# NOTES ON COOPERATIVE GAME THEORY AND THE CORE

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## 1. Introduction

Cooperative game theory is fundamentally different from the types of games we have studied so far, which we will now refer to as *noncooperative* games. In noncooperative games, actions are taken by individual players, and the outcome of the game is described by the action taken by each player, along with the payoff that each player achieves. In contrast, cooperative games consider the set of joint actions that any group of players can take. The outcome of a cooperative game will be specified by which group of players forms, and the joint action that that group takes. We refer to these groups of players as *coalitions.*

Before we continue, let's establish the following notation for analyzing an $n$-person cooperative game.

*Notation* 1.1. We will denote by $N$ the coalition consisting of all $n$ players, which we will also refer to as the *grand coalition.* $S$ will denote any sub-coalition consisting of some subset of these $n$ players. Finally, $|S|$ will denote the *size* of $S$, e.g. if $S = \{2, 3, 5, 7\}$ then $|S| = 4$.

There are many real world situations that are modeled quite naturally as cooperative games. The following are taken from [5], where detailed analyses of each can be found.

*Example* 1.2. **Land-owner and workers.** Let player 1 be a land-owner, and players $\{2, \ldots, n\}$ be workers. The land-owner needs at least one worker to produce any output. Likewise any coalition of workers needs the land-owner's participation to have land to work on, otherwise no production will occur. We see that it is natural to model this situation as a coalition game. If $|S| = 1$ then the formation of $S$ will result in zero output. If $S$ consists of workers taken from the set $\{2, \ldots, n\}$ and does not contain the land-owner then $S$ will

again produce zero output. Any coalition with $|S| \geq 2$ which does contain the land-owner will result in some positive level of output.

*Example* 1.3. **Housing swap.** Consider a game with $n$ players living in the same neighborhood. Each player owns one house initially and each will have a different preference ordering on houses in the neighborhood, depending on the characteristics of the various houses. If a given coalition, $S$, forms, the set of joint actions that $S$ can take consists of all the possible ways to redistribute among the members of $S$ the set of houses that the members of $S$ owned initially.

The preceding examples demonstrate two important types of cooperative games: those with *transferable* payoff, and those with *nontransferable* payoff. A game with transferable payoff only considers how much total payoff is achievable by the joint action of a given coalition, not how that coalition will distribute the payoff amongst its members. The idea is that the coalition is free to transfer payoff within the coalition. Such games can be characterized completely by specifying how much payoff is achievable by any given coalition, referred to as the *worth* of the coalition.

*Notation* 1.4. Denote the worth of the coalition $S$ by $w(S)$. For instance, in example (1.2) $w(S) = 0$ for any $S$ with $|S| = 1$.

Example (1.2) is an instance of a cooperative game with transferable utility because we never specified how the production output would be distributed among the members of any coalition containing both the land-owner and at least one worker. In this example the worth of any one player coalition, or any coalition not containing the land-owner, will be zero. The worth of any coalition containing the land-owner and at least one other player (a worker) would be some positive number. Example (1.3) (assuming the houses are indivisible) demonstrates a cooperative game with *nontransferable* payoff. In order to analyze this game we need to consider which house each member of a given coalition is assigned. The total payoff achievable by a given coalition depends completely on how the houses are allocated

within the coalition since each player has a different preference ordering on the set of available houses.

## 2. The Core

When we analyzed noncooperative games, our objective was usually to find a set of actions from which no individual player has an incentive to deviate. We will establish a similar solution concept that is applicable to cooperative games. Our goal in analyzing a cooperative $n$-person game will be to find a payoff vector $(u_1, \dots, u_n)$ for all $n$ players, from which no smaller coalition of players has an incentive to deviate. Such vectors are said to make up the *core* of the $n$-person game. The precise definition of the core will depend on whether we are considering a game with transferable or nontransferable payoff.

**Definition 2.1.** For an $n$-person game with transferable payoff, the core is the set of payoff vectors,

$$\left\{ \mathbf{u} = (u_1, \dots, u_n) : \sum_{i=1}^{n} u_i \leq w(N) \text{ and } \sum_{i \in S} u_i \geq w(S) \text{ for all subsets } S \right\}$$

The condition that $\sum_{i=1}^{n} u_i \leq w(N)$ tells us that the payoff vector $\mathbf{u}$ is feasible, and the condition that $\sum_{i \in S} u_i \geq w(S)$ tells us that none of the coalitions $S$ have an incentive to form and deviate from the grand coalition.

**Definition 2.2.** We say that the coalition $S$ *blocks* the payoff vector $\mathbf{u} = (u_1, \dots, u_n)$ if $S = (i_1, \dots, i_k)$ can achieve a payoff vector $\mathbf{y} = (y_{i_1}, \dots, y_{i_k})$ such that $y_{i_j} > u_{i_j}$ for all $i_j \in S$. In other words, every member of $S$ prefers the payoff vector $\mathbf{y}$ to the payoff vector $\mathbf{u}$. For an $n$-person game with nontransferable payoff, the core is made up of the payoff vectors, $\mathbf{u}$, such that $\sum_{i=1}^{n} u_i \leq w(N)$ and $\mathbf{u}$ can not be blocked by any coalition $S$.

As we can see from definition (2.2), we are starting to have to deal with multiple indices when articulating precise definitions involving coalitions. Let's introduce some more notation that will simplify what follows.

*Notation* 2.3. If $\mathbf{u} = (u_1, \ldots, u_n)$ is a vector in $\mathbb{R}^n$, then let $\mathbf{u}_S$ denote the projection of $\mathbf{u}$ onto the vector space, $\mathbb{R}^{|S|}$, whose coordinates are indexed by the members of $S$. For example, if $\mathbf{u} = (6, 5, 3, 7, 9) \in \mathbb{R}^5$, with $N = \{1, 2, 3, 4, 5\}$ and $S = \{2, 4, 5\}$, then $\mathbf{u}_S = (5, 7, 9) \in \mathbb{R}^3$. Furthermore, for each coalition $S$, we define $V_S$ to be the set of payoff vectors that can be achieved by the coalition $S$. Clearly $V_S$ is a subset of $\mathbb{R}^{|S|}$.

With this notation, the vector $\mathbf{u}$ is blocked by the coalition $S$ if there is a vector $\mathbf{y} \in V_S$ with $\mathbf{y} > \mathbf{u}_S$ in each coordinate of $\mathbf{y}$ and $\mathbf{u}_S$. Furthermore, a vector $\mathbf{u}$ is in the core if $\mathbf{u} \in V_N$ and $\mathbf{u}$ cannot be blocked by any coalition $S$. We will make the following assumptions on the sets $V_S$:

- $V_S$ is a closed set for all $S$. This assumption is necessary because we will later make use of a limiting argument involving the sets $V_S$.
- Let $\mathbf{u} \in V_S$ and $\mathbf{y} \in \mathbb{R}^{|S|}$ be such that $\mathbf{y} \leq \mathbf{u}$ in each coordinate. Then $\mathbf{y} \in V_S$. It seems counterintuitive at the moment that we would want to consider payoff vectors that are not necessarily on the boundary of our sets $V_S$, but the purpose of this assumption will become clear in the following section.
- The set of vectors in $V_S$ in which each member of $S$ receives no less than the maximum than he can obtain by him/herself is a nonempty, bounded set. This assumption is simply saying that the game is truly cooperative in nature, in that players will not in general have an incentive to break away and form a single member coalition.

We also make the following general assumptions about the cooperative games we will consider. Each assumption is quite reasonable and we accept them as given without further justification.

- The outcomes achievable by the coalition $S$ are not effected by actions taken by players who do not belong to $S$.
- The games we analyze are *cohesive*:

**Definition 2.4.** A coalition game is said to be *cohesive* if disjoint coalitions always have an incentive to combine. In other words, the grand coalition always forms. For games

with transferable payoff, the game is cohesive if $w(S \cup T) \geq w(S) + w(T)$ for any disjoint coalitions $S$ and $T$. For games with nontransferable payoff, the game is cohesive if for disjoint coalitions, $S$ and $T$, if $u_T \in V_T$ and $u_S \in V_S$, we also have that $u_{S \cup T} \in V_{S \cup T}$.

The example (1.3) above is clearly cohesive, since any allocation of houses achievable by the coalition $S$ is still achievable by the grand coalition $N$. Example (1.2) is also cohesive if we make the additional assumption that the output achievable by a coalition $S$ increases as the size of $S$ increases. This is a reasonable assumption to make since we would expect a higher number of workers to yield a higher output in general. Even if we suppose that the land only allows for at most $k$ workers, then $w(\{1, 2, \ldots, k+1\}) = \alpha$ for some positive number $\alpha$, and $w(\{k+2, \ldots, n\}) = 0$ since the coalition $\{k+2, \ldots, n\}$ does not contain player 1, the land-owner. Nevertheless we still have the necessary condition for cohesiveness since $w(N) = \alpha$ (assuming the additional $n-(k+2)$ workers do not contribute any output), and $w(N) = \alpha \geq w(\{1, 2, \ldots, k+1\}) + w(\{k+2, \ldots, n\}) = \alpha + 0 = \alpha$.

Given the preceding discussion, we see that the assumption that our cooperative game is cohesive will certainly facilitate finding a feasible payoff vector in the core. However, as we will see in the following example, cohesiveness alone is not enough to guarantee a nonempty core.

*Example* 2.5. **3-player majority (from** [6]**).** Consider a game with 3 players, so that $N = \{1, 2, 3\}$ and $S$ is any subset of the set $N$. In this example, we will set $w(N) = 1$, $w(S) = \alpha$ for any $S$ with $|S| = 2$, and $w(S) = 0$ for any $S$ with $|S| = 1$. Assume that $0 < \alpha < 1$. With these condition, the core of this game is the set $\{\mathbf{u} = (u_1, u_2, u_3) : \mathbf{u}(N) = 1 \text{ and } \mathbf{u}(S) \geq \alpha \;\; \forall \;\; S \text{ with } |S| = 2\}$ where $\mathbf{u}(S) = \sum_{i \in S} u_i$. In this simple example, it is only a matter of solving a system of linear inequalities that will allow us to determine a condition under which the core is nonempty. Writing out the condition that $\mathbf{u}(S) \geq \alpha$ for

all $S$ of size 2, we obtain the system

$$\begin{aligned} u_1 + u_2 &\geq \alpha \\ u_2 + u_3 &\geq \alpha \\ u_1 + u_3 &\geq \alpha \end{aligned}$$

Adding these three inequalities yields $u_1+u_2+u_3 \geq 3\alpha/2$ and solving for $\alpha$ we have $\alpha \leq 2/3$. Hence, in the game modeled above the core is nonempty if and only if $\alpha \in (0, 2/3]$.

This was an example of a game with transferable payoff since we did not take into consideration how the payoff to $S$ would be distributed among the members of $S$. In this case it was quite simple to analyze the core of the game, and we saw that this analysis reduced to solving a system of linear inequalities. Also note that although the game was cohesive (since we assumed $\alpha$ to be less than 1), cohesiveness alone is not enough to assure a non-empty core. If, for instance, $\alpha = 3/4$ the game is still cohesive, but as our analysis above shows, the core will be non-empty. Thus even in the simple case of a coalition game with transferable payoff, we will need to put a stronger condition on our game in order to guarantee a non-empty core.

## 3. Balanced Games

We need to introduce the concept of balancedness in order to describe mathematically a condition under which we are guaranteed a non-empty core.

**Definition 3.1.** Let $T = \{S\}$ be a collection of coalitions. $T$ is called a *balanced collection* if it is possible to find nonnegative weights, $\delta_S$, one for each $S \in T$, such that for each $i$,

$$\sum_{S \in T, S \ni i} \delta_S = 1$$

Note that for each $i$, we sum over the $S \in T$ which contain $i$.

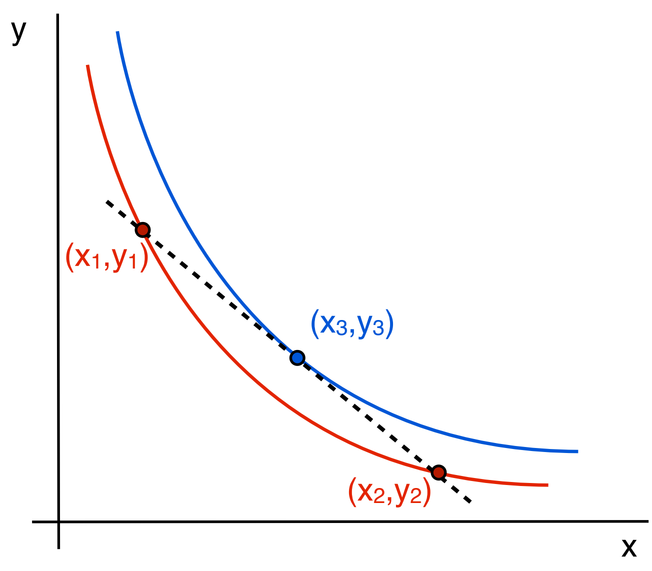
*Example* 3.2. Consider a three player coalition game, so $N = \{1, 2, 3\}$. Let $T$ be the collection of all two player coalitions: $T = \{(1\ 2), (1\ 3), (2\ 3)\}$. Then $T$ is a balanced collection, since we can find the appropriate weights, $\delta_{(1\ 2)} = \delta_{(1\ 3)} = \delta_{(2\ 3)} = 1/2$, which satisfy 3.1.

It is not trivial to tell whether a given collection of coalitions is balanced, since doing so involves finding a set of weights $\delta_S$ that will satisfy definition (3.1). Intuitively, we can think of this condition as each player having one unit of time to divide however (s)he wishes amongst the coalitions to which (s)he belongs. In order for a given coalition to form, each member of the coalition must devote the same fraction of their one time unit to that coalition. These fractions are exactly the weights $\delta_S$. We can now define the concept of a balanced game, which will play a central role in all that follows.

**Definition 3.3.** We say that an $n$ person coalition game is *balanced* if for every balanced collection $T$, the payoff vector $\mathbf{u}$ is in $V_N$ if $\mathbf{u}_S \in V_S$ for all $S \in T$.

Interestingly, the concept of a balanced collection has a rather concrete motivation from economics. It is standard in economics to assume that consumers have convex preferences. This means that consumers prefer diversified commodity bundles. Any point on the graph in figure (1) represents a different commodity bundle made of up goods $x$ and $y$.

FIGURE 1. Convex Preferences



We have plotted two utility curves which are the level sets of a consumer's utility function $u : \mathbb{R}^2 \to \mathbb{R}$. The consumer is indifferent between commodity bundles that lie on the same utility curve. The condition that these utility curves be convex then means that any linear combination of two points on the same utility curve will yield a point on a higher utility curve. For instance $(x_3, y_3)$ is a linear combination of $(x_1, y_1)$ and $(x_2, y_2)$, and the blue utility curve represents a higher level set of the consumer's utility function.

In [3] it is shown that a market game, without the assumption of convex preferences, can fail to have a nonempty core. This motivates Scarf in [7] to reformulate the condition of convex preferences into a condition stated in terms of the sets $V_S$. Consider a three-player market game modeling an exchange economy. Each player $i$ has commodity bundle $c_i$ initially, and as a result of the game receives the commodity bundle $x_i$ with utility $u_i(x_i)$. Then, keeping in mind the assumptions we've made on the sets $V_S$, we have the following descriptions of the sets $V_S$:

$$\begin{aligned} V_{(1\ 2\ 3)} &= \{(u_1, u_2, u_3)\ :\ u_i \leq u_i(x_i) \text{ for some } (x_1, x_2, x_3) \text{ with } x_1 + x_2 + x_3 = c_1 + c_2 + c_3\} \\ V_{(1\ 2)} &= \{(u_1, u_2)\ :\ u_i \leq u_i(x_i) \text{ for some } (x_1, x_2) \text{ with } x_1 + x_2 = c_1 + c_2\} \end{aligned}$$

The sets $V_S$ for the other possible coalitions $S$ are defined similarly. Now assume that the payoff vector $\mathbf{u} = (u_1, u_2, u_3)$ satisfies $(u_i, u_j) \in V_{(i\ j)}$ for all pairs $i$ and $j$. The fact that $(u_1, u_2) \in V_{(1\ 2)}$ means exactly that there exists a commodity bundle $(x_1, x_2)$ with $x_1 + x_2 = c_1 + c_2$ and $u_1(x_1) \geq u_1$ and $u_2(x_2) \geq u_2$. Likewise for the other pairs, (2 3) and (1 3). So there exist commodity bundles $(x_1, x_2)$, $(y_2, y_3)$, and $(z_1, z_3)$ such that

$$\begin{aligned} x_1 + x_2 &= c_1 + c_2 \quad \text{and} \quad u_1(x_1) \geq u_1\ ,\ \ u_2(x_2) \geq u_2 \\ y_2 + y_3 &= c_2 + c_3 \quad \text{and} \quad u_2(y_2) \geq u_2\ ,\ \ u_3(y_3) \geq u_3 \\ z_1 + z_3 &= c_1 + c_3 \quad \text{and} \quad u_1(z_1) \geq u_1\ ,\ \ u_3(z_3) \geq u_3 \end{aligned}$$

Notice that we can take the average of these commodity bundles for each player and

$$\frac{x_1 + z_1}{2} + \frac{x_2 + y_2}{2} + \frac{y_3 + z_3}{2} = c_1 + c_2 + c_3,$$

so this averaging represents a feasible exchange that the three players could make. Now assume that these three players have convex preferences. This will imply, for example, that player 1 will be at least as happy with the commodity bundle that is a linear combination of $x_1$ and $z_1$ as (s)he was with either bundle $x_1$ or bundle $z_1$. Likewise for players 2 and 3:

$$\begin{aligned} u_1\left(\frac{x_1 + z_1}{2}\right) &\geq \min[u_1(x_1), u_1(z_1)] \geq u_1 \\ u_2\left(\frac{x_2 + y_2}{2}\right) &\geq \min[u_2(x_2), u_2(y_2)] \geq u_2 \\ u_3\left(\frac{y_3 + z_3}{2}\right) &\geq \min[u_3(y_3), u_3(z_3)] \geq u_3 \end{aligned}$$

This set of inequalities are exactly what we need to conclude that $(u_1, u_2, u_3) \in V_{(1\,2\,3)}$. To summarize: we began with a vector $\mathbf{u}$ which we assumed had the property that $\mathbf{u}_S \in V_S$ for any $S$ with $|S| = 2$. We then used the assumption of convex preferences to conclude that $\mathbf{u} \in V_{(1\,2\,3)}$. As noted previously, the collection of all two player coalitions forms a balanced collection, so we could have replaced the condition that the players have convex preferences with the condition that our game is balanced in order to reach the same conclusion: that $\mathbf{u} \in V_{(1\,2\,3)}$.

In light of the motivation we are given for the definition of a balanced game, it is reassuring to know that a market game with convex preferences will always be balanced. It is a simple matter to prove this; details can be found in [7]. Also note that the condition that a game is balanced is stronger than the condition that the game is cohesive. Any balanced game is necessarily cohesive: if $T$ is a collection of disjoint coalitions $S_k$, we can find the necessary weights by setting $\delta_{S_k} = 1$ for each $S_k$, since each player will belong to at most one coalition $S_k$.

## 4. Scarf's Algorithm

We can now state the main result of Scarf's paper, *The Core of an N-person Game*[7]:

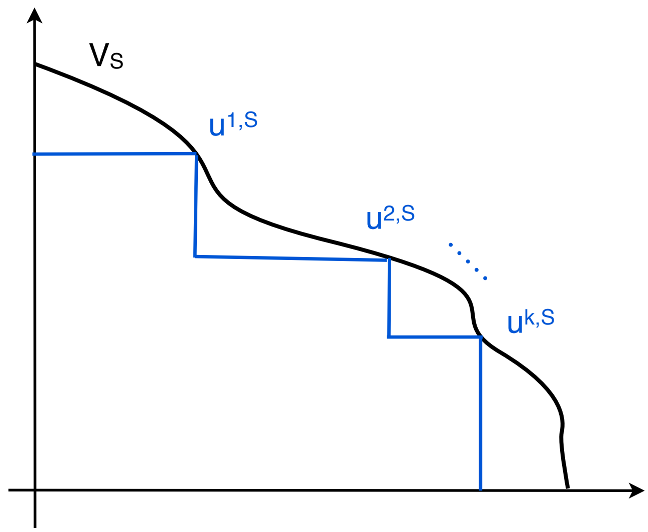
**Theorem 4.1.** *A balanced $n$-person game has a non-empty core.*

As expected, proving theorem (4.1) is much more straightforward in the case of a game with transferable utility. In fact, a game with transferable utility has a nonempty core if and only if it is balanced. The proof of this amounts to solving a linear programming problem and its dual: a payoff vector $\mathbf{u}$ which is in the core will be the solution to an appropriately specified linear programming problem, and the weights $\delta_S$ will comprise the solution of the dual problem. I will refer you again to [7] for the details.

The proof of theorem (4.1) in the case of cooperative games with nontransferable payoff requires a different set of techniques. The proof given by Scarf in [7] proceeds according to the following steps:

(1) Approximate each set $V_S$ by a set with finitely many corners, given by the vectors $\mathbf{u}^{1,S}, \ldots, \mathbf{u}^{k,S}$, each of which gives no player in $S$ less than the maximum (s)he can achieve as a single member coalition. See figure (2).
(2) For balanced games, an algorithm is given which will yield a payoff vector in the set $V_N$ which can not be blocked by an proper coalition using a vector from the aforementioned finite list of vectors to block.
(3) Let the number of corners (and hence the number of approximating vectors) tend to infinity and pass to the limit to obtain a vector in $V_N$ which can not be blocked by any proper coalition $S$.

FIGURE 2. Approximation of the set $V_S$




We will demonstrate the algorithm referred to in step (2) by looking at a specific example of a three-player cooperative game where we approximate the three 2-player coalitions by two payoff vectors each.

*Example* 4.2. Assume that each single player coalition achieves a payoff of 0. The payoff vectors approximating the 2-player coalitions are taken as given. We organize these payoff vectors into a matrix, including a dash (-) in the $ij^{\text{th}}$ entry whenever player $i$ is not a member of the coalition whose payoff vector makes up column $j$.

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} \mathbf{u}^{1,(1)} & \mathbf{u}^{1,(2)} & \mathbf{u}^{1,(3)} & \mathbf{u}^{1,(1\,2)} & \mathbf{u}^{2,(1\,2)} & \mathbf{u}^{1,(1\,3)} & \mathbf{u}^{2,(1\,3)} & \mathbf{u}^{1,(2\,3)} & \mathbf{u}^{2,(2\,3)}) \end{array} \\
C = & \begin{pmatrix} 0 & - & - & 6 & 2 & 12 & 3 & - & - \\ - & 0 & - & 6 & 8 & - & - & 7 & 2 \\ - & - & 0 & - & - & 2 & 8 & 5 & 9 \end{pmatrix}
\end{array}
$$

A key part of the algorithm will involve choosing a certain square submatrix of $C$ so that

when we define

$$u_i = \min_j c_{ij}, \tag{1}$$

where we take the minimum over all the columns $j$ making up the square submatrix, we obtain a vector in the core. It is evident that we do not want the entries of $C$ currently held by dashes to influence the choice of the $u_i$, so we will pick arbitrary large constants, $M_j$, to replace these blank spaces in our matrix $C$. In order to avoid degeneracy, we can without loss of generality choose the $M_j$ to be distinct and satisfying $M_1 > M_2 > \ldots > M_9$.

$$\begin{array}{cc} & \begin{array}{ccccccccc} \mathbf{u}^{1,(1)} & \mathbf{u}^{1,(2)} & \mathbf{u}^{1,(3)} & \mathbf{u}^{1,(1\ 2)} & \mathbf{u}^{2,(1\ 2)} & \mathbf{u}^{1,(1\ 3)} & \mathbf{u}^{2,(1\ 3)} & \mathbf{u}^{1,(2\ 3)} & \mathbf{u}^{2,(2\ 3)}) \end{array} \\ C = & \left( \begin{array}{ccccccccc} 0 & M_2 & M_3 & 6 & 2 & 12 & 3 & M_8 & M_9 \\ M_1 & 0 & M_3 & 6 & 8 & M_6 & M_7 & 7 & 2 \\ M_1 & M_2 & 0 & M_4 & M_5 & 2 & 8 & 5 & 9 \end{array} \right) \end{array}$$

Take for example the submatrix made of of the vectors in columns 4, 7, and 8. Then according to equation (1), $\mathbf{u} = (3, 6, 5)$. If $\mathbf{u}$ were blocked by $S$ using any vector from our finite list to block, there would be a column $j$ with $c_{ij} > u_i$ for all $i = 1, 2, 3$. No such column exists in $C$, as can be checked by the reader. Furthermore, the coalitions whose representative columns make up this square submatrix form a balanced collection (i.e. the collection of all 2-player coalitions). As a result, we can conclude that if the game is balanced, then $\mathbf{u} \in V_{(1\ 2\ 3)}$ since

$$\begin{aligned} (3,6,5)_{(1\ 2)} &= (3,6) \leq (6,6) \in V_{(1\ 2)} \\ (3,6,5)_{(1\ 3)} &= (3,5) \leq (3,8) \in V_{(1\ 3)} \\ (3,6,5)_{(2\ 3)} &= (6,5) \leq (7,5) \in V_{(2\ 3)} \end{aligned}$$

The above inequalities will always hold precisely because of how $\mathbf{u}$ is defined. Somehow we have done well by choosing this particular basis of $C$ which yields a payoff vector that can not be blocked by any vector on our finite list, and which at the same time is generated by columns associated to a balanced collection of coalitions. The question of how to locate

this (or another) well chosen basis is what the algorithm given in [7] addresses. The goal of the algorithm is then to locate a basis of $C$ such that the payoff vector defined by equation (1) satisfies two conditions:

- $\mathbf{u} \in V_N$ (this will be satisfied if $\mathbf{u}$ is generated by vectors corresponding to a balanced collection of coalitions).
- $\mathbf{u}$ can not be blocked by any coalition $S$.

In order make sure that our $\mathbf{u}$ satisfies the first condition, we define another matrix, the *incidence matrix* associated to the payoff matrix $C$ (the $ij^{\text{th}}$ entry is 1 if $i$ is a member of the coalition which column $j$ represents, otherwise 0):

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Then satisfying first condition above means exactly that we want to solve linear system $A\mathbf{x} = (1, 1, \ldots, 1)$ for a nonnegative solution with $x_j = 0$ if the $j^{\text{th}}$ column is not in the basis which generates $\mathbf{u}$. Then we can take $\delta_S = \sum_j x_j$ where we sum over all the $j$ for which column $j$ represents a payoff vector for $S$. These will be the weights that are needed to show that the chosen basis arises from a balanced collection of coalitions. In other words, our problem has now been reduced to that of choosing a feasible basis for

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

such that the corresponding basis for $C$ satisfies the second condition above. We will now describe the algorithm that is detailed in [7].

**Definition 4.2.** Let $A$ and $C$ be $n \times m$ matrices of the following forms:

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & a_{1,n+1} & \cdots & a_{1,m} \\ 0 & 1 & \cdots & 0 & a_{2,n+1} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n,n+1} & \cdots & a_{n,m} \end{pmatrix}$$

$$C = \begin{pmatrix} c_{1,1} & \cdots & c_{1,n} & c_{1,n+1} & \cdots & c_{1,m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,n} & c_{n,n+1} & \cdots & c_{n,m} \end{pmatrix}$$

We say that $A$ and $C$ are in *standard form* if

(i) For each row $i$, $c_{ii}$ is the minimum over all the elements in row $i$, and

(ii) For each nondiagonal element $c_{ij}$ in the first $n$ columns, and for each column $k > n$, we have $c_{ij} \geq c_{ik}$.

It is easily verified that the matrices $A$ and $C$ that result from the set-up described above of a 3-player cooperative game are indeed in standard form so long as we ensure that every payoff vector making up $C$ gives each player in $S$ no less than the maximum payoff (s)he can achieve alone, and the constant $M_j$ are chosen to be decreasing in $j$.

**Theorem 4.3.** *Let $A$ and $C$ be matrices in standard form. Let $\boldsymbol{b}$ be a nonnegative vector such that the set $\{\boldsymbol{x} : x_i \geq 0$ and $A\boldsymbol{x} = \boldsymbol{b}\}$ is bounded. Then there exists a feasible basis for the linear equations $A\boldsymbol{x} = \boldsymbol{b}$ such that $u_i = \min_j c_{ij}$ satisfies the condition that for every column $k$, there exists an $i$ with $u_i \geq c_{ik}$.*

Given the preceding discussion, we see that an algorithm demonstrating theorem (4.3) will solve our problem of finding a vector that is generated by a balanced collection of coalitions,

and can not be blocked by any vector in the finite list making up $C$. This algorithm will alternate between pivots to find a feasible basis for $A$ and a so-called *ordinal basis* for $C$.

**Definition 4.4.** An *ordinal basis* for $C$ consists of $n$ columns, $j_1, \ldots, j_n$ such that if $u_i = \min(c_{ij_1}, \ldots, c_{ij_n})$ then for each $k$ there exists an $i$ with $u_i \geq c_{ik}$.

Of course, this is just restating the requirement that the vector $\mathbf{u}$ generated by our basis of $C$ not be blocked by any of the payoff vectors making up $C$. The following lemma tells us that it is possible to carry out *ordinal pivot steps*, i.e. a change of basis process that will always result in another ordinal basis.

**Lemma 4.5.** *Let $\{j_1, \ldots, j_n\}$ be an ordinal basis for $C$ with $j_1$ arbitrary and $j_2, \ldots, j_n$ not all taken from the first $n$ columns of $C$. Then if no two elements in the same row of $C$ are equal, and $C$ is in standard form, then there exists a unique column $j^* \neq j_1$ such that $j^*, j_2, \ldots, j_n$ is again an ordinal basis for $C$.*

The procedure to carry out an ordinal pivot step is as follows:

- Choose one column to be removed from the original ordinal basis for $C$.
- Of the remaining columns, exactly one will have two row minimizers: one entry which was a row minimizers for the original basis, and one entry that is a row minimizer after the chosen column has been removed.
- Label the row associated to the former of these two row minimizers by $i^*$.
- Consider the columns in $C$ for which $c_{ik} > \min\{c_{ij} : j \text{ is in the original ordinal basis}\}$ holds for all $i \neq i^*$.
- Of these columns, choose the one that maximizes $c_{i^*k}$ to introduce into the new ordinal basis for $C$.

The idea is now to start with a feasible basis for $A$ and an ordinal basis for $C$ that are not identical, but which are close enough so that our algorithm is definitive about how to proceed. We will use the first $n$ columns of $A$, just an $n \times n$ identity matrix, for our feasible basis of $A$ and use

$$\begin{pmatrix} c_{1j} & c_{12} & \cdots & c_{1n} \\ c_{2j} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{nj} & c_{n2} & \cdots & c_{nn} \end{pmatrix}$$

as our ordinal basis for $C$. Here we select $j$ from the columns with $j > n$ to maximize $c_{1j}$. Notice the following relationship between the bases for $A$ and $C$: the $A$ basis contains column 1 and $(n-1)$ other columns, while the $C$ basis contains these same $(n-1)$ columns, along with one other column besides column 1. The algorithm will proceed by alternating between feasible pivots for $A$ and ordinal pivots for $C$ such that this relationship between the bases is maintained at every step of the algorithm. Note that if the $C$ basis contained the same $(n-1)$ columns as the $A$ basis, and also contained column 1, then the algorithm would be finished as we would have a feasible basis for $A$ that is simultaneously an ordinal basis for $C$.

Let's now return to analyzing example (4.2). According to the steps outlined above, we choose the initial basis for $A$ to be the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the initial basis for $C$ to be the matrix

$$\begin{pmatrix} M_8 & M_2 & M_3 \\ 7 & 0 & M_3 \\ 5 & M_2 & 0 \end{pmatrix}$$

Here the only option available if we are to maintain the desired relationship between the two bases is to pivot column 8 (which appears in the $C$ basis but not in the $A$ basis) into the basis for $A$. First we perturb the matrix $A$ slightly to avoid degeneracy. Choose $0 < \epsilon_1 < \epsilon_2 < \epsilon_3$

and define

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1+\epsilon_1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1+\epsilon_2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1+\epsilon_3 \end{pmatrix}$$

Then pivoting column 8 into the feasible $A$ basis yields the following matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1+\epsilon_1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1+\epsilon_2 \\ 0 & -1 & 1 & -1 & -1 & 1 & 1 & 0 & 0 & \epsilon_3-\epsilon_2 \end{pmatrix}$$

So the new feasible $A$ basis is made up of columns 1, 3, and 8. Since column 2 was removed from the $A$ basis, we next do an ordinal pivot step to remove column 2 from the $C$ basis. Upon doing so, we see that column 8 has two row minimizers, the old one in row 1 and a new one in row 2. So we set $i^* = 1$ according to where the old row minimizer appears. We now exam all the columns $k$ of $C$ which satisfy both $c_{2k} > 7$ and $c_{3k} > 0$. We wish to pick the column that maximizes $c_{1k}$, so from columns 1, 5, 6, and 7 we choose column 6. Hence the new ordinal basis for $C$ consists of columns 6, 3 and 8. Continuing the algorithm we see that we must now introduce column 6 into the feasible $A$ basis. This results in the following matrix.

$$\begin{pmatrix} 1 & 1 & -1 & 2 & 2 & 0 & 0 & 0 & 0 & 1+\epsilon_1-\epsilon_3+\epsilon_2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1+\epsilon_2 \\ 0 & -1 & 1 & -1 & -1 & 1 & 1 & 0 & 0 & \epsilon_3-\epsilon_2 \end{pmatrix}$$

We see that column 3 has been removed from the $A$ basis, which is now made up of columns 1, 6, and 8. Thus we wish to remove column 3 from the ordinal $C$ basis as well. Doing so we see that column 6 has two row minimizers, an old one in the first row and a new one in

row 3, making $i^* = 1$. The columns $k$ in $C$ with both $c_{2k} > 7$ and $c_{3k} > 2$ are columns 1, 5, and 7. Among these, column 7 maximizes $c_{1k}$ so we bring column 7 into the ordinal $C$ basis, which now consists of columns 7, 6, and 8.

The next step is to then bring column 7 into the feasible $A$ basis which will then consist of columns 1, 7, and 8. Since column 6 was removed from the feasible $A$ basis, we also remove column 6 from the ordinal $C$ basis. Now column 8 has two row minimizers, the old one in the second row, and the new one in the third row, making $i^* = 2$. The columns $k$ with both $c_{1k} > 3$ and $c_{3k} > 5$ are columns 2, 4, and 9. Among these, column 4 maximizes $c_{2k}$ so we bring column 4 into the ordinal $C$ basis, which now consists of columns 4, 7, and 8.

The next step is to bring column 4 into the feasible $A$ basis. This pivot results in column 1 being removed from the feasible $A$ basis, which now consists of columns 4, 7, and 8. The algorithm terminates here as we have found a set of columns which are simultaneously a feasible basis for $A$ and an ordinal basis for $C$. Considering the submatrix of $C$ defined by columns 4, 7, and 8 and setting $u_i = \min_j c_{ij}$ for $j = 4, 7, 8$ we obtain a payoff vector $\mathbf{u} = (3, 6, 5)$ which is generated by columns corresponding to a balanced collection of coalitions and which can not be blocked by any column in $C$.

This demonstrates theorem (4.1) in the case where each set $V_S$ has finitely many corners. For the general case, let the number of approximating vectors for each $V_S$ increase to infinity to obtain a sequence of vectors in $V_N$, any limit point of which can not be blocked by any coalition $S$. Note that the resulting "solution" vector might not be Pareto optimal. That is, it is possible that the algorithm results in a payoff vector $\mathbf{u} \in V_N$ which, although it can not be blocked by any coalition $S$, could be improved upon for every member of the grand coalition $N$ without making any member worse off. In this case, any Pareto optimal payoff vector achievable by $V_N$ and greater than or equal to $\mathbf{u}$ in each coordinate will also be in the core.

## 5. Conclusion

From an economic standpoint, Scarf's result is very interesting as it provides an algorithmic proof of the existence of competitive equilibria. If the number of players is allowed to tend to infinity, and some additional assumptions are made, then the core actually approaches the set of competitive equilibria in an exchange economy. I will also remark that the concept of the core as a solution to an $n$-person cooperative game has been tested empirically with success, at least in the case of games with transferable utility. In [2], Berl et. al. run experiments involving 5-person and 3-person games, and find that "both series of experiments strongly support the core as a solution concept when it exists."

If you would like to read more about applications of Scarf's algorithm to problems in combinatorics, you may refer to [4] and [1] - two papers that Professor Vetta located for me, but which I did not have time to incorporate into my talk.

## References

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