions of the linear equation (54) with $\mathcal{F}'_w$ = const correspond to the degenerate solutions of the functional equation (53).

Remark. It may be reasonable to look for more complicated solutions of equation (54) of the form

$$w = w(z), \quad z = \varphi(\xi) + \psi(t), \quad \xi = x + at.$$

Substituting these expressions into equation (54) yields the functional equation S.5.5-2 again, in which (x must be replaced by ξ)

$$f(t) = -\psi'_t, \quad g(\xi) = \varphi''_{\xi\xi} - a\varphi'_\xi, \quad h(\xi) = (\varphi'_\xi)^2, \quad Q(z) = w''_{zz}/w'_z, \quad R(z) = f(w(z))/w'_z.$$

Further, one should follow the same procedure of constructing the solution as in Example 13.

Example 14. Likewise, one can analyze the more general equation

$$\frac{\partial w}{\partial t} = a(x)\frac{\partial^2 w}{\partial x^2} + b(x)\frac{\partial w}{\partial x} + \mathcal{F}(w). \tag{58}$$

It arises in convective heat/mass exchange problems (a = const and b = const), problems of heat transfer in inhomogeneous media ($b = a'_x \neq$ const), and spatial heat transfer problems with axial or central symmetry (a = const and b = const $/x$).

Searching for exact solutions of equation (58) in the form (55) leads to the functional equation S.5.5-2 in which

$$f(t) = -\psi'_t, \quad g(x) = a(x)\varphi''_{xx} + b(x)\varphi'(x), \quad h(x) = a(x)(\varphi'_x)^2, \quad Q(z) = w''_{zz}/w'_z, \quad R(z) = f(w(z))/w'_z.$$

Substituting these expressions into (51)–(53) yields a system of ordinary differential equations for the unknowns.

Remark. In Examples 13 and 14, different equations were all reduced to the same functional equation. This demonstrates the utility of the isolation and independent analysis of individual types of functional equations, as well as the expedience of developing methods for solving functional equations with a composite argument.

S.5.5-3. The functional equation $f(t) + g(x)Q(z) + h(x)R(z) = 0$, where $z = \varphi(x) + \psi(t)$.

Differentiating with respect to x yields the two-argument functional-differential equation

$$g'_x Q + g\varphi'_x Q'_z + h'_x R + h\varphi'_x R'_z = 0, \tag{59}$$

which coincides with equation (51) in Subsection S.4.4, up to notation.

Nondegenerate case. Equation (59) can be solved using formulas (52) in Subsection S.4.4. In this way, we arrive at the system of ordinary differential equations

$$\begin{aligned} g'_x &= (A_1 g + A_2 h)\varphi'_x, \\ h'_x &= (A_3 g + A_4 h)\varphi'_x, \\ Q'_z &= -A_1 Q - A_3 R, \\ R'_z &= -A_2 Q - A_4 R, \end{aligned} \tag{60}$$

where $A_1, \dots, A_4$ are arbitrary constants.

The solution of equation (60) is given by

$$\begin{aligned} g(x) &= A_2 B_1 e^{k_1\varphi} + A_2 B_2 e^{k_2\varphi}, \\ h(x) &= (k_1 - A_1) B_1 e^{k_1\varphi} + (k_2 - A_1) B_2 e^{k_2\varphi}, \\ Q(z) &= A_3 B_3 e^{-k_1 z} + A_3 B_4 e^{-k_2 z}, \\ R(z) &= (k_1 - A_1) B_3 e^{-k_1 z} + (k_2 - A_1) B_4 e^{-k_2 z}, \end{aligned} \tag{61}$$

where $B_1, \dots, B_4$ are arbitrary constants and k_1 and k_2 are roots of the quadratic equation

$$(k - A_1)(k - A_4) - A_2 A_3 = 0. \tag{62}$$

In the degenerate case $k_1 = k_2$, the terms $e^{k_2\varphi}$ and $e^{-k_2 z}$ in (61) must be replaced by $\varphi e^{k_1\varphi}$ and $z e^{-k_1 z}$, respectively. In the case of purely imaginary or complex roots, one should extract the real (or imaginary) part of the roots in solution (61).

On substituting (61) into the original functional equation, one obtains conditions that must be met by the free coefficients and identifies the function $f(t)$, specifically,

$$\begin{aligned} B_2 = B_4 = 0 \quad &\Longrightarrow \quad f(t) = [A_2 A_3 + (k_1 - A_1)^2] B_1 B_3 e^{-k_1\psi}, \\ B_1 = B_3 = 0 \quad &\Longrightarrow \quad f(t) = [A_2 A_3 + (k_2 - A_1)^2] B_2 B_4 e^{-k_2\psi}, \\ A_1 = 0 \quad &\Longrightarrow \quad f(t) = (A_2 A_3 + k_1^2) B_1 B_3 e^{-k_1\psi} + (A_2 A_3 + k_2^2) B_2 B_4 e^{-k_2\psi}. \end{aligned} \tag{63}$$

Solution (61), (63) involves arbitrary functions $\varphi = \varphi(x)$ and $\psi = \psi(t)$.

Degenerate case. In addition, the functional equation has two degenerate solutions,

$$f = B_1 B_2 e^{A_1\psi}, \quad g = A_2 B_1 e^{-A_1\varphi}, \quad h = B_1 e^{-A_1\varphi}, \quad R = -B_2 e^{A_1 z} - A_2 Q,$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $Q = Q(z)$ are arbitrary functions, A_1, A_2, B_1, and B_2 are arbitrary constants; and

$$f = B_1 B_2 e^{A_1\psi}, \quad h = -B_1 e^{-A_1\varphi} - A_2 g, \quad Q = A_2 B_2 e^{A_1 z}, \quad R = B_2 e^{A_1 z},$$

where $\varphi = \varphi(x)$, $\psi = \psi(t)$, and $g = g(x)$ are arbitrary functions, and A_1, A_2, B_1, and B_2 are arbitrary constants. The degenerate solutions can be obtained immediately from the original equation or its consequence (59) using formulas (53) in Subsection S.4.4.

Example 15. For the first-order nonlinear equation

$$\frac{\partial w}{\partial t} = \mathcal{F}(w)\left(\frac{\partial w}{\partial x}\right)^2 + \mathcal{G}(x),$$

the search for exact solutions in the form (55) leads to the functional equation S.5.5-3 in which

$$f(t) = -\psi'_t, \quad g(x) = (\varphi'_x)^2, \quad h(x) = \mathcal{G}(x), \quad Q(z) = \mathcal{F}(w)w'_z, \quad R(z) = 1/w'_z, \quad w = w(z).$$

Example 16. For the nonlinear heat equation (14) [see Example 10 in S.5.3-2] the search for exact solutions in the form $w = w(z)$, where $z = \varphi(x) + \psi(t)$, leads to the functional equation (16), which coincides with equation S.5.5-3 if

$$f(t) = -\psi'_t, \quad g(x) = \varphi''_{xx}, \quad h(x) = (\varphi'_x)^2, \quad Q(z) = \mathcal{F}(w), \quad R(z) = \frac{[\mathcal{F}(w)w'_z]'_z}{w'_z}, \quad w = w(z).$$

S.5.5-4. The equation $f_1(x)+f_2(y)+g_1(x)P(z)+g_2(y)Q(z)+R(z)=0, \quad z=\varphi(x)+\psi(y)$.

Differentiating with respect to y and dividing the resulting relation by $\psi'_y P'_z$ and differentiating with respect to y again, one arrives at the functional equation with two arguments, y and z, that is discussed in Subsections S.4.2–S.4.4 [see equation (21) and its solutions (48)].

Example 17. Consider the following equation of steady-state heat transfer in an anisotropic inhomogeneous medium with a nonlinear source:

$$\frac{\partial}{\partial x}\left[a(x)\frac{\partial w}{\partial x}\right]+\frac{\partial}{\partial y}\left[b(y)\frac{\partial w}{\partial y}\right]=\mathcal{F}(w). \tag{64}$$

The search for exact solutions in the form $w=w(z)$, $z=\varphi(x)+\psi(y)$, leads to the functional equation S.5.5-4 in which

$$f_1(x)=a(x)\varphi''_{xx}+a'_x(x)\varphi'_x,\quad f_2(y)=b(y)\psi''_{yy}+b'_y(y)\psi'_y,\quad g_1(x)=a(x)(\varphi'_x)^2,\quad g_2(y)=b(y)(\psi'_y)^2,$$
$$P(z)=Q(z)=w''_{zz}/w'_z,\quad R(z)=-\mathcal{F}(w)/w'_z,\quad w=w(z).$$

Here we confine ourselves to studying functional separable solutions existing for arbitrary right-hand side $\mathcal{F}(w)$.

With the change of variable $z=\zeta^2$, we look for solutions of equation (64) in the form

$$w=w(\zeta),\quad \zeta^2=\varphi(x)+\psi(y). \tag{65}$$

Taking into account that $\frac{\partial\zeta}{\partial x}=\frac{\varphi'_x}{2\zeta}$ and $\frac{\partial\zeta}{\partial y}=\frac{\psi'_y}{2\zeta}$, we find from (64)

$$\left[(a\varphi'_x)'_x+(b\psi'_y)'_y\right]\frac{w'_\zeta}{2\zeta}+\left[a(\varphi'_x)^2+b(\psi'_y)^2\right]\frac{\zeta w''_{\zeta\zeta}-w'_\zeta}{4\zeta^3}=\mathcal{F}(w),\qquad \mathcal{F}(w)=\mathcal{F}\bigl(w(\zeta)\bigr). \tag{66}$$

For this functional-differential equation to be solvable we require that the expressions in square brackets be functions of ζ:

$$(a\varphi'_x)'_x+(b\psi'_y)'_y=M(\zeta),\quad a(\varphi'_x)^2+b(\psi'_y)^2=N(\zeta).$$

Differentiating the first relation with respect to x and y yields the equation $(M'_\zeta/\zeta)'_\zeta=0$, whose general solution is $M(\zeta)=C_1\zeta^2+C_2$. Likewise, we find $N(\zeta)=C_3\zeta^2+C_4$. Here, $C_1,\dots,C_4$ are arbitrary constants. Consequently, we have

$$(a\varphi'_x)'_x+(b\psi'_y)'_y=C_1(\varphi+\psi)+C_2,\quad a(\varphi'_x)^2+b(\psi'_y)^2=C_3(\varphi+\psi)+C_4.$$

The separation of variables results in a system of ordinary differential equations for $\varphi(x)$, $a(x)$, $\psi(y)$, and $b(y)$:

$$(a\varphi'_x)'_x-C_1\varphi-C_2=k_1,\qquad (b\psi'_y)'_y-C_1\psi=-k_1,$$
$$a(\varphi'_x)^2-C_3\varphi-C_4=k_2,\qquad b(\psi'_y)^2-C_3\psi=-k_2.$$

This system is always integrable by quadrature and can be rewritten as

$$\begin{aligned}&(C_3\varphi+C_4+k_2)\varphi''_{xx}+(C_1\varphi+C_2+k_1-C_3)(\varphi'_x)^2=0,\quad && a=(C_3\varphi+C_4+k_2)(\varphi'_x)^{-2};\\ &(C_3\psi-k_2)\psi''_{yy}+(C_1\psi-k_1-C_3)(\psi'_y)^2=0, && b=(C_3\psi-k_2)(\psi'_y)^{-2}.\end{aligned} \tag{67}$$

Here, the equations for φ and ψ do not involve a and b and, hence, can be solved independently. Without full analysis of system (67), we note a special case where the system can be solved in explicit form.

For $C_1=C_2=C_4=k_1=k_2=0$ and $C_3=C\neq 0$, we find

$$a(x)=\alpha e^{\mu x},\quad b(y)=\beta e^{\nu y},\quad \varphi(x)=\frac{Ce^{-\mu x}}{\alpha\mu^2},\quad \psi(y)=\frac{Ce^{-\nu y}}{\beta\nu^2},$$

where α, β, μ, and ν are arbitrary constants. Substituting these expressions into (66) and taking into account (65), we obtain the ordinary differential equation for $w(\zeta)$

$$w''_{\zeta\zeta}-\frac{1}{\zeta}w'_\zeta=\frac{4}{C}\mathcal{F}(w).$$

System (67) has other solutions as well; these lead to various expressions of $a(x)$ and $b(y)$. Table 19 lists the cases where these functions can be written in explicit form (the traveling-wave solution, which corresponds to $a=\text{const}$ and $b=\text{const}$, is omitted). In general, the solution of system (67) enables one to represent $a(x)$ and $b(y)$ in parametric form.

⊙ *References for Subsection* S.5.5: V. F. Zaitsev and A. D. Polyanin (1996), A. D. Polyanin and A. I. Zhurov (1998), A. D. Polyanin (2002, Supplement B), A. D. Polyanin and V. F. Zaitsev (2002).

TABLE 19
Functional separable solutions of the form $w = w(\zeta)$, $\zeta^2 = \varphi(x) + \psi(y)$, for heat equations in an anisotropic inhomogeneous medium with an arbitrary nonlinear source.
Notation: C, α, β, μ, ν, n, and k are free parameters ($C \neq 0$, $\mu \neq 0$, $\nu \neq 0$, $n \neq 2$, and $k \neq 2$)

Heat equation	Functions $\varphi(x)$ and $\psi(y)$	Equation for $w = w(\zeta)$
$\frac{\partial}{\partial x}\Big(\alpha x^m \frac{\partial w}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(\beta y^n \frac{\partial w}{\partial y}\Big) = \mathcal{F}(w)$	$\varphi = \frac{Cx^{2-m}}{\alpha(2-m)^2}$, $\psi = \frac{Cy^{2-n}}{\beta(2-n)^2}$	$w''_{\zeta\zeta} + \frac{4-mn}{(2-m)(2-n)}\frac{1}{\zeta}w'_\zeta = \frac{4}{C}\mathcal{F}(w)$
$\frac{\partial}{\partial x}\Big(\alpha e^{\mu x} \frac{\partial w}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(\beta e^{\nu y} \frac{\partial w}{\partial y}\Big) = \mathcal{F}(w)$	$\varphi = \frac{C}{\alpha\mu^2}e^{-\mu x}$, $\psi = \frac{C}{\beta\nu^2}e^{-\nu y}$	$w''_{\zeta\zeta} - \frac{1}{\zeta}w'_\zeta = \frac{4}{C}\mathcal{F}(w)$
$\frac{\partial}{\partial x}\Big(\alpha e^{\mu x} \frac{\partial w}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(\beta y^n \frac{\partial w}{\partial y}\Big) = \mathcal{F}(w)$	$\varphi = \frac{C}{\alpha\mu^2}e^{-\mu x}$, $\psi = \frac{Cy^{2-n}}{\beta(2-n)^2}$	$w''_{\zeta\zeta} + \frac{n}{2-n}\frac{1}{\zeta}w'_\zeta = \frac{4}{C}\mathcal{F}(w)$
$\frac{\partial}{\partial x}\Big(\alpha x^2 \frac{\partial w}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(\beta y^2 \frac{\partial w}{\partial y}\Big) = \mathcal{F}(w)$	$\varphi = \mu \ln\lvert x\rvert$, $\psi = \nu \ln\lvert y\rvert$	Equation (66); both expressions in square brackets are constant
$\alpha\frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial y}\Big(\beta y^2 \frac{\partial w}{\partial y}\Big) = \mathcal{F}(w)$	$\varphi = \mu x$, $\psi = \nu \ln\lvert y\rvert$	Equation (66); both expressions in square brackets are constant

S.6. Generalized Similarity Reductions of Nonlinear Equations

S.6.1. Clarkson–Kruskal Direct Method: a Special Form for Similarity Reduction

S.6.1-1. Simplified scheme. Examples of constructing exact solutions.

Prior to giving a description of the Clarkson–Kruskal direct method in the general case, consider a simplified scheme.

The basic idea of the method is the following: for an equation with the unknown function $w = w(x,t)$, an exact solution is sought in the form

$$w = f(t)u(z) + g(x,t), \qquad z = \varphi(t)x + \psi(t). \tag{1}$$

The functions $f(t)$, $g(x,t)$, $\varphi(t)$, and $\psi(t)$ are found in the subsequent analysis and are chosen in such a way that, ultimately, the function $u(z)$ would satisfy a single ordinary differential equation.

Below we consider some cases in which it is possible to construct exact solutions of nonlinear equations of the form (1).

Example 1. Consider the generalized Burgers–Korteweg–de Vries equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^n w}{\partial x^n} + bw\frac{\partial w}{\partial x}. \tag{2}$$

We seek its exact solution in the form (1). Inserting (1) into (2), we obtain

$$af\varphi^n u_z^{(n)} + bf^2\varphi uu'_z + f(bg\varphi - \varphi'_t x - \psi'_t)u'_z + (bfg_x - f'_t)u + ag_x^{(n)} + bgg_x - g_t = 0. \tag{3}$$

Equating the functional coefficients of $u_z^{(n)}$ and uu'_z in (3), we get

$$f = \varphi^{n-1}. \tag{4}$$

Further, equating the coefficient of u'_z to zero, we obtain

$$g = \frac{1}{b\varphi}(\varphi'_t x + \psi'_t). \tag{5}$$

Inserting the expressions (4) and (5) into (3), we arrive at the relation

$$\varphi^{2n-1}(au_z^{(n)} + buu'_z) + (2-n)\varphi^{n-2}\varphi'_t u + \frac{1}{b\varphi^2}\left[(2\varphi_t^2 - \varphi\varphi_{tt})x + 2\varphi_t\psi_t - \varphi\psi_{tt}\right] = 0.$$

Dividing each term by φ^{2n-1} and then eliminating x with the help of the relation $x = (z - \psi)/\varphi$, we obtain

$$au_z^{(n)} + buu_z' + (2-n)\varphi^{-n-1}\varphi_t' u + \frac{1}{b}\varphi^{-2n-2}(2\varphi_t^2 - \varphi\varphi_{tt})z + \frac{1}{b}\varphi^{-2n-2}(\varphi\psi\varphi_{tt} - \varphi^2\psi_{tt} + 2\varphi\varphi_t\psi_t - 2\psi\varphi_t^2) = 0. \quad (6)$$

Let us require that the functional coefficient of u and the last term be constant,

$$\varphi^{-n-1}\varphi_t' = -A, \qquad \varphi^{-2n-2}(\varphi\psi\varphi_{tt} - \varphi^2\psi_{tt} + 2\varphi\varphi_t\psi_t - 2\psi\varphi_t^2) = B,$$

where A and B are arbitrary. As a result, we arrive at the following system of ordinary differential equations for φ and ψ:

$$\begin{aligned} \varphi_t &= -A\varphi^{n+1}, \\ \psi_{tt} + 2A\varphi^n\psi_t + A^2(1-n)\varphi^{2n}\psi &= -B\varphi^{2n}. \end{aligned} \quad (7)$$

Using (6) and (7), we obtain an equation for $u(z)$,

$$au_z^{(n)} + buu_z' + A(n-2)u + \frac{A^2}{b}(1-n)z + \frac{B}{b} = 0. \quad (8)$$

For $A \neq 0$, the general solution of equations (7) has the form

$$\begin{aligned} \varphi(t) &= (Ant + C_1)^{-\frac{1}{n}}, \\ \psi(t) &= C_2(Ant + C_1)^{\frac{n-1}{n}} + C_3(Ant + C_1)^{-\frac{1}{n}} + \frac{B}{A^2(n-1)}, \end{aligned} \quad (9)$$

where C_1, C_2, and C_3 are arbitrary constants.

Formulas (1), (4), (5), and (9), together with equation (8), describe an exact solution of the generalized Burgers–Korteweg–de Vries equation (2).

In the special case of $n = 3$ and $a = b = -1$, the solution constructed above turns into the solution obtained by Clarkson and Kruskal (1989).

Example 2. Consider the Boussinesq equation

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial x}\left(w\frac{\partial w}{\partial x}\right) + a\frac{\partial^4 w}{\partial x^4} = 0. \quad (10)$$

Just as in Example 1, we seek its solutions in the form (1), where the functions $f(t)$, $g(x,t)$, $\varphi(t)$, and $\psi(t)$ are found in the subsequent analysis. Substituting (1) into (10) yields

$$\begin{aligned} af\varphi^4 u'''' + f^2\varphi^2 uu'' + f(z_t^2 + g\varphi^2)u'' + f^2\varphi^2(u')^2 + (fz_{tt} + 2fg_x\varphi + 2f_t z_t)u' \\ + (fg_{xx} + f_{tt})u + g_{tt} + gg_{xx} + g_x^2 + ag_x^{(4)} = 0. \end{aligned} \quad (11)$$

Equating the functional coefficients of u'''' and uu'', we get

$$f = \varphi^2. \quad (12)$$

Equating the functional coefficient of u'' to zero and taking into account (12), we obtain

$$g = -\frac{1}{\varphi^2}(\varphi_t' x + \psi_t')^2. \quad (13)$$

Substituting the expressions (12) and (13) into (11), we arrive at the relation

$$\varphi^6(au'''' + uu'' + u'^2) + \varphi^2(x\varphi_{tt} + \psi_{tt})u' + 2\varphi\varphi_{tt}u - \left[\varphi^{-2}(\varphi_t x + \psi_t)^2\right]_{tt} + 6\varphi^{-4}\varphi_t^2(\varphi_t x + \psi_t)^2 = 0.$$

Let us perform the double differentiation of the expression in square brackets and then divide all terms by φ^6. Excluding x with the help of the relation $x = (z - \psi)/\varphi$, we get

$$au'''' + uu'' + (u')^2 + \varphi^{-5}(\varphi_{tt}z + \varphi\psi_{tt} - \psi\varphi_{tt})u' + 2\varphi^{-5}\varphi_{tt}u + \cdots = 0. \quad (14)$$

Let us require that the functional coefficient of u' be a function of only one variable, z, i.e.,

$$\varphi^{-5}(\varphi_{tt}z + \varphi\psi_{tt} - \psi\varphi_{tt}) = \varphi^{-5}\varphi_{tt}z + \varphi^{-5}(\varphi\psi_{tt} - \psi\varphi_{tt}) \equiv Az + B,$$

where A and B are arbitrary constants. Hence, we obtain the following system of ordinary differential equations for the functions φ and ψ:

$$\begin{aligned} \varphi_{tt} &= A\varphi^5, \\ \psi_{tt} &= (A\psi + B)\varphi^4. \end{aligned} \quad (15)$$

Let us eliminate the second and the third derivatives of the functions φ and ψ from (14). As a result, we arrive at the following ordinary differential equation for the function $u(z)$:

$$au'''' + uu'' + (u')^2 + (Az + B)u' + 2Au - 2(Az + B)^2 = 0. \quad (16)$$

Formulas (1), (12), and (13), together with equations (15)–(16), describe an exact solution of the Boussinesq equation (10).

S.6.1-2. Description of the Clarkson–Kruskal method. A special form for similarity reduction.

1°. The basic idea of the method is the following: for an equation with the unknown function $w = w(x,t)$, an exact solution is sought in the form

$$w(x,t) = f(x,t)u(z) + g(x,t), \qquad z = z(x,t). \tag{17}$$

The functions $f(x,t)$, $g(x,t)$, and $z(x,t)$ are determined in the subsequent analysis, so that ultimately one obtains a single ordinary differential equation for the function $u(z)$.

2°. Inserting (17) into a nonlinear partial differential equation with a quadratic or a power nonlinearity, we obtain

$$\Phi_1(x,t)\Pi_1[u] + \Phi_2(x,t)\Pi_2[u] + \cdots + \Phi_m(x,t)\Pi_m[u] = 0. \tag{18}$$

Here, the $\Pi_k[u]$ are differential forms that are the products of nonnegative integer powers of the function u and its derivatives u'_z, u''_{zz}, etc., and the $\Phi_k(x,t)$ depend on the functions $f(x,t)$, $g(x,t)$, and $z(x,t)$ and their partial derivatives with respect to x and t. Suppose that the differential form $\Pi_1[u]$ contains the highest-order derivative with respect to z. Then the function $\Phi_1(x,t)$ is used as a normalizing factor. This means that the following relations should hold:

$$\Phi_k(x,t) = \Gamma_k(z)\Phi_1(x,t), \qquad k = 1, \dots, m, \tag{19}$$

where the $\Gamma_k(z)$ are functions to be determined.

3°. The representation of a solution in the form (17) has "redundant" generality and the functions f, g, u, and z are ambiguously determined. In order to remove the ambiguity, we use the following three degrees of freedom in the determination of the above functions:

(*a*) if $f = f(x,t)$ has the form $f = f_0(x,t)\Omega(z)$, then we can take $\Omega \equiv 1$, which corresponds to the replacement $u(z) \to u(z)/\Omega(z)$;

(*b*) if $g = g(x,t)$ has the form $g = g_0(x,t) + f(x,t)\Omega(z)$, then we can take $\Omega \equiv 0$, which corresponds to the replacement $u(z) \to u(z) - \Omega(z)$;

(*c*) if $z = z(x,t)$ is determined by an equation of the form $\Omega(z) = h(x,y)$, where $\Omega(z)$ is any invertible function, then we can take $\Omega(z) = z$, which corresponds to the replacement $z \to \Omega^{-1}(z)$.

4°. Having determined the functions $\Gamma_k(z)$, we substitute (19) into (18) to obtain an ordinary differential equation for $u(z)$,

$$\Pi_1[u] + \Gamma_2(z)\Pi_2[u] + \cdots + \Gamma_m(z)\Pi_m[u] = 0. \tag{20}$$

Below we illustrate the main points of the Clarkson–Kruskal direct method by an example.

Example 3. We seek a solution of the Boussinesq equation (10) in the form (17). We have

$$afz_x^4 u'''' + a(6fz_x^2 z_{xx} + 4f_x z_x^3)u''' + f^2 z_x^2 uu'' + \cdots = 0. \tag{21}$$

Here, we have written out only the first three terms and have omitted the arguments of the functions f and z. The functional coefficients of u'''' and uu'' should satisfy the condition [see (19)]:

$$f^2 z_x^2 = afz_x^4 \Gamma_3(z),$$

where $\Gamma_3(z)$ is a function to be determined. Hence, using the degree of freedom mentioned in Item 3°(*a*), we choose

$$f = z_x^2, \qquad \Gamma_3(z) = 1/a. \tag{22}$$

Similarly, the functional coefficients of u'''' and u''' must satisfy the condition

$$6fz_x^2 z_{xx} + 4f_x z_x^3 = fz_x^4 \Gamma_2(z), \tag{23}$$

where $\Gamma_2(z)$ is another function to be determined. Hence, with (22), we find

$$14\, z_{xx}/z_x = \Gamma_2(z) z_x.$$

Integrating with respect to x yields

$$\ln z_x = I(z) + \ln \widetilde{\varphi}(t), \qquad I(z) = \frac{1}{14}\int \Gamma_2(z)\,dz,$$

where $\widetilde{\varphi}(t)$ is an arbitrary function. Integrate again to obtain

$$\int e^{-I(z)}\,dz = \widetilde{\varphi}(t)x + \widetilde{\psi}(t),$$

where $\widetilde{\psi}(t)$ is another arbitrary function. We have a function of z on the left, and therefore, using the degree of freedom mentioned in Item $3^\circ(c)$, we obtain

$$z = x\varphi(t) + \psi(t), \tag{24}$$

where $\varphi(t)$ and $\psi(t)$ are to be determined.

From formulas (22)–(24) it follows that

$$f = \varphi^2(t), \qquad \Gamma_2(z) = 0. \tag{25}$$

Substituting (24) and (25) into (17), we obtain a solution of the form (1) with the function f defined by (12). Thus, the general approach based on the representation of a solution in the form (17) ultimately leads us to the same result as the approach based on the more simple formula (1).

Remark 1. In a similar way, it can be shown that formulas (1) and (17) used for the construction of an exact solution of the generalized Burgers–Korteweg–de Vries equation (2) lead us to the same result.

Remark 2. The above examples clearly show that it is more reasonable to perform the initial analysis of specific equations on the basis of the simpler formula (1) rather than the general formula (17).

⊙ *References for Subsection* S.6.1: P. A. Clarkson and M. D. Kruskal (1989), D. Arrigo, P. Broadbridge, and J. M. Hill (1993), P. A. Clarkson, D. K. Ludlow, and T. J. Priestley (1997), D. K. Ludlow, P. A. Clarkson, and A. P. Bassom (1999, 2000).

S.6.2. Clarkson–Kruskal Direct Method: the General Form for Similarity Reduction

S.6.2-1. General form of solutions.

The basic idea of the method is the following: for an equation with the unknown function $w = w(x,t)$, an exact solution is sought in the form

$$w(x,t) = F\bigl(x,t,u(z)\bigr), \qquad z = z(x,t). \tag{26}$$

The functions $F(x,t,u)$ and $z(x,t)$ should be chosen so as to obtain ultimately a single ordinary differential equation for $u(z)$. Unlike formulas (1) and (17), the relationship between the functions w and u in (26) can be nonlinear.

Below we illustrate the main features of the Clarkson–Kruskal direct method by examples.

S.6.2-2. Examples with applications of the Clarkson–Kruskal direct method.

Example 4. Consider once again the Boussinesq equation (10). Substituting (26) into (10), we get

$$aF_u z_x^4 u'''' + 4aF_{uu} z_x^4 u' u''' + a(4F_{xu} z_x^3 + 6F_u z_x^2 z_{xx})u''' + \cdots = 0. \tag{27}$$

Here, we have written out only the first three principal terms and omitted the arguments of the functions F and z. In order to ensure that (27) is reducible to an ordinary differential equation for $u = u(z)$, the ratios of the functional coefficients of $u'u'''$, u''', $\ldots$ to the coefficient of the highest-order derivative u'''' must be functions of z and u, i.e.,

$$\frac{4aF_{uu}z_x^4}{aF_u z_x^4} = \Gamma_2(z,u), \qquad \frac{a(4F_{xu}z_x^3 + 6F_u z_x^2 z_{xx})}{aF_u z_x^4} = \Gamma_3(z,u), \qquad \ldots$$

From the first relation we have

$$F_{uu}/F_u = \Gamma_2(z,u).$$

Integrating twice with respect to u yields

$$F(x,t,u) = f(x,t)\Theta(z,u) + g(x,t), \tag{28}$$

where $f(x,t)$ and $g(x,t)$ are arbitrary functions of two arguments, $\Theta = \int \exp\bigl(\int \Gamma_2\,du\bigr)\,du$.

Denoting $\Theta(z,u(z)) = U(z)$ in (28) and using the representation (26), we arrive at a solution which, up to notation, coincides with (17). Therefore, if we seek a similarity reduction of the Boussinesq equation (10) in the general form (26), we are naturally led to the special form (17).

Example 5. Consider the Harry–Dym equation

$$\frac{\partial w}{\partial t} + 2\frac{\partial^3}{\partial x^3}\frac{1}{\sqrt{w}} = 0. \tag{29}$$

Let us seek a similarity reduction in the form (26). Inserting the expression (26) into (29), we arrive at the relation

$$-F^{-3/2}F_u z_x^3 u''' + \left(-3F^{-3/2}F_{uu} + \tfrac{9}{2}F^{-5/2}F_u^2\right) z_x^3 u'u'' + \cdots = 0.$$

The ratio of the functional coefficients of $u'u''$ and u''' must be a function of z and u, i.e.,

$$3\frac{F_{uu}}{F_u} - \frac{9}{2}\frac{F_u}{F} = \Gamma(z,u).$$

The double integration yields

$$F^{-1/2}(x,t,u) = f(x,t)\Theta(z,u) + g(x,t), \tag{30}$$

where $f(x,t)$ and $g(x,t)$ are arbitrary functions of two arguments, $\Theta = -\int \exp\left(\frac{1}{3}\int \Gamma\,du\right) du$. From (26) and (30) it follows that one can seek similarity reductions of the Harry-Dym equation (29) in the form

$$w^{-1/2}(x,t) = f(x,t)U(z) + g(x,t), \qquad z = z(x,t).$$

⊙ *References for Subsection* S.6.2: P. A. Clarkson and M. D. Kruskal (1989), D. Arrigo, P. Broadbridge, and J. M. Hill (1993), D. Levi and P. Winternitz (1989), P. Olver (1994).

S.6.3. Some Modifications and Generalizations

S.6.3-1. Similarity reductions based on the ideas of the generalized separation of variables.

1°. The Clarkson–Kruskal direct method based on the representation of solutions in the forms (17) and (26) attaches particular significance to the function $u = u(z)$, because the choice of the other functions is meant to ensure a *single ordinary differential equation* for $u(z)$. However, in some cases it is reasonable to combine these methods with the ideas of the generalized and functional separation of variables, with all determining functions being regarded as equally important. Then, the function $u(z)$ is described by an *overdetermined system of equations*.

2°. Exact solutions of nonlinear partial differential equations with quadratic or power nonlinearities may be sought in the form (1) with $g(x,t) = g_1(t)x + g_0(t)$. Substituting (1) into an equation under consideration, we replace x by the expression $x = [z - \psi(t)]/\varphi(t)$. As a result, we obtain a functional-differential equation with two arguments, t and z. Its solution can sometimes be obtained by the differentiation and splitting methods outlined in Subsections S.4.2–S.4.4.

Example 6. Consider the equation of an axisymmetric steady hydrodynamic boundary layer

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = a\frac{\partial}{\partial y}\left(y\frac{\partial^2 w}{\partial y^2}\right) + \mathcal{F}(x), \tag{31}$$

which, obviously, coincides with equation 9.3.1.3 in suitable notation.

Its solution is sought in the form (for convenience, we introduce a coefficient a)

$$w(x,y) = af(x)u(z) + ag(x), \qquad z = \varphi(x)y + \psi(x). \tag{32}$$

Let us substitute this expression into equation (31) and eliminate y, using the relation $\varphi(x)y = z - \psi(x)$. After the division by $a^2\varphi^2 f$, we arrive at the functional-differential equation

$$(zu''_{zz})'_z - \psi u'''_{zzz} + f'_x u u''_{zz} + g'_x u''_{zz} - \frac{(f\varphi)'_x}{\varphi}(u'_z)^2 + \frac{\mathcal{F}}{a^2 f\varphi^2} = 0. \tag{33}$$

General methods for solving such equations are outlined in Section S.4. Here we use a simplified scheme for the construction of exact solutions. Assume that the functional coefficients of uu''_{zz}, u''_{zz}, $(u'_z)^2$, and 1 are linear combinations of the coefficients 1 and ψ of the highest-order terms $(zu''_{zz})'_z$ and u'''_{zzz}, respectively. We have

$$\begin{aligned} f'_x &= A_1 + B_1\psi,\\ g'_x &= A_2 + B_2\psi,\\ -(f\varphi)'_x/\varphi &= A_3 + B_3\psi,\\ \mathcal{F}/(a^2 f\varphi^2) &= A_4 + B_4\psi, \end{aligned} \tag{34}$$

where the A_k and B_k are arbitrary constants. Let us substitute the expressions of (34) into (33) and sum up the terms proportional to ψ (it is assumed that $\psi \neq \text{const}$). Equating the functional coefficient of ψ to zero, we obtain the following overdetermined system

$$(zu''_{zz})'_z + A_1 uu''_{zz} + A_2 u''_{zz} + A_3 (u'_z)^2 + A_4 = 0, \tag{35}$$

$$-u'''_{zzz} + B_1 uu''_{zz} + B_2 u''_{zz} + B_3 (u'_z)^2 + B_4 = 0. \tag{36}$$

Case 1. Let

$$A_1 = A_3 = A_4 = 0, \quad A_2 = -n. \tag{37}$$

Then, the solution of equation (35) has the form

$$u(z) = \frac{C_1}{n(n+1)} z^{n+1} + C_2 z + C_3, \tag{38}$$

where C_1, C_2, and C_3 are integration constants. The solution (38) of equation (35) can be a solution of equation (36) only if the following conditions are satisfied:

$$n = -2, \quad B_1 = B_3, \quad C_1 = -4/B_1, \quad C_2^2 = -B_4/B_1, \quad C_3 = -B_2/B_1. \tag{39}$$

Let us insert the coefficients (37), (39) into system (34). Integrating yields

$$g(x) = 2x - C_3 f, \quad \varphi = \frac{C_4}{f^2}, \quad \psi = -\frac{C_1}{4} f'_x, \quad \mathcal{F} = -(aC_2C_4)^2 \frac{f'_x}{f^3}, \tag{40}$$

where $f = f(x)$ is an arbitrary function.

Formulas (32), (38), (40) define an exact solution of the axisymmetric boundary layer equation (31).

Case 2. For

$$B_1 = B_3 = B_4 = 0, \quad B_2 = -\lambda, \quad A_2 = 0, \quad A_3 = -A_1, \quad A_4 = \lambda^2/A_1 \tag{41}$$

a common solution of system (35), (36) can be written in the form

$$u(z) = \frac{1}{A_1}(C_1 e^{-\lambda z} + \lambda z - 3). \tag{42}$$

A solution of system (34) with coefficients (41) is described by the formulas

$$f = A_1 x + C_2, \quad \varphi = C_3, \quad \psi = -\frac{1}{\lambda} g'_x, \quad \mathcal{F} = \frac{(aC_3\lambda)^2}{A_1}(A_1 x + C_2), \tag{43}$$

where C_1, C_2, and C_3 are arbitrary constants and $g = g(x)$ is an arbitrary function.

Formulas (32), (42), (43) define an exact solution of the axisymmetric boundary layer equation (31).

Case 3. System (35)–(36) also admits solutions of the form

$$u(z) = C_1 z^2 + C_2 z + C_3,$$

with constants C_1, C_2, and C_3 related to the A_n and B_n. For the corresponding solutions of equation (31), see 9.3.1.3.

⦿ *References*: G. I. Burde (1994, 1995), A. D. Polyanin and V. F. Zaitsev (2002).

Example 7. Consider the equation with a cubic nonlinearity

$$\frac{\partial w}{\partial t} + \sigma w \frac{\partial w}{\partial x} = a \frac{\partial^2 w}{\partial x^2} + b_3 w^3 + b_2 w^2 + b_1 w + b_0. \tag{44}$$

Let us seek its solution in the form

$$w(x,t) = f(x,t)u(z) + \lambda, \quad z = z(x,t), \tag{45}$$

where the functions $f = f(x,t)$, $z = z(x,t)$, and $u = u(z)$, as well as the constant λ, are to be determined. Substituting (45) into the equation, we obtain

$$\begin{aligned} &afz_x^2 u'' - \sigma f^2 z_x uu' + (afz_{xx} + 2af_x z_x - \sigma\lambda f z_x - f z_t)u' + b_3 f^3 u^3 \\ &\quad + (3b_3\lambda f^2 + b_2 f^2 - \sigma f f_x)u^2 + (3b_3\lambda^2 f + 2b_2\lambda f + b_1 f + af_{xx} - \sigma\lambda f_x - f_t)u \\ &\quad + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0. \end{aligned} \tag{46}$$

From the overdetermined system of ordinary differential equations resulting from the condition of proportionality of the three functions u'', uu', and u^3 and that of the two functions u' and u^2, it follows that

$$u(z) = 1/z, \tag{47}$$

where the constant factor is taken equal to unity [this factor can be included in f, since formula (45) contains the product of u and f]. Let us substitute (47) into (46) and represent the resulting expression as a finite expansion in negative powers of z. Equating the functional coefficient of z^{-3} to zero, we obtain

$$f = \beta z_x, \tag{48}$$

where β is a root of the quadratic equation

$$b_3\beta^2 + \sigma\beta + 2a = 0. \tag{49}$$

Equating the functional coefficients of the other powers of z to zero and taking into account (48), we find that

$$\begin{aligned}
z_t - (3a + \beta\sigma)z_{xx} + (\sigma\lambda + \beta b_2 + 3\beta b_3\lambda)z_x &= 0 \qquad \text{(coefficient of } z^{-2}),\\
z_{xt} - az_{xxx} + \sigma\lambda z_{xx} - (b_1 + 2\lambda b_2 + 3b_3\lambda^2)z_x &= 0 \qquad \text{(coefficient of } z^{-1}),\\
b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 &= 0 \qquad \text{(coefficient of } z^{0}).
\end{aligned} \tag{50}$$

Here, the first two linear partial differential equations form an overdetermined system for the function $z(x,t)$, while the last cubic equation serves for the determination of the constant λ.

Using (45), (47), and (48), we can write out a solution of equation (44) in the form

$$w(x,t) = \frac{\beta}{z}\frac{\partial z}{\partial x} + \lambda. \tag{51}$$

Let β be a root of the quadratic equation (49), and λ be a root of the last (cubic) equation in (50). According to the value of the constant b_3, one should consider two cases.

1°. *Case* $b_3 \neq 0$. From the first two equations in (50), one obtains

$$\begin{aligned}
z_t + p_1 z_{xx} + p_2 z_x &= 0,\\
z_{xxx} + q_1 z_{xx} + q_2 z_x &= 0,
\end{aligned}$$

where

$$p_1 = -\beta\sigma - 3a, \quad p_2 = \lambda\sigma + \beta b_2 + 3\beta\lambda b_3, \quad q_1 = -\frac{\beta b_2 + 3\beta\lambda b_3}{\beta\sigma + 2a}, \quad q_2 = -\frac{3b_3\lambda^2 + 2b_2\lambda + b_1}{\beta\sigma + 2a}.$$

Four situations are possible.

1.1. For $q_2 \neq 0$ and $q_1^2 \neq 4q_2$, we have

$$\begin{gathered}
z(x,t) = C_1\exp(k_1x + s_1t) + C_2\exp(k_2x + s_2t) + C_3,\\
k_n = -\tfrac{1}{2}q_1 \pm \tfrac{1}{2}\sqrt{q_1^2 - 4q_2}, \quad s_n = -k_n^2p_1 - k_np_2,
\end{gathered}$$

where C_1, C_2, and C_3 are arbitrary constants; $n = 1,\ 2$.

1.2. For $q_2 \neq 0$ and $q_1^2 = 4q_2$,

$$\begin{gathered}
z(x,t) = C_1\exp(kx + s_1t) + C_2(kx + s_2t)\exp(kx + s_1t) + C_3,\\
k = -\tfrac{1}{2}q_1, \quad s_1 = -\tfrac{1}{4}p_1q_1^2 + \tfrac{1}{2}p_2q_1, \quad s_2 = -\tfrac{1}{2}p_1q_1^2 + \tfrac{1}{2}p_2q_1.
\end{gathered}$$

1.3. For $q_2 = 0$ and $q_1 \neq 0$,

$$z(x,t) = C_1(x - p_2t) + C_2\exp[-q_1x + q_1(p_2 - p_1q_1)t] + C_3.$$

1.4. For $q_2 = q_1 = 0$,

$$z(x,t) = C_1(x - p_2t)^2 + C_2(x - p_2t) - 2C_1p_1t + C_3.$$

2°. *Case* $b_3 = 0$, $b_2 \neq 0$. The solutions are determined by (51), where

$$\beta = -\frac{2a}{\sigma}, \quad z(x,t) = C_1 + C_2\exp\left[Ax + A\left(\frac{b_1\sigma}{2b_2} + \frac{2ab_2}{\sigma}\right)t\right], \quad A = \frac{\sigma(b_1 + 2b_2\lambda)}{2ab_2},$$

and $\lambda = \lambda_{1,2}$ are roots of the quadratic equation $b_2\lambda^2 + b_1\lambda + b_0 = 0$.

⊙ *References*: M. C. Nucci and P. A. Clarkson (1992), N. A. Kudryashov (1993).

S.6.3-2. Similarity reductions in equations with three or more independent variables.

The procedure of the construction of exact solutions to nonlinear equations with three or more independent variables sometimes involves (at intermediate stages) the solution of functional-differential equations considered in Subsections S.4.2–S.4.4.

Example 8. Consider the nonlinear nonstationary wave equation anisotropic in one of the directions

$$\frac{\partial^2 w}{\partial t^2} = a\frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial y}\left[(bw + c)\frac{\partial w}{\partial y}\right]. \tag{52}$$

Let us seek its solution in the form

$$w = U(z) + f(x,t), \qquad z = y + g(x,t). \tag{53}$$

Substituting (53) into equation (52), we get

$$[(bU + ag_x^2 - g_t^2 + bf + c)U_z']_z' + (ag_{xx} - g_{tt})U_z' + af_{xx} - f_{tt} = 0.$$

Suppose that the functions f and g satisfy the following overdetermined system of equations:

$$af_{xx} - f_{tt} = C_1, \tag{54}$$

$$ag_{xx} - g_{tt} = C_2, \tag{55}$$

$$ag_x^2 - g_t^2 + bf = C_3, \tag{56}$$

where C_1, C_2, and C_3 are arbitrary constants. Then the function $U(z)$ is determined by the autonomous ordinary differential equation

$$[(bU + c + C_3)U'_z]'_z + C_2U'_z + C_1 = 0. \tag{57}$$

The general solutions of equations (54)–(55) are expressed as

$$f = \varphi_1(\xi) + \psi_1(\eta) - \tfrac{1}{2}C_1t^2,$$
$$g = \varphi_2(\xi) + \psi_2(\eta) - \tfrac{1}{2}C_2t^2,$$
$$\xi = x + t\sqrt{a}, \quad \eta = x - t\sqrt{a}.$$

Let us insert these expressions into equation (56) and then eliminate t with the help of the formula $t = \dfrac{\xi - \eta}{2\sqrt{a}}$. After simple transformations, we obtain a functional-differential equation with two arguments,

$$b\varphi_1(\xi) + C_2\xi\varphi_2'(\xi) - k\xi^2 - C_3 + b\psi_1(\eta) + C_2\eta\psi_2'(\eta) - k\eta^2 + \psi_2'(\eta)[4a\varphi_2'(\xi) - C_2\xi] + \eta[2k\xi - C_2\varphi_2'(\xi)] = 0, \tag{58}$$

where

$$k = \frac{1}{8a}(bC_1 + 2C_2^2).$$

Equation (58) can be solved by the splitting method described in Section S.4. According to the simplified scheme, set

$$\begin{aligned} b\varphi_1(\xi) + C_2\xi\varphi_2'(\xi) - k\xi^2 - C_3 &= A_1,\\ 4a\varphi_2'(\xi) - C_2\xi &= A_2,\\ 2k\xi - C_2\varphi_2'(\xi) &= A_3, \end{aligned} \tag{59}$$

where A_1, A_2, and A_3 are constants. The common solution of system (59) has the form

$$\varphi_1(\xi) = -\frac{C_2^2}{8ab}\xi^2 - \frac{BC_2}{b}\xi + \frac{A_1 + C_3}{b}, \quad \varphi_2(\xi) = \frac{C_2}{8a}\xi^2 + B\xi \tag{60}$$

and corresponds to the following values of the constants:

$$A_1 \text{ is arbitrary}, \quad A_2 = 4aB, \quad A_3 = -BC_2, \quad B \text{ is arbitrary}, \quad C_1 = -\frac{C_2^2}{b}, \quad C_2 \text{ and } C_3 \text{ are arbitrary}, \quad k = \frac{C_2^2}{8a}. \tag{61}$$

From (58) and (59) we obtain an equation that establishes a relation between the functions ψ_1 and ψ_2,

$$A_1 + b\psi_1(\eta) + C_2\eta\psi_2'(\eta) - k\eta^2 + A_2\psi_2'(\eta) + A_3\eta = 0. \tag{62}$$

Hence, taking into account (61), we get

$$\psi_1(\eta) = -\frac{1}{b}(C_2\eta + 4aB)\psi_2'(\eta) + \frac{1}{b}\left(\frac{C_2^2}{8a}\eta^2 + BC_2\eta - A_1\right), \quad \psi_2(\eta) \text{ is an arbitrary function.}$$

Ultimately, we find the functions that determine solution (53):

$$f(x,t) = -\frac{C_2^2}{2\sqrt{a}\,b}xt + \frac{C_2^2}{2b}t^2 - \frac{2\sqrt{a}\,BC_2}{b}t + \frac{C_3}{b} - \frac{1}{b}(C_2\eta + 4aB)\psi_2'(\eta),$$
$$g(x,t) = \frac{C_2}{8a}\left(x^2 + 2\sqrt{a}\,xt - 3at^2\right) + B(x + \sqrt{a}\,t) + \psi_2(\eta).$$

Remark 1. For other solutions of this equation, see 4.1.3.1.

Remark 2. In the special case of $a = 1$, $b < 0$, and $c > 0$, equation (52) describes spatial transonic flows of an ideal polytropic gas (Pokhozhaev, 1989).

S.7. Group Analysis Methods

S.7.1. Classical Method for Symmetry Reductions

The group analysis methods (also referred to as Lie group methods) suggest a regular procedure for identifying symmetries of an equation and allow us to find the following:

(i) transformations under which the equation is invariant (i.e., turns into exactly the same equation after these transformations),

(ii) new variables (both dependent and independent), in which the structure of the equation is much simplified.

The transformations mentioned in item (i) map a solution of the equation into the same or another solution of the same equation. In the former case, we have an invariant solution, which can be found by reducing the original equation to another equation in fewer new variables. In the second case, noninvariant solutions can be "multiplied," so as to form a family of solutions.

Remark. The methods of group analysis may be regarded as a wide generalization of the similarity methods described in Section S.3.

S.7.1-1. Local one parameter Lie group of transformations. Invariance condition.

We will consider transformations of the following second-order partial differential equation:

$$F\left(x_i, w, \frac{\partial w}{\partial x_i}, \frac{\partial^2 w}{\partial x_i \partial x_j}\right) = 0, \qquad i, j = 1, \ldots, n, \tag{1}$$

where $x = (x_1, \ldots, x_n)$ are independent variables and w is a dependent variable (unknown function).

Consider a set of transformations of the $(n+1)$-dimensional Euclidean space

$$\mathrm{T}_\varepsilon = \begin{cases} \bar{x}_i = \varphi_i(x, w, \varepsilon), & \bar{x}_i|_{\varepsilon=0} = x_i, \\ \bar{w} = \psi(x, w, \varepsilon), & \bar{w}|_{\varepsilon=0} = w, \end{cases} \tag{2}$$

where the φ_i and ψ are smooth functions of their arguments and ε is a real parameter. This set of transformations is called a one-parameter continuous point Lie group of transformations, G, if for all ε_1 and ε_2, we have $\mathrm{T}_{\varepsilon_1} \circ \mathrm{T}_{\varepsilon_2} = \mathrm{T}_{\varepsilon_1+\varepsilon_2}$, i.e., the successive application of two transformations of the form (1) with parameters ε_1 and ε_2 is equivalent to a single transformation of the same form with parameter $\varepsilon_1 + \varepsilon_2$.

Let G be a group of transformations of a set M in the $(n+1)$-dimensional Euclidean space, and let $u = (x, w)$ be a point of that set. The set $G(u)$ formed by all images $\mathrm{T}u$, as T ranges within the entire group G, is called the orbit of the point u. The set M is called invariant under a group of transformations if the orbit of each point u of M belongs to M, i.e., $G(M) = M$. In other words, any point of an invariant set remains in that set under arbitrary transformations of the group, i.e., the set is mapped into itself.

Below, we consider local one-parameter continuous point Lie groups of transformations (briefly, point groups) that correspond to the infinitesimal transformation (2) as $\varepsilon \to 0$. Expanding the functions $\bar{x}$ and $\bar{w}$ from (2) into the Taylor series in powers of the parameter ε about the point $\varepsilon = 0$ and neglecting the second- and higher-order terms, we obtain

$$\bar{x}_i \simeq x_i + \xi_i(x, w)\varepsilon, \quad \bar{w} \simeq w + \zeta(x, w)\varepsilon, \tag{3}$$

where

$$\xi_i(x, w) = \frac{\partial \varphi_i(x, w, \varepsilon)}{\partial \varepsilon}\bigg|_{\varepsilon=0}, \qquad \zeta(x, w) = \frac{\partial \psi(x, w, \varepsilon)}{\partial \varepsilon}\bigg|_{\varepsilon=0}.$$

The vector (ξ, ζ) is tangent (at the point (x, w)) to the curve formed by the transformed points $(\bar{x}, \bar{w})$.

The first-order linear differential operator

$$\mathrm{X} = \xi_i(x, w)\frac{\partial}{\partial x_i} + \zeta(x, w)\frac{\partial}{\partial w}, \tag{4}$$

corresponding to the infinitesimal transformation (3), is called the infinitesimal operator (or infinitesimal generator) of the group (here and in what follows, summation over repeated indices is assumed).

By definition, a universal invariant (briefly, invariant) of the group (2) and the operator (4) is a function $I_0(x,w)$ satisfying the condition $I_0(\bar{x},\bar{w}) = I_0(x,w)$. The expansion in powers of the small parameter ε yields the following linear partial differential equation for I_0:

$$\mathrm{X}I_0 = \xi_i(x,w)\frac{\partial I_0}{\partial x_i} + \zeta(x,w)\frac{\partial I_0}{\partial w} = 0. \tag{5}$$

It follows that the group (2) and the operator (4) have n functionally independent universal invariants. This means that any function $F(x,w)$ which is invariant under the group (2) can be represented as a function of n invariants, which play the role of new variables.

In the new variables (2), the first and second derivatives take the form

$$\begin{aligned}\frac{\partial\bar{w}}{\partial\bar{x}_i} &\simeq \frac{\partial w}{\partial x_i} + \zeta_i\varepsilon,\\ \frac{\partial^2\bar{w}}{\partial\bar{x}_i\partial\bar{x}_j} &\simeq \frac{\partial^2 w}{\partial x_i\partial x_j} + \zeta_{ij}\varepsilon.\end{aligned} \tag{6}$$

Here, the coordinates of the first and second prolongations ζ_i and ζ_{ij} are defined by

$$\begin{aligned}\zeta_i &= D_i(\zeta) - p_j D_i(\xi_j),\\ \zeta_{ij} &= D_j(\zeta_i) - q_{ik} D_j(\xi_k),\end{aligned} \tag{7}$$

where the following brief notation is used for the partial derivatives: $p_i = \dfrac{\partial w}{\partial x_i}$, $q_{ij} = \dfrac{\partial^2 w}{\partial x_i\partial x_j}$; $D_i = \dfrac{\partial}{\partial x_i} + p_i\dfrac{\partial}{\partial w} + q_{ij}\dfrac{\partial}{\partial p_j} + \cdots$ is the operator of total differentiation with respect to x_i.

Let us prove the first set of formulas (6) for the coordinates of the first prolongation. For simplicity, consider the case of two independent variables x and y. Then formulas (3) can be written as

$$\bar{x} \simeq x + \xi_1(x,y,w)\varepsilon, \quad \bar{y} \simeq y + \xi_2(x,y,w)\varepsilon, \quad \bar{w} \simeq w + \zeta(x,y,w)\varepsilon. \tag{8}$$

Obviously,

$$\bar{w}_x = \bar{w}_{\bar{x}}\bar{x}_x + \bar{w}_{\bar{y}}\bar{y}_x, \quad \bar{w}_y = \bar{w}_{\bar{x}}\bar{x}_y + \bar{w}_{\bar{y}}\bar{y}_y. \tag{9}$$

Differentiating relations (8) with respect to x and y, we obtain

$$\begin{aligned}&\bar{x}_x = 1 + D_x\xi_1\varepsilon, &&\bar{x}_y = D_y\xi_1\varepsilon,\\ &\bar{y}_x = D_x\xi_2\varepsilon, &&\bar{y}_y = 1 + D_y\xi_2\varepsilon,\\ &\bar{w}_x = w_x + D_x\zeta\varepsilon, &&\bar{w}_y = w_y + D_y\zeta\varepsilon.\end{aligned} \tag{10}$$

In order to calculate $\bar{w}_{\bar{x}}$, we eliminate $\bar{w}_{\bar{y}}$ from (9) and then replace the derivatives $\bar{x}_x$, $\bar{x}_y$, $\bar{y}_x$, $\bar{y}_y$, $\bar{w}_x$, $\bar{w}_y$ by the corresponding expressions from (10) to obtain

$$\bar{w}_{\bar{x}} = \frac{w_x + \varepsilon(D_x\zeta + w_x D_y\xi_2 - w_y D_x\xi_2) + \varepsilon^2(D_x\zeta D_y\xi_2 - D_x\xi_2 D_y\zeta)}{1 + \varepsilon(D_x\xi_1 + D_y\xi_2) + \varepsilon^2(D_x\xi_1 D_y\xi_2 - D_x\xi_2 D_y\xi_1)}.$$

Using the expansion in powers of ε, we find that

$$\bar{w}_{\bar{x}} \simeq w_x + \zeta_1\varepsilon, \quad \zeta_1 = D_x\zeta - w_x D_x\xi_1 - w_y D_x\xi_2,$$

as required. In a similar way, one can calculate ζ_2 and the coordinates of the second prolongation ζ_{ij}.

Let us require that equation (1) be invariant (i.e., preserve its form) under the transformations in question,

$$F\left(\bar{x}_i, \bar{w}, \frac{\partial\bar{w}}{\partial\bar{x}_i}, \frac{\partial^2\bar{w}}{\partial x_i\partial\bar{x}_j}\right) = 0.$$

Let us expand this expression into a series in powers of the small parameter $\varepsilon \to 0$. Taking into account that the leading term of the expansion (1) is zero, using (3) and (6), and retaining only the first-order terms, we obtain

$$\underset{2}{\mathrm{X}}F\left(x_i, w, \frac{\partial w}{\partial x_i}, \frac{\partial^2 w}{\partial x_i\partial x_j}\right)\bigg|_{F=0} = 0, \tag{11}$$

where $\underset{2}{\mathrm{X}}$ is the twice prolonged operator,

$$\underset{2}{\mathrm{X}} = \xi_i(x,w)\frac{\partial}{\partial x_i} + \zeta(x,w)\frac{\partial}{\partial w} + \zeta_i\frac{\partial}{\partial p_i} + \zeta_{ij}\frac{\partial}{\partial q_{ij}}.$$

Relation (11) is called the invariance condition.

Remark. The invariant I_0, which is a solution of equation (5), also satisfies the equation $\underset{2}{\mathrm{X}}I_0 = 0$.

S.7.1-2. Group analysis of second-order nonlinear equations in two independent variables.

Consider the second-order equation in two independent variables

$$\frac{\partial^2 w}{\partial y^2} = H\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}\right). \tag{12}$$

In this case, the infinitesimal operator (4) has the form

$$\mathrm{X} = \xi(x, y, w)\frac{\partial}{\partial x} + \eta(x, y, w)\frac{\partial}{\partial y} + \zeta(x, y, w)\frac{\partial}{\partial w},$$

where we have used the notation $\xi = \xi_1$ and $\eta = \xi_2$.

The coordinates of the first prolongation are given by

$$\zeta_1 = D_x(\zeta) - w_x D_x(\xi) - w_y D_x(\eta),$$
$$\zeta_2 = D_y(\zeta) - w_x D_y(\xi) - w_y D_y(\eta),$$

which, after suitable calculations, become

$$\begin{aligned}
\zeta_1 &= \zeta_x + (\zeta_w - \xi_x)w_x - \eta_x w_y - \xi_w w_x^2 - \eta_w w_x w_y,\\
\zeta_2 &= \zeta_y - \xi_y w_x + (\zeta_w - \eta_y)w_y - \xi_w w_x w_y - \eta_w w_y^2.
\end{aligned} \tag{13}$$

The coordinates of the second prolongation are expressed as

$$\zeta_{11} = D_x(\zeta_1) - w_{xx} D_x(\xi) - w_{xy} D_x(\eta),$$
$$\zeta_{12} = D_y(\zeta_1) - w_{xx} D_y(\xi) - w_{xy} D_y(\eta),$$
$$\zeta_{22} = D_y(\zeta_2) - w_{xy} D_y(\xi) - w_{yy} D_y(\eta),$$

or, after calculations,

$$\begin{aligned}
\zeta_{11} &= \zeta_{xx} + (2\zeta_{wx} - \xi_{xx})w_x - \eta_{xx} w_y + (\zeta_{ww} - 2\xi_{wx})w_x^2 - 2\eta_{wx} w_x w_y\\
&\quad - \xi_{ww} w_x^3 - \eta_{ww} w_x^2 w_y + (\zeta_w - 2\xi_x - 3\xi_w w_x - \eta_w w_y)w_{xx} - 2(\eta_x + \eta_w w_x)w_{xy},\\
\zeta_{12} &= \zeta_{xy} + (\zeta_{wy} - \xi_{xy})w_x + (\zeta_{wx} - \eta_{xy})w_y - \xi_{wy} w_x^2\\
&\quad - (\zeta_{ww} - \xi_{wx} - \eta_{wy})w_x w_y - \eta_{wx} w_y^2 - \xi_{ww} w_x^2 w_y - \eta_{ww} w_x w_y^2\\
&\quad - (\xi_y + \xi_w w_y)w_{xx} + (\zeta_w - \xi_x - \eta_y - 2\xi_w w_x - 2\eta_w w_y)w_{xy} - (\eta_x + \eta_w w_x)w_{yy},\\
\zeta_{22} &= \zeta_{yy} - \xi_{yy} w_x + (2\zeta_{wy} - \eta_{yy})w_y - 2\xi_{wy} w_x w_y + (\zeta_{ww} - 2\eta_{wy})w_y^2\\
&\quad - \xi_{ww} w_x w_y^2 - \eta_{ww} w_y^3 - 2(\xi_y + \xi_w w_y)w_{xy} + (\zeta_w - 2\eta_y - \xi_w w_x - 3\eta_w w_y)w_{yy}.
\end{aligned} \tag{14}$$

The invariance condition (11) for equation (12) reads

$$\zeta_{22} = \xi\frac{\partial H}{\partial x} + \eta\frac{\partial H}{\partial y} + \zeta\frac{\partial H}{\partial w} + \zeta_1\frac{\partial H}{\partial w_x} + \zeta_2\frac{\partial H}{\partial w_y} + \zeta_{11}\frac{\partial H}{\partial w_{xx}} + \zeta_{12}\frac{\partial H}{\partial w_{xy}}, \tag{15}$$

and in the expressions (13) and (14) of the coordinates of the first and second prolongations, ζ_i and ζ_{ij}, the derivative $\frac{\partial^2 w}{\partial y^2}$ should be replaced by the function H, in accordance with equation (12). The resulting equation can be rewritten as a polynomial in the "independent variables" represented by the remaining derivatives (w_x, w_y, w_{xx}, and w_{xy} in our case):

$$\sum A_{k_1k_2k_3k_4}(w_x)^{k_1}(w_y)^{k_2}(w_{xx})^{k_3}(w_{xy})^{k_4} = 0, \tag{16}$$

where the functional coefficients $A_{k_1k_2k_3k_4}$ depend only on $x, y, w, \xi, \eta, \zeta$ and the derivatives of the functions ξ, η, ζ and are independent of the derivatives of w. Relation (16) holds if all $A_{k_1k_2k_3k_4} = 0$. Thus, the invariance condition is split and can be rewritten as an overdetermined determining system, which is obtained by equating to zero the functional coefficients of the "independent variables" represented by the remaining derivatives w_x, w_y, w_{xx}, w_{xy}, of which the unknown functions ξ, η, ζ are independent.

It should be noted that the functional coefficients $A_{k_1k_2k_3k_4}$ and the determining system are linear with respect to the desired quantities ξ, η, ζ.

Below we illustrate the above procedure by examples.

Example 1. Consider the two-dimensional stationary heat equation with a nonlinear source

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - f(w) = 0, \tag{17}$$

which corresponds to the right-hand side $H = f(w) - w_{xx}$ of equation (12).

Let us insert $H = f(w) - w_{xx}$ into the invariance condition (15), taking into account the expressions (13) and (14) for the coordinates of the first and second prolongations. Now, replacing w_{yy} by $f(w) - w_{xx}$ [a consequence of equation (17)] and equating the coefficients of the remaining derivatives to zero, we obtain the following system:

$$\begin{aligned}
&w_x w_{xx}\colon && \xi_w = 0,\\
&w_y w_{xx}\colon && \eta_w = 0,\\
&w_{xx}\colon && \xi_x - \eta_y = 0,\\
&w_{xy}\colon && \xi_y + \eta_x = 0,\\
&w_x^2\colon && \zeta_{ww} - 2\xi_{wx} = 0,\\
&w_x w_y\colon && \eta_{wx} + \xi_{wy} = 0,\\
&w_x\colon && 2\zeta_{wx} - \xi_{xx} - \xi_{yy} - \xi_w f(w) = 0,\\
&w_y^2\colon && \zeta_{ww} - 2\eta_{wy} = 0,\\
&w_y\colon && 2\zeta_{wy} - \eta_{xx} - \eta_{yy} - 3\eta_w f(w) = 0,\\
&1\colon && \zeta_{xx} + \zeta_{yy} - f'(w)\zeta + f(w)(\zeta_w - 2\eta_y) = 0.
\end{aligned}$$

Here, the first column contains combinations of derivatives and the second column contains the corresponding coefficients (up to constant factors); the coefficients of $w_y w_{xy}$, $w_x w_{xy}$, w_x^3, $w_x^2 w_y$, $w_x w_y^2$, and w_y^3 are omitted, since these coincide with some of the equations of the system or are their differential consequences. Using the first, the second, and the fifth equations, we find that $\xi = \xi(x,y)$, $\eta = \eta(x,y)$, $\zeta = aw + b(x,y)$, and $a = \text{const}$. Ultimately, the system becomes

$$\begin{aligned}
&\xi_x - \eta_y = 0,\\
&\xi_y + \eta_x = 0,\\
&b_{xx} + b_{yy} - awf'(w) - bf'(w) + f(w)(a - 2\eta_y) = 0.
\end{aligned} \tag{18}$$

Obviously, for an arbitrary function f, we have $a = b = \eta_y = 0$, and therefore, $\xi = C_1 y + C_2$, $\eta = -C_1 x + C_3$, and $\zeta = 0$. Successively, taking one of the constants equal to unity and the others equal to zero, we find that the original equation admits three operators

$$\mathrm{X}_1 = \partial_x, \quad \mathrm{X}_2 = \partial_y, \quad \mathrm{X}_3 = y\partial_x - x\partial_y. \tag{19}$$

The first two operators correspond to all possible translations along the axes x and y, and the third operator corresponds to a rotation.

Consider more closely the third equation of system (18). If

$$(aw + b)f'(w) - f(w)(a - 2\eta_y) = 0, \tag{20}$$

then there may exist other solutions of system (18) which lead to operators other than (19). We should investigate two cases: $a \neq 0$ and $a = 0$.

Case 1. Solving equation (20) for $a \neq 0$, we get

$$f(w) = C(aw + b)^{1 - \frac{2\gamma}{a}},$$

where $\gamma = \eta_y = \text{const}$ and $b = \text{const}$. Therefore, for $f(w) = w^k$ equation (17) admits an additional operator

$$\mathrm{X}_4 = x\partial_x + y\partial_y + \frac{2}{1-k} w\partial_w,$$

which describes nonuniform scaling.

Case 2. For $a = 0$, the solution has the form

$$f(w) = Ce^{\lambda w},$$

where $\lambda = \text{const}$. Then $b = -2\eta_y/\lambda$ and the functions ξ and η satisfy the first two equations in (18), which coincide with the Cauchy–Riemann equations for analytic functions. These conditions hold for the real and the imaginary parts of any analytic function $f(z) = \xi(x,y) + i\eta(x,y)$ of the complex variable $z = x + iy$. In particular, for $b = \text{const}$ and $f(w) = e^w$, the following additional operator is admitted:

$$\mathrm{X}_4 = x\partial_x + y\partial_y - 2\partial_w,$$

which corresponds to scaling in x and y combined with a translation in w.

Example 2. Consider the nonlinear heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]. \tag{21}$$

The invariance condition is obtained by applying the operator $\underset{2}{\mathrm{X}} = \xi\partial_x + \eta\partial_t + \zeta\partial_w + \zeta_1\partial_{w_x} + \zeta_2\partial_{w_t} + \zeta_{11}\partial_{w_{xx}}$ to the equation

$$w_t - f(w)w_{xx} - f'(w)(w_x)^2 = 0.$$

Using the expressions (13) and (14) for the coordinates of the first and the second prolongations ζ_1 and ζ_{11} for $y = t$, and replacing w_t in the invariance condition by the right-hand side of equation (21), let us equate to zero the coefficients of different powers of the remaining derivatives. We obtain the following system:

$$\begin{array}{ll}
w_x w_{xx}\colon & 2f(w)\big(\eta_{wx}f(w) + \xi_w\big) + f'(w)\eta_x = 0,\\
w_{xx}\colon & \zeta f'(w) - f^2(w)\eta_{xx} - f(w)(2\xi_x - \eta_t) = 0,\\
w_x w_{xt}\colon & f(w)\eta_w = 0,\\
w_{xt}\colon & f(w)\eta_x = 0,\\
w_x^4\colon & f'(w)\eta_w + f(w)\eta_{ww} = 0,\\
w_x^3\colon & 2[f'(w)]^2\eta_x + f(w)\xi_{ww} + f'(w)\xi_w + 2f(w)f'(w)\eta_{wx} = 0,\\
w_x^2\colon & f(w)\zeta_{ww} + f''(w)\zeta - 2f(w)\xi_{wx} - f'(w)(2\xi_x - \eta_t) + f'(w)\zeta_w - f(w)f'(w)\eta_{ww} = 0,\\
w_x\colon & 2f(w)\zeta_{wx} + 2f'(w)\zeta_x - f(w)\xi_{xx} + \xi_t = 0,\\
1\colon & \zeta_t - f(w)\zeta_{xx} = 0.
\end{array}$$

Here, the first column lists combinations of derivatives and the second column contains the corresponding functional coefficients (up to a constant factor); identical expressions and those obtained by differentiation are omitted. Since $f(w) \not\equiv 0$, the third and the fourth equations of the system imply that $\eta = \eta(t)$. Then, from the first and the second equations we have

$$\xi = \xi(x,t), \qquad \zeta = \frac{f(w)(2\xi_x - \eta_t)}{f'(w)}.$$

Taking into account the relations obtained above, we can rewrite the system in the form

$$\begin{aligned}
&[ff'f''' - f(f'')^2 + (f')^2 f''](2\xi_x - \eta_t) = 0,\\
&f[4ff'' - 7(f')^2\xi_{xx} - (f')^2\xi_t = 0,\\
&2f\xi_{xxx} - 2\xi_{xt} + \eta_{tt} = 0
\end{aligned}$$

(the equations have been divided by common factors which are always nonzero). In the general case, for arbitrary $f(w)$, the first equation implies that $2\xi_x - \eta_t = 0$, and the second equation implies that $\xi_t = 0$. From the third equation, we get $\xi = C_1 + C_2 x$, and therefore, $\eta = 2C_2 t + C_3$. It follows that for arbitrary $f(w)$, equation (21) admits three operators:

$$\mathrm{X}_1 = \partial_x, \quad \mathrm{X}_2 = \partial_t, \quad \mathrm{X}_3 = 2t\partial_t + x\partial_x.$$

Likewise, it can be shown that for the following specific f there arise additional operators:

1. $f = e^w$: $\mathrm{X}_4 = x\partial_x + 2\partial_w$.
2. $f = w^k$, $k \neq 0, -4/3$: $\mathrm{X}_4 = kx\partial_x + 2w\partial_w$.
3. $f = w^{-4/3}$: $\mathrm{X}_4 = 2x\partial_x - 3w\partial_w$, $\quad \mathrm{X}_5 = x^2\partial_x - 3xw\partial_w$.

Example 3. Consider the nonlinear wave equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]. \tag{22}$$

Let us use the invariance condition (15) for $y = t$ and $H = f(w)w_{xx} + f'(w)(w_x)^2$. We substitute the expressions (13) and (14) of the coordinates of the first and the second prolongations, at $y = t$, and replace w_{tt} in the invariance condition by the right-hand side of equation (22), and then equate the coefficients of different powers of the remaining derivatives to zero. Thus, we obtain the following system (identical expressions and those obtained by differentiation are omitted):

$$\begin{array}{ll}
w_x w_{xx}\colon & f(w)\xi_w = 0,\\
w_t w_{xx}\colon & f(w)\eta_w = 0,\\
w_{xx}\colon & f'(w)\zeta + 2f(w)(\eta_t - \xi_x) = 0,\\
w_{xt}\colon & f(w)\eta_x - \xi_t = 0,\\
w_x^3\colon & f'(w)\xi_w + f(w)\xi_{ww} = 0,\\
w_x^2 w_t\colon & f(w)\eta_{ww} - f'(w)\eta_w = 0,\\
w_x^2\colon & f(w)\zeta_{ww} + f'(w)\zeta_w + f''(w)\zeta - 2f(w)\xi_{wx} - 2f'(w)(\xi_x - \eta_t) = 0,\\
w_x w_t\colon & 2f'(w)\eta_x + 2f(w)\eta_{wx} - 2\xi_{wt} = 0,\\
w_x\colon & 2f'(w)\zeta_x - f(w)\xi_{xx} + 2f(w)\zeta_{wx} + \xi_{tt} = 0,\\
w_t^2\colon & \zeta_{ww} - 2\eta_{wt} = 0,\\
w_t\colon & f(w)\eta_{xx} + 2\zeta_{wt} - \eta_{tt} = 0,\\
1\colon & \zeta_{tt} - f(w)\zeta_{xx} = 0.
\end{array}$$

Since $f(w) \neq$ const, the first two equations yield $\xi = \xi(x,t)$, $\eta = \eta(x,t)$. Therefore, the tenth equation of the system takes the form $\zeta_{ww} = 0$ and we obtain the expression $\zeta = a(x,t)w + b(x,t)$. As a result, there remain the following equations of the system:

$$\begin{aligned}
&wf'(w)a(x,y) + f'(w)b(x,y) + 2f(w)(\eta_t - \xi_x) = 0,\\
&f'(w)a(x,y) + wf''(w)a(x,y) + f''(w)b(x,y) - 2f'(w)(\xi_x - \eta_t) = 0,\\
&2f'(w)(a_x w + b_x) - f(w)\xi_{xx} + 2f(w)a_x = 0,\\
&2a_t - \eta_{tt} = 0,\\
&a_{tt}w + b_{tt} - f(w)(a_{xx}w + b_{xx}) = 0.
\end{aligned}$$

For an arbitrary function $f(w)$, we obtain $a = b = 0$, $\eta_{tt} = 0$, and $\xi_x - \eta_t = 0$. The integration yields three operators:

$$\mathrm{X}_1 = \partial_x, \quad \mathrm{X}_2 = \partial_t, \quad \mathrm{X}_3 = x\partial_x + t\partial_t.$$

Likewise, it can be shown that for the following specific f, there are additional operators:

1. $f = e^w$: $\mathrm{X}_4 = x\partial_x + 2\partial_w$.
2. $f = w^k$, $k \neq 0, -4/3, -4$: $\mathrm{X}_4 = kx\partial_x + 2w\partial_w$.
3. $f = w^{-4/3}$: $\mathrm{X}_4 = 2x\partial_x - 3w\partial_w$, $\mathrm{X}_5 = x^2\partial_x - 3xw\partial_w$.
4. $f = w^{-4}$: $\mathrm{X}_4 = x\partial_x - w\partial_w$, $\mathrm{X}_5 = t^2\partial_x + tw\partial_w$.

S.7.1-3. Finding exact solutions with the help of an admissible group. Invariant solutions.

1°. Suppose that we know a solution w of an equation under investigation. Then every admissible group generates a one-parameter family of solutions, namely the orbit $\mathrm{T}w$, except for the case in which the solution is transformed into itself under the action of the group transformations (see Item 2°).

2°. A solution $w = w(x,y)$ of equation (12) is called invariant under a group G if the corresponding orbit $\mathrm{T}w$ is an invariant set.

Let G be a one-parameter group admitted by equation (12) and let $I_1 = I_1(x,y)$ and $I_2 = I_2(x,y,w)$ be two functionally independent invariants of the group G.

Invariant solutions are sought in the form

$$I_2 = \Phi(I_1), \tag{23}$$

where Φ is a function to be determined. Solving (23) for w and substituting the result into (12), we obtain an ordinary differential equation for the function Φ.

A well-known and very important special class of invariant solutions is represented by self-similar solutions which are constructed on the basis of invariants of extension groups.

For the sake of illustration, the general scheme of the construction of invariant solutions of second-order evolution equations is represented in Figure 3. Here, we omit the first-order partial differential equation for the determination of the group invariants (because we can proceed directly to the corresponding characteristic system of ordinary differential equations).

Example 4. Again, consider the stationary heat equation with a nonlinear source

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(w).$$

1°. Let us examine the case $f = w^k$, in which the equation admits an additional operator (see Example 1):

$$\mathrm{X}_4 = x\partial_x + y\partial_y + \frac{2}{1-k}w\partial_w.$$

In order to find invariants of this operator, one should consider the linear first-order partial differential equation $\mathrm{X}_4 I = 0$ which can be written out in complete form as

$$x\frac{\partial I}{\partial x} + y\frac{\partial I}{\partial y} + \frac{2}{1-k}w\frac{\partial I}{\partial w} = 0.$$

The corresponding characteristic system of ordinary differential equations

$$\frac{dx}{x} = \frac{dy}{y} = \frac{1-k}{2}\frac{dw}{w}$$

admits the first integrals

$$y/x = C_1, \quad x^{2/(k-1)}w = C_2,$$

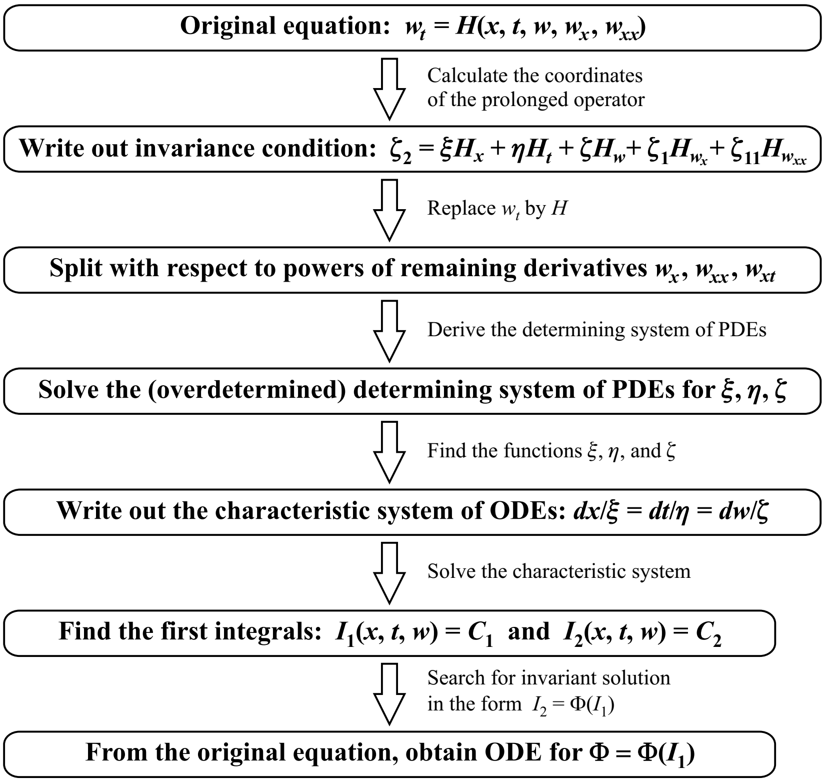

Figure 3. Algorithm for the construction of invariant solutions of second-order evolution equations. Here, ODE stands for ordinary differential equation and PDE for partial differential equation; $\xi = \xi(x,t,w)$, $\eta = \eta(x,t,w)$, and $\zeta = \zeta(x,t,w)$

where C_1 and C_2 are arbitrary constants. Therefore, $I_1 = y/x$ and $I_2 = x^{2/(k-1)}w$ are invariants of the operator X_4.

Setting $I_2 = \Phi(I_1)$ and expressing w, we find that

$$w = x^{-2/(k-1)}\Phi(y/x), \tag{24}$$

where $\Phi(z)$ is a function to be determined in the further analysis. Substituting (24) into the original equation (17), we obtain a second-order ordinary differential equation that determines a two-parameter family of invariant solutions

$$(k-1)^2(z^2+1)\Phi''_{zz} + 2(k^2-1)z\Phi'_z + 2(k+1)\Phi - (k-1)^2\Phi^k = 0,$$

where $z = y/x$. Its general solution can be found by quadrature (in parametric form):

$$\begin{cases} z = \tan Q, \\ \Phi = \tau(\tan^2 Q + 1)^{1/(1-k)}, \end{cases} \quad \text{where} \quad Q = (k^2-1)\int \frac{d\tau}{\sqrt{2(k-1)^2\tau^{k+1} - 4(k+1)\tau^2 + A_1}} + A_2,$$

A_1 and A_2 are arbitrary constants and τ is the parameter.

2°. The functions $u = x^2+y^2$ and w are invariants of the operator X_3 of (19) for the nonlinear heat equation in question. The substitution $w = w(u)$, $u = x^2+y^2$, yields an ordinary differential equation which describes rotationally invariant solutions of the original equation,

$$uw''_{uu} + w'_u = \tfrac{1}{4}f(w).$$

Remark. In applications, one often takes the polar radius $r = \sqrt{x^2+y^2}$ to be an invariant instead of $u = x^2+y^2$.

Example 5. Consider the nonlinear heat equation (21).

1°. For an arbitrary function $f(w)$, the equation admits the operator (see Example 2)

$$\mathrm{X}_3 = 2t\partial_t + x\partial_x.$$

The invariants are found from the linear first-order partial differential equation $\mathrm{X}_3 I = 0$, which, in complete form, reads

$$2t\frac{\partial I}{\partial t} + x\frac{\partial I}{\partial x} + 0\frac{\partial I}{\partial w} = 0.$$

The corresponding characteristic system of ordinary differential equations

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{dw}{0}$$

admits the first integrals

$$xt^{-1/2} = C_1, \qquad w = C_2,$$

where C_1 and C_2 are arbitrary constants. Therefore, the functions $I_1 = xt^{-1/2}$ and $I_2 = w$ are invariants of the operator X_3.

Taking $I_2 = \Phi(I_1)$, we get

$$w = \Phi(z), \qquad z = xt^{-1/2}, \tag{25}$$

where $\Phi(z)$ is a function to be determined in the further analysis. Substituting (25) into the original equation (21), we arrive at the second-order ordinary differential equation

$$2[f(\Phi)\Phi'_z]'_z + z\Phi'_z = 0,$$

which describes an invariant (self-similar) solution.

2°. Let us examine the case $f(w) = w^k$, in which the equation admits the operator

$$X_4 = kx\partial_x + 2w\partial_w.$$

The invariants are described by the linear first-order partial differential equation $X_4 I = 0$, which, in complete form, reads

$$0\frac{\partial I}{\partial t} + kx\frac{\partial I}{\partial x} + 2w\frac{\partial I}{\partial w} = 0.$$

The corresponding characteristic system of ordinary differential equations

$$\frac{dt}{0} = \frac{dx}{kx} = \frac{dw}{2w}$$

admits the first integrals

$$t = C_1, \qquad x^{-2/k}w = C_2,$$

where C_1 and C_2 are arbitrary constants. Therefore, the functions $I_1 = t$ and $I_2 = x^{-2/k}w$ are invariants of the operator X_4.

Setting $I_2 = \Psi(I_1)$ and expressing w, we find that

$$w = x^{2/k}\Psi(t), \tag{26}$$

where $\Psi(t)$ is a function to be determined in the further analysis. Substituting (26) into the original equation (21), we arrive at the first-order ordinary differential equation

$$2k\Psi'_t = 2a(k+2)\Psi^{k+1}.$$

Integrating yields

$$\Psi(t) = \left[A - \frac{2a(k+2)}{k}t\right]^{-1/k},$$

where A is an arbitrary constant. Thus, the scaling-invariant solution of equation (21) for $f(w) = w^k$ has the form

$$w(x,t) = x^{2/k}\left[A - \frac{2a(k+2)}{k}t\right]^{-1/k}.$$

Example 6. Consider the nonlinear wave equation (22). For an arbitrary $f(w)$, this equation admits the operator (see Example 3)

$$X_3 = t\partial_t + x\partial_x.$$

The invariants are found from the linear first-order partial differential equation $X_3 I = 0$, which, in complete form, reads as follows:

$$t\frac{\partial I}{\partial t} + x\frac{\partial I}{\partial x} + 0\frac{\partial I}{\partial w} = 0.$$

The corresponding characteristic system of ordinary differential equations

$$\frac{dx}{x} = \frac{dt}{t} = \frac{dw}{0}$$

admits the first integrals

$$xt^{-1} = C_1, \qquad w = C_2,$$

where C_1 and C_2 are arbitrary constants. Therefore, the functions $I_1 = xt^{-1}$ and $I_2 = w$ are invariants of the operator X_3.

Taking $I_2 = \Phi(I_1)$, we have

$$w = \Phi(y), \qquad y = xt^{-1}. \tag{27}$$

The function $\Phi(y)$ is sought by substituting (27) into the original equation (22) to obtain the ordinary differential equation

$$[f(\Phi)\Phi'_y]'_y = (y\Phi'_y)'_y,$$

which determines an invariant (self-similar) solution. Obviously, the last equation admits the first integral $f(\Phi)\Phi'_y = y\Phi'_y + C$.

⊙ *References for Subsection* S.7.1: L. V. Ovsiannikov (1962, 1982), G. W. Bluman and J. D. Cole (1974), J. M. Hill (1982, 1992), N. H. Ibragimov (1985, 1994), P. J. Olver (1986, 1995), D. H. Sattinger and O. L. Weaver (1986), G. W. Bluman and S. Kumei (1989), H. Stephani (1989), W. I. Fushchich, V. M. Stelen, and N. I. Serov (1993), G. Gaeta (1994), A. M. Vinogradov and I. S. Krasilshchik (1997), G. Baumann (2000), P. A. Clarkson (2000), P. E. Hydon (2000), P. P. Kiryakov, S. I. Senashov, and A. N. Yakhno (2001), B. J. Cantwell (2002), D. M. Klimov and V. F. Zhuravlev (2002).

S.7.2. Nonclassical Method for Symmetry Reductions

S.7.2-1. Description of the method. Invariant surface condition.

Consider a second-order equation in two independent variables of the form

$$\frac{\partial^2 w}{\partial y^2} = H\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}\right). \tag{28}$$

The results of the classical group analysis (see Subsection S.7.1) can be substantially extended if, instead of finding invariants of an admissible infinitesimal operator X by means of solving the characteristic system of equations

$$\frac{dx}{\xi(x, y, w)} = \frac{dy}{\eta(x, y, w)} = \frac{dw}{\zeta(x, y, w)},$$

one imposes the corresponding invariant surface condition (Bluman and Cole, 1969)

$$\xi(x, y, w)\frac{\partial w}{\partial x} + \eta(x, y, w)\frac{\partial w}{\partial y} = \zeta(x, y, w). \tag{29}$$

Equation (28) and condition (29) are supplemented by the invariance condition

$$\underset{2}{\mathrm{X}}\left[\frac{\partial^2 w}{\partial y^2} - H\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}\right)\right] = 0, \tag{30}$$

which coincides with equation (15) from Subsection S.7.1.

All three equations (28)–(30) are used for the construction of exact solutions of the original equation (28). It should be observed that in this case, the determining equations obtained for the unknown functions $\xi(x, y, w)$, $\eta(x, y, w)$, and $\zeta(x, y, w)$ by the splitting procedure are nonlinear. The symmetries determined by the invariant surface (29) are called nonclassical.

Figure 4 is intended to clarify the general scheme for constructing of exact solutions of a second-order evolution equation by the nonclassical method on the basis of the invariant surface condition (29).

S.7.2-2. Examples: the Fitzhugh–Nagumo equation and a nonlinear wave equation.

Example 1. Consider the Fitzhugh–Nagumo equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + w(1 - w)(w - a). \tag{31}$$

Without loss of generality, we set $\eta = 1$ in the invariant surface condition (29) with $y = t$, thus assuming that $\eta \neq 0$. We have

$$\frac{\partial w}{\partial t} + \xi(x, t, w)\frac{\partial w}{\partial x} = \zeta(x, t, w). \tag{32}$$

The invariance condition is obtained by a procedure similar to the classical algorithm (see Subsection S.7.1). Namely, we apply the operator $\underset{2}{\mathrm{X}} = \xi\partial_x + \eta\partial_t + \zeta\partial_w + \zeta_1\partial_{w_x} + \zeta_2\partial_{w_t} + \zeta_{11}\partial_{w_{xx}}$ to equation (31) and take into account the expressions (13) and (14) for the coordinates of the first and the second prolongations ζ_1 and ζ_{11} for $y = t$. Next, we substitute w_{xx} from (31) and then w_t from (32) into the invariance condition. Consequently, there remains only one "independent" variable, w_x. Splitting with respect to powers of this variable yields the following determining system of only four equations:

$$\begin{aligned}
&w_x^3: &&\xi_{ww} = 0,\\
&w_x^2: &&\zeta_{ww} - 2(\xi_{wx} - \xi\xi_w) = 0,\\
&w_x: &&2\zeta_{wx} - 2\xi_w\zeta - 3w(w - a)(w - 1)\xi_w - \xi_{xx} + 2\xi\xi_x + \xi_t = 0,\\
&1: &&\zeta_t - \zeta_{xx} + 2\xi_x\zeta + (2\xi_x - \zeta_w)w(w - a)(w - 1) + [3w^2 - 2(a + 1)w + a]\zeta = 0.
\end{aligned}$$

It can be seen that the employment of the invariant surface condition (29) substantially increases our freedom in choosing the coordinates ξ, η, ζ.

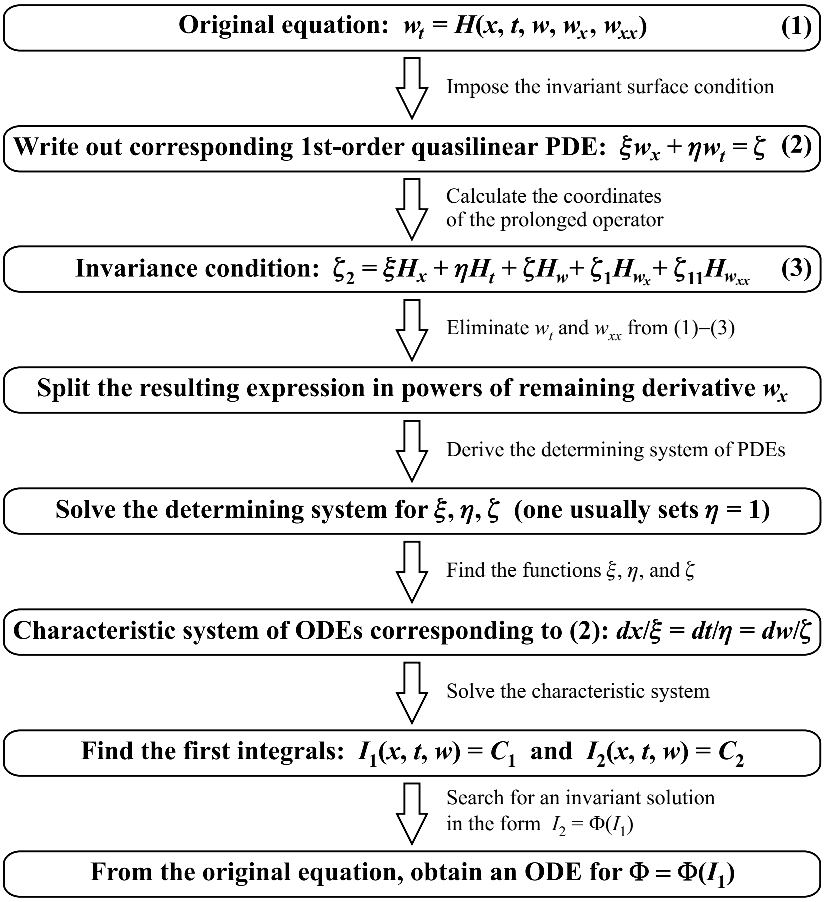

Figure 4. Algorithm for the construction of exact solutions by a nonclassical method for second-order evolution equations. Here, ODE stands for ordinary differential equation and PDE for partial differential equation.

1°. Let $a = -1$. In this case, equation (31) reduces to the Newell–Whitehead equation

$$w_t = w_{xx} + w - w^3.$$

Computing the coordinates yields

$$\xi = \alpha(x,t), \quad \eta = 1, \quad \zeta = -\alpha_x w,$$

where the function $\alpha(x,t)$ satisfies the system

$$\alpha_t - 3\alpha_{xx} + 2\alpha\alpha_x = 0, \quad \alpha_{xt} - \alpha_{xxx} + 2\alpha_x^2 + 2\alpha_x = 0, \tag{33}$$

and the associated invariant surface condition is

$$w_t + \alpha w_x + \alpha_x w = 0. \tag{34}$$

The transformation $\alpha = -3(\ln\varphi)_x$ reduces the equations of (33) into the linear equations

$$\varphi_t = 3\varphi_{xx}, \quad \varphi_{xt} = \varphi_{xxx} + \varphi_x,$$

respectively. The solution that satisfies the two equations simultaneously is expressed as

$$\alpha(x,t) = -\frac{3}{\sqrt{2}}\,\frac{C_1\exp\bigl[\frac{1}{2}(\sqrt{2}\,x + 3t)\bigr] - C_2\exp\bigl[\frac{1}{2}(-\sqrt{2}\,x + 3t)\bigr]}{C_1\exp\bigl[\frac{1}{2}(\sqrt{2}\,x + 3t)\bigr] + C_2\exp\bigl[\frac{1}{2}(-\sqrt{2}\,x + 3t)\bigr] + C_3}.$$

Finally, the invariant surface condition (34) gives the exact solution

$$w(x,t) = \bigl\{C_1\exp\bigl[\tfrac{1}{2}(\sqrt{2}\,x + 3t)\bigr] - C_2\exp\bigl[\tfrac{1}{2}(-\sqrt{2}\,x + 3t)\bigr]\bigr\}\,h\bigl(z;\tfrac{\sqrt{2}}{2}\bigr), \tag{35}$$

where

$$z = C_1\exp\bigl[\tfrac{1}{2}(\sqrt{2}\,x + 3t)\bigr] + C_2\exp\bigl[\tfrac{1}{2}(-\sqrt{2}\,x + 3t)\bigr] + C_3,$$

the function $h(z;k)$ is the Jacobi elliptic function satisfying the ordinary differential equation

$$(h'_z)^2 = h^4 + (2k^2 - 1)h^2 + k^2(k^2 - 1), \tag{36}$$

and C_1, C_2, and C_3 are arbitrary constants.

2°. Let $a = 1/2$. Calculating the coordinates yields

$$\xi = \alpha(x,t), \quad \eta = 1, \quad \zeta = -\alpha_x\left(w - \tfrac{1}{2}\right),$$

where the function $\alpha(x,t)$ satisfies the system

$$\alpha_t - 3\alpha_{xx} + 2\alpha\alpha_x = 0, \quad 2\alpha_{xt} - 2\alpha_{xxx} + 4\alpha_x^2 + \alpha_x.$$

In exactly the same manner, we arrive at the exact solution

$$w(x,t) = \tfrac{1}{2}\left\{C_1 \exp\left[\tfrac{1}{8}(2\sqrt{2}\,x + 3t)\right] - C_2 \exp\left[\tfrac{1}{8}(-2\sqrt{2}\,x + 3t)\right]\right\} h\left(z; \tfrac{\sqrt{2}}{2}\right),$$

where

$$z = C_1 \exp\left[\tfrac{1}{8}(2\sqrt{2}\,x + 3t)\right] + C_2 \exp\left[\tfrac{1}{8}(-2\sqrt{2}\,x + 3t)\right] + C_3,$$

the function $h(z;k)$ is the Jacobi elliptic function satisfying the ordinary differential equation (36); C_1, C_2, and C_3 are arbitrary constants.

3°. Let $a = 2$. Calculating the coordinates yields

$$\xi = \alpha(x,t), \quad \eta = 1, \quad \zeta = -\alpha_x\left(w - \tfrac{1}{2}\right),$$

where the function $\alpha(x,t)$ satisfies system (33). In this case, we obtain solution (35).

4°. Let a be an arbitrary constant. Calculating the coordinates yields

$$\xi = \tfrac{\sqrt{2}}{2}(3w - a - 1), \quad \eta = 1, \quad \zeta = -\tfrac{3}{2}w(w-a)(w-1).$$

The associated invariant surface condition becomes

$$w_t + \tfrac{\sqrt{2}}{2}(3w - a - 1)w_x + \tfrac{3}{2}w(w-a)(w-1) = 0. \tag{37}$$

Eliminating w_t from (31) and (37), we obtain the equation

$$w_{xx} = \tfrac{\sqrt{2}}{2}(a + 1 - 3w)w_x - \tfrac{1}{2}w(w-a)(w-1), \tag{38}$$

which, by the substitution $w = \sqrt{2}\,(\ln\varphi)_x$, is reduced to the linear equation

$$2\varphi_{xxx} - \sqrt{2}\,(1+a)\varphi_{xx} + a\varphi_x = 0.$$

Solving this equation, we arrive at a solution of equation (38) in the form

$$w(x,t) = \frac{a\psi_1(t)\exp\left(\frac{\sqrt{2}}{2}ax\right) + \psi_2(t)\exp\left(\frac{\sqrt{2}}{2}x\right)}{\psi_1(t)\exp\left(\frac{\sqrt{2}}{2}ax\right) + \psi_2(t)\exp\left(\frac{\sqrt{2}}{2}x\right) + \psi_3(t)},$$

where the functions $\psi_i(t)$, $i = 1, 2, 3$, are found by the substitution of the expression of $w(x,t)$ into (37). Finally, we obtain a solution of equation (31),

$$w(x,t) = \frac{aC_1 \exp\left[\frac{1}{2}(\sqrt{2}\,ax + a^2t)\right] + C_2(t)\exp\left[\frac{1}{2}(\sqrt{2}\,x + t)\right]}{C_1 \exp\left[\frac{1}{2}(\sqrt{2}\,ax + a^2t)\right] + C_2(t)\exp\left[\frac{1}{2}(\sqrt{2}\,x + t)\right] + C_3\exp(at)},$$

where C_1, C_2, and C_3 are arbitrary constants.

5°. Let a be an arbitrary constant. Another set of coordinates is possible (it differs from that of Item 4° by the sign of ξ), namely,

$$\xi = -\tfrac{\sqrt{2}}{2}(3w - a - 1), \quad \eta = 1, \quad \zeta = -\tfrac{3}{2}w(w-a)(w-1),$$

and the associated invariant surface condition is

$$w_t - \tfrac{\sqrt{2}}{2}(3w - a - 1)w_x + \tfrac{3}{2}w(w-a)(w-1) = 0.$$

A similar procedure yields a solution of equation (31),

$$w(x,t) = \frac{aC_1 \exp\left[\frac{1}{2}(\sqrt{2}\,ax + a^2t)\right] + C_2(t)\exp\left[\frac{1}{2}(\sqrt{2}\,x + t)\right]}{C_1 \exp\left[\frac{1}{2}(\sqrt{2}\,ax + a^2t)\right] + C_2(t)\exp\left[\frac{1}{2}(\sqrt{2}\,x + t)\right] + C_3\exp\left[\frac{1}{2}(\sqrt{2}\,(a+1)x + at)\right]},$$

where C_1, C_2, and C_3 are arbitrary constants.

6°. Let a be an arbitrary constant and $\eta \equiv 0$. Calculating the coordinates yields

$$\xi = 1, \quad \eta = 0, \quad \zeta = \zeta(x,t,w),$$

where ζ satisfies the equation

$$2\zeta\zeta_{xw} + \zeta^2\zeta_{ww} + \zeta_{xx} + w(w-a)(w-1)\zeta_w - \zeta_t - [3w^2 - 2(a+1)w + a]\zeta = 0, \tag{39}$$

and the associated invariant surface condition is

$$w_x = \zeta. \tag{40}$$

Eliminating w_x and w_{xx} from (31) and (40), we obtain

$$w_t = \zeta\zeta_w + \zeta_x - w(w-a)(w-1). \tag{41}$$

Whenever a solution of equation (39) is known, we can integrate equation (40) to obtain exact solutions of the original equation (31).

Example 2. Consider the nonlinear wave equation

$$\frac{\partial^2 w}{\partial t^2} = w\frac{\partial^2 w}{\partial x^2}. \tag{42}$$

Let us supplement this equation with the invariant surface condition (32) with $y=t$. The invariance condition can be obtained from (15) by taking into account formulas for the coordinates of the prolonged operator (13)–(14) and the relations $y=t$, $\eta=1$, and $H=ww_{xx}$. Next, we insert w_{tt} of (42) and then w_t of (32) into the invariance condition. As a result, there remain two "independent" variables: w_x and w_{xx}. The splitting in powers of these variables yields the following determining system:

$$\begin{array}{ll}
w_x w_{xx}: & (\xi^2 - w)\xi_w = 0,\\
w_{xx}: & 2\xi\xi_t + 2w\xi_x + 2\xi\xi_w\zeta - \zeta = 0,\\
w_x^3: & (\xi^2 - w)\xi_{ww} = 0,\\
w_x^2: & (\xi^2 - w)\zeta_{ww} + 2w\xi_{wx} + 2\xi\xi_{wt} + 2\xi\xi_{ww}\zeta = 0,\\
w_x: & w\xi_{xx} - 2w\zeta_{wx} - 2\xi_{wt}\zeta - \xi_{ww}\zeta - 2\xi\zeta_{wt} - 2\xi\zeta\zeta_{ww} - \xi_{tt} = 0,\\
1: & \zeta_{tt} - w\zeta_{xx} + 2\zeta\zeta_{wt} + \zeta^2\zeta_{ww}.
\end{array}$$

From the first equation it follows that either (i) $\xi=\xi(x,t)$ or (ii) $\xi=\sqrt{w}$. Case (ii) corresponds to $\zeta=0$ and is not considered in what follows. In case (i), it turns out that the third and the fourth equations are satisfied and the second equation implies that $\zeta = 2w\xi_x + 2\xi\xi_t$. The further substitution of the obtained functions into the fifth and the sixth equations, after the splitting in powers of w, yields

$$\xi = \alpha t + \beta, \quad \zeta = 2\alpha(\alpha t + \beta),$$

where α and β are arbitrary constants. To be specific, we take $\alpha=2$ and $\beta=0$ and write out the characteristic system of ordinary differential equations:

$$\frac{dt}{1} = \frac{dx}{2t} = \frac{dw}{8t}.$$

Consequently, first integrals are: $C_1 = x - t^2$ and $C_2 = w - 4t^2$. According to the scheme represented in Figure 4, we seek an invariant solution in the form $w - 4t^2 = \Phi(x - t^2)$. Inserting

$$w = \Phi(z) + 4t^2, \quad z = x - t^2, \tag{43}$$

into (42), we obtain an autonomous ordinary differential equation for $\Phi = \Phi(z)$:

$$\Phi\Phi''_{zz} + 2\Phi'_z = 8.$$

This equation is easy to integrate, since its order can be reduced, upon which it turns into a separable equation. As a result, we can find an exact solution of equation (42) of the form (43).

⊙ *References for Subsection* S.7.2: G. W. Bluman and J. D. Cole (1969), P. J. Olver and Ph. Rosenau (1987), D. Levi and P. Winternitz (1989), M. C. Nucci and P. A. Clarkson (1992), D. Arrigo, P. Broadbridge, and J. M. Hill (1993), P. A. Clarkson, D. K. Ludlow, and T. J. Priestley (1997).

S.8. Differential Constraints Method

S.8.1. Description of the Method

S.8.1-1. Preliminary remarks. A simple example.

In Subsections S.4.1 and S.4.3, we have considered examples of additive separable solutions of nonlinear equations in the form

$$w(x,y) = \varphi(x) + \psi(y). \tag{1}$$

At the initial stage, the functions $\varphi(x)$ and $\psi(y)$ are assumed arbitrary and are to be determined in the subsequent analysis.

Differentiating the expression (1) with respect to y, we obtain

$$\frac{\partial w}{\partial y} = f(y) \qquad (f = \psi'_y). \tag{2}$$

Conversely, relation (2) implies a representation of the solution in the form (1).

Further, differentiating (2) in x, we get

$$\frac{\partial^2 w}{\partial x \partial y} = 0. \tag{3}$$

Conversely, from (3) we obtain a representation of the solution in the form (1).

Thus, the problem of finding exact solutions of the form (1) for a specific partial differential equation may be replaced by an equivalent problem of finding exact solutions of the given equation supplemented with the condition (2) or (3). Such supplementary conditions in the form of one or several differential equations will be called *differential constraints*.

Prior to giving a general description of the differential constraints method, we demonstrate its features by a simple example.

Example 1. Consider the third-order nonlinear equation

$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} + a\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = b\frac{\partial^3 w}{\partial y^3}, \tag{4}$$

which, for $a = -1$, occurs in the theory of the hydrodynamic boundary layer (see Subsection 9.3.1). Let us seek a solution of equation (4) satisfying the linear first-order differential constraint

$$\frac{\partial w}{\partial x} = \varphi(y). \tag{5}$$

Here, the function $\varphi(y)$ cannot be arbitrary, in general, but must satisfy the condition of compatibility of equations (4) and (5). The compatibility condition is a differential equation for $\varphi(y)$ and is a consequence of equations (4), (5) and those obtained by their differentiation.

Successively differentiating (5) with respect to different variables, we calculate the derivatives

$$w_{xx} = 0, \quad w_{xy} = \varphi'_y, \quad w_{xxy} = 0, \quad w_{xyy} = \varphi''_{yy}, \quad w_{xyyy} = \varphi'''_{yyy}. \tag{6}$$

Differentiating (4) with respect to x yields

$$w_{xy}^2 + w_y w_{xxy} + a w_{xx} w_{yy} + a w_x w_{xyy} = b w_{xyyy}. \tag{7}$$

Substituting the derivatives of the function w from (5) and (6) into (7), we obtain the following third-order ordinary differential equation for φ:

$$(\varphi'_y)^2 + a\varphi\varphi''_{yy} = b\varphi'''_{yyy}, \tag{8}$$

which represents the compatibility condition for equations (4) and (5).

In order to construct an exact solution, we integrate equation (5) to obtain

$$w = \varphi(y)x + \psi(y). \tag{9}$$

The function $\psi(y)$ is found by substituting (9) into (4) and taking into account the condition (9). As a result, we arrive at the ordinary differential equation

$$\varphi'_y\psi'_y + a\varphi\psi''_{yy} = b\psi'''_{yyy}. \tag{10}$$

Finally, we obtain an exact solution of the form (9), with the functions φ and ψ described by equations (8) and (10).

Remark 1. It is easier to obtain the above solution by directly substituting expression (9) into the original equation (4).

Remark 2. The above results can be extended to a more general case of equation (4) containing arbitrary functions $a = a(y)$ and $b = b(y)$.

S.8.1-2. General description of the differential constraints method.

The procedure of the construction of exact solutions to nonlinear equations of mathematical physics by the differential constraints method consists of several steps described below.

1°. In the general case, the identification of particular solutions of the equation

$$F\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}, \dots\right) = 0 \tag{11}$$

is performed by supplementing this equation with an additional differential constraint

$$G\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}, \dots\right) = 0. \tag{12}$$

The form of the differential constraint (12) may be prescribed on the basis of: (i) a priori considerations (for instance, it may be required that the constraint should represent a solvable equation); (ii) certain properties of the equation under consideration (for instance, it may be required that the constraint should follow from symmetries of the equation or the corresponding conservation laws).

2°. In general, the thus obtained overdetermined system (11)–(12) requires a compatibility analysis. If the differential constraint (12) is specified on the basis of a priori considerations, it should allow for sufficient freedom in choosing functions (i.e., involve arbitrary determining functions). The compatibility analysis of system (11)–(12) should provide conditions that specify the structure of the determining functions. These conditions (*compatibility conditions*) are written as a system of ordinary differential equations (or a system of partial differential equations).

In simplest cases,* the compatibility analysis is performed by means of differentiating (possibly, several times) equations (11) and (12) with respect to x and y and eliminating the highest-order derivatives from the resulting differential relations and equations (11)–(12) (see Examples 1 and 3). As a result, one arrives at an equation involving powers of lower-order derivatives. Equating the coefficients of all powers of the derivatives to zero, one obtains compatibility conditions connecting the functional coefficients of equations (11) and (12).

3°. One solves the system of differential equations obtained in Item 2° for the determining functions. Then these functions are substituted into the differential constraint (12) to obtain an equation of the form

$$g\left(x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}, \dots\right) = 0. \tag{13}$$

A differential constraint (13) that is consistent with equation (11) under consideration is called an *invariant manifold* of equation (11).

4°. One should find the general solution of: (i) equation (13) or (ii) some equation that follows from equations (11) and (13). The solution thus obtained will involve some arbitrary functions $\{\varphi_m\}$ (these may depend on x and y, as well as w). Note that in some cases, one can use, instead of the general solution, some particular solutions of equation (13) or equations that follow from (13).

5°. The solution obtained in Item 4° should be substituted into the original equation (11). As a result, one arrives at a functional-differential equation from which the functions $\{\varphi_m\}$ should be found. Having found the $\{\varphi_m\}$, one should insert these functions into the solution from Item 4°. Thus, one obtains an exact solution of the original equation (11).

Remark 1. Should the choice of a differential constraint be inadequate, equations (11) and (12) may happen to be incompatible (having no common solutions).

Remark 2. There may be several differential constraints of the form (12).

Remark 3. At the last three steps of the differential constraints method, one has to solve various equations (systems of equations). If no solution can be constructed at one of those steps, one fails to construct an exact solution of the original equation.

For the sake of clarity, the general scheme of the differential constraints method is represented in Figure 5.

⊙ *References for Subsection* S.8.1: N. N. Yanenko (1964), A. F. Sidorov, V. P. Shapeev, and N. N. Yanenko (1984), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

S.8.2. First Order Differential Constraints

S.8.2-1. Second-order evolution equations.

Consider a general second-order evolution equation solved for the highest-order derivative:

$$\frac{\partial^2 w}{\partial x^2} = \mathcal{F}\left(x, t, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}\right). \tag{14}$$

* In the general case, for the investigation of overdetermined systems one should utilize methods based on: (i) the Cartan algorithm or (ii) the Janet–Spenser–Kuranishi algorithm. A description of these algorithms and other relevant information regarding the theory of overdetermined systems can be found, for instance, in the works of M. Kuranishi (1967), J. F. Pommaret (1978), A. F. Sidorov, V. P. Shapeev, and N. N. Yanenko (1984).

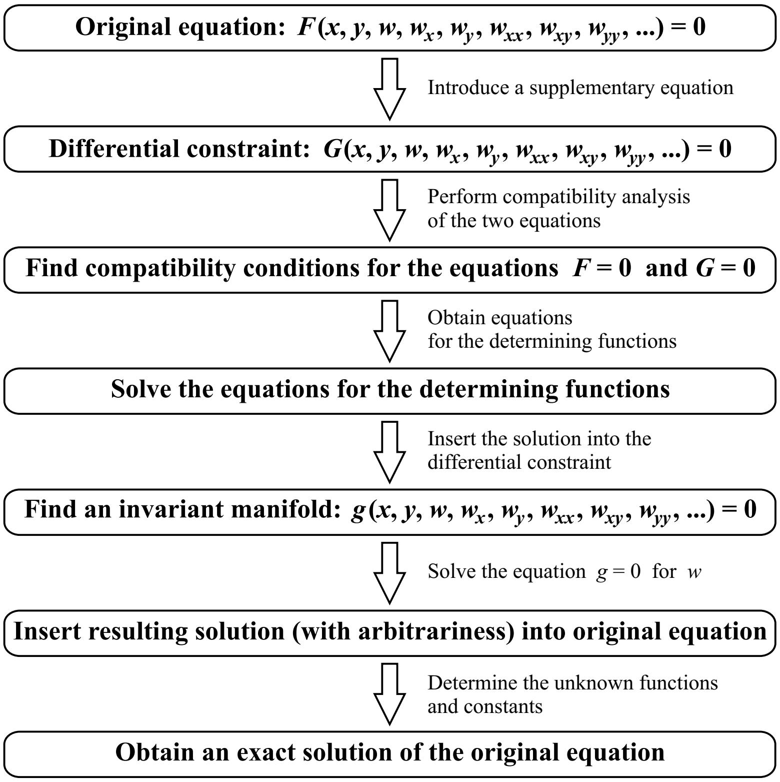

Figure 5. Algorithm for the construction of exact solutions by the differential constraints method

Let us supplement this equation with a first-order differential constraint

$$\frac{\partial w}{\partial t} = \mathcal{G}\left(x, t, w, \frac{\partial w}{\partial x}\right). \tag{15}$$

The condition of compatibility of these equations is obtained by differentiating (14) with respect to t once and differentiating (15) with respect to x twice, and then equating the two resulting expressions for the third derivatives w_{xxt}:

$$\mathrm{D}_t \mathcal{F} = \mathrm{D}_x^2 \mathcal{G}. \tag{16}$$

Here, D_t and D_x are the total differentiation operators with respect to t and x:

$$\mathrm{D}_t = \frac{\partial}{\partial t} + w_t \frac{\partial}{\partial w} + w_{xt} \frac{\partial}{\partial w_x} + w_{tt} \frac{\partial}{\partial w_t}, \qquad \mathrm{D}_x = \frac{\partial}{\partial x} + w_x \frac{\partial}{\partial w} + w_{xx} \frac{\partial}{\partial w_x} + w_{xt} \frac{\partial}{\partial w_t}. \tag{17}$$

The partial derivatives w_t, w_{xx}, w_{xt}, and w_{tt} in (17) should be expressed in terms of x, t, w, and w_x by means of the relations (14), (15) and those obtained by differentiation of (14), (15). As a result, we get

$$\begin{aligned} &w_t = \mathcal{G}, \quad w_{xx} = \mathcal{F}, \quad w_{xt} = \mathrm{D}_x \mathcal{G} = \frac{\partial \mathcal{G}}{\partial x} + w_x \frac{\partial \mathcal{G}}{\partial w} + \mathcal{F} \frac{\partial \mathcal{G}}{\partial w_x}, \\ &w_{tt} = \mathrm{D}_t \mathcal{G} = \frac{\partial \mathcal{G}}{\partial t} + \mathcal{G} \frac{\partial \mathcal{G}}{\partial w} + w_{xt} \frac{\partial \mathcal{G}}{\partial w_x} = \frac{\partial \mathcal{G}}{\partial t} + \mathcal{G} \frac{\partial \mathcal{G}}{\partial w} + \left(\frac{\partial \mathcal{G}}{\partial x} + w_x \frac{\partial \mathcal{G}}{\partial w} + \mathcal{F} \frac{\partial \mathcal{G}}{\partial w_x}\right) \frac{\partial \mathcal{G}}{\partial w_x}. \end{aligned} \tag{18}$$

In the expression for $\mathcal{F}$, the derivative w_t should be replaced by $\mathcal{G}$ by virtue of (15).

Example 2. From the class of nonlinear heat equations with a source

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w) \frac{\partial w}{\partial x}\right] + g(w), \tag{19}$$

let us single out equations possessing invariant manifolds of the simplest form

$$\frac{\partial w}{\partial t} = \varphi(w). \tag{20}$$

Equations (19) and (20) are special cases of (14) and (15) with

$$\mathcal{F} = \frac{w_t - f'(w)w_x^2 - g(w)}{f(w)} = \frac{\varphi(w) - g(w) - f'(w)w_x^2}{f(w)}, \qquad \mathcal{G} = \varphi(w).$$

The functions $f(w)$, $g(w)$, and $\varphi(w)$ are unknown in advance and are to be determined in the subsequent analysis.

Using (18) and (17), we find partial derivatives and the total differentiation operators:

$$\begin{aligned} &w_t = \varphi, \quad w_{xx} = \mathcal{F}, \quad w_{xt} = \varphi' w_x, \quad w_{tt} = \varphi\varphi', \\ &\mathrm{D}_t = \frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial w} + \varphi' w_x \frac{\partial}{\partial w_x} + \varphi\varphi' \frac{\partial}{\partial w_t}, \quad \mathrm{D}_x = \frac{\partial}{\partial x} + w_x\frac{\partial}{\partial w} + \mathcal{F}\frac{\partial}{\partial w_x} + \varphi' w_x \frac{\partial}{\partial w_t}. \end{aligned}$$

We insert the expressions of D_x and D_t into the compatibility conditions (16) and rearrange terms to obtain

$$f\left[\frac{(f\varphi)'}{f}\right]' w_x^2 + \frac{\varphi - g}{f}\varphi' - \varphi\left(\frac{\varphi - g}{f}\right)' = 0.$$

In order to ensure that this equality holds true for any w_x, one should take

$$\left[\frac{(f\varphi)'}{f}\right]' = 0, \qquad \frac{\varphi - g}{f}\varphi' - \varphi\left(\frac{\varphi - g}{f}\right)' = 0. \tag{21}$$

Nondegenerate case. Assuming that the function $f = f(w)$ is given, we obtain a three-parameter solution of equations (21) for the functions $g(w)$ and $\varphi(w)$:

$$g(w) = \frac{a + cf}{f}\left(\int f\,dw + b\right), \qquad \varphi(w) = \frac{a}{f}\left(\int f\,dw + b\right), \tag{22}$$

where a, b, and c are arbitrary constants.

We substitute $\varphi(w)$ of (22) into equation (20) and integrate to obtain

$$\int f\,dw = \theta(x)e^{at} - b. \tag{23}$$

Differentiating (23) with respect to x and t, we get $w_t = ae^{at}\theta/f$ and $w_x = e^{at}\theta'_x/f$. Substituting these expressions into (19) and taking into account (22), we arrive at the equation $\theta''_{xx} + c\theta = 0$, whose general solution is given by

$$\theta = \begin{cases} C_1 \sin\bigl(x\sqrt{c}\bigr) + C_2 \cos\bigl(x\sqrt{c}\bigr) & \text{if } c > 0, \\ C_1 \sinh\bigl(x\sqrt{-c}\bigr) + C_2 \cosh\bigl(x\sqrt{-c}\bigr) & \text{if } c < 0, \\ C_1 x + C_2 & \text{if } c = 0, \end{cases} \tag{24}$$

where C_1 and C_2 are arbitrary constants. Formulas (23)–(24) describe exact solutions (in implicit form) of equation (19) with $f(w)$ arbitrary and $g(w)$ given by (22).

Degenerate case. There also exists a two-parameter solution of equations (21) for the functions $g(w)$ and $\varphi(w)$ (as above, f is assumed arbitrary):

$$g(w) = \frac{b}{f} + c, \qquad \varphi(w) = \frac{b}{f},$$

where b and c are arbitrary constants. This solution can be obtained from (22) by renaming variables, $b \to b/a$ and $c \to ac/b$, and letting $a \to 0$. After simple calculations, we obtain the corresponding solution of equation (19) in implicit form:

$$\int f\,dw = bt - \frac{1}{2}cx^2 + C_1 x + C_2.$$

⊙ *Reference*: V. A. Galaktionov (1994).

The example given below shows that calculations may be performed without the use of the general formulas (16)–(18).

Example 3. Consider the problem of finding second-order nonlinear equations

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f_1(w)\frac{\partial w}{\partial x} + f_0(w) \tag{25}$$

admitting first-order invariant manifolds of the form

$$\frac{\partial w}{\partial t} = g_1(w)\frac{\partial w}{\partial x} + g_0(w). \tag{26}$$

Equations (25) and (26) are special cases of (14) and (15) for $\mathcal{F} = w_t - f_1(w)w_x - f_0(w)$ and $\mathcal{G} = g_1(w)w_x + g_0(w)$. The functions $f_1(w)$, $f_0(w)$, $g_1(w)$, and $g_0(w)$ are unknown in advance and are to be determined in the subsequent analysis.

First, we calculate derivatives. Equating the right-hand sides of (25) and (26), we get

$$w_{xx} = h_1 w_x + h_0, \quad \text{where} \quad h_1 = g_1 - f_1, \quad h_0 = g_0 - f_0. \tag{27}$$

Here and in what follows, the argument of the functions f_1, f_0, g_1, g_0, h_1, and h_0 is omitted. Differentiating (26) with respect to x twice and using the expression (27) for w_{xx}, we find the mixed derivatives

$$\begin{aligned} w_{xt} &= g_1 w_{xx} + g_1' w_x^2 + g_0' w_x = g_1' w_x^2 + (g_1 h_1 + g_0')w_x + g_1 h_0, \\ w_{xxt} &= g_1'' w_x^3 + (g_1 h_1' + 3g_1' h_1 + g_0'')w_x^2 + (g_1 h_0' + 3g_1' h_0 + g_1 h_1^2 + g_0' h_1)w_x + (g_1 h_1 + g_0')h_0, \end{aligned} \tag{28}$$

where the prime denotes a derivative with respect to w. Differentiating (27) with respect to t and using the expressions (26) and (28) for w_t and w_{xt}, we obtain

$$w_{xxt} = h_1 w_{xt} + h_1' w_x w_t + h_0' w_t = (g_1 h_1' + g_1' h_1)w_x^2 + (g_1 h_1^2 + g_0' h_1 + g_0 h_1' + g_1 h_0')w_x + g_1 h_0 h_1 + g_0 h_0'. \tag{29}$$

We equate the expressions for the third derivative w_{xxt} from (28) and (29) and collect terms with the same power of w_x to obtain an invariance condition in the form

$$g_1'' w_x^3 + (2g_1' h_1 + g_0'')w_x^2 + (3g_1' h_0 - g_0 h_1')w_x + g_0' h_0 - g_0 h_0' = 0. \tag{30}$$

For condition (30) to hold we require that the coefficients of like powers of w_x be zero:

$$g_1'' = 0, \quad 2g_1' h_1 + g_0'' = 0, \quad 3g_1' h_0 - g_0 h_1' = 0, \quad g_0' h_0 - g_0 h_0' = 0.$$

The general solution of this system of ordinary differential equations is given by the following formulas:

$$g_1 = C_1 w + C_2, \quad g_0 = -C_1^2 C_3 w^3 - C_1 C_4 w^2 + C_5 w + C_6, \quad h_1 = 3C_1 C_3 w + C_4, \quad h_0 = C_3 g_0, \tag{31}$$

where $C_1, \ldots, C_6$ are arbitrary constants. Using formulas (27) for h_0 and h_1 together with (31), we find the unknown functions involved in equations (25) and (26):

$$\begin{aligned} &f_1(w) = C_1(1 - 3C_3)w + C_2 - C_4, \quad f_0(w) = (-C_1^2 C_3 w^3 - C_1 C_4 w^2 + C_5 w + C_6)(1 - C_3), \\ &g_1(w) = C_1 w + C_2, \quad g_0(w) = -C_1^2 C_3 w^3 - C_1 C_4 w^2 + C_5 w + C_6. \end{aligned} \tag{32}$$

Let us dwell on the special case of

$$C_1 = -k, \quad C_2 = C_4 = 0, \quad C_3 = -1/k, \quad C_5 = ak, \quad C_6 = bk$$

in (32), where a, b, and k are arbitrary constants ($k \neq 0$). The corresponding equation (25) and the invariant manifold (26) have the form

$$w_t = w_{xx} - (k + 3)ww_x + (k + 1)(w^3 + aw + b), \tag{33}$$

$$w_t = -kww_x + k(w^3 + aw + b). \tag{34}$$

The general solution of the first-order quasilinear equation (34) can be written out in implicit form; it involves the integral $I(w) = \int w(w^3 + aw + b)^{-1}\,dw$ and its inversion. Due to its complex structure, this solution is inconvenient for the construction of exact solutions of equation (33).

In this situation, instead of (34) one can use equations obtained from (33) and (34) by eliminating the derivative w_t:

$$w_{xx} = 3ww_x - w^3 - aw - b. \tag{35}$$

This ordinary differential equation coincides with (27), where h_1 and h_0 are expressed by (31). The substitution $w = -U_x/U$ transforms (35) into a third-order linear equation with constant coefficients,

$$U_{xxx} + aU_x - b = 0, \tag{36}$$

whose solutions are determined by the roots of the cubic equation $\lambda^3 + a\lambda - b = 0$. In particular, if all its roots λ_n are real, then the general solutions of equations (35) and (36) are given by

$$w = -U_x/U, \quad U = r_1(t)\exp(\lambda_1 x) + r_2(t)\exp(\lambda_2 x) + r_3(t)\exp(\lambda_3 x). \tag{37}$$

The functions $r_n(t)$ are found by substituting (37) into equation (33) or (34).

Note that equation (33) was studied in more detail by another method in Subsection S.6.3 (see Example 7 with $a = 1$ and $b_2 = 0$).

Remark 1. In the general case, for a given function $\mathcal{F}$, the compatibility condition (16) is a nonlinear partial differential equation for the function $\mathcal{G}$. This equation admits infinitely many solutions (by the theorem about the local existence of solutions). Therefore, the second-order partial differential equation (14) admits infinitely many compatible first-order differential constraints (15).

Remark 2. In the general case, the solution of the first-order partial differential equation (15) reduces to the solution of a system of ordinary differential equations; see Kamke (1965) and Polyanin, Zaitsev, and Moussiaux (2002).

S.8.2-2. Second-order hyperbolic equations.

In a similar way, one can consider second-order hyperbolic equations of the form

$$\frac{\partial^2 w}{\partial x \partial t} = \mathcal{F}\left(x, t, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}\right), \tag{38}$$

supplemented by a first-order differential constraint (15). Assume that $\mathcal{G}_{w_x} \neq 0$.

The compatibility condition for these equations is obtained by differentiating (38) with respect to t and (15) with respect to t and x, and then equating the resulting expressions of the third derivative w_{xtt} to one another:

$$\mathrm{D}_t \mathcal{F} = \mathrm{D}_x[\mathrm{D}_t \mathcal{G}]. \tag{39}$$

Here, D_t and D_x are the total differential operators of (17) in which the partial derivatives w_t, w_{xx}, w_{xt}, and w_{tt} must be expressed in terms of x, t, w, and w_x with the help of relations (38) and (15) and those obtained by differentiating (38) and (15).

Let us show how the second derivatives can be calculated. We differentiate (15) with respect to x and replace the mixed derivative by the right-hand side of (38) to obtain the following expression for the second derivative with respect to x:

$$\frac{\partial \mathcal{G}}{\partial x} + w_x \frac{\partial \mathcal{G}}{\partial w} + w_{xx} \frac{\partial \mathcal{G}}{\partial w_x} = \mathcal{F}\left(x, t, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}\right) \quad \Longrightarrow \quad \frac{\partial^2 w}{\partial x^2} = \mathcal{H}_1\left(x, t, w, \frac{\partial w}{\partial x}\right). \tag{40}$$

Here and in what follows, we have taken into account that (15) allows us to express the derivative with respect to t through the derivative with respect to x. Further, differentiating (15) with respect to t yields

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial \mathcal{G}}{\partial t} + w_t \frac{\partial \mathcal{G}}{\partial w} + w_{xt} \frac{\partial \mathcal{G}}{\partial w_x} = \frac{\partial \mathcal{G}}{\partial t} + \mathcal{G} \frac{\partial \mathcal{G}}{\partial w} + \mathcal{F} \frac{\partial \mathcal{G}}{\partial w_x} \quad \Longrightarrow \quad \frac{\partial^2 w}{\partial t^2} = \mathcal{H}_2\left(x, t, w, \frac{\partial w}{\partial x}\right). \tag{41}$$

Replacing the derivatives w_t, w_{xt}, w_{xx}, and w_{tt} in (17) by their expressions from (15), (38), (40), and (41), we find the total differential operators D_t and D_x, which are required for the compatibility condition (39).

Example 4. Consider the nonlinear equation

$$\frac{\partial^2 w}{\partial x \partial t} = f(w) \tag{42}$$

with two different first-order differential constraints.

Case 1. Let us supplement (42) with a quasilinear differential constraint of the form

$$\frac{\partial w}{\partial t} = g(w) \frac{\partial w}{\partial x}.$$

Simple calculations combined with the compatibility conditions (39), where $\mathcal{F} = f(w)$ and $\mathcal{G} = g(w) w_x$, lead us to the expression

$$3 f g' w_x + [g g'' - (g')^2] w_x^3 = 0.$$

Equating the coefficients of like powers of w_x to zero, we find that $g = \text{const}$. This corresponds to a traveling-wave solution of equation (42), $w = w(kx + \lambda t)$.

Case 2. Now let us supplement equation (42) by a differential constraint with a quadratic nonlinearity in derivatives,

$$\frac{\partial w}{\partial t} \frac{\partial w}{\partial x} = g(w). \tag{43}$$

Calculations with the help of the compatibility condition (39), where $\mathcal{F} = f(w)$ and $\mathcal{G} = g(w)/w_x$, lead us to an expression relating the functions $f = f(w)$ and $g = g(w)$:

$$g g'' - (g')^2 - 2 f' g + 3 f g' - 2 f^2 = 0. \tag{44}$$

It can be shown that the differential constraint (43), together with the compatibility condition (44), yields a self-similar solution $w = w(xt)$ of equation (42); here, x and t can be replaced by $x + C_1$ and $t + C_2$.

S.8.2-3. Second-order equations of general form.

Consider a second-order hyperbolic equation of the general form

$$\mathcal{F}(x,t,w,w_x,w_t,w_{xx},w_{xt},w_{tt})=0 \tag{45}$$

with a first-order differential constraint

$$\mathcal{G}(x,t,w,w_x,w_t)=0. \tag{46}$$

Let us successively differentiate equations (38) and (39) with respect to both variables so as to obtain differential relations involving second and third derivatives. We get

$$\mathrm{D}_x\mathcal{F}=0,\quad \mathrm{D}_t\mathcal{F}=0,\quad \mathrm{D}_x\mathcal{G}=0,\quad \mathrm{D}_t\mathcal{G}=0,\quad \mathrm{D}_x[\mathrm{D}_x\mathcal{G}]=0,\quad \mathrm{D}_x[\mathrm{D}_t\mathcal{G}]=0,\quad \mathrm{D}_t[\mathrm{D}_t\mathcal{G}]=0. \tag{47}$$

The compatibility condition for (45) and (46) can be found by eliminating the derivatives w_t, w_{xx}, w_{xt}, w_{tt}, w_{xxx}, w_{xxt}, w_{xtt}, and w_{ttt} from the nine equations of (45)–(47). In doing so, we obtain an expression of the form

$$\mathcal{H}(x,t,w,w_x)=0. \tag{48}$$

If the left-hand side of (48) is a polynomial in w_x, then the compatibility conditions result from equating the functional coefficients of the polynomial to zero.

⊙ *References for Subsection* S.8.2: A. F. Sidorov, V. P. Shapeev, and N. N. Yanenko (1984), V. A. Galaktionov (1994), P. J. Olver (1994), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

S.8.3. Second and Higher Order Differential Constraints

Constructing exact solutions of nonlinear partial differential equations with the help of second- and higher-order differential constraints requires finding exact solutions of these differential constraints. The latter is generally rather difficult or even impossible. For this reason, one employs some special differential constraints that involve derivatives with respect to only one variable. In practice, one considers second-order ordinary differential equations in, say, x and the other variable, t, is involved implicitly or is regarded as a parameter, so that integration constants depend on t.

The problem of compatibility of a second-order evolution equation

$$\frac{\partial w}{\partial t}=\mathcal{F}_1\left(x,t,w,\frac{\partial w}{\partial x},\frac{\partial^2 w}{\partial x^2}\right)$$

with a similar differential constraint

$$\frac{\partial w}{\partial t}=\mathcal{F}_2\left(x,t,w,\frac{\partial w}{\partial x},\frac{\partial^2 w}{\partial x^2}\right)$$

may be reduced to a problem with the first-order differential constraint considered in Subsection S.8.2-1. To that end, one should first eliminate the second derivative w_{xx} from the equations. Then, the resulting first-order equation is examined together with the original equation (or the original differential constraint).

Example 5. From the class of nonlinear heat equations with a source

$$\frac{\partial w}{\partial t}=\frac{\partial}{\partial x}\left[f_1(w)\frac{\partial w}{\partial x}\right]+f_2(w) \tag{49}$$

one singles out equations that admit invariant manifolds of the form

$$\frac{\partial^2 w}{\partial x^2}=g_1(w)\left(\frac{\partial w}{\partial x}\right)^2+g_2(w). \tag{50}$$

The functions $f_2(w)$, $f_1(w)$, $g_2(w)$, and $g_1(w)$ are to be determined in the further analysis.

Eliminating the second derivative from (49) and (50), we obtain

$$\frac{\partial w}{\partial t}=\varphi(w)\left(\frac{\partial w}{\partial x}\right)^2+\psi(w), \tag{51}$$

where

$$\varphi(w) = f_1(w)g_1(w) + f_1'(w), \quad \psi(w) = f_1(w)g_2(w) + f_2(w). \tag{52}$$

The condition of invariance of the manifold (50) under equation (49) is obtained by differentiating (50) with respect to t:

$$w_{xxt} = 2g_1 w_x w_{xt} + g_1' w_x^2 w_t + g_2' w_t.$$

The derivatives w_{xxt}, w_{xt}, and w_t should be eliminated from this relation with the help of equations (50) and (51) and those obtained by their differentiation. As a result, we get

$$(2\varphi g_1^2 + 3\varphi' g_1 + \varphi g_1' + \varphi'')w_x^4 + (4\varphi g_1 g_2 + 5\varphi' g_2 + \varphi g_2' - g_1\psi' - \psi g_1' + \psi'')w_x^2 + 2\varphi g_2^2 + \psi' g_2 - \psi g_2' = 0.$$

Equating the coefficients of like powers of w_x to zero, one obtains three equations, which, for convenience, may be written in the form

$$\begin{gathered}(\varphi' + \varphi g_1)' + 2g_1(\varphi' + \varphi g_1) = 0,\\ 4g_2(\varphi' + \varphi g_1) + (\varphi g_2 - \psi g_1)' + \psi'' = 0,\\ \varphi = -\tfrac{1}{2}(\psi/g_2)'.\end{gathered} \tag{53}$$

The first equation can be satisfied by taking $\varphi' + \varphi g_1 = 0$. The corresponding particular solution of system (53) has the form

$$\varphi = -\frac{1}{2}\mu', \quad \psi = \mu g_2, \quad g_1 = -\frac{\mu''}{\mu'}, \quad g_2 = \left(2C_1 + \frac{C_2}{\sqrt{|\mu|}}\right)\frac{1}{\mu'}, \tag{54}$$

where $\mu = \mu(w)$ is an arbitrary function.

Taking into account (52), we find the functional coefficients of the original equation (49) and the invariant set (50):

$$f_1 = \left(C_3 - \frac{1}{2}w\right)\mu', \quad f_2 = (\mu - f_1)g_2, \quad g_1 = -\frac{\mu''}{\mu'}, \quad g_2 = \left(2C_1 + \frac{C_2}{\sqrt{|\mu|}}\right)\frac{1}{\mu'}. \tag{55}$$

Equation (50), together with (55), admits the first integral

$$w_x^2 = \left[4C_1\mu + 4C_2\sqrt{|\mu|} + 2\sigma_t'(t)\right]\frac{1}{(\mu')^2}, \tag{56}$$

where $\sigma(t)$ is an arbitrary function. Let us eliminate w_x^2 from (51) by means of (56) and substitute the functions φ and ψ from (54) to obtain the equation

$$\mu' w_t = -C_2\sqrt{|\mu|} - \sigma_t'(t). \tag{57}$$

Let us dwell on the special case $C_2 = C_3 = 0$. Integrating equation (57) and taking into account that $\mu_t = \mu' w_t$ yields

$$\mu = -\sigma(t) + \theta(x), \tag{58}$$

where $\theta(x)$ is an arbitrary function. Substituting (58) into (56) and taking into account the relation $\mu_x = \mu' w_x$, we obtain

$$\theta_x^2 - 4C_1\theta = 2\sigma_t - 4C_1\sigma.$$

Equating both sides of this equation to zero and integrating the resulting ordinary differential equations, we find the functions on the right-hand side of (58):

$$\sigma(t) = A\exp(2C_1 t), \quad \theta(x) = C_1(x + B)^2, \tag{59}$$

where A and B are arbitrary constants. Thus, an exact solution of equation (49) with the functions f_1 and f_2 from (55) can be represented in implicit form for $C_2 = C_3 = 0$ as follows:

$$\mu(w) = -A\exp(2C_1 t) + C_1(x + B)^2.$$

In the solution and the determining relations (55), the function $\mu(w)$ can be chosen arbitrary.

Example 6. Consider the problem of finding nonlinear second-order equations

$$\frac{\partial w}{\partial t} = f_2(w)\frac{\partial^2 w}{\partial x^2} + f_1(w)\frac{\partial w}{\partial x} + f_0(w)$$

admitting invariant manifolds of the form

$$\frac{\partial^2 w}{\partial x^2} = g_1(w)\frac{\partial w}{\partial x} + g_0(w).$$

The compatibility analysis of these equations leads us to the following relations for the determining functions:

$$\begin{aligned}&f_2(w) \text{ is an arbitrary function,}\\ &f_1(w) = C_1 w + C_2 - (3C_1C_3 w + C_4)f_2(w),\\ &f_0(w) = (-C_1^2C_3w^3 - C_1C_4w^2 + C_5w + C_6)[1 - C_3 f_2(w)],\\ &g_1(w) = 3C_1C_3w + C_4,\\ &g_0(w) = C_3(-C_1^2C_3w^3 - C_1C_4w^2 + C_5w + C_6),\end{aligned}$$

where $C_1, \ldots, C_6$ are arbitrary constants.

Section S.8.4 contains examples of second- and third-order differential constraints that are essentially equivalent to most common structures of exact solutions.

Note that third- or higher-order differential constraints are rarely used, since they lead to cumbersome computations and rather complex equations (often, the original equations are simpler).

⊙ *References for Subsection* S.8.3: A. F. Sidorov, V. P. Shapeev, and N. N. Yanenko (1984), V. A. Galaktionov (1994), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

TABLE 20
Second-order differential constraints corresponding to some classes of exact solutions representable in explicit form

No.	Type of solution	Structure of solution	Differential constraints
1	Additive separable solution	$w=\varphi(x)+\psi(y)$	$w_{xy}=0$
2	Multiplicative separable solution	$w=\varphi(x)\psi(y)$	$ww_{xy}-w_x w_y=0$
3	Generalized separable solution	$w=\varphi(x)y^2+\psi(x)y+\chi(x)$	$w_{yy}-f(x)=0$
4	Generalized separable solution	$w=\varphi(x)\psi(y)+\chi(x)$	$w_{yy}-f(y)w_y=0$ $w_{xy}-g(x)w_y=0$
5	Functional separable solution	$w=f(z),\ z=\varphi(x)y+\psi(x)$	$w_{yy}-g(w)w_y^2=0$
6	Functional separable solution	$w=f(z),\ z=\varphi(x)+\psi(y)$	$ww_{xy}-g(w)w_x w_y=0$

TABLE 21
Third-order differential constraints corresponding to some classes of exact solutions representable in explicit form

Type of solution	Structure of solution	Differential constraint
Generalized separable	$w=\varphi(x)y^2+\psi(x)y+\chi(x)$	$w_{yyy}=0$
Generalized separable	$w=\varphi(x)\psi(y)+\chi(x)$	$w_y w_{xyy}-w_{xy}w_{yy}=0$
Functional separable	$w=f\bigl(\varphi(x)y+\psi(x)\bigr)$	$w_y(w_x w_{yyy}-w_y w_{xyy})=2w_{yy}(w_x w_{yy}-w_y w_{xy})$
Functional separable	$w=f\bigl(\varphi(x)+\psi(y)\bigr)$	$w_x w_y w_{xyy}-w_y w_{xxy}=w_{xy}(w_x^2 w_{yy}-w_y^2 w_{xx})$

S.8.4. Connection Between the Differential Constraints Method and Other Methods

The differential constraints method is one of the most general methods for the construction of exact solutions to nonlinear partial differential equations. Many other methods can be treated as its particular cases.*

S.8.4-1. Generalized and functional separation of variables versus differential constraints.

Table 20 lists examples of second-order differential constraints which are essentially equivalent to most common forms of separable solutions. For functional separable solutions (rows 5 and 6), the function g can be expressed through f.

Table 21 lists examples of third-order differential constraints which may be regarded as equivalent to direct specification of most common forms of functional separable solutions.

* The basic difficulty of applying the differential constraints method is due to the great generality of its statements and the necessity of selecting differential constraints suitable for specific classes of equations. This is why for the construction of exact solutions of nonlinear equations, it is often preferable to use more simple (but less general) methods.

Searching for a generalized separable solution of the form $w(x,y) = \varphi_1(x)\psi_1(y) + \cdots + \varphi_n(x)\psi_n(y)$, with $2n$ unknown functions, is equivalent to prescribing a differential constraint of order $2n$; in general, the number of unknown functions $\varphi_i(x)$, $\psi_i(y)$ corresponds to the order of the differential equation representing the differential constraint.

For the types of solutions listed in Tables 20 and 21, it is preferable to use the methods of generalized and functional separation of variables, since these methods require less steps where it is necessary to solve intermediate differential equations. Furthermore, the method of differential constraints is ill-suited for the construction of exact solutions of higher (arbitrary) order equations.

S.8.4-2. Generalized similarity reductions and differential constraints.

Consider a generalized similarity reduction based on a prescribed form of the desired solution,

$$w(x,t) = F\big(x,t,u(z)\big), \qquad z = z(x,t), \tag{60}$$

where $F(x,t,u)$ and $z(x,t)$ should be selected so as to obtain ultimately a single ordinary differential equation for $u(z)$; see Subsection S.6.2.

Let us show that employing the solution structure (60) is equivalent to searching for a solution with the help of a first-order quasilinear differential constraint

$$\xi(x,t)\frac{\partial w}{\partial t} + \eta(x,t)\frac{\partial w}{\partial x} = \zeta(x,t,w). \tag{61}$$

Indeed, first integrals of the characteristic system of ordinary differential equations

$$\frac{dt}{\xi(x,t)} = \frac{dx}{\eta(x,t)} = \frac{dw}{\zeta(x,t,w)}$$

have the form

$$z(x,t) = C_1, \qquad \varphi(x,t,w) = C_2, \tag{62}$$

where C_1 and C_2 are arbitrary constants. Therefore, the general solution of equation (61) can be written as follows:

$$\varphi(x,t,w) = u\big(z(x,t)\big), \tag{63}$$

where $u(z)$ is an arbitrary function. On solving (63) for w, we obtain a representation of the solution in the form (60).

⊙ *Reference*: P. J. Olver (1994).

S.8.4-3. Group analysis and differential constraints.

The group analysis method for differential equations can be restated in terms of the differential constraints method. This can be demonstrated by the following example with a general second-order equation

$$F\left(x,y,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y},\frac{\partial^2 w}{\partial x^2},\frac{\partial^2 w}{\partial x\partial y},\frac{\partial^2 w}{\partial y^2}\right) = 0. \tag{64}$$

Let us supplement equation (64) with two differential constraints

$$\xi\frac{\partial w}{\partial x} + \eta\frac{\partial w}{\partial y} = \zeta, \tag{65}$$

$$\xi\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial y} + \zeta\frac{\partial F}{\partial w} + \zeta_1\frac{\partial F}{\partial w_x} + \zeta_2\frac{\partial F}{\partial w_y} + \zeta_{11}\frac{\partial F}{\partial w_{xx}} + \zeta_{12}\frac{\partial F}{\partial w_{xy}} + \zeta_{22}\frac{\partial F}{\partial w_{yy}} = 0, \tag{66}$$

where $\xi = \xi(x,y,w)$, $\eta = \eta(x,y,w)$, and $\zeta = \zeta(x,y,w)$ are unknown functions, and the coordinates of the first and the second prolongations ζ_i and ζ_{ij} are defined by formulas (13) and (14) of

Subsection S.7.1. The differential constraint (66) coincides with the invariance condition for equation (64); see (11) in Subsection S.7.1.

The method for the construction of exact solutions to equation (64) based on using the first-order partial differential equation (65) and the invariance condition (66) corresponds to the nonclassical method of group analysis (see Subsection S.7.2).

Remark. When the classical schemes of group analysis are employed, one first considers two equations, (64) and (66). From these, one eliminates one of the highest-order derivatives, say w_{yy}, while the remaining derivatives (w_x, w_y, w_{xx}, and w_{xy}) are assumed "independent." The resulting expression splits into powers of independent derivatives (see Subsection S.7.1). As a result, one arrives at an overdetermined system of equations, from which the functions ξ, η, and ζ are found. Then, these functions are inserted into the quasilinear first-order equation (65), whose solution allows us to determine the general form of a solution (this solution contains some arbitrary functions). Next, using (64), one can refine the structure of the solution obtained on the preceding step.

The classical scheme may result in the loss of some solutions, since at the first step of splitting it is assumed that the first derivatives w_x and w_y are independent, whereas these derivatives are in fact linearly dependent due to equation (65).

⊙ *References for Subsection* S.8.4: S. V. Meleshko (1983), V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov (1999).

S.9. Painlevé Test for Nonlinear Equations of Mathematical Physics*

S.9.1. Movable Singularities of Solutions of Ordinary Differential Equations

1°. The connection between the structure of differential equations and singularities of their solutions was established more than a hundred years ago. The singularities of solutions of linear ordinary differential equations are completely determined by singularities of the coefficients of the equations. Since the position of such singularities does not depend on integration constants, they are called fixed singularities. In the case of nonlinear equations, their solutions may also possess movable singularities, whose position depends on the initial conditions (integration constants).

Below, we give simplest examples of first-order ordinary differential equations and their solutions with movable singularities.

Equation	*Solution*	*Type of singularity of the solution*
$u'_z = -u^2$	$u = 1/(z - z_0)$	movable pole
$u'_z = 1/u$	$u = 2\sqrt{z - z_0}$	algebraic branch point
$u'_z = e^{-u}$	$u = \ln(z - z_0)$	logarithmic branch point
$u'_z = -u \ln^2 u$	$u = \exp[1/(z - z_0)]$	essentially singular point

Algebraic branch points, logarithmic branch points, and essentially singular points are called "critical singular points."

2°. In 1884, L. L. Fuchs established the following fact: the first-order nonlinear differential equation

$$u'_z = R(z, u),$$

where the function R is rational in the second argument and analytic with respect to the first, admits solutions without movable critical points (other than movable poles) only if it coincides with the general Riccati equation $u'_z = A_0(z) + A_1(z)u + A_2(z)u^2$.

* Section S.9 was written by V. G. Baydulov and V. A. Gorodtsov.

3°. The second-order ordinary differential equations (in the complex plane) of the form

$$u''_{zz} = R(z, u, u'_z),$$

where $R = R(z, u, w)$ is a rational function of u and w and is analytic in z, were classified by P. Painlevé (1900) and B. Gambier (1910). These authors showed that all equations of this form whose solutions have no movable critical points (other than fixed singular points and movable poles) can be divided into 50 classes. The equations of 44 out of these classes can be integrated by quadrature or their order can be reduced. The remaining 6 classes, in canonical form, are irreducible and are called Painlevé equations (their solutions are called Painlevé transcendents).

4°. The first Painlevé equation has the form

$$u''_{zz} = 6u^2 + z.$$

The equation has a movable pole z_0; in its neighborhood, the solutions can be represented by the series

$$u = \frac{1}{(z - z_0)^2} + \sum_{n=2}^{\infty} a_n (z - z_0)^n,$$
$$a_2 = -\tfrac{1}{10} z_0, \quad a_3 = -\tfrac{1}{6}, \quad a_4 = C, \quad a_5 = 0, \quad a_6 = \tfrac{1}{300} z_0^2,$$

where z_0 and C are arbitrary constants; the coefficients a_n ($n \geq 7$) are uniquely determined by z_0 and C.

The second Painlevé equation is expressed as

$$u''_{zz} = 2u^3 + zu + a.$$

In a neighborhood of the movable pole z_0, its solutions admit the following expansions:

$$u = \frac{m}{z - z_0} + \sum_{n=1}^{\infty} b_n (z - z_0)^n,$$
$$b_1 = -\tfrac{1}{6} m z_0, \quad b_2 = -\tfrac{1}{4}(m + \alpha), \quad b_3 = C, \quad b_4 = \tfrac{1}{72} z_0 (m + 3\alpha),$$
$$b_5 = \tfrac{1}{3024} \left[(27 + 81\alpha^2 - 2z_0^3) m + 108\alpha - 216 C z_0 \right],$$

where $m = \pm 1$; z_0 and C are arbitrary constants; and the coefficients b_n ($n \geq 6$) are uniquely determined by z_0 and C.

More detailed information about the Painlevé equations can be found in the literature cited at the end of this subsection. It should be observed that the solution of the fourth Painlevé equation has a movable pole, while the solutions of the third, the fifth, and the sixth Painlevé equations have fixed logarithmic branch points.

Remark. In 1888, S. V. Kowalevskaya succeeded in integrating the equations of motion of a rigid body having a fixed point and subject to gravity, in a case previously unknown. She examined solutions of a system of six first-order nonlinear ordinary differential equations. Solutions were sought in the form of series expansions in powers of each unknown quantity with movable poles,

$$u = (z - z_0)^{-n} \left[a_0 + a_1 (z - z_0) + \cdots \right].$$

The generality of the solution was ensured by a suitable (corresponding to the order of the system) number of arbitrary coefficients in the expansions and the free parameter z_0.

It should be mentioned that the studies of S. V. Kowalevskaya preceded the works of Painlevé on the classification of second-order ordinary differential equations, where similar expansions were used.

⊙ *References for Subsection* S.9.1: V. V. Golubev (1950), G. M. Murphy (1960), A. R. Its and V. Yu. Novokshenov (1986), M. Tabor (1989), V. I. Gromak and N. A. Lukashevich (1990), A. R. Chowdhury (2000), V. I. Gromak (2002), A. D. Polyanin and V. F. Zaitsev (2003).

S.9.2. Solutions of Partial Differential Equations with a Movable Pole. Description of the Method

By analogy with ordinary differential equations, solutions of partial differential equations may be sought in the form of power series expansions with movable pole singularities. The position of the pole is given by an arbitrary function.

For simplicity of exposition, we consider equations of mathematical physics in two independent variables x, t and a dependent variable w, assuming that the equations do not explicitly depend on x or t.

1°. *Simplest scheme.* A solution is sought near a singular manifold $x - x_0(t) = 0$ as the following series (Jimbo, Kruskal, and Miwa, 1982):

$$w(x,t) = \frac{1}{\varepsilon^\alpha} \sum_{n=0}^{\infty} w_n(t)\varepsilon^n, \qquad \varepsilon = x - x_0(t). \tag{1}$$

Here, the exponent α is a positive integer (this ensures that the movable singularity is of the pole type), and the function $x_0(t)$ is assumed arbitrary.

The expression (1) is substituted into the equation under consideration. First, by equating the leading singular terms, one finds the exponent α and the leading term $u_0(t)$ of the series. Then, the terms with the same powers of ε are collected. Equating the resulting coefficients of the same powers of ε to zero, one obtains a system of ordinary differential equations for the functions $w_n(t)$.

The thus obtained solutions are general, provided that series (1) contains arbitrary functions whose number is equal to the order of the equation under consideration.

2°. *General scheme. The Painlevé test.* A solution of a partial differential equation is sought in a neighborhood of the singular manifold $\varepsilon(x,t) = 0$ in the form of a generalized series symmetric with respect to the independent variables (Weiss, Tabor, and Carnevalle, 1983):

$$w(x,t) = \frac{1}{\varepsilon^\alpha} \sum_{n=0}^{\infty} w_n(x,t)\varepsilon^n, \qquad \varepsilon = \varepsilon(x,t), \tag{2}$$

where $\varepsilon_t \varepsilon_x \neq 0$. Here and in what follows, the subscripts x and t denote the corresponding partial derivatives.

Series (1) is a special case of the expansion (2), provided the equation of the singular manifold, $\varepsilon(x,t) = 0$, is solvable for the variable x.

The requirement that there are no movable critical points implies that α is a positive integer. The solution will be general if the total number of arbitrary functions among the $w_n(x,t)$ and $\varepsilon(x,t)$ coincides with the order of the equation.

Substituting (2) into the equation, collecting terms with the same powers of ε, and equating them to zero, we obtain the following recurrence relations for the expansion coefficients:

$$P_N(n)w_n = f_n(w_0, w_1, \ldots, w_{n-1}, \varepsilon_t, \varepsilon_x, \ldots).$$

Here, the $P_N(n)$ is a polynomial of degree N of the integer argument n,

$$P_N(n) = (n+1)(n-j_1)(n-j_2)\ldots(n-j_{N-1}),$$

and N is the order of the equation under consideration.

If the roots of the polynomial $j_1, j_2, \ldots, j_{N-1}$ (called resonances) are nonnegative integers and the compatibility conditions

$$f_{n=j_k} = 0 \qquad (k = 1, 2, \ldots, N-1)$$

hold, then one says that the conditions of the Painlevé test hold for the equation under consideration. Equations satisfying these conditions are often regarded as integrable equations (this is confirmed by the fact that in many known cases, such equations can be reduced to linear equations).

3°. For the initial verification of the Painlevé conditions for a specific equation, it is convenient to use a simplified scheme based on the expansion (1). The relations $(w_n)_x = 0$ and $\varepsilon_x = 1$ ensure some important simplifications of technical character, as compared with the expansion (2).

The more general expansion (2) entails more cumbersome but more informative computations. It can be effectively used at the second step of the investigation, after the conditions of the Painlevé test have been verified. This helps to clarify many important properties of the equations and their solutions and find the form of the Bäcklund transformation that linearizes the original equation.

⊙ *References for Subsection* S.9.2: M. Jimbo, M. D. Kruskal, and T. Miwa (1982), J. Weiss, M. Tabor, and G. Carnevalle (1983), J. Weiss (1983, 1984, 1985), W.-H. Steeb and N. Euler (1988), R. Conte (1989, 1999), R. Conte and M. Musette (1989, 1993), M. Tabor (1989), M. Musette (1998).

S.9.3. Examples of the Painlevé Test Applications

In this section, we consider some examples of equations of mathematical physics. For their analysis, we first resort to the simplest and then the general scheme of the Painlevé test application based on series (1) and (2) from Section S.9.2.

Example 1. Consider the Burgers equation

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = \nu\frac{\partial^2 w}{\partial x^2}.$$

1°. Substituting the leading term of the expansion (1) into this equation, we obtain

$$\frac{w_0'}{(x-x_0)^\alpha} + \frac{\alpha w_0 x_0'}{(x-x_0)^{\alpha+1}} - \frac{\alpha w_0^2}{(x-x_0)^{2\alpha+1}} = \frac{\nu\alpha(\alpha+1)w_0}{(x-x_0)^{\alpha+2}},$$

where $x_0 = x_0(t)$ and $w_0 = w_0(t)$; the prime denotes a derivative with respect to t. Retaining the leading singular terms (omitting the first two terms on the left), we find that

$$\alpha = 1, \quad w_0 = -2\nu \qquad (n = 0).$$

The Burgers equation, upon the insertion of series (1) in it and the collection of terms with the same powers of $\varepsilon = x - x_0(t)$, takes the form

$$w_t + ww_x - \nu w_{xx} = \sum_{n=0}^{\infty} E_n(t)\varepsilon^{n-3} = 0, \quad \text{where} \quad E_n(t) = -(n+1)(n-2)\nu w_n + \cdots.$$

Here, in the expression for $E_n(t)$, the terms containing $w_0, \dots, w_{n-1}$ and $x_0(t)$ are omitted.

It is clear that there is a single resonance, $n = 2$; the compatibility condition holds only in this case (the sum of the terms with lowest-subscript coefficients in the recurrence relation vanishes) and the function $w_2(t)$ remains arbitrary. This can be seen from the structure of the following recurrence relations:

$$\begin{aligned} -E_0/w_0 &= w_0 + 2\nu = 0 && (n = 0),\\ -E_1/w_0 &= w_1 + \varepsilon_t = 0 && (n = 1),\\ E_2 &= (w_0)_t = 0 && (n = 2). \end{aligned}$$

The relation for $n = 2$ is a consequence of the preceding relations and does not contain w_2.

Thus, the Burgers equation satisfies the conditions of the Painlevé test, and its solution contains two arbitrary functions, as required. Collecting terms with like powers of $x - x_0(t)$, we can write out the solution in the form

$$w(x,t) = -\frac{2\nu}{x - x_0(t)} + x_0'(t) + w_2(t)[x - x_0(t)]^2 + \cdots,$$

where $x_0(t)$ and $w_2(t)$ are arbitrary functions.

2°. For the purpose of subsequent analysis of the Burgers equation, let us take advantage of the general expansion (2), where $w_n = w_n(x,t)$ and $\varepsilon = \varepsilon(x,t)$. From the condition of balance of the leading terms, we obtain

$$\alpha = 1, \quad w_0 = -2\nu\varepsilon_x \qquad (n = 0).$$

The recurrence relations for the next three terms of the expansion have the form

$$\begin{aligned} w_1\varepsilon_x - \nu\varepsilon_{xx} + \varepsilon_t &= 0 \quad (n = 1),\\ (w_1\varepsilon_x - \nu\varepsilon_{xx} + \varepsilon_t)_x &= 0 \quad (n = 2),\\ (w_1)_t - \nu(w_1)_{xx} + w_1(w_1)_x + (w_0w_2)_x + (\varepsilon_t - \nu\varepsilon_{xx})w_2 &\\ - 2\nu\varepsilon_x(w_2)_x + \varepsilon_x(w_1w_2 + w_0w_3) - 2\nu\varepsilon_x^2 w_3 &= 0 \quad (n = 3). \end{aligned}$$

Setting $w_2 = w_3 = 0$ in these formulas, we obtain a consistent truncated series of (2) with zero higher-order coefficients ($w_k = 0$ for $k \ge 2$). The remaining relations allow us to represent the solution in the form

$$w = \frac{w_0}{\varepsilon} + w_1, \qquad w_0 = -2\nu\varepsilon_x,$$
$$\varepsilon_t + w_1\varepsilon_x = \nu\varepsilon_{xx}, \quad (w_1)_t + w_1(w_1)_x = \nu(w_1)_{xx}.$$

These relations represent a Bäcklund transformation and allow us to use solutions $w_1 = w_1(x,t)$ of the Burgers equation for the construction of its other solutions $w = w(x,t)$. Taking, for example, $w_1 = 0$ to be the initial solution, we obtain the well-known Cole–Hopf transformation

$$w = -2\nu\frac{\varepsilon_x}{\varepsilon},$$

which reduces the nonlinear Burgers equation to the linear heat equation

$$\varepsilon_t = \nu\varepsilon_{xx}.$$

Example 2. Consider the Korteweg–de Vries equation

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} = 0.$$

1°. Substituting the leading term of the expansion (1) into this equation yields

$$\frac{w_0'}{(x-x_0)^\alpha} + \frac{\alpha w_0 x_0'}{(x-x_0)^{\alpha+1}} - \frac{\alpha w_0^2}{(x-x_0)^{2\alpha+1}} - \frac{\alpha(\alpha+1)(\alpha+2)w_0}{(x-x_0)^{\alpha+3}} = 0,$$

where $x_0 = x_0(t)$ and $w_0 = w_0(t)$. From the condition of balance of the leading terms, we find that

$$\alpha = 2, \quad w_0 = -12 \qquad (n = 0).$$

Upon the insertion of the expansion (1), the Korteweg–de Vries equation can be represented in the form

$$w_t + ww_x + w_{xxx} = \sum_{n=0}^{\infty} E_n(t)\varepsilon^{n-5} = 0, \quad \text{where} \quad E_n(t) = (n+1)(n-4)(n-6)w_n + \cdots.$$

From the expression for $E_n(t)$, it follows that there are two resonances, $n = 4$ and $n = 6$. Writing out explicitly the first seven equations for the coefficients in the expansion (1), we see that the compatibility condition holds for the resonances,

$$\begin{aligned}
w_0 + 12 &= 0 \qquad (n = 0),\\
w_1 &= 0 \qquad (n = 1),\\
\varepsilon_t + w_2 &= 0 \qquad (n = 2),\\
w_3 &= 0 \qquad (n = 3),\\
(w_1)_t &= 0 \qquad (n = 4),\\
\varepsilon_{tt} + 6w_5 &= 0 \qquad (n = 5),\\
(w_3)_t + w_3^2 + 2w_1w_5 &= 0 \qquad (n = 6).
\end{aligned}$$

The relations for $n = 4$ and $n = 6$ are consequences of the preceding ones and do not contain w_4 and w_6. Therefore, the Korteweg–de Vries equation satisfies the conditions of the Painlevé test. The three arbitrary functions $w_4(t)$, $w_6(t)$, and $x_0(t)$ ensure the required generality of the solution of the third-order equation.

2°. Now, let us obtain a consequence of the general expansion by truncating series (2). Inserting the truncated series with $w_3 = w_4 = \cdots = 0$ into the Korteweg–de Vries equation, we arrive at the Bäcklund transformation

$$\begin{aligned}
&w = \frac{w_0}{\varepsilon^2} + \frac{w_1}{\varepsilon} + w_2 = 12(\ln\varepsilon)_{xx} + w_2,\\
&\varepsilon_t\varepsilon_x + w_2\varepsilon_x^2 + 4\varepsilon_x\varepsilon_{xxx} - 3\varepsilon_{xx}^2 = 0,\\
&\varepsilon_{xt} + w_2\varepsilon_{xx} + \varepsilon_{xxx} = 0,\\
&(w_2)_t + w_2(w_2)_x + (w_2)_{xxx} = 0.
\end{aligned}$$

Eliminating w_2 from the second and the third equations, we obtain an equation for the function ε, which can be reduced to a system of linear equations by means of several transformations.

⊙ *References*: J. Weiss, M. Tabor, and G. Carnevalle (1983), M. Tabor (1989), J. Weiss (1993).

Example 3. Consider the Kadomtsev–Petviashvili equation

$$\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3}\right) + a\frac{\partial^2 w}{\partial y^2} = 0,$$

which can be regarded as an integrable generalization of the Korteweg–de Vries equation of a higher dimension and a higher order.

1°. In many-dimensional cases, one utilizes an analogue of the expansion (1):

$$w(x,y,t)=\frac{1}{\varepsilon^{\alpha}}\sum_{n=0}^{\infty}w_n(y,t)\varepsilon^n,\quad \varepsilon=x-x_0(y,t). \tag{3}$$

Equating the leading singular terms for the Kadomtsev–Petviashvili equation, we obtain the same result as that for the Korteweg–de Vries equation,

$$\alpha=2,\quad w_0=-12\qquad (n=0).$$

Substituting the expansion (3) into the original equation, we obtain

$$w_{tx}+ww_{xx}+w_x^2+w_{xxxx}+aw_{yy}=\sum_{n=0}^{\infty}\varepsilon^{n-6}E_n(y,t)=0,$$
$$E_n(y,t)=(n+1)(n-4)(n-5)(n-6)w_n+\cdots.$$

It is apparent that there are three resonances: $n=4,\,5,\,6$. In order to verify the conditions of the Painlevé test, let us write out recurrence relations for the first seven terms of the expansion,

$$\begin{aligned}
&E_0=10w_0(w_0+12)=0 && (n=0),\\
&E_1=12w_1(w_0+2)=0 && (n=1),\\
&E_2=3[2(\varepsilon_t+a\varepsilon_y^2+w_2)w_0+w_1^2]=0 && (n=2),\\
&E_3=a(w_1)_{yy}-2(w_0)_t-4a(w_0)_y\varepsilon_y-2[aw_0\varepsilon_{yy}-(\varepsilon_t+a\varepsilon_y^2+w_2)w_1-w_3w_0]=0 && (n=3),\\
&E_4=a(w_0)_{yy}-(w_1)_t-2a(w_1)_y\varepsilon_y-aw_1\varepsilon_{yy}=0 && (n=4),\\
&E_5=a(w_1)_{yy}=0 && (n=5),\\
&E_6=a(w_2)_{yy}+\bigl\{(w_3)_t+2a(w_3)_y\varepsilon_y+aw_3\varepsilon_{yy})\\
&\qquad\qquad +2[(\varepsilon_t+a\varepsilon_y^2+w_2w_4+\tfrac{1}{2}w_3^2+w_5w_1+(w_0+12)w_6]\bigr\}=0 && (n=6).
\end{aligned}$$

The last three relations (corresponding to resonances), in view of the preceding relations, hold identically and do not contain w_4, w_5, w_6. There are four arbitrary functions $(\varepsilon, w_4, w_5, w_6)$ in the solution of the forth-order equation under consideration, which indicates that the Painlevé property holds.

2°. The utilization of the general expansion, with the series truncated so that $w_n=0$ for $n>2$, leads us to the Bäcklund transformation (for simplicity, we set $a=1$)

$$\begin{aligned}
&w=12(\ln\varepsilon)_{xx}+w_2,\\
&\varepsilon_t\varepsilon_x+4\varepsilon_x\varepsilon_{xxx}-3\varepsilon_{xx}^2+\varepsilon_y^2+w_2\varepsilon_x^2=0,\\
&\varepsilon_{xt}+\varepsilon_{xxxx}+\varepsilon_{yy}-w_2\varepsilon_{xx}=0,\\
&(w_2)_{tx}+w_2(w_2)_{xx}+(w_2)_x^2+(w_2)_{xxxx}+(w_2)_{yy}=0.
\end{aligned}$$

Eliminating w_2 from the second and the third equations, we obtain an equation for the function ε, which allows us to pass to a solution of a system of linear equations.

Example 4. Consider the model system of equations (Gorodtsov, 1998, 2000)

$$\begin{aligned}
&\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial x}=-\frac{1}{2}\frac{\partial c^2}{\partial x}+\nu\frac{\partial^2 w}{\partial x^2},\\
&\frac{\partial c}{\partial t}+\frac{\partial (wc)}{\partial x}=\chi\frac{\partial^2 c}{\partial x^2}
\end{aligned}$$

that describes convective mass transfer of an active substance in a viscous fluid in the case where the flow is affected by the substance through the pressure quadratically dependent on its concentration. Such equations are used for describing one-dimensional flows of electrically conducting fluids in a magnetic field with high magnetic pressure.

1°. By analogy with the expansion (1), let us represent the desired quantities in the form

$$w(x,t)=\frac{1}{\varepsilon^{\alpha}}\sum_{n=0}^{\infty}w_n(t)\varepsilon^n,\quad c(x,t)=\frac{1}{\varepsilon^{\beta}}\sum_{n=0}^{\infty}c_n(t)\varepsilon^n,\quad \varepsilon\equiv x-x_0(t).$$

Equating the leading singular terms of the equations, we find that

$$\alpha=\beta=1,\quad w_0=-\chi,\quad c_0^2=\chi(2\nu-\chi).$$

Let us write the recurrence relations for the series terms in matrix form

$$\begin{pmatrix}-(n-2)[\chi+\nu(n-1)] & (n-2)c_0\\ (n-2)c_0 & -(n-2)n\chi\end{pmatrix}\begin{pmatrix}w_n\\ c_n\end{pmatrix}=\begin{pmatrix}f_{n-1}\\ g_{n-1}\end{pmatrix}.$$

The quantities f_{n-1}, g_{n-1} depend on the functions $w_0, \ldots, w_{n-1}, c_0, \ldots, c_{n-1}, x_0$. The condition of unique solvability of the matrix equation for the specified higher-order coefficients is violated if the characteristic determinant is equal to zero (the case of degenerate matrix), and then these coefficients may turn out to be arbitrary. Thus, the resonances are determined from the condition

$$\nu\chi(n+1)(n-2)^2(n-2+\chi/\nu) = 0.$$

All these resonances are positive integers (except for the special resonance $n = -1$) only if the Prandtl number is equal to unity, $\Pr \equiv \nu/\chi = 1$. One resonance, $n = 1$, is simple, and the other, $n = 2$, is multiple, so that the overall number of resonances is equal to four.

Writing out the first three recurrence relations

$$\begin{aligned}
&c_0^2 + w_0(w_0 + 2\nu) = 0, && w_0 + \nu = 0 && (n = 0),\\
&c_0c_1 + w_0(\varepsilon_t + w_1) = 0, && w_0c_1 + c_0(\varepsilon_t + w_1) = 0 && (n = 1),\\
&(w_0)_t = 0, && (c_0)_t = 0 && (n = 2),
\end{aligned}$$

we see that the compatibility condition holds for the resonance $n = 1$, since the two relations for $n = 1$ coincide by virtue of the expressions for $n = 0$ ($w_0 = \pm c_0$). The multiple resonance $n = 2$ also satisfies the compatibility condition, since both coefficients w_0, c_0 are constant. Therefore, the Painlevé property takes place for the equations of a fluid with an active substance (for $\nu/\chi = 1$).

2°. Using the general expansion with the series truncated so that $w_2 = w_3 = \cdots = 0$ and $c_2 = c_3 = \cdots = 0$, we obtain a Bäcklund transformation for the equations of a fluid with an active substance

$$w = \frac{w_0}{\varepsilon} + w_1, \qquad c = \frac{c_0}{\varepsilon} + c_1,$$
$$w_0 = -\nu\varepsilon_x, \quad c_0 = \pm\nu\varepsilon_x, \quad \varepsilon_t + (w_1 \mp c_1)\varepsilon_x = \nu\varepsilon_{xx},$$
$$(w_1)_t + w_1(w_1)_x = -c_1(c_1)_x + \nu(w_1)_{xx}, \qquad (c_1)_t + (w_1c_1)_x = \nu(c_1)_{xx}.$$

Comparing this with the Bäcklund transformation for the Burgers equation, we see that if, in the above transformation, we pass to the new variables equal to the sum and the difference of the original variables, we obtain identical equations. Indeed, passing to such variables in the original equations with unit Prandtl number, we obtain a pair of identical Burgers equations,

$$s_t + ss_x = \nu s_{xx}, \qquad s = w + c,$$
$$r_t + rr_x = \nu r_{xx}, \qquad r = w - c,$$

each of which reduces to the linear heat equation (see Example 1).

Numerous investigations show that many known integrable nonlinear equations of mathematical physics possess the Painlevé property. Some new equations with this property have also been found. During the verification of the conditions of the Painlevé test for more complex equations and systems, resonances with higher n may arise. In such situations, analytical solution becomes more and more difficult. However, the Painlevé test is highly adapted for algorithmization and allows for the utilization of symbolic computation methods. For example, the Maple software has been successfully used to obtain a complete classification of integrable cases of the equations of shallow water with dissipation and dispersion of lower orders [see Klimov, Baydulov, and Gorodtsov (2001)].

⊙ *References for Subsection* S.9.3: M. Jimbo, M. D. Kruskal, and T. Miwa (1982), J. Weiss, M. Tabor, and G. Carnevalle (1983), J. Weiss (1983, 1984, 1985), W.-H. Steeb and N. Euler (1988), R. Conte (1989, 1999), R. Conte and M. Musette (1989, 1993), M. Tabor (1989), M. Musette (1998).

S.10. Inverse Scattering Method

S.10.1. Lax Pair Method

S.10.1-1. Basic idea of the method. The Lax pair.

Consider the nonlinear evolution equation

$$\frac{\partial w}{\partial t} = \mathrm{F}(w), \tag{1}$$

with the right-hand side $\mathrm{F}(w)$ depending on the function w and its derivatives in x.

The basic idea of the method consists in representing equation (1) in the form

$$\frac{\partial \mathrm{L}}{\partial t} = \mathrm{LM} - \mathrm{ML}. \tag{2}$$

Here, L and M are linear differential operators in x (these operators are called the Lax pair) whose coefficients depend on w and its derivatives with respect to x. The right-hand side of equation (2) is the commutator of the operators L and M. This commutator will be denoted by [L, M].

Suppose that the operators L and M satisfy equation (2). Consider two auxiliary linear differential equations. The first corresponds to an eigenvalue problem and contains derivatives with respect to the spatial variable x alone,

$$\mathrm{L}\varphi = \lambda\varphi \tag{3}$$

(here, the variable t is involved implicitly through the function w and is regarded as a parameter). The second auxiliary equation describes the time-evolution of an eigenfunction,

$$\frac{\partial\varphi}{\partial t} = -\mathrm{M}\varphi. \tag{4}$$

The operator equation (2) may be regarded as the compatibility condition for equations (3) and (4), provided that the eigenvalues λ are independent of time t. Indeed, differentiating (3) with respect to t, we get $\mathrm{L}_t\varphi + \mathrm{L}\varphi_t = \lambda\varphi_t$. Substituting (4) into this expression, we obtain $\mathrm{L}_t\varphi - \mathrm{LM}\varphi = -\lambda\mathrm{M}\varphi$. Next, taking into account the relations $\lambda\mathrm{M}\varphi = \mathrm{M}(\lambda\varphi)$ and $\lambda\varphi = \mathrm{L}\varphi$, we arrive at the equation $\mathrm{L}_t\varphi = \mathrm{LM}\varphi - \mathrm{ML}\varphi$, which is equivalent to (2).

The above procedure shows how the analysis of the original nonlinear equation (1) can be reduced to the examination of two simpler linear equations (3) and (4).

Example 1. Let us show that a Lax pair for the Korteweg–de Vries equation

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} - 6w\frac{\partial w}{\partial x} = 0 \tag{5}$$

can be defined as

$$\mathrm{L} = \frac{\partial^2}{\partial x^2} - w, \quad \mathrm{M} = 4\frac{\partial^3}{\partial x^3} - 6w\frac{\partial}{\partial x} - 3\frac{\partial w}{\partial x}. \tag{6}$$

It is easy to verify that

$$\begin{aligned}
&\mathrm{LM}(\varphi) = 4\varphi_{xxxxx} - 10w\varphi_{xxx} - 15w_x\varphi_{xx} + (6w^2 - 12w_{xx})\varphi_x + (3ww_x - 3w_{xxx})\varphi,\\
&\mathrm{ML}(\varphi) = 4\varphi_{xxxxx} - 10w\varphi_{xxx} - 15w_x\varphi_{xx} + (6w^2 - 12w_{xx})\varphi_x + (9ww_x - 4w_{xxx})\varphi,\\
&\mathrm{LM}(\varphi) - \mathrm{ML}(\varphi) = (w_{xxx} - 6ww_x)\varphi,
\end{aligned} \tag{7}$$

where $\varphi(x)$ is an arbitrary function. From (6) and (7) it follows that

$$\mathrm{L}_t = -w_t, \quad \mathrm{LM} - \mathrm{ML} = w_{xxx} - 6ww_x.$$

Substituting these expressions into (2), we arrive at the Korteweg–de Vries equation (5).

S.10.1-2. The Cauchy problem.

The procedure for solving the Cauchy problem for equation (1) with the initial condition

$$w = w_0(x) \quad \text{at} \quad t = 0 \tag{8}$$

involves four steps outlined below.

1°. First, one finds the Lax pair representation (2) for the evolution equation (1), which is often the most difficult part of the calculation.

2°. Using the initial condition (8), one evaluates the operator L at $t = 0$ and substitutes it into equation (3). Then the resulting equation is employed to find the eigenvalues λ_n and the initial values of the eigenfunctions $\varphi_n(x, 0)$. Note that the spectrum of the Sturm–Liouville problem determined by equation (3) consists, in general, of two components: continuous part and several discrete eigenvalues.

3°. One finds the time-evolution of the eigenfunctions $\varphi_n(x, t)$ by solving (4).

4°. One determines $w(x,t)$ by solving an inverse problem and taking into account that the eigenfunctions $\varphi_n(x,t)$ satisfy equation (3) for $t > 0$.

Remark. The procedure for solving the Cauchy problem for various nonlinear equations is detailed in the literature cited below. For the solution of the Cauchy problem for the Korteweg–de Vries equation (5), see Subsection 9.1.1, Item 10°.

⊙ *References for Subsection* S.10.1: P. D. Lax (1968), V. E. Zakharov and A. B. Shabat (1972), M. J. Ablowitz and H. Segur (1981), F. Calogero and A. Degasperis (1982), R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris (1982), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), L. D. Faddeev and L. A. Takhtajan (1987), K. Chadan, D. Colton, L. Paivarinta, and W. Rundell (1997), M. J. Ablowitz and P. A. Clarkson (1991), R. Pike and P. Sabatier (2002).

S.10.2. Method Based on the Compatibility Condition for Two Linear Equations

Consider two linear equations

$$\boldsymbol{\varphi}_x = \mathbf{A}\boldsymbol{\varphi}, \tag{9}$$

$$\boldsymbol{\varphi}_t = \mathbf{B}\boldsymbol{\varphi}, \tag{10}$$

where $\boldsymbol{\varphi}$ is an n-dimensional vector and $\mathbf{A}$, $\mathbf{B}$ are $n \times n$-matrices. Let us differentiate equations (9) and (10) in t and x, respectively, and eliminate the mixed derivative $\boldsymbol{\varphi}_{xt}$ from the resulting equations. Next, replacing the derivatives $\boldsymbol{\varphi}_x$ and $\boldsymbol{\varphi}_t$ by the right-hand sides of (9) and (10), we obtain

$$\mathbf{A}_t - \mathbf{B}_x + [\mathbf{A}, \mathbf{B}] = 0, \tag{11}$$

where $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$. It turns out that for a given $\mathbf{A}$, there is a simple deductive procedure for finding $\mathbf{B}$. As a result of that procedure, the compatibility condition (11) turns into a nonlinear evolution equation (see Ablowitz and Segur, 1981).

In what follows, we restrict our investigation to the special case of a two-component vector-valued function $\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$.

Assume that the matrix $\mathbf{A}$ has the form

$$\mathbf{A} = i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \qquad i^2 = -1, \tag{12}$$

where λ is a spectral parameter, and q and r are (complex-valued) functions of two real variables, x and t. The matrix $\mathbf{B}$ should be chosen so that (11) could be reduced to given partial differential equations.

Example 2. Choosing $\mathbf{B}$ in the form

$$\mathbf{B} = 2i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i\lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \begin{pmatrix} 0 & q_x \\ -r_x & 0 \end{pmatrix} - i \begin{pmatrix} qr & 0 \\ 0 & -qr \end{pmatrix},$$

we see that (11) is equivalent to the following system of equations:

$$ir_t + r_{xx} + 2qr^2 = 0,$$

$$iq_t - q_{xx} - 2qr^2 = 0.$$

Hence, taking $q = \overline{r}$ or $q = -\overline{r}$ (the bar over a symbol denotes the complex conjugate), we obtain the nonlinear Schrödinger equations

$$ir_t + r_{xx} + 2|r|^2 r = 0 \qquad (\text{if } q = \overline{r}),$$

$$ir_t + r_{xx} - 2|r|^2 r = 0 \qquad (\text{if } q = -\overline{r}).$$

Example 3. Take $r = q = \frac{1}{2} w_x$ in (12) and consider the matrix

$$\mathbf{B} = \frac{1}{4i\lambda} \begin{pmatrix} \cos w & -i \sin w \\ i \sin w & -\cos w \end{pmatrix}.$$

For the function w from (11) we obtain the sine-Gordon equation:

$$w_{xt} = \sin w.$$

Example 4. Take $r = -q = -\frac{1}{2}w_x$ in (12) and consider the matrix

$$\mathbf{B} = \frac{1}{4i\lambda}\begin{pmatrix} \cosh w & -i\sinh w \\ -i\sinh w & -\cosh w \end{pmatrix}.$$

In this case, (11) is reduced to the sinh-Gordon equation:

$$w_{xt} = \sinh w.$$

⊙ *References for Subsection* S.10.2: M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur (1974), M. J. Ablowitz and H. Segur (1981), F. Calogero and A. Degasperis (1982), R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris (1982), S. P. Novikov, S. V. Manakov, L. B. Pitaevskii, and V. E. Zakharov (1984), K. Chadan, D. Colton, L. Paivarinta, and W. Rundell (1997), M. J. Ablowitz and P. A. Clarkson (1991), R. Pike and P. Sabatier (2002).

S.10.3. Method Based on Linear Integral Equations

Below we outline the approach proposed by Zakharov and Shabat (1974) based on using linear integral equations of the form

$$K(x,y) = F(x,y) + \int_x^\infty K(x,z)N(x;z,y)\,dz, \quad y \ge x, \tag{13}$$

where the functions F, N, and K can depend on some additional parameters other than the specified arguments. In each specific case, the function N is explicitly expressed through F.

Define an operator A_x such that

$$\mathrm{A}_x f(y) = \begin{cases} \int_x^\infty f(z)N(x;z,y)\,dz & \text{if } y \ge x, \\ 0 & \text{if } y < x \end{cases}$$

and assume that for each chosen N, it is possible to prove that the operator $\mathrm{I} - \mathrm{A}_x$ is invertible and its inverse, $(\mathrm{I} - \mathrm{A}_x)^{-1}$, is continuous, where I is the identity operator. The following three steps represent an algorithm for finding a nonlinear equation that can then be solved by the inverse scattering method.

1°. The function F satisfies the following two linear ordinary (or partial) differential equations:

$$\mathrm{L}_i F = 0, \quad i = 1,\ 2. \tag{14}$$

2°. The function K is related to F by equation (13), which can be rewritten as

$$(\mathrm{I} - \mathrm{A}_x)K = F. \tag{15}$$

3°. Applying the operators L_i involved in (14) to equation (15), we obtain

$$\mathrm{L}_i(\mathrm{I} - \mathrm{A}_x)K = 0, \quad i = 1,\ 2.$$

This equation can be rewritten in the form

$$(\mathrm{I} - \mathrm{A}_x)(\mathrm{L}_i K) = \mathrm{R}_i, \quad i = 1,\ 2,$$

where R_i contains all nonzero terms of the commutator $[\mathrm{L}_i, (\mathrm{I} - \mathrm{A}_x)]$. Moreover, (13) and (14) should be chosen so that R_i could be represented in the form

$$\mathrm{R}_i = (\mathrm{I} - \mathrm{A}_x)\mathrm{M}_i(K), \quad i = 1,\ 2,$$

where $\mathrm{M}_i(K)$ is a nonlinear functional of K. But the operator $\mathrm{I} - \mathrm{A}_x$ is invertible, and therefore, the function K satisfies the nonlinear differential equations

$$\mathrm{L}_i K - \mathrm{M}_i(K) = 0, \quad i = 1, 2. \tag{16}$$

It follows that each solution of the linear integral equation (13) is a solution of the nonlinear differential equation (16).

Example 5. Let us consider the integral equation

$$K(x,y) = F(x,y) + \int_x^\infty K(x,z)F(z,y)\,dz \tag{17}$$

and write out some identities to be used in the sequel,

$$\partial_x^n \int_x^\infty K(x,z)F(z,y)\,dz = \int_x^\infty F(z,y)\partial_x^n K(x,z)\,dz + A_n, \tag{18}$$

$$\int_x^\infty K(x,z)\partial_x^n F(z,y)\,dz = (-1)^n \int_x^\infty F(z,y)\partial_z^n K(x,z)\,dz + B_n, \tag{19}$$

where the A_n are defined by the recurrence relations

$$A_1 = -K(x,x)F(x,y), \quad A_n = (A_{n-1})_x - F(x,y)[\partial_x^{n-1}K(x,z)]_{z=x},$$

and

$$B_1 = -K(x,x)F(x,y), \quad B_2 = -K(x,x)\partial_x F(x,y) + [\partial_z K(x,z)]_{z=x}F(x,y), \quad \ldots$$

Let us introduce an operator L_1 and require that F satisfy the linear equation

$$L_1 F \equiv (\partial_x^2 - \partial_y^2)F(x,y) = 0. \tag{20}$$

Applying the operator L_1 to (17) and taking into account (18), (19), we obtain

$$(\partial_x^2 - \partial_y^2)K(x,y) = \int_x^\infty F(x,z)(\partial_x^2 - \partial_y^2)K(x,z)\,dz - 2F(x,y)\frac{d}{dx}K(x,x).$$

Using the equation $F = (\mathrm{I} - \mathrm{A}_x)K$ and taking into account that the operator $\mathrm{I} - \mathrm{A}_x$ is invertible, we finally get

$$(\partial_x^2 - \partial_y^2)K(x,y) + u(x)K(x,y) = 0, \tag{21}$$

where the function $u(x)$ is defined by

$$u(x) = 2\frac{d}{dx}K(x,x). \tag{22}$$

Require that F satisfy the linear equation

$$L_2 F = (\partial_t + (\partial_x + \partial_y)^3)F = 0 \tag{23}$$

and apply the operator L_2 to (17) to obtain

$$\bigl(\partial_t + (\partial_x + \partial_y)^3\bigr)K(x,y) = \bigl(\partial_t + (\partial_x + \partial_y)^3\bigr)\int_x^\infty K(x,z)F(z,y)\,dz.$$

A procedure similar to the above calculations for the operator L_1 yields

$$K_t + (\partial_x + \partial_y)^3 K + 3u(\partial_x + \partial_y)K = 0. \tag{24}$$

For the characteristic $y = x$, equation (24) can be rewritten in terms of $u = 2(d/dx)K(x,x)$. Differentiating (24) with respect to x and rearranging terms, we arrive at the Korteweg–de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

Any function F satisfying the linear equations (20), (23) and rapidly decaying as $x \to +\infty$ generates a solution of the Korteweg–de Vries equation. To this end, one should solve the linear integral equation (17) for function K and express u through K by formula (22).

Example 6. Consider the integral equation

$$K(x,y) = F(x,y) + \frac{\sigma}{4}\int_x^\infty \int_x^\infty K(x,z)F(z,u)F(u,y)\,dz\,du, \tag{25}$$

where $\sigma = \pm 1$. Here and in what follows, the coefficients are chosen with a view to simplifying the calculations. Let the operator L_1 have the form

$$L_1 F = (\partial_x - \partial_y)F = 0, \tag{26}$$

which implies that

$$F(x,y) = F\Bigl(\frac{x+y}{2}\Bigr).$$

Shifting the lower limit of integration to zero, we rewrite equation (25) in the form

$$K(x,y) = F\Bigl(\frac{x+y}{2}\Bigr) + \frac{\sigma}{4}\int_0^\infty \int_0^\infty K(x,x+\zeta)F\Bigl(\frac{2x+\zeta+\eta}{2}\Bigr)F\Bigl(\frac{x+\eta+y}{2}\Bigr)\,d\zeta\,d\eta, \tag{27}$$

or, equivalently,

$$[(\mathrm{I} - \sigma\mathrm{A}_x)K](x,y) = F\Bigl(\frac{x+y}{2}\Bigr),$$

where the operator A_x is defined by

$$\mathrm{A}_x f(y) = \frac{1}{4}\int_0^\infty \int_0^\infty f(\zeta)F\Big(\frac{2x+\zeta+\eta}{2}\Big)F\Big(\frac{x+\eta+y}{2}\Big)\,d\zeta\,d\eta.$$

Introducing the function

$$K_2(x,z) = \int_0^\infty K(x,x+\zeta)F\Big(\frac{x+\zeta+z}{2}\Big)\,d\zeta \tag{28}$$

we can rewrite equation (25) as

$$K(x,y) = F\Big(\frac{x+y}{2}\Big) + \frac{\sigma}{4}\int_0^\infty K_2(x,x+\eta)F\Big(\frac{x+\eta+y}{2}\Big)\,d\eta. \tag{29}$$

Applying the operator L_1 of (26) to equation (29), and the operator $\partial_x+\partial_z$ to (28), and taking into account the invertibility of $\mathrm{I}-\sigma\mathrm{A}_x$, we find, after appropriate calculations, that

$$(\partial_x+\partial_y)K_2(x,y) = -2K(x,x)K(x,y), \tag{30}$$

$$(\partial_x-\partial_y)K(x,y) = -\frac{\sigma}{2}K(x,x)K_2(x,y). \tag{31}$$

Applying the operator $\partial_x+\partial_y$ to (27), we get

$$F'\Big(\frac{x+y}{2}\Big) = (\mathrm{I}-\sigma\mathrm{A}_x)\Big[(\partial_x+\partial_y)K(x,y)+\frac{\sigma}{2}K_2(x,x)K(x,y)\Big]. \tag{32}$$

Let us require that the function F satisfy the second linear equation

$$L_2F = (\partial_t + (\partial_x+\partial_y)^3)F = 0. \tag{33}$$

Applying the operator L_2 to equation (27) and taking into account the above auxiliary relations (30)–(32), we ultimately find that

$$[\partial_t + (\partial_x+\partial_y)^3]K(x,y) = 3\sigma K(x,x)K(x,y)\partial_x K(x,x) + 3\sigma K^2(x,x)(\partial_x+\partial_y)K(x,y) \tag{34}$$

for $y \ge x$. Now, by setting $q(x,t) = K(x,x;t)$, we rewrite equation (34), for $y=x$, in terms of the dependent variable q to obtain the modified Korteweg–de Vries equation

$$q_t + q_{xxx} = 6\sigma q^2 q_x. \tag{35}$$

Thus, each solution of the equations $\mathrm{L}_i F = 0$, $i=1,2$, with a sufficiently fast decay rate as $x\to\infty$ determines a solution of equation (35). Note that we have to solve the linear integral equation (25) at an intermediate step.

⊙ *References for Subsection* S.10.3: V. E. Zakharov and A. B. Shabat (1974), M. J. Ablowitz and H. Segur (1981), M. J. Ablowitz and P. A. Clarkson (1991).

S.11. Conservation Laws

S.11.1. Basic Definitions and Examples

Consider a partial differential equation with two independent variables

$$F\Big(x,t,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial t},\frac{\partial^2 w}{\partial x^2},\frac{\partial^2 w}{\partial x\partial t},\frac{\partial^2 w}{\partial x^2},\dots\Big) = 0. \tag{1}$$

A conservation law for this equation has the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0, \tag{2}$$

where

$$T = T\Big(x,t,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial t},\dots\Big), \qquad X = X\Big(x,t,w,\frac{\partial w}{\partial x},\frac{\partial w}{\partial t},\dots\Big). \tag{3}$$

The left-hand side of the conservation law (2) must vanish for all (sufficiently smooth) solutions of equation (1). In simplest cases, the substitution of relations (3) into the conservation law (2) followed by differentiation and elementary transformations leads to a relation that coincides with (1) up to a functional factor. The quantities T and X in (2) are called a density and a flow, respectively.

If the total variation of the quantity X on the interval $a \le x \le b$ is equal to zero, i.e., $X(a) = X(b)$, then the following "integral of motion" takes place:

$$\int_a^b T\,dx = \text{const} \qquad \text{(for all } t\text{)}. \tag{4}$$

For many specific equations, relations of the form (4) have a clear physical meaning and are used for approximate analytical solution of the corresponding problems, as well as for the verification of results obtained by numerical methods.

For nonstationary equations with n spatial variables $x_1, \dots, x_n$, conservation laws have the form

$$\frac{\partial T}{\partial t} + \sum_{k=1}^{n} \frac{\partial X_k}{\partial x_k} = 0.$$

Partial differential equations can have several (sometimes infinitely many) conservation laws or none at all.

Example 1. The Korteweg–de Vries equation

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} + 6w\frac{\partial w}{\partial x} = 0$$

admits infinitely many conservation laws of the form (2). The first three are determined by

$$\begin{aligned}
&T_1 = w, && X_1 = 3w^2 + w_{xx};\\
&T_2 = w^2, && X_2 = 4w^3 + 2ww_{xx} - w_x^2;\\
&T_3 = 2w^3 - w_x^2, && X_3 = 9w^4 + 6w^2 w_{xx} - 12ww_x^2 - 2w_x w_{xxx} + w_{xx}^2,
\end{aligned}$$

where the subscripts denote partial derivatives with respect to x.

Example 2. The sine-Gordon equation

$$\frac{\partial^2 w}{\partial x \partial t} - \sin w = 0$$

also has infinitely many conservation laws. The first three are described by the formulas

$$\begin{aligned}
&T_1 = w_x^2, && X_1 = 2\cos w;\\
&T_2 = w_x^4 - 4w_{xx}^2, && X_2 = 4w_x^2 \cos w;\\
&T_3 = 3w_x^6 - 12w_x^2 w_{xx}^2 + 16w_x^3 w_{xxx} + 24w_{xxx}^2, && X_3 = (2w_x^4 - 24w_{xx}^2)\cos w.
\end{aligned}$$

Example 3. The Monge–Ampère equation

$$\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = \frac{1}{x^4} f\left(\frac{y}{x}\right),$$

where $f(z)$ is an arbitrary function, admits the conservation law

$$\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y} + \frac{1}{x^3}\int_C^{y/x} f(z)\,dz\right) = 0.$$

⊙ *References for Subsection* S.11.1: G. B. Whitham (1965), R. M. Miura, C. S. Gardner, and M. D. Kruskal (1968), M. D. Kruskal, R. M. Miura, C. S. Gardner, and N. J. Zabusky (1970), A. C. Scott, F. Y. Chu, and D. W. McLaughlin (1973), J. L. Lamb (1974), R. K. Dodd and R. K. Bullough (1977), P. J. Olver (1986), N. H. Ibragimov (1994), S. E. Harris (1996), A. M. Vinogradov and I. S. Krasilshchik (1997), A. N. Kara and F. M. Mahomed (2002), B. J. Cantwell (2002).

S.11.2. Equations Admitting Variational Formulation. Noetherian Symmetries

Here, we consider second-order equations in two independent variables, x and y, and an unknown function, $w = w(x, y)$. We will deal with equations admitting the variational formulation of minimizing a functional of the form

$$Z[w] = \int_S L(x, y, w, w_x, w_y)\,dx\,dy. \tag{5}$$

The function $L = L(x, y, w, w_x, w_y)$ is called a Lagrangian.

It is well known that a minimum of the functional (5) corresponds to the Euler–Lagrange equation

$$\frac{\partial L}{\partial w} - D_x\left(\frac{\partial L}{\partial w_x}\right) - D_y\left(\frac{\partial L}{\partial w_y}\right) = 0, \tag{6}$$

where D_x and D_y are the total differential operators in x and y. Therefore, the original equation must be a consequence of equation (6).

A symmetry that preserves the differential form $\Omega = L(x, y, w, w_x, w_y)\,dx\,dy$ is called a Noetherian symmetry of the Lagrangian L. In order to obtain Noetherian symmetries, one should find point transformations

$$\bar{x} = f_1(x, y, w, \varepsilon), \quad \bar{y} = f_2(x, y, w, \varepsilon), \quad \bar{w} = g(x, y, w, \varepsilon) \tag{7}$$

such that preserve the differential form, $\bar{\Omega} = \Omega$, i.e.,

$$\bar{L}\,d\bar{x}\,d\bar{y} = L\,dx\,dy. \tag{8}$$

Calculating the differentials $d\bar{x}$, $d\bar{y}$ and taking into account (7), we obtain

$$d\bar{x} = D_x f_1\,dx, \quad d\bar{y} = D_y f_2\,dy,$$

and therefore, relation (8) can be rewritten as

$$(L - \bar{L} D_x f_1 D_y f_2)\,dx\,dy = 0,$$

which is equivalent to

$$L - \bar{L} D_x f_1 D_y f_2 = 0. \tag{9}$$

Let us associate the point transformation (7) with the prolongation operator

$$X = \xi\partial_x + \eta\partial_y + \zeta\partial_w + \zeta_1\partial_{w_x} + \zeta_2\partial_{w_y},$$

where the coordinates of the first prolongation, ζ_1 and ζ_2, are defined by formulas (13) from Subsection S.7.1. Then, by the usual procedure, from (9) one obtains the invariance condition in the form

$$X(L) + L(D_x\xi + D_y\eta) = 0. \tag{10}$$

Noetherian symmetries are determined by (10).

Each Noetherian symmetry operator X generates a conservation law,

$$D_x\left(L\xi + (\zeta - \xi w_x - \eta w_y)\frac{\partial L}{\partial w_x}\right) + D_y\left(L\eta + (\zeta - \xi w_x - \eta w_y)\frac{\partial L}{\partial w_y}\right) = 0.$$

Example 4. The equation of minimal surfaces

$$(1 + w_y^2)w_{xx} - 2w_x w_y w_{xy} + (1 + w_x^2)w_{yy} = 0$$

corresponds to the functional

$$Z[w] = \int_S \sqrt{1 + w_x^2 + w_y^2}\,dx\,dy$$

with Lagrangian $L = \sqrt{1 + w_x^2 + w_y^2}$. The admissible point operators

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x + y\partial_y + w\partial_w, \quad X_4 = y\partial_x - x\partial_y, \quad X_5 = \partial_w$$

are found by the procedure described in detail in Section S.7.1-2. These operators determine Noetherian symmetries and correspond to conservation laws:

$$\begin{aligned}
&X_1: \quad D_x\left(L - w_x\frac{\partial L}{\partial w_x}\right) + D_y\left(-w_x\frac{\partial L}{\partial w_y}\right) = 0,\\
&X_2: \quad D_x\left(-w_y\frac{\partial L}{\partial w_x}\right) + D_y\left(L - w_y\frac{\partial L}{\partial w_y}\right) = 0,\\
&X_3: \quad D_x\left(Lx + (w - xw_x - yw_y)\frac{\partial L}{\partial w_x}\right) + D_y\left(Ly + (w - xw_x - yw_y)\frac{\partial L}{\partial w_y}\right) = 0,\\
&X_4: \quad D_x\left(Ly + (yw_x - xw_y)\frac{\partial L}{\partial w_x}\right) + D_x\left(-Ly + (yw_x - xw_y)\frac{\partial L}{\partial w_y}\right) = 0,\\
&X_5: \quad D_x\left(\frac{w_x}{\sqrt{1 + w_x^2 + w_y^2}}\right) + D_y\left(\frac{w_y}{\sqrt{1 + w_x^2 + w_y^2}}\right) = 0.
\end{aligned}$$

⊙ *References for Subsection* S.11.2: A. M. Vinogradov (1984), P. J. Olver (1986), J. A. Cavalcante and K. Tenenblat (1988), N. H. Ibragimov (1994), A. M. Vinogradov and I. S. Krasilshchik (1997).

S.12. Hyperbolic Systems of Quasilinear Equations*

S.12.1. Conservation Laws. Some Examples

The main mathematical models in continuum mechanics and theoretical physics have the form of systems of conservation laws. Usually mass, momentum, and energy for phases and/or components are conserved.

We consider systems of conservation laws with the form

$$\frac{\partial \mathbf{G}(\mathbf{u})}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{u})}{\partial x} = 0, \tag{1}$$

where $\mathbf{u} = \mathbf{u}(x,t)$ is a vector function of two scalar variables, and $\mathbf{F} = \mathbf{F}(\mathbf{u})$ and $\mathbf{G} = \mathbf{G}(\mathbf{u})$ are vector functions,

$$\mathbf{u} = (u_1, \dots, u_n)^{\mathrm{T}}, \quad u_i = u_i(x,t);$$
$$\mathbf{F} = (F_1, \dots, F_n)^{\mathrm{T}}, \quad F_i = F_i(\mathbf{u});$$
$$\mathbf{G} = (G_1, \dots, G_n)^{\mathrm{T}}, \quad G_i = G_i(\mathbf{u}).$$

Here and henceforth, $(u_1, \dots, u_n)^{\mathrm{T}}$ stands for a column vector with components $u_1, \dots, u_n$.

For any **F** and **G** system (1) admits the following particular solutions:

$$\mathbf{u} = \mathbf{C},$$

where **C** is an arbitrary constant vector.

Example 1. Consider a single quasilinear equation of the special form

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0, \tag{2}$$

which is a special case of (1) with $n = 1$, $G(u) = u$, and $F = F(u)$.

Equation (2) represents a law of conservation of mass (or another quantity) and is often encountered in gas dynamics, fluid mechanics, wave theory, acoustics, multiphase flows, and chemical engineering. This equation is a model for numerous processes of mass transfer, including sorption and chromatography, two-phase flows in porous media, flow of water in river, road traffic development, flow of liquid films along inclined surfaces, etc. The independent variables t and x in equation (2) usually play the role of time and the spatial coordinate, respectively, $u = u(x,t)$ is the density of the quantity being transferred, and $F(u)$ is the flux of u.

Example 2. A one-dimensional ideal adiabatic (isentropic) gas flow is governed by the system of two equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0, \tag{3}$$

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2 + p(\rho))}{\partial x} = 0. \tag{4}$$

Here, $\rho = \rho(x,t)$ is the density, $v = v(x,t)$ is the velocity, and p is the pressure. Equation (3) represents the law of conservation of mass in fluid mechanics and is referred to as a continuity equation. Equation (4) represents the law of conservation of momentum. The equation of state is given in the form $p = p(\rho)$. For an ideal polytropic gas, $p = A\rho^{\gamma}$, where the constant γ is the adiabatic exponent.

Remark. System (3)–(4) with $\rho = h$ and $p(\rho) = \frac{1}{2}gh^2$, where v is the horizontal velocity averaged over the height h of the water level and g is the acceleration due to gravity, governs the dynamics of shallow water.

The origin of hyperbolic systems of conservation laws as mathematical models for physical phenomena is discussed extensively in the literature. The classical treatises by Courant and Hilbert (1989), Landau and Lifshitz (1987), and Whitham (1974) and also a recent comprehensive monograph by Dafermos (2000) should be mentioned. Conservation law systems for various gas flow regimes in Eulerian and Lagrangian coordinates are treated in the monographs Courant and Friedrichs (1985), Landau and Lifshitz (1987), Logan (1994), and Zel'dovich and Raizer (1968). Gas flows with chemical reactions (combustion and phase transitions) are discussed in the books by Zel'dovich and Raizer (1966, 1967), Zel'dovich, Barenblatt, Librovich, and Makhviladze (1985). Hyperbolic

* Section S.12 was written by P. G. Bedrikovetsky and A. P. Pires.

systems for chromatography are dealt with in Rhee, Aris, and Amundson (1970, 1986, 1989). The monographs Hanyga (1985) and Kulikovskii and Sveshnikova (1995) give a comprehensive presentation of the theory of elastic media. Both Barenblatt, Entov, and Ryzhik (1991) and Bedrikovetsky (1993) discuss hyperbolic systems for two-phase multi-component flows in porous media describing oil recovery processes. Traffic flow and shallow water mechanics are treated in Logan (1994) and Whitham (1974).

Methods of analytical integration for self-similar Riemann problems are presented in the monographs by Smoller (1983) and Dafermos (2000); non-self-similar problem integration methods for wave interactions are given in Glimm (1989), LeVeque (2002), and Bedrikovetsky (1993).

S.12.2. Cauchy Problem, Riemann Problem, and Initial Boundary Value Problem

Cauchy problem ($t \ge 0$, $-\infty < x < \infty$). Find a function $\mathbf{u} = \mathbf{u}(x,t)$ that solves system (1) for $t > 0$ and satisfies the initial condition

$$\mathbf{u} = \boldsymbol{\varphi}(x) \quad \text{at} \quad t = 0, \tag{5}$$

where $\boldsymbol{\varphi}(x)$ is a prescribed vector function. The Cauchy problem is also often referred to as an initial value problem.

Riemann problem ($t \ge 0$, $-\infty < x < \infty$). Find a function $\mathbf{u} = \mathbf{u}(x,t)$ that solves system (1) for $t > 0$ and satisfies the following initial condition of a special form:

$$\mathbf{u} = \begin{cases} \mathbf{u}_{\mathrm{L}} & \text{if } x < 0 \\ \mathbf{u}_{\mathrm{R}} & \text{if } x > 0 \end{cases} \quad \text{at} \quad t = 0. \tag{6}$$

Here, $\mathbf{u}_{\mathrm{L}}$ and $\mathbf{u}_{\mathrm{R}}$ are two prescribed constant vectors.

Initial-boundary value problem ($t \ge 0$, $x \ge 0$). Find a function $\mathbf{u} = \mathbf{u}(x,t)$ that solves system (1) for $t > 0$ and $x > 0$ and satisfies the following conditions:

$$\begin{aligned} \mathbf{u} &= \boldsymbol{\varphi}(x) \quad \text{at} \quad t = 0 \quad \text{(initial condition)}, \\ \mathbf{u} &= \boldsymbol{\psi}(t) \quad \text{at} \quad x = 0 \quad \text{(boundary condition)}. \end{aligned}$$

Here, $\boldsymbol{\varphi}(x)$ and $\boldsymbol{\psi}(t)$ are prescribed vector functions.

S.12.3. Characteristic Lines. Hyperbolic Systems. Riemann Invariants

Let us show that some systems of conservation laws can be represented as systems of ordinary differential equations along curves $x = x(t)$ called characteristic curves.

Differentiating both sides of system (1) yields

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{u}}{\partial x} = 0, \tag{7}$$

where $\mathbf{A} = \widetilde{\mathbf{G}}^{-1}(\mathbf{u})\widetilde{\mathbf{F}}(\mathbf{u})$, $\widetilde{\mathbf{F}}(\mathbf{u})$ is the matrix with entries $\frac{\partial F_i}{\partial u_j}$, $\widetilde{\mathbf{G}}(\mathbf{u})$ is the matrix with entries $\frac{\partial G_i}{\partial u_j}$, and $\widetilde{\mathbf{G}}^{-1}$ is the inverse of the matrix $\widetilde{\mathbf{G}}$.

Let us multiply each scalar equation in (7) by $b_i = b_i(\mathbf{u})$ and take the sum. On rearranging terms under the summation sign, we obtain

$$\sum_{i=1}^{n} b_i \frac{\partial u_i}{\partial t} + \sum_{i,j=1}^{n} b_j a_{ji} \frac{\partial u_i}{\partial x} = 0, \tag{8}$$

where the $a_{ij} = a_{ij}(\mathbf{u})$ are the entries of the matrix $\mathbf{A}$.

If $\mathbf{b} = (b_1, \dots, b_n)$ is a left eigenvector of the matrix $\mathbf{A}(\mathbf{u})$ that corresponds to an eigenvalue $\lambda = \lambda(\mathbf{u})$, so that

$$\sum_{j=1}^{n} b_j a_{ji} = \lambda b_i,$$

then equation (8) can be rewritten in the form

$$\sum_{i=1}^{n} b_i \left(\frac{\partial u_i}{\partial t} + \lambda \frac{\partial u_i}{\partial x} \right) = 0. \qquad (9)$$

Thus, system (7) is transformed to a linear combination of total derivatives of the unknowns u_i with respect to t along the direction $(\lambda, 1)$ on the plane (x, t), i.e., the total time derivatives are taken along the trajectories having the velocity λ:

$$\sum_{i=1}^{n} b_i \frac{du_i}{dt} = 0, \qquad \frac{dx}{dt} = \lambda, \qquad (10)$$

where

$$b_i = b_i(\mathbf{u}), \quad \lambda = \lambda(\mathbf{u}), \quad x = x(t), \quad \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + \frac{dx}{dt}\frac{\partial u_i}{\partial x}.$$

Equations (10) are called differential relations on characteristics. The second equation in (10) explains why an eigenvalue λ is called a characteristic velocity.

The system of quasilinear equations (7) is called hyperbolic if the following two conditions are satisfied:

1°. All eigenvalues $\lambda_k = \lambda_k(\mathbf{u})$ $(k = 1, \dots, n)$ of the matrix $\mathbf{A}(\mathbf{u})$ are real.

2°. There is a basis $\{\mathbf{b}^1, \dots, \mathbf{b}^n\} \subset E^n$ formed by n left eigenvectors of $\mathbf{A}(\mathbf{u})$ and subjected to a normalization condition; the symbol E^n stands for the n-dimensional Euclidean space.

Let us assume that the $n \times n$ hyperbolic system (7) has n distinct eigenvalues $\lambda_k(\mathbf{u})$, $k = 1, \dots, n$. A trajectory $x(t)$ with velocity $\lambda_k(\mathbf{u})$ that is a solution of system (10) is called the kth characteristic direction. The eigenvectors $\mathbf{b}^k(\mathbf{u})$ that correspond to the eigenvalues $\lambda_k(\mathbf{u})$, respectively, are linearly independent.

If all eigenvalues are distinct for any $\mathbf{u} = (u_1, \dots, u_n)^{\mathrm{T}} \subset \mathbb{R}^n$, they can be enumerated in order of increasing values, so that $\lambda_1(\mathbf{u}) < \dots < \lambda_n(\mathbf{u})$, and system (7) is called strictly hyperbolic.

If all characteristic velocities $\lambda = \lambda_k$ of the hyperbolic system (7) are positive, the following initial-boundary value problem can be posed:

$$\mathbf{u} = \mathbf{u}_{\mathrm{L}} \quad \text{at} \quad t = 0, \qquad \mathbf{u} = \mathbf{u}_{\mathrm{R}} \quad \text{at} \quad x = 0.$$

Remark 1. If the hyperbolic system (7) is linear and the coefficients of the matrix $\mathbf{A}$ are constant, then the eigenvalues λ_k are constant and the characteristic lines in the (x, t) plane become straight lines:

$$x = \lambda_k t + \text{const}.$$

Since all eigenvalues λ_k are different, the general solution of system (7) can be represented as the sum of particular solutions as follows:

$$\mathbf{u} = \phi_1(x - \lambda_1 t)\mathbf{r}^1 + \dots + \phi_n(x - \lambda_n t)\mathbf{r}^n, \qquad (11)$$

where the $\phi_k(\xi_k)$ are arbitrary functions, $\xi_k = x - \lambda_k t$, and $\mathbf{r}^k$ is the right eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ_k, $k = 1, \dots, n$. The particular solutions $\mathbf{u}_k = \phi_k(x - \lambda_k t)\mathbf{r}^k$ are called traveling wave solutions. Each of these solutions represents a wave that travels in the $\mathbf{r}^k$-direction with velocity λ_k.

Remark 2. The characteristic form (9) of the hyperbolic system (7) forms the basis for the numerical characteristics method which allows the solution of system (7) in its domain of continuity.

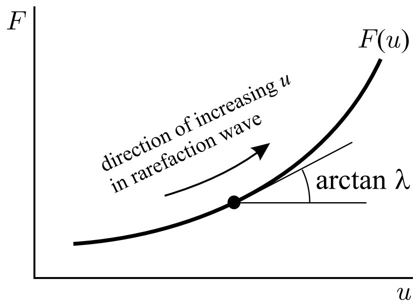

Figure 6. Characteristic velocity for a single quasilinear (hyperbolic) equation (2).

Suppose that we already have a solution $\mathbf{u}(x,t)$ for all values of x and a fixed time t. To construct a solution at a point $(x,\ t+\Delta t)$, we find the points $(x-\lambda_k\Delta t,\ t)$ from which the characteristics arrive at the point $(x,\ t+\Delta t)$. Since the $\mathbf{u}(x-\lambda_k\Delta t,\ t)$ are known, relations (10) can be regarded as a system of n linear equations in the n unknowns $\mathbf{u}(x,t+\Delta t)$. Thus, a solution for the time $t+\Delta t$ can be found.

Consider small perturbations of a solution to system (7). Substitute $\mathbf{u}=\mathbf{u}_0+\delta\mathbf{u}$ into (7), where $\mathbf{u}_0=\mathbf{u}_0(x,t)$ is a solution of system (7) and $\delta\mathbf{u}=(\delta u_1,\dots,\delta u_n)^{\mathrm{T}}$ is a small perturbation, $|\mathbf{u}_0|\gg|\delta\mathbf{u}|$. Neglecting the terms of higher order than the first term in $|\delta\mathbf{u}|$, we obtain a system of linear equations in the form

$$\frac{\partial\delta u_i}{\partial t}+\sum_{j=1}^{n}a_{ij}\frac{\partial\delta u_j}{\partial x}=-\sum_{j,k=1}^{n}\frac{\partial a_{ij}}{\partial u_k}\frac{\partial u_j}{\partial x}\delta u_k,\qquad i=1,\dots,n, \tag{12}$$

where the $a_{ij}=a_{ij}(\mathbf{u}_0)$ are the entries of the matrix $\mathbf{A}$ at the point $\mathbf{u}_0$. If $\mathbf{u}_0$ is a constant vector, then the right-hand side of the linearized equation (12) is zero and its general solution can be represented as a superposition of n traveling waves; see formula (11).

Example 3. For the case of a single hyperbolic equation (2), relations (10) become

$$\frac{du}{dt}=0,\qquad \frac{dx}{dt}=F'(u). \tag{13}$$

It has been taken into account here that $\lambda=F'(u)$; the prime denotes the derivative with respect to u.

The second equation in (13) shows that the characteristic velocity equals the tangent to the flux function at the point $u=u(x,t)$; see Fig. 6. There is one characteristic velocity for one equation, and the unknown function is constant along the characteristic (first equation in (13)). Therefore, the characteristic velocity is also constant (second equation in (13)), and the characteristic is a straight line. This allows the construction of an exact solution to a Cauchy problem for (2) whenever the characteristic velocity of the initial condition (5) increases monotonically in x, $[F'(\varphi(x))]'=F''(\varphi)\varphi'(x)>0$. In this case, a unique characteristic straight line crosses an arbitrary point (x,t), and the solution is constant along this line. As a result, the solution can be represented in the parametric form

$$\begin{aligned}x&=\zeta+F'\big(\varphi(\zeta)\big)t,\\ u&=\varphi(\zeta).\end{aligned} \tag{14}$$

The first equation in (14) is a transcendental equation in the unknown $\zeta=\zeta(x,t)$, and the second one allows the calculation of the unknown $u=u(x,t)$ from the initial condition (5).

Example 4. Adiabatic gas flow is governed by the system of equations (3)–(4). The vector $\mathbf{u}$ and the matrix $\mathbf{A}(\mathbf{u})$, which arise in the transformed system (7), become

$$\mathbf{u}=\begin{pmatrix}\rho\\ v\end{pmatrix},\qquad \mathbf{A}=\begin{pmatrix}v & \rho\\ p'/\rho & v\end{pmatrix},$$

where $p'=p'(\rho)$. The eigenvalues and the corresponding left eigenvectors are

$$\lambda=v\pm\sqrt{p'},\qquad \mathbf{b}=\big(\sqrt{p'},\ \pm\rho\big).$$

The linear combination of the equations (3)–(4) with coefficients b_i is:

$$\sqrt{p'}\,\frac{d\rho}{dt}\pm\rho\frac{dv}{dt}\equiv\pm\rho\frac{d}{dt}\left(v\pm\int\frac{\sqrt{p'}}{\rho}\,d\rho\right)=0.$$

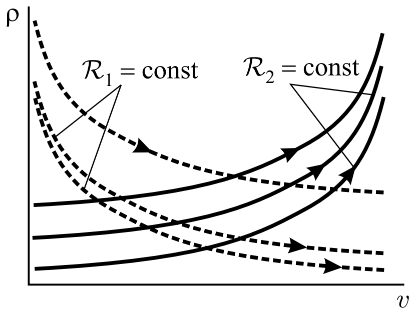

Figure 7. Loci of points where Riemann invariants are constant.

For an ideal polytropic gas, with $p = A\rho^\gamma$, the eigenvalues and the corresponding left eigenvectors are:

$$\lambda = v \pm \sqrt{A\gamma\rho^{\gamma-1}}, \quad \mathbf{b} = \left(\sqrt{A\gamma\rho^{\gamma-1}},\ \pm\rho\right).$$

In this case, the differential relations on the characteristics (10) become

$$\begin{aligned} &\sqrt{A\gamma\rho^{\gamma-1}}\,\frac{d\rho}{dt} \pm \rho\frac{dv}{dt} \equiv \pm\rho\frac{d}{dt}\left(v \pm \frac{2\sqrt{A\gamma\rho^{\gamma-1}}}{\gamma-1}\right) = 0,\\ &\frac{dx}{dt} = v \pm \sqrt{A\gamma\rho^{\gamma-1}}. \end{aligned} \tag{15}$$

The relations on the characteristics (10) can be simplified if system (7) admits Riemann invariants. Consider the differential $b_i^k(\mathbf{u})\,du_i$, where $\mathbf{b}^k(\mathbf{u})$ is a left eigenvector corresponding to the eigenvalue $\lambda_k(\mathbf{u})$. Assume that this differential admits an integrating multiplier $\mu^k(\mathbf{u})$ or, in other words, the differential can be represented in the form

$$\sum_{i=1}^{n} b_i^k(\mathbf{u})\,du_i = \mu^k(\mathbf{u})\,d\mathcal{R}_k(\mathbf{u}).$$

The function $\mathcal{R}_k(\mathbf{u})$ is called the kth Riemann invariant. The integrating multiplier $\mu^k(\mathbf{u})$ can be found from Maxwell's relations:

$$\frac{\partial}{\partial u_j}\left(\frac{b_i^k}{\mu^k}\right) = \frac{\partial}{\partial u_i}\left(\frac{b_j^k}{\mu^k}\right).$$

From (10) it follows that each Riemann invariant is constant along the corresponding characteristic curve.

Two Riemann invariants can always be constructed for a system of two equations, since the differential of two variables always admits an integrating multiplier. In this case, the change of variables

$$u_i = u_i(\mathcal{R}_1, \mathcal{R}_2), \quad i = 1,\ 2,$$

brings the hyperbolic system to

$$\frac{\partial \mathcal{R}_i}{\partial t} + \lambda_i(\mathcal{R}_1, \mathcal{R}_2)\frac{\partial \mathcal{R}_i}{\partial x} = 0, \quad i = 1,\ 2. \tag{16}$$

Example 5. As follows from (13), the Riemann invariant for the single equation (2) is the density $u(x,t)$, which is constant along characteristics.

Example 6. Let us consider an adiabatic gas flow (see Example 4). From (15) it follows that the Riemann invariants are:

$$\mathcal{R} = v \pm \int \frac{\sqrt{p'}}{\rho}\,d\rho.$$

For an ideal polytropic gas, with $p = A\rho^\gamma$, the Riemann invariants are constant along characteristics:

$$\mathcal{R} = v \pm \frac{2\sqrt{A\gamma\rho^{\gamma-1}}}{\gamma-1} = \text{const} \quad \text{along} \quad \frac{dx}{dt} = v \pm \sqrt{A\gamma\rho^{\gamma-1}}. \tag{17}$$

Figure 7 shows lines of $\mathcal{R}_i$ = const on the phase plane (v, ρ).

Example 7. The system of equations describing one-dimensional longitudinal oscillations of an elastic bar consists of the equations of balance of mass and momentum:

$$\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} = 0,$$
$$\frac{\partial v}{\partial t} - \frac{\partial \sigma(u)}{\partial x} = 0.$$

Here, u is the deformation gradient (strain), v is the strain rate, and $\sigma(u)$ is the stress.

The eigenvalues and the corresponding left eigenvectors are given by

$$\lambda = \pm\sqrt{\sigma'(u)}, \qquad \mathbf{b} = \left(\sqrt{\sigma'(u)},\ \mp 1\right).$$

The Riemann invariants are constant along characteristics:

$$\mathcal{R} = v \mp \int \sqrt{\sigma'(u)}\, du = \text{const} \quad \text{along} \quad \frac{dx}{dt} = \pm\sqrt{\sigma'(u)}.$$

⊙ *References for Subsection* S.12.3: I. M. Gelfand (1959), P. Lax (1973), G. B. Whitham (1974), A. Jeffrey (1976), F. John (1982), B. L. Rozhdestvenskii and N. N. Yanenko (1983), J. Smoller (1983), R. Courant and D. Hilbert (1985), D. Serre (1996), C. M. Dafermos (2000), A. G. Kulikovskii, N. V. Pogorelov, and A. Yu. Semenov (2001), R. J. LeVeque (2002).

S.12.4. Self Similar Continuous Solutions. Rarefaction Waves

The transformation $(x,t) \to (kx,kt)$, where k is any positive number, changes neither system (1) nor the initial conditions (6). From the uniqueness of the Riemann problem solution it follows that the unknown $\mathbf{u}(x,t)$ depends on a single variable, $\xi = x/t$. Without loss of generality, we consider the case $\mathbf{G}(\mathbf{u}) = \mathbf{u}$. The substitution of the self-similar form $\mathbf{u}(x,t) = \mathbf{u}(\xi)$ into (1) yields

$$\left(\widetilde{\mathbf{F}} - \xi\mathbf{I}\right)\frac{d\mathbf{u}}{d\xi} = 0, \tag{18}$$

where $\widetilde{\mathbf{F}} = \widetilde{\mathbf{F}}(\mathbf{u})$ is the matrix with entries $F_{ij} = \frac{\partial F_i}{\partial u_j}$ and $\mathbf{I}$ is the identity matrix.

Hence, the velocity vector for the continuous solution $\mathbf{u}(\xi)$ is a right eigenvector of the matrix $\widetilde{\mathbf{F}}$ for any point $\mathbf{u}$, and the corresponding eigenvalue equals the self-similar coordinate:

$$\xi = \lambda_k, \qquad \frac{d\mathbf{u}}{d\xi} = \alpha\mathbf{r}^k. \tag{19}$$

Here, $\lambda_k = \lambda_k(\mathbf{u})$ is a root of the algebraic equation $\det|\widetilde{\mathbf{F}} - \lambda\mathbf{I}| = 0$, $\mathbf{r}^k = \mathbf{r}^k(\mathbf{u})$ is a solution of the corresponding degenerate linear system of equations $\left(\widetilde{\mathbf{F}} - \lambda\mathbf{I}\right)\mathbf{r} = 0$, and $\alpha = \alpha(\mathbf{u})$ is a positive function, which will be defined below.

Differentiating both sides of the first equation (19) with respect to ξ yields

$$\alpha = \frac{1}{\langle\nabla\lambda_k, \mathbf{r}^k\rangle}, \qquad \langle\nabla\lambda_k, \mathbf{r}^k\rangle = \frac{\partial\lambda_k}{\partial u_1}r_1^k + \cdots + \frac{\partial\lambda_k}{\partial u_n}r_n^k.$$

Here, $\langle\mathbf{x}, \mathbf{y}\rangle$ stands for the scalar product of two vectors $\mathbf{x}$ and $\mathbf{y}$ in the n-dimensional Euclidean space.

Any $n\times n$ hyperbolic system allows for n continuous solutions (of system (18)) corresponding to n characteristic velocities $\lambda = \lambda_k$. The continuous solutions are determined by n systems of ordinary differential equations. Each system is represented by a phase portrait in the n-dimensional $\mathbf{u}$-space. A solution/trajectory which corresponds to a characteristic velocity λ_k is called a kth rarefaction wave.

Example 8. On calculating the multiplier α for equation (2), one obtains a rarefaction wave expression:

$$\xi = F'(u), \qquad \frac{du}{d\xi} = \frac{1}{F''(u)}. \tag{20}$$

Equations (20) show that the self-similar coordinate ξ is an eigenvalue, which is equal to the tangent to the flux function at the point $u = u(\xi)$; see Fig. 6, where $\lambda = \xi$. The trajectory $(u(\xi), F(u(\xi)))$ in the plane (u, F) lies on the graph of the flux

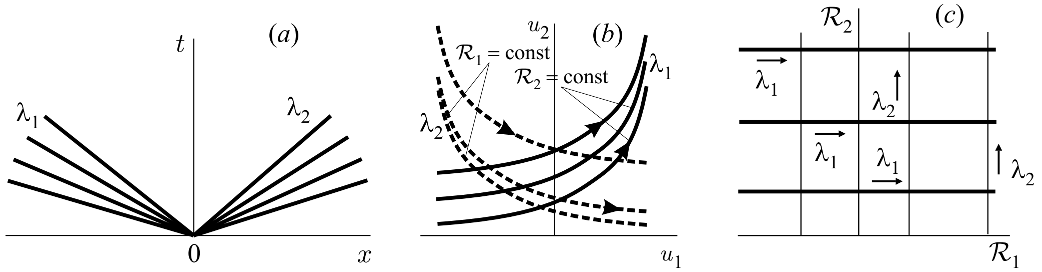

Figure 8. Mapping from the plane (x,t) to the hodograph plane (u_1,u_2) and further to the plane of Riemann invariants $(\mathcal{R}_1,\mathcal{R}_2)$: (a) characteristics in two centered rarefaction waves; (b) trajectories of two families of rarefaction waves; (c) Riemann invariants are constant along the characteristics and along rarefaction waves for 2×2 systems.

function $F = F(u)$. As follows from (20), $u = u(\xi)$ increases in the intervals of concavity of the curve $F = F(u)$, $F''(u) > 0$; see Fig. 6.

Let us show that the Riemann invariants are constant along the rarefaction waves for 2×2 hyperbolic systems. The substitution of the self-similar solution form $\mathbf{u} = \mathbf{u}(\xi)$, $\xi = x/t$, into system (16) results in the following system of two ordinary differential equations:

$$\big[\xi - \lambda_i(\mathcal{R}_1,\mathcal{R}_2)\big]\frac{d\mathcal{R}_i}{d\xi} = 0, \qquad i = 1,\,2. \tag{21}$$

The equality $\xi = \lambda_1(\mathcal{R}_1,\mathcal{R}_2)$ takes place along the first rarefaction wave. Hence, the first factor in the second equation of (21) is nonzero. Therefore, the second factor in the second equation of (21) is zero. It follows that $\mathcal{R}_2 = \text{const}$ along the first rarefaction wave. Along the second rarefaction wave, $\mathcal{R}_1$ is constant. Figure 8a shows two rarefaction wave families that correspond to speeds λ_1 and λ_2. A continuous solution of a 2×2 system $u_i = u_i(x,t)$, $i = 1,\,2$, realizes the mapping $(x,t)\to(u_1,u_2)$. The inverse of a characteristic with speed λ_i is the curve $\mathcal{R}_i(u_1,u_2) = \text{const}$ (see Fig. 8b). The expressions $\mathcal{R}_i = \mathcal{R}_i(u_1,u_2)$ realize the mapping $(u_1,u_2)\to(\mathcal{R}_1,\mathcal{R}_2)$. The inverse of a characteristic with speed λ_i is a set of straight lines that are parallel to the $\mathcal{R}_i$-axis (see Fig. 8c).

Example 9. For an adiabatic gas flow [see system (3)–(4)], the rarefaction waves are found from (19) by calculating the right eigenvectors of the matrix $A(\mathbf{u})$ and the function $\alpha(\mathbf{u})$ (see Example 4) to obtain

$$\frac{d\rho}{d\xi} = \pm\frac{\rho}{\rho(\sqrt{p'}\,)' + \sqrt{p'}}, \qquad \frac{dv}{d\xi} = \frac{\sqrt{p'}}{\rho(\sqrt{p'}\,)' + \sqrt{p'}}, \qquad p = p(\rho). \tag{22}$$

Here, the upper sign corresponds to the first eigenvalue and the lower sign, to the second eigenvalue.

Eliminating ξ from system (22), we obtain the first-order separable equation

$$\frac{dv}{d\rho} = \pm\frac{\sqrt{p'(\rho)}}{\rho}. \tag{23}$$

Integrating (23) yields

$$v \mp \int \frac{\sqrt{p'(\rho)}}{\rho}\,d\rho = \text{const}\,. \tag{24}$$

The left-hand side of (24) taken with the minus sign is equal to the second Riemann invariant, and that taken with the plus sign is equal to the first Riemann invariant. Hence, the second Riemann invariant is constant along the first rarefaction wave and the first Riemann invariant is constant along the second rarefaction wave.

The expressions for Riemann invariants for an ideal polytropic gas are given by formula (17). The trajectories of the rarefaction waves are given by the lines where the Riemann invariants are constant. Figure 7 presents the rarefaction waves for the first characteristic speed (solid lines), where the second Riemann invariant is constant. The dashed lines show the rarefaction waves of the second characteristic speed, where the first Riemann invariant is constant. The arrows show the directions of increasing the self-similar coordinate. Both v and ρ increase along the first rarefaction in the direction shown in Fig. 7, and the first eigenvalue $\lambda_1 = v + \sqrt{A\gamma\rho^{\gamma-1}}$ also increases. Along the second rarefaction, v increases and ρ decreases, and, hence, the second eigenvalue $\lambda_2 = v - \sqrt{A\gamma\rho^{\gamma-1}}$ increases.

⊙ *References for Subsection* S.12.4: P. Lax (1973), G. B. Whitham (1974), A. Jeffrey (1976), F. John (1982), B. L. Rozhdestvenskii and N. N. Yanenko (1983), R. Courant and R. Friedrichs (1985), R. J. LeVeque (2002).

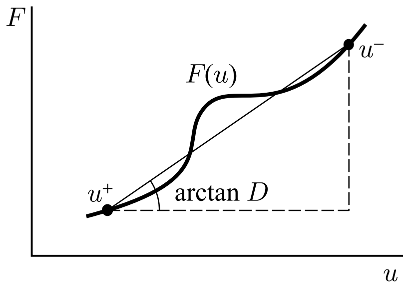

Figure 9. Illustration to Rankine–Hugoniot and Lax conditions for a shock wave in a scalar conservation law (2).

S.12.5. Shock Waves. Rankine–Hugoniot Jump Conditions

In general, the basic hyperbolic equations (1) are obtained from continuity equations, i.e., balances of mass, momentum, and energy for continuous flows. Continuous solutions of these equations were studied in Subsections S.12.3 and S.12.4. We now derive balance conditions on shocks.

Let us consider a discontinuity along a trajectory $x_{\rm f}(t)$ and obtain the mass balance condition along a discontinuity (shock wave). The region $x > x_{\rm f}(t)$ is conventionally assumed to lie ahead of the shock, and the region $x < x_{\rm f}(t)$ is assumed to lie behind the shock. The shock speed D is determined by the relation

$$D = \frac{dx_{\rm f}}{dt}.$$

To refer the value of a quantity, A, behind the shock, the minus superscript will be used, A^-, since this value corresponds to negative x in the initial value formulation. Likewise, the value of A ahead of the shock will be denoted A^+. In particular, the density and the velocity ahead of the shock are denoted ρ^+ and v^+, while those behind the shock are ρ^- and v^-.

For an abitrary system of the form (1), the balance equations for a shock can be represented as

$$[G_i(\mathbf{u})]D = [F_i(\mathbf{u})], \qquad i = 1, \dots, n, \tag{25}$$

where $[A] = A^+ - A^-$ stands for the jump of a quantity A at the shock. The equations of (25) are called the Rankine–Hugoniot jump conditions.

Example 10. The Rankine–Hugoniot condition for the single equation (2) reads as follows:

$$[u]D = [F]. \tag{26}$$

The shock speed is equal to the slope of the line connecting the points $(u^-, F(u^-))$ and $(u^+, F(u^+))$ in the (u, F) plane (see Fig. 9).

Example 11. The Rankine–Hugoniot conditions for an isentropic gas flow (3)–(4) follow from (25). We have

$$\begin{aligned} [\rho]D &= [\rho v], \\ [\rho v]D &= [\rho v^2 + p(\rho)]. \end{aligned} \tag{27}$$

Eliminating the shock speed D from (27), we obtain the equation

$$[v] = \pm\sqrt{\frac{[\rho][p(\rho)]}{\rho^-\rho^+}}. \tag{28}$$

Each of the signs before the square root in (28) corresponds to a branch of the locus of points that can be connected with a given point (v^-, ρ^-) by a shock (see Fig. 10a). For an ideal polytropic gas ($p = A\rho^\gamma$), relations (28) can be transformed to

$$v^+ - v^- = \pm\sqrt{\frac{A(\rho^+ - \rho^-)\big((\rho^+)^\gamma - (\rho^-)^\gamma\big)}{\rho^-\rho^+}}. \tag{29}$$

Let us determine the set of states (v^+, ρ^+) reachable by a shock from a given point (v^-, ρ^-). Express the point (v^+, ρ^+) ahead of the shock via the solution of the transcendental system (27) to obtain

$$v^+ = v^+(v^-, \rho^-, D), \qquad \rho^+ = \rho^+(v^-, \rho^-, D). \tag{30}$$

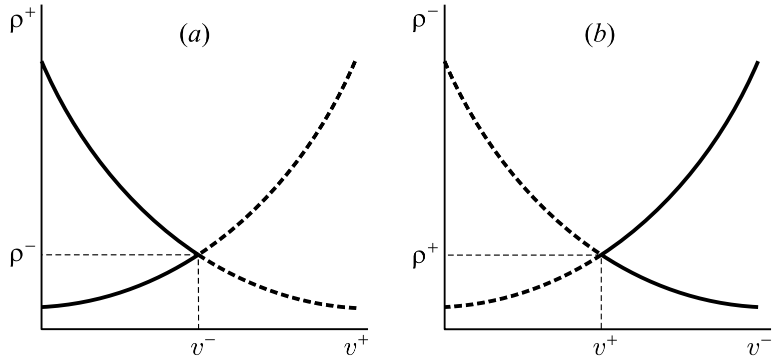

Figure 10. Loci of points that can be connected by a shock wave: (*a*) with a given state (v^-, ρ^-) and (*b*) with a given state (v^+, ρ^+). The solid lines correspond, respectively, to the minus and plus sign in formula (29) before the radical for cases (*a*) and (*b*).

The graphs of the solution determined by (29), or (28), are shown in Figs. 10*a* and 10*b*. The solid lines correspond to the minus sign before the radical and represent stable (evolutionary) shocks, while the dashed lines correspond to the plus sign and represent unstable (nonevolutionary) shocks; see the next subsection.

Consider the locus of points $\mathbf{u}^+$ and a rarefaction wave trajectory near a point $\mathbf{u}^-$ in the space $\mathbf{u} = (u_1, \dots, u_n)^{\mathrm{T}}$. These two curves have the same tangent vector at the point $\mathbf{u}^-$. In order to prove this fact, let us consider small-amplitude shocks. Setting $G_i(\mathbf{u}) = u_i$ in (25) and retaining only the leading term in the expansion in powers of $|\mathbf{u}^+ - \mathbf{u}^-| \ll 1$, we obtain

$$(\widetilde{\mathbf{F}} - D\mathbf{I})[\mathbf{u}] = 0,$$

where the same notation as in (18) is used. Hence, the vector $[\mathbf{u}]$ is a right eigenvector of the matrix $\widetilde{\mathbf{F}} = \widetilde{\mathbf{F}}(\mathbf{u})$. Therefore, it coincides with the rarefaction wave vector. The shock speed D tends to an eigenvalue at the point $\mathbf{u}^+$ (or $\mathbf{u}^-$).

The set of points, or the locus of states, $\mathbf{u}^+ = \mathbf{u}^+(\mathbf{u}^-, D)$ is a solution of the transcendental system of n equations (25). In general, the transcendental system has n roots. It follows that there should exist n shock curves $\mathbf{u}^+ = \mathbf{u}^+(\mathbf{u}^-, D)$. We call a curve the ith shock if it is tangent to the ith rarefaction wave at $\mathbf{u}^-$.

⊙ *References for Subsection* S.12.5: O. A. Oleinik (1957), I. M. Gelfand (1959), P. Lax (1973), A. G. Kulikovskii (1979), C. M. Dafermos (1983), B. L. Rozhdestvenskii and N. N. Yanenko (1983), J. Smoller (1983), R. Courant and R. Friedrichs (1985), L. D. Landau and E. M. Lifshitz (1987), D. J. Logan (1994), E. Godlewski and P.-A. Raviart (1996), A. G. Kulikovskii, N. V. Pogorelov, and A. Yu. Semenov (2001), A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

S.12.6. Evolutionary Shocks. Lax Condition (Various Formulations)

In general, discontinuities of solutions are surfaces where conditions are imposed that relate the quantities on both sides of the surfaces. For hyperbolic systems in the conservation-law form (1), these relations have the form (25) and involve the discontinuity velocity D.

The evolutionary conditions are necessary conditions for unique solvability of the problem of the discontinuity interaction with small perturbations depending on the x-coordinate normal to the discontinuity surface. For hyperbolic systems, a one-dimensional small perturbation can be represented as a superposition of n waves, each being a traveling wave propagating at a characteristic velocity $\lambda_i^{\mp}$. This allows us to classify all these waves into incoming and outgoing ones, depending on the sign of the difference $\lambda_i^{\mp} - D$. Incoming waves are fully determined by the initial conditions, while outgoing ones must be determined from the linearized boundary conditions at the shock.

We consider below the stability of a shock with respect to a small perturbation. This kind of stability is determined by incoming waves. For this reason, we focus below on incoming waves.

Let m^+ and m^- be the numbers of incoming waves from the right and left of the shock, respectively. It can be shown that if the relation

$$m^+ + m^- - 1 = n \tag{31}$$

holds, the problem of the discontinuity interaction with small perturbations is uniquely solvable. Relation (31) is called the Lax condition. If (31) holds, the corresponding discontinuity is called evolutionary; otherwise, it is called nonevolutionary. For evolutionary discontinuities, small incoming perturbations generate small outgoing perturbations and small changes in the discontinuity velocity.

For a single equation (2), it follows from (31) that the two waves on both sides of an evolutionary discontinuity must be incoming.

If

$$m^+ + m^- - 1 > n,$$

then either such discontinuities do not exist or the perturbed quantities cannot be uniquely determined (the given conditions are underdetermined).

If

$$m^+ + m^- - 1 < n,$$

then the problem of the discontinuity interaction with small perturbations has no solution in the linear approximation. Previous studies of various physical problems have shown that the interaction of nonevolutionary discontinuities with small perturbations results in their disintegration into two or more evolutionary discontinuities.

The evolutionary condition (31) can be rewritten in the form of inequalities relating the shock speed D and the velocities $\lambda_i^{\mp}$ of small disturbances. Let us enumerate the characteristic velocities on both sides of the discontinuity so that

$$\lambda_1(\mathbf{u}) \le \lambda_2(\mathbf{u}) \le \cdots \le \lambda_n(\mathbf{u}).$$

A shock is called a k-shock if both kth characteristics are incoming:

$$\begin{aligned} &\text{if } \ i > k, \ \text{ then } \ D < \lambda_i^{\mp};\\ &\text{if } \ i < k, \ \text{ then } \ D > \lambda_i^{\mp};\\ &\text{if } \ i = k, \ \text{ then } \ \lambda_i^{+} < D < \lambda_i^{-}. \end{aligned} \tag{32}$$

Below is another, equivalent statement of the Lax condition: $n+1$ inequalities out of the $2n$ inequalities

$$\lambda_k^+ \le D \le \lambda_k^- \qquad (k = 1, \dots, n) \tag{33}$$

must hold.

Example 12. The Lax condition (33) for a single equation (2) takes the form

$$F'(u^+) \le D \le F'(u^-). \tag{34}$$

From the Rankine–Hugoniot condition for one scalar equation (26) it follows that the shock speed D on the plane (u, F) is equal to the slope of the line segment connecting the "plus" and "minus" points.

The graphical interpretation of condition (34) in the plane (u, F) is as follows: the slope of the segment connecting the points with coordinates $(u^-, F(u^-))$ and $(u^+, F(u^+))$ is less than the slope of the flux curve $F(u)$ at the point $(u^-, F(u^-))$ and greater than the slope of $F(u)$ at the point $(u^+, F(u^+))$ (see Fig. 9).

Example 13. The Lax condition for the adiabatic gas flow equations (3)–(4) are obtained by substituting the eigenvalue expressions $\lambda = v \pm \sqrt{p'(\rho)}$ (see Example 4) into inequalities (33). We have

$$v^+ \pm \sqrt{p'(\rho^+)} < D < v^- \pm \sqrt{p'(\rho^-)}. \tag{35}$$

The shock evolutionarity requires that three of the four inequalities in (35) hold. Substituting the equation of state for a polytropic ideal gas, $p = Ar^\gamma$, into (35), we obtain the following evolutionarity criterion:

$$v^+ \pm \sqrt{A\gamma(\rho^+)^{\gamma-1}} < D < v^- \pm \sqrt{A\gamma(\rho^-)^{\gamma-1}}. \tag{36}$$

For the adiabatic gas flow system (3), (4), the solution vector is $\mathbf{u} = (\rho, v)^{\mathrm{T}}$. Figure 10*a* shows the locus of points $\mathbf{u}^{+}$ that can be connected by a shock to the point $\mathbf{u}^{-}$; it is divided into the evolutionary part (solid line) and nonevolutionary part (dashed line). It can be shown that a shock issuing from the point (v^{-}, ρ^{-}) and passing through any point (v^{+}, ρ^{+}) of the solid part of the locus of first-family shocks obeys the Lax conditions (36). The shock speed D of the first family decreases from $\lambda_1(\mathbf{u}^{-})$, for points $\mathbf{u}^{+}$ tending to point $\mathbf{u}^{-}$, to v^{-} as $\rho^{+} \to 0$ and $v^{+} \to -\infty$. Along the locus of the second-family shocks, the speed decreases from $\lambda_2(\mathbf{u}^{-})$, for points $\mathbf{u}^{+}$ tending to $\mathbf{u}^{-}$, to $-\infty$ as $\rho^{+} \to \infty$ and $v^{+} \to -\infty$.

Figure 10*b* depicts the locus of points $\mathbf{u}^{-}$ that can be connected by a shock to the point $\mathbf{u}^{+}$. The evolutionary part of the locus is shown by a solid line; the dashed line shows the nonevolutionary part. The shock speed D of the first family increases from $\lambda_1(\mathbf{u}^{+})$, for points $\mathbf{u}^{-}$ tending to $\mathbf{u}^{+}$, to ∞ as $\rho^{-} \to \infty$ and $v^{-} \to \infty$. Along the locus of the second family shocks, the speed increases from $\lambda_2(\mathbf{u}^{+})$, for points $\mathbf{u}^{-}$ tending to $\mathbf{u}^{+}$, to v^{+} as $\rho^{-} \to 0$ and $v^{-} \to \infty$.

⊙ *References for Subsection* S.12.6: O. A. Oleinik (1957), I. M. Gelfand (1959), P. Lax (1973), A. G. Kulikovskii (1979), C. M. Dafermos (1983), B. L. Rozhdestvenskii and N. N. Yanenko (1983), L. D. Landau and E. M. Lifshitz (1987), D. J. Logan (1994), E. Godlewski and P.-A. Raviart (1996), A. G. Kulikovskii, N. V. Pogorelov, and A. Yu. Semenov (2001).

S.12.7. Solutions for the Riemann Problem

In this section, we consider system (1) having a special form, with $\mathbf{G}(\mathbf{u}) = \mathbf{u}$. The solution of the corresponding Riemann problem (1), (6) is self-similar:

$$\mathbf{u} = \mathbf{u}(\xi), \qquad \xi = x/t. \tag{37}$$

The substitution of (37) into system (1) leads to the system of ordinary differential equations (18) with the following boundary conditions:

$$\mathbf{u} \to \mathbf{u}_{\mathrm{L}} \quad \text{as} \quad \xi \to -\infty, \qquad \mathbf{u} \to \mathbf{u}_{\mathrm{R}} \quad \text{as} \quad \xi \to \infty.$$

A trajectory of solution (37) in the space $\mathbf{u} = (u_1, \dots, u_n)^{\mathrm{T}}$ is called a solution path. The path is parametrized by the self-similar coordinate ξ. The path connects the point $\mathbf{u} = \mathbf{u}_{\mathrm{L}}$ with the point $\mathbf{u} = \mathbf{u}_{\mathrm{R}}$. The self-similar coordinate ξ monotonically increases along the path varying from $-\infty$ at $\mathbf{u} = \mathbf{u}_{\mathrm{L}}$ to $+\infty$ at $\mathbf{u} = \mathbf{u}_{\mathrm{R}}$. The path consists of continuous segments representing solutions of the ordinary differential equation (18) (rarefaction waves), line segments that connect two points $\mathbf{u}^{-}$ and $\mathbf{u}^{+}$ satisfying the Rankine–Hugoniot conditions (25) and evolutionary conditions (33), and rest points $\mathbf{u}(\xi) = \text{const}$.

Consider an example of a solution consisting of two shocks and one rarefaction. The structural formula* for the solution path is $\mathbf{u}_{\mathrm{L}} \to 1 — 2 \to \mathbf{u}_{\mathrm{R}}$; specifically,

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}_{\mathrm{L}} & \text{if } -\infty < x/t < D_1, \\ \mathbf{u}_1 & \text{if } D_1 < x/t < \lambda_2(\mathbf{u}_1), \\ \mathbf{u}^{(2)}(\xi) & \text{if } \lambda_2(\mathbf{u}_1) < x/t < \lambda_2(\mathbf{u}_2), \\ \mathbf{u}_2 & \text{if } \lambda_2(\mathbf{u}_2) < x/t < D_2, \\ \mathbf{u}_{\mathrm{R}} & \text{if } D_2 < x/t < \infty. \end{cases}$$

The shock speed D_1 (resp., D_2) can be found from the Hugoniot condition by setting $\mathbf{u}^{-} = \mathbf{u}_{\mathrm{L}}$ and $\mathbf{u}^{+} = \mathbf{u}_1$ (resp., $\mathbf{u}^{-} = \mathbf{u}_2$ and $\mathbf{u}^{+} = \mathbf{u}_{\mathrm{R}}$). Points 1 and 2 are located on the same rarefaction curve. The vector $\mathbf{u}^{(2)}(\xi)$ is a second-family rarefaction wave, which is described by the system of ordinary differential equations (18) with $\xi = \lambda_2(\mathbf{u})$.

Figure 11*a* depicts a sequence of rarefactions and shocks in the plane (x,t). Figure 11*b* shows the profile of the solution component u_i along the x-axis. The self-similar curves $\mathbf{u} = \mathbf{u}(\xi)$ coincide with the profiles $\mathbf{u}(x, t = 1)$. For $t > 1$, the graphs of $\mathbf{u}(x,t)$ are obtained from the self-similar curves by extending them along the axis x by a factor of t.

Example 14. Let us discuss the solution to the Riemann problem for a single equation (2) for various forms of the flux function. For concave flux function, with $F''(u) > 0$, any shock $u_{\mathrm{L}} \to u_{\mathrm{R}}$ with $u_{\mathrm{L}} > u_{\mathrm{R}}$ satisfies the Lax condition (34). Hence, the solution to the Riemann problem (2), (6) is given by

$$u(x,t) = \begin{cases} u_{\mathrm{L}} & \text{if } -\infty < x/t < D, \\ u_{\mathrm{R}} & \text{if } D < x/t < \infty, \end{cases} \qquad D = \frac{F(u_{\mathrm{L}}) - F(u_{\mathrm{R}})}{u_{\mathrm{L}} - u_{\mathrm{R}}}.$$

* In structural formulas like $\mathbf{u}_{\mathrm{L}} \to 1 — 2 \to \mathbf{u}_{\mathrm{R}}$, the symbol "$\to$" stands for a shock wave and "—" stands for a rarefaction.

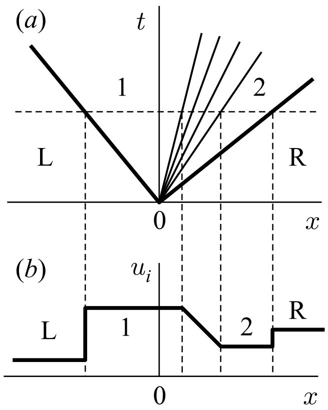

Figure 11. Solution for the Riemann problem: (*a*) centered waves in the (x,t) plane; (*b*) the u_i profile.

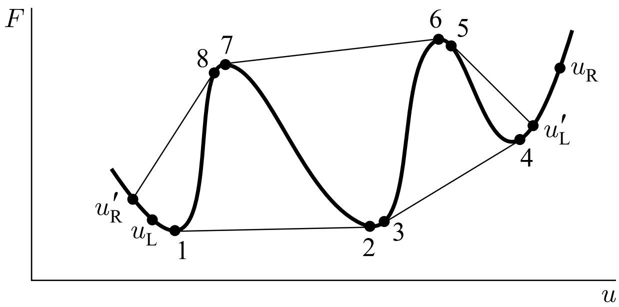

Figure 12. Graphical solution to the Riemann problem for a single conservation law (2).

For convex flux function, $F''(u) < 0$, any shock $u_L \to u_R$ with $u_L > u_R$ does not satisfy the Lax condition (34). The solution to the Riemann problem is given by a rarefaction wave:

$$u(x,t) = \begin{cases} u_L & \text{if } -\infty < x/t < D_L = F'(u_L), \\ x/t = F'(u) & \text{if } D_L < x/t < D_R = F'(u_R), \\ u_R & \text{if } D_R < x/t < \infty. \end{cases}$$

Note that here the solution in the intermediate region is defined implicitly: $x/t = F'(u)$.

For convex flux function, $F''(u) < 0$, the solution to the Riemann problem with $u_L > u_R$ is given by a rarefaction wave; the solution for the case $u_L < u_R$ is given by a shock $u_L \to u_R$. The solution to the Riemann problem (2), (6) for arbitrary flux function corresponds to the convex envelope of the curve $F(u)$ inside the interval $[u_L, u_R]$ for the case $u_L < u_R$. Shocks correspond to line segments between tangent points (e.g., points 1 and 2, 3 and 4, u'_L and 5, 6 and 7, and 8 and u'_R in Fig. 12). Rarefactions correspond to segments of the density function between tangent points (e.g., points u_L and 1, 2 and 3, and 4 and u_R in Fig. 12).

The solution to the Riemann problem can be expressed by structural formulas where an arrow stands for a jump and a dash stands for a rarefaction wave. The solution for the case $u_L < u_R$, which corresponds to a convex envelope, in Fig. 12 can be expressed by the following structural formula: $u_L — 1 \to 2 — 3 \to 4 — u_R$. The solution is given by

$$u(x,t) = \begin{cases} u_L, & -\infty < x/t < F'(u_L), \\ g(x/t), & F'(u_L) < x/t < F'(u_1), \\ g(x/t), & F'(u_1) = F'(u_2) < x/t < F'(u_3), \\ g(x/t), & F'(u_3) = F'(u_4) < x/t < F'(u_R), \\ u_R, & F'(u_R) < x/t < \infty, \end{cases}$$

where the function $u = g(\xi)$ is determined by the inversion of the relation $\xi = F'(u)$.

For the case $u'_L > u'_R$, the solution corresponds to the concave envelope (Fig. 12). The corresponding structural formula is: $u'_L \to 5 — 6 \to 7 — 8 \to u'_R$.

Example 15. The solution to the Riemann problem for strictly hyperbolic systems of two equations with arbitrary initial data can be obtained graphically from the phase portrait for two families of rarefactions (Fig. 7) and for loci of shocks (Figs. 10*a* and 10*b*). There are four types of solutions shown in Fig. 13 and outlined below.

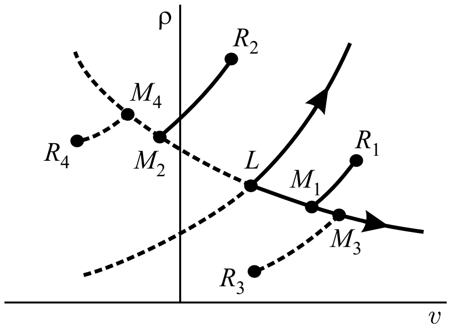

Figure 13. Four different cases for the evolution of a discontinuity in gas dynamics; point L has the coordinates (v^-, ρ^-) and points R_n have the coordinates (v^+, ρ^+).

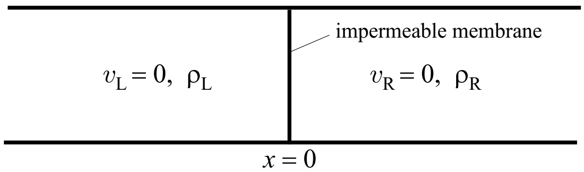

Figure 14. Shock tube problem: the initial distributions of the gas velocities and densities in the tube.

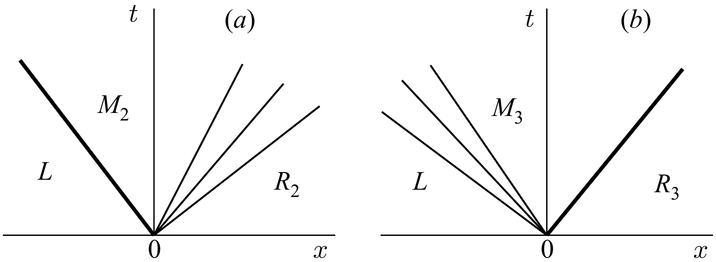

Figure 15. Decay of density discontinuity in the shock tube: (*a*) the gas density on the left is lower than that on the right; (*b*) the gas density on the right is lower than that on the left.

1°. If the right point R, with coordinates (v^+, ρ^+), lies above the locus of the second rarefaction and below the first rarefaction ($R = R_1$), the solution is given by two rarefaction waves: L — M_1 — R_1, where M_1 is the intersection point of the loci of the rarefactions through points L and R_1.

2°. If point R lies above both the locus of the first rarefaction and that of the second shock ($R = R_2$), the solution is given by the second shock and the first rarefaction: $L \to M_2$ — R_2, where M_2 is the intersection point of the locus of the first rarefaction that passes through point R_2 and of the locus of the second shock through point L.

3°. If point R is located below both the locus of the second rarefaction and that of the first shock ($R = R_3$), the solution is given by the second rarefaction and the first shock: L — $M_3 \to R_3$, where M_3 is the intersection point of the locus of the second rarefaction that passes through point L and the locus of the first shock through point R_3.

4°. If point R lies below the locus of the second shock and above the locus of the first shock ($R = R_4$), the solution is given by two shocks: $L \to M_4 \to R_4$, where M_4 is the intersection point of the shocks loci that pass through points L and R_4. This solution is given by

$$\rho = \begin{cases} \rho_L & \text{if } -\infty < \xi < D_1, \\ \rho_{M_4} & \text{if } D_1 < \xi < D_2, \\ \rho_{R_4} & \text{if } D_2 < \xi < +\infty, \end{cases} \qquad v = \begin{cases} v_L & \text{if } -\infty < \xi < D_1, \\ v_{M_4} & \text{if } D_1 < \xi < D_2, \\ v_{R_4} & \text{if } D_2 < \xi < +\infty, \end{cases}$$

where $\xi = x/t$; the shock speeds D_1, D_2 and the intermediate point (ρ_{M_4}, v_{M_4}) are calculated from the Hugoniot conditions in the form (28) or (29).

Problem 1. Let us consider the so-called shock tube problem (see Fig. 14). An impermeable membrane separates the two parts of the tube and it is suddenly removed at the time $t = 0$. The gas is at rest in the initial state, $v_L = v_R = 0$. The sequence of a shock and a rarefaction on the plane (x, t) is shown in Fig. 15*a* for the case $\rho_L < \rho_R$. It corresponds to the case where the pressure in the tube on the left ($x < 0$) is lower than that on the right ($x > 0$). The shock races into a quiescent low

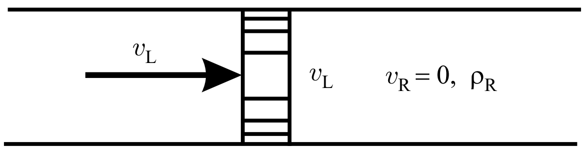

Figure 16. Constant velocity piston motion in a tube.

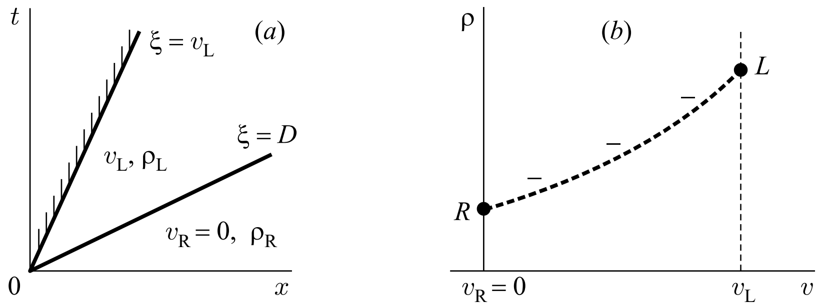

Figure 17. Solution of the piston problem: (a) shock wave in the (x, t) plane; (b) shock locus in the phase plane.

pressure gas. The solution is of the type 2° above:

$$\rho = \begin{cases} \rho_L & \text{if } 0 < \xi < D, \\ \rho_1 & \text{if } D < \xi < v_1 + \sqrt{A\gamma\rho_1^{\gamma-1}}, \\ \widetilde{\rho}(\xi) & \text{if } v_1 + \sqrt{A\gamma\rho_1^{\gamma-1}} < \xi < v_R + \sqrt{A\gamma\rho_R^{\gamma-1}}, \\ \rho_R & \text{if } v_R + \sqrt{A\gamma\rho_R^{\gamma-1}} < \xi < +\infty, \end{cases} \qquad v = \begin{cases} v_L & \text{if } 0 < \xi < D, \\ v_1 & \text{if } D < \xi < v_1 + \sqrt{A\gamma\rho_1^{\gamma-1}}, \\ \widetilde{v}(\xi) & \text{if } v_1 + \sqrt{A\gamma\rho_1^{\gamma-1}} < \xi < v_R + \sqrt{A\gamma\rho_R^{\gamma-1}}, \\ v_R & \text{if } v_R + \sqrt{A\gamma\rho_R^{\gamma-1}} < \xi < +\infty, \end{cases}$$

where (ρ_1, v_1) is the intersection point of the locus of the first rarefaction passing through point R and that of the second shock passing through point L, and the functions $\widetilde{\rho} = \widetilde{\rho}(\xi)$ and $\widetilde{v} = \widetilde{v}(\xi)$ are determined by solving the algebraic equations

$$\xi = \widetilde{v} + \sqrt{A\gamma\widetilde{\rho}^{\gamma-1}}, \qquad \widetilde{v} - \frac{2}{\gamma-1}\sqrt{A\gamma\widetilde{\rho}^{\gamma-1}} = v_R - \frac{2}{\gamma-1}\sqrt{A\gamma\rho_R^{\gamma-1}}.$$

The functions $\widetilde{\rho}(\xi)$ and $\widetilde{v}(\xi)$ can be represented in explicit form.

A type 3° solution occurs in the case $v_L = v_R = 0$ and $\rho_L > \rho_R$. The wave motion is shown in Fig. 15b and the solution is given by

$$\rho = \begin{cases} \rho_L & \text{if } 0 < \xi < v_L - \sqrt{A\gamma\rho_L^{\gamma-1}}, \\ \widetilde{\rho}(\xi) & \text{if } v_L - \sqrt{A\gamma\rho_L^{\gamma-1}} < \xi < v_1 - \sqrt{A\gamma\rho_1^{\gamma-1}}, \\ \rho_1 & \text{if } v_1 - \sqrt{A\gamma\rho_1^{\gamma-1}} < \xi < D, \\ \rho_R & \text{if } D < \xi < \infty, \end{cases} \qquad v = \begin{cases} v_L & \text{if } 0 < \xi < v_L - \sqrt{A\gamma\rho_L^{\gamma-1}}, \\ \widetilde{v}(\xi) & \text{if } v_L - \sqrt{A\gamma\rho_L^{\gamma-1}} < \xi < v_1 - \sqrt{A\gamma\rho_1^{\gamma-1}}, \\ v_1 & \text{if } v_1 - \sqrt{A\gamma\rho_1^{\gamma-1}} < \xi < D, \\ v_R & \text{if } D < \xi < \infty, \end{cases}$$

where (ρ_1, v_1) is the intersection point of the locus of the second rarefaction passing through point L and that of the first shock through point R, and the functions $\widetilde{\rho} = \widetilde{\rho}(\xi)$ and $\widetilde{v} = \widetilde{v}(\xi)$ are determined by solving the algebraic equations

$$\xi = \widetilde{v} - \sqrt{A\gamma\widetilde{\rho}^{\gamma-1}}, \qquad \widetilde{v} + \frac{2}{\gamma-1}\sqrt{A\gamma\widetilde{\rho}^{\gamma-1}} = v_L + \frac{2}{\gamma-1}\sqrt{A\gamma\rho_R^{\gamma-1}}.$$

Problem 2. Now let us discuss an adiabatic gas flow in a tube in front of an impermeable piston moving with a velocity v_L (see Fig. 16). The initial state is defined by prescribing initial values of the velocity and density:

$$v = 0, \quad \rho = \rho_R \quad \text{at} \quad \xi = \infty. \tag{38}$$

The piston is impermeable; therefore, the gas velocity in front of the shock is equal to the piston velocity (Fig. 17a):

$$v = v_L \quad \text{at} \quad \xi = v_L. \tag{39}$$

The gas density in front of the piston is unknown in this problem.

Figure 17b shows the locus of points that can be connected by a shock to the point $(v_R = 0, \rho_R)$. This locus is a first-family shock. The intersection of the locus with the line $v = v_L$ defines the value ρ_L. Hence, ρ_L can be found from the equation

$$v_L^2 = \frac{[p(\rho_L) - p(\rho_R)](\rho_L - \rho_R)}{\rho_L\rho_R}, \tag{40}$$

which has been obtained by taking the square of equation (28). There exists a root ρ_L of (40) such that $\rho_L > \rho_R$. Hence, the gas is compressed ahead of the piston (Fig. 17*a*). The expression for the shock speed can be found from the first Hugoniot condition (27):

$$D = \frac{\rho_L v_L}{\rho_L - \rho_R} > v_L. \tag{41}$$

The shock speed exceeds the piston velocity of (41) for $\rho_L > \rho_R$. Both characteristics of the first family as well as the characteristic ahead of the shock from the second family arrive at the shock, so that the Lax condition is satisfied. It can be proved that there are no other configurations that satisfy the initial-boundary value conditions (38), (39).

⊙ *References for Subsection* S.12.7: I. M. Gelfand (1959), P. Lax (1973), T. P. Liu (1974), C. M. Dafermos (1983, 2000), B. L. Rozhdestvenskii and N. N. Yanenko (1983), J. Smoller (1983), H. Rhee, R. Aris, and N. R. Amundson (1986, 1989), D. J. Logan (1994).

S.12.8. Initial Boundary Value Problems of Special Form

Consider an initial-boundary value problem for hyperbolic system (1) with initial and boundary conditions of the special form:

$$\mathbf{u} = \mathbf{u}_\mathrm{i} \quad \text{at} \quad t = 0, \qquad \mathbf{u} = \mathbf{u}_\mathrm{b} \quad \text{at} \quad x = 0. \tag{42}$$

Here, $\mathbf{u}_\mathrm{i}$ and $\mathbf{u}_\mathrm{b}$ are prescribed constant vectors ($x \ge 0$, $t \ge 0$).

The transformation $(x,t) \to (kx, kt)$ with any positive k preserves both system (1) and conditions (42). Therefore, the solution of the initial-boundary value problem (1), (42) is self-similar:

$$\mathbf{u} = \mathbf{u}(\xi), \qquad \xi = x/t. \tag{43}$$

The substitution of (43) into system (1) with $\mathbf{G}(\mathbf{u}) = \mathbf{u}$ yields the system of ordinary differential equations (18) with the following boundary conditions:

$$\mathbf{u} = \mathbf{u}_\mathrm{b} \quad \text{at} \quad \xi = 0, \qquad \mathbf{u} \to \mathbf{u}_\mathrm{i} \quad \text{as} \quad \xi \to \infty. \tag{44}$$

The solution to problem (18), (44) can be constructed in a similar way as the solution to the Riemann problem and consists of portions with constant $\mathbf{u}$, shocks, and rarefaction waves.

Example 16. Consider equation (2) with the initial and boundary conditions (42) where $\mathbf{u} = u$. In petroleum engineering, this problem is used as a model for the displacement of oil by water in reservoirs. Here u is the water saturation (volumetric water fraction in pore space), $F(u)$ is the dimensionless water flux (so-called fractional flow function which represents the water flux fraction in the total two-phase flux). The initial condition of (42) corresponds to the initial water saturation in the reservoir and the boundary condition means that only water flows through the inlet cross-section.

We assume that the function $F(u)$ satisfies the conditions

$$F(u_\mathrm{i}) = 0, \qquad F(u_\mathrm{b}) = 1, \qquad F'(u) > 0 \quad \text{for} \quad u_\mathrm{i} < u < u_\mathrm{b}, \qquad F''(u_\mathrm{i}) > 0, \qquad F''(u_\mathrm{b}) < 0.$$

The solution of problem (2), (42) consists of constant value segments, $u = u_\mathrm{b}$ and $u = u_\mathrm{i}$, a rarefaction wave, and a shock:

$$u(x,t) = \begin{cases} u_\mathrm{b} & \text{if } 0 < x/t < D_\mathrm{b} = F'(u_\mathrm{b}), \\ g(x/t) & \text{if } D_\mathrm{b} < x/t < D_\mathrm{f} = F(u_\mathrm{f})/(u_\mathrm{f} - u_\mathrm{i}), \\ u_\mathrm{i} & \text{if } D_\mathrm{f} < x/t < \infty, \end{cases}$$

where u_f is determined by the transcendental equation

$$F(u_\mathrm{f}) = (u_\mathrm{f} - u_\mathrm{i})F'(u_\mathrm{f})$$

and the continuous solution $u = g(\xi)$ is obtained by the inversion of the relation $\xi = F'(u)$. Note that the above solution is defined implicitly in the intermediate region (rarefaction wave region).

⊙ *References for Subsection* S.12.8: S. E. Buckley and M. C. Leverett (1942), G. B. Whitham (1974), B. L. Rozhdestvenskii and N. N. Yanenko (1983), P. G. Bedrikovetsky (1993), A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux (2002).

S.12.9. Examples of Nonstrict Hyperbolic Systems

We now consider several examples of nonstrict hyperbolic systems of the form (7), for which the matrix $\mathbf{A}$ has coincident eigenvalues, $\lambda_i(\mathbf{u}) = \lambda_j(\mathbf{u})$ for $i \ne j$, in some domain.

Example 17. Let us discuss the Riemann problem for the 2×2 hyperbolic system

$$\frac{\partial s}{\partial t} - \frac{\partial}{\partial x}(s - c - 2)^2 = 0, \tag{45}$$

$$\frac{\partial (cs)}{\partial t} + \frac{\partial}{\partial x}\bigl\{c[1 - (s - c - 2)^2]\bigr\} = 0 \tag{46}$$

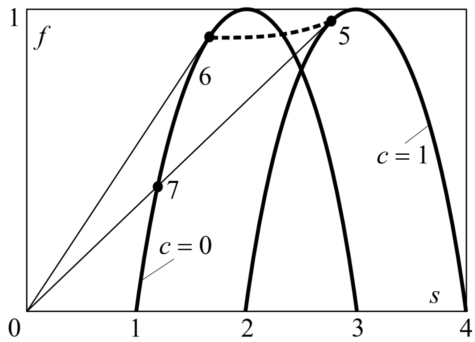

Figure 18. Graphical solution for the Riemann problem (45)–(47).

with initial conditions

$$s = \begin{cases} 4 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases} \quad c = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0, \end{cases} \quad \text{at} \quad t = 0. \tag{47}$$

This is a model system for a two-phase multicomponent flow through porous media in the gravitational field.

We are going first to classify the elementary waves for system (45), (46) and then to construct the solution of the problem (45)–(47) from these elements.

1°. Differentiating both sides of system (45), (46), we find the 2×2 matrix $\mathbf{A}$ of (7) in the form

$$\mathbf{A} = \begin{pmatrix} -2(s-c-2) & 2(s-c-2) \\ 0 & \dfrac{1-(s-c-2)^2}{s} \end{pmatrix}.$$

The eigenvalues of $\mathbf{A}$ are:

$$\lambda_1 = -2(s-c-2), \quad \lambda_2 = \frac{1-(s-c-2)^2}{s}. \tag{48}$$

Figure 18 shows the graphs of the function $f(s,c) = 1-(s-c-2)^2$ for two fixed values of c: $c = 0$ and $c = 1$. From (48) it follows that the first eigenvalue is equal to the slope of a curve $f = f(s,\ c = \text{const})$. The second eigenvalue is equal to the slope of the line segment linking the point (s, f) with the origin of coordinates. Points 5 and 6 are the points of tangency of the curves $c = 1$ and $c = 0$ and the straight lines through the origin of coordinates, respectively. The eigenvalues λ_1 and λ_2 of (48) are equal at points 5 and 6. The locus of points with equal eigenvalues (48) for $0 < c < 1$ is shown in Fig. 18 by the dashed line linking points 5 and 6. The first eigenvalue is higher than the second one in the area below the dashed curve, while in the area above the dashed curve, the inequality $\lambda_1 < \lambda_2$ holds.

From the Rankine–Hugoniot conditions (25) it follows that system (45), (46) allows for two types of shocks: shocks without jumps of c (so-called s-shocks),

$$D_s = \frac{(s^+ - c - 2)^2 - (s^- - c - 2)^2}{s^- - s^+}, \quad c^+ = c^- = c, \tag{49}$$

and shocks with jumps of c (so-called c-shocks),

$$D_c = \frac{1-(s^+ - c^+ - 2)^2}{s^+} = \frac{1-(s^- - c^- - 2)^2}{s^-}. \tag{50}$$

The calculation of the right eigenvector (19) for the first eigenvalue (48) shows that c is constant along the first-family rarefactions. These rarefactions are called s-waves. The calculation of (19) for the second-family rarefactions shows that they degenerate into c-shocks.

Hence, system (45), (46) allows for three elementary waves: an s-shock, a c-shock, and a rarefaction s-wave.

The solution of the problem (45)–(47) is self-similar, i.e., can be found in the form (37). The initial conditions (47) for the self-similar coordinate $\xi = x/t$ become

$$s = 4, \quad c = 1 \quad \text{as} \quad \xi \to -\infty; \qquad s = 1, \quad c = 0 \quad \text{as} \quad \xi \to \infty. \tag{51}$$

2°. Let us calculate several values that will be helpful for solving problem (45), (46), (51).

The values of λ_1 at points 1 and 4 (denote them by D_1 and D_4) can be calculated from (48): $D_1 = 2$ and $D_4 = -2$. The coordinate s_5 of point 5 follows from the condition of equality of the two eigenvalues of (48) on the curve $c = 1$: $s_5 = 2\sqrt{2}$. The slope D_5 of the curve $c = 1$ at point 5 is equal to λ_1 at this point: $D_5 = 6 - 4\sqrt{2}$. Let us plot the intersection point (point 7) of the straight line 0–5 and the curve $c = 0$. The coordinate of point 7 is: $s_7 = 3\sqrt{2} - 3$, and the slope D_7 of the straight line 7–1 is: $D_7 = 8 - 5\sqrt{2}$.

The solution of the problem (51) must connect point 4 with point 1. Both the s-shock and the c-shock from point 4 are unstable, so it is possible to exit from point 4 just by the s-wave. The point that would be reached by s-wave from point 4 could be located before or after point 5. In the former case, the c-shock from the curve $c = 1$ to the curve $c = 0$ is unstable for any point behind the shock located between points 4 and 5. In the latter case, the c-shock from the curve $c = 1$ to the curve

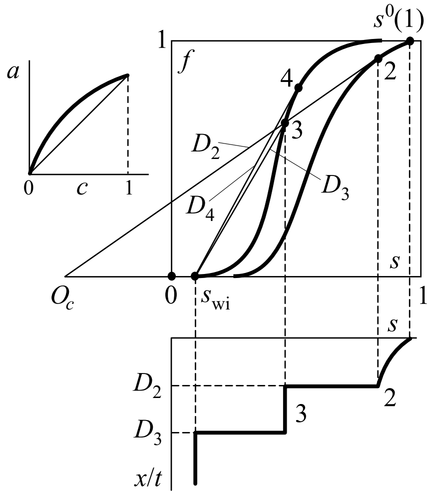

Figure 19. Construction of a graphical solution for the displacement of oil by a chemical additive.

$c = 0$ is stable if the point behind the shock lies between points 5 and 2; nevertheless, the value ξ in this interval exceeds the shock speed, so the sequence of an s-wave and a forthcoming c-shock is not allowed. The only possibility left is a c-shock from tangent point 5.

The solution consists of an s-wave, a c-shock, and an s-wave; the structural formula is: 4 — 5 → 7 — 1.

Finally, we can write out the solution in the form

$$s(x,t) = \begin{cases} 4 & \text{if } -\infty < x/t < D_4, \\ s^1(\xi) & \text{if } D_4 < x/t < D_5, \\ s_7 & \text{if } D_5 < x/t < D_7, \\ s^2(\xi) & \text{if } D_7 < x/t < D_1, \\ 1 & \text{if } D_1 < x/t < \infty; \end{cases} \qquad c(x,t) = \begin{cases} 1 & \text{if } -\infty < x/t < D_5, \\ 0 & \text{if } D_5 < x/t < \infty, \end{cases} \tag{52}$$

where $s^1(\xi) = 3 - \frac{1}{2}\xi$, $s^2(\xi) = 2 - \frac{1}{2}\xi$, and $\xi = x/t$.

System (45), (46) is not strictly hyperbolic, and consequently both an s-wave and an s-shock are present in the solution of the Riemann problem (47).

Example 18. A two-phase immiscible flow of oil and water with a chemical additive in water is governed by a 2×2 system

$$\begin{aligned} &\frac{\partial s}{\partial t} + \frac{\partial f(s,c)}{\partial x} = 0, \\ &\frac{\partial \big(cs + a(c)\big)}{\partial t} + \frac{\partial \big(cf(s,c)\big)}{\partial x} = 0. \end{aligned} \tag{53}$$

Here, s is the water saturation, c is the additive concentration, $f(s,c)$ is the water flux, and $a(c)$ is the adsorbed chemical concentration, the so-called sorption isotherm.

The function $f(s,c)$ satisfies the following conditions:

$$f(s,c) = 0 \quad \text{for} \quad 0 < s < s_{\rm i}; \qquad f'_s(s,c) > 0, \quad f'_c(s,c) < 0 \quad \text{for} \quad s_{\rm i} < s < s^0(c); \qquad f(s,c) = 1 \quad \text{for} \quad s^0(c) < s < 1.$$

The graphs of $f(s,c)$ at $c = 1$ and $c = 0$ are presented in Fig. 19.

The dependence $f = f(s,c)$ allows us to choose either (s,c) or (s,f) to be the unknown functions in system (53).

The problem of oil displacement by an aqueous solution of a chemical admixture is described by system (53) and the following initial and boundary conditions:

$$\begin{aligned} &s = s_{\rm i}, \qquad c = 0 \quad \text{at} \quad t = 0, \\ &s = s^0(1), \quad c = 1 \quad \text{at} \quad x = 0. \end{aligned} \tag{54}$$

The solution of problem (53), (54) is obtained by the same method as in Example 17.

The initial-boundary value problem (54) can be transformed to the following boundary value problem for the self-similar coordinate:

$$s = s^0(1), \quad c = 1 \quad \text{at} \quad \xi = 0; \qquad s = s_{\rm i}, \quad c = 0 \quad \text{at} \quad \xi \to \infty. \tag{55}$$

The point at $\xi = 0$ lies on the curve $c = 1$, the point at $x \to \infty$ is located on the curve $c = 0$ (Fig. 19).

The self-similar path $(s(\xi), f(\xi))$ should connect the points $(s^0(1), 1)$ and $(s_{\rm i}, 0)$ on the plane (s, f); see Fig. 19.

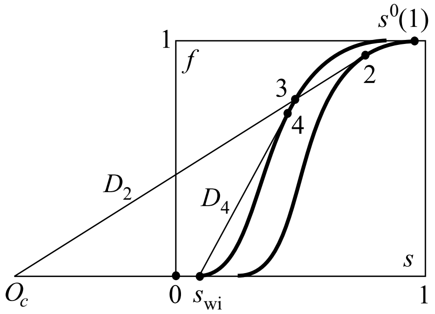

Figure 20. Graphical construction of the solution for the displacement of oil by a chemical additive in the case of high sorption of the additive.

System (53) can be reduced to an equivalent system of the form (7) with matrix

$$\mathbf{A} = \begin{pmatrix} f'_s(s,c) & f'_c(s,c) \\ 0 & \dfrac{f(s,c)}{s+a'(c)} \end{pmatrix}. \tag{56}$$

The eigenvalues of system (53) are evaluated as

$$\lambda_1(s,c) = f'_s, \quad \lambda_2(s,c) = \frac{f}{s+a'(c)}. \tag{57}$$

Right eigenvectors corresponding to the first and second eigenvalues (57) are given by

$$r_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad r_2 = \begin{pmatrix} \dfrac{f}{s+a'} - f'_s \\ f'_c \end{pmatrix}. \tag{58}$$

Let us take the unknown s in the ordinary differential equations for rarefaction waves (19) to be the independent variable, so we look for a solution of (19) in the form $f = f(s)$, $\xi = \xi(s)$. The equations for the s-waves and c-waves (first and second families of rarefactions) read

$$\frac{df}{ds} = f'_s = \xi, \tag{59}$$

$$\frac{df}{ds} = \frac{f}{s+a'} = \xi. \tag{60}$$

From (59) it follows that the first eigenvalue is equal to the slope of the curve $f = f(s, c = \text{const})$; see Fig. 19. The second eigenvalue (60) is equal to the slope of the line segment connecting the points (s, f) and $(-a'(c), 0)$.

From the Rankine–Hugoniot conditions (25) it follows that system (53) admits two types of shocks: shocks without jumps of c (so-called s-shocks),

$$D = \frac{f(s^+, c) - f(s^+, c)}{s^+ - s^-}, \qquad c^+ = c^- = c, \tag{61}$$

and shocks with c-jumps (so-called c-shocks),

$$D = \frac{f(s^+, c^+)}{s^+ + \sigma} = \frac{f(s^-, c^-)}{s^- + \sigma}, \qquad \sigma = \frac{a(c^+) - a(c^-)}{c^+ - c^-}. \tag{62}$$

The case of a convex sorption isotherm is presented in Fig. 19. Point 2 is the tangent point of the curve $c = 1$ and the straight line through point O_c with coordinates $(-[a]/[c], 0)$. The shock $2 \longrightarrow 3$ is evolutionary.

Let us plot the tangent point 4 of the curve $c = 0$ and the straight line through s_i. Figure 19 shows the case where point 4 is located above point 3, which corresponds to low sorption.

The solution consists of an s-wave and two shocks; the structural formula is: $s^0(1)$ — $2 \longrightarrow 3 \longrightarrow s_\text{i}$. The speeds D_2 and D_3 of the shocks $2 \longrightarrow 3$ and $3 \longrightarrow s_\text{i}$ are calculated by formulas (62) and (61), respectively. The solution is given by

$$s(x,t) = \begin{cases} s^1(\xi) & \text{if } 0 < x/t < D_2, \\ s_3 & \text{if } D_2 < x/t < D_3, \\ s_\text{i} & \text{if } D_3 < x/t < \infty; \end{cases} \qquad c(x,t) = \begin{cases} 1 & \text{if } 0 < x/t < D_2, \\ 0 & \text{if } D_2 < x/t < \infty; \end{cases} \tag{63}$$

where the function $s^1(\xi)$ is determined by the inversion of the relation $\xi = f'_s(s^1)$.

Figure 19 shows the correspondence between the solution image on the planes (s, f) and (s, ξ), $\xi = x/t$. The continuous curve $s = s^1(\xi)$ of (63) corresponds to the motion along the curve $c = 1$ from the point $s^0(1)$ to point 2; the slope of the curve $c = 1$ at a point s is equal to the coordinate ξ that corresponds to the value s of the curve $s = s^1(\xi)$. The shocks $2 \longrightarrow 3$ and $3 \longrightarrow s_\text{i}$ on the plane (s, f) correspond to discontinuities in the curve $s = s(\xi)$ at the points $\xi = D_2$ and $\xi = D_3$. If sorption is high, and point 4 is located below point 3, as in Fig. 20, the structural formula for the solution is: $s^0(1)$ — $2 \longrightarrow 3$ — $4 \longrightarrow s_\text{i}$.

Example 19. If the sorption isotherm in (53) is concave, the transition from $c = 1$ to $c = 0$ occurs by a c-wave. The structural formula is: $s^0(1)$ — 2 — $3 \longrightarrow s_\text{i}$.

If the sorption isotherm in (53) has inflection points, the transition from $c = 1$ to $c = 0$ occurs by a sequence of c-shocks and c-waves that correspond to a concave envelope of the sorption isotherm (see Fig. 21).

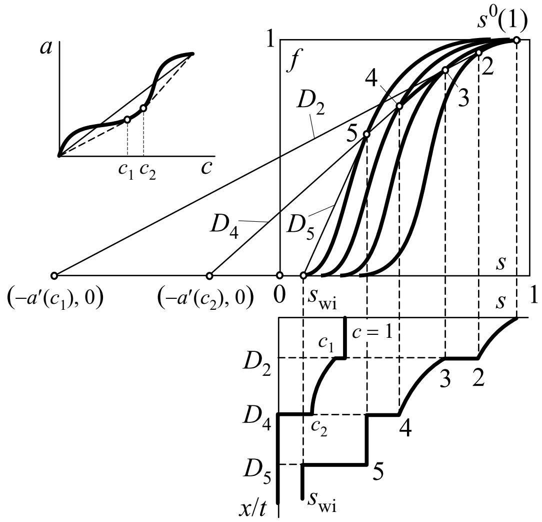

Figure 21. Graphical solution for the displacement of oil by a chemical solution in the case of the sorption isotherm having inflection points.

Example 20. A two-phase (liquid–gas) three-component incompressible flow in porous media is governed by the system

$$\begin{aligned}&\frac{\partial C}{\partial t}+\frac{\partial U}{\partial x}=0,\\&\frac{\partial}{\partial t}\big[\alpha(g_2)C+\beta(g_2)\big]+\frac{\partial}{\partial x}\big[\alpha(g_2)U+\beta(g_2)\big]=0.\end{aligned}\tag{64}$$

Here, the following notation is adopted:

$$C=l_1 s+g_1(1-s),\;\; U=l_1 f(s,g_2)+g_1[1-f(s,g_2)],\;\; \alpha(g_2)=\frac{l_2-g_2}{l_1-g_1},\;\; \beta(g_2)=g_2-\alpha g_1,\;\; l_n=l_n(g_2),\;\; g_1=g_1(g_2),\tag{65}$$

where $s=s(x,t)$ is the liquid saturation, l_n and g_n are the volume concentrations of the nth component in the liquid and gas phases respectively, and $f(s,g_2)$ is the liquid phase flux. The independent concentration in this system is g_2, the other concentrations are functions of g_2. The unknowns in system (64) are s and g_2 or C and g_2.

The functions f, U, l_1, and g_1 satisfy the following conditions:

$$f(0,g_2)=0,\quad f(1,g_2)=1,\quad \tfrac{\partial f}{\partial s}(s,g_2)>0,\quad \tfrac{\partial U}{\partial g_2}(C,g_2)<0,\quad g_1'(g_2)>0,\quad l_1'(g_2)<0.$$

System (64) is analogous to system (53) analysed in Example 18.

The problem of the displacement of oil with composition A by gas with composition B corresponds to the initial and boundary conditions

$$C=C_{\rm A},\quad g_2=g_{2\rm A}\quad \text{at}\quad t=0;\qquad C=0,\quad g_2=g_{2\rm B}\quad \text{at}\quad x=0.\tag{66}$$

The solution of problem (64)–(66) is expressed as

$$C(x,t)=\begin{cases}0 & \text{if } 0<\xi<D_2,\\ C(\xi) & \text{if } D_2<\xi<D_3,\\ C_4 & \text{if } D_3<\xi<D_4,\\ C_{\rm A} & \text{if } D_4<\xi<\infty,\end{cases}\qquad g_2(x,t)=\begin{cases}g_{2\rm B} & \text{if } 0<\xi<D_3,\\ g_{2\rm A} & \text{if } D_3<\xi<\infty,\end{cases}\tag{67}$$

where $\xi=x/t$, the function $C(\xi)$ is the inverse of the function $\xi=\frac{\partial U}{\partial C}(C,g_{2\rm B})$, and the constants D_2, D_3, D_4, C_4 (and also C_2 and C_3) are determined by the following transcendental equations:

$$\begin{aligned}D_2&=\frac{\partial U}{\partial C}(C_2,g_{2\rm B})=\frac{U(C_2,g_{2\rm B})}{C_2};\\ D_3&=\frac{\partial U}{\partial C}(C_3,g_{2\rm B})=\frac{U(C_3,g_{2\rm B})+k}{C_3+k}=\frac{U(C_4,g_{2\rm A})+k}{C_4+k},\qquad k=\frac{\beta(g_{2\rm A})-\beta(g_{2\rm B})}{\alpha(g_{2\rm A})-\alpha(g_{2\rm B})};\\ D_4&=\frac{U(C_4,g_{2\rm A})-U(C_A,g_{2\rm A})}{C_4-C_{\rm A}}.\end{aligned}$$

The structural formula for solution (67) is: $(0,g_{2\rm B})\to(C_2,g_{2\rm B})$ — $(C_3,g_{2\rm B})\to(C_4,g_{2\rm A})\to(C_{\rm A},g_{2\rm A})$.

⊙ *References for Subsection* S.12.9: C. Wachmann (1964), L. W. Lake (1989), P. G. Bedrikovetsky and M. L. Chumak (1992*a*, 1992*b*).