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REGULARIZATION AND RENORMALIZATION OF GAUGE FIELDS

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Abstract: A new regularization and renormalization procedure is presented. It is particularly well suited for the treatment of gauge theories. The method works for theories that were known to be renormalizable as well as for Yang-Mills type theories. Overlapping divergencies are disentangled. The procedure respects unitarity, causality and allows shifts of integration variables. In non-anomalous cases also Ward identities are satisfied at all stages. It is transparent when anomalies, such as the Bell-Jackiw-Adler anomaly, may occur.

1. INTRODUCTION

Recently it has been shown [1] that it is possible to formulate renormalizable theories of charged massive vector bosons. The derived Feynman rules involve ghost particles, and in order to establish unitarity and causality of the S-matrix Ward identities are needed. The necessary combinatorial techniques were given in ref. [2], in the treatment of massless Yang-Mills fields. It was emphasized that these same techniques work also in the case of massive vector boson theories obtained from the massless theory by means of the Higgs-Kibble [3] mechanism. Stated somewhat differently: the manifestly renormalizable set ** of Feynman rules involving ghosts may be transformed into a set of manifestly unitary and causal Feynman rules by means of Ward identities. Actually these manifestly unitary and causal Feynman rules are quite meaningless in view of the occurring divergencies, and a direct proof of unitary and causality starting from the manifestly renormalizable rules is to be preferred. This is precisely the program carried through in refs. [1, 2].

However, even with a set of manifestly renormalizable rules one cannot be sure that a consistent theory results unless a suitable cut-off and subtraction procedure has been defined. In particular, since unitarity depends crucially on the validity of the Ward indentities one must have a procedure that respects those Ward identities. In ref. [2] the existence of such a procedure was proven for diagrams containing at

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** i.e. renormalizable with respect to power counting.

most one closed loop; it is the aim of the present paper to extend the argument to arbitrary order of perturbation theory.

In connection with the question of renormalizability there is the problem of overlapping divergencies. This problem is of course not peculiar to gauge theories, and since it has been solved in other cases one could perhaps consider this as a not very urgent problem. However, our treatment deviates in several respects from the conventional procedures, and for this reason we have also considered this problem. It turns out that the techniques given below are particularly well suited to this purpose, and it will be shown that no difficulties arise.

The procedure suggested in ref. [2] was based on the observation that the Ward identities do hold irrespective of the dimension of the space involved. By introducing a fictitious fifth dimension and a very large fifth component of momentum inside the loop suitable gauge invariant regulator diagrams could be formulated. This procedure breaks down for diagrams containing two or more closed loops because then the "fifth" component of loop momentum may be distributed over the various internal lines. It was guessed that more dimensions would have to be introduced, and thus the idea of continuation in the number of dimensions suggested itself. This is the basic idea employed in this paper *.

In sect. 2 we define an analytic continuation of the S-matrix elements in the complex n-plane, where n is a variable that for positive integer values equals the dimension of the space involved with respect to loop quantities. The physical situation is obtained for $n = 4$. This definition is such that for finite diagrams the limit for $n = 4$ equals the conventional result. It turns out that the generalized S-matrix elements so defined are analytic in n and the infinities of perturbation theory manifest here as poles for $n = 4$.

Renormalization amounts to subtraction of these poles, and one must show that this subtraction procedure does not violate unitarity etc. In fact, in sect. 3 it will be shown that the generalized S-matrix elements satisfy Ward identities, unitarity and causality for all n. In sect. 4 we consider the question of renormalization and overlapping divergencies.

Since the whole subject of this paper is rather involved and technical, we have stripped the argument as much as possible of non-essential details. The arguments of sects. 2 to 4 are valid for theories containing scalar, vector etc. particles; in sect. 5 the extension to include fermions is indicated.

Sect. 6 is devoted to a discussion of the limitations of the method. It is indicated where there arise conflicts between the method and Ward identities; there seem to be no limitations with respect to the other properties. It appears that such conflicts happen only where there really are troubles, i.e. in cases where anomalies [4–6] occur. Even then the method is very suitable for practical evaluation of the anomalies, which is demonstrated in this section.

* Independently C.G. Bollini and J.J. Giambiagi [12] have also advanced and pursued the idea of continuation in the number of dimensions.

Infrared difficulties associated with zero mass particles are not considered in this paper.

For completeness we note that our method bears some resemblance to the method of analytical continuation [7]. The analytic continuation in the exponents of the propagators, as suggested by Bollini et al., amounts in the actual calculations to almost the same as continuation in the number of dimensions (see for instance eq. (A5) in appendix A). In several crucial respects, however, continuation in n gives less deformation of the structure of perturbation theory.

2. DEFINITIONS

As an example we take a photon interacting with charged pions. Vertices:

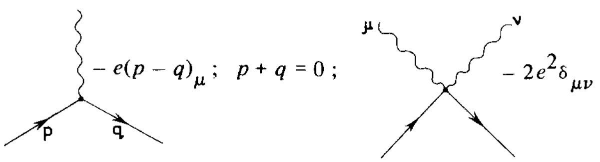

Fig. 1.

The arrows denote the direction of charge flow. Momenta are taken in the direction of the vertices.

The lowest order photon self-energy diagrams are:

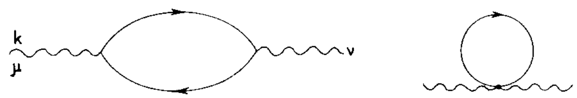

Fig. 2.

Assuming n component loop momentum p we have

$$e^2\int \mathrm{d}_n p\left[\frac{(2p+k)_\mu(2p+k)_\nu}{(p^2+m^2)((p+k)^2+m^2)}-\frac{2\delta_{\mu\nu}}{p^2+m^2}\right]. \tag{1}$$

Evaluating the integral without worrying about divergencies:

$$=e^2\int_0^1 \mathrm{d}x\int \mathrm{d}_n p\,\frac{4p_\mu p_\nu+2p_\mu k_\nu+2k_\mu p_\nu+k_\mu k_\nu-2((p+k)^2+m^2)\delta_{\mu\nu}}{(p^2+2pkx+k^2x+m^2)^2}.$$

Using the formulae of appendix A (note that in the end terms odd in $(1-2x)$ may be dropped):

$$(1)=e^2 i\pi^{\frac{1}{2}n}\,\Gamma(2-\tfrac{1}{2}n)\int_0^1 \mathrm{d}x\,\frac{(1-2x)^2(k_\mu k_\nu-k^2\delta_{\mu\nu})}{(m^2+k^2x(1-x))^{2-\frac{1}{2}n}}. \tag{2}$$

This expression is manifestly gauge invariant. In the complex n plane there are simple poles for $n = 4, 6, 8$ etc. Note that the n-dependence is such that gauge invariance holds for any n. This is the property referred to in the introduction: Ward identities do not involve the dimensionality of space.

In order to carry through renormalization we subtract from (2) the pole and its residue for $n = 4$

$$e^2 i\pi^2 \frac{2}{4-n}(k_\mu k_\nu - k^2\delta_{\mu\nu}) \int_0^1 dx(1-2x)^2 . \tag{3}$$

This subtraction term is a polynomial in the external momentum, and of course gauge invariant. Subtracting (3) from (2) and taking the limit $n = 4$ gives the customary result:

$$\begin{aligned} &-ie^2\pi^2(k_\mu k_\nu - k^2\delta_{\mu\nu}) \int_0^1 dx(1-2x)^2 \ln(m^2 + k^2x(1-x)) \\ &+ C(k_\mu k_\nu - k^2\delta_{\mu\nu}) , \end{aligned} \tag{4}$$

where C is a constant related to the n dependence other then in the exponent of the denominator. Actually C is undetermined, which may be seen as follows. Suppose that in (2) we replace e^2 by e^2M^{4-n}, where M is an arbitrary mass. This gives (2) an n independent dimension of $(\text{mass})^2$. However C in (4) is changed by a term proportional to $\ln M$. It may be noted that the same arbitrariness results if one evaluates (1) with the help of Pauli-Villars regulators.

The above heuristic derivation shows many of the features of the method advocated in this paper. In practical calculations for one loop diagrams this provides a very simple scheme for computing gauge invariant results. It could for instance be used to show cancellation of divergencies in the manifestly unitary set of Feynman rules mentioned in the introduction and investigated by several authors [8].

There are several serious objections to the above manipulations. First of all, our starting point eq. (1) is meaningless for $n \geqslant 2$. In order to obtain a sensible result one must (i) change the Feynman rules such that for non-integer n all diagrams give rise to well-defined expressions, and (ii) define a suitable limiting procedure for $n = 4$, which restores originally convergent diagrams to their original values while originally divergent diagrams are given a meaning consistent with unitarity etc. Thus first of all a redefinition of the S-matrix is in order.

Consider again eq. (1). First we split the n-dimensional space in a 4 dimensional (physical) and an n–4 dimensional subspace:

$$\int d_n p \to \int d_4 \underline{p} \int d_{n-4} P . \tag{5}$$

Multiplying (1) with two arbitrary physical four vectors $e_{1\mu}$ and $e_{2\nu}$ we see that (1) depends on the direction of $\underline{p}$ but not on the direction of P. Introducing polar coordinates in P space and integrating over angles one finds:

$$(1) = \int d_4\underline{p} \int d\omega\, \omega^{n-5} \frac{2(\pi)^{\frac{1}{2}(n-4)}}{\Gamma(\frac{1}{2}(n-4))} f(\underline{p}, \omega^2)\,, \tag{6}$$

where ω is the length of P in the $n-4$ dimensional subspace. The dependence on the external vectors e_1, e_2 and k is not shown explicitly. Note that $(pk) = (\underline{p}k)$, $(e_1 p) = (e_1 \underline{p})$, $(e_2 p) = (e_2 \underline{p})$ and $p^2 = \underline{p}^2 + \omega^2$. (6) is still quite meaningless, and we continue our formal manipulations until we arrive at an expression that can be given a meaning. Note that the second integral in (6) contains an infrared divergence for $n \leqslant 4$. This divergence is superficial and may be removed by partial integration (throwing away surface terms):

$$\int_0^\infty d\omega\, \omega^{n-5} f(\underline{p}, \omega^2) = -\frac{2}{n-4} \int_0^\infty d\omega\, \omega^{n-3} \frac{\partial}{\partial\omega^2} f(\underline{p}, \omega^2)\,. \tag{7}$$

Doing this λ times on (6) gives

$$\frac{\pi^{\frac{1}{2}(n-4)}\,2}{\Gamma(\frac{1}{2}(n-4)+\lambda)} \int d_4\underline{p} \int_0^\infty d\omega\, \omega^{n-5+2\lambda} \left(-\frac{\partial}{\partial\omega^2}\right)^{\lambda} f(\underline{p}, \omega^2) \tag{8}$$

For the (in 4 dimensional space) quadratically divergent diagrams of fig. 2 this is a well defined formula for $4 - 2\lambda < n < 2$. Note that $\omega \geqslant 0$. *Eq.* (8) *with sufficiently large* λ *defines the contribution of one loop diagrams to the generalized S-matrix elements in a finite region of the complex n-plane. This region is the domain of convergence of the integrals in* (8).

By taking a sufficiently large λ the domain of convergence extends to arbitrarily small n. Furthermore, the degree of convergence is $2-n$ as far as ultraviolet behaviour is concerned and $n-4+2\lambda$ for the infrared behaviour. Clearly, by choosing a suitable λ and n one has a representation of the generalized diagrams in some region of the n-plane in terms of arbitrarily convergent integrals.

If a diagram is convergent in 4-dimensional space then the redefinition (8) exists for $n < n_0$ with $n_0 > 4$. Moreover, for $n = 4$ (8) equals the result evaluated in the conventional way, as may be seen by taking $\lambda = 1$ and setting $n = 4$. Thus for finite diagrams our prescription gives the conventional result in the limit $n = 4$.

For divergent diagrams (8) will be meaningless for $n = 4$. However, as will be shown, (8) may be continued in the complex n-plane to larger n values. The result will in general have a pole at $n = 4$. In order to make sense in the limit $n = 4$ one must introduce counterterms in the perturbation expansion, and those counterterms must be chosen such that the poles at $n = 4$ disappear. Whether this can be done in a consistent manner is a separate problem, to be tackled in sect. 4.

For values of n outside the domain of convergence of the integrals in (8) *the contribution to the generalized S-matrix elements is defined as the analytic continuation of* (8).

It turns out to be possible to construct explicity this analytic continuation towards larger n values. The method is as follows. By means of partial integration, valid inside the domain of convergence of (8) we derive a new formula, identical to (8) inside the domain of convergence of (8), but analytic in n in an enlarged domain. By the principles of analytic continuation this new formula is then equal to the analytic continuation of (8) in this enlarged domain.

In view of the importance of this construction we will try to formulate it as clearly as possible. Consider the integral:

$$I = \int d_\kappa p \frac{p_a^{\lambda_1} p_b^{\lambda_2} \dots p_c^{\lambda_j}}{((p+k_1)^2 + m_1^2)^{\alpha_1} ((p+k_2)^2 + m_2^2)^{\alpha_2} \dots ((p+k_l)^2 + m_l^2)^{\alpha_l}} \tag{9}$$

p_a, p_b etc. are components a, b etc. of p. The exponents $\lambda_i \dots \lambda_j$ are not necessarily integer. (8) is of the form (9) with $\kappa = 5$, where the integration over p_5 in (9) is nothing but the ω-integration in (8). Thus p_5 occurs with an n-dependent exponent in the numerator. Also $\underline{p}_1$, $\underline{p}_2$ etc. may occur in the numerator, they are contained in (8) in the function f. The differentiations with respect to ω^2 in (8) have as net effect an increase of the exponents of the factors in the denominator; the α_i will be integer, but they can be larger than 1.

The integral in (9) will be convergent if

$$\begin{aligned} &\lambda_1 > -1, \lambda_2 > -1, \; \dots, \lambda_j > -1 \\ &\kappa + \lambda_1 + \lambda_2 + \dots \lambda_j - 2(\alpha_1 + \alpha_2 + \dots \alpha_l) < 0 \,. \end{aligned} \tag{10}$$

Next we insert in (9) the expression, identical to unity:

$$\frac{1}{\kappa} \sum_{i=1}^{\kappa} \frac{\partial p_i}{\partial p_i}. \tag{11}$$

Within the region (10) we may perform partial integration. After some trivial algebra we obtain:

$$I = -\frac{\lambda_1 + \lambda_2 + \dots \lambda_j}{\kappa} I + \frac{2(\alpha_1 + \alpha_2 + \dots \alpha_l)}{\kappa} I - \frac{1}{\kappa} I'$$

with

$$\begin{aligned} I' = \int d_\kappa p p_a{}^{\lambda_1} \dots p_c^{\lambda_j} &\left[\frac{2\alpha_1 (m_1^2 + k_1^2 + (pk_1))}{((p+k_1)^2 + m_1^2)^{\alpha_1 + 1} (\quad)^{\alpha_2} \dots (\quad)^{\alpha_l}} \right. \\ &\left. + \frac{2\alpha_2 (m_2^2 + k_2^2 + (pk_2))}{(\quad)^{\alpha_1} (\quad)^{\alpha_2 + 1} \dots (\quad)^{\alpha_l}} + \dots + \frac{2\alpha_l (m_l^2 + k_l^2 + (pk_l))}{(\quad)^{\alpha_1} (\quad)^{\alpha_2} \dots (\quad)^{\alpha_l + 1}} \right] \end{aligned} \tag{12}$$

or

$$I = -\frac{1}{(\kappa + \lambda_1 + \lambda_2 + \dots \lambda_j - 2\alpha_1 - 2\alpha_2 - \dots - 2\alpha_l)} I' \,. \tag{13}$$

The integral I' converges if

$$\lambda_1 > -1\,, \qquad \lambda_2 > -1\,, \ldots, \lambda_j > -1$$

$$\kappa + \lambda_1 + \lambda_2 + \ldots \lambda_j - 2(\alpha_1 + \alpha_2 + \ldots \alpha_l) < 1\,. \tag{14}$$

This is a larger domain than (10), and the right hand side of (13) is the explicit representation of the analytic continuation of I into this domain.

For one loop diagrams the variable n appears linearly in some exponent λ. In that case one obtains an analytic continuation over a region of magnitude one in the direction of positive n.

The above operation will be called *partial p*. By repeated application of partial p and for sufficiently large λ one obtains an explicit representation valid in an arbitrarily large domain in the complex n-plane. Or, in a given region of the n-plane a representation in terms of arbitrarily convergent integrals.

With this prescription one may now evaluate the integrals in the example (1). The result is of course precisely (2).

For diagrams with two closed loops one may proceed in a similar way. There will be two n-fold integrals, and one writes:

$$\int \mathrm{d}_n p \int \mathrm{d}_n p' \to \int \mathrm{d}_4 \underline{p} \int \mathrm{d}_4 \underline{p}' \int \mathrm{d}_{n-4}\, P \int \mathrm{d}_{n-4}\, P'\,. \tag{15}$$

In the P' integral the fifth axis is taken in the direction of the $(n-4)$ vector P:

$$(15) \to \int \mathrm{d}_4 \underline{p} \int \mathrm{d}_4 \underline{p}' \int \mathrm{d}_{n-4}\, P \int \mathrm{d}p_5' \int \mathrm{d}_{n-5}\, P'\,.$$

The integrands will be independent of the P and P' directions. The integration over angles may be performed:

$$(15) \to \frac{4\pi^{\frac{1}{2}(2n-9)}}{\Gamma(\frac{1}{2}(n-4))\,\Gamma(\frac{1}{2}(n-5))} \int \mathrm{d}_4 \underline{p} \int \mathrm{d}_4 \underline{p}' \int\limits_0^\infty \mathrm{d}\omega\, \omega^{n-5} \int\limits_{-\infty}^\infty \mathrm{d}p_5' \int\limits_0^\infty \mathrm{d}\omega'\, \omega'^{n-6}\,. \tag{16}$$

The argument of such integrals will be a function of the components $\underline{p}$ and $\underline{p}'$, of ω^2, $p_5'^2 + \omega'^2$ and $(p_5' + \omega)^2 + \omega'^2$ (arising from p^2, p'^2 and $(p+p')^2$).

(16) may be written in an elegant form by introducing a twodimensional space, and the vectors

$$q = \begin{pmatrix} \omega \\ 0 \end{pmatrix}, \qquad q' = \begin{pmatrix} p_5' \\ \omega' \end{pmatrix}. \tag{17}$$

With ϵ_{ij} the completely antisymmetric tensor in two dimensions ($\epsilon_{12} = 1$) we have:

$$\int d_n p \int d_n p' \, f(p^2, p'^2, (p+p')^2)$$
$$= \frac{2\pi^{\frac{1}{2}(2n-9)-1}}{\Gamma(\frac{1}{2}(n-4))\,\Gamma(\frac{1}{2}(n-5))} \int d_4\underline{p} \int d_4\underline{p}' \int d_2 q \int d_2 q' \, (\epsilon_{ij} q_i q_j')^{n-6} \theta(\epsilon_{ij} q_i q_j')$$
$$f(q^2, q'^2, (q+q')^2) \,. \tag{18}$$

The step-function θ is needed because of the lower limit 0 in the ω' integration in (16). We have dropped explicit indication of the dependence on the components of $\underline{p}$ and $\underline{p}'$.

The equivalent of (8) is also obtained by partial integration. To this purpose one observes that

$$(\epsilon_{ij} q_i q_j')^\alpha = \frac{1}{(\alpha+2)(\alpha+1)} \epsilon_{ab} \frac{\partial}{\partial q_a} \frac{\partial}{\partial q_b'} (\epsilon_{ij} q_i q_j')^{\alpha+1} \,. \tag{19}$$

λ times application of (19) and subsequent partial integration leads to an expression similar to (8):

$$\frac{2\pi^{\frac{1}{2}(2n-11)}}{\Gamma(\frac{1}{2}(n-4)+\lambda)\,\Gamma(\frac{1}{2}(n-5)+\lambda)} \int d_4\underline{p} \int d_4\underline{p}' \int d_2 q \int d_2 q' \, (\epsilon_{ij} q_i q_j')^{n-6+2\lambda} \,.$$
$$\times \theta(\epsilon_{ij} q_i q_j') \left(\frac{\partial^2}{\partial q^2 \, \partial q'^2} + \frac{\partial^2}{\partial q^2 \, \partial (q+q')^2} + \frac{\partial^2}{\partial q'^2 \, \partial (q+q')^2} \right)^\lambda$$
$$\times f(q^2, q'^2, (q-q')^2) \,. \tag{20}$$

Again, (20) and its analytic continuation to larger n define the contribution of the two-loop diagrams to the generalized S-matrix elements. Explicit representations for large n may be obtained by operations similar to partial p in the one loop case. We need four such operations in the two loop case, and we will discuss them in sect. 4.

Definitions similar to (20) may be given for the three or more closed loop cases.

The above prescription applies if all loop particles are scalars. To complete our prescription to cover vector fields we note that indices that are part of the propagators contained in the loops now take the values 1 to n for integer n. This is because polarization vectors corresponding to internal lines become n-component vectors. The only practical consequence of this fact is that in doing the vector algebra of all occurring loop indices one must use the rule

$$\delta_{\mu\mu} = n \,. \tag{21}$$

After that one has expressions that can be used to define the diagrams for non-integer n. In establishing Ward-identities one notes an interplay between these factors

n and the factors n occurring in association with averaging over all directions in p space of factors like $p_\mu p_\nu$. See eqs. (A7), (A8). By virtue of (21) equations like

$$p_\mu(\delta_{\mu\nu}p^2 - p_\mu p_\nu) = 0$$

remain true even for continuous n in the sense defined above.

3. WARD IDENTITIES, UNITARITY, CAUSALITY

We must now establish that our generalized amplitudes satisfy Ward identities. These Ward identities involve (i) vector algebra and (ii) shifting of integration variables. In the sense defined in sect. 2 it is easy to see that the vector algebra goes through unchanged for any n. For example, consider the photon-pion vertex of fig. 1. One requires that this vertex when multiplied by the photon four momentum equals the difference of two inverse pion propagators. Thus, in 4 dimensions with the vertex of fig. 1 where $q = -p - k$:

$$k_\mu \{-(2p+k)_\mu\} = (p+k-p)_\mu \{-(p+k+p)_\mu\}$$
$$= -(p+k)^2 - m^2 + (p^2+m^2)\,.$$

In the n dimensional formulation, with the notation of eq. (5):

$$k_\mu \{-(2\underline{p}+k)_\mu\} = -(k, 2\underline{p}+k) = -(k, 2p+k)$$
$$= -(k+p-p, k+p+p) = -(p+k)^2 - m^2 + (p^2+m^2)\,,$$

with $(p+k)^2 = (\underline{p}+k)^2 + P^2$ and $p^2 = \underline{p}^2 + P^2$.

It is this rather trivial type of vector algebra that is involved in proving the gauge invariance of eq. (1). In the case of vector particles things are slightly more complicated, and the rule (21) comes in. In that case one must demonstrate for instance that eq. (A7) from appendix A can be obtained from (A8) on multiplication with $\delta_{\mu\nu}$, which indeed happens to be the case. In general one must show that the vector algebra that must hold for the left hand side of eqs. (A5) – (A10) (and their generalizations) actually holds for the right hand sides for any n. One easily convinces oneself that this property holds keeping in mind that all necessary equations can be obtained from (A5) by differentiation with respect to k.

As for point (ii) the shifting of integration variables, we first note that any shift over an external (= physical) four vector is certainly allowed since we have kept the integrations over the first four components unchanged. Nothing else is required in the one loop case. In the two loop case we must have invariance for shifts like $p \to p + p'$, where p and p' are both loop momenta. From formula (20) this is evidently correct, due to the fact that the ϵ-product is invariant for such shifts.

Establishing unitarity and causality, or more precisely cutting rules [9, 10] is also very easy. With the usual $\pm i\epsilon$ prescriptions one needs only to establish the validity of the "largest time" equation of ref. [11] (or (2.8) of ref. [10]) which involves only the time components of the n-vectors, or after fourier transformation, the energy components. Since, as far as these components are concerned, we have not changed the structure of the propagators and integrations (as is evident from (8) and (20)), and since we can always take sufficiently convergent representations in some region of the n-plane, it is obvious that cutting rules hold in some region of the n-plane. Because any term in the cutting equations can be continued analytically to smaller and larger n values by means of the methods of sect. 3 we have the result that the cutting rules hold for any n.

It must be noted that in these rules the integration over intermediate states involves n-space. If all poles for $n = 4$ have been removed then in the limit $n = 4$ this integration over intermediate states reduces to the required integral over physical phase space. It is essential in this context that the phase space integrations are themselves finite, and do not introduce new poles.

In considering cut diagrams some care in handling the δ-functions is necessary. The following remarks may be of help in this respect;

(i) in (8) (and analogously in (20)) the various factors ω^2 occur in denominators simply in addition to the masses of the loop particles. E.g. $p^2 + m^2 = \underline{p}^2 + \omega^2 + m^2$. Thus one can see (8) as a superposition of diagrams where internal masses m_i^2 have been replaced by $\omega^2 + m_i^2$, with weight function

$$\omega^{n-5-2\lambda}\left(-\frac{\partial}{\partial\omega^2}\right)^{\lambda}.$$

One may go further and exchange the ω and $\underline{p}$ integrations in (8). For cut diagrams, where the $\underline{p}$ integration is convergent also one may further exchange differentiation with respect to ω^2 and the $\underline{p}$ integration. Then the calculation of cut diagrams becomes identical to the conventional calculation followed by differentiation and integration with respect to ω.

(ii) If one chooses a very small n the above mentioned weight function may induce strong threshold singularities.

4. RENORMALIZATION

In order to obtain a consistent theory it must be shown that the poles for $n = 4$ can be removed, order by order in perturbation theory. In a given order any new subtraction terms to be introduced must satisfy a stringent criterion: they may not have an imaginary part. This follows very simple from the fact that in a given order the imaginary part, through unitarity, is determined unambigeously by the lower order results. In practice this means that new subtraction terms must be finite polynomials in the external momenta. The demonstration that this is possible includes treatment of the overlapping divergencies. It is perhaps worthwhile to mention the fact that Ward identities and the problem of overlapping divergencies have nothing to do with each other, even though in quantum electrodynamics Ward identities have been of technical assistance in unraveling the divergence structure of perturbation theory.

In this section we will treat one and two closed loop diagrams. The generalization to more closed loops is obvious. In order to keep the discussion transparant we will omit most numerator factors as for instance occurring when there are vector particles. Yet the treatment will be such that these factors can be written without interference with the main argument.

The definitions (8) and (20) will form the basis of our discussion. Consider first one loop diagrams, eq. (8). We must show that the residues of eventual poles for $n = 4$ are finite polynomials in the external momenta. If the diagram is convergent in 4-space the expression (8) will be non-singular for $n = 4$. If the diagram is divergent than (8) will be well-defined only in some region to the left of $n = 4$. Subsequent analytic continuation by means of partial p shows a single pole for $n = 4$ multiplied by a finite and well-defined expression. On the other hand, (8) may be evaluated explicitly by means of Feynman parameters and the equations of appendix A. One obtains an expression of the form:

$$\Gamma(j - \tfrac{1}{2}n) \int \mathrm{d}y_1 \ldots \int \mathrm{d}y_i \frac{1}{(M^2)^{j-\frac{1}{2}n}} P(y, m, k) \tag{22}$$

independent of λ. In here j is some integer and M^2 and P are polynomials in the Feynman parameters y_i, the masses and the external momenta. The eventual pole for $n = 4$ is explicit in the Γ function; one has poles for $n \geqslant 2j$ in agreement with the results obtained by means of partial p. All this is to ensure that there is no trouble hidden in the Feynman parameter integrations *.

The residue of a pole for n an even integer $\geqslant 2j$ is:

$$C \int \mathrm{d}y_1 \ldots \int \mathrm{d}y_i (M^2)^l \, P(y, m, k) \tag{23}$$

with $l = \frac{1}{2}n - j \geqslant 0$. Obviously (23) is a finite polynomial, there are no terms of the form $\ln k^2$.

Thus, up to one closed loop, eventual poles for $n = 4$ (or for any other n) have as residue polynomials in the external momenta, and may be subtracted.

Next we turn to two closed loops. We assume that counter-terms of the form

$$\frac{1}{n-4} P(k, m)$$

* This is by no means trivial. For instance infrared divergencies are usually hidden in the Feynman parameter integrations. For two or more closed loops ultra-violet divergencies may also be transferred from the momentum integrations to the Feynman parameter integrations. For this reason one must be very careful in taking together propagators belonging to different loops by means of Feynman parameters.

have been introduced such as to make all one loop diagrams finite. Consider the general two loop diagram:

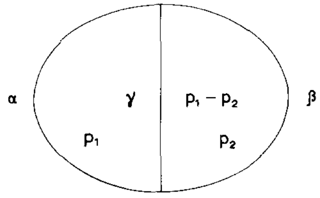

Fig. 3. $\alpha\beta\gamma$ diagram.

We have omitted all external lines. The corresponding expression is:

$$\int \mathrm{d}_n p_1\, \mathrm{d}_n p_2 \frac{1}{(p_1^2+m_1^2)^\alpha\,(p_2^2+m_2^2)^\beta\,((p_1-p_2+k)^2+M^2)^\gamma} \tag{24}$$

and we will speak of the $\alpha\,\beta\,\gamma$ diagram.

In here we have assumed that all propagators that depend on p_1 and external momenta have been taken together by means of the Feynman parameter method. Similarly for p_2 and $p_1 - p_2$ dependent propagators. Furthermore we have suppressed all numerators, except insofar as power counting is concerned; thus

$$\frac{p_{1_\mu}\, p_{1_\nu}}{(p_1^2+m_1^2)^5}$$

is in the above represented as

$$\frac{1}{(p_1^2+m_1^2)^4}\,.$$

Here, and in what follows, we write the integrals as if we are operating in an n-dimensional space with positive integer n, but this must be understood as symbolic for integrals like in (20) with sufficiently large λ. We work in a region of small n sufficiently far to the left of $n = 4$.

There are three one-loop diagrams contained in the above two-loop diagram. We will call them the $\alpha\gamma$, $\beta\gamma$ and $\alpha\beta$ sub-diagrams respectively:

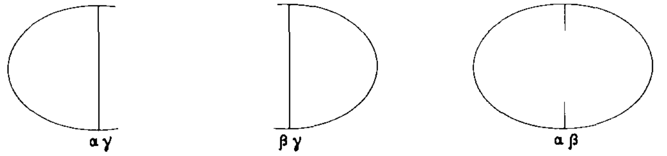

Fig. 4.

If any of these subdiagrams diverge we have counterterms associated with them. In addition to the $\alpha\beta\gamma$ diagram we must therefore also consider the subtraction diagrams:

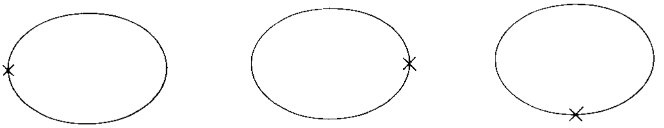

Fig. 5.

The crossed vertex in the first diagram refers to the pole with polynomial coefficient to be subtracted from the $\alpha\gamma$ diagram. Similarly for the other diagrams. It is clear that the subtraction diagrams contain double poles (1 pole from the crossed vertex and 1 pole from the loop integration) as well as single poles. In particular we have the single pole from the vertex multiplying the finite part of the loop integration. This will lead to terms of the form

$$\frac{1}{n-4}\ln(k^2)\,.$$

Such terms cannot be renormalized away, and the theory will be renormalizable only if these terms cancel against similar terms coming from the two loop $(\alpha\beta\gamma)$ diagram. This means that we must unravel the pole structure of the $\alpha\beta\gamma$ diagram, eq. (24). In this way presents itself here the problem of the overlapping divergencies.

The expression (24) is well defined for sufficiently low n. The continuation to large n is slightly more complicated then in the one loop case. There are four operations that may be performed. First one may insert in (24) the expression, equal to unity:

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\partial p_{1_i}}{\partial p_{1_i}}$$

and perform partial integration. The result is an equation similar to (13) showing a pole for $n = 2(\alpha+\gamma)$. Note that the denominators with the exponents α and γ involve p_1. We will call this operation partial (α, γ). Similarly one may define an operation partial (β, γ) by partial integration with respect to p_2. Further there is the operation partial (α, β) obtained by partial integration with respect to p_1 after the substitution $p_2' = p_1 - p_2$. Finally there is the operation partial (α, β, γ) showing a pole at $n = \alpha + \beta + \gamma$, and obtained by insertion of

$$\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{\partial p_{1_i}}{\partial p_{1_i}} + \frac{\partial p_{2i}}{\partial p_{2i}}\right).$$

The explicit expression obtained after applying partial (α, β, γ) to (24) is:

$$(24) = -\frac{1}{(2n-2\alpha-2\beta-2\gamma)}\, I' \tag{25}$$

with

$$I' = \int \mathrm{d}_n p_1\, \mathrm{d}_n p_2 \Bigg[\frac{2\alpha\, m_1^2}{(\ )^{\alpha+1}(\ )^{\beta}(\ )^{\gamma}} + \frac{2\beta\, m_2^2}{(\ )^{\alpha}(\ )^{\beta+1}(\ )^{\gamma}}$$
$$+ \frac{2\gamma(M^2+k^2+(p_1-p_2,k))}{(\ )^{\alpha}(\ )^{\beta}(\ )^{\gamma+1}}\Bigg] \tag{26}$$

where the explicit form of the denominators is as in (24).

The procedure for continuation of (24) to large n is now clear. (24) has poles for $n = 2(\alpha+\beta)$, $n = 2(\alpha+\gamma)$, $n = 2(\beta+\gamma)$ and $n = \alpha+\beta+\gamma$. The integral representation will be valid for n less than the minimum of these four. Applying partial (α, β, γ) eventually followed by partial (α, β), (α, γ) and/or (β, γ) gives the desired continuation.

It would appear that (24) could contain four coincident poles, but actually there are at most second order poles.

We are now interested in poles with non-polynomial residues. *Definition.* A harmless pole is a pole with as residue a polynomial of finite order in the external momenta. *Definition.* The subintegral α, β is said to be divergent or convergent according to whether $\alpha+\beta \leqslant 2$ or $\alpha+\beta > 2$. *Definition.* The $\alpha\beta\gamma$ diagram (24) is said to be overall convergent or overall divergent according to whether $\alpha+\beta+\gamma > 4$ or $\alpha+\beta+\gamma \leqslant 4$.

More specifically we may call a subintegral α, β logarithmically, linearly etc. divergent if $\alpha+\beta = 2, \frac{3}{2}$, etc. Similar for the overall divergent cases $\alpha+\beta+\gamma = 4, 3, \ldots$ · etc.

First we will consider a simple situation, namely the case that the α, γ subintegral is logarithmically divergent while all other subintegrals are convergent. Then there is only one subtraction diagram, and we must show that (24) together with this subtraction diagram contains only harmless poles. More specifically consider the case $\alpha+\gamma = 2$, $\beta = 2$. Taking together the α and γ denominators of (24) by means of Feynman parameters and performing the integration over p_1 we obtain an expression of the form

$$\int \mathrm{d}_n p\, \frac{\Gamma(\alpha+\gamma-\frac{1}{2}n)}{(p^2+m_3^2)^{\alpha+\gamma-\frac{1}{2}n}}\, \frac{1}{(p^2+2pq+m_4^2)^{\beta}}\,, \tag{27}$$

where we have written p instead of p_2 and suppressed various irrelevant factors. From this we must subtract the expression corresponding to the subtraction diagram

$$\int \mathrm{d}_n p\, \frac{2}{2\alpha+2\gamma-n}\, \frac{1}{(p^2+2pq+m_4^2)^{\beta}}\,. \tag{28}$$

Main Theorem. The difference of (27) and (28) for $\alpha + \gamma = 2, \beta \geqslant 1$ contains only harmless poles. Proof: see appendix B.

This theorem is really the key theorem to our problem, because the general case may be reduced by means of partial operations to the case considered in the main theorem.

The general case may now be treated systematically. The case that any of the exponents α, β or γ is zero or negative is trivial and corresponds to a diagram containing no overlapping divergencies. Example:

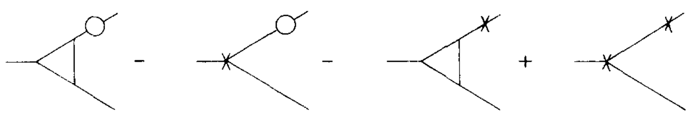

Fig. 6.

Such cases do not require new subtractions. Thus we assume now $\alpha > 0, \beta > 0, \gamma > 0$.
Theorem 1. If the integral (24) is logarithmically divergent and contains no divergent subintegrals than it contains a harmless single pole at $n = 4$.
Proof: by means of partial (α, β, γ) it is seen that the integral behaves at $n = 4$ as a single pole times a finite function. Actual calculation of (24) with the help of Feynman parameters exhibits this pole:

$$(24) \to \Gamma(\alpha + \beta + \gamma - n) \int \mathrm{d}y_1 \int \mathrm{d}y_2\, f(y_1, y_2)$$

and the integrals over the Feynman parameters $y_1\, y_2$ are well defined and finite. By actually setting $n = 4$ in those integrals one obtains the desired result.
Theorem 2. If the integral (24) is overall divergent and contains no divergent subintegrals then it contains a harmless single pole at $n = 4$.
Proof: by means of partial (α, β, γ) this case may be reduced to the case of theorem 1.
Theorem 3. If the integral (24) is overall convergent or logarithmically divergent then it contains at most one divergent sub-integral. The denominator not involved in the sub-integral has an exponent $\geqslant 2$.
Proof: if $\alpha + \beta + \gamma \geqslant 4$ and for instance $\alpha + \beta \leqslant 2$ then $\gamma \geqslant 2$.
Theorem 4. If the integral (24) is overall convergent or logarithmically divergent and contains one divergent sub-integral then the difference with the subtraction diagram containing the pole subtraction term corresponding to the divergent sub-integral has only harmless poles.
Proof: let α, β be the divergent subintegral. By means of partial (α, β) the divergent subintegral may be reduced to a logarithmically divergent subintegral. Since the remaining denominator has an exponent $\geqslant 2$ we are then in the case considered in the main theorem.

Finally we must consider the case that the integral (24) is linearly, quadratically etc. divergent. By means of partial (α, β, γ) this case can be reduced to the case of an overall convergent or logarithmically divergent integral, which has been considered above. This terminates the proof of renormalizability up to and including two loop diagrams.

As an example consider the case $\alpha = \beta = \gamma = 1$.
Thus:

$$I = \int \mathrm{d}_n p_1 \, \mathrm{d}_n p_2 \frac{1}{(p_1^2 + m_1^2)(p_2^2 + m_2^2)((p_1 - p_2 - k)^2 + m_3^2)} . \tag{29}$$

The integral is overall quadratically divergent, and $\alpha\gamma$, $\beta\gamma$ and $\alpha\beta$ subintegrals are all logarithmically divergent. Thus we have three subtraction diagrams, and for instance the subtraction diagram corresponding to the $\alpha\gamma$ subintegral is:

$$I^{\mathrm{s}}_{\alpha\gamma} = -\int \mathrm{d}_n p_2 \, PP\left\{\int \mathrm{d}_n p_1 \frac{1}{(p_1^2 + m_1^2)((p_1 - p_2 - k)^2 + m_3^2)} \frac{1}{(p_2^2 + m_2^2)}\right\} \tag{30}$$

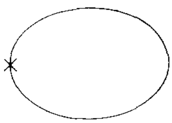

Fig. 7.

where $PP\{\ \}$ means pole part for $n = 4$. Because we have a logarithmic divergence the polynomial multiplying the pole $1/(n-4)$ will be simply a constant. Similarly for the other two subintegrals.

Applying partial (α, β, γ) to (29) we obtain:

$$I = -\frac{1}{2n-6} I'$$

$$I' = \int \mathrm{d}_n p_1 \, \mathrm{d}_n p_2 \left\{ \frac{2m_1^2}{(p_1^2 + m_1^2)^2 (p_2^2 + m_2^2)((p_1 - p_2 - k)^2 + m_3^2)} \right.$$

$$+ \frac{2m_2^2}{(p_1^2 + m_1^2)(p_2^2 + m_2^2)^2 ((p_1 - p_2 - k)^2 + m_3^2)}$$

$$\left. + \frac{2m_3^2 + 2k^2 - 2(p_1 - p_2, k)}{(p_1^2 + m_1^2)(p_2^2 + m_2^2)((p_1 - p_2 - k)^2 + m_3^2)^2} \right\} . \tag{31}$$

Applying partial integration with respect to p_2 in (30) we obtain:

$$I^{\mathrm{s}}_{\alpha\gamma} = +\frac{1}{n-2}\int \mathrm{d}_n p_2\, PP\{\ \}\, \frac{2m_2^2}{(p_2^2+m_2^2)^2}\,. \tag{32}$$

In the limit $n = 4$ the second term of (31) combines together with the subtraction diagram such as to have precisely the case discussed in the main theorem. Similarly for the other terms, to be combined with the other subtraction diagrams.

This example demonstrates how the operation partial (α, β, γ) neatly separates out the various overlapping divergencies. After that the main theorem guarantees the absence of unwanted poles.

5. EXTENSION TO FERMIONS

The extension to fermions is straightforward and based on the following observation. Everything may be formulated such that only traces of strings of γ-matrices occur. If there are external fermion lines this may be done through the use of suitable projection operators. These traces must than be evaluated according to the rules (see appendix C)

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\delta_{\mu\nu}\,, \tag{33}$$

$$\mathrm{Tr}\,(S) = 0 \text{ if } S \text{ is an odd string of } \gamma\text{'s}\,. \tag{34}$$

$$\mathrm{Tr}\,(I) = 4\,. \tag{35}$$

Remember that $\delta_{\mu\mu} = n$. Any Ward identity relying as far as γ-matrices is concerned only on (33) (as in quantum-electrodynamics) will be satisfied for every n.

Note that there is no place for the pseudo-scalar γ^5 (in conventional notation) in (33), as there is no place for the pseudo tensor $\epsilon_{\mu\nu\alpha\beta}$. See sect. 6.

The rule (35) can be satisfied by finite matrices only for $n = 4$, but this is of no importance because we are only interested in a consistent algebra for $n \neq 4$. Or, in n-dimensional space one will have

$$\mathrm{Tr}\,(I) = f(n)$$

where f is a function of n only. We need only $f(n) = 4$, and the deviations of $f(n)$ from $f(4)$ are never important for Ward identities because one always compares diagrams with an equal number of traces.

As an example we consider the lowest order photon selfenergy diagram in quantum electrodynamics:

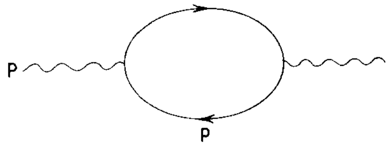

Fig. 8.

$$\int \mathrm{d}_n p \, \frac{\mathrm{Tr}\,[\gamma^\mu (i\gamma\,(p+k)+m)\gamma^\nu (i\gamma p + m)]}{((p+k)^2+m^2)(p^2+m^2)} \, .$$

The trace may be evaluated using (33), (34) and (35). Taking denominators together one obtains:

$$4\int_0^1 \mathrm{d}x \int \mathrm{d}_n p \, \frac{\delta_{\mu\nu}(m^2+p^2+pk) - 2p_\mu p_\nu - p_\mu k_\nu - k_\mu p_\nu}{(p^2+2pkx+k^2x+m^2)^2} \, .$$

Using the equations of appendix A:

$$= \frac{4i\pi^{\frac{1}{2}n}}{\Gamma(2)} \Gamma(2-\tfrac{1}{2}n) \int_0^1 \mathrm{d}x \, \frac{2x(1-x)(k_\mu k_\nu - \delta_{\mu\nu}k^2)}{(m^2+k^2x(1-x))^{2-\frac{1}{2}n}}$$

which is manifestly gauge invariant.

6. LIMITATIONS OF THE METHOD

The method fails if in the Ward identities there appear quantities that have the desired properties only in four dimensional space. An example is the completely antisymmetric tensor $\epsilon_{\mu\nu\alpha\beta}$. If the particular properties of this tensor are vital for the Ward identities to hold our method will fail because we cannot generalize $\epsilon_{\mu\nu\alpha\beta}$ to a tensor satisfying the required properties for non-integer n. Similarly for γ^5. One can write:

$$\gamma^5 = \frac{1}{4!}\,\epsilon_{\mu\nu\alpha\beta}\,\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta \, ,$$

insert this whenever γ^5 occurs and take the ϵ-tensor outside of the expression to be generalized to non-integer n. However, if we are dealing with Ward identities that rely on

$$\{\gamma^5, \gamma^\alpha\} = 0 \qquad \text{for} \qquad \alpha = 1, \ldots, n$$

$$\mathrm{Tr}\,(\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4\epsilon_{\mu\nu\alpha\beta}$$

than this method breaks down. This is precisely the case of the well-known anomalies; to see this consider the vertex

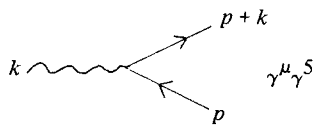

Fig. 9.

which is part of a linearly divergent loop with p as integration variable. One of the required Ward identities is that multiplication with k_μ should give something proportional to the fermion mass. In the combinatorial proof one writes:

$$\gamma k \gamma^5 = (\gamma(p+k) - \gamma p)\gamma^5 = \gamma(p+k)\gamma^5 + \gamma^5 \gamma p$$

where p is the loop momentum.

This is incorrect if $n \neq 4$. It is easily seen that the breakdown of the Ward identity is proportional to $\underline{p} - p$, and after integration to $n - 4$. When multiplied by a pole arising from the loop integration a non-zero part remains in the limit $n = 4$.

To exhibit all this explicitly we evaluate the anomaly for the well-known triangle graph. In the notation of Bell and Jackiw [4] the triangle graph is:

$$\Gamma^{\alpha\mu\nu} = \int \mathrm{d}_n r \, \frac{i \operatorname{Tr}[\gamma^5 \gamma^\alpha \{i\gamma(p+r) - m\} \, \gamma^\mu (i\gamma r - m)\gamma^\nu \{i\gamma(r-q) - m\}]}{((p+r)^2 + m^2)(r^2 + m^2)((r-q)^2 + m^2)} \tag{36}$$

where p and q are the photon momenta. Multiplying by $k_\alpha = (p+q)_\alpha$ one writes:

$$\begin{aligned}\gamma^5 i\gamma k &= \gamma^5 \, \{-i\gamma(p+r) - m + i\gamma(r-q) - m + 2m\} \\ &= - \, \{i\gamma(r-q) + m\} \, \gamma^5 - \gamma^5 \, \{i\gamma(p+r) + m\} \, + 2m\gamma^5 \\ &\quad + 2i\gamma^5 \gamma(r - \underline{r})\end{aligned}$$

where $\underline{r}$ coincides with r in the first four components, but is zero otherwise. The last term is the anomaly which we will evaluate. The four-vector $s = r - \underline{r}$ has the first four components zero. This greatly simplifies computation of the trace if one everywhere sets $r = \underline{r} + s$. The expression for the anomaly becomes

$$2i \int \mathrm{d}_n r \, \frac{\operatorname{Tr}[\gamma s \, \{i\gamma(p + \underline{r}) + i\gamma s - m\} \, \gamma^\mu \, \{i\gamma \underline{r} + i\gamma s - m\} \, \gamma^\nu \, \{i\gamma(\underline{r} - q) + i\gamma s - m\} \gamma^5]}{((p+r)^2 + m^2)(r^2 + m^2)((r-q)^2 + m^2)}$$

Remembering that $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$, and that all vectors except s are physical (have zero components beyond the fourth) one reduces the trace to $4\, i s^2 \epsilon_{\lambda\mu\nu\kappa} p_\lambda q_\kappa$ where one uses the fact that γs anticommutes with all other γ-matrices. Finally we must evaluate the integral

$$\int \mathrm{d}_n r \frac{(r - \underline{r}, r - \underline{r})}{(r^2 + m^2)^3}$$

where we have omitted p and q in the denominator since we need only the pole for $n = 4$. With the help of (A8) multiplied with $\delta_{\mu\nu}$ where now the indices are taken to run from 5 to n only we obtain for this integral:

$$\frac{i\pi^{\frac{1}{2}n}}{(m^2)^{2-\frac{1}{2}n}} \frac{\Gamma(2 - \frac{1}{2}n)}{\Gamma(3)} \tfrac{1}{2}(n - 4) .$$

In the limit $n = 4$, and taking into account that there are two graphs (the second obtained by the interchange $\mu \leftrightarrow \nu$, $p \leftrightarrow q$) we find for the anomaly:

$$8 i\pi^2 \epsilon_{\lambda\mu\nu\kappa} p_\lambda q_\kappa$$

which agrees with the results of Bell and Jackiw [4], eq. (3.12c), and Adler [4], eq. (22).

Note that in our formulation the anomaly has nothing to do with the pecularities of shifts of integration variables.

The usual ambiguity of the choice of integration variables is replaced in our formalism by the ambiguity of the location of γ^5 in the trace in (36). If before generalization to $n \neq 4$ is done, γ^5 is (anti) commuted to the right, a different result emerges.

7. CONCLUSIONS

The method presented essentially completes the proof of the renormalizability of the theories presented in ref. [1]. The method fails if quantities particular to four-dimensional space, such as γ^5 or the tensor $\epsilon_{\mu\nu\alpha\beta}$ or scaling behaviour play an essential role in the Ward identities.

We have not considered infrared problems. Here we only wish to remark that the generalized S-matrix elements will have additional poles for integer n-values if infrared divergencies are present.

The authors are indebted to the participants of the discussion meeting at Orsay, Jan. 1972, for inspiring and constructive criticism.

APPENDIX A. SOME USEFUL FORMULAE

$$\int \mathrm{d}_n x f(x) = \int f(x) r^{n-1}\, \mathrm{d}r \sin^{n-2}\theta_{n-1}\, \mathrm{d}\theta_{n-1} \sin^{n-3}\theta_{n-2}\, \mathrm{d}\theta_{n-2} \ldots \mathrm{d}\theta_1 \quad \text{(A1)}$$

with $0 \leqslant \theta_i \leqslant \pi$, except $0 \leqslant \theta_1 \leqslant 2\pi$.

If $f(x)$ depends only on $r = \sqrt{x_1^2 + \ldots x_n^2}$ one may perform the integration over angles using

$$\int_0^\pi \sin^m \theta \, \mathrm{d}\theta = \sqrt{\pi} \frac{\Gamma(\frac{1}{2}(m+1))}{\Gamma(\frac{1}{2}(m+2))} , \tag{A2}$$

leading to

$$\int \mathrm{d}_n x f(r) = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)} \int f(r)\, r^{n-1} \, \mathrm{d}r \tag{A3}$$

$$\int_0^\infty \mathrm{d}x \frac{x^\beta}{(x^2+M^2)^\alpha} = \frac{1}{2} \frac{\Gamma(\frac{1}{2}(\beta+1))\, \Gamma(\alpha - \frac{1}{2}(\beta+1))}{\Gamma(\alpha)\,(M^2)^{\alpha - \frac{1}{2}(\beta+1)}} . \tag{A4}$$

Keeping the prescriptions and definitions of sect. 3 in mind, the following equations hold for arbitrary n:

$$\int \mathrm{d}_n p \frac{1}{(p^2+2kp+m^2)^\alpha} = \frac{i\pi^{\frac{1}{2}n}}{(m^2-k^2)^{\alpha-\frac{1}{2}n}} \frac{\Gamma(\alpha-\frac{1}{2}n)}{\Gamma(\alpha)} , \tag{A5}$$

$$\int \mathrm{d}_n p \frac{p_\mu}{(p^2+2kp+m^2)^\alpha} = \frac{i\pi^{\frac{1}{2}n}}{(m^2-k^2)^{\alpha-\frac{1}{2}n}} \frac{\Gamma(\alpha-\frac{1}{2}n)}{\Gamma(\alpha)} (-k_\mu) , \tag{A6}$$

$$\int \mathrm{d}_n p \frac{p^2}{(p^2+2kp+m^2)^\alpha} = \frac{i\pi^{\frac{1}{2}n}}{(m^2-k^2)^{\alpha-\frac{1}{2}n}} \cdot \frac{1}{\Gamma(\alpha)} \{\Gamma(\alpha-\tfrac{1}{2}n) k^2$$
$$+ \Gamma(\alpha - 1 - \tfrac{1}{2}n)\tfrac{1}{2}n(m^2-k^2)\} , \tag{A7}$$

$$\int \mathrm{d}_n p \frac{p_\mu p_\nu}{(p^2+2kp+m^2)^\alpha} = \frac{i\pi^{\frac{1}{2}n}}{(m^2-k^2)^{\alpha-\frac{1}{2}n}} \frac{1}{\Gamma(\alpha)} \{\Gamma(\alpha-\tfrac{1}{2}n) k_\mu k_\nu$$
$$+ \Gamma(\alpha-1-\tfrac{1}{2}n)\tfrac{1}{2}\delta_{\mu\nu}(m^2-k^2)\} , \tag{A8}$$

$$\int \mathrm{d}_n p \frac{p_\mu p_\nu p_\lambda}{(p^2+2kp+m^2)^\alpha} = \frac{i\pi^{\frac{1}{2}n}}{(m^2-k^2)^{\alpha-\frac{1}{2}n}} \frac{1}{\Gamma(\alpha)} \{-\Gamma(\alpha-\tfrac{1}{2}n) k_\mu k_\nu k_\lambda$$
$$- \Gamma(\alpha-1-\tfrac{1}{2}n)\tfrac{1}{2}(\delta_{\mu\nu}k_\lambda + \delta_{\mu\lambda}k_\nu + \delta_{\nu\lambda}k_\mu)(m^2-k^2)\} , \tag{A9}$$

$$\int d_n p \frac{p^2 p_\mu}{(p^2+2kp+m^2)^\alpha} = \frac{i\pi^{\frac{1}{2}n}}{(m^2-k^2)^{\alpha-\frac{1}{2}n}} \frac{1}{\Gamma(\alpha)} (-k_\mu) \{\Gamma(\alpha-\tfrac{1}{2}n)k^2$$

$$+ \Gamma(\alpha - \tfrac{1}{2}n - 1) \tfrac{1}{2}(n+2)(m^2-k^2)\} . \qquad \text{(A10)}$$

The above equations contain indices μ, ν, λ. These indices are understood to be contracted with arbitrary n-vectors q_1, q_2 etc. In computing the integrals one first integrates over the part of n-space orthogonal to the vectors k, q_1, q_2 etc., using (A1) – (A4). After that the expressions are meaningful also for non-integer n. Note that formally (A6) – (A10) may be obtained from (A5) by differentiation with respect to k, or by using $p^2 = (p^2 + 2pk + m^2) - 2pk - m^2$. The Feynman parameter method for non-integer exponents:

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(ax+b(1-x))^{\alpha+\beta}} \qquad \text{(A11)}$$

valid for $\alpha > 0, \beta > 0$. If one needs this formula for α in the neighbourhood of 0 one may write

$$\frac{1}{a^\alpha b^\beta} = \frac{a}{a^{\alpha+1} b^\beta}$$

and then use (A11).

The generalization of eq. (A11) for many factors:

$$\frac{1}{a_1^{\alpha_1} a_2^{\alpha_2} \dots a_m^{\alpha_m}} = \frac{\Gamma(\alpha_1+\alpha_2+\dots\alpha_m)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\dots\Gamma(\alpha_m)} \cdot \int_0^1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{m-2}} dx_{m-1}$$

$$\times \frac{x_{m-1}^{\alpha_1-1}(x_{m-2}-x_{m-1})^{\alpha_2-1}\dots(1-x_1)^{\alpha_m-1}}{[a_1 x_{m-1} + a_2(x_{m-2}-x_{m-1}) + \dots\dots + a_m(1-x_1)]^{\alpha_1+\alpha_2+\dots\alpha_m}} . \qquad \text{(A12)}$$

APPENDIX B. THE MAIN THEOREM

Consider the integral:

$$\int d_n p \left\{ \frac{\Gamma(\frac{1}{2}(n_0-n))}{(p^2+m^2)^{\frac{1}{2}(n_0-n)}} - \frac{2}{n_0-n} \right\} \frac{A}{(p^2+2pk+M^2)^\alpha} . \qquad \text{(B1)}$$

We must prove that (B1) contains only harmless poles for $n = n_0$ (where n_0 is the dimension of physical space, i.e. $n_0 = 4$) and $\alpha \geq 1$. For $\alpha > \frac{1}{2}n_0$ the theorem is trival because then the only poles for $n = n_0$ are those already explicit in (B1), and they obviously cancel if the rest is nonsingular for $n = n_0$. The nontrivial cases are $\alpha = \frac{1}{2}(n_0 - 1)$ and $\alpha = \frac{1}{2}n_0$.

Formula (B1) is symbolic insofar that the exponent α stands in fact for the difference of the powers of p in the numerator A and those in the denominator. We will prove (B1) for the case $\alpha = \frac{1}{2}n_0$ with $A = 1$; the case $A = (pq)$ and exponent $\alpha = \frac{1}{2}n_0$ in the denominator may be proven similarly, and the case that there are two or more p's in A may be reduced to the case of no or one p in the numerator.

Thus consider the special case:

$$\int \mathrm{d}_n p \left\{ \frac{\Gamma(\frac{1}{2}(n_0 - n))}{(p^2 + m^2)^{\frac{1}{2}(n_0 - n)}} - \frac{2}{n_0 - n} \right\} \frac{1}{(p^2 + 2pk + M^2)^{\frac{1}{2}n_0}} . \tag{B2}$$

The first term of (B2) may be worked out (see (A11)):

$$\Gamma(\tfrac{1}{2}(n_0 - n)) \int \mathrm{d}_n p \frac{p^2 + m^2}{(p^2 + m^2)^{\frac{1}{2}(n_0 - n + 2)} (p^2 + 2pk + M^2)^{\frac{1}{2}n_0}}$$

$$= \frac{\Gamma(\frac{1}{2}(n_0 - n))\, \Gamma(n_0 + 1 - \frac{1}{2}n)}{\Gamma(\frac{1}{2}(n_0 + 2 - n))\, \Gamma(\frac{1}{2}(n_0))} \int_0^1 \mathrm{d}x\, x^{\frac{1}{2}(n_0 - 2)} (1 - x)^{\frac{1}{2}(n_0 - n)} .$$

$$\int \mathrm{d}_n p \frac{p^2 + m^2}{(p^2 + 2pkx + M^2 x + m^2(1 - x))^{n_0 + 1 - \frac{1}{2}n}} . \tag{B3}$$

Use of eqs. (A5) and (A7) gives:

$$(\mathrm{B3}) = \mathrm{i}\pi^{\frac{1}{2}n} \frac{\Gamma(\frac{1}{2}(n_0 - n))\, \Gamma(n_0 + 1 - \frac{1}{2}n)}{\Gamma(\frac{1}{2}(n_0 + 2 - n))\, \Gamma(\frac{1}{2}(n_0))} \int_0^1 \mathrm{d}x\, x^{\frac{1}{2}(n_0 - 2)} (1 - x)^{\frac{1}{2}(n_0 - n)}$$

$$\times \left[\frac{\Gamma(n_0 + 1 - n)}{\Gamma(n_0 + 1 - \frac{1}{2}n)} \frac{m^2 + k^2 x^2}{(M^2 x + m^2(1 - x) - k^2 x^2)^{n_0 + 1 - n}} \right.$$

$$\left. + \frac{\Gamma(n_0 - n)}{2\Gamma(n_0 + 1 - \frac{1}{2}n)} \frac{n}{(M^2 x + m^2(1 - x) - k^2 x^2)^{n_0 - n}} \right] . \tag{B4}$$

The only singularities for $n = n_0$ are now located in the two Γ-functions. We are only interested in possible non-polynomial residues of the poles at $n = n_0$. To this purpose we may set everywhere $n = n_0$ except in the two Γ-functions and in the last exponent in (B4). Next, writing

$$x^{\frac{1}{2}(n_0 - 2)} = \frac{2}{n_0} \frac{\mathrm{d}}{\mathrm{d}x} x^{\frac{1}{2}n_0}$$

and performing partial integration with respect to x in the second term we obtain:

$$(B4) \Rightarrow i\pi^{\frac{1}{2}n_0} \frac{\Gamma(\frac{1}{2}(n_0 - n))}{\Gamma(\frac{1}{2}n_0)} \int_0^1 dx\, x^{\frac{1}{2}(n_0-2)} \cdot \frac{1}{(M^2 x + m^2(1-x) - k^2 x^2)^{n_0 - n}}$$

$$+ i\pi^{\frac{1}{2}n_0} \frac{\Gamma(\frac{1}{2}(n_0 - n))\,\Gamma(n_0 - n)}{\Gamma(\frac{1}{2}n_0)} \cdot \frac{1}{(M^2 - k^2)^{n_0 - n}} . \tag{B5}$$

The second term in (B5) is the surface term arising from the partial integration.

The first term in (B5) displays a harmless single pole.

The second term contains the single pole

$$i\pi^{\frac{1}{2}n_0} \frac{\Gamma(\frac{1}{2}(n_0 - n))\,\Gamma(n_0 - n)}{\Gamma(\frac{1}{2}n_0)} (n_0 - n) \ln (M^2 - k^2) \tag{B6}$$

in addition to harmless poles.

Consider next the contribution of the second term in (B2):

$$\frac{2}{n_0 - n} \int d_n p \frac{1}{(p^2 + 2pk + M^2)^{\frac{1}{2}n_0}} =$$

$$= i\pi^{\frac{1}{2}n_0} \frac{2}{n_0 - n} \frac{\Gamma(\frac{1}{2}(n_0 - n))}{\Gamma(\frac{1}{2}n_0)} \frac{1}{(M^2 - k^2)^{\frac{1}{2}(n_0 - n)}} . \tag{B7}$$

(B7) contains harmless poles as well as the pole

$$i\pi^{\frac{1}{2}n_0} \frac{\Gamma(\frac{1}{2}(n_0 - n))}{\Gamma(\frac{1}{2}n_0)} \frac{2}{n_0 - n} \tfrac{1}{2}(n_0 - n) \ln (M^2 - k^2) . \tag{B8}$$

Since $\Gamma(x)$ behaves as $1/x$ for x in the neighbourhood of zero we see that the dangerous pole in (B6) is cancelled by (B8).

It may be noted that the difference of (B6) and (B8) contains a harmless double pole. Thus in general we may expect to need double pole counterterms at the two loop level.

APPENDIX C. TRACES MAY BE COMPUTED BY MEANS OF THE EQUATION

$$\mathrm{Tr}\,(\gamma^{\mu_1}\gamma^{\mu_2} \ldots \gamma^{\mu_m}) = -\,\mathrm{Tr}\,(\gamma^{\mu_m}\gamma^{\mu_1} \ldots \gamma^{\mu_{m-1}})$$

$$+ 2 \sum_{i=1}^{m-1} (-1)^{i+1}\, \mathrm{Tr}\,(\gamma^{\mu_1} \ldots \gamma^{\mu_{i-1}}\gamma^{\mu_{i+1}} \ldots \gamma^{\mu_{m-1}})\, \delta_{\mu_i \mu_m} \tag{C1}$$

valid for even m. This equation is based solely on eq. (33). Since the first term on the right hand side equals minus the term on the left hand side we have a recursion formula relating traces of m γ-matrices to traces of $m-2$ matrices.

The requirement

$$\mathrm{Tr}\,(\gamma^{\mu_1}\ldots\gamma^{\mu_m}) = 0 \text{ for odd } m$$

excludes for instance for $n = 3$ the choice

$$\gamma^i = \sigma^i\,, \qquad i = 1, 2, 3$$

where the σ^i are the well-known Pauli spin matrices. Instead one may simply use the 4×4 matrices γ^1, γ^2 and γ^3. In fact, if one has a set of γ-matrices for some large n one may for lower n always use a subset of that set.

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