Lie Algebra and Representation of SU(4)

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Abstract: In this paper we give a new representation for the Lie unimodular group SU(4). Both the nonzero structure constant $f_{k\mu\nu}$ and the symmetric invariant tensor $d_{k\mu\nu}$ one calculated. We also bring out a new representation of SU(4) in terms of Pauli matrices constructed. Finally, the weight of the first fundamental representation of SU(4) is also obtained.

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1. Introduction

The group of all unitary matrices of order n is known as U(n), whereas the group of all unitary matrices of order n with determinant +1 is denoted by SU(n)[1, 2]. It is clear that the SU(n) group is a subgroup of U(n), continuous, connected, lie group, and possesses $n^2 - 1$ real independent parameters.

The unitary unimodular group SU(2) in two complex dimensions is the simplest non-trivial example of a non abelian group which describes the two level quantum system, at the same time SU(2) is homomorphic on SO(3), which have three generators

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

which are a set of three independent traceless hermitian matrices of order 2.

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The next group after SU(2) is SU(3) which possesses eight dimensional unimodular matrices in three complex dimension. The description of three level systems[3, 4, 5] in general quantum mechanics also involves SU(3). It has been discussed a generalization of SU(3)[6],their commutation and anticommutation relations, structure constant $f_{k\mu\nu}$ and the symmetric invariant tensor $d_{k\mu\nu}$.

The next group after SU(3) is SU(4). In SU(4) we have 15 generators $(n^2 - 1)$ of SU(4) as 4×4 matrices. The group SU(4) which can be broken to anther subgroup such that SU(4) ⊃ SU(2)×SU(2), often exploits the canonical subgroup chain SU(4) ⊃ SU(3) ⊃ SU(2), also it is useful to study the members of the baryon 20_M of SU(4). In the quark model there are at least four quarks, the four quarks are members of the first fundamental quartet of SU(4).

This paper is organized as follows. Section II deals with the definition of the group SU(4) and the generators λ-matrices in the defining representation. For their commutation relations, we constructed the structure constant $f_{k\mu\nu}$ and for their anticommutator, we constructed the symmetric invariant tensor $d_{k\mu\nu}$. The nonzero structure constant $f_{k\mu\nu}$ and symmetric invariant $d_{k\mu\nu}$ are listed. Section III deals with another kind of generators which are obtained from the $\lambda_\mu(\mu = 1 \cdots 15)$, and from these generators we will find the weights of the first fundamental representation of SU(4).

2. Generators of SU(4) and λ-matrices

The group SU(4) is represented as

$$\mathrm{SU}(4) = \{ \ \mathrm{A} = 4 \times 4 \text{ comlex matrix} \mid \ \mathrm{A}^\dagger \mathrm{A} = 1, \ \det(\mathrm{A}){=}1 \ \} \tag{2}$$

The hermitian matrix generators of SU(4), analogous to the pauli matrices of SU(2) and the Gell Mann matrices of SU(3) which are:

$$\left.\begin{array}{l}
\lambda_1 = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0&-i&0&0\\ i&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \\
\lambda_4 = \begin{pmatrix} 0&0&1&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0&0&-i&0\\ 0&0&0&0\\ i&0&0&0\\ 0&0&0&0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&0 \end{pmatrix} \\
\lambda_7 = \begin{pmatrix} 0&0&0&0\\ 0&0&-i&0\\ 0&i&0&0\\ 0&0&0&0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-2&0\\ 0&0&0&0 \end{pmatrix}, \lambda_9 = \begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{pmatrix} \\
\lambda_{10} = \begin{pmatrix} 0&0&0&-i\\ 0&0&0&0\\ 0&0&0&0\\ i&0&0&0 \end{pmatrix}, \lambda_{11} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&1&0&0 \end{pmatrix}, \lambda_{12} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&-i\\ 0&0&0&0\\ 0&i&0&0 \end{pmatrix} \\
\lambda_{13} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}, \lambda_{14} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-i\\ 0&0&i&0 \end{pmatrix}, \lambda_{15} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&-3 \end{pmatrix}
\end{array}\right\} \quad (3)$$

The λ_μ matrices are orthogonal and satisfy

$$\mathrm{Tr}\,(\lambda_\mu)^2 = 2; \quad \mu = 1 \cdots 15 \quad (4)$$

These matrices λ_μ obey both the characteristic commutation and the Jacobi identity relations

$$[[\lambda_\nu, \lambda_\mu], \lambda_k] + [[\lambda_\mu, \lambda_k], \lambda_\nu] + [[\lambda_k, \lambda_\nu], \lambda_\mu] = 0 \tag{5}$$

$$[\lambda_k, \lambda_\mu] = 2i \sum_\mu f_{k\mu\nu} \lambda_\nu; \quad k, \mu, \nu = 1 \cdots 15 \tag{6}$$

$$\mathrm{Tr}\left(\lambda_k \left[\lambda_\mu, \lambda_\nu\right]\right) = 4i f_{k\mu\nu} \tag{7}$$

where [ ] stands for the commutation relation. For SU(4) the number of commutation relations worked out is 105 and $f_{k\mu\nu}$ is denoted by the structure constant. The independent non-vanishing structure constants $f_{k\mu\nu}$ which are completely antisymmetric under the permutation of any two indices satisfying $k < \mu < \nu$ are given in the table (1).

For each case of anti-commutation relation { a, b}, where {a, b}= ab+ba, the calculation is more complicated. The matrices λ_μ obey characteristic anti-commutation relation

$$\{\lambda_k, \lambda_\mu\} = 2 \sum_\nu d_{k\mu\nu} \lambda_\nu + (\mathrm{A}_{k\mu})_{4\times4} \tag{8}$$

$$\mathrm{Tr}(\lambda_k \{\lambda_\mu, \lambda_\nu\}) = 4 d_{k\mu\nu} \tag{9}$$

$$\mathrm{Tr}(\lambda_k \lambda_\mu \lambda_\nu) = 2i f_{k\mu\nu} + 2 d_{k\mu\nu} \tag{10}$$

where $\mathrm{A}_{k\mu}$ is defined by 4×4 matrix which satisfy

$$\mathrm{A}_{k\mu} = \begin{cases} (0)_{4\times4} \ \text{ if } k \neq \mu;\ k, \mu = 1 \cdots 15 \\ (\mathrm{A})_{4\times4} \text{ if } k = \mu;\ k, \mu = 1 \cdots 15 \end{cases} \tag{11}$$

This matrix $\mathrm{A}_{k\mu}$is characterized by all the matrix elements is zero except the diagonal as listed in table (2). In this table we have given the independent non-vanishing components of the completely symmetric $d_{k\mu\nu}$ under the permutation of any two indices. It is clear from tables (1) and (2) that if k, μ and ν are less than or equal to eight, the $f_{k\mu\nu}$ and $d_{k\mu\nu}$ are identical to the SU(3) structure constants $f_{k\mu\nu}$ and $d_{k\mu\nu}$. It should be noted here that the equations (5,6,7,9 and 10) also properties for the three dimensional λ matrices SU(3). However for all group SU(n), $n \geq 3$, nothing is new apart except that the anti- commutators as it is clear in equation (8).

3. Weight of SU(4)

In the language of quantum mechanics there are always three spin operators in three dimensional space xyz, we shall call these operators H_1 ,H_2 and H_3. The vector ψ_m^j is an eigenstate of H_3

$$H_3 \psi_m^j = m \psi_m^j \tag{12}$$

The eigenvalue m is called a weight. In this representation, one can express the Lie algebra of SU(4) in term of the Pauli spin matrices. We dened these generators as follows

$$H_i = \frac{1}{2\sqrt{2}}\sigma_z \tag{13}$$

$$E_i = \frac{\sigma_x + i\sigma_y}{4} \tag{14}$$

$$E_{-i} = \frac{\sigma_x - i\sigma_y}{4} \tag{15}$$

using the above equations one can obtain the generators in the standard form in term of λ_k $(k = 1 \cdots 15)$. These are given by

$$\left.\begin{array}{lll}
H_1 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & H_2 = \frac{1}{2\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & H_3 = \frac{1}{4\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \\
E_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & E_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & E_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
E_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & E_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & E_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
E-1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & E_{-2} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & E_{-3} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
E_{-4} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & E_{-5} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & E_{-6} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{array}\right\} \tag{16}$$

where

$$E_{-i} = E_i^{\dagger}, \qquad i = 1 \cdots 6 \tag{17}$$

If we operate on the four dimensional basis vector

$$U_1\left(\frac{3}{2}\right)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\quad U_2\left(\frac{1}{2}\right)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix};\quad U_3\left(\frac{-1}{2}\right)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix};\quad U_4\left(\frac{-3}{2}\right)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{18}$$

with the diagonal vector operator $H=(H_1,H_2,H_3)$, we obtain

$$HU_1=\left(\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{6}},\frac{1}{4\sqrt{3}}\right)U_1 \tag{19}$$

$$HU_2=\left(\frac{-1}{2\sqrt{2}},\frac{1}{2\sqrt{6}},\frac{1}{4\sqrt{3}}\right)U_2 \tag{20}$$

$$HU_3=\left(0,\frac{-1}{\sqrt{6}},\frac{1}{4\sqrt{3}}\right)U_3 \tag{21}$$

$$HU_4=\left(0,0,\frac{-\sqrt{3}}{4}\right)U_4 \tag{22}$$

Thus the four weight of the first representation of SU(4) are given by

$$m(1)=\left(\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{6}},\frac{1}{4\sqrt{3}}\right) \tag{23}$$

$$m(2)=\left(\frac{-1}{2\sqrt{2}},\frac{1}{2\sqrt{6}},\frac{1}{4\sqrt{3}}\right) \tag{24}$$

$$m(3)=\left(0,\frac{-1}{\sqrt{6}},\frac{1}{4\sqrt{3}}\right) \tag{25}$$

$$m(4)=\left(0,0,\frac{-\sqrt{3}}{4}\right) \tag{26}$$

From equations (23) and (24) the weight of the second fundamental representation are just the negative of the weight of the first.

Conclusion

In this paper, matrices of the Lie algebra of SU(4) have been represented, from these matrices the calculation of the commutation and anti-commutation relations has been carried out with the given independent non-vanishing components of $f_{k\mu\nu}$ and $d_{k\mu\nu}$, as it is clear from tables (1) and (2), it is shown that if k,μ and ν are less than or equal to eight, the invariant tensors $f_{k\mu\nu}$ and $d_{k\mu\nu}$ are identical to SU(3) structure constants. As it is obvious from our analysis $n\geq 4$, nothing is new a part in the commutation and anti-commutation relations except that as mention in equation 8, which is represented by a new matrix $A_{k\mu}$, this matrix is diagonal matrix which is symmetric under permutation of any two indices of $d_{k\mu\nu}$.

In the parallel direction, we have constructed another representation of SU(4) in term of Palui spin matrices, from which we calculated the weight. For any representation of j of the group, there are $2j+1$ different weights which belong to a different representation and if we know one weight we can obtain another weight of the representation.
Thus, it is expected that our method can be systematically extended to higher dimensional group as SU(6).

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Table 1 Non-zero structure constant $f_{k\mu\nu}$ of SU(4)

k	μ	ν	$d_{k\mu\nu}$
1	2	3	1
1	4	7	$\frac{1}{2}$
1	5	6	$\frac{1}{-2}$
2	4	6	$\frac{1}{2}$
2	5	7	$\frac{1}{2}$
3	4	5	$\frac{1}{2}$
3	6	7	$\frac{1}{-2}$
4	5	8	$\frac{\sqrt{3}}{2}$
6	7	8	$\frac{\sqrt{3}}{2}$
1	9	12	$\frac{1}{2}$
1	10	11	$\frac{1}{-2}$
2	9	11	$\frac{1}{2}$
2	10	12	$\frac{1}{2}$
3	9	10	$\frac{1}{2}$
3	11	12	$\frac{1}{-2}$
4	9	14	$\frac{1}{2}$
4	10	13	$\frac{1}{-2}$
5	9	13	$\frac{1}{2}$
5	10	14	$\frac{1}{2}$
6	11	14	$\frac{1}{2}$
6	12	13	$\frac{1}{-2}$
7	11	13	$\frac{1}{2}$

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Table 1 - continued from previous page

k	μ	ν	$d_{k\mu\nu}$
7	12	14	$\frac{1}{2}$
8	9	10	$\frac{1}{2\sqrt{3}}$
8	11	12	$\frac{1}{2\sqrt{3}}$
8	13	14	$\frac{-1}{2\sqrt{3}}$
9	10	15	$\frac{1}{2\sqrt{3}}$
11	12	15	$\frac{1}{2\sqrt{3}}$
13	14	15	$\frac{1}{2\sqrt{3}}$

Table 2 Non-zero independent element of the tensor $d_{k\mu\nu}$ and the matrix $A_{k\mu}$

k	μ	ν	$d_{k\mu\nu}$	$A_{k\mu}$
1	4	6	$\frac{1}{2}$	(0)
1	5	7	$\frac{1}{2}$	(0)
1	1	8	$\frac{1}{\sqrt{3}}$	$\frac{4}{3}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$
1	1	15	$\frac{1}{\sqrt{6}}$	$\frac{1}{6}\begin{pmatrix}5&0&0&0\\0&5&0&0\\0&0&-1&0\\0&0&0&3\end{pmatrix}$
1	10	12	$\frac{1}{2}$	(0)
1	9	11	$\frac{1}{2}$	(0)
2	2	8	$\frac{1}{\sqrt{3}}$	$\frac{4}{3}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$
2	2	15	$\frac{1}{\sqrt{6}}$	$\frac{4}{3}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$
2	4	7	$\frac{-1}{2}$	(0)
2	5	6	$\frac{1}{2}$	(0)
2	9	12	$\frac{-1}{2}$	(0)

continue on next page

Table 2 D continued from previous page

k	μ	ν	$d_{k\mu\nu}$	$A_{k\mu}$
2	10	11	$\frac{1}{2}$	(0)
3	3	8	$\frac{1}{\sqrt{3}}$	$\frac{4}{3}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$
3	3	15	$\frac{1}{\sqrt{6}}$	$\frac{4}{3}\begin{pmatrix}5&0&0&0\\0&5&0&0\\0&0&-1&0\\0&0&0&3\end{pmatrix}$
3	4	4	$\frac{1}{2}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&0\end{pmatrix}$
3	5	5	$\frac{1}{2}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&0\end{pmatrix}$
3	6	6	$\frac{-1}{2}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&0\end{pmatrix}$
3	7	7	$\frac{-1}{2}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&2\\0&0&0&0\end{pmatrix}$

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Table 2 D continued from previous page

k	μ	ν	$d_{k\mu\nu}$	$A_{k\mu}$
3	9	9	$\frac{1}{2}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$
3	10	10	$\frac{1}{2}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$
3	11	11	$\frac{-1}{2}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$
3	12	12	$\frac{-1}{2}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$
4	4	8	$\frac{-1}{2\sqrt{3}}$	$\frac{-1}{2\sqrt{3}}\begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
4	4	15	$\frac{1}{\sqrt{6}}$	$\frac{1}{3}\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$
4	9	13	$\frac{1}{2}$	(0)
4	10	14	$\frac{1}{2}$	(0)
5	5	8	$\frac{-1}{2\sqrt{3}}$	$\frac{1}{3}\begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
5	9	14	$\frac{-1}{2}$	(0)

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Table 2 D continued from previous page

k	μ	ν	$d_{k\mu\nu}$	$\mathrm{A}_{k\mu}$
5	10	13	$\frac{1}{2}$	(0)
5	5	15	$\frac{1}{\sqrt{6}}$	$\frac{1}{3}\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$
6	6	8	$\frac{-1}{2\sqrt{3}}$	$\frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
6	6	15	$\frac{1}{\sqrt{6}}$	$\frac{1}{3}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$
6	11	13	$\frac{1}{2}$	(0)
6	12	14	$\frac{1}{2}$	(0)
7	7	8	$\frac{-1}{2\sqrt{3}}$	$\frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
7	7	15	$\frac{1}{\sqrt{6}}$	$\frac{1}{3}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$
7	11	14	$\frac{-1}{2}$	(0)
7	12	13	$\frac{1}{2}$	(0)

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Table 2 D continued from previous page

k	μ	ν	$d_{k\mu\nu}$	$\mathrm{A}_{k\mu}$
8	8	8	$\frac{-1}{\sqrt{3}}$	$\frac{4}{3}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$
8	8	15	$\frac{1}{\sqrt{6}}$	$\frac{1}{3}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&7&0\\0&0&0&3\end{pmatrix}$
8	9	9	$\frac{1}{2\sqrt{3}}$	$\frac{1}{3}\begin{pmatrix}5&0&0&0\\0&-1&0&0\\0&0&2&0\\0&0&0&6\end{pmatrix}$
8	10	10	$\frac{1}{2\sqrt{3}}$	$\frac{1}{3}\begin{pmatrix}5&0&0&0\\0&-1&0&0\\0&0&2&0\\0&0&0&6\end{pmatrix}$
8	11	11	$\frac{1}{2\sqrt{3}}$	$\frac{1}{3}\begin{pmatrix}-1&0&0&0\\0&5&0&0\\0&0&2&0\\0&0&0&6\end{pmatrix}$
8	12	12	$\frac{1}{2\sqrt{3}}$	$\frac{1}{3}\begin{pmatrix}-1&0&0&0\\0&5&0&0\\0&0&2&0\\0&0&0&6\end{pmatrix}$
8	13	13	$\frac{-1}{\sqrt{3}}$	$\frac{2}{3}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&3\end{pmatrix}$

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Table 2 D continued from previous page

k	μ	ν	$d_{k\mu\nu}$	$\mathrm{A}_{k\mu}$
8	14	14	$\frac{-1}{\sqrt{3}}$	$\frac{2}{3}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&3\end{pmatrix}$
9	9	15	$\frac{-1}{\sqrt{6}}$	$\frac{1}{3}\begin{pmatrix}7&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&3\end{pmatrix}$
10	10	15	$\frac{-1}{\sqrt{6}}$	$\frac{1}{3}\begin{pmatrix}7&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&3\end{pmatrix}$
11	11	15	$\frac{-1}{\sqrt{6}}$	$\frac{1}{3}\begin{pmatrix}1&0&0&0\\0&7&0&0\\0&0&1&0\\0&0&0&3\end{pmatrix}$
12	12	15	$\frac{-1}{\sqrt{6}}$	$\frac{1}{3}\begin{pmatrix}1&0&0&0\\0&7&0&0\\0&0&1&0\\0&0&0&3\end{pmatrix}$

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Table 2 D continued from previous page

k	μ	ν	$d_{k\mu\nu}$	$\mathrm{A}_{k\mu}$
13	13	15	$\frac{-1}{\sqrt{6}}$	$\frac{1}{3}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&7&0\\0&0&0&3\end{pmatrix}$
14	14	15	$\frac{-1}{\sqrt{6}}$	$\frac{1}{3}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&7&0\\0&0&0&3\end{pmatrix}$
8	14	14	$\frac{-1}{\sqrt{3}}$	$\frac{2}{3}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&3\end{pmatrix}$