LECTURE NOTES 16

THE STRUCTURE OF SPACE-TIME

Lorentz Transformations Using Four-Vectors:

Space-time {as we all know…} has four dimensions:
1 time dimension & 3 {orthogonal} spatial dimensions: $(t, \vec{r} = x\hat{x} + y\hat{y} + z\hat{z})$.

Einstein's Theory of (Special) Relativity:

1-D time and 3-D space are placed on an ***equal/symmetrical*** footing with each other.

We use 4-vector/tensor notation for ***relativistic kinematics*** and ***relativistic electrodynamics*** because the mathematical description of the physics takes on a simpler, and more elegant appearance; the principles and physical consequences of the physics are also made clearer/more profound!

Lorentz Transformations Expressed in 4-Vector Notation:

Note the ***contravariant superscripts***, here!

We define ***any*** 4-vector: $x^\mu \equiv (x^0, x^1, x^2, x^3)$

Where, by convention: the 0th component of the 4-vector, x^0 = is the ***temporal*** (***time-like***), {*i.e.* ***scalar***} component of the 4-vector x^μ, and (x^1, x^2, x^3) are the (x, y, z) ***spatial*** (***space-like***) {*i.e.* 3-***vector***} components of the 4-vector x^μ, respectively.

n.b. Obviously, the physical SI units of a 4-vector components ***must all*** be the ***same***!!!

For ***space-time*** 4-vectors, we define ***contravariant/superscript*** x^μ as:

$$\left.\begin{array}{l} x^0 \equiv ct \\ x^1 \equiv x \\ x^2 \equiv y \\ x^3 \equiv z \end{array}\right\} \quad x^\mu \equiv (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

Then the Lorentz transformation of space-time quantities in IRF(S) to IRF(S'), the latter of which is moving *e.g.* with velocity $\vec{v} = +v\hat{x}$ relative to IRF(S) is given by:

Original 4-vector Notation:			New/Tensor 4-vector Notation
	$ct' = \gamma(ct - \beta x)$	$\Rightarrow$	$x'^0 = \gamma(x^0 - \beta x^1)$
	$x' = \gamma(x - \beta ct)$	$\Rightarrow$	$x'^1 = \gamma(x^1 - \beta x^0)$
	$y' = y$	$\Rightarrow$	$x'^2 = x^2$
	$z' = z$	$\Rightarrow$	$x'^3 = x^3$

Where: $\beta \equiv v/c$ and: $\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}$

We can also write these four equations (either version) in matrix form as:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad \text{or:} \quad \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Each of the four above equations of the RHS representation can also be written compactly and elegantly in tensor notation as:

$$\boxed{x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\nu}}$$ where: $\mu = 0, 1, 2, 3$ and: $\Lambda \equiv$ Lorentz Transformation Matrix

$$\Lambda \equiv \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \Lambda^0_0 & \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 \\ \Lambda^2_0 & \Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 \\ \Lambda^3_0 & \Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 \end{pmatrix}$$

← ***row*** index

↓ ***column*** index

superscript, $\mu = 0, 1, 2, 3$ = ***row*** index

where: $\Lambda^{\mu}_{\nu} = \Lambda^{row}_{column} = (\mu - \nu)^{th}$ element of Λ

subscript, $\nu = 0, 1, 2, 3$ = ***column*** index

We explicitly write out each of the four equations associated with $x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\nu}$ for $\mu = 0, 1, 2, 3$:

$$\boxed{x'^0 = \sum_{\nu=0}^{3} \Lambda^0_{\nu} x^{\nu} = \Lambda^0_0 x^0 + \Lambda^0_1 x^1 + \Lambda^0_2 x^2 + \Lambda^0_3 x^3 = \gamma\left(x^0 - \beta x^1\right)}$$

$$\boxed{x'^1 = \sum_{\nu=0}^{3} \Lambda^1_{\nu} x^{\nu} = \Lambda^1_0 x^0 + \Lambda^1_1 x^1 + \Lambda^1_2 x^2 + \Lambda^1_3 x^3 = \gamma\left(x^1 - \beta x^0\right)}$$

$$\boxed{x'^2 = \sum_{\nu=0}^{3} \Lambda^2_{\nu} x^{\nu} = \Lambda^2_0 x^0 + \Lambda^2_1 x^1 + \Lambda^2_2 x^2 + \Lambda^2_3 x^3 = x^2}$$

$$\boxed{x'^3 = \sum_{\nu=0}^{3} \Lambda^3_{\nu} x^{\nu} = \Lambda^3_0 x^0 + \Lambda^3_1 x^1 + \Lambda^3_2 x^2 + \Lambda^3_3 x^3 = x^3}$$

We can write this relation even more compactly using the ***Einstein*** ***summation*** ***convention***: Repeated indices are ***always*** summed over:

$$\boxed{x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\nu}}$$

The RHS of this equation has repeated index ν, which ***implicitly*** means we are to sum over it, *i.e.*

Thus: $\boxed{x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}}$ is simply ***shorthand*** ***notation*** for: $\boxed{x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\nu}}$

People (including Einstein) got / get tired of explicitly writing all of the summation symbols $\sum_{\nu=0}^{3}$ all the time / everywhere….

The nature/composition of the Lorentz transformation matrix Λ (a rank-two, 4 x 4 = 16 component tensor) defines the space-time structure of our universe, *i.e.* specifies the rules for transforming from one IRF to another IRF.

Generally speaking mathematically, one can define a 4-vector a^μ to be ***anything*** one wants, however for ***special relativity*** and ***Lorentz transformations*** between one IRF and another, our 4-vectors are ***only*** those which transform from one IRF to another IRF as:

$$\boxed{a'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu a^\nu} \quad \Rightarrow \quad \boxed{a'^\mu = \Lambda^\mu_\nu a^\nu}$$

This compact relation mathematically defines the space-time nature/structure of our universe!

For a Lorentz transformation along the $\hat{1} = \hat{x}$ axis, with: $\boxed{\vec{v} = +v\hat{x}}$ and thus: $\boxed{\vec{\beta} = \beta\hat{x}}$, $\boxed{\vec{\beta} = \vec{v}/c}$ for a 4-vector $\boxed{a^\mu = \left(a^0, a^1, a^2, a^3\right)}$, where a^0 is the ***temporal/scalar*** component and $\boxed{\vec{a} = \left(a^1, a^2, a^3\right) = \left(a_x, a_y, a_z\right)}$ are the $\left(\hat{x}, \hat{y}, \hat{z}\right)$ ***spatial/3-vector*** components of the 4-vector a^μ, then $\boxed{a'^\mu = \Lambda^\mu_\nu a^\nu}$ written out in matrix form is:

$$\begin{pmatrix} a'^0 \\ a'^1 \\ a'^2 \\ a'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = \begin{pmatrix} \gamma\left(a^0 - \beta a^1\right) \\ \gamma\left(a^1 - \beta a^0\right) \\ a^2 \\ a^3 \end{pmatrix}$$

Dot Products with 4-Vectors:

In "standard" 3-D space-type vector algebra, we have the familiar scalar product / dot product:

$$\boxed{\vec{a}\bullet\vec{b} \equiv \left(a_x\hat{x} + a_y\hat{y} + a_z\hat{z}\right)\bullet\left(b_x\hat{x} + b_y\hat{y} + b_z\hat{z}\right) = a_x b_x + a_y b_y + a_z b_z}$$ = ***scalar*** quantity (*i.e.* = pure #)

$\exists$ A ***relativistic*** 4-vector analog of this, but it is ***NOT*** simply the sum of like components.

Instead, the ***zeroth*** component product of a ***relativistic*** 4-vector dot product has a ***minus*** sign:

$\boxed{-a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3}$ $\Leftarrow$ four-dimensional scalar product / dot product (= pure #)

Just as an ordinary / "normal" 3-D vector product $\vec{a}\bullet\vec{b}$ is ***invariant*** (*i.e.* unchanged) under ***3-D space rotations*** ($\vec{a}\bullet\vec{b}$ is the length of vector $\vec{b}$ projected onto $\vec{a}$ {and/or vice versa} – a length does ***not*** change under a 3-D space rotation), the four-dimensional scalar product between two relativistic 4-vectors is ***invariant*** (*i.e.* unchanged) under any/all Lorentz transformations, from one IRF(S) to another IRF(S').

i.e. The scalar product/dot product of ***any*** two relativistic 4-vectors is a ***Lorentz invariant quantity***.
$\Rightarrow$ The scalar product/dot product of ***any*** two relativistic 4-vectors has the ***same numerical value*** in ***any/all*** IRFs !!!

Thus: $\boxed{-a'^0b'^0 + a'^1b'^1 + a'^2b'^2 + a'^3b'^3} = \boxed{-a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3} = \text{pure \#}$

In IRF(S') In IRF(S)

In order to keep track of the minus sign associated with the ***temporal*** component of a 4-vector, especially when computing a scalar/dot product, we introduce the notion of ***contravariant*** and ***covariant*** 4-vectors.

What we have been using thus far in these lecture notes are ***contravariant*** 4-vectors a^μ, denoted by the ***superscript*** μ:

$$\boxed{a^\mu = \left(a^0, a^1, a^2, a^3\right)} = \textbf{contravariant 4-vector:} \quad \boxed{a^\mu = a^{row} = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}}$$

A ***covariant*** 4-vector a_μ is denoted by its ***subscript*** μ:

$$\boxed{a_\mu = \left(a_0, a_1, a_2, a_3\right)} = \textbf{covariant 4-vector:} \quad \boxed{a_\mu = a_{column} = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \end{pmatrix}}$$

The temporal/zeroth component {only} of ***covariant*** a_μ ***differs*** from that of ***contravariant*** a^μ by a ***minus*** sign:

$$\boxed{\begin{aligned} a_0 &= -a^0 \\ a_1 &= +a^1 \\ a_2 &= +a^2 \\ a_3 &= +a^3 \end{aligned}}$$

← Note the – sign difference !!!

a_1, a_2, a_3: ***Same*** sign for ***both*** contravariant ***and*** covariant 4-vectors

Thus, ***raising*** {or ***lowering***} the index μ of a 4-vector, *e.g.* $\boxed{a_\mu \to a^\mu}$ or $\boxed{a^\mu \to a_\mu}$ changes the ***sign*** of the ***zeroth*** (*i.e.* ***temporal/scalar***) component of the 4-vector {***only***}.

⇒ That's why we have to pay ***very*** close attention to ***subscripts*** *vs*. ***superscripts*** here !!!

Thus, a 4-vector scalar/dot product (= a Lorentz invariant quantity) may be written using contravariant and covariant 4-vectors as:

$$\boxed{\sum_{\mu=0}^{3} a_\mu b^\mu = \sum_{\mu=0}^{3} a^\mu b_\mu} = \boxed{-a_0b^0 + a_1b^1 + a_2b^2 + a_3b^3} = \boxed{-a^0b_0 + a^1b_1 + a^2b_2 + a^3b_3} = \text{pure \#}$$

This {again} can be written more compactly / elegantly / succinctly using the Einstein summation convention (*i.e.* summing over repeated indices) as:

$$\boxed{a_\mu b^\mu = a^\mu b_\mu} = \boxed{-a_0b^0 + a_1b^1 + a_2b^2 + a_3b^3} = \boxed{-a^0b_0 + a^1b_1 + a^2b_2 + a^3b_3} = \text{pure \#}$$

We define the 4-D "flat" space-time ***metric*** $g^{\mu\nu} = g_{\mu\nu}$:

$$\boxed{g^{\mu\nu} = g_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \text{with: } \boxed{g_{\alpha\gamma}g^{\gamma\beta} = \delta_\alpha^\beta}$$

δ_α^β is the 4-D space-time version of the 3-D Kroenecker δ-function δ_{ij}, *i.e.* $\delta_\alpha^\beta = 0$ if $\alpha \neq \beta$ and $\delta_\alpha^\alpha = 1$ for $\alpha = 0,1,2,3$.

Note: the above definition of the metric $g^{\mu\nu} = g_{\mu\nu}$ is ***not*** universal in the literature/textbooks…

The space-time metric $g^{\mu\nu} = g_{\mu\nu}$ is very useful *e.g.* for changing/converting a ***contravariant*** 4-vector to a ***covariant*** 4-vector (and ***vice-versa***): $\boxed{a_\mu = g_{\mu\nu} a^\nu}$ and: $\boxed{a^\mu = g^{\mu\nu} a_\nu}$.

There is an interesting parallel between relativistic Lorentz transformations (to/from different IRF's in space-time) and spatial rotations in 3-D space:

A spatial rotation in 3-dimensional Euclidean space (*e.g.* for a rotation about the $\hat{z}$ -axis) can be written in matrix form as:

$$\underbrace{\begin{pmatrix} a'_x \\ a'_y \\ a'_z \end{pmatrix}}_{\vec{a}' \text{ or } a'^\mu} = \underbrace{\begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\vec{R} \text{ or } R^\mu_\nu} \underbrace{\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}}_{\vec{a} \text{ or } a^\mu} = \begin{pmatrix} a_x \cos\varphi + a_y \sin\varphi \\ -a_x \sin\varphi + a_y \cos\varphi \\ a_z \end{pmatrix}$$

= 2nd rank, 3×3 = 9 component tensor

$\Rightarrow$ $\boxed{\vec{a}' = \vec{R} \bullet \vec{a}}$ (in 3-D vector notation) or: $\boxed{a'^\mu = R^\mu_\nu a^\nu}$ (in tensor notation, $\mu, \nu = 1:3$ {***here***})

Compare this to the Lorentz transformation (*e.g.* along the $\hat{x}$ -axis) for 4-vectors in space-time:

$$\underbrace{\begin{pmatrix} a'^0 \\ a'^1 \\ a'^2 \\ a'^3 \end{pmatrix}}_{a'^\mu} = \underbrace{\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\Lambda^\mu_\nu} \underbrace{\begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}}_{a^\mu} = \begin{pmatrix} \gamma a^0 - \gamma\beta a^1 \\ -\gamma\beta a^0 + \gamma a^1 \\ a^2 \\ a^3 \end{pmatrix} \quad \Rightarrow \quad \boxed{a'^\mu = \Lambda^\mu_\nu a^\nu}$$

$\mu, \nu = 0:3$

Comparing the matrix for $\vec{R}$ or R^μ_ν with that of Λ^μ_ν, we can see that a Lorentz transformation from one IRF to another is analogous to/has similarities to a physical rotation in 3-D Euclidean space – *i.e.* a Lorentz transformation is a certain kind of rotation in space-time – where the rotation is between the ***longitudinal*** space dimension (= the direction of the Lorentz boost, *a.k.a.* the "boost axis") and time!

In order to make this parallel somewhat sharper, we introduce a new kinematic variable, known as the ***rapidity*** (ζ), which is defined as:

$\boxed{\zeta \equiv \tanh^{-1}\beta = \tanh^{-1}(v/c)}$ or: $\boxed{\tanh\zeta \equiv \beta = v/c}$ where: $\boxed{-1 \le \beta(=v/c) \le +1}$ thus: $\boxed{-\infty \le \zeta \le +\infty}$

Since: $\boxed{\tanh\zeta = \dfrac{\sinh\zeta}{\cosh\zeta}}$ and: $\boxed{\cosh^2\zeta - \sinh^2\zeta = 1}$

Then: $\boxed{\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}} = \dfrac{1}{\sqrt{1-\tanh^2\zeta}} = \dfrac{\cosh\zeta}{\sqrt{\cosh^2\zeta - \sinh^2\zeta}} = \cosh\zeta}$ *i.e.* $\boxed{\gamma = \cosh\zeta}$ with: $\boxed{1 \le \gamma \le \infty}$

$\therefore$ $\boxed{\gamma\beta = \cosh\zeta \tanh\zeta = \cancel{\cosh\zeta}\dfrac{\sinh\zeta}{\cancel{\cosh\zeta}} = \sinh\zeta}$ *i.e.* $\boxed{\gamma\beta = \sinh\zeta}$ and: $\boxed{\beta = \tanh\zeta}$

Since: $\boxed{\cosh^2\zeta - \sinh^2\zeta = 1}$ we also see that: $\boxed{\gamma^2 - \gamma^2\beta^2 = 1}$. {Obvious, since: $\boxed{\gamma^2 = \dfrac{1}{1-\beta^2}}$}

Thus, the Lorentz transformation (along the $\hat{x}$-axis) $\boxed{a'^{\mu} = \Lambda^{\mu}_{\nu} a^{\nu}}$ of a 4-vector a^{μ} can be written {using $\boxed{\beta = \tanh\zeta}$, $\boxed{\gamma = \cosh\zeta}$ and $\boxed{\gamma\beta = \sinh\zeta}$} as:

$$\begin{pmatrix} a'^0 \\ a'^1 \\ a'^2 \\ a'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} = \begin{pmatrix} a^0\cosh\zeta - a^1\sinh\zeta \\ -a^0\sinh\zeta + a^1\cosh\zeta \\ a^2 \\ a^3 \end{pmatrix}$$

Again, compare this with the 3-D space rotation of a 3-D space-vector $\vec{a}$ about the $\hat{z}$-axis:

$$\begin{pmatrix} a'_x \\ a'_y \\ a'_z \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} a_x\cos\varphi + a_y\sin\varphi \\ -a_x\sin\varphi + a_y\cos\varphi \\ a_z \end{pmatrix}$$

We see that the above Lorentz transformation is ***similar*** (but ***not*** ***identical***) to the expression for the 3-D Euclidean geometry spatial rotation!

However, because of the $\sinh\zeta$ and $\cosh\zeta$ nature associated with the Lorentz transformation we see that the Lorentz transformation is in fact a ***hyperbolic*** rotation in space-time – *i.e.* the transformation of the ***longitudinal*** space dimension associated with the axis parallel to the Lorentz boost direction and time is that of a ***hyperbolic-type*** rotation!!!

The use of the rapidity variable, ζ has benefits *e.g.* for the **Einstein Velocity Addition Rule:**

If $\vec{u} = d\vec{x}/dt$ = the velocity of a particle as seen by an observer in IRF(S) and $\vec{u}' = d\vec{x}'/dt'$ = the velocity of the particle as seen by an observer in IRF(S') and $\vec{v} = v\hat{x}$ = the ***relative*** velocity between IRF(S) and IRF(S'), then u' is related to u by:

$$\boxed{u' = \frac{u - v}{1 - uv/c^2}} \Leftarrow \textbf{Einstein Velocity Addition Rule (1-D Case)}$$

We can re-write this as: $\boxed{u'/c = \dfrac{u/c - v/c}{1 - uv/c^2}} \Rightarrow \boxed{\beta'_{u'} = \dfrac{\beta_u - \beta}{1 - \beta_u\beta}}$

Then since: $\boxed{\beta \equiv \tanh\zeta}$ we can similarly define: $\boxed{\beta_u \equiv \tanh\zeta_u}$ and $\boxed{\beta'_{u'} \equiv \tanh\zeta'_{u'}}$.

Then: $\boxed{\beta'_{u'} = \dfrac{\beta_u - \beta}{1 - \beta_u\beta}} \Rightarrow \boxed{\tanh\zeta'_{u'} = \dfrac{\tanh\zeta_u - \tanh\zeta}{1 - \tanh\zeta_u \tanh\zeta} \equiv \tanh(\zeta_u - \zeta)} \Leftarrow$ See *e.g.* CRC Handbook *r.e.* trigonometric identities for hyperbolic functions!

Thus: $\boxed{\tanh\zeta'_{u'} = \tanh(\zeta_u - \zeta)}$

Or: $\boxed{\zeta'_{u'} = \zeta_u - \zeta} \Leftarrow$ ***Rapidity*** form of the **Einstein Velocity Addition Rule (1-D Case)**

$\Rightarrow$ Rapidities $\boxed{\zeta \equiv \tanh^{-1}\beta}$ are ***additive*** quantities in going from one IRF to another IRF !!!

Rapidity Addition Law: $\zeta'_{u'} = \zeta_u - \zeta$

$$\zeta'_{u'} = \tanh^{-1}\left(\frac{u'}{c}\right)$$

$$\zeta_u = \tanh^{-1}\left(\frac{u}{c}\right)$$

$$\zeta = \tanh^{-1}\left(\frac{v}{c}\right) \equiv \tanh^{-1}(\beta)$$

Velocities (certainly) are ***not*** additive in going from one IRF to another.

However: $\zeta = \tanh^{-1}(\beta)$ rapidities ***are*** additive in this regard.

We explicitly show that 4-vector "dot products" $x_\mu x^\mu$ and $x'_\mu x'^\mu$ are Lorentz invariant quantities:

$$x_\mu = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} -ct & x & y & z \end{pmatrix} \qquad x'_\mu = \begin{pmatrix} x'_0 & x'_1 & x'_2 & x'_3 \end{pmatrix} = \begin{pmatrix} -ct' & x' & y' & z' \end{pmatrix}$$

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \qquad x'^\mu = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

$$x_\mu x^\mu = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} -ct & x & y & z \end{pmatrix}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$
$$= x_0 x^0 + x_1 x^1 + x_2 x^2 + x_3 x^3 = -(ct)^2 + x^2 + y^2 + z^2 = x^\mu x_\mu$$

$$x'_\mu x'^\mu = \begin{pmatrix} x'_0 & x'_1 & x'_2 & x'_3 \end{pmatrix}\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} -ct' & x' & y' & z' \end{pmatrix}\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$
$$= x'_0 x'^0 + x'_1 x'^1 + x'_2 x'^2 + x'_3 x'^3 = -(ct')^2 + x'^2 + y'^2 + z'^2 = x'^\mu x'_\mu$$

But: $x'^\mu = \Lambda^\mu_\nu x^\nu$ and: $x'_\mu = \Lambda^\nu_\mu x_\nu$

For a Lorentz transform (*a.k.a.* Lorentz "boost") along the $\hat{x}$ direction:

$$x'^\mu = \Lambda^\mu_\nu x^\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma(ct - \beta x) \\ \gamma(x - \beta ct) \\ y \\ z \end{pmatrix}$$

And:
$$x'_\mu = \Lambda_\mu^\nu x_\nu = \begin{pmatrix} -ct & x & y & z \end{pmatrix} \begin{pmatrix} +\gamma & +\gamma\beta & 0 & 0 \\ +\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\gamma(ct-\beta x) & \gamma(x-\beta ct) & y & z \end{pmatrix}$$

Thus:
$$\begin{aligned} x'_\mu x'^\mu &= \begin{pmatrix} -\gamma(ct-\beta x) & \gamma(x-\beta ct) & y & z \end{pmatrix} \begin{pmatrix} \gamma(ct-\beta x) \\ \gamma(x-\beta ct) \\ y \\ z \end{pmatrix} \\ &= -\gamma^2(ct-\beta x)^2 + \gamma^2(x-\beta ct)^2 + y^2 + z^2 \\ &= -\gamma^2\left[(ct)^2 - 2\beta xct + \beta^2 x^2\right] + \gamma^2\left[x^2 - 2\beta xct + \beta^2(ct)^2\right] + y^2 + z^2 \\ &= -\gamma^2(ct)^2 + \cancel{2\gamma^2\beta xct} - \gamma^2\beta^2 x^2 + \gamma^2 x^2 - \cancel{2\gamma^2\beta xct} + \gamma^2\beta^2(ct)^2 + y^2 + z^2 \\ &= -\gamma^2(ct)^2 + \gamma^2\beta^2(ct)^2 + \gamma^2 x^2 - \gamma^2\beta^2 x^2 + y^2 + z^2 \\ &= -\gamma^2(1-\beta^2)(ct)^2 + \gamma^2(1-\beta^2)x^2 + y^2 + z^2 \end{aligned}$$

But: $\gamma^2 \equiv 1/1-\beta^2$

$$\therefore \quad x'^\mu x'_\mu = -\left(\frac{\cancel{1-\beta^2}}{\cancel{1-\beta^2}}\right)(ct)^2 + \left(\frac{\cancel{1-\beta^2}}{\cancel{1-\beta^2}}\right)x^2 + y^2 + z^2 = -(ct)^2 + x^2 + y^2 + z^2$$

i.e.
$$x'^\mu x'_\mu = x'_\mu x'^\mu = -(ct')^2 + x'^2 + y'^2 + z'^2 = -(ct)^2 + x^2 + y^2 + z^2 = x^\mu x_\mu = x_\mu x^\mu$$

$\therefore$ $\boxed{x'_\mu x'^\mu = x'^\mu x'_\mu}$ = $\boxed{x_\mu x^\mu = x^\mu x_\mu}$ are ***indeed*** Lorentz invariant quantities!

Lorentz Transformations from the Lab Frame IRF(S) to a Moving Frame IRF(S'):

1.) 1-D Lorentz Transform / "Boost" along the $\hat{x}$ direction: $\beta_x \equiv \frac{v_x}{c}$ $\gamma = \frac{1}{\sqrt{1-\beta_x^2}}$

$$x'^\mu = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda_\nu^\mu x^\nu = \begin{pmatrix} \gamma & -\gamma\beta_x & 0 & 0 \\ -\gamma\beta_x & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma(ct-\beta_x x) \\ \gamma(x-\beta_x ct) \\ y \\ z \end{pmatrix}$$

2.) <u>1-D Lorentz Transform / "Boost" along the $\hat{y}$ direction</u>: $\beta_y \equiv \frac{v_y}{c}$ $\gamma = \frac{1}{\sqrt{1-\beta_y^2}}$

$$x'^{\mu} = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda^{\mu}_{\nu} x^{\nu} = \begin{pmatrix} \gamma & 0 & -\gamma\beta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta_y & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma(ct - \beta_y y) \\ x \\ \gamma(y - \beta_y ct) \\ z \end{pmatrix}$$

3.) <u>1-D Lorentz Transform / "Boost" along the $\hat{z}$ direction</u>: $\beta_z \equiv \frac{v_z}{c}$ $\gamma = \frac{1}{\sqrt{1-\beta_z^2}}$

$$x'^{\mu} = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda^{\mu}_{\nu} x^{\nu} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta_z & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma(ct - \beta_z z) \\ x \\ y \\ \gamma(z - \beta_z ct) \end{pmatrix}$$

4.) <u>3-D Lorentz Transform / "Boost" along arbitrary $\hat{r}$ direction</u>: $\vec{\beta} \equiv \frac{\vec{v}}{c}$ $\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$

First, we define: $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$ In IRF(S) $v = |\vec{v}| = \sqrt{v_x^2+v_y^2+v_z^2}$

$\vec{\beta} \equiv \frac{\vec{v}}{c}$ $\vec{\beta} = \beta_x\hat{x} + \beta_y\hat{y} + \beta_z\hat{z}$ $\beta \equiv |\vec{\beta}| = \sqrt{\beta_x^2+\beta_y^2+\beta_z^2}$

Then:

$$x'^{\mu} = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda^{\mu}_{\nu} x^{\nu} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+\frac{(\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & 1+\frac{(\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 1+\frac{(\gamma-1)\beta_z^2}{\beta^2} \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

{*n.b.* By inspection of this 3-D Λ-matrix for 1-D motion (*i.e.* only along $\hat{x}$, $\hat{y}$, or $\hat{z}$) it is easy to show that this expression reduces to the appropriate 1-D Lorentz transformation 1.) – 3.) above.}

Or: $$x'^{\mu} = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda^{\mu}_{\nu} x^{\nu} = \begin{pmatrix} \gamma\left(ct - \vec{\beta}\bullet\vec{r}\right) \\ x + \frac{(\gamma-1)}{\beta^2}\left(\vec{\beta}\bullet\vec{r}\right)\beta_x - \gamma\beta_x ct \\ y + \frac{(\gamma-1)}{\beta^2}\left(\vec{\beta}\bullet\vec{r}\right)\beta_y - \gamma\beta_y ct \\ z + \frac{(\gamma-1)}{\beta^2}\left(\vec{\beta}\bullet\vec{r}\right)\beta_z - \gamma\beta_z ct \end{pmatrix}$$ with: $$\vec{\beta} = \beta_x\hat{x} + \beta_y\hat{y} + \beta_z\hat{z}, \quad \beta \equiv \left|\vec{\beta}\right| = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2}, \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

n.b. The $x'^0 = ct'$ equation follows trivially from $x'^0 = ct'$ in 1.) through 3.) above.

The 3-D spatial part can be written vectorially as: $\vec{r}' = \vec{r} + \frac{(\gamma-1)}{\beta^2}\left(\vec{\beta}\bullet\vec{r}\right)\vec{\beta} - \gamma\vec{\beta}ct$

which may appear to be a more complicated expression, but it's really only sorting out components of $\vec{r}$ and $\vec{r}'$ that are $\perp$ and $||$ to $\vec{v}$ for separate treatment.

Thus, we can write the 3-D Lorentz transformation from the ***lab*** frame IRF(S) to the ***moving*** frame IRF(S') along an ***arbitrary*** direction $\hat{r}$ with relative velocity $\vec{v} = v\hat{r}$ elegantly and compactly as:

$$\begin{pmatrix} ct' \\ \vec{r}' \end{pmatrix} = \begin{pmatrix} \gamma\left(ct - \vec{\beta}\bullet\vec{r}\right) \\ \vec{r} + \frac{(\gamma-1)}{\beta^2}\left(\vec{\beta}\bullet\vec{r}\right)\vec{\beta} - \gamma\vec{\beta}ct \end{pmatrix}$$

See J.D. Jackson's "Electrodynamics", 3rd Edition, *p*. 525 & *p*. 547 for more information.

Inverse Lorentz Transformations from a Moving Frame IRF(S') to the Lab Frame IRF(S):

1′.) 1-D Lorentz Transform / "Boost" along the $\hat{x}'$ direction: $\beta'_x \equiv \frac{v'_x}{c} = -\frac{v_x}{c} = -\beta_x$ $\gamma' = \frac{1}{\sqrt{1-\beta'^2_x}} = \gamma$

$$x^{\mu} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \Lambda'^{\mu}_{\nu} x'^{\nu} = \begin{pmatrix} \gamma & +\gamma\beta_x & 0 & 0 \\ +\gamma\beta_x & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma\left(ct' + \beta_x x'\right) \\ \gamma\left(x' + \beta_x ct'\right) \\ y' \\ z' \end{pmatrix}$$

2′.) 1-D Lorentz Transform / "Boost" along the $\hat{y}'$ direction: $\beta'_y \equiv \frac{v'_y}{c} = -\frac{v_y}{c} = -\beta_y$ $\gamma' = \frac{1}{\sqrt{1-\beta'^2_y}} = \gamma$

$$x^{\mu} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \Lambda'^{\mu}_{\nu} x'^{\nu} = \begin{pmatrix} \gamma & 0 & +\gamma\beta_y & 0 \\ 0 & 1 & 0 & 0 \\ +\gamma\beta_y & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma\left(ct' + \beta_y y'\right) \\ x' \\ \gamma\left(y' + \beta_y ct'\right) \\ z' \end{pmatrix}$$

3′.) 1-D Lorentz Transform / "Boost" along the $\hat{z}'$ direction: $\beta'_z \equiv \frac{v'_z}{c} = -\frac{v_z}{c} = -\beta_z$

$$\gamma' = \frac{1}{\sqrt{1-\beta_z'^2}} = \gamma$$

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \Lambda'^{\mu}_{\nu} x'^{\nu} = \begin{pmatrix} \gamma & 0 & 0 & +\gamma\beta_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\gamma\beta_z & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma(ct' + \beta_z z') \\ x' \\ y' \\ \gamma(z' + \beta_z ct') \end{pmatrix}$$

4′.) 3-D Lorentz Transform / "Boost" along arbitrary $\hat{r}$ direction: $\vec{\beta}' \equiv \frac{\vec{v}'}{c} = -\frac{\vec{v}}{c} = -\vec{\beta}$ $\quad \gamma' = \frac{1}{\sqrt{1-\beta'^2}} = \gamma$

First, we define:

$$\vec{r}' = x'\hat{x}' + y'\hat{y}' + z'\hat{z}'$$

$$\vec{v}' = v'_x\hat{x}' + v'_y\hat{y}' + v'_z\hat{z}' = -\vec{v}$$

$$\vec{\beta}' = \beta'_x\hat{x}' + \beta'_y\hat{y}' + \beta'_z\hat{z}' = -\vec{\beta} = -\beta_x\hat{x} - \beta_y\hat{y} - \beta_z\hat{z}$$

In IRF(S')

$$r' = |\vec{r}'| = \sqrt{x'^2 + y'^2 + z'^2}$$

$$v' = |\vec{v}'| = \sqrt{v_x'^2 + v_y'^2 + v_z'^2} = v$$

$$\beta' \equiv |\vec{\beta}'| = \sqrt{\beta_x'^2 + \beta_y'^2 + \beta_z'^2} = \beta \equiv |\vec{\beta}| = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2}$$

$$\vec{\beta}' \equiv \frac{\vec{v}'}{c} = -\frac{\vec{v}}{c} = -\vec{\beta}$$

Then:

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \Lambda'^{\mu}_{\nu} x'^{\nu} = \begin{pmatrix} \gamma & +\gamma\beta_x & +\gamma\beta_y & +\gamma\beta_z \\ +\gamma\beta_x & 1+\frac{(\gamma-1)\beta_x^2}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} \\ +\gamma\beta_y & \frac{(\gamma-1)\beta_x\beta_y}{\beta^2} & 1+\frac{(\gamma-1)\beta_y^2}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} \\ +\gamma\beta_z & \frac{(\gamma-1)\beta_x\beta_z}{\beta^2} & \frac{(\gamma-1)\beta_y\beta_z}{\beta^2} & 1+\frac{(\gamma-1)\beta_z^2}{\beta^2} \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

{*n.b.* By inspection of this 3-D Λ'-matrix for 1-D motion (*i.e.* only along $\hat{x}'$, $\hat{y}'$, or $\hat{z}'$) it is easy to show that this expression reduces to the appropriate 1-D Lorentz transformation 1′.) – 3′.) above.}

Or:
$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \Lambda'^{\mu}_{\nu} x'^{\nu} = \begin{pmatrix} \gamma(ct' + \vec{\beta}\bullet\vec{r}') \\ x' + \frac{(\gamma-1)}{\beta^2}(\vec{\beta}\bullet\vec{r}')\beta_x + \gamma\beta_x ct' \\ y' + \frac{(\gamma-1)}{\beta^2}(\vec{\beta}\bullet\vec{r}')\beta_y + \gamma\beta_y ct' \\ z' + \frac{(\gamma-1)}{\beta^2}(\vec{\beta}\bullet\vec{r}')\beta_z + \gamma\beta_z ct' \end{pmatrix}$$

with:
$$\vec{\beta}' = \beta'_x\hat{x}' + \beta'_y\hat{y}' + \beta'_z\hat{z}' = -\vec{\beta}$$
$$\beta' \equiv |\vec{\beta}'| = \sqrt{\beta_x'^2 + \beta_y'^2 + \beta_z'^2} = \beta$$
$$\gamma' \equiv \frac{1}{\sqrt{1-\beta'^2}} = \gamma$$

n.b. The $\boxed{x^0 = ct}$ equation follows trivially from $\boxed{x^0 = ct}$ in 1′.) through 3′.) above.

The 3-D spatial part can be written vectorially as: $\boxed{\vec{r} = \vec{r}' + \frac{(\gamma-1)}{\beta^2}\left(\vec{\beta}\bullet\vec{r}'\right)\vec{\beta} + \gamma\vec{\beta}ct'}$

which may appear to be a more complicated expression, but it's really only sorting out components of $\vec{r}'$ and $\vec{r}$ that are $\perp$ and $||$ to $\vec{v}'$ for separate treatment.

Thus, we can write the 3-D Lorentz transformation from the ***moving*** frame IRF(S') to the ***lab*** frame IRF(S) along an ***arbitrary*** direction $\hat{r}'$ with relative velocity $\vec{v}' = -v\hat{r}'$ elegantly and compactly as:

$$\boxed{\begin{pmatrix} ct \\ \vec{r} \end{pmatrix} = \begin{pmatrix} \gamma\left(ct' + \vec{\beta}\bullet\vec{r}'\right) \\ \vec{r}' + \frac{(\gamma-1)}{\beta^2}\left(\vec{\beta}\bullet\vec{r}'\right)\vec{\beta} + \gamma\vec{\beta}ct' \end{pmatrix}}$$

Note also that: $\boxed{x^\mu = \Lambda'^\mu_\nu x'^\nu}$ but: $\boxed{x'^\nu = \Lambda^\nu_\tau x^\tau}$. $\therefore$ $\boxed{x^\mu = \Lambda'^\mu_\nu x'^\nu = \Lambda'^\mu_\nu \Lambda^\nu_\tau x^\tau}$

The quantity: $\boxed{\Lambda'^\mu_\nu \Lambda^\nu_\tau = 1^\mu_\tau}$ = identity (*i.e.* unit) 4×4 matrix = $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ⇐ *n.b.* Lorentz-invariant quantity!!!

Thus: $\boxed{x^\mu = \Lambda'^\mu_\nu x'^\nu = \Lambda'^\mu_\nu \Lambda^\nu_\tau x^\tau = 1^\mu_\tau x^\tau = x^\mu}$ *i.e.* $\boxed{\Lambda'^\mu_\nu \Lambda^\nu_\tau = \Lambda^\nu_\tau \Lambda'^\mu_\nu = 1^\mu_\tau}$

We define the ***relativistic*** ***space-time*** ***interval*** between two "events" as the

Space-time difference: $\boxed{\Delta x^\mu \equiv x^\mu_A - x^\mu_B}$ ← known as the space-time ***displacement*** 4-vector

Event A occurs at space-time coordinates: $\boxed{x^\mu_A = \begin{pmatrix} x^0_A \\ x^1_A \\ x^2_A \\ x^3_A \end{pmatrix} = \begin{pmatrix} ct_A \\ x_A \\ y_A \\ z_A \end{pmatrix}}$

Event B occurs at space-time coordinates: $\boxed{x^\mu_B = \begin{pmatrix} x^0_B \\ x^1_B \\ x^2_B \\ x^3_B \end{pmatrix} = \begin{pmatrix} ct_B \\ x_B \\ y_B \\ z_B \end{pmatrix}}$

The scalar 4-vector product of $\boxed{\Delta x_\mu \Delta x^\mu = \Delta x^\mu \Delta x_\mu}$ is a ***Lorentz-invariant quantity***,
= ***same*** numerical value in ***any*** IRF, also known as the ***interval*** I between two events:

Lorentz-Invariant Interval: $\boxed{I \equiv \Delta x^{\mu}\Delta x_{\mu} = \Delta x_{\mu}\Delta x^{\mu}}$ = ***same*** numerical value in ***all*** IRF's.

$$I = -\left(\Delta x^0\right)^2 + \left(\Delta x^1\right)^2 + \left(\Delta x^2\right)^2 + \left(\Delta x^3\right)^2 = -c^2\underbrace{\left(t_A - t_B\right)}_{\Delta t_{AB}}^2 + \underbrace{\left(x_A - x_B\right)}_{\Delta x_{AB}}^2 + \underbrace{\left(y_A - y_B\right)}_{\Delta y_{AB}}^2 + \underbrace{\left(z_A - z_B\right)}_{\Delta z_{AB}}^2$$
$$= -c^2\Delta t_{AB}^2 + \Delta x_{AB}^2 + \Delta y_{AB}^2 + \Delta z_{AB}^2$$

Define the usual 3-D spatial distance: $\boxed{d_{AB} \equiv \sqrt{\Delta x_{AB}^2 + \Delta y_{AB}^2 + \Delta z_{AB}^2}}$

$\therefore$ $\boxed{I \equiv \Delta x^{\mu}\Delta x_{\mu} = \Delta x_{\mu}\Delta x^{\mu} = -c^2\Delta t_{AB}^2 + \Delta x_{AB}^2 + \Delta y_{AB}^2 + \Delta z_{AB}^2 = -c^2\Delta t_{AB}^2 + d_{AB}^2}$ $\leftarrow$ ***Lorentz-invariant quantity***, ***same*** numerical value in ***all*** IRF's

Thus, when Lorentz transform from one IRF(*S*) to another IRF(*S′*):

In IRF(*S*): $\boxed{I \equiv \Delta x^{\mu}\Delta x_{\mu} = \Delta x_{\mu}\Delta x^{\mu} = -c^2\Delta t_{AB}^2 + d_{AB}^2}$

In IRF(*S′*): $\boxed{I' \equiv \Delta x'^{\mu}\Delta x'_{\mu} = \Delta x'_{\mu}\Delta x'^{\mu} = -c^2\Delta t'^2_{AB} + d'^2_{AB}}$

Because the interval I is a ***Lorentz-invariant quantity***, then:

$$\boxed{I = \Delta x^{\mu}\Delta x_{\mu} = \Delta x_{\mu}\Delta x^{\mu}} = \boxed{I' = \Delta x'^{\mu}\Delta x'_{\mu} = \Delta x'_{\mu}\Delta x'^{\mu}}$$

Or: $$\boxed{-c^2\Delta t_{AB}^2 + d_{AB}^2} = \boxed{-c^2\Delta t'^2_{AB} + d'^2_{AB}}$$

Work this out / prove to ***yourselves*** that it ***is*** true $\rightarrow$ follow procedure / same as on pages 7-8 of these lecture notes.

Note that: $\boxed{\underset{[IRF(S)]}{\Delta t_{AB}} \neq \underset{[IRF(S')]}{\Delta t'_{AB}}}$ and $\boxed{\underset{[IRF(S)]}{d_{AB}} \neq \underset{[IRF(S')]}{d'_{AB}}}$

Time dilation in IRF(*S′*) relative to IRF(*S*) is ***exactly*** compensated by spatial Lorentz contraction in IRF(*S′*) relative to IRF(*S*), keeping the interval *I* the same (*i.e.* Lorentz invariant) in all IRF's !

$\Rightarrow$ ***Profound*** aspect / nature of space-time!

Depending on the details of the two events (*A* & *B*), the interval $\boxed{I \equiv \Delta x^{\mu}\Delta x_{\mu} = \Delta x_{\mu}\Delta x^{\mu} = -c^2\Delta t_{AB}^2 + d_{AB}^2}$ can be ***positive***, ***negative***, or ***zero***:

$I < 0$: Interval *I* is ***time-like***: $\boxed{c^2\Delta t_{AB}^2 > d_{AB}^2}$

e.g. If two events *A* & *B* occur at ***same spatial*** location, then: $\boxed{\vec{r}_A = \vec{r}_B} \rightarrow \boxed{d_{AB} = 0}$,

$\Rightarrow$ The two events *A* & *B* ***must*** have occurred at ***different*** times, thus: $\boxed{\Delta t_{AB} \neq 0}$.

$I > 0$: Interval *I* is ***space-like***: $\boxed{c^2\Delta t_{AB}^2 < d_{AB}^2}$

e.g. If two events *A* & *B* occur ***simultaneously***, then: $\boxed{t_A = t_B} \rightarrow \boxed{\Delta t_{AB} = 0}$,

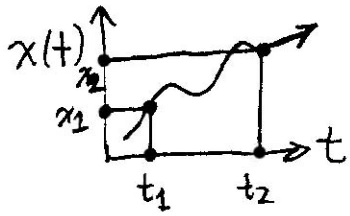
$\Rightarrow$ The two events *A* & *B* ***must*** have occurred at ***different*** spatial locations, thus: $\boxed{d_{AB} \neq 0}$.

$I = 0$: Interval *I* is ***light-like***: $\boxed{c^2\Delta t_{AB}^2 = d_{AB}^2}$

e.g. The two events *A* & *B* are connected by a signal traveling at the speed of light (in vacuum).

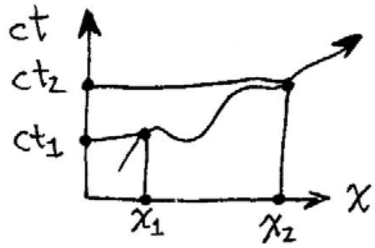
<u>Space-time Diagrams</u> ≡ <u>Minkowski Diagrams</u>:

On a "normal"/Galilean space-time diagram, we plot $x(t)$ *vs*. t:



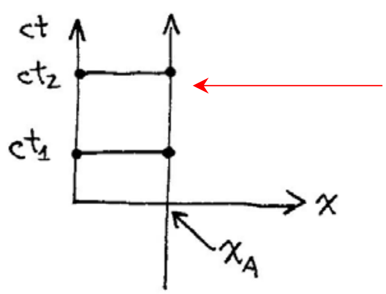
speed, $v(t)$ = local slope $\left(\left.\dfrac{dx(t)}{dt}\right|_t\right)$ of $x(t)$ *vs*. t graph at time t.

In relativity, we {instead} plot ct *vs*. x (danged theorists!!!) for the ***space-time*** diagram: (*a.k.a.* Minkowski diagram)



Dimensionless speed β: $\left(\dfrac{v}{c}\right)=\beta=\dfrac{1}{\text{slope}}=\dfrac{1}{\left(\left.\dfrac{d(ct)}{dx}\right)\right|_x}$ of ct *vs*. x graph at point x.

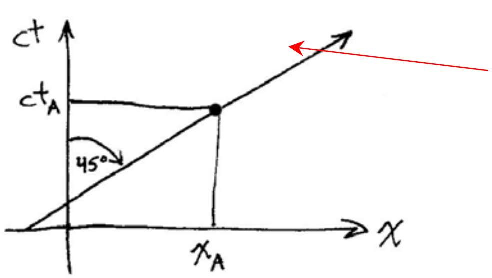
A particle at ***rest*** in an IRF is represented by a ***vertical*** line on the relativistic space-time diagram:



slope $\boxed{\left.\dfrac{d(ct)}{dx}\right|_{x=x_A}=\infty} \Rightarrow \boxed{\beta=\dfrac{1}{slope}=\dfrac{1}{\infty}=0}$

The "trajectory" of a particle in the space-time diagram makes an angle $\theta = 0^{\circ}$ with respect to ***vertical*** (ct) axis.

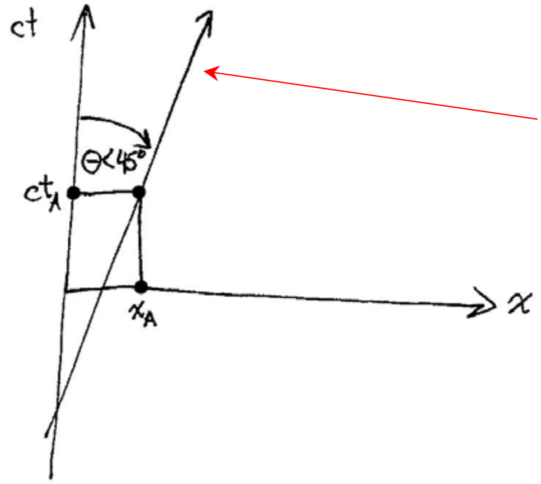
A photon traveling at $v = c$ is represented by a straight line at 45° with respect to the vertical (ct) axis:



slope $\boxed{\left.\dfrac{d(ct)}{dx}\right|_{x=x_A}=1} \Rightarrow \boxed{\beta_{photon}=\dfrac{1}{slope}=\dfrac{1}{1}=1}$

(*i.e.* $v_{photon}=c$)

A particle traveling at constant speed $v < c$ $(\beta < 1)$ is represented by a straight line making an angle $\theta < 45^{\rm o}$ with respect to the vertical (ct) axis:



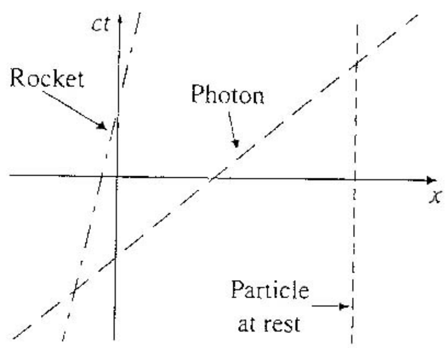
$$\text{slope } \left. \frac{d(ct)}{dx} \right|_{x=x_A} > 1 \Rightarrow \beta_{particle} = \frac{1}{slope} = \frac{1}{>1} < 1$$

i.e. $v_{\text{particle}} < c$

$$\left(\beta_{particle} = \frac{v_{particle}}{c} < c \right)$$

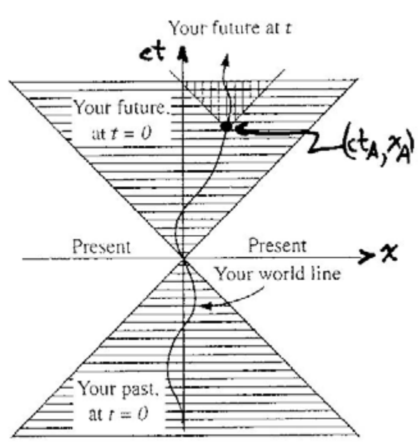
The ***trajectory*** {*i.e.* locus of space-time points} of a particle on a relativistic space-time / Minkowski diagram is known as the ***world line*** of the particle.

All three of the above situations superimposed together on the Minkowski/space-time diagram:



Suppose you set out from $t = 0$ at the origin of your ***own*** Minkowski diagram.
Because your speed can ***never*** exceed c ($v \le c$, *i.e.* $\beta \le 1$), your trajectory (your world line) in the ct *vs.* x space-time diagram can ***never*** have |slope| = $|d(ct)/dx| < 1$, anywhere along it.

⇒ Your "motion" in the Minkowski diagram is ***restricted*** to the wedge-shaped region bounded by the two $45^{\rm o}$ lines (with respect to vertical (ct) axis) as shown in the figure below:



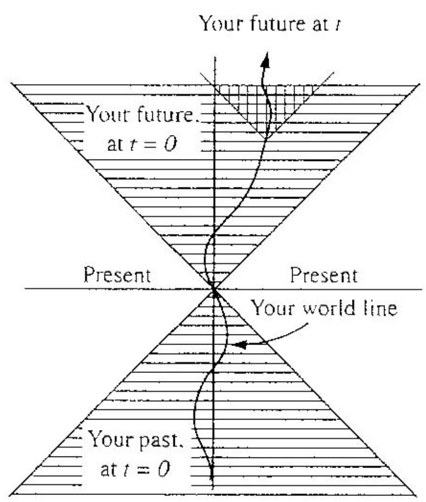
- The ±45° wedge-shaped region ***above*** the horizontal *x*-axis ($ct > 0$) is your ***future*** at $t = 0$ = locus of all space-time points ***potentially*** accessible to you.
- Of course, as time goes on, as you ***do*** progress along your world line, your "options" progressively ***narrow*** – your future at any moment $t > 0$ is the ±45° wedge constructed from / at whatever space-time point (ct_A, x_A) you are at, at that point in space (x_A) at the time t_A.
- The backward ±45° wedge ***below*** the horizontal *x*-axis ($ct < 0$) is your ***past*** at $t = 0$ = locus of all points ***potentially*** accessed by you in the past.
- The space-time regions ***outside*** the ***present*** and ***past*** ±45° wedges in the Minkowski diagram are ***inaccessible*** to you, because you would have to travel faster than speed of light *c* to be in such regions!
- A space-time diagram with one time dimension (vertical axis) and 3 space dimensions (3 horizontal axes: *x*, *y* and *z*) is a **4**-***dimensional*** diagram – can't draw it on 2-D paper!
- In a 4-D Minkowski Diagram, ±45° wedges become 4-D "hypercones" (*aka* light cones).
  "future" = contained within the ***forward*** light cone.
  "past" = contained within the ***backward*** light cone.

The ***slope*** of the world line/the trajectory connecting two events on a space-time diagram tells you at a glance whether the invariant interval $\boxed{I \equiv \Delta x^{\mu}\Delta x_{\mu} = \Delta x_{\mu}\Delta x^{\mu}}$ is:

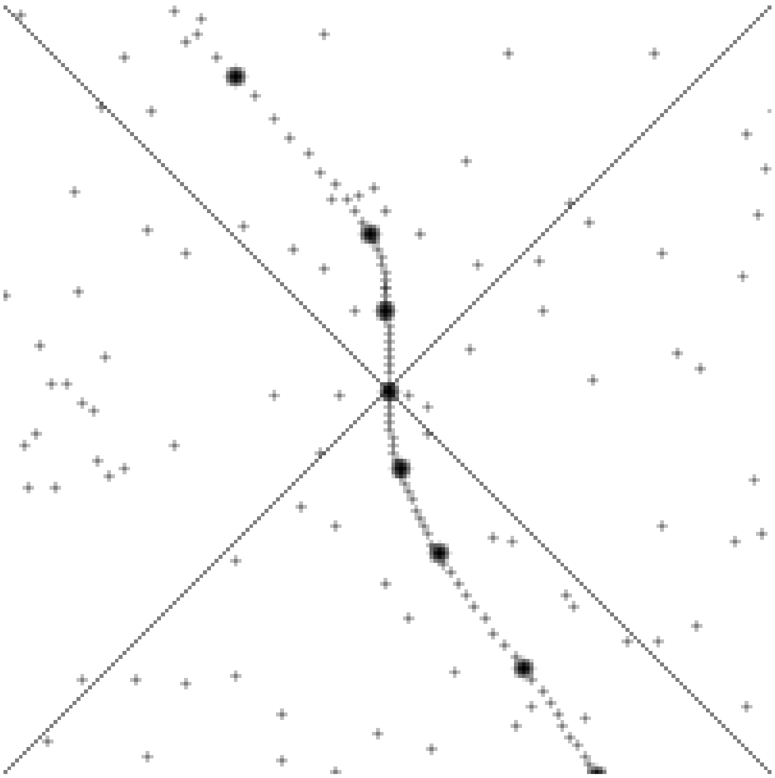
a) Time-like (slope $\frac{d(ct)}{dx} > 1$) (all points in your ***future*** and your ***past*** are ***time-like***)

b) Space-like (slope $\frac{d(ct)}{dx} < 1$) (all points in your ***present*** are ***space-like***)

c) Light-like (slope $\frac{d(ct)}{dx} \equiv 1$) (all points on your ***light*** cone(s) are ***light-like***)



Changing Views of Relativistic Space-time Along the World line of a Rapidly Accelerating Observer

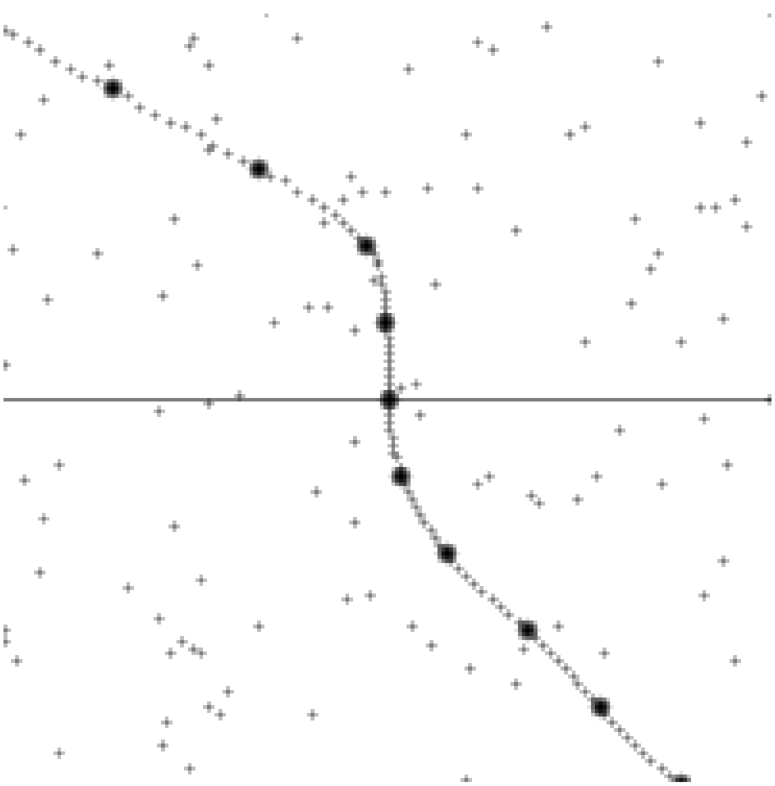


For relativistic space-time, the vertical axis is *c*×time, the horizontal axis is distance; the dashed line is the space-time trajectory ("world line") of the observer. The small dots are arbitrary events in space-time.

The ***lower quarter*** of the diagram (within the light cone) shows events (dots) in the past that were visible to the user, the ***upper quarter*** (within the light cone) shows events (dots) in the future that the observer will be able to see.

The ***slope*** of the world line (deviation from vertical) gives the relative speed to the observer. Note how the view of relativistic space-time changes when the observer accelerates {see relativistic animation}.

Changing Views of Galilean Space-time Along the World Line of a Slowly Accelerating Observer



In non-relativistic Galilean/ Euclidean space, the vertical axis is *c*×time, the horizontal axis is distance; the dashed line is the space-time trajectory ("world line") of the observer. The small dots are arbitrary events in space-time.

The ***lower half*** of the diagram shows (past) events that are "earlier" than the observer, the ***upper half*** shows (future) events that are "later" than the observer.

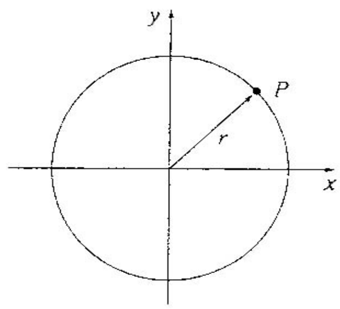
The ***slope*** of the world line (deviation from vertical) gives the relative speed to the observer. Note how the view of Galilean / Euclidean space-time changes when the observer accelerates {see Galilean animation}.

Note that time in space-time is ***not*** "just another coordinate" (like *x*, *y*, *z*) – its "mark of distinction" is the ***minus sign*** in the ***invariant interval***:

$$I \equiv \Delta x^{\mu} \Delta x_{\mu} = \Delta x_{\mu} \Delta x^{\mu} = -(c\Delta t)^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

The minus sign in the invariant interval (arising from / associated with time dimension) imparts a ***rich*** structure to sinh, cosh, tanh . . . the ***hyperbolic geometry*** of ***relativistic space-time*** versus the ***circular geometry*** of ***Euclidean 3-dimensional space***.

In Euclidean 3-D space, a ***rotation*** {*e.g.* about the $\hat{z}$ -axis} of a point *P* in the *x*-*y* plane describes a ***circle*** – the locus of all points at a fixed distance $r = \sqrt{x^2 + y^2}$ from the origin:

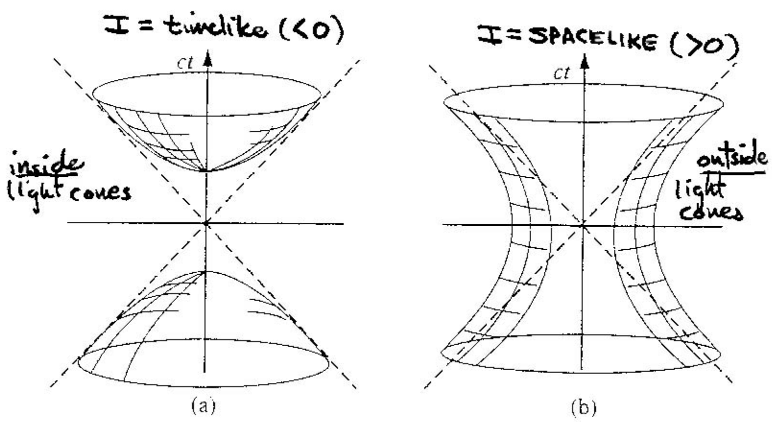


r = constant (*i.e.* is ***invariant***) under a ***rotation*** in Euclidean / 3-D space.

For a ***Lorentz transformation*** in ***relativistic space-time***, the ***interval*** $I \equiv x^{\mu} x_{\mu} = x_{\mu} x^{\mu} = -(c\Delta t)^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$ is a ***Lorentz-invariant quantity*** (*i.e.* is preserved under ***any/all*** Lorentz transformations from one IRF to another).

The locus of all points in space-time with a given / specific value of *I* is a ***hyperbola*** (for *ct* and Δx (*i.e.* 1 space dimension) only): $I = -(c\Delta t)^2 + \Delta x^2$

If we include *e.g.* the $\hat{y}$ -axis, the locus of all points in space-time with a given / specific value of $I = -(c\Delta t)^2 + \Delta x^2 + \Delta y^2$ is a ***hyperboloid of revolution***:



When the ***invariant interval*** *I* is ***time-like*** ($I < 0$) → surface is a **hyperboloid of *two* sheets**.

When the ***invariant interval*** *I* is ***space-like*** ($I > 0$) → surface is a **hyperboloid of *one* sheet**.

- When carrying out a Lorentz transformation from IRF(S) to IRF(S') (where IRF(S') is moving with respect to IRF(S) with velocity $\vec{v}$) the space-time coordinates (x, ct) of a given event will change (via appropriate Lorentz transformation) to (x', ct').

- The new coordinates (x', ct') will lie on the ***same*** hyperbola as (x, ct) !!!

- By appropriate combinations of Lorentz transformations ***and*** rotations, a ***single*** space-time point (x, ct) can ***generate*** the ***entire*** surface of a given ***hyperboloid*** (*i.e.* but only the hyperboloid that the ***original*** space-time point (x, ct) is on).

- ∃ ***no*** Lorentz transformations from the upper → lower sheet of the ***time-like*** ($I < 0$) hyperboloid of two sheets (and vice versa).

- ∃ ***no*** Lorentz transformations from the upper or lower sheet of the ***time-like*** ($I < 0$) hyperboloid of two sheets to the ***space-like*** ($I > 0$) hyperboloid of one sheet (and vice versa).

- In discussion(s) of the ***simultaneity*** of events, ***reversing*** the ***time-ordering*** of events is {in general} ***not*** always possible.

⇒ If the ***invariant interval*** $\boxed{I = -(c\Delta t)^2 + d^2 < 0}$ (*i.e.* is ***time-like***) the ***time-ordering*** is ***absolute*** (*i.e.* the time-ordering ***cannot*** be changed).

⇒ If the ***invariant interval*** $\boxed{I = -(c\Delta t)^2 + d^2 > 0}$ (*i.e.* is ***space-like***) the ***time-ordering*** of events depends on the IRF in which they are observed.

In terms of the space-time/Minkowski diagram for ***time-like*** invariant intervals $\boxed{I = -(c\Delta t)^2 + d^2 < 0}$:

- An event on the ***upper*** sheet of a ***time-like*** hyperboloid (*n.b.* lies ***inside*** of light cone) ***definitely*** occurred ***after*** time $t = 0$.

- An event on ***lower*** sheet of a ***time-like*** hyperboloid (*n.b.* also lies ***inside*** of light cone) ***definitely*** occurred ***before*** time $t = 0$.

- For an event occurring on a ***space-like*** hyperboloid, ***invariant interval*** $\boxed{I = -(c\Delta t)^2 + d^2 > 0}$, the ***space-like*** hyperboloid lies ***outside*** of the light cone) the event can occur ***either*** at ***positive*** or ***negative*** time t – it depends on the IRF from which the event is viewed!

- This rescues the notion of ***causality***!
  To an observer in one IRF: "Event A ***caused*** event B"
  To another "observer" (outside of light cone, in another IRF) could say: "B ***preceded*** A".

- If two events are ***time-like*** separated (within the light cone) → they ***must*** obey causality.

- If the ***invariant interval*** $\boxed{I = x_\mu x^\mu = x^\mu x_\mu = -(c\Delta t)^2 + d^2 < 0}$ (*i.e.* is ***time-like***) between two events (*i.e.* they lie ***within*** the light cone) then the ***time-ordering*** is same ∀ (for all) observers – *i.e.* ***causality*** ***is*** obeyed.

- Causality is IRF-dependent for the ***space-like invariant interval*** $I = x_\mu x^\mu = x^\mu x_\mu = -(c\Delta t)^2 + d^2 > 0$ between two events (*i.e.* they lie ***outside*** the light cone). Temporal-ordering is IRF-dependent / ***not*** the same for all observers.

- ***We*** don't live ***outside*** the light cone (*n.b.* outside the light cone $\to \beta > 1$).

Another Perspective on the Structure of Space-Time:

Mathematician Herman Minkowski (1864-1909) in 1907 introduced the notion of 4-D space-time (not just space and time separately). In his mathematical approach to special relativity and inertial reference frames, space and time Lorentz transform (*e.g.* along the $\hat{x}$ direction) as given above, however in his scheme the contravariant x^μ and covariant x_μ 4-vectors that he advocated using were:

$$x^\mu = \begin{pmatrix} ict \\ x \\ y \\ z \end{pmatrix} \quad \text{and} \quad x_\mu = \begin{pmatrix} ict & x & y & z \end{pmatrix}$$

It can be readily seen that the Lorentz invariant quantity $x_\mu x^\mu = -(ct)^2 + x^2 + y^2 + z^2$ is the same as always, but here the – ve sign in the temporal (0) index is generated by $i * i = -1$.

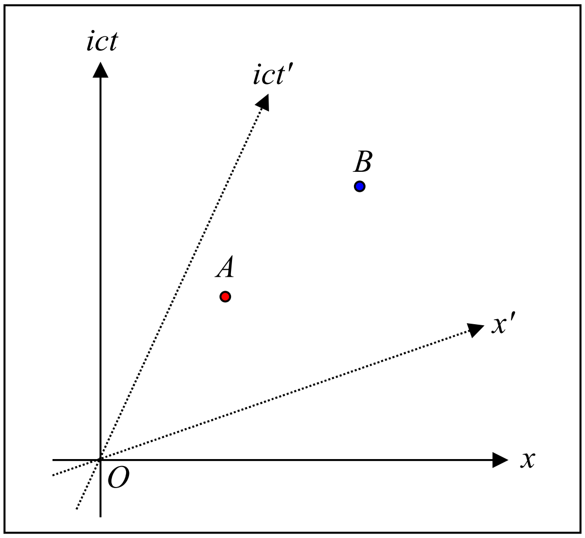
Thus, in Minkowski's notation $x'^\mu = \Lambda^\mu_\nu x^\nu$ for a 1-D Lorentz transform along the $\hat{x}$-direction is:

$$x'^\mu = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} ict' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda^\mu_\nu x^\nu = \begin{pmatrix} \gamma & -\gamma\beta_x & 0 & 0 \\ -\gamma\beta_x & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ict \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma(ict - \beta_x x) \\ \gamma(x - \beta_x ict) \\ y \\ z \end{pmatrix} \quad \underline{\text{with}}: \quad \beta_x \equiv \frac{v_x}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta_x^2}}$$

The physical interpretation of the "ict" temporal component *vs.* the *x*, *y*, *z* spatial components of the four-vectors x^μ and x_μ is that there exists a ***complex***, 90° ***phase relation*** between space and time in ***special*** relativity – *i.e.* ***flat space-time***.

We've seen this before, *e.g.* for {zero-frequency} ***virtual*** photons, where the relation for the relativistic total energy associated with a virtual photon is $E_{\gamma*}^2 = p_{\gamma*}^2 c^2 + m_{\gamma*}^2 c^4 = hf_{\gamma*} = 0 \Rightarrow$ $p_{\gamma*} c = \pm i m_{\gamma*} c^2$.

In the ***flat*** space-time of ***special*** relativity, graphically this means that Lorentz transformations from one IRF to another are related to each other *e.g.* via the {***flat***} space-time diagram as shown in the figure below:



This formalism works fine in ***flat*** space-time/***special*** relativity, but in ***curved*** space-time / ***general*** relativity, it is cumbersome to work with – the complex phase relation between time and space is ***no longer*** 90°, it depends on the ***local curvature*** of space-time!

Imagine taking the above ***flat*** space-time 2-D surface and ***curving*** it *e.g.* into ***potato-chip*** shape!!! Then imagine taking the **4-D** ***flat*** space-time and ***curving/warping*** it per the ***curved*** **4-D** space-time *e.g.* in proximity to a supermassive black hole or a neutron star!!!

Thus, for people working in ***general*** relativity, the use of the modern 4-vector notation *e.g.* for contravariant x^μ and covariant x_μ is ***strongly*** preferred, *e.g.*

$$\boxed{x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}} \text{ and } \boxed{x_\mu = \begin{pmatrix} -ct & x & y & z \end{pmatrix}}$$

In ***flat*** space-time/***special*** relativity, the modern mathematical notation works equally well and then also facilitates people learning the mathematics of curved space-time/general relativity.

Using the rule for the temporal (0) component of covariant x_μ that $\boxed{x_0 = -x^0}$, then Lorentz invariant quantities such as $\boxed{x_\mu x^\mu = -(ct)^2 + x^2 + y^2 + z^2}$ are "automatically" calculated properly.

However, the physical interpretation of the complex phase relation between time and space (and the temporal-spatial components of {all} other 4-vectors) often gets lost in the process…. which is why we explicitly mention it here…