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Maximal Area of a Bicentric Quadrilateral

Martin Josefsson

Abstract. We prove an inequality for the area of a bicentric quadrilateral in terms of the radii of the two associated circles and show how to construct the quadrilateral of maximal area.

1. Introduction

A bicentric quadrilateral is a convex quadrilateral that has both an incircle and a circumcircle, so it is both tangential and cyclic. Given two circles, one within the other with radii r and R (where $r < R$), then a necessary condition that there can be a bicentric quadrilateral associated with these circles is that the distance δ between their centers satisfies Fuss' relation

$$\frac{1}{(R+\delta)^2} + \frac{1}{(R-\delta)^2} = \frac{1}{r^2}.$$

A beautiful elementary proof of this was given by Salazar (see [8], and quoted at [1]). According to [9, p.292], this is also a sufficient condition for the existence of a bicentric quadrilateral. Now if there for two such circles exists one bicentric quadrilateral, then according to Poncelet's closure theorem there exists infinitely many; any point on the circumcircle can be a vertex for one of these bicentric quadrilaterals [11]. That is the configuration we shall study in this note. We derive a formula for the area of a bicentric quadrilateral in terms of the inradius, the circumradius and the angle between the diagonals, conclude for which quadrilateral the area has its maximum value in terms of the two radii, and show how to construct that maximal quadrilateral.

2. More on the area of a bicentric quadrilateral

In [4] and [3, §6] we derived a few new formulas for the area of a bicentric quadrilateral. Here we will prove another area formula using properties of bicentric quadrilaterals derived by other authors.

Theorem 1. *If a bicentric quadrilateral has an incircle and a circumcircle with radii r and R respectively, then it has the area*

$$K = r\left(r + \sqrt{4R^2 + r^2}\right)\sin\theta$$

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where θ is the angle between the diagonals.

Proof. We give two different proofs. Both of them uses the formula

$$K = \tfrac{1}{2}pq\sin\theta \tag{1}$$

which gives the area of a convex quadrilateral with diagonals p, q and angle θ between them.

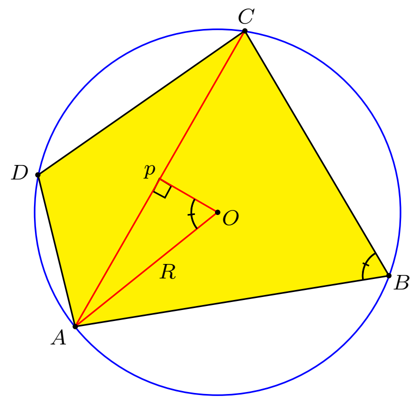

Figure 1. Using the inscribed angle theorem

First proof. In a cyclic quadrilateral it is easy to see that the diagonals satisfy $p = 2R\sin B$ and $q = 2R\sin A$ (see Figure 1). Inserting these into (1) we have that a cyclic quadrilateral has the area [1]

$$K = 2R^2 \sin A \sin B \sin\theta. \tag{2}$$

In [13] Yun proved that in a bicentric quadrilateral $ABCD$ (which he called a double circle quadrilateral),

$$\sin A \sin B = \frac{r^2 + r\sqrt{4R^2 + r^2}}{2R^2}.$$

Inserting this into (2) proves the theorem.

Second proof. In [2, pp.249, 271–275] it is proved that the inradius in a bicentric quadrilateral is given by

$$r = \frac{pq}{2\sqrt{pq + 4R^2}}.$$

Solving for the product of the diagonals gives

$$pq = 2r\left(r + \sqrt{4R^2 + r^2}\right)$$

where we chose the solution of the quadratic equation with the plus sign since the product of the diagonals is positive. Inserting this into (1) directly yields the theorem. $\square$

[1] A direct consequence of this formula is the inequality $K \le 2R^2$ in a cyclic quadrilateral, with equality if and only if the quadrilateral is a square.

Remark. According to [12, p.164], it was Problem 1376 in the journal Crux Mathematicorum to derive the equation

$$\frac{pq}{4r^2} - \frac{4R^2}{pq} = 1$$

in a bicentric quadrilateral. Solving this also gives the product pq in terms of the radii r and R.

Corollary 2. *If a bicentric quadrilateral has an incircle and a circumcircle with radii r and R respectively, then its area satisfies*

$$K \leq r\left(r + \sqrt{4R^2 + r^2}\right)$$

where there is equality if and only if the quadrilateral is a right kite.

Proof. There is equality if and only if the angle between the diagonals is a right angle, since $\sin\theta \leq 1$ with equality if and only if $\theta = \frac{\pi}{2}$. A tangential quadrilateral has perpendicular diagonals if and only if it is a kite according to Theorem 2 (i) and (iii) in [5]. Finally, a kite is cyclic if and only if two opposite angles are right angles since it has a diagonal that is a line of symmetry and opposite angles in a cyclic quadrilateral are supplementary angles. □

We also have that the semiperimeter of a bicentric quadrilateral satisfies

$$s \leq r + \sqrt{4R^2 + r^2}$$

where there is equality if and only if the quadrilateral is a right kite. This is a direct consequence of Corollary 2 and the formula $K = rs$ for the area of a tangential quadrilateral. To derive this inequality was a part of Problem 1203 in Crux Mathematicorum according to [10, p.39]. Another part of that problem was to prove that in a bicentric quadrilateral, the product of the sides satisfies

$$abcd \leq \tfrac{16}{9}r^2(4R^2 + r^2).$$

It is well known that the left hand side gives the square of the area of a bicentric quadrilateral (a short proof is given in [4, pp.155–156]). Thus the inequality can be restated as

$$K \leq \tfrac{4}{3}r\sqrt{4R^2 + r^2}.$$

This is a weaker area inequality than the one in Corollary 2, which can be seen in the following way. An inequality between the two radii of a bicentric quadrilateral is $R \geq \sqrt{2}r$.[2] From this it follows that $4R^2 \geq 8r^2$, and so

$$3r \leq \sqrt{4R^2 + r^2}.$$

Hence, from Theorem 1, we have

$$\frac{K}{r} \leq r + \sqrt{4R^2 + r^2} \leq \tfrac{4}{3}\sqrt{4R^2 + r^2}$$

so the expression in Corollary 2 gives a sharper upper bound for the area.

[2]References to several different proofs of this inequality are given at the end of [6], where we provided a new proof of an extension to this inequality.

3. Construction of the maximal bicentric quadrilateral

Given two circles, one within the other, and assuming that a bicentric quadrilateral exist inscribed in the larger circle and circumscribed around the smaller, then among the infinitely many such quadrilaterals that are associated with these circles, Corollary 2 states that the one with maximal area is a right kite. Since a kite has a diagonal that is a line of symmetry, the construction of this is easy. Draw a line through the two centers of the circles. It intersect the circumcircle at A and C. Now all that is left is to construct tangents to the incircle through A. This is done by constructing the midpoint M between the incenter I and A, and drawing the circle with center M and radius MI according to [7]. This circle intersect the incircle at E and F. Draw the tangents AE and AF extended to intersect the circumcircle at B and D. Finally connect the points $ABCD$, which is the right kite with maximal area of all bicentric quadrilaterals associated with the two circles having centers I and O.

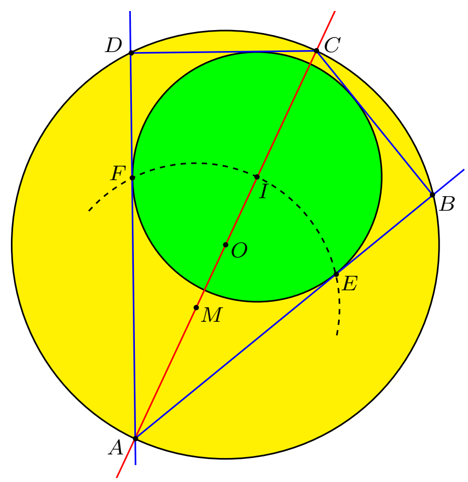

Figure 2. Construction of the right kite $ABCD$

References

[1] A. Bogomolny, Fuss' Theorem, *Interactive Mathematics Miscellany and Puzzles*, `http://www.cut-the-knot.org/Curriculum/Geometry/Fuss.shtml`

[2] H. Fukagawa and T. Rothman, *Sacred Mathematics, Japanese Temple Geometry*, Princeton university press, 2008.

[3] M. Josefsson, Calculations concerning the tangent lengths and tangency chords of a tangential quadrilateral, *Forum Geom.*, 10 (2010) 119–130.

[4] M. Josefsson, The area of a bicentric quadrilateral, *Forum Geom.*, 11 (2011) 155–164.

[5] M. Josefsson, When is a tangential quadrilateral a kite?, *Forum Geom.*, 11 (2011) 165–174.

[6] M. Josefsson, A new proof of Yun's inequality for bicentric quadrilaterals, *Forum Geom.*, 12 (2012) 79–82.

[7] *Math Open Reference*, Tangents through an external point, 2009, `http://www.mathopenref.com/consttangents.html`

[8] J. C. Salazar, Fuss' theorem, *Math. Gazette*, 90 (2006) 306–307.

[9] M. Saul, *Hadamard's Plane Geometry, A Reader's Companion*, Amer. Math. Society, 2010.
[10] E. Specht, *Inequalities proposed in "Crux Mathematicorum"*, 2007, available at http://hydra.nat.uni-magdeburg.de/math4u/ineq.pdf
[11] E. W. Weisstein, Bicentric Polygon, *MathWorld* – A Wolfram Web Resource, Accessed 22 April 2012, http://mathworld.wolfram.com/BicentricPolygon.html
[12] P. Yiu, *Notes on Euclidean Geometry*, Florida Atlantic University Lecture Notes, 1998.
[13] Z. Yun, Euler's inequality revisited, *Mathematical Spectrum*, 40 (2008) 119–121.

Martin Josefsson: Västergatan 25d, 285 37 Markaryd, Sweden
E-mail address: martin.markaryd@hotmail.com