Another Solution to the Polyhedron in Dürer's *Melencolia*: A Visual Demonstration of the Delian Problem

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Introduction

The engraving *Melencolia* (1514) (Fig. 1) by Albrecht Dürer (1471-1528) has been interpreted in various ways, but is not yet completely solved. It is filled with marvelous instruments, and among them a large stone polyhedron in the left center has attracted both art historians and scientists, and the efforts to clarify the significance of the polyhedron have been made by both sides.

The preliminary sketch of the polyhedron (Fig. 2), which Dürer drew in perspective, indicating the center of vision by an eye, shows all edges of the polyhedron transparently. And each vertex bears a mark.

This polyhedron (Fig. 3) has been regarded as a truncated rhombohedron, which consists of 6 rhombuses. Cut off two corners along the planes perpendicular to the line AH, the solid now consists of 6 irregular pentagons and 2 equilateral triangles. One side of the upper triangular face and one side of the bottom triangular face are perpendicular to the picture plane.

The engraving is also drawn in perspective. Lines that are perpendicular to the picture plane, such as the side cornice of the building and the arm of the balance, converge at the vanishing point on the horizon of the sea. Dürer set the mirror image of the polyhedron of the sketch in such a manner that the vanishing points and horizons of both planes correspond with each other, so that the polyhedron may be regarded as correct in perspective on the engraving.

The polyhedron is so large-sized that it must have some significance, which has been often argued over, and is still in controversy. Art historians were apt to regard intuitively the stone polyhedron as some symbol in the context of their interpretation, for example, as a symbol of geometry, perspective, material, and so on. Meanwhile, scientists attempted to clarify the structure of the polyhedron by determining the acute angle of the rhombus before truncation, that is, the most acute angle of the pentagon. Since Dürer is prominent both as an artist and a geometer, it is necessary to examine this issue from both sides.

Examination of Preceding Treatises

Preceding treatises dealing with the structure of this polyhedron are listed in Table 1. While some insist that the polyhedron is a truncated cube without any rational grounds [1], the rest are divided into two groups, the one that determined the acute angle to be about 72° and the other about 80° [2].

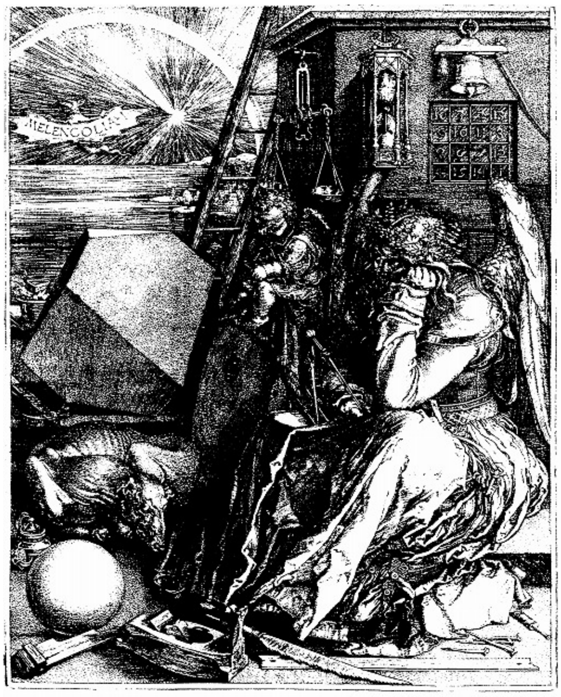

Fig. 1: Albrecht Dürer, *Melencolia*, 1514, Engraving, 23.9×18.6 cm

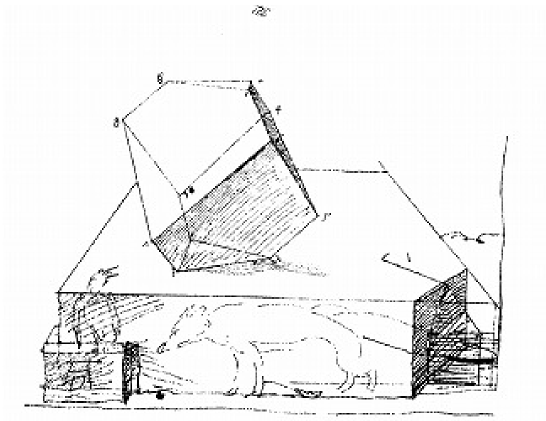

Fig. 2: Albrecht Dürer, *Drawing*, 1514, Pen and brown and black ink, 20.2×19.3 cm, Dresden Landesbibliothek

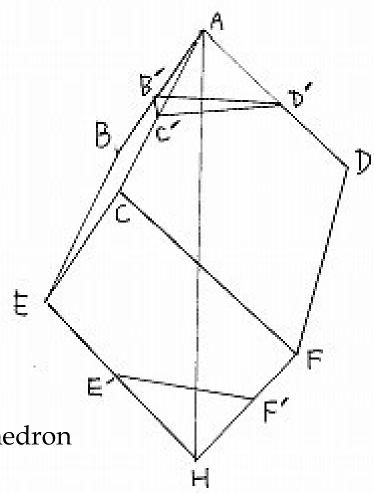

Fig. 3: Model of the Polyhedron (drafted by Ishizu)

Table 1

year	name	angle	method	reason / interpretation
c.1900	Niemann	80°	by use of graphics	intended to be a cube
1922	Nagel	(90°)		enlarged a cube obliquely
1955	Grodzinski	72°	making models	visually most alike among 60°, 72°, 90°
1957	Richter	79°36′	based on perspective	
1970	Rösch	72°	admitted Grodzinski's idea	as it's connected with the golden section
1972	Harnest	80°	reconstruction	
1976	Wangart	90°	by use of graphics	
1978	Deckwitz	76°	making models	as it's connected with pyramids
1979	Enomoto	72°	making models	as it's connected with the golden section
1980	Schröder	(81°47′)	analysis	the ratio between two diagonals is 2: $\sqrt{3}$
1981	MacGillavry	79° ±1°	based on perspective	crystal of calcite → mineralogy
1982	Lynch	80~83°	making models	three-dimensional anamorphosis
1990	Sixel	about 80°	admitted Richter's notion	
1993	Engelhardt	80°	calculation	crystal of calcite
1999	Schreiber	72°		as it's connected with the golden section

Those who insist on an angle 72° founded their theory on the relation of this angle to equilateral pentagons, and moreover, to the golden section, as a result of looking for a meaningful rhombohedron [3]. In 1955, Grodzinski stated that the acute angle of the rhombus was 72° based on making models. He made three models of the truncated rhombohedron with rhombuses having acute angles 60°, 72°, and 90°,—the angles which seemed somehow significant. Among them, one with an angle 72° seemed to him to be most appropriate visually. In 1970 approved Rösch this angle, because of its connection with the golden section. In addition, he insisted that several parts of *Melencolia* were constructed based on the golden section. In 1999, Schreiber stated that Dürer had drawn this polyhedron based on a ground plan and an elevation, and had determined the angle to be 72° as it is connected with the golden section. As stated above, an angle 72° is a very convincing angle which attracts many scientists.

On the other hand, the original polyhedron can be reconstructed as it is drawn in perspective. Derived from reconstruction by use of graphics and calculation, the acute angle of the rhombus, the face of the rhombohedron before truncation, is identified as about 80°. The quadrilateral AEHD in Fig. 3 is parallel to the picture plane, that is, it is similar to the original figure. The sides AD and EH of this parallelogram are projections of the sides of the original rhombus. The sides AE and DH are projections of the longer diagonals of the rhombus. Thus, for example, we can construct the original rhombus, and obtain the acute angle about 80°. At the request of Gielow, Niemann arrived at an angle 80°, although he was sure that Dürer had intended to draw a truncated cube. Then, Richter (1957), Harnest (1972), MacGillavry (c. 1980), Schröder (c. 1980), and Engelhardt (1993) also identified the acute angle as about 80°. Schröder tried to find some significance relating to the ratio between the two diagonals of the rhombus which he set at $2 : \sqrt{3}$. In this case, the acute angle can be calcu-lated to be about 82°. Lynch derived an angle 80° by use of both reconstruction and making a model. But 80° cannot cause any important geometric meaning. Lynch insisted that this polyhedron represented a three-dimensional anamorphosis, which appears to be a rhombohedron set on a pentagonal face, and appears to be a cube set on a triangular face. MacGillavry and Engelhardt stated that it represented a crystal of calcite, which does not seem persuasive enough, as there is no reason why calcite should appear so large in the engraving. Schröder and Harnest identified the acute angle to be 80°, while they did not mention further interpretation on it.

As stated above, the preceding attempts to determine somehow logically the acute angle of the rhombus, the face of the rhombohedron before truncation, to be 72° or about 80° seem both unsatisfactory. An angle 72° is attractive, while it does not correspond with the result of the analysis, having a large margin of error. And the case is possible only when Dürer lacked the ability to draft. On the other hand, an angle 80°, which was derived from the analysis, on the assumption that he was skilled in drafting, seems to be meaningless. In addition, it should be noted that their arguments are restricted only to geometric area, and that they do not refer to the vague image on the largest pentagonal face of the solid in the engraving which is too noticeable to be ignored.

New Hypothesis

Now I set up a new hypothesis, which corresponds with the result of the geometric analysis, is meaningful enough, explains about the patches like a vague figure on the pentagonal face, and explains also why and how Dürer drew the solid in the engraving in such a way. I'm going to argue below: 1) the solid had been primarily a truncated cube, which Dürer had drawn in perspective, 2) and then he enlarged the drawing vertically in a certain ratio; 3) the enlargement ratio is an approximate solution to the duplication of the cube, or the so-called Delian problem, which originated in ancient Greece. 4) He was interested in this problem and gave the full treatment of it in his book *Underweysung der Messung*. 5) It was such an important knowledge of geometry that he visualized in his work.

1) Significance of the Vague Figure on the Polyhedron

As mentioned above, there are some shadowy patches on the large pentagonal face of the polyhedron in the engraving. That the vague figure represents a skull or a phantom has been often reported [4]. As this shadowy image has been engraved on a hard copper plate, it cannot be a product of chance, like a stain caused by a drop or a rub on a watercolor. It must be meant to represent something especially. In both Dürer's *Knight, Death and Devil* (1513) and *St. Jerome in His Study* (1514), which along with *Melencolia* are collectively referred to as Dürer's three master prints, a skull is engraved. So it seems natural to consider that a skull exists in *Melencolia*, too. However, this skull is not represented in a normal way but as an anamorphic figure. We can recognize a skull when we view the engraving obliquely from the lower side. I drew a skull I found on an obliquely taken photograph of the polyhedron so that you exactly see what I mean (Fig. 4). So large that it completely occupies the pentagonal face. The skull I recognize has similar features to that of his *Coat-of-Arms with a Skull* (1503) (Fig.5), such as ear holes so dark as eye sockets, a waving hollow over the eyebrows, and two vertical lines carved

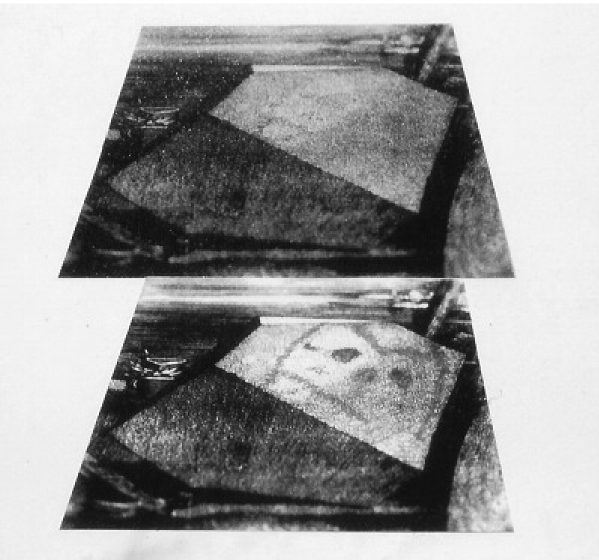

Fig. 4: Polyhedron viewed obliquely, with a sketch on it by Ishizu

Fig. 5: Albrecht Dürer, Detail of *Coat-of-Arms with a Skull* (1503, Engraving, 22.0×15.9 cm)

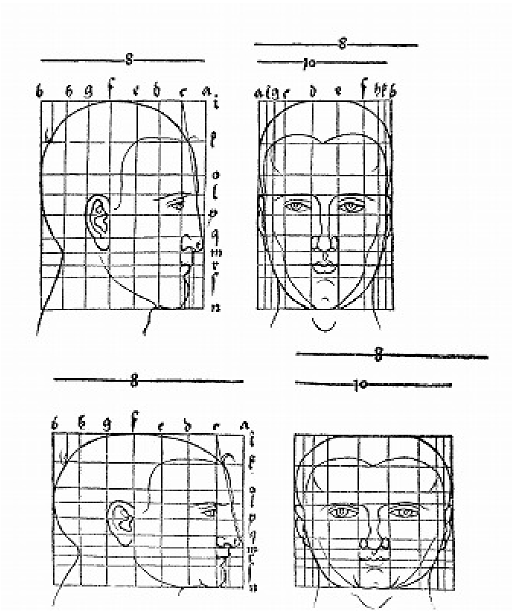

Fig. 6: Albrecht Dürer, *Transformation of a Head Inscribed in a Cube*, Illustration from *Vier Bücher von Menschlicher Proportion,* 1528, Woodcut

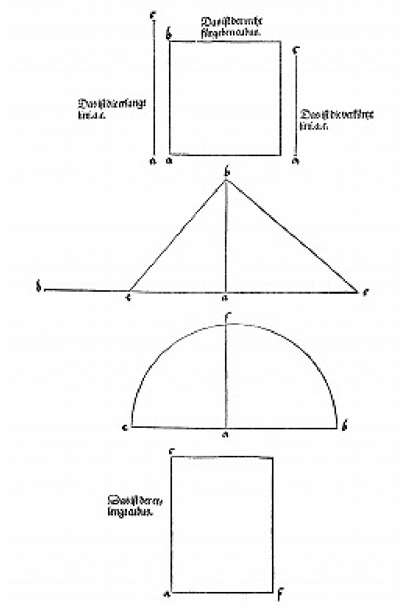

Fig. 7: Albrecht Dürer, *Geometric Method of Transformation*, Illustration from *Vier Bücher von Menschlicher Proportion,* 1528, Woodcut

on a cheekbone. As a way of viewing cannot be supported by logical argument, I must leave the judgment to the viewers. But even if my conviction is false, the question about the figure remains without a fitter answer. So it seems appropriate to regard the patches as a skull.

And when we see the engraving from the left lower side, just as we regard the patches as a skull, the polyhedron appears to be a truncated cube (see Figs. 4 and 14). That the patches appear to be a skull, and the polyhedron appears to be a cube at the same time, seems natural, as Dürer often associated a human head with a cube. In around 1514 Dürer used to draw a human head enclosed in a rectangle or a square as preliminary sketches for his work *Vier Bücher von Menschlicher Proportion* (Nuremberg, 1528). The book was an attempt to measure the proportion of human bodies numerically and to construct them with geometric figures. He tried to determine a face by measuring the proportion of each part — eyes, nose, and mouth — lengthwise and crosswise, which results in caging a head in a grid. Furthermore, he drafted a human head inscribed in a cube. Fig. 6 shows the transformation of a cube with an inscribed head into a rectangular parallelepiped with the constant volume. Fig. 7 shows its geometric method. The earliest manuscript of this transformation is dated c.1513 [5]. This proves that a cube with an inscribed human head was one of Dürer's basic concepts then, and so was a cube with an inscribed skull.

Therefore it can be inferred that the polyhedron and the vague pattern on its face are a cube and an inscribed skull, represented in an anamorphic manner, the latter being an indicator to guide us to a cube. In addition I quote a description of this polyhedron from an essay on the engraving *Melencolia* that is contained in *Elementa rhetoricae* (1541) by Joachim Camerarius (1500-1574), a contemporary of Dürer. He called the polyhedron "quadratum saxum", i.e. "a square block of stone" in it [6]. We may infer from these words that he might

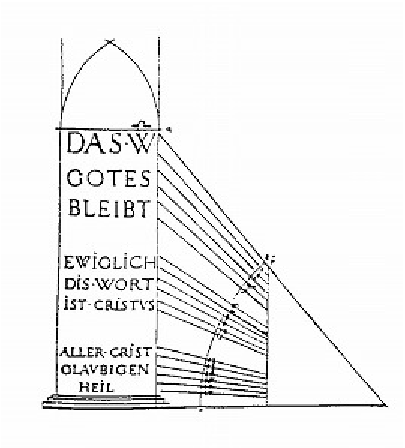

Fig. 8: Albrecht Dürer, *Letters on a Building*, Illustration from *Underweysung der Messung,* 1525, Woodcut

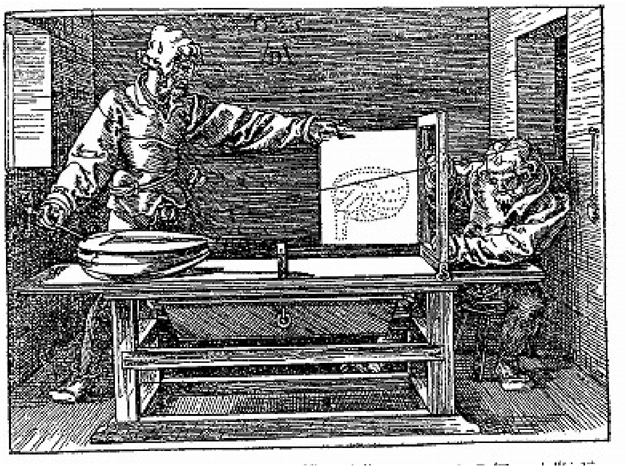

Fig. 9: Albrecht Dürer, *A Man Drawing a Lute*, Illustration from *Underweysung der Messung,* 1525, Woodcut

have been informed by Dürer that it had been primarily a square block, a cube. His description is reassuring support for my hypothesis. In general, woodcuts around 1525 by Erhard Schön, Dürer's pupil, and a painting titled *The Ambassadors* (1533) by Holbein the Younger are regarded as the earliest examples of anamorphosis. And Dürer is regarded to have produced no anamorphic work. Nevertheless, since he was so skilled in perspective, we cannot deny the possibility that he could have used anamorphosis partly in his work prior to them, as it is the reverse method of perspective.

In his book *Underweysung der Messung* (Nuremberg, 1525), Dürer indicates a method to write letters on a tower so that they would appear to be of the same size when viewed from the ground (Fig. 8). This shows that perspective and anamorphic drawing are reverse practices of the same principle. In the same book he writes a long explanation of perspective and shows an apparatus that helps to draw in perspective correctly (Fig. 9), the sketch of which dates back to 1514 [7]. So, we can conclude that he must have mastered not only perspective but also anamorphosis, when he engraved *Melencolia* using perspective in 1514. The abovementioned inference is fully reasonable.

2) Vertical Enlargement

But the transformation he used when he engraved *Melencolia* cannot be a method to draw a geometrically correct anamorphosis, as the polyhedron seems correct in perspective, without any distortion through the method [8]. The only possible method in this case is to enlarge the drawn figure vertically in a certain ratio. In this way the solid will turn to a longer shape, but it remains nevertheless "correct in perspective", because, as mentioned later, intersection points of lines don't become off horizontally—x-coordinates remain constant in xy-plane, in mathematical terms. And the original figure appears when viewed obliquely. This method was also used sometimes to draw anamorphic figures.

In order to explain the transformation in detail, I must mention some matters caused by the rules of perspective. Fig. 10 (a) is a model showing the principle of linear perspective. The

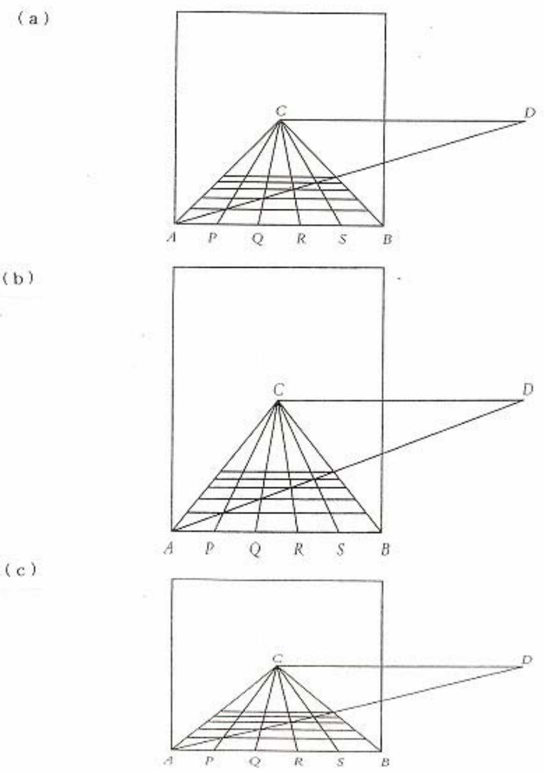

Fig.10: Model of perspective, (a) Fig. 2.8 of Field, J.V., *The Invention of Infinity*, Oxford, New York, Tokyo, 1997, p. 30, (b) Vertically enlarged figure of (a) to 125% size, (c) Vertically reduced figure of (a) to 80% size, (b and c transformed by Ishizu)

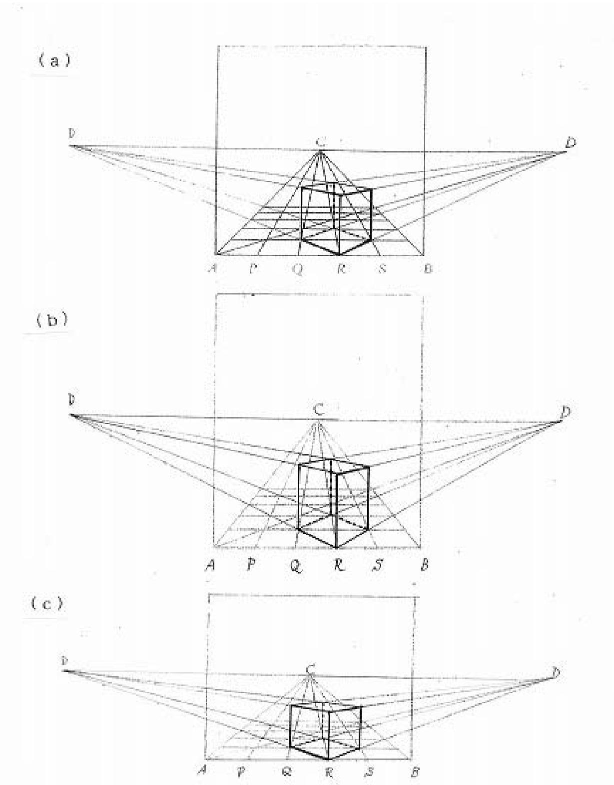

Fig.11: (a) Model of perspective with a cube, (b) Vertically enlarged figure of (a) to 125% size, (c) Vertically reduced figure of (a) to 80% size (all by Ishizu)

horizontal gauge is fixed at even intervals, and C is the center of vision, that is, the vanishing point of lines that are perpendicular to the picture plane. D is the distance point, that is, the vanishing point of lines that are parallel to the ground plane and cross the picture plane at a 45° angle. (b) is a vertically enlarged figure of (a) (to 125% size of (a)), (c) a vertically reduced figure of (a) (to 80% size of (a)). We see that intersections of vertical or oblique lines and transversals won't slide horizontally, that is, the x-coordinates of the intersections remain constant, even if the figure is vertically enlarged or reduced in a certain ratio. So, Fig. 10 (b) can be identified as a perspective model with the same horizontal gauge, the same distance as (a), because the length between the center of vision and the distance point remains constant, but with a higher eye-level than (a), 1.25 times higher than (a). In the same way Fig. 10 (c) can be regarded as the case viewed from a lower view point, while the distance and the horizontal gauge are the same as (a). In Fig.11 (a), adding an obliquely set cube at a 45° angle, we can see similarly as above, that the horizontal gauge remains constant, and that a square on a horizontal plane continues to be a square, the upper face and the base of the cube in this case, when the whole figure is vertically enlarged or reduced, because the vanishing points of the extensions of the sides on the horizon line remain constant. But the height of the cube changes and the cube turns to a rectangular parallelepiped. In Fig. 11 (b), vertically enlarged to 125% of (a), the height of the cube is also increased to 125% of (a), just like the height of the eye-level.

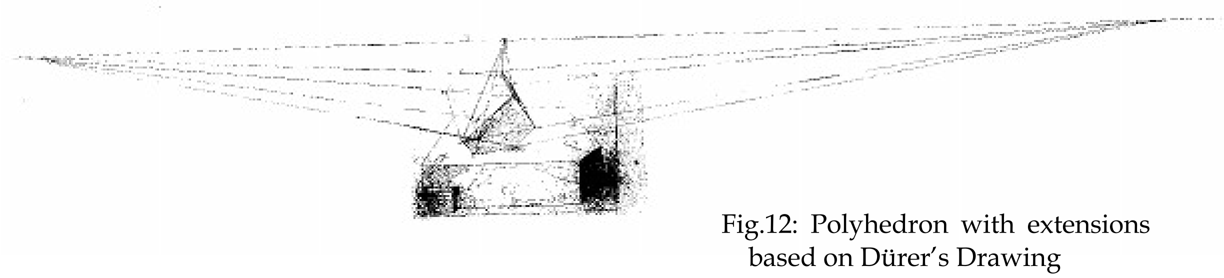

Fig.12: Polyhedron with extensions based on Dürer's Drawing (drafted by Ishizu)

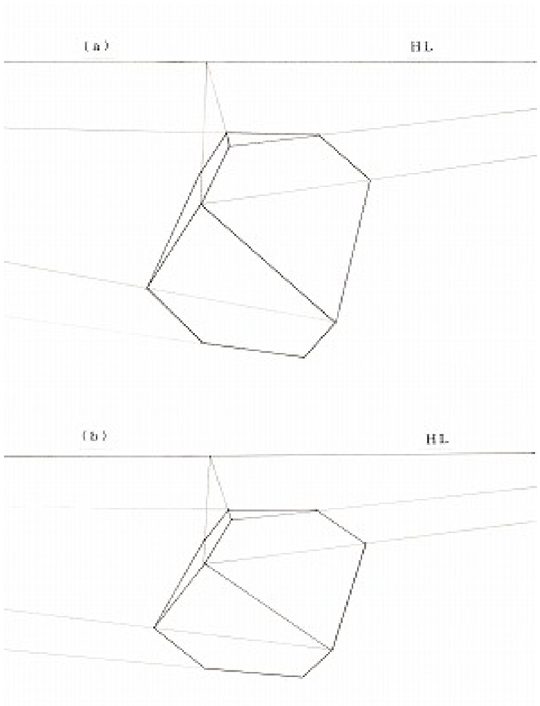

Fig.13: Model of the center part of Fig. 12 and a vertically shortened figure of it to 79% (by Ishizu)

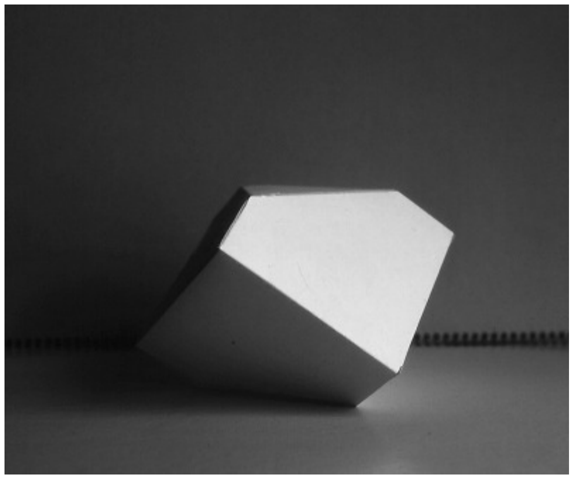

Fig.14: Truncated cube (by Ishizu)

The same is realized when reduced. In conclusion we can point out some characters about figures drawn in perspective: plane figures drawn on a horizontal plane continue to be regarded as the same figures, when the whole picture plane is regularly enlarged (or reduced) in the vertical direction, but the height of the eye-level and the height of a drawn solid turn longer (or shorter) depending on the ratio of enlargement (or reduction).

Let us now apply this transformation to our polyhedron. Fig. 12 shows the polyhedron with extensions of its edges and diagonals based on the preliminary sketch [9]. Every group of parallel lines converges at each vanishing point. Lines that are perpendicular to the picture plane converge at the center of vision. Lines that meet the picture plane at a 30° angle converge at their vanishing point on the horizon line, right and left far off. Fig. 13 (a) is the mirror image of the central part of Fig. 12. Reduced vertically to 79% size it turns to Fig. 13 (b). In this figure, the upper face and the base are both recognized as equilateral triangles, as the extensions of the sides converge at three vanishing points on the horizon line just as those in Fig. 13 (a). And the polyhedron will be recognized to be correctly drawn in perspective with the same distance point and horizontal gauge as Fig. 13 (a). This figure actually appears to be a truncated cube (see Fig. 14). And if we enlarge it vertically, it turns to a truncated rhombohedron as Fig. 13 (a). In conclusion, we can presume two cases how Dürer may have drawn the polyhedron, both of which are equally possible. One is simply to draw a truncated rhombohedron correctly in perspective, which has been accepted as an obvious premise. The other is to draw a truncated cube first, and then to enlarge it vertically, as argued here. We just cannot determine in which way he actually did [10]. From neither the engraving nor the sketch we can trace it.

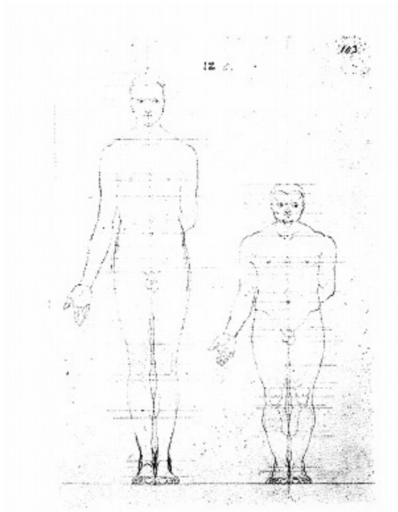

Fig.15: Albrecht Dürer, *Tall and Short Man Compared, Constructed,* 1514, Drawing, Dresden Landesbibliothek

Of course Dürer could practice such a transformation skillfully then. Fig. 15 is a preliminary sketch for an illustration in his book *Vier Bücher von Menschlicher Proportion*, and dated 1514. It shows how to draw a thin tall man and a fat short man based on an average body. Dürer divides a human body into about 30 components with horizontal lines, and then enlarges the body vertically, maintaining the vertical proportion of each part, by a method named "verkerer"(Verkehrer) [11]. Meanwhile, there is no horizontal deformation, the width of each part remains constant. Thus he constructed a thin tall man. And a fat short man on the other hand by reducing a standard model by the same method. As for this polyhedron, he only had to care about 12 vertexes and a vanishing point shown by an eye, that is, 13 points in all, the transformation is easier than that of a human body by far. So it is appropriate to infer that Dürer drew thus, by enlarging vertically, a new solid which seems different from a truncated cube, and that he drew an anamorphic skull on it, so that its original substance would not be missed. And this transformation, to enlarge a figure in one direction, horizontally or vertically, keeping the proportion of each part constant, was sometimes used as a method of drawing anamorphosis in the sixteenth and seventeenth cen-turies, which Baltrušaitis calls "old-fashioned" anamorphosis [12]. In the preliminary sketch of the polyhedron, each vertex bears a mark, which leads some to conclude that Dürer had drafted the polyhedron based on a plan and an elevation. Meanwhile, others insist that he drew it with the help of a convenient apparatus for perspective drawing which was introduced in his book *Underweysung der Messung*, estimating his ability of perspective draft not high enough [13]. But it is not our present concern to clarify in which way he drew it, a truncated cube in my view, as this is irrelevant to our argument. In any case, the marks can be also regarded as the remains of the transformation, which seems highly possible.

3) Ratio of Vertical Enlargement

As argued above, a hypothesis that "the solid is an anamorphic figure of a truncated cube", which was drawn from the skull-like patches on the large pentagon, does not contradict the fact that "the solid was drawn correctly in perspective." Hence, it is feasible to set up a hypothesis that the polyhedron is a vertically enlarged truncated cube.

But why on earth did he transform a cube in such a way? My answer is that the enlargement ratio indicates the solution to the Delian problem, a famous geometric problem which originated in ancient Greece. We can calculate the enlargement ratio easily. See Fig. 16. If we enlarge a cube in the direction of the line ah to make a rhombohedron whose constituent faces are rhombuses with the angles 80° and 100°, the ratio of enlargement is about 1.277. The procedure of calculation is as follows.

When we enlarge the figure of the cube only vertically in a certain ratio, an equilateral triangle $\triangle$bcd on a horizontal plane turns to $\triangle$BCD in the transformed rhombohedron, and both triangles are regarded as congruent. (i.e. $\triangle bcd \equiv \triangle BCD$.) Similarly, $\triangle efg \equiv \triangle EFG$. So we are to calculate the ratio of the segment AH to the segment ah, provided that bc =BC.

Let the length of each edge of the cube be 1, then the length of the diagonal of each square is $\sqrt{2}$, and the length of the segment ah is $\sqrt{3}$. Fix BC of the rhombus with an acute angle 80° to be $\sqrt{2}$. By the equality tan 50° = AM／BM, and BM=$\sqrt{2}/2$, it follows that AM = $\sqrt{2}/2 \times \tan 50° \approx 0.8427$.

On the other hand, in $\triangle$BCD, DM = $\sqrt{2} \times \sqrt{3}/2$, MN = $\sqrt{6}/6$.

In $\triangle$AMN, by applying Pythagorean theorem, and by substituting numbers obtained above, it follows,

$AN^2 = AM^2 - MN^2 \approx (0.8427)^2 - (\sqrt{6}/6)^2$, AN $\approx$ 0.7372.

$\therefore$ AH = 3AN $\approx$ 2.2116.

So we obtain the ratio of AH to ah as

$2.2116/\sqrt{3} \approx 1.277$.

So we may regard that the rhombohedron as 1.277 times the enlarged cube, in the direction of the line ah.

This number 1.277 per se seems apparently to have no significant meaning. But it suggests a very important problem famous in mathematical history which originated in ancient Greece. I regard this number as an approximate solution to the Delian problem, namely the duplication of the cube, that is, to obtain the edge of a cube which has twice the volume of the given one. To solve the Delian problem algebraically is to obtain x such that $x^3=2$, and its approximate value is 1.26. (i.e. $1.26^3 \approx 2$.) I represent the solution with a number as an expedient for today readers to grasp it easily. But in ancient Greece and also in Dürer’s time, the problem was regarded only as a geometric construction problem, which was to be solved by a compass and a straightedge.

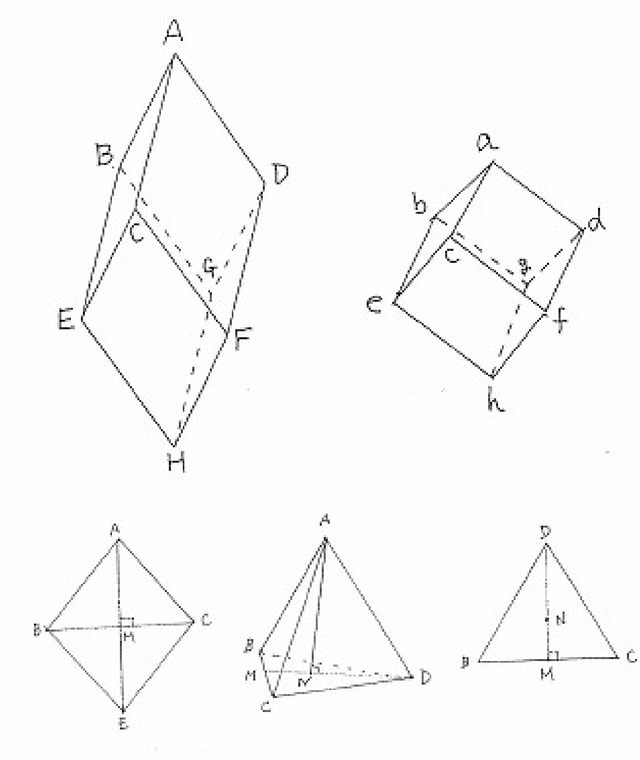

Fig.16: Rhombohedron, cube, rhombus (each face of the rhombohedron), triangular pyramid ABCD, and equilateral triangle BCD (all drafted by Ishizu)

Let us now verify the account in a

reverse way. See Fig. 16. If we enlarge the cube in the direction of the line ah 1.26 times, the number being an approximation of the solution to the Delian problem, keeping triangles △bcd and △efg in the original space unchanged, the cube turns to a rhombohedron, and in this case, the acute angle of each rhombus face is calculated to be about 80°34′ [14]. So we see that the abovementioned argument is quite reasonable. If we, or Dürer himself, draw a cube and enlarge it in the ratio, which approximates the solution to the Delian problem, a rhombohedron whose faces are about 80°−100° rhombuses appears as a result.

Although the numbers we obtained do not match quite accurately, we may regard the slight difference as a tolerance, which does not disturb to hypothesize that the ratio of the vertically enlarged polyhedron to the cube represents the solution to the Delian problem. I may remark a cube itself suggests the Delian problem potentially, because this problem seems to be strictly associated with cubes and squares in Dürer's mind then, as mentioned later. He could have indicated the problem with a cube, and its solution by the ratio of enlargement. How can the problem be otherwise visualized? To put two cubes one of which has double the volume of the other? Or to enlarge a normally set cube vertically only to show a rectangular parallelepiped? Both aren't good enough. We may admit the superb polyhedron is a unique device to demonstrate the Delian problem.

Assuming the above, we can put forward an interpretation of the stone polyhedron which explains its geometric construction based on measurement, the vague patches on a face in the engraving, and the geometric significance, all without contradiction. Fig.17 is a vertically reduced figure of the polyhedron to 79% size of the original, which in fact appears to be a truncated cube, as the reciprocal of 1.277 is about 0.783, of 1.26 about 0.794. It's sufficient to compare it with the cube (Fig. 14) in order to confirm the argument stated above visually.

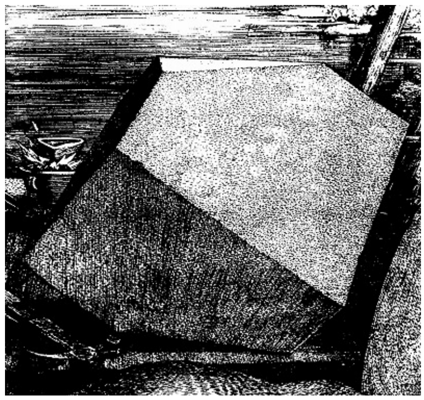

Fig.17: Polyhedron Vertically Reduced to 79% (by Ishizu)

4) The Delian Problem

As proved above, it is an objective fact that the polyhedron is a vertically enlarged figure of a truncated cube, and that the enlargement ratio indicates the solution to the Delian problem. Now we only have to search for Dürer's motivation for visualizing it. First, I explain about the problem of the duplication of the cube [15]. It is named the Delian problem after a Greek legend about the oracle for Delians to construct a cubical alter to Apollo which has double the volume of the existing one, in order to get rid of an epidemic sweeping over the country, but they could not succeed in it. The problem is renowned as one of the "three problems of geometric construction", along with the quadrature of the circle and the trisection of the angle, which also originated in ancient Greece. They are now called the three Greek problems of impossible construction, as they have been already proved to be impossible to solve with a compass and a ruler in the nineteenth century, after they have attracted mathematicians for far more than 2000 years. To solve the Delian problem, we only need, when a and b are given, to obtain x and y such that a : x = x : y = y : b, which leads to $a^3 : x^3 = a : b$, upon the elimination of y. Since Hippocrates of Chios found it in the fifth century BC, the Delian problem has been regarded as equivalent to the problem of finding two proportional means between two given lines in the abovementioned form. Provided that b=2a, it'll be the duplication of a cube, to obtain x such that $x^3=2a^3$. More than ten ancient Greek mathematicians had solved the problem in different ways, which are known to us as Eutocius (sixth century AD) reported them in his commentary on Archimedes' *On the Sphere and Cylinder*. The works of Archimedes were newly translated (from Greek into Latin) in the fifteenth century and became widely known again through publications in the sixteenth century, and incidentally the Delian problem mentioned in commentaries on them drew mathematicians' attention, too [16]. We may name for instance *De expetendis et fugiendis rebus opus* (Venice, 1501) translated by Giorgio Valla as an early example of such publications. In Nuremberg, Johannes Werner, a mathematician and an acquaintance of Dürer, introduced eleven solutions to the problem in his book *In hoc opera haec continentur. . .* (Nuremberg, 1522), which is an adaptation of Valla's book. Heinrich Schreiber (Grammateus) also introduced one solution and the original legend of the problem in his book *Ayn new kunstlich Buech* (Nuremberg, 1518). Such publications tell that the problem was popular and significant then. And Dürer also deals with this problem in his book on geometry *Underweysung der Messung*, in the fourth chapter which is devoted to solid geometry and perspective. He employs about ten pages for the problem, describing the original episode and three different solutions, while he briefly introduces only one construction for each of the other two of the "three problems of geometric construction" in the second chapter, which is devoted to plane geometry. Thus the Delian problem proves to be a matter of utmost concern for him. He writes indeed so proudly as follows: "This is a very useful skill for all workmen. This method is kept secret and hidden by scholars, but I wish to bring it to light and teach it. ...Therefore, let all artisans pay heed, because to this day, as far as I know, no one has explained it in the German language."[17] This shows his enthusiasm for the problem and also contemporary concern for it. (But it was not actually the first published explanation in German, as Schreiber's book appeared earlier. We can see that Dürer wrote the draft before 1518.) He adopts the solution of Sporus, or of Pappus (Fig. 18), as both are alike, to explain constructions and variations, and then shows applications and how to put them to practical use based on the obtained solution. He restricts his description only to intelligible cases of concrete multiples by natural numbers such as double, three times, and so on, all the while showing illustrations connected with cubes and squares. But in the other two solutions, one attributed falsely to

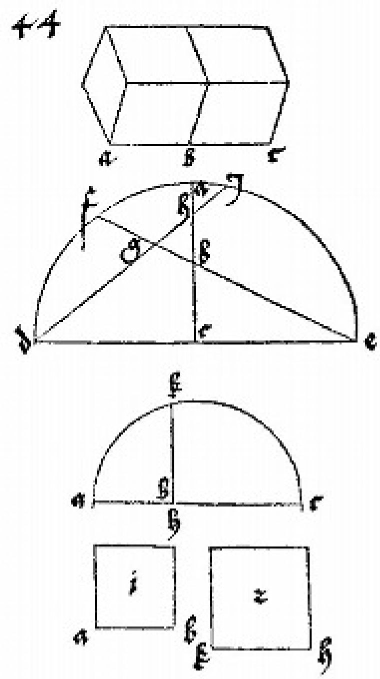

Fig.18: Albrecht Dürer, *Illustration of a Solution to the Delian Problem*, Illustration from *Underweysung der Messung*, 1525, Woodcut

Plato and the other of Heron, he deals with the problem as that of proportional means in a generalized and rather abstract form. They are essentially different from other parts of description that are concrete and plain for the laity, and seem to be inserted in the whole plot afterwards. Except for these two solutions, we can see that he handled the problem within the scope of cubes on the whole, from introduction to the last description of the practical use, including 7 illustrations among 9. This allows us to infer that he could have visualized the problem with a cube. And we also see that the problem was worthy and striking for him, as we can read from the introductory narration and the long pages that he felt obliged to extend this knowledge widely, being convinced of its usefulness.

5) Visual Demonstration of the Delian Problem

So it can be stated that a cube alludes to the Delian problem potentially, which was a summit of geometry for Dürer. As he sets the problem at the end of solid geometry, that is, at the end of one-, two-, and three-dimensional geometry in his book [18], it is reasonable to consider that he might have regarded the problem as a superior representation of solid geometry, and furthermore, of geometry as a whole. Besides, the problem seems to have been up-to-date in his time, as some publications prove, which was remarkable and appealing enough to be visualized in the elaborate engraving for him, and also for those the engraving was aimed at.

Perhaps Dürer had abovementioned concept already when he engraved *Melencolia* in 1514. Though his *Underweysung* was published in 1525, it is recognized that he had reserved its manuscripts beforehand [19]. As for the Delian problem, besides some pre-publication drafts, some early manuscripts remain: the figures of the first solution and its application to obtain cubes that are a half, four and eight times the size of the given one (1513/15), and some description of the second solution with a figure (1513) [20]. As he must have drafted the main part of the problem before the second and the third solutions were added, he probably had mastered the problem by 1513. It follows that we can confirm that he regarded it highly already then, which is revealed in his book. In *Melencolia*, many subjects of mathematics such as a compass, a ruler, and a magic square on the wall are depicted. They allow us to associate the engraving with mathematics. Among them, the particularly large polyhedron has a significant meaning, the important knowledge of geometry, as logically inferred above. Dürer addressed the knowledge "all workmen" as he wrote in his book [21]. Beside the polyhedron lies a hammer, and a melting pot on a fire together with pincers are set behind. They remind us of carving and casting, where the knowledge is most useful. In fact he chose the case of casting cannonballs as a practical application of the Delian problem in his book. It seems as if he emphasized the application of the knowledge to practical field with these instruments.

We may presume that Dürer learned the solution to the problem from the library of Regiomontanus (1436-1476). The second and the third solutions are just a direct translation (from Latin into German) of those contained in *De arte mensurandi*, written by Johannes de Muris in the fourteenth century, as was pointed out by Marshall Clagett [22]. A copy of the work belonged to Regiomontanus' library and was regarded as his own work under the title *Commensurator* until the mid-twentieth century. As for the first solution, Clagett named the following three possible sources at the time Dürer's *Underweisung* appeared in 1525. Valla's book mentioned above, a manuscript of another translation of the same work of Archimedes with Eutocius' commentary which Regiomontanus possessed, and Werner's book mentioned above. As Dürer's earliest manuscript of the first solution dates back to 1513, and Werner's book appeared in 1522, the former two are valid. While it's not clear from when Valla's book, the source of Werner's book, was in Nuremberg, the manuscript which Regiomontanus possessed had been in Nuremberg since he moved to the city in 1471. Regiomontanus established his own print shop there to publish many books on astronomy and mathematics. He notified the publication of the said work of Archimedes in an advertising leaflet printed at his own printing office in 1473, and its manuscript is listed in both of the inventories compiled in 1512 and 1522, which proves that it had been in Nuremberg all along [23]. As the library of Regiomontanus was under the control of his student Bernhard Walther, a friend of Dürer's father, Dürer is regarded to have investigated it at will. There is a strong possibility that he learned the solution from the library, as he does not seem to have missed any chance.

Regiomontanus, the most prominent astronomer and mathematician in the fifteenth century, moved to Nuremberg because it was convenient to communicate with scholars in distant places there, being the mercantile center of Europe. And because precision instruments for astronomical observation were crafted there. Such a background seems to have prepared good grounding in accepting an expert knowledge of mathematics for the educated class including Dürer, that is, the creator and the appreciators of *Melencolia*.

Notes

[1] Strauss, Walter L., *The Complete Drawings of Albrecht Dürer, Vol. 3. 1510-1519*, New York, 1974, p. 1436. De Haas, Karel, *Albrecht Dürer's Engraving Melencolia §I: A Symbolic Memorial to the Scientist Johann Müller (Regiomontanus)*, Rotterdam, 1951.

Apart from them Wangart has measured by use of graphics the longest segment between two opposite vertexes of the polyhedron, on the assumption that each face of it before truncation is a square, to be $\sqrt{3}$ times as long as the side of the reconstructed square, in order to prove that the polyhedron is a cube. But his conclusion may be a coincidence resulting from identifying two different lines that appear to be one on the draft. (Wangart, Adolf, "Der Geometrische Körper in Dürers 'Melencolia'", *Humanismus und Technik* 20, 1976, p. 16-27.)

[2] Those who insist that the acute angle is 72° include the following. Grodzinski, Paul, "Diamond-Geometry", *Industrial Diamond Review*, Vol. 15, 1955, p. 66-76. Rösch, Siegfried, "Die Bedeutung des Polyeders in Dürers Kupferstich 'Melancholia I' (1514)", *Fortschritte der Mineralogie 48. Beiheft,* 1970, p. 83-85. Enomoto, Kazuko, "Dürer no tamentai ni tsukarete", *Mizue*, no. 891, 1979, p. 88-95. —, "'Infinite Vision: The Octahedron' A. Dürer 'Melencolia I'" (Translated by Reiko Tomii), *The 18th Exhibition Homage to Shuzi Takiguchi*, 1998. Schreiber, Peter, "A New Hypothesis on Dürer's Enigmatic Polyhedron in His Copper Engraving 'Melencolia I'", *Historia Mathematica* 26, 1999, p. 369-377.

Those who insist that the acute angle is about 80° include the following. Richter, David Hch., "Perspektive und Proportionen in Albrecht Dürers 'Melancholie'", *Zeitschrift für Vermessungswesen* 86, 1957, p. 284-288, p. 350-357. Harnest, Joseph, "Theorie und Ausführung in der perspektivischen Raumdarstellung Albrecht Dürers", *Festschrift Luitpold Dußler*, München, 1972, p. 189-204. Schröder, Eberhard, *Dürer Kunst und Geometrie*, Basel, Boston, Stuttgart, 1980, p.64-75.

MacGillavry, Caroline H., "The Polyhedron in A. Dürer's Melencolia I: An over 450 years old puzzle solved?", *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen. Series. B, Palaeontology, Geology, Physics and Chemistry*. Vol. 79, no. 3, 1981, p. 287-294. Lynch, Terence, "The Geometric Body in Dürer's Engraving Melencolia I", *Journal of the Warburg and Courtauld Institutes*, XLV, 1982, p. 226-232. Sixel, Gustav, "Zum Prismatoid in Dürers 'Melancholie'", *Zeitschrift für Vermessungswesen* 115, 1990, p. 23-28. Engelhardt, Wolf von, "Dürers Kupferstich >Melencolia I<", *Städel- Jahrbuch,* 14, 1993, p. 173-198. For Niemann's idea, see Klibansky, R., Panofsky, E., and Saxl, F., *Saturn and Melancholy: Studies in the History of Natural Philosophy, Religion and Art*, Nendeln, 1979 (Reprint. First published in London in 1964), p. 400-402.

By the way, Deckwitz insists that the angle is 76° which is associated with pyramids. But I cannot discuss the justification of his argument. (Deckwitz, Franz, *Dürer's MELENCOLIA with Compasses and Ruler*, Amsterdam, 1978.)

[3] A diagonal of an equilateral pentagon is divided by another diagonal in the ratio of the golden section. Two isosceles triangles made of two sides and a diagonal of a equilateral pentagon (attached at diagonals) form a rhombus whose acute angle is 72°.

[4] Reutersward, Patrik, "Sinn und Nebensinn bei Dürer. Randbemerkungen zur Melencolia I", *Gestalt und Wirklichkeit. Festgabe für Ferdinand Weinhandl*, Berlin, 1967, p. 411-436. Lanckorońska, Maria, "Die zeitgeschichtliche Komponente in Dürers Stichpaar >Hieronymus im Gehäus< und >Melencolia I<", *Gutenberg Jahrbuch*, 1975, p. 253-273. Büchsel, Martin, "Die gescheiterte >Melancholia Generosa< MELENCOLIA, I", *Städel- Jahrbuch*, 9. 1983, p. 89-114. Schuster, Peter-Klaus, *MELENCOLIA I Dürers Denkbild*, Berlin, 1991, p.151, p.158, p.181.

Kilian suggests that it is a phantom's face. (Kilian, Werner G., "Eine Studie zu Dürers Melencolia", *Wissenschaftliche Zeitschrift der Karl-Marx-Universität Leipzig, Gesellschafts- und Sprachwissenschaftliche Reihe*, Jg. 10, 1961. p. 617-634.)

[5] For sketches of a head in a grid, see, for example, Strauss, *The Complete Drawings of Albrecht Dürer, Vol. 5, Human Proportions*, New York, 1974, p. 2469 (dated 1512). For the transformation of a cube with an inscribed head, see *Dürer Schriftlicher Nachlass II*, Herausgegeben von Hans Rupprich, Berlin, 1966, p.447-454.

[6] Heckscher, William S., "Melancholia (1541). An Essay in the Rhetoric of Description by Joachim Camerarius*", Joachim Camerarius (1500-1574),* Herausgegeben und eingeleitet von Frank Baron, München, 1978, p. 31-120. For Camerarius' words, see p.32-33.

[7] Strauss, *The Complete Drawings of Albrecht Dürer, Vol. 3, 1510-1519,* p.1472-1473.

[8] For the method of geometrically correct anamorphosis, see Baltrušaitis, Jurgis, *Anamorphic Art*, translated by W.J. Strachan, Cambridge, 1977, p. 37-60.

[9] On the assumption that the polyhedron is exactly in perspective, one vertex must be about 3 mm higher in the sketch, which is in the opposite area, and we cannot see it in the engraving.

[10] Nagel assumed that Dürer had enlarged a truncated cube in the direction parallel to the ladder. But in this case the axis of the rhombohedron becomes oblique, parallel to the ladder, which does not correspond with the fact that it is vertical. (Nagel, Friedrich Augst, *Der Kristall auf Dürers Melancholie*, Nürnberg, 1922.)

[11] For Verkehrer, see Albrecht Dürer, *Hierinn sind begriffen vier Bücher von menschlicher Proportion*, Neudruck der Ausgabe Nürnberg 1528, dritte Auflage, Nördllingen, 1996.

[12] Baltrušaitis, p. 35, p. 36.

[13] Those who insist on the draft based on a plan and an elevation are Richter, Schröder, Lynch, Schreiber, in note [2], while Wangart and Harnest insist on the use of apparatus.

[14] See Fig.16. Let α be an acute angle of each rhombus, a face of a rhombohedron such that: the length of the shorter diagonal of a rhombus is $\sqrt{2}$, and the length of the segment AH in the rhombohedron is $1.26 \times \sqrt{3}$. Then we see, $AN = \sqrt{3}/3 \times 1.26$. In $\triangle AND$, $AD^2=AN^2+DN^2 = 1/3 \times 1.26^2 + (\sqrt{6}/3)^2 \approx 1.19587$. So, it follows that $AD \approx 1.0936$ (=AB). So, $\sin \alpha/2 = BM/AB$

≈ 0.646612 . We obtain $\alpha \approx 80°34'30''$ from the table of sines.

[15] For the Delian problem, see Boyer, Carl B., *A History of Mathematics*, second edition, revised by Merzbach, Uta C., New York, 1991, p. 64-65, p. 71-72, and Heath, Thomas L., *A Manual of Greek Mathematics*, New York, 2003 (first published by Oxford University Press in 1931), p. 131, p. 154-170.

[16] For the translation and publication of the works of Archimedes in the fifteenth century, see Clagett, Marshall, *Archimedes in the Middle Ages, Volume Three, The Fate of the Medieval Archimedes 1300 to 1565*, Philadelphia, 1978, p. 297-475. In particular, for the Delian problem or the problem of proportional means, see p. 1163-1179.

[17] *The Painter's Manual by Albrecht Dürer*, translated and with a Commentary by Walter L. Strauss, New York, 1977, p. 346-347.

[18] The book contains four chapters (called four books). The first is devoted to linear geometry, the second to plane geometry, the third to architecture and engineering, and the fourth to solid geometry and perspective. The Delian problem is placed between some description of solids and that of perspective.

[19] Strauss, *The Painter's Manual*, p. 9.

[20] *Dürer Schriftlicher Nachlass III*, Herausgegeben von Hans Rupprich, Berlin, 1969, p. 355, p. 359-360, Tafel 82, Nr. 275, 276, Tafel 83, Nr. 278, 279.

[21] Strauss, *The Painter's Manual*, p. 346-347.

[22] Clagett, p. 346-347, p.1169-1170.

[23] Zinner, Ernst, *Regiomontanus*: *His Life and Work*, Translated by E. Brown, Amsterdam, New York, Oxford, Tokyo, 1990 (*Leben und Wirken des Joh. Müller von Königsberg, genannt Regiomontanus*, Osnabrück, 1968), p. 216-217.

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