Perturbation Theory for Polynomials

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January 22, 2013

1 Regular Perturbations

We begin with a very simple example. Suppose you are solving the equation

$$x^2 + \epsilon x - 1 = 0 \tag{1.1}$$

where $\epsilon << 1$. Although we can solve this equation exactly, we will solve it using a simple perturbation expansion. If we assume that

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ... \tag{1.2}$$

we see that collecting the zeroeth order terms in ϵ, we conclude that

$$x_0^2 - 1 = 0 \tag{1.3}$$

This means that $x_0 = \pm 1$. We will choose the root $x_0 = 1$. If we collect the first order terms in the expansion we get

$$2x_0 x_1 + x_0 = 0 \tag{1.4}$$

This implies that

$$x_1 = -\frac{1}{2} \tag{1.5}$$

and hence to first order we have

$$x(\epsilon) = 1 - \frac{\epsilon}{2} + O(\epsilon^2) \tag{1.6}$$

Note that we could have used the quadratic formula to get

$$x = \frac{-\epsilon + \sqrt{\epsilon^2 + 4}}{2} \tag{1.7}$$

If we write $\sqrt{\epsilon^2 + 4} = 2\sqrt{1 + \epsilon^2/4}$, we can then use the Taylor series for $\sqrt{1+x}$ to get

$$\sqrt{\epsilon^2 + 4} = 2\left(1 + \epsilon^2/8\right) + O(\epsilon^4) \tag{1.8}$$

To order ϵ this will give us the same result as in Eqn. (1.6).

Here are a few comments about this example.

- Though it is almost trivial, it gets you familar with the procedure of sytematically collecting terms in a perturbation expansion.
- We could apply the same result to similar equations where we could not so easily find the expansion by other means. For example $x^3 - \epsilon x - 1 = 0$.

- We could have carried out this expansion by doing repeated implicit differentiation. This is in fact what I would recommend for such problems.
- The fact that we could expand in powers of ϵ follows from the implicit function theorem.

2 A Singularly Perturbed Example

We now consider the example

$$\epsilon x^2 + x + 1 = 0 \tag{2.1}$$

where we once again assume tht $\epsilon << 1$. We could once again carry out a perturbation expansion assuming that $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + ...$ Collecting the zeroeth order terms in ϵ we get

$$x_0 = -1 \tag{2.2}$$

Collecting the first order terms in ϵ we get

$$x_1 = -x_0^2 = -1 \tag{2.3}$$

This shows us that

$$x = -1 - \epsilon + O(\epsilon^2) \tag{2.4}$$

We could once again confirm this using the quadratic equation. The solution using the quadratic equation gives us

$$x = \frac{-1 \pm \sqrt{1 - 4\epsilon}}{2\epsilon} \tag{2.5}$$

Using the approximation $\sqrt{1+x} = 1 + x/2 - x^2/8 + ..$, we have

$$\sqrt{1 - 4\epsilon} = 1 - 2\epsilon - 2\epsilon^2 + O(\epsilon^2) \tag{2.6}$$

If we take the positive sign in the square root, this gives us

$$x = \frac{-2\epsilon - 2\epsilon^2}{2\epsilon} = -1 - \epsilon + ... \tag{2.7}$$

This agrees with Eqn. (2.4). If we take the negative sign in the square root we get

$$x = \frac{-2 + 2\epsilon + 2\epsilon^2}{2\epsilon} = -\frac{1}{\epsilon} + 1 + O(\epsilon) \tag{2.8}$$

Our simple perturbation expansion allowed us to get the first of these roots, but there was no evidence of the second root. We would like to understand how we could have arrived at this second root by doing a simple perturbation expansion.

The problem arises from the fact that it may be good to look for a change of variable of the form

$$x = \epsilon^\alpha z \tag{2.9}$$

where α is a paramter we would like to determine. If we express Eqn. (2.5) in terms of z we will get the equation

$$\epsilon^{1+2\alpha} z^2 + \epsilon^\alpha z + 1 = 0 \tag{2.10}$$

We would like to determine α so that at least two of the three terms in our equation are the same order as ϵ goes to zero. We would like these two terms to be bigger as ϵ goes to zero than the remaining term. If this is the case, we say we have a distinguished limit.

To see why a distinguished limit is good, lets take an example that is not a distinguished limit. Suppose we let $\alpha = -2$. After multiplying by ϵ^3 this will give us the equation

$$z^2 + \epsilon z + \epsilon^3 = 0 \tag{2.11}$$

If we let $\epsilon = 0$ in this equation this will give us $z = 0$. When doing a perturbation expansion, when the lowest order term is zero, this is giving you no information. It is merely saying that x is much smaller than $1/\epsilon^2$. Further evidence that this is giving us little information is that if we try to carry out a regular perturbation expansion expanding z in powers of ϵ we cannot carry out the expansion. This is because the conditions of the implicit function theorem do not hold. That is if $f(z,\epsilon) = z^2 + \epsilon z + \epsilon^3$, then $f'(0,0) = 0$.

We will give one more example of a bad scaling. Suppose we set $\alpha = 1$. In this case, we get the equation

$$\epsilon^3 z^2 + \epsilon z + 1 = 0 \tag{2.12}$$

If we let $\epsilon = 0$ this gives us the equation $1 = 0$, which we cannot satisfy.

Unless two of the terms in the equation are of the same order as each other, we will either end up saying that the leading order term in the expansion is zero (which is giving us no information), or we will get that we cannot satisfy the equation at all to leading order.

With this in mind, lets look for some distinguished limits of Eqn. (2.10). If we choose α so that the first two terms in Eqn. (2.10) have the same order, this requires that $1 + 2\alpha = \alpha$, this gives us $\alpha = -1$, which after multiplying by ϵ will give us the equation

$$z^2 + z + \epsilon = 0 \tag{2.13}$$

This is in fact the scaling we are looking for (we will discuss other possibilities later). If we set $\epsilon = 0$ this gives us

$$z^2 + z = 0 \tag{2.14}$$

This has the roots $z = 0$ and $z = -1$. The root $z = 0$ is associated with the root we already found by using Eqn. (2.5) in its primitive form. We can actually do a regular perturbtion expansion abut $z = 0$ to get this previous root.

On the other hand Eqn. (2.14) has the root $z = -1$. This root is giving us some non-trivial information, and we will have no problems carrying out a perturbation expansion about this solution. In particular, if we assume that $z = z_0 + \epsilon z_1 + \epsilon^2 z_2 + ..$ then collecting the zeroeth order terms in ϵ we get

$$z_0(z_0 + 1) = 0 \tag{2.15}$$

which we already noted has the solution $z_0 = -1$. The first order terms can be written as

$$2z_0 z_1 + z_1 + 1 = 0 \tag{2.16}$$

Using $z_0 = -1$, this gives us $z_1 = 1$. This gives us

$$z = -1 + \epsilon + O(\epsilon^2) \tag{2.17}$$

Puttng this back in terms of x, this gives us

$$x = -\frac{1}{\epsilon} + 1 + O(\epsilon) \tag{2.18}$$

This agrees with Eqn. (2.8).

It is instructive to consider other scalings. Suppose that we scale our equations so that in Eqn. (2.10) the first and third terms balance each other. In this case we must have $1 + 2\alpha = 0$, and hence $\alpha = -1/2$. After multiplying by $\sqrt{\epsilon}$ this would give us the equation

$$\epsilon^{1/2} z^2 + z + \epsilon^{1/2} = 0 \tag{2.19}$$

Though the first and the third terms are of the same order, they are not the leading order terms. Thus this is not a distinguished limit. The other possibility is to have the second and third terms in Eqn. (2.10) be the same order. We can do this by letting $\alpha = 0$, in which case we get our original scaling.

3 Multi-variate Polynomials-Puiseaux Series

We now give a singularly perturbed example involving mulivariate polynomials.

Suppose we have the equations

$$\epsilon y^2 + xy + x + y + 1 = 0 \tag{3.1}$$

$$2\epsilon xy + x^2 + y + 2 = 0 \tag{3.2}$$

We are interested in finding the solutions for small values of ϵ. If we let $\epsilon = 0$, we get the equations

$$xy + x + y + 1 = 0 \tag{3.3}$$

$$x^2 + y + 2 = 0 \tag{3.4}$$

The second of these equations shows that $y = -2 - x^2$. If we substitute this into the first equation (still assuming $\epsilon = 0$) this will give us

$$-x(2 + x^2) + x - 2 - x^2 + 1 = 0 \tag{3.5}$$

which is a cubic equation in x. The three roots of this equation will give us three roots as ϵ goes to zero of Eqns. (3.1) and (3.2). We would like to see if there are any other roots we have missed that are similar to the one we missed at first in the last section. We will suppose that

$$x = \epsilon^{\alpha}\xi, y = \epsilon^{\beta}\eta \tag{3.6}$$

If we substitute this into the equations (3.1) and (3.2) we get

$$\epsilon^{1+2\beta}\eta^2 + \epsilon^{\alpha+\beta}\xi\eta + \epsilon^{\alpha}\xi + \epsilon^{\beta}\eta + 1 = 0 \tag{3.7}$$

$$2\epsilon^{1+\alpha+\beta}\xi\eta + \epsilon^{2\alpha}\xi^2 + \epsilon^{\beta}\eta + 2 = 0 \tag{3.8}$$

We will have a distinguished limit if we can choose α and β so that two of the terms in the first equation are of the same order, and two of the terms in the second equation are the same order. We also require that these terms that are of the same order in each equation are the leading order terms. This is a much more challenging problem than in our one dimensional example. It is good to understand conceptually how to solve this problem, but in prqactice it is good to apply some cleverness.

In this particular example, a bit of playing around shows that we can choose $\alpha = -1$, $\beta = -2$. After multiplying Eqn. (3.7) by ϵ^3 and Eqn. (3.8) by ϵ^2 we get

$$\eta^2 + \xi\eta + \epsilon^2\xi + \epsilon\eta + \epsilon^3 = 0 \tag{3.9}$$

$$2\xi\eta + \xi^2 + \eta + 2\epsilon^2 = 0 \tag{3.10}$$

When $\epsilon = 0$ the first equation gives $\eta(\eta + \xi) = 0$, This either has $\eta = 0$ or $\xi = -\eta$. If $\eta = 0$, the second equation gives $\xi = 0$. This is a trivial anwer that we ignore. On the other hand, if $\xi = -\eta$, the second equation gives us $\eta^2 = \eta$, which has the solution $\eta = 0$ or $\eta = 1$. The souton $\eta = 0$ will be trivial, and it follows that the solution $\eta = 1$ is the one we are looking for. This will give us the leading order behavior

$$x = -\frac{1}{\epsilon} + .. \tag{3.11}$$

$$y = \frac{1}{\epsilon^2} + ... \tag{3.12}$$