Primality & Prime Number Generation

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OUTLINE

The Problem

- Given an integer n, test whether it is prime.
- Easy Solution: Divide n by all numbers between 2 and $(n-1)$.
- What is the deal about primality testing then ??

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Efficiently Solving a Problem

- Given n we want a polynomial time primality test, one that runs in atmost $(\log n)^c$ steps.
- Note that practically $(\log n)^{\log\log\log n}$ steps is efficient enough for the prime numbers we encounter in real life!
- Nevertheless, the notion of polynomial time elegantly captures the theoretical complexity of a problem.

Notation:

- $(\log n)$ is logarithm base 2. $(\ln n)$ is natural log.
- $O^{\sim}(\log^c n)$ denotes $\log^c n \cdot (\log\log n)^{O(1)}$.

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Eratosthenes Sieve

Proposed by Eratosthenes (ca. 300 BC).

1. List all numbers from 2 to n in a sequence.
2. Take the smallest uncrossed number and cross out all its multiples (except itself).
3. At the end all the uncrossed numbers are primes.

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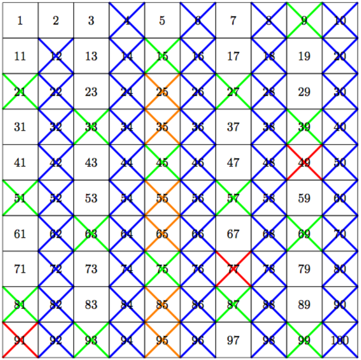

Time Complexity

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Density of primes

- Suppose we want a prime number *close* to n.
- Eratosthenes sieve is a way to generate it. But it's slow.
- Fortunately, the primes are abundant in nature. If $\pi(x)$ is the number of primes below x then *precise* estimates on $\pi(x)/x$ are known.

Rosser (1941)

showed that $\frac{1}{\ln x+2} < \frac{\pi(x)}{x} < \frac{1}{\ln x-4}$, for $x \geq 55$.

- Thus, if we randomly pick a $(\log n)$-bit number N, then with high probability it will be prime!

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Ring based primality tests

- All the advanced primality tests associate a ring R to n and study its properties.
- The favorite rings are:
  1. $\mathbb{Z}_n$ – Integers modulo n.
  2. $\mathbb{Z}_n[\sqrt{3}]$ – Quadratic extensions.
  3. $\mathbb{Z}_n[x,y]/(y^2 - x^3 - ax - b)$ – Elliptic curves.
  4. $\mathbb{Z}_n[x]/(x^r - 1)$ – Cyclotomic rings.

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Fermat's Little Theorem (FLT)

Theorem (Fermat, 1660s)

If n is prime then for every a, $a^n = a \pmod{n}$.

- Basically, for all $a \in \mathbb{Z}_n^*$, $a^{n-1} = 1$.
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- But it is the starting point!

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Theorem (Lucas, 1876)

n is prime iff $\exists a \in \mathbb{Z}_n$ such that $a^{n-1} = 1$ and $a^{\frac{n-1}{p}} \neq 1$ for all primes $p|(n-1)$.

- Suppose $(n-1)$ is smooth and we know its prime factors.
- Do the above test for a random a.

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Theorem (Pocklington, 1914)

If $\exists a \in \mathbb{Z}_n$ such that $a^{n-1} = 1$ and $gcd(a^{\frac{n-1}{p_i}} - 1, n) = 1$ for any distinct primes $p_1, \ldots, p_t | (n-1)$. Then any divisor of n is of the form $1 + kp_1 \cdots p_t$.

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Solovay-Strassen: First randomized test

Theorem (Strengthening FLT)

An odd number n is prime iff for all $a \in \mathbb{Z}_n$, $a^{\frac{n-1}{2}} = \left(\frac{a}{n}\right)$.

- Jacobi symbol $\left(\frac{a}{n}\right)$ is computable in time $O^{\sim}(\log^2 n)$.
- Solovay-Strassen (1977) check the above equation for a random a.
- This gives a randomized test that takes time $O^{\sim}(\log^2 n)$.
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Strengthening FLT further [Miller, 1975]

An odd number $n = 1 + 2^s \cdot t$ (odd t) is prime iff for all $a \in \mathbb{Z}_n$, the sequence $a^{2^{s-1}\cdot t}$, $a^{2^{s-2}\cdot t}$, $\ldots, a^t$ has either a -1 or all 1's.

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Riemann Hypothesis and Primality

Generalized Riemann Hypothesis [Piltz, 1884]

Let Dirichlet L-function be the analytic continuation of $L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$. For every Dirichlet character χ and every complex number s with $L(\chi, s) = 0$: if $\mathrm{Re}(s) \in (0, 1]$ then $\mathrm{Re}(s) = \frac{1}{2}$.

- By taking χ to be the character modulo n it can be shown: the GRH implies that there exists an $a \leq 2\log^2 n$ such that $\left(\frac{a}{n}\right) \neq 1$ (Ankeny 1952; Miller 1975; Bach 1980s).
- This magical small a would be a witness of the compositeness of n.
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- This magical small a would be a witness of the compositeness of n.
- Thus, GRH derandomizes both Solovay-Strassen and Miller-Rabin primality tests.

This a also factors Carmichael numbers!

Outline

Lucas-Lehmer Test

This is a test specialized for Mersenne primes $M_k = 2^k - 1$.

Theorem (Lucas-Lehmer, 1930)

M_k is prime iff $(2+\sqrt{3})^{\frac{M_k+1}{2}} = -1$ in $\mathbb{Z}_n[\sqrt{3}]$.

- This yields a deterministic polynomial time primality test for Mersenne primes.
- Generalization: Whenever $(n+1)$ has small prime factors one can test n for primality by working in $\mathbb{Z}_n[\sqrt{D}]$ where $\left(\frac{D}{n}\right) = -1$.
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Outline

Elliptic Curve Based Tests

- An elliptic curve over $\mathbb{Z}_n$ is the set of points:

$$E_{a,b}(\mathbb{Z}_n) = \left\{(x,y) \in \mathbb{Z}_n^2 \mid y^2 = x^3 + ax + b\right\}$$

- When n is prime: $E_{a,b}(\mathbb{Z}_n)$ is an abelian group.
- $\#E_{a,b}(\mathbb{Z}_n)$ can be computed in deterministic polynomial time (Schoof 1985).
- When n is prime: number of points on a random elliptic curve is uniformly distributed in the interval $[(\sqrt{n}-1)^2, (\sqrt{n}+1)^2]$ (Lenstra 1987).

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Goldwasser-Kilian Test

1. Pick a random elliptic curve E over $\mathbb{Z}_n$ and a random point $A \in E$.
2. Compute $\#E(\mathbb{Z}_n)$. If $\#E(\mathbb{Z}_n)$ is odd then output COMPOSITE.
3. Let $\#E(\mathbb{Z}_n) =: 2q$. Prove the primality of q recursively.
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Proof of Correctness:

- Firstly, note that conjecturally there are "many" numbers between $[(\sqrt{n}-1)^2, (\sqrt{n}+1)^2]$ that are twice a prime and for a random E, $\#E(\mathbb{Z}_n)$ will hit such numbers whp when n is prime.
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- This is the first randomized test that never errs when n is composite (1986).
- Time complexity (Atkin-Morain 1993): $O^{\sim}(\log^4 n)$.
- But its proof assumed a conjecture about the density of primes in the interval $\left[\frac{n+1-2\sqrt{n}}{2}, \frac{n+1+2\sqrt{n}}{2}\right]$.
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Adleman-Pomerance-Rumeli Test

- Recall how Lucas-Lehmer-Williams tested n for primality when $(n-1), (n+1), (n^2-n+1)$ or (n^2+n+1) was smooth.
- What can we do when (n^m-1) is smooth? Maybe go to some m-th extension of $\mathbb{Z}_n$?
- This question inspired the APR test (1980). Speeded up by Cohen and Lenstra (1981).
- Deterministic algorithm with time complexity $\log^{O(\log\log\log n)} n$.
- Is conceptually the most complex algorithm of all.
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Agrawal-Kayal-S (AKS) Test

Theorem (A Generalization of FLT)

If n is a prime then for all $a \in \mathbb{Z}_n$, $(x+a)^n = (x^n + a) \pmod{n, x^r - 1}$.

- This was the basis of the AKS test proposed in 2002.
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AKS Test

1. If n is a prime power, it is composite.
2. Select an r such that $\text{ord}_r(n) > 4\log^2 n$ and work in the ring $R := \mathbb{Z}_n[x]/(x^r - 1)$.
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AKS Test: The Proof

- Suppose all the congruences hold and p is a prime factor of n.
- The group $I := \langle n, p \pmod{r} \rangle$. $t := \# I \geq \mathrm{ord}_r(n) \geq 4 \log^2 n$.
- The group $J := \langle (x+1), \ldots, (x+\ell) \pmod{p, h(x)} \rangle$ where $h(x)$ is an irreducible factor of $\frac{x^r-1}{x-1}$ modulo p.
  $\# J \geq 2^{\min\{t,\ell\}} > 2^{2\sqrt{t}\log n} \geq n^{2\sqrt{t}}$.
- *Proof:* Let $f(x), g(x)$ be two different products of $(x+a)$'s, having degree $< t$. Suppose $f(x) = g(x) \pmod{p, h(x)}$.
- The test tells us that $f(x^{n^i \cdot p^j}) = g(x^{n^i \cdot p^j}) \pmod{p, h(x)}$.
- But this means that $f(z) - g(z)$ has atleast t roots in the field $\mathbb{F}_p[x]/(h(x))$, which is a contradiction.

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- Each congruence $(x+a)^n = (x^n + a) \pmod{n, x^r - 1}$ can be tested in time $O^{\sim}(r \log^2 n)$.
- The algorithm takes time $O^{\sim}(r^{\frac{3}{2}} \cdot \log^3 n)$.
- Recall that r is the least number such that $\mathrm{ord}_r(n) > 4\log^2 n$.
- Prime number theorem gives $r = O(\log^5 n)$ and thus, time $O^{\sim}(\log^{10.5} n)$.
- **Proof:** Stare at the product:

$$\Pi := (n-1)(n^2-1)\cdots(n^{\lfloor 4\log^2 n \rfloor}-1)$$

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Theorem (Fouvry 1985)

$$\#\left\{ prime\ p \le x \mid \exists\ prime\ q \ge p^{\frac{2}{3}}, q|(p-1) \right\} \sim \frac{x}{\log x}.$$

- Fouvry's theorem gives $r = O(\log^3 n)$ and thus, time $O^{\sim}(\log^{7.5} n)$.
- **Proof:** A "Fouvry prime" $r = O^{\sim}(\log^3 n)$ with $\text{ord}_r(n) \le 4\log^2 n$ has to divide the product:

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- Original AKS test took time $O^{\sim}(\log^{12} n)$. The above improvement used ideas from Hendrik Lenstra Jr.
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Outline

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Let $r > \log n$ be a prime number that does not divide $(n^3 - n)$. Then $(x-1)^n \equiv (x^n - 1) \pmod{n, x^r - 1}$ iff n is prime.

Evidence:

- Even for $r = 5$ the above conjecture holds for all $n \leq 10^{11}$.
- The above conjecture holds for all primes $r \leq 100$ and $n \leq 10^{10}$.

Could this test be used for *factoring* integers?

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Could this test be used for *factoring* integers?

Thank you!

Questions

Can we reduce the number of a's for which the test is performed?

Conjecture: (Bhattacharjee-Pandey 2001; AKS 2004)

Let $r > \log n$ be a prime number that does not divide $(n^3 - n)$. Then $(x-1)^n \equiv (x^n - 1) \pmod{n, x^r - 1}$ iff n is prime.

Evidence:

- Even for $r = 5$ the above conjecture holds for all $n \leq 10^{11}$.
- The above conjecture holds for all primes $r \leq 100$ and $n \leq 10^{10}$.

Could this test be used for *factoring* integers?

Thank you!