VILNIAUS UNIVERSITETAS
MATEMATIKOS ir INFORMATIKOS FAKULTETAS

DIRICHLET SERIES, ASSOCIATED WITH THUE-MORSE SEQUENCE

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Preprintas 2001-15

Dirichlė eilutė, asocijuota su Tue-Morso seka

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2001

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Dirichlet series, associated with Thue-Morse sequence

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Mathematical subject classification:**11M41, 11A63**

Abstract

In this work Dirichlet series $\kappa(s)=\sum\limits_{n=1}^{\infty}\frac{w(n)}{n^s}$, associated with Thue -Morse sequence $w(n)$ is considered. It is known, that this function has an analytical continuation to a whole complex plane as entire function with trivial zeros on negative real line and imaginary line. The function $\Lambda(t)$, satisfying integral equation $\frac{1}{2}\Lambda(\frac{t}{2})=\int\limits_{0}^{t}\Lambda(u)du$, naturally appears in the representation of the function $\kappa(s)$. The main result of this article is two representations of $\kappa(s)\Gamma(s)$, one of whom is defined for $\Re s<0$ and provides Dirichlet series with complex exponents, multiplied by a function $2^{s^2/2+s/2}$. As a corollary we prove, that $\kappa(s)$ is entire function of order 2.

Keywords: Dirichlet series, Thue-Morse sequence.

1 Introduction

Thue-Morse sequence is defined inductively by

$$w(0)=1, w(2n)=w(n), w(2n+1)=-w(n). \tag{1}$$

That is, it is a sequence 1,-1,-1,1,-1,1,1,-1...The other definition of this sequence is following. Let the binary expansion of the natural number n be

$n = \sum_{i \geq 0} c_{n,i} \cdot 2^i$, where $c_{n,i} = 0$ or 1. Then $w(n) = (-1)^{\sum_{i \geq 0} c_{n,i}}$. The third way do define a sequence is generated power function, which is $\sum_{n=0}^{\infty} w(n)x^n = \prod_{n=0}^{\infty}(1 - x^{2^n})$. It is easy to check the equivalence of these three definitions. Thue-Morse sequence provides 2-multiplicative and 2-automatic function. This sequence was introduced by Thue [6] and by Morse [5], with connection to geodesics on the surface of negative curvature. This sequence is non-periodic, and if this sequence is divided into blocks of length 2^k from the beginning, there are blocks only of two kind, and if denoted by 1 and -1, we get the same sequence 1,-1,-1,1,...,that is, $w(n)$. This and means, that this sequence is 2-automatic. This property will be of the great importance in the future, when we'll consider analytical continuation. General q-multiplicative functions were studied in detail in [2] and [3].

Our aim is to investigate the function

$$\kappa(s) = \sum_{n=1}^{\infty} \frac{w(n)}{n^s} \tag{2}$$

which is defined for complex number $s = \sigma + it$. The Dirichlet series of this kind were studied in [1]. It was shown, that the function

$$f(s) = \sum_{n=0}^{\infty} \frac{w(n)}{(n+1)^s}$$

which is equal to (see lemma 1 bellow) $\frac{1-2^s}{1+2^s}\kappa(s)$, satisfies functional equation

$$f(s) = \sum_{k=1}^{\infty} C_{s+k-1}^{k} 2^{-s-k} f(s+k).$$

The series for $\kappa(s)$ is convergent for $\sigma > 0$ (relativilly for $0 < \sigma < 1$) and defines the analytical function in this half plane. As was shown in [1] and [4], this function has an analytical continuation to a whole complex plane as entire function with trivial zeros $\kappa(-n) = 0$, $n \in \mathbf{N}$. We'll do the same in other, much more simple way, since in the future we will need the function $\Lambda(t)$, which will appear in the analytical continuation. All technique used in this work is elementary and can be found in any book of complex variable.

2 Analytical continuation

To continue analytically we can use partial summation, but next method gives besides and trivial zeros of the function $\kappa(s)$.

Lemma 1. For $\sigma > 0$ $\sum\limits_{n=1}^{\infty} \frac{w(n-1)}{n^s} = \frac{1-2^s}{1+2^s}\kappa(s)$.

Proof. In fact, every natural number n has a unique representation of the form $n = 2^k(2m+1)$, $k \geq 0$, $m \geq 0$, and so for $\sigma > 1$ we have:

$$\sum_{n=1}^{\infty} \frac{w(n-1)}{n^s} = \sum_{k \geq 0, m \geq 0} \frac{w(2^k(2m+1)-1)}{2^{ks}(2m+1)^s} = \sum_{k \geq 0, m \geq 0} \frac{(-1)^k w(m)}{2^{ks}(2m+1)^s} =$$

$$\frac{2^s}{2^s+1} \sum_{m \geq 0} \frac{w(m)}{(2m+1)^s} = \frac{1-2^s}{1+2^s}\kappa(s)$$

From analytical continuation we get, that this is valid for $\sigma > 0$, and thus lemma 1 is proved.

Now define a function $\rho_0(t) = w([t])$, where $[t]$ means integral part of the real number t. Hence from (2) and lemma 1 we have

$$s\int_1^{\infty} \frac{\rho_0(t)}{t^{s+1}}dt = \kappa(s) - \sum_{n \geq 2} \frac{w(n-1)}{n^s} = \frac{2^{s+1}}{1+2^s}\kappa(s) + 1 \tag{3}$$

Since integral on the left is convergent for $\sigma > -1$ (by Abel-Dirichlet principle), and uniformly in any angle $|Arg(s+1-\delta)| < \frac{\pi}{2} - \varepsilon$ with positive δ and ε, this gives analytical continuation to this region. In the future, when we'll encounter with a representation of a function as an integral, we will not mention, that convergence is uniform in regions, which cover the specified region, thus having, that this function is analytic (if not mention the contrary). Here we deduce, that for $s = i\frac{\pi+2\pi l}{\ln 2}, l \in \mathbf{Z}$, on the left of (3) we have a finite number, so on the right it also must be, hence these s are zeros or of $\kappa(s)$. Also we deduce, that $\kappa(0) = -1$.Note that $s\int\limits_0^1 \frac{\rho_0(t)}{t^{s+1}}dt = -1$, when $\sigma < 0$, hence, if we denote $\kappa_0(s) = \frac{2^{s+1}}{1+2^s}\kappa(s)$

$$\kappa_0(s) = s\int_0^{\infty} \frac{\rho_0(t)}{t^{s+1}}dt, -1 < \sigma < 0 \tag{4}$$

The equation (4) will be the basis of analytical continuation by integrating by part. Define $\rho_1(t) = \int\limits_0^t \rho_0(u)du$.

Since $\rho_1(0) = \rho_1(2) = 0$, and as was noted above , blocks of the length 2 form the same sequence, it is clear, that $\rho_1(t) = \rho_0(\frac{t}{2})\overline{\rho_1}(t)$, where $\overline{\rho}_1(t)$ is periodic function with period 2. Now define inductively $\rho_{k+1}(t) = \int\limits_0^t \rho_k(u)du$. By induction, $\rho_k(t) = \rho_0(\frac{t}{2^k})\overline{\rho_k}(t)$, where $\overline{\rho_k}(t)$ is periodic function of period 2^k.Now from integral expression it is easy to check inductive preposition. Note, that $\rho_k(t) = \frac{t^k}{k!}$, for $0 \le t \le 1$, and $\rho_k(t)$ is positive in the interval $0 < t < 2^k$.Hence integrating the equation (4) by part k times, we obtain

$$\kappa_0(s) = s(s+1)...(s+k)\int\limits_0^\infty \frac{\rho_k(t)}{t^{s+k+1}}dt \tag{5}$$

This is valid for $-1 < \sigma < 0$, but the integral is convergent for $-k-1 < \sigma < 0$, hence it provides with analytical continuation of the function $\kappa_0(s)$ for this region. Note, that in the last equation taking $s = -k$, we get $\kappa_0(-k) = 0, k > 0$ and this gives trivial zeros of the function $\kappa(s)$.

3 Function $\Lambda(t)$

We now investigate functions $\rho_k(t)$ and prove that this sequence if normed and scaled, is uniformly convergent to a certain function $\Lambda(t)$. Namely, we'll prove that for constants $c_k = 2^{(k-2)(k-1)/2}$ the functions

$$\Lambda_k(t) = c_k^{-1}\rho_k(2^k t) \tag{6}$$

uniformly converge to a certain function $\Lambda(t)$. First note, that $\rho_k(t) = \rho_k(2^k - t)$ for $0 \le t \le 2^k$. For $k = 0$ it is clear, and if it's true for k, it's true for $k+1$,since

$$\rho_{k+1}(t) = \int\limits_0^t \rho_k(u)du = -\int\limits_{2^{k+1}-t}^{2^{k+1}} \rho_k(u)du = \int\limits_0^{2^{k+1}-t} \rho_k(u)du = \rho_k(2^{k+1}-t).$$

Next, note that $\rho_k(t) + \rho_k(2^{k-1} - t) = \rho_k(2^{k-1})$ for $k \ge 1$ and $0 \le t \le 2^{k-1}$. For $k = 1$ it is checked directly, and if it is true for k, it is true for $k+1$, since

$$\rho_{k+1}(t) + \rho_{k+1}(2^k - t) = \int\limits_0^t \rho_k(u)du + \int\limits_0^{2^k-t} \rho_k(u)du = \int\limits_0^{2^k} \rho_k(u)du = \rho_{k+1}(2^k).$$

We now calculate the maximum of the function $\rho_{k+1}(t)$. As can be concluded from the above, the maximum is obtained at $t = 2^k$. We now can calculate this maximum:

$$\begin{aligned}\rho_{k+1}(2^k) &= 2\rho_{k+1}(2^{k-1}) = 2\int\limits_0^{2^{k-1}} \rho_k(u)du = \\ &= \int\limits_0^{2^{k-1}} (\rho_k(u) + \rho_k(2^{k-1}-u))du = 2^{k-1}\rho_k(2^{k-1}).\end{aligned}$$

Since $\rho_1(1) = 1$, we get $\rho_k(2^{k-1}) = 2^{(k-2)(k-1)/2}$, for $k \geq 1$. Now if we substitute $\rho_k(t)$ by $c_k\Lambda_k(\frac{t}{2^k})$, we get, that $\Lambda_k(0) = \Lambda_k(1) = 0$, $\Lambda_k(\frac{1}{2}) = 1$ and that $\int\limits_0^t \Lambda_k(u)du = \frac{1}{2}\Lambda_{k+1}(\frac{t}{2})$. Now define $r_k(t) = \Lambda_k(t) - \Lambda_{k+1}(t)$. Note, that

$$r_k(t) \geq 0\ , 0 \leq t \leq \frac{1}{4}, \frac{3}{4} \leq t \leq 1, and\ r_k(t) \leq 0, \frac{1}{4} \leq t \leq \frac{3}{4}$$

This can be deduced directly from the properties of the function $\rho_k(t)$, and so from the properties of $\Lambda_k(t)$. Hence, $\sup|r_{k+1}(t)| \leq 2\cdot\frac{1}{4}\sup|r_k(t)|$ and $\sup|r_k(t)| \leq 2^{-k+1}\sup|r_1(t)|$. So the series $\Lambda_1(t) + \sum\limits_{k=1}^{\infty}(\Lambda_{k+1}(t) - \Lambda_k(t))$ converges uniformly, and so we have, that the limit function $\Lambda(t)$ satisfy integral equation $\int\limits_0^t \Lambda(u)du = \frac{1}{2}\Lambda(\frac{t}{2})$, which can be written as $\Lambda'(t) = 4\Lambda(2t)$, and this gives the simplest type of differential equation with delayed argument.

Since from calculation $\Lambda_k(t) = \frac{2^{k^2/2+3k/2-1}}{k!}x^k$ in the interval $\left[0, \frac{1}{2^k}\right]$, and in the interval $\left[0, \frac{1}{4}\right]$ r_k is negative, then $\Lambda_k(t)$ uniformly decreases in the same interval, hence

$$\Lambda(\frac{1}{2^k}) \leq \frac{2^{-k^2/2+3k/2-1}}{k!}, k \geq 2 \tag{7}$$

It is convenient to replace $\rho_k(t)$ in the expression of $\kappa_0(s)$ by $c_k\Lambda_k(\frac{t}{2^k})$. Hence we have

$$\frac{\kappa_0(s)\Gamma(s)}{\Omega(s)} = 2\frac{\Gamma(s+k+1)}{\Omega(s+k)}\int\limits_0^{\infty}\frac{\Lambda_k(t)}{t^{s+k+1}}dt, for - k - 1 < \sigma < 0. \tag{8}$$

,where $\Omega(s) = 2^{s^2/2+3s/2}$.

4 Integral representation

We have from (8):

$$h_k(s) = \frac{\kappa_0(s-k-1)\Gamma(s-k-1)\Omega(s-1)}{2\Omega(s-k-1)} = \Gamma(s)\int_0^\infty \frac{\Lambda_k(t)}{t^s}dt =$$
$$= \int_0^\infty e^{-x}x^{s-1}dx \int_0^\infty \Lambda_k(t)t^{-s}dt, 0 < \sigma < k+1 \qquad (9)$$

Since $e^{-x}x^{s-1}\Lambda_k(t)t^{-s}$ is integrable function in the first quarter for $1 < \sigma < k+1$ (for $0 < \sigma < 1$ it is not), we can write the above as double integral by Fubinni theorem. Changing variables $t = t, x = \alpha t$ we get from (9)

$$h_k(s) = \int_0^\infty f_k(\alpha)\alpha^{s-1}d\alpha = \ln 2 \int_{-\infty}^\infty S_k(\alpha)2^{\alpha s}d\alpha \qquad (10)$$

where $f_k(\alpha) = \int_0^\infty e^{-\alpha t}\Lambda_k(t)dt$ and $S_k(\alpha) = f_k(2^\alpha)$. Hence, $f_k(\alpha)$ is a Laplace's transform of $\Lambda_k(t)$, and $h_k(s)$ is a Mellin's transform of $f_k(\alpha)$. We have $S_k(\alpha) = \sum_{i=0}^\infty e^{-2^\alpha i}\int_0^1 e^{-2^\alpha t}w(i)\Lambda_k(t)dt = \sum_{i=0}^\infty w(i)e^{-2^\alpha i}\cdot$ $\cdot\int_0^1 e^{-2^\alpha t}\Lambda_k(t)dt = \prod_{i=0}^\infty(1-e^{-2^{\alpha+i}})\int_0^1 e^{-2^\alpha t}\Lambda_k(t)dt$. Denote $\int_0^1 e^{-2^\alpha t}\Lambda_k(t)dt$ by $F_k(2^\alpha)$. The function $S_k(\alpha)$ is defined for α, for which $\left|e^{-2^\alpha}\right| < 1$, since generated power series of coefficients $w(n)$ converge only for $|z| < 1$, and has the unit circle as its natural bound. That is, $S_k(\alpha)$ is defined for α, for whom $\Re 2^\alpha > 0$. Now consider the function $S(\alpha) = \int_0^\infty e^{-2^\alpha t}\Lambda(t)dt =$ $\int_0^\infty e^{-2^\alpha t}d\frac{1}{2}\Lambda(\frac{t}{2}) = 2^\alpha\int_0^\infty e^{-2^{\alpha+1}t}\Lambda(t)dt = 2^\alpha S(\alpha+1)$. Hence $S(\alpha) = 2^{-\alpha^2/2+\alpha/2}q(\alpha)$, where q is an analytical function, satisfying relation $q(\alpha) = q(\alpha+1)$, that is, periodic function, defined for those α, for whom $\Re 2^\alpha > 0$, that is, for $-\frac{\pi}{2} + 2\pi l < \Im\alpha\ln 2 < \frac{\pi}{2} + 2\pi l, l \in \mathbf{Z}$. Further, as in the previous example, $S(\alpha) = \prod_{i=0}^\infty(1-e^{-2^{\alpha+i}})\int_0^1 e^{-2^\alpha t}\Lambda(t)dt$. Denote $\int_0^1 e^{-2^\alpha t}\Lambda(t)dt$ by $F(2^\alpha)$.Now for $S_k(\alpha)$ we get the more convenient expression

$$S_k(\alpha) = 2^{-\alpha^2/2+\alpha/2}q(\alpha)\frac{F_k(2^\alpha)}{F(2^\alpha)}$$

Denote $\frac{F_k(2^\alpha)}{F(2^\alpha)}$ by $\phi_k(2^\alpha)$. So we have

$$h_k(s) = \ln 2 \int_{-\infty}^{\infty} 2^{-\alpha^2/2+\alpha/2} q(\alpha)\phi_k(2^\alpha) 2^{\alpha s} d\alpha \tag{11}$$

Equation $S_{k-1}(\alpha - 1) = 2^{\alpha-1} S_k(\alpha)$ written in the terms of F_k have appearance $F_k(2^\alpha) = \frac{1-e^{-2^{\alpha-1}}}{2^{\alpha-1}} F_{k-1}(2^{\alpha-1})$. And in the same manner $F(2^\alpha) = \frac{1-e^{-2^{\alpha-1}}}{2^{\alpha-1}} F(2^{\alpha-1})$.Since $F(2^{\alpha-k}) \to \frac{1}{2}$, as $k \to \infty$, we have

$$F_k(2^\alpha) = (\frac{1-e^{-2^{\alpha-k}}}{2^{\alpha-k}}) \prod_{i=1}^{k} (\frac{1-e^{-2^{\alpha-i}}}{2^{\alpha-i}}), F(2^\alpha) = \frac{1}{2} \prod_{i=1}^{\infty} (\frac{1-e^{-2^{\alpha-i}}}{2^{\alpha-i}})$$

and so

$$\phi_k(2^\alpha) = (\frac{1-e^{-2^{\alpha-k}}}{2^{\alpha-k-1}}) \cdot \prod_{i=k+1}^{\infty} (\frac{2^{\alpha-i}}{1-e^{-2^{\alpha-i}}}) \tag{12}$$

Note, that $\phi_k(z) = \phi(z 2^{-k-1})$, where

$$\phi(z) = (\frac{1-e^{-2z}}{z}) \cdot \prod_{i=0}^{\infty} (\frac{z2^{-i}}{1-e^{-z2^{-i}}})$$

In the future we will need the following lemma:

Lemma 2. $\phi_k(z)$ is even function.

Proof.

$$\phi_k(z) = \frac{\int_0^1 e^{-zt}\Lambda_k(t)dt}{\int_0^1 e^{-zt}\Lambda(t)dt} = \frac{e^z \int_0^1 e^{-zt}\Lambda_k(t)dt}{e^z \int_0^1 e^{-zt}\Lambda(t)dt} = \frac{\int_0^1 e^{zt}\Lambda_k(1-t)dt}{\int_0^1 e^{zt}\Lambda(1-t)dt} = \phi_k(-z).$$

In the last equation we use symmetry of both functions $\Lambda_k(t)$ and $\Lambda(t)$ with respect to a point $t = \frac{1}{2}$.

For the completeness we can give explicit expression of $q(\alpha)$. From the above can be deduced, that

$$q(\alpha) = 2^{\alpha^2/2-\alpha/2-1} \prod_{i=0}^{\infty} (1 - e^{-2^{\alpha+i}}) \prod_{i=1}^{\infty} (\frac{1-e^{-2^{\alpha-i}}}{2^{\alpha-i}})$$

5 Representation in terms of Dirichlet series with weights

For $|\Im\alpha| < \frac{\pi}{2\ln 2}$ we have Fourier expansion $q(\alpha) = \sum\limits_{n\in\mathbf{Z}} c_n e^{2\pi i n\alpha}$. Then we can integrate (11) term by term, having $h_k(s) = \ln 2 \sum\limits_{n\in\mathbf{Z}} c_n h_k(s,n)$, where

$$h_k(s,n) = \int\limits_{-\infty}^{\infty} 2^{-\alpha^2/2+\alpha/2} e^{2\pi i n\alpha} \phi_k(2^\alpha) 2^{\alpha s} d\alpha \tag{13}$$

Note, that $1 < \sigma < k+1$. If not stated the contrary, in this section will be $|\ t\ln 2\ | < \frac{\pi}{2}$. To base the integrating term by term we have to evaluate $\phi_k(z)$.

Lemma 3. For $\Re z > 1$ and $|Arg(z)| < \frac{\pi}{2} - \varepsilon$ $|\phi_k(z)| <$ $< C_{k,\varepsilon} \cdot 2^{\log_2^2|z|/2-(k+3/2)\log_2|z|}$. For $\Re z \le 1$ and $|Arg(z)| < \frac{\pi}{2} - \varepsilon$ $\phi_k(z)$ is bounded by constant $D_{k,\varepsilon}$.

Proof. $\phi_k(z) = (\frac{1-e^{-z2^{-k}}}{z2^{-k-1}}) \cdot \prod\limits_{i=k+1}^{\infty} (\frac{z2^{-i}}{1-e^{-z2^{-i}}})$ (see (12)). Second statement is obviuos. Function $\prod\limits_{i=0}^{\infty} (\frac{z2^{-i}}{1-e^{-z2^{-i}}})$ for z in the region $|z| \le 1$ is bounded by a constant E. For $\Re z > 2^{k+1}$ let $l = \{\log_2|z|\}$ and $j = [\log_2|z|]$. Note, that $j \ge k+1$. Then $\left|\prod\limits_{i=j+1}^{\infty} (\frac{z2^{-i}}{1-e^{-z2^{-i}}})\right| < E$. Further, $\left|(\frac{1-e^{-z2^{-k}}}{z2^{-k-1}})\right| \cdot \left|\prod\limits_{i=k+1}^{j} (\frac{z2^{-i}}{1-e^{-z2^{-i}}})\right| <$ $F_{k,\varepsilon} \cdot |z|^{j-k-1} \cdot 2^{-j^2/2-j/2}$ (We use fact, that $\frac{\Re z}{|z|} > ctg(\frac{\pi}{2} - \varepsilon)$, and that infinite product $\prod\limits_{i=0}^{\infty}(1 - e^{-2^i ctg(\frac{\pi}{2}-\varepsilon)})$ converges). Hence $|\phi_k(z)| < E \cdot F_{k,\varepsilon} \cdot 2^{-l^2/2+l/2} \cdot$ $2^{\log_2^2|z|/2-(k+3/2)\log_2|z|}$ and the lemma 3 is proved.

Since $\phi_k(z)$ is even function, we get the same bound for $\Re z < -1$ and $|\pi - Arg z| < \frac{\pi}{2} - \varepsilon$.

Hence integral (13) converges for $\sigma < k+1$. So for these values of s $|c_n h_k(s,n)| \le |c_n| \int\limits_{-\infty}^{\infty} 2^{-\alpha^2/2+\alpha/2} \cdot |\phi_k(2^\alpha)| \cdot 2^{\alpha\sigma} d\alpha$. Since the series $\sum\limits_{n=-\infty}^{\infty} |c_n|$ converges, we can apply the Lebesgue theorem, and integrating term by term is based. Changing variables in (13) $\alpha = \frac{2}{\ln 2}\pi i n + s + \frac{1}{2} + \alpha'$,we get

$$h_k(s,n) = 2^{s^2/2+s/2} \cdot 2^{-\frac{2}{\ln^2 2}\pi^2 n^2 + \frac{1}{8}} \cdot (-1)^n \cdot e^{2\pi i n s} \int\limits_{-\infty-\frac{2}{\ln 2}\pi i n}^{\infty-\frac{2}{\ln 2}\pi i n} 2^{-\alpha^2/2} \phi_k(2^{s+\frac{1}{2}+\alpha}) d\alpha \tag{14}$$

(In fact, limits should be $-\infty - it - \frac{2}{\ln 2}\pi in$ to $\infty - it - \frac{2}{\ln 2}\pi in$, but since integrable function has no poles in the considered strip (since $\mid t\ln 2 \mid < \frac{\pi}{2}$) , and from easy calculations vertical integrals tend to zero, we have right to make such a substitution).

Function $\phi_k(2^z)$ has poles (see (12)) at points $z_{l,m} = (\frac{\ln\pi + \ln l}{\ln 2}) + i(\frac{\pi/2+\pi m}{\ln 2})$, when $ord_2(l) \geq k+3, l \in \mathbf{N}, m \in \mathbf{Z}$, and these poles are of order $ord_2(l)-k-2$, where $ord_2(l)$ means the biggest power of 2, dividing l. The equation (14) is valid for $1 < \sigma < k+1$ and $|t\ln 2| < \frac{\pi}{2}$. We now can use Couchy theorem about integrals and residues. Now take any natural number l with property $\left\{\frac{l}{2^k}\right\} \in [\frac{1}{4}, \frac{3}{4}]$ for $k < \log_2 l$ (take, for example, $l = c_j := \frac{4^j-1}{3}$) and take $T_j = \log_2 c_j + \log_2 \pi - \sigma - \frac{1}{2}$. From calculation for such number T_j we have (integral is taken by a straight line) $\int\limits_{T_j - \frac{\pi i m}{\ln 2}}^{T_j - \frac{\pi i(m-1)}{\ln 2}} \mid \cdot 2^{-\alpha^2/2}\phi_k(2^{s+\frac{1}{2}+\alpha}) \mid d\alpha = O(e^{\frac{\pi^2 m^2}{2\ln 2}} \cdot 2^{(\sigma-k-1)2j})$, and tends to zero, as $j \to \infty$, and $\sigma < k+1$. For negative T we have this property trivially. Now evaluation of the following integral for $\mid t\ln 2 \mid < \frac{\pi}{2}$ directly follows from the lemma 3:

$$\int\limits_{T_j - \frac{\pi m i}{\ln 2}}^{T_{j+1} - \frac{\pi m i}{\ln 2}} 2^{-\alpha^2/2}\phi_k(2^{s+\frac{1}{2}+\alpha})d\alpha = O_t(e^{\frac{\pi^2 m^2}{2\ln 2}} \cdot 2^{(\sigma-k-1)2j})$$

If so, then, for positive n and $|t\ln 2| < \frac{\pi}{2}$

$$\int\limits_{-\infty - \frac{2}{\ln 2}\pi in}^{\infty - \frac{2}{\ln 2}\pi in} 2^{-\alpha^2/2}\phi_k(2^{s+\frac{1}{2}+\alpha})d\alpha = \int\limits_{-\infty}^{\infty} 2^{-\alpha^2/2}\phi_k(2^{s+\frac{1}{2}+\alpha})d\alpha +$$

$$+2^{-s^2/2-s/2} \cdot \sum_{m=-1}^{-2n}\sum_{j=0}^{\infty}\Big(\sum_{l=c_j}^{c_{j+1}-1} P^k_{l,m}(s)2^{z_{l,m}s}\Big) \tag{15}$$

For negative n we obtain the same integral and sum, with bounds $\sum\limits_{m=0}^{-2n-1}$,and for $n = 0$ an empty sum. The third sum means, that only those $l \in \mathbf{N}$ are counted, for whom $ord_2 l \geq k+3$. The polynomial $P_{l,m}$ is of degree $k+3-ord_2 l$. Since both integrals converges absolutely, the double sum converges absolutely in the region $\sigma < k+1$ and $|t\ln 2| < \frac{\pi}{2}$. We now can extend the (15) to the whole region $\sigma < k+1$. Both integrals converges absolutely for $t\ln 2 \neq \frac{\pi}{2} + \pi k$, hence, double sum converges absolutely for the same values of t. Now shifting the lines of integration down for small number, we can extend equality (15) for the rest values of t, thus obtaining, that the double sum converges absolutely for $\sigma < k+1$.

Now we can evaluate $\sum_{l=c_j}^{c_{j+1}-1} P^k_{l,m}(s)2^{z_{l,m}s}$. It can be easily deduced, that

$$\sum_{l=c_j}^{c_{j+1}-1} P^k_{l,m}(s)2^{z_{l,m}s} = \int_C 2^{-\alpha^2/2}\phi_k(2^{s+\frac{1}{2}+\alpha})d\alpha \tag{16}$$

where contour C forms a rectangular with adjacent vertices $T_j - \frac{\pi m i}{\ln 2}$ and $T_{j+1} - \frac{\pi(m-1)i}{\ln 2}$. Hence, we have, that this integral is $O_t(e^{\frac{\pi^2 m^2}{2\ln 2}} \cdot 2^{(\sigma-k-1)2j})$. Shifting a contour of integration down for a small number we can make this bound to be independent of t.

Now denote by $b_n = 4\ln 2 \cdot c_n \cdot 2^{-\frac{2}{\ln^2 2}\pi^2 n^2 + \frac{1}{8}} \cdot (-1)^n$. Define by $p(e^{2\pi i s}) = \sum_{n\in\mathbf{Z}} b_n e^{2\pi i n s}$. Note, that since $q(\alpha)$ has its natural bound the lines t=$-\frac{\pi}{2\ln 2} + \frac{2\pi l}{\ln 2}$ and $t = \frac{\pi}{2\ln 2} + \frac{2\pi l}{\ln 2}$, so there are infinity many non-zero coefficients for $q(\alpha)$ for both positive and negative n, so this is also true for $p(e^{2\pi i s})$. The Fourier series for $p(e^{2\pi i s})$ is convergent for all complex s and defines periodic analytical function. Taking all results in one place, we get

$$\frac{\kappa_0(s-k-1)\Gamma(s-k-1)}{\Omega(s-k-1)} = p(e^{2\pi i s}) \int_{-\infty}^{\infty} 2^{-\alpha^2/2}\phi_k(2^{s+\frac{1}{2}+\alpha})d\alpha \quad +$$

$$+2^{-s^2/2-s/2} \cdot (\pi i)^s \sum_{n\in\mathbf{Z}} b_n e^{2\pi i n s} \sum_m \sum_j \Big(\sum_{l=c_j}^{c_{j+1}-1} P^k_{l,m}(s) l^s e^{i\pi m s}\Big) \tag{17}$$

where second sum means as above sums for positive and negative n, and empty sum for $n=0$, and fourth sum means for $l \in \mathrm{N}, ord_2(l) \geq k+3$. We want to sum by n and m. Note, that defining $u = 2n+m$, we can change the first two sums into $\sum_{u\in\mathrm{N}_0}\sum_{n>\frac{u}{2}} + \sum_{u\in-\mathrm{N}}\sum_{n\leq\frac{u}{2}}$.Thus we obtain

$$\frac{\kappa_0(s-k-1)\Gamma(s-k-1)}{\Omega(s-k-1)} = p(e^{2\pi i s}) \int_{-\infty}^{\infty} 2^{-\alpha^2/2}\phi_k(2^{s+\frac{1}{2}+\alpha})d\alpha +$$

$$+2^{-s^2/2-s/2}(\pi i)^s \cdot \sum_{u\in\mathbf{Z}} \sum_{j=0}^{\infty} \Big(\sum_{l=c_j}^{c_{j+1}-1} R^k_{l,u}(s) e^{\pi i u s} l^s\Big) \tag{18}$$

The double sum converges absolutely for $\sigma < k+1$. From the evaluation of the sum (16)and coefficients b_n we obtain:

$$\sum_{l=c_j}^{c_{j+1}-1} R^k_{l,u}(s) l^s = O(e^{-\frac{\pi^2 u^2}{2\ln 2}} 2^{(\sigma-k-1)2j}) \tag{19}$$

Translation in (18) $s \mapsto s+k+1$ and changing the function $\phi_k(z)$ into $\phi(z)$, we obtain (noticing, that second sum is for $l \in \mathbf{N}$,$\mathrm{ord}_2(l) \geq k+3$ and denoting $R^k_{2^{k+3}l,u}(s+k+1)$ by $R^k_{l,u}(s)$)

$$\frac{\kappa_0(s)\Gamma(s)}{\Omega(s)} = p(e^{2\pi i s}) \int_{-\infty}^{\infty} 2^{-\alpha^2/2}\phi(2^{s+\frac{1}{2}+\alpha})d\alpha + \tag{20}$$

$$+2^{-s^2/2-s/2+k/2-k^2/2}(8\pi i)^s \cdot \sum_{u\in\mathbf{Z}} \sum_{j=0}^{\infty} (\sum_{l=c_j}^{c_{j+1}-1} R^k_{l,u}(s) e^{\pi i u(s+k+1)} l^{s+k+1})$$

This is valid for $\sigma < 0$ and $|t \ln 2| < \frac{\pi}{2}$. Note, that for $z = 2^s$ the integral on the right in (20) $\int_{-\infty}^{\infty} 2^{-\alpha^2/2}\phi(z \cdot 2^{\frac{1}{2}+\alpha})d\alpha$ for $\sigma < 0$ and $|t\ln 2| < \frac{\pi}{2}$ defines analytical function $Z(z)$ which is from (13) with $n=0$ is continued analytically to the whole region $\sigma < 0$. From lemma 2 we get, that $Z(z) = X(z^2)$. The distribution of zeros of every function is important question. For more detailed study of the function $p(e^{2\pi i s})$ we'll prove the following lemma:

Lemma 4. Function $p(e^{2\pi i s})$ has no zeros in the region $|t| \leq \frac{\pi}{2\ln 2}$.

Proof.Remark. This gives alternative proof of ability to extend function $X(z^2)$ analytically to a whole region $|z| < 1$. In fact, $p(e^{2\pi i s}) \cdot X(4^s)$ is continued analytically from (20) to the region $\sigma < 0$, so $X(4^s)$ is finite everywhere, except for a zeros of $p(e^{2\pi i s})$, and if s is irregularity of $X(4^s)$, then also and $s + \frac{\pi i}{\ln 2}$, so it suffices to prove, that $p(e^{2\pi i s})$ has no zeros in the region $|t| \leq \frac{\pi}{2\ln 2}$. Making the same calculations for the integral

$$\int_0^{\infty} \frac{\Lambda(x)}{x^s} dx \tag{21}$$

as for (9), we find, that it is equal to $\frac{2^{s^2/2+s/2}}{\Gamma(s)} p(e^{2\pi i s})$ up to a constant multiplier. The integral (21) absolutely converges for $\sigma > 1$. Let s be zero of (21) in this region $\sigma > 1$ and $|t| \leq \frac{\pi}{2\ln 2}$, then also and $s+n$, $n \in \mathbf{N}$, is a zero.

We suppose, that $t > 0$. The case $t < 0$ is analogous, and the case $t = 0$ is a more simple one. Making substitution $x \to \frac{1}{x}$ and $s \to s + 2$, we get, that imaginary part is also zero:

$$\int_0^\infty \Lambda(\frac{1}{x}) x^\sigma \sin(t \ln x) dx = 0 \tag{22}$$

Since integral is additive, then for all polynomials $P(x)$ from (22) we have

$$\int_0^\infty \Lambda(\frac{1}{x}) x^\sigma P(x) \sin(t \ln x) dx = 0 \tag{23}$$

Our plan is following: we'll construct polynomials, for which (23) is not satisfied, thus proving lemma.

$\sin(t \ln x)$ for $x > 1$ has simple zeros at the points $\exp(\frac{\pi k}{t}), k \in \mathbf{N}$. Now define a function $R(x)$ and polynomials $P_T(x), T \in \mathbf{N}$:

$$R(x) = \prod_{k=1}^{\infty} (1 - x \cdot \exp(-\frac{\pi k}{t})), P_T(x) = \prod_{k=1}^{T} (1 - x \cdot \exp(-\frac{\pi k}{t})) \tag{24}$$

Note, that infinite product converges, thus the function $R(x)$ is defined correctly. Since $R(x)$ is strongly positive in the interval $[0, 1]$ and finite, $0 < R(x) \leq c$ in this interval.Trivially $|P_T(x)| \leq 1$ for $0 \leq x \leq 1$. Since $R(x \exp(-\frac{\pi T}{t})) \cdot P_T(x) = R(x)$, hence

$$|P_T(x)| \geq c^{-1} |R(x)|, 0 \leq x \leq \exp(\frac{\pi T}{t}) \tag{25}$$

Let $\delta(T)$ be any natural number. We'll chose later $\delta(T)$ so that $\delta(T) \to \infty$. Note, that $|P_T(x)| < d_T x^T$ for positive x,where $d_T = \exp(-\frac{\pi(T^2+T)}{t})$. Now evaluate the following integral (we use (7)):

$$\left| \int_{\exp(\frac{\pi T}{t})}^{\infty} \Lambda(\frac{1}{x}) x^{\sigma+\delta(T)} \sin(t \ln x) P_T(x) dx \right| < d_T \int_{\exp(\frac{\pi T}{t})}^{\infty} \Lambda(\frac{1}{x}) x^{\sigma+\delta(T)+T} dx <$$

$$< d_T \sum_{k=\left[\frac{\pi T}{t \ln 2}\right]}^{\infty} \frac{2^{(k+1)(\sigma+\delta(T)+T)-k^2/2+3k/2+k-1}}{k!} \tag{26}$$

Since $|t| \leq \frac{\pi}{2\ln 2}$, then $k > 2T-1$, and also $kT - k^2/2 < k/2$, and we can continue (26):

$$\left|\int\limits_{\exp(\frac{\pi T}{t})}^{\infty} ...dx\right| < d_T 2^{T-1+\sigma+\delta(T)} \sum_{k>2T-1} \frac{2^{k(\sigma+\delta(T)+3)}}{k!} <$$

$$< \exp(-\frac{\pi(T^2+T)}{t}) 2^{T-1+\sigma+\delta(T)} \exp(2^{\sigma+\delta(T)+3}) \qquad (27)$$

If we now choose $\delta(T) = [\log_2 T]$ in (27), we will have

$$\left|\int\limits_{\exp(\frac{\pi T}{t})}^{\infty} \Lambda(\frac{1}{x}) x^{\sigma+\delta(T)} \sin(t\ln x) P_T(x) dx\right| \to 0, as T \to \infty. \qquad (28)$$

Next, evaluation of the integral (23) for $P(x) = x^{\delta(T)} P_T(x)$ in the interval $[0,1]$ is easy:

$$\left|\int\limits_0^1 \Lambda(\frac{1}{x}) x^{\sigma+\delta(T)} P_T(x) \sin(t\ln x) dx\right| < \frac{1}{\sigma+\delta(T)+1} \to 0, as T \to \infty. \qquad (29)$$

And at last, since $\sin(t\ln x) x^{\delta(T)} P_T(x)$ and $x^{\delta(T)} R(x)$ is of constant sign in the interval $\left[1, \exp(\frac{\pi T}{t})\right]$, we have (using (25)):

$$\left|\int\limits_1^{\exp(\frac{\pi T}{t})} \Lambda(\frac{1}{x}) x^{\sigma+\delta(T)} P_T(x) \sin(t\ln x) dx\right| \geq$$

$$c^{-1} \left|\int\limits_1^{\exp(\frac{\pi T}{t})} \Lambda(\frac{1}{x}) x^{\sigma+\delta(T)} R(x) \sin(t\ln x) dx\right| \to \infty, T \to \infty \qquad (30)$$

Now (28), (29) and (30) are inconsistent with (23). Lemma 3 is proved.

Making in (20) $k = 1$ and denoting $X(4^s) = \frac{1}{2} X(4^s), R_{l,u}(s) = \frac{1}{2} l^2 R^1_{l,u}(s)$ we have a following main theorem:

Theorem 1. There exist function $X(z)$, holomorphic for $|z| < 1$, function $p(z)$, holomorphic in the whole complex plane, except for a point

$z = 0$, and polynomials $R_{l,u}(s)$, for which

$$\frac{\kappa(s)\Gamma(s)}{1+2^s} = 2^{s^2/2+s/2} \cdot p(e^{2\pi is}) \cdot X(4^s) + (8\pi i)^s \cdot \sum_{u\in\mathbf{Z}} \sum_{j=0}^{\infty} (\sum_{l=c_j}^{c_{j+1}-1} R_{l,u}(s) e^{\pi ius} l^s) \tag{31}$$

The double sum on the right converges absolutely for $\sigma < 0$.

Corollary. Function $\kappa(s)$ is the entire function of order 2.

Proof. From (3) by partial integration we get, that for $\sigma > -1$

$$|\kappa(s)| \leq (\frac{|s|\,(|s|+1)}{\sigma+1} + 1) 2^{-\sigma}\, |2^s + 1|$$

Thus it suffices to evaluate $\kappa(s)$ in the region $\sigma < -\frac{1}{2}$. From the evaluation of (19) and b_n, we obtain, that $p(e^{2\pi is}) = O(2^{t^2/2})$, the double sum for $\sigma < -\frac{1}{2}$ is $O(2^{t^2/2})$. The order of the entire function $F(z)$ is the number $\rho = \limsup_{r\to\infty} \frac{\ln\ln M(r)}{\ln r}$, where $M(r) = \sup_{|z|=r} |F(z)|$. $(\Gamma(s))^{-1}$ is entire function of order 1. For the function on the right of (31) $M(r) = O(2^{r^2/2})$ and evaluation can't be better on the negative real line, hence $\kappa(s)$ is of order 2.

6 Other representation

Note, that from (14)

$$h_k(s,n) = 2^{s^2/2+s/2} \cdot 2^{\frac{1}{8}} \cdot (-1)^n \cdot e^{2\pi ins} \int_{-\infty}^{\infty} 2^{-\alpha^2/2} \cdot e^{2\pi in\alpha} \cdot \phi_k(2^{s+\frac{1}{2}+\alpha}) d\alpha \tag{32}$$

Integral is defined for $\sigma < k+1$ and $|\ t\ln 2\ | < \frac{\pi}{2}$. From (13) it is obvious, that this function is continued to the region $\sigma < k+1$ as analytical function. Since ϕ_k is even, the integral with multiplier $\frac{1}{2}\ln 2 c_n (-1)^n 2^{\frac{1}{8}}$ can be defined as $Y_n(4^s)$. Note, that $|\ Y_n(4^s)\ | \leq C\ |\ c_n\ |\ \int_{-\infty}^{\infty}\ |\ 2^{-\alpha^2/2}\phi_k(2^{s+\frac{1}{2}+\alpha})\ |\ d\alpha$. Hence, the series $\sum_{n\in\mathbf{Z}} e^{2\pi ins} \cdot Y_n(4^s)$ converges absolutely in the region $\sigma < k+1$ and $|\ t\ln 2\ | < \frac{\pi}{2}$,since $q(\alpha)$ is defined and absolutely convergent in this region. Now note, that for every s we can find integer number w, so that $|\ t + \frac{w\pi}{\ln 2}\ | < \frac{\pi}{2\ln 2}$, so we can apply (32). Now denoting $Y_n(4^{s+k+1})$ by the same $Y_n(4^s)$, and defining the m-th coefficient of Taylor expansion at the point $z = 0$ of the $Y_n(z)$ by $d_{n,m}$ and define $D_{w,n} = e^{\pi iws} \cdot 2^{-\frac{\pi^2 w^2}{2\ln^2 2}} \cdot (-1)^w \cdot e^{-\frac{2nw\pi^2}{\ln 2}}$, define $d_{n,m,w} = d_{n,m} \cdot D_{w,n}$ then we obtain the following theorem:

Theorem 2. There exist coefficients $d_{n,m,w}, n \in \mathbf{Z}, w \in \mathbf{Z}, m \in \mathbf{N_0}$, for which

$$\frac{\kappa(s)\Gamma(s)}{1+2^s} = 2^{s^2/2+s/2} \cdot \sum_{n\in\mathbf{Z}} \sum_{m\in\mathbf{N_0}} d_{n,m,w} \cdot e^{2\pi ins} \cdot 4^{ms}$$

The double sum converges absolutely for $\sigma < 0$ and $\mid (t + \frac{w\pi}{\ln 2}) \mid < \frac{\pi}{2\ln 2}$.

7 Integral equation

First investigate the function, which we have already encountered. Let for $\sigma > 1$

$$F(s) = \int_0^\infty \frac{\Lambda(x)}{x^s} dx = \int_0^\infty x^{-s} d\frac{1}{2}\Lambda(\frac{x}{2}) = \frac{s}{2^s} F(s+1)$$

Thus, for $\sigma > 1$ $|F(s+1)| \leq \frac{2^\sigma}{t} G(\sigma)$, where $G(\sigma) = \int\limits_0^\infty |\Lambda(x)|\, x^{-\sigma} dx$. In the same manner for $\sigma > 2$ we have

$$|F(s+1)| \leq \frac{2^{2\sigma-1}}{t^2} G(\sigma - 1) \tag{33}$$

In previous section we got, that

$$F(s) = \frac{2^{s^2/2+s/2}}{\Gamma(s)} p(e^{2\pi is})$$

Note, that for $\sigma > 0$ $\int\limits_0^\infty \Lambda(\frac{1}{x}) x^{s-1} dx = F(s+1)$ and $\Lambda(\frac{1}{x}) x^{\delta-1} \in L(0, \infty)$ for $\delta > 0$, thus, from Mellin's inversion formula, we have

$$\Lambda(x) = \frac{1}{2\pi i} (v.p.) \int_{\delta-i\infty}^{\delta+i\infty} F(s+1) x^s ds$$

Since for $\delta > 2$ from (33) we get, that integral absolutely converges, then

$$\Lambda(x) = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} F(s+1) x^s ds, \delta > 2 \tag{34}$$

Note, that $\sum_{n=0}^{\infty} \frac{\Lambda(n+\frac{1}{2})}{(n+\frac{1}{2})^u} = \kappa(u)(1-2^u) = \kappa_1(u)$ by definition, and $\sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^u} = \zeta(u)(2^u-1) = \zeta_1(u)$, where ζ is Riemann zeta function. Making in equation (34) $x = n + \frac{1}{2}$, dividing it by $(n+\frac{1}{2})^u$ and with assumption $\Re u > \delta + 1$ summing it with limits $n = 0$ to ∞, we obtain

$$\kappa_1(u) = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} F(s+1)\zeta_1(u-s)ds$$

Each summand multiplying by $w(n)$ and summing, we in the same manner for $\Re u > \delta + 1$ obtain

$$\zeta_1(u) = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} F(s+1)\kappa_1(u-s)ds$$

Now define $\Psi(u) = \kappa_1(u) + \zeta_1(u) = (2^u-1)(\zeta(u) - \kappa(u))$.Then summing the last two equations, we obtain:

$$\Psi(u) = \frac{1}{\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{2^{s^2/2+3s/2}}{\Gamma(s+1)} p(e^{2\pi i s})\Psi(u-s)ds, \delta > 2, \Re u > \delta + 1.$$

This gives integral equation of convolution type for the function $\Psi(u)$.

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