The AdS/CMT manual for plumbers and electricians

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I. INTRODUCTION

Is the history of science in the making, in a quite unexpected corner of physics? A small community of mostly theoretical physicists is in the grip of a dream encoded by "AdS-CMT": the Anti-de-Sitter to Condensed-Matter-Theory correspondence.

It is inter-disciplinarity at its best. It involves on the one hand the string theory community where the subject is at present very popular. String theory is a quite old subject. It started in particle physics as early as the 1960's and its traditional focus was on the fundamentals: what is space time, where is the standard model coming from, how do gravity and gauge quantum field theory hang together? All along, it has been energized by the powers of mathematics, embodied by Witten and other masters of the art. At present the status of string theory as source of inspiration in modern pure mathematics is undisputed, but in physics it is different. It takes quite an investment, but invariably everybody who learns the trade eventually gets in the grip of the idea that it *has* to relate to reality in one or the other way. The tragedy is that in the forty years of its existence string theory has had no consequence in the empirical realms.

Perhaps one should blame more the high energy experimenters: string theory has produced crisp productions like the production of mini black holes, proving that we are living in a universe with many more dimensions than we are aware off. That these mini black holes are not observed at the LHC might well mean that one needs a hundred times more energy, translating into a hundred times more euro expenditure in this experimental paradigm.

Although less glamorous for the laymen, there is another area in physics which is to the eye of the string theorist no less fundamental than the high energy stuff: hard condensed matter physics. In a way it is the mirror image of high energy physics. The experimental techniques to study the universes formed from strongly interacting electrons in solids, cold atoms an so forth, have leaped forward in the last two decades. It has an established track record as "paradise of serendipity", when the "telescopes of quantum matter" looked deeper and farther in these worlds, uncovering time after time again phenomena that nobody expected. Although the condensed matter theorists might find it uneasy, fact is that in this field theoretical activity has moved increasingly to the side line. Although the experimental observations are very suggestive regarding a beautiful and deep mathematical narrative at work, the theorists were embarrassed time after time by their lack of success of foreseeing the experimental surprises.

The condensed matter theorists have a place to hide. As in the laboratory, one needs first new machines before one can make progress with physics. These are now mathematical machines and the equipments at disposal to the condensed matter theorists are rather antiquated. It amounts largely to the perturbative fermiology and the Hartree-Fock mean field theory uncovered in the 1950's, augmented by a number of spin-offs of the 1970's "bosonic" non-perturbative quantum field theory revolution, like the Luttinger liquids of one dimensions and the Chern-Simons topological field theory describing topological order. But none of these shed any light on the main stream great mysteries like the strange metals, the origin of high Tc superconductivity, the pseudogap physics and so forth.

The dream is that the awesome mathematical machinery of string theory might actually be the magic bullet explaining to us the profundity behind the experimental observations.

It is still very fresh. The first contact was established in 2007, by Sachdev, Son and others, realizing that a central result of string theory (the AdS/CFT correspondence) has a bearing on the finite temperature physics of quantum critical states of electron matter. This inspired the string theorists to zoom in on the specific circumstances met in the condensed matter systems and amazing discoveries followed in a rapid succession: holographic superconductivity, the holographic "strange metals" and the emergence of stable Fermi-liquids. Among the string theorists there is a consensus that merely asking the condensed matter questions has been quite beneficial for the further development of the AdS-CFT correspondence which is a theoretical construct still littered with mystery. On the other hand, among the few condensed matter physicists who are equipped to understand what it is about it is perceived as a form of conceptual enlightenment. It is the counterintuitive powers of mathematics at its best, given the freedom to think in completely different ways about the physics. Perhaps most importantly, looking at the physical world with this particular "mathematical eye glass" strange questions for experiment come to mind one would never contemplate departing from the conventional paradigm.

This is the state of the art: it is suggestive but not decisive. Given the gravity of the theoretical claims, in order to land this in the history books high quality, unique "smoking gun" predictions are required, which are subsequently triumphantly confirmed by experiment. We firmly believe that there is quite a potential for such a development but to let it happen the experimental community should be mobilized and this is at present the bottleneck. An appropriate metaphor is the story of Einstein and Eddington. Einstein had discovered the role model of mathematical theory in physics (general relativity), yielding through its counterintuitive powers (space-time as fabric) the strange question to experiment that light gets deflected in a particular way in a gravitational field. Given his specific expertise as practicing astronomer, Eddington figured out how to test the prediction by observing star light during a solar eclipse. To let it all happen, he had to exploit his political clout, force of personality, business instincts whatever, to get the funding and resources to organize an expedition involving a big vessel to execute this very non main-stream experiment. Clearly, Eddington had a thorough understanding of the theory itself, as required to design the observations putting Einstein on the real axis of empirical reality. Perhaps more importantly, his obsessive attitudes were required to generate the resilience to bring the business side of this affair to a happy end.

Right now, the experimental community at large is staring away from AdS/CMT. This is not necessarily because of conservatism and lack of intellectual reach. The main hurdle is that the language of string theory is so far removed from the typical theoretical baggage of a condensed matter physicist: the AdS/CMT literature at large is effectively impenetrable. What remains are superficial impressions and these are at first sight quite outrageous indeed. After all, the central claim of AdS/CMT is that the strange metals and high T_c superconductors as realized in pieces of mundane rusted copper are actually post-modern black holes in disguise, showing their face in the form of fancied-up Hawking black hole radiation coming form a world with an extra dimension ...

Strangely enough, after getting acquainted with the world of quite orderly mathematics behind this outrageous claim, the notion becomes just quite reasonable and assuring. At the same time, one becomes increasingly nervous about the shaky fundaments of the fermiology standard lore and so forth, knowing how radically different systems of strongly interacting fermions can behave in principle. How to get this mind set across to the condensed matter community at large, realizing that one cannot possibly demand from these busy experimental physicists and young theorists looking for a tenure track job to invest two years of their career in learning string theory?

It is at the least an interesting challenge to try to construct a maximally user friendly, accessible tour guide of the holographic universe aimed at condensed matter physicists. The remainder of this text is just an attempt to construct a first sketch of such a user guide, hoping to get feedback from the customers and the professional string theorists which will be used to upgrade this affair to a more perfect state. There is already a substantial literature of excellent tutorials and reviews written by the professionals [1–5]. The tutorials are however invariably aiming a level higher, explaining how the computations are actually done to a an audience of string theory graduate students. We will aim here as well to make this literature accessible, but our main pursuit will be to construct the equivalent of a lonely planet guide for tourists visiting for the first time the holographic universe.

In part we will attempt to achieve this tour guide spirit by the use of the time-tested *boxes* trick of high school text books. Although the material that is found in these boxes does contain crucial and penetrating insights, it is heavy stuff that needs a considerable attention span to master. The lazy reader who is just after a feeling what matters are about can however safely skip the boxes, of course realizing that he/she missing some essence that however will not prohibit him/her to get an impression of what is going in the sections following the box. The casual reader might even find parts of the un-boxed text here and there on the heavy side. We also did our best to first tell the story in a very descriptive language before we start throwing the equations substantiating the case. We aimed at making it readable even for the very casual reader who wants to jump over the passages which are dense with equations. However, in some of these equation-dense passages key insights are explained which have to be grasped by even the laziest species of readers. To alert them we invented "dual boxes", containing one liners in a bold face print which summarizes the punchline. We find this an appropriate terminology in the present context: the boxes are for the strongly interacting reader and self evidently the dual variety suffices for the weakly coupled crowd: the extremely casual reader should use these as warning signs, inspecting the text in the immediate vicinity of the dual boxes to get an idea why it is of such a gravity.

However, foremost we did our best to tell the story using a language that should come natural to the condensed matter physicist. The language in which string theory was originally written is that of mathematics, where the discipline is at its best in terms of rigor and beauty. Here and there we will let this shimmer through in the boxed passages. However, perhaps the greatest significance of the recent developments has been that it has become increasingly meaningful to view these insights in the language of physics. And this physics is rather of the kind that is felt as natural by those with a condensed matter training. AdS/CFT can be viewed as a post-modern form of statistical physics, revolving around the principles governing *emergence*. It is about theories which are deeply phenomenological in their intent, in precisely the very positive meaning that these adjectives have in the condensed matter tradition. We perceive this as an experiment by itself: cutting corners here and there with regard to the mathematical underpinnings, we will force ourselves to tell the AdS/CMT narrative as much as possible speaking with the tongue of Landau. Given our intent to get somewhere as high school text book writers there should be no moral hazard in doing so. However, we still look forward to the opinions of string theory professionals in this regard. We are ourselves a bit surprised of how far we get, even given the present level of acceptance in the string theory community of the emergence side.

Let us finish this introduction with a short overview of the plot of the story that follows. Section II is devoted to a very descriptive perspective on what AdS/CFT is, resting strongly on general wisdoms which are common place among well trained condensed matter physicists. Section III is perhaps the most important part of this text, but likely also the most scary one for the condensed matter reader. This is devoted to the mathematical structure of the AdS/CFT correspondence, culminating in the "dictionary" prescribing in a precise way how to translate the physics of the boundary field theory into the gravitational affair realized in the bulk. The hurdle to overcome is in the knowledge of general relativity. After all, the actual computations all the time revolve around GR. You just need to refresh your basic GR knowledge in case you lost the wisdoms of the introductory course you might have followed in graduate school (if at all). In fact, the use of GR by the professional holographist is cutting edge. You don't need however their profound skills in this regard in order to consume their inventions. It is a different story when you have the ambition to figure out your own holographic dualities: the only way out is then to join the string theory freshmen in following one year of intense graduate school courses. When you are just interested in the products of this mathematical machine, the knowledge acquired in an introductory GR course suffices most of the time. You will find here and there however crucial passages that do require a *deep* insight in the peculiarities of that strange theory of Einstein. When you cannot figure it out, continue reading and a next time you bump into a string theorist ask him/her to explain this to you. No cause for embarrassment: invariably, members of this tribe love to explain these deep thoughts. For those who have lost it all together, or never learned it: before starting reading further, first delve into an introductory GR text book with a physics orientation. There are plenty of such books available which are excellent *e.g.* [6–9].

The very short section IV is dedicated to a discussion of the main uncertainty related to the application of AdS/CFT to condensed matter problems. Very deeply hard wired in the construction, there is a need for a huge number of degrees of freedom in the field theory in order to get away with the classical gravity in the bulk. More precisely, the gravitational dual becomes classical only when one is dealing with the large N limit of a *matrix* large N theory. Although the *vector* large N limit has some credibility in condensed matter physics, embodied by the slave theories, here we are dealing with theories of the QCD variety where one wants to see a large number of gauge colors. But this requirement is associated with the shortest distances: surely the chemistry of oxidized copper and so forth is not of this kind, but the central mystery of holography is whether it matters. In the best phenomenological tradition of condensed matter physics, it is widely believed among the string theorists that the so-called "UV independence" is in effect. AdS/CFT is suspected to be "generating functional" for the *structure* of the theories describing the highly collective long wavelength while only the numbers in this realm are sensitive to the specifics of the short distance physics. The unresolved mystery is whether "N" is also part of the UV independence.

From section V onwards, it is just a report on workings of this machinery in the context of real physics. There are now a lot more boxes because you can skip the derivations when you are just interested what this machinery delivers in your condensed matter reality. We will first highlight the astonishing success of the AdS/CFT correspondence in addressing the universal description of thermal matter: thermodynamics and hydrodynamics. This revolves around Schwarzschild black holes in AdS, and it is perhaps not so surprising that the highlights of classic black hole complementarity such as the Hawking radiation and Beckenstein black hole entropy are recovered. After all, these have played an important guiding role in the discovery of AdS/CFT. However, AdS/CFT takes this a step further by its capacity to describe phases of matter and phase transitions in terms of gravitationally stable space times, which destabilize at the transitions. This will be a prevalent theme in the later chapters, but we introduce the idea in section V with a particularly subtle and striking example: the "Hawking-Page" black hole game in AdS that dualizes in a surprising insight in the nature of confinement transitions in conformal Yang-Mills theories, as pointed out first by Witten. Another important lesson from this example is that the mysterious field theory stuff arising from the presence of black hole horizons in AdS had dealings with deconfined degrees of freedom. Perhaps the most stunning success of AdS/CFT is the way that the GR equations describing the jitters of space and time near the black hole horizon dualize into the Navier-Stokes equations describing the classical fluids at macroscopic times, that form in the boundary from the quantum critical states when temperature is finite. This is highlighted in section VI, where it is also discussed why such finite temperature quantum critical liquids have an entropy producing habit that approaches closely an upper bound determined by Planck's constant. The rise of AdS/CFT in empirical realms actually started by the prediction of the so-called minimal viscosity, a typical manifestation of this "Planckian dissipation", which was after the fact confirmed by measurements of the quark gluon plasma as created at the relativistic heavy ion collider, followed by observations in the unitary fermion gas of cold atom physics.

In section VII we turn to the core business: what has AdS/CFT to say about the physics of quantum matter, matter realized at zero temperature while the density is finite? The king of the hill in this landscape is undoubtedly the "Reissner-Nordstrom" strange metal which is introduced in section VII. This departs from the simplest, most natural gravitational object one can imagine that encodes for finite density in the field theory: the Reissner-Nordstrom black hole, a black hole carrying electrical charge as constructed by Reissner and Nordstrom shorty after the discovery of general relativity. Its extremal version is representative for the zero temperature limit of the boundary matter, and here one encounters a form of gravity with most spectacular consequence for its field theoretical dual. Its near horizon "AdS_2" geometry describes an emergent scale invariance tied to the finite density in the field theory, which only acts along the time direction. This strange metal state shows therefore a "local quantum criticality", of a kind that is strikingly similar to what the condensed matter experimentalists have been measuring since the late 1980's. Is this literal, or just a coincidence? This is perhaps more than anything else right now the hundred thousand dollar question in this business.

This "RN" strange metal is also characterized by a zero temperature entropy indicating that it is a very unstable "quantum frustrated" stuff. The good news is that this unstable nature also shows its face on the gravitational side. Upon lowering the temperature of the field theory, the RN black hole has a strong tendency to, in essence, un-collapse in a star like object! Such "stars" in turn represent the gravitational encoding of stable, orderly states of the finite density quantum matter. Generically, bosonic symmetry breaking transitions are encoded by fields that spontaneously switch on in the gravitational bulk. Although at present seemingly all forms of such conventional symmetry breaking have been identified, the fruit fly in this regard is the "holographic superconductor" as discussed in section VIII. This involves a Higgs field in the bulk that due to the strong curvature near the RN black hole decides to switch on in the form of "scalar hair", as an "atmosphere" surrounding the black hole. This "hair" dualizes in the boundary in the form of a meat-and-potato superconducting order parameter, which can even be used to construct Josephson junctions encoded in terms of colorful black hole "hair cuts". Intriguingly, looking at it with the tools of experimental condensed matter physics it could be easily confused with a conventional BCS superconductor. It is about fermion pairs that form at T_c, with a gap that is growing in a BCS fashion when temperature decreases, while even perfect Bogoliubov fermions appear at low temperatures. This is quite like what is observed in the best high T_c superconductors: could it be that here holographic superconductivity is at work? The big difference with the BCS variety is that holographic superconductivity is born from a non Fermi liquid strange metal state and we will discuss experimental strategies to observe the differences.

The RN type strange metals are the first true non-Fermi liquids with a real mathematical description that have been identified by humanity. The strong signals in this regard appeared in 2009, when AdS/CFT was mobilized to compute photoemission spectra. We were ourselves involved in this pursuit, being stunned when we discovered that in certain parameter regimes one does find Fermi surfaces, and even signs of marginal Fermi liquids and so forth. These "fermions" jump started the AdS/CMT pursuit in full, given the resemblances of the results with modern condensed matter experimentation. This is the subject matter of the last two sections. In Section IX we will start out discussing the early "probe fermion" results. However, soon thereafter it became clear that again the fundamental instability of the RN black hole is lurking around the corner: when the Fermi-surface gets seriously sturdy the probe fermions actually signal that the black hole wants to uncollapse in an "electron star". This is literally a star formed from a fermion gas much like the neutron star, with the difference that the fermions are now electrically charged. Astoundingly, when the star forms the matter in the boundary gives birth to a stable Fermi-liquid! This is again a precedent in theoretical physics. It has been understood for a long time that a Fermi-liquid is some form of cohesive, stable state of matter, much like the symmetry broken bosonic and classical states of matter. Different from the latter, however, it was entirely in the dark how the Fermi-liquid could emerge (like an order parameter) from a disordered state via a (quantum) phase transition. AdS/CFT supplies an answer: there is no difference of principle between the fermions and bosons on the gravitational side, since in both cases the "emergence of order" involves the un-collapse of the black hole in a star like object. On the field theory side, one finds that the first signs of the proximity to the Fermi-liquid instability appear in the form of a "relaxational Fermi-surface" which is quite similar to the relaxational peak seen in order parameter response functions approaching a symmetry breaking transition.

Still, the precise formulation of the electron star in AdS is still not quite completely settled. The simplest way to implement the star is in a Thomas-Fermi style fluid limit: the classic Tolman-Oppenheimer approach to describe neutron stars. It appears however that in the holographic context this limit is to a degree pathological. These stars need to be re-quantized and this is subject of active research that will be discussed in further detail in especially section X. Although this has become a fairly bulky text, it still represents only a selection of topics that are presently investigated using holography, where we just focussed on the best developed and in a way most elementary themes. We will finish in section XI with just a short overview of what else is looked at at the present time, with the explicit aim to guide the more ambitious readers into the research literature. The convention of the manual is in the Tab. I.

label	**physical meaning**
d	the space dimension of the field theory
D	the spacetime dimension of the field theory, thus $D = d + 1$
$D + 1$	the spacetime dimension of the gravity theory
r	AdS radius, $r = 0(r_h)$ is the horizon and $r = \infty$ is the boundary
z	(i) AdS radius, $z = \infty(z_h)$ is the horizon and $z = 0$ is the boundary; (ii) Lifshitz scaling
$x_i(i = 1, ..., D - 1)$	spatial coordinate of the field theory

TABLE I: The convention of the manual.

II. ADS/CFT: THE GRAND UNIFIED GAUGE/GRAVITY DUALITY, ALSO KNOWN AS HOLOGRAPHY

For many condensed matter physicists the concept of duality rings a bell. The simplest form is the familiar particle-wave duality of quantum mechanics, but in the present context the notion of field-theoretical "weak-strong" or "Kramers-Wannier" duality is on the foreground. This was discovered in the 1940's by Kramers and Wannier in the context of the statistical physics of the Ising model in 2 space dimensions, as the perfect incarnation of the Yin-Yang principle of eastern mystique in very precise mathematical physics terms. The Ising model has a low temperature ordered- and high temperature disordered phase. The long wavelength physics of the ordered phase can be enumerated in the language of the Landau effective theory governed by the order parameter and its fluctuations ("scalar ϕ^4"), while naively one thinks about the high temperature state as a entropy dominated, featureless affair. The crucial observation is however that the objects responsible for the destruction of the order are unique topological excitations (the domain walls), to find out that the high temperature state can be as well precisely enumerated as an ordered state of these "disorder fields". The high temperature state of 2D Ising is a condensate of domain walls, which is also described by an Ising model but now with a inverted coupling constant $J/(k_B T)$: the 2D Ising model is self-dual. Although such a powerful self-duality is rare, it appears that duality per se is ubiquitous at least for the simple bosonic field theories as of relevance to condensed matter physics. Especially in 2+1D quantum- or 3D classical systems "global-local" dualities are the rule. For instance the 3D Ising model is dual to Ising gauge theory. Another famous example is the superfluid-Bose Mott insulator duality in 2+1D: the superfluid is ruled by the spontaneous breaking of the global $U(1)$ symmetry associated with global number conservation. The quantum disordered superfluid turns out to be described as a relativistic "Higgs phase of $U(1)/U(1)$" superconductor. The "vortex pancake" particles which are proliferated destroying the superfluid Bose condense, but these have long range interactions which are precisely like the electromagnetic interactions between electrically charged particles. This is therefore a gauged system, and since it is relativistic it is just like the abelian Higgs model written down by Higgs in 1964. This carries the "massive vector bosons" and these are just coincident with the "holons" and "doublons" of the Bose Mott insulator.

Next to being conceptually very pleasing, such dualities can also be quite practical when one wants to compute things. Deep in the ordered state theorists have an easy life computing Goldstone bosons and rare "order violating" fluctuations like small closed loops of vortex-antivortex excitations ("weak coupling"). On the (quantum) disordered side one is dealing with a brew of vigorously fluctuating field configurations in the original "order representation". However, in the dual representation it is again a tranquil, orderly affair in terms of the vortex superconductor. All one needs is a "dictionary" translating the easy to compute properties on the dual side into the observable properties expressed in the original order degrees of freedom – this translation is typically available for simple field theories.

Above all, AdS/CFT has to be understood as such a weak-strong duality, taking however the notion much further. It relates quantum field theories of an extremely strongly interacting kind to a weakly interacting dual world that is actually no longer only quantum field theoretical, but also gravitational. Remarkably, it is achieving this goal by dualizing the renormalization group structure of the strongly interacting theory into a manifestly geometrical edifice, through an extra space dimension in the gravitational theory. This is in turn in perfect balance with the holographic principle of quantum gravity, descending from black hole physics. This insist that gravitational universes are less dense in information than quantum field theoretical worlds living in a flat space time, to the degree that the former can be encoded in a "holographic screen" with one less dimension supporting the field theory. The holograms one finds in theme parks are a very appropriate metaphor: the quantum field theory is quite like the interference fringes on the photographic plate, that reconstruct in a three dimensional image of some familiar object upon shining laser light through the plate: the gravitational place.

A very short history of string theory

Nowadays even high school students are aware that string theory wants to be the theory of everything. It appears to describe all the interactions in nature, including gravity. But during the first years after the birth of string theory this was not quite realized, even not by the inventors of string theory. In the late 1960s, the precursor of string theory was an attempt to formulate a model to describe the strong interaction physics as showed up in experiments at that time. This was superseded by the discovery of the standard model in the early 1970's, with as a highlight the theory quantum chromodynamics (QCD) that proved to be the correct description of the strong interactions. However, in the same era it was realized that a massless spin-2 particle exists in string theory which can be used to describe gravity. The "magical" connections between general relativity and quantum field theory as revealed by string theory have been a major motivation for this pursuit since then.
The most elementary object in string theory is no longer a point particle as in conventional quantum field theories but instead it departs from the quantum physics of a one-dimensional string with a tiny length, satisfying

fundamental symmetries like conformal invariance and general covariance. These occur both as open- and closed strings, where the former are associated with the standard model forces while the closed strings naturally describe gravitational interactions. To avoid that quantum fluctuations wreck the conformal invariance of the string world sheet (the stringy analogue of a worldline) the "bosonic" string theory carrying only bosonic excitations has to live in 26 dimensions. Given that one also wants to address the origin of fermions in nature, one needs to import supersymmetry and the supersymmetric version of string theory has to live in 10 dimensions. To get rid of the extra dimensions, the old device of Kaluza-Klein compactification was imported: one rolls up the extra dimensions to a small size with as added benefit that gauge forces and particles emerge as left overs from the purely geometrical theory. In order to account for the structure standard model particular ingenious ways of compactifying the extra dimensions in the form of Calabi-Yau manifolds were developed.

The further history of the subject is characterized by two periods of frantic activity. The so called "first string theory revolution" unfolded in the mid 1980's when this community was in the grip of the reductionist dream of finding a single unique theory of everything describing nature. It turns out that the quantum theory of strings is strongly constrained: just imposing some fundamental symmetry demands not much is possible. However, when the dust settled it turned out that the pursuit was suffering from the embarrassment of the riches. No less than five consistent superstring theories appeared to be consistent: the type I, type IIA, type IIB superstring theories, and $SO(32)$ and $E_8 \times E_8$ heterotic string theories. This depression came to an end due to a talk by Witten at a string conference that completely changed the general perspective on the meaning of string theory. He re-identified the existing string theories as the quantum mechanics of isolated excitations associated with semi-classical limits of a single underlying, all-encompassing theory: the "M-theory", where the "M" stands nowadays more than ever before for "mystery". This act of Witten embodied a major paradigm shift from the reductionist view towards the emergence perspective, In a way this second string revolution is emergence principle at best. It turns out that the different string theories by weak-strong ("S") dualities, and also the T-dualities which are special to strings, associated with the exchange of relative- to center of mass coordinates on compactified manifolds. These dualities between the different string theories, as well as the 11-dimensional $\mathcal{N} = 1$ supergravity, form a "web" encircling the M theory in the middle leading to the very unusual conclusion that M theory has to exist although it is nearly completely in the dark what this theory actually is. Witten himself believes that M-theory is not governed by the action principle, while nobody can tell what the controlling principle of this theory is instead.

With this new understanding of the real meaning of string theory, Polchinski discovered that it had to be extended by including extra non-perturbative objects. These $p-$branes are spatially extended objects in supergravity and superstring theories (D-branes), typically with higher dimensions, required to make the spectrum of superstring theories consistent. These in turn also shifted the physical interpretation of the theory. Open strings should end on the branes where their end point acts as sources of the gauge fields, while closed strings responsible for gravity van get "off the brane". This lead to the notion of braneworlds: our universe can be viewed as a three space-dimensional brane "floating" in the ten dimensional fundamental space-time of string theory. The gauge particles cannot get off the brane but the gravitons can This leaves room for the compactification radius of the extra dimensions to be "large" (like millimeters) and this should have as observable consequence that gravity is modified at short distances because the gravitons can explore the large dimensional universe. These "large extra dimensions" would also make it possible that mini-black holes are created at high energy collisions. Unfortunately there is no sign of these mini black holes yet at the large hadron collider of CERN. A final round in this development aimed at the very fundamental was the discovery of "flux stabilization". This is about the idea of Kaluza-Klein compactification. Although ver elegant, it was shot down by Einstein himself on the ground that it appeared to be impossible to find out a circumstance in GR such that the compactification radius of the compact dimension becomes stationary – in GR the circle either grows or shrinks in time indefinitely. It was early this century found out how to actually obtain stationary compact dimensions invoking fluxes that arise in string theory. Given that this can be done in a myriad of ways, this let to the idea of the "string landscape": the notion that the landscape of ground states/vacua of string theory could look like that of a glass with a nearly infinite number of equivalent potential, minima each corresponding with a different universe with different values for the natural constants. The anthropic principle was subsequently mobilized to explain why our universe is as it is: just to make a pleasant place for humans to live. This "string landscaping" continues to be very controversial until the present day.

The AdS/CFT becoming a big business.

At least according to the old masters of the art, the development of the string theory fundamentals finds itself at present in a log jam. Apparently M theory needs completely new principle, and the cosmological branch tends to loose itself in gross, inconsequential speculation. However, as a form of mathematics string theory has been very successful, at present enriching the live in pure mathematics departments. Perhaps AdS/CFT should be in first viewed as such a piece of mathematics, a gross generalization of the Fourier transformation with benefits in unexpected corners of empirical Science. Different from the string landscaping and so forth, one does not need to speculate: stick in a question in the machinery and it delivers a no-nonsense answer, whether it is desired or not. AdS/CFT is actually a direct outcome of the second string revolution which rests on the notion of the D-branes. As the epilogue of the second superstring revolution, Maldacena proposed in 1997 the conjecture that type IIB superstring theory in $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory in 3+1 dimensions. To understand the origin of Maldacena's idea, let us first inspect in more detail the nature of D-branes. As already mentioned in the above, D-branes are extended objects in type II superstring theories. D_p-branes in perturbative string theories can be defined as $p + 1$-space-time dimensional hypersurfaces where open strings can end. For example, D_0-branes are particle like solitons and D_1-branes are string like solitons in superstring theories. There are two ways to describe the dynamics of D-branes, either using open strings or closed strings. The excitations of open strings that end on the branes can describe the oscillations of the D-branes. Dealing with N coincident D-branes, the two end points of a single open string can reside on two different D-branes, and therefore the excitations of open strings carry two indices that run from one to N. Thus the low energy dynamics of this N D-branes system can be described by a $U(N)$ gauge field theory. From the other dual viewpoint, in the weak coupling limit of superstring theory (i.e. supergravity theory), D-branes carry energy densities and charges which can source curvature and flux, which can be viewed as coming from closed string modes. If we consider a set of N coincident D_3-branes the near horizon geometry that these branes source turns out to be $AdS_5 \times S^5$. On the other hand, as mentioned above the low energy dynamics on their worldvolume is governed by a U(N) gauge theory with $\mathcal{N} = 4$ supersymmetry.

From the equivalence of these two pictures, we can arrive at the conjecture [10] that type IIB superstring theory in $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory in 3+1 dimensions. Note that nearly invariably we are interested in the low energy regimes on both sides of the duality. In this low energy limit, the IIB superstring theory in $AdS_5 \times S^5$ reduces to $10D$ type IIB supergravity theory in $AdS_5 \times S^5$. We call it a conjecture because these two pictures of D-branes are perturbatively valid for different regimes in the space of possible coupling constants. Perturbative field theory is valid when $g_s N$ is small, while the low-energy gravitational description is perturbatively valid when the radius of curvature is much larger than the string scale, which turns out to imply that $g_s N$ should be very large. Thus AdS/CFT duality is a strong-weak duality in the sense that when the gravity description is valid the dual field theory is in a strong coupling regime that is generally not under control. It a conjectural since it is not possible to decypher the strong coupling regime on the field theory side. There are some interesting physical quantities that are independent of the coupling which can be computed on both sides. These were checked and shown to work out, supporting the strong conjecture. Since 1997 a large number of further nontrivial evidences supporting the conjecture have been found See for a detailed discussion of these checks ref. [11]: at present the strong conjecture has a similar status as, say, the path integral. Although the strict mathematical prove has still to be delivered, there is no doubt that it is true.

Another well studied example is AdS_4/CFT_3 [12] which underlies in the physics of multiple membranes: M theory in $AdS_4 \times S^7/Z_k$ is the same as three-dimensional $\mathcal{N} = 6$ supersymmetric Chern-Simons-matter theories (ABJM theory) with gauge group $U(N) \times U(N)$ and CS level k. One can also compactify the string theory on other, more complicated manifolds to reduce supersymmetry or add probe branes to introduce flavor degrees of freedom an so forth to obtain more interesting field theories and their corresponding gravity duality.

Maldacena discovered this structure as a particular limiting case of the more general open-closed string dualities of full string theory. Although mathematically still unproven, literally thousands of follow papers added further circumferential evidence that it has to be correct ("strong conjecture"). A portfolio of further generalizations of such so-called "top-down" correspondences were identified, including the Dp/Dq brane set ups and the consistent truncations of 11 (or 10) dimensional supergravity/Sasaki-Einstein compactifications of M theory (or string theory) which are at present quite relevant for the AdS/CMT pursuit. These all share the trait that the physics on the gravitational side is reasonably transparant: it is in first instance 10 dimensional classical supergravity on a special background manifold like $AdS_5 \times S^5$, and operationally this boils down to fanciful general relativity with added scalar-, fermion-, gauge- fields as well as branes (higher D gravitating manifolds) in overall stationary space times. However, the field theoretical duals are very remote from anything we have figured out in condensed matter systems. In the original Maldacene AdS/CFT these correspond with four dimensional maximally supersymmetric Yang-Mills gauge theories, where one has to take simultaneously the large N- and the large 't Hooft limit.

These are just generalizations of the *matrix* Yang-Mills theory describing the gluons of QCD, now also involving the supersymmetric fermionic partners of the gluons, for the case that one has not 3 different colors, but nearly an infinity (large N). The parameter that governs the effective strength of the interaction is the 't Hooft coupling, and the fact that this is also to be taken large is a-priori pleasant since this means that the field theory is in a strong coupling limit where one can look for non-perturbative physics. It is not all as bad as it might appear at first sight. Supersymmetry is not essential: top-downs have been identified where supersymmetry is explicitly broken, showing that with regard to typical condensed matter questions (associated with finite density) it makes little or no difference. One aspect one should keep in mind is that due to the "non-renormalization theorems" (cancellation of bosonic- and fermionic quantum radiative corrections) such supersymmetric theories have an attitude that they are always quantum critical/conformal regardless the coupling constants. Instead of the fine tuning to critical coupling constants as familiar from bosonic theories, the "superconformal" theories are quantum critical by default.

Even the large 't Hooft coupling requirement can be loosened to a degree: the string theoretical embedding is telling that one can get away from this limit by incorporating the so-called α' (string tension) corrections on the gravitational side, boiling down to adding higher derivative corrections to the Einstein-Hilbert action in the gravitational "bulk", something that can be done systematically at least to a degree. The killer is the large N limit in the field theory: to get away from this limit one has to study corrections associated with the finite string coupling constant, which is for a string theorists equivalent to be deported to hell. This is the way that serious quantum gravity is supposed to show up in string theory and despite an enormous effort still very little can be done. We will be after corners of the correspondence telling us about highly phenomenological theories describing "deep emergent" physics which might be rather independent of the specifics of the short distance physics. Even in these corners the large N limit seems to exert a rather undesired influence: most importantly, it imposes a mean field attitude to everything, even overruling the number of space dimensions.

Soon after the discovery of Maldacena, Gubser, Klebanov and Polyakov [13] and independently Witten [14] argued that the gauge-gravity duality can be formulated on very general grounds extending beyond the top-down constructions explicitly derived from string theory. They managed to formulate a presumably universal "dictionary": the rules translating the quantities of the boundary gauge theory into the the gravitational bulk and vice versa. This claim of a greater universality of gauge-gravity duality is called the "weak conjecture", since there is much less mathematical support for it than for the top downs. From the viewpoint of the mathematically inclined string theorist, this is where the relevancy of the present flirtation with condensed matter physics resides: the dream is that condensed matter experiment might be used as an analogue quantum computer to test the weak conjecture under circumstances where one does not know how to proceed mathematically.

This is also the subject matter as of primary interest for the condensed matter physicists. Much of the present AdS/CMT activity rests on the weak conjecture, in the form of the "bottom-up" approach. Our interest is in the behavior of an infinite number of strongly interacting quantum degrees of freedom, and "holography" appears as a "generating functional" that is supposedly extremely powerful in revealing the principles controlling "deep emergence" physics, translating it into phenomenological theories of a Landauesque quality. Surely matters are completely untractable on the field theory side, but the dictionary rules are amazingly constraining regarding the construction of the gravitational dual. This is magic: at least for equilibrium problems, one ends up studying minimalistic and very natural GR problems of a kind that especially excites the professional relativists. GR was flourishing in the 1960's-1970's by the discovery of black hole physics, in terms of Hawking radiation, no hair theorems and so forth. This has entered a second youth, due to the discovery of a number of novel black holes and other graviational structures, inspired by the condensed matter questions holographically translating into great homework assignments for the relativists.

The remainder of this tutorial will be dedicated to the cause of trying to get across some of this magic to the condensed matter physicist who might barely remember the course he/she took in GR as a starting graduate student. The focus will be entirely on the bottom up approaches dealing with equilibrium in the field theory. After refreshing elementary GR, at least to understand what is going on is relatively easy. Although the top-downs have the last word in this business, these are much harder to grasp than the bottom-ups since one needs a thorough understanding of string theory itself. String theory proper takes much more effort to learn, but actually quite a number of professional holographists no longer want to be identified with string theory. This should be a relief to the condensed matter reader: you can just get away ignoring string theory-proper, as long as you are willing to submit to the absolute authority of the top-down specialists when interpretational ambiguities in holography have to be scrutinized. After all, the top downs are themselves invariably ruled by the dictionary as well.

III. THE ADS/CFT CORRESPONDENCE AS COMPUTATIONAL DEVICE: THE DICTIONARY RULES

At present the applications of holography to condensed matter physics are focussed on circumstances where any control on the field theory side is lacking. There are two gross classes of many-particle quantum problems where the existing field theoretical methods fall short fundamentally. In the first place, except for the simplest free fields anything that is driven seriously out of equilibrium can be regarded as uncomputable. In fact, there are high hopes that in a near future AdS/CMT will be mobilized to make a big difference in this regard. The reason is the first dictionary rule:

Dictionary rule 1: the holographic dual of equilibrium in the boundary field theory is a gravitationally stationary spacetime in the bulk. Non-equilibrium time evolutions in the field theory correspond with non-stationary general relativity in the bulk.

This is good news, but it is also much more of a challenge in holography because non stationary GR is very hard, and little is known in general especially regarding the typical problems one encounters as the dual of meaningful questions associated with non-equilibrium in the field theory. Only very recently it became possible to numerically simulate serious time evolutions like black hole mergers. There are high hopes that much will be learned by combining numerical relativity and holography, but this lies still largely in the future. Except for the near-equilibrium Navier-Stokes hydrodynamics of section VI B, we will ignore this activity completely in the remainder.

With the limitation of stationary GR, there is still much to study since also equilibrium many particle quantum physics is littered with mysteries, especially so dealing with the behavior of compressible strongly interacting quantum matter at a finite density. Before we delve into this core business (sections VII-X) you should first learn to appreciate why the holographists have so much confidence in their trade. This is based on the experience with zero density quantum conformal states of matter where we do know enough at least qualitatively about the field theory to judge the correspondence. This is about stuffs which are subjected to scale invariance and (effective) Lorentz invariance of their quantum dynamics at zero temperature. To be precise, conformal symmetry in space-time also means that angles stay the same under scale transformations (the special conformal transformations). This seems however always to be coincident with condensed matter systems undergoing continuous quantum phase transitions at zero density. Modulo large N hassles, this is quite directly applicable to systems like graphene undergoing a Mott metal-insulator transition, the superfluid- to Bose Mott insulator transition "at zero chemical potential", or the antiferromagnet-incompressible singlet phase transition in a bilayer Heisenberg model. We assume that the readership has some familiarity with these simple examples of quantum criticality.

What follow is the introduction of AdS/CFT in several stages: first, you have to appreciate how the renormalization group transformations in the field theory are encoded in the anti-de-Sitter geometry of the gravitational bulk. This is followed by the dictionary entries telling how currents associated with conserved charges in the boundary field theory are dualized into gauge fields, and how fermions in the boundary turn into other fermions in the bulk and so forth. The highlight is to find out how the two-point propagators of the various currents as "fixed by kinematics" are encoded in terms of free photons and fermions on the gravitational side.

A. The scaling direction as an extra dimension

The point of departure is the outcome of Maldacena's top down: the field theory lives in a flat space time in d spacetime dimensions, and this can be viewed as the boundary of a "bulk" space with an Anti-de-Sitter (AdS) geometry with one extra dimension where the weakly coupled gravitational physics is at work. As we will see later, the requirement of an AdS geometry has to be fulfilled "near" the boundary and it can be violated in the interior of this AdS spacetime. The point of departure is in the demonstrate that the renormalization of the field theory as governed by conformal invariance can be geometrized in terms of a space-time with one extra dimension and an Anti-de-Sitter geometry, which is governed by general covariance. The extra "radial direction" connecting the boundary and the deep interior of AdS turns out to be coincident with the scaling direction of the field theory where the near boundary region is associated with short distances (UV), and the deep interior with the long distance (IR) physics of the field theory. To formulate the dictionary it is necessary that at short distances the field theory is conformal/quantum critical while the bulk has near its boundary an AdS geometry. At present there is a large research effort unfolding to find out whether one can generalize the holographic correspondence to other UV's, like the non-relativistic Schroedinger and Lifshitz cases. As we will see later, one has plenty of room to fool around in the interior of AdS, by plugging in black holes and so forth which change the IR physics of the field theory in something that is more interesting than the CFT.

Knowing the answer from Maldacena it is interesting to find out why the renormalization flow of the field theory can be "geometrized" in terms of the gravitational bulk theory. The point of departure of Einstein theory is the symmetry requirement of general covariance. Distances are measured in terms of the measure of length (metric) $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. The metric tensor $g_{\mu\nu}$ corresponds with a particular "gauge choice" associated with a preferred coordinate frame. The Einstein space time should be invariant under the choice of frame (general covariance, or diffeomorphism), while the isometries of the space-time are those choices of frames that describe the same overall geometry. Dealing with a dynamical space time, Einstein theory emerges as the minimal theory which is invariant under such transformations.

Let us now turn to the field theory. We assume that this is a non-gravitational affair and that it lives a flat, non-dynamical Minkowski space time,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + d\vec{x}^2. \tag{III.1}$$

so that it is subjected to *global* Lorentz transformation leading to a conservation of its energy and momentum. However, the field theory is subjected to renormalization. Departing from the central wisdom of the Wilson renormalization group (RG) we learn that by integrating out short distance degrees of freedom the coupling constants g of the theory change under scale transformations according to differential equations which are local in the RG scale u,

$$u\frac{\partial g(u)}{\partial u} = \beta(g(u)) \tag{III.2}$$

Right at the critical point where the physics becomes scale invariant the beta functions have to vanish, $\beta = 0$: the field theory becomes invariant under conformal transformations. Assuming also Lorentz invariance, this means that the scale transformation $x^\mu \to \lambda x^\mu$ is a symmetry ($\mu = 0$ is the time direction, $\mu = 1, \cdots, D-1$ are space directions).

The RG scale u is of course a non-geometrical entity that is just telling how the field behave when the scale is changed. However, what happens when we insist that it could as well be interpret it as an extra dimension, in a different description of the conformal field theory which lives in a $D+1$ dimensional space-time? This operation is conceptually (and mathematically [15, 16]) analogous to the standard path integral construction where one goes from the Hamiltonian language to the Lagriangian language where one transforms the quantum problem as described by operators in space, to the "holistic" view on worldhistories in space-time. Dimensional analysis insists that the RG (energy) scale ("scaling direction") u should transform as $u \to u/\lambda$ under the scale transformation.

One now discovers that the global scale transformations $x^\mu \to \lambda x^\mu$ of the field theory can be combined with those of the scaling direction into a $D+1$ space time with the metric,

$$ds^2 = \left(\frac{u}{L}\right)^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^2}{u^2} du^2. \tag{III.3}$$

In fact, these simple observations have a profound mathematical meaning: the transformations encoding the conformal symmetry of the field theory in D dimensions have acquired a geometrical meaning in D+1 dimensions:

$$x^\mu \to \lambda x^\mu \text{ and } u \to u/\lambda \tag{III.4}$$

are called the "isometries" in the D+1 dimensional AdS space since these leave ds^2 invariant. Moreover, the local isometry of (III.5) forms a group $SO(2, D)$ which matches exactly to the conformal symmetry of a d spacetime dimentional CFT.

This describes a spacetime with an anti-de-Sitter AdS_{D+1} geometry. It is a family of copies of Minkowski spaces parametrized by the "radial coordinate" u, whose size is shrinking when u decreases from the boundary at $u \to \infty$ (boundary) to the interior $u \to 0$ (Poincare horizon). See fig. 1, and notice that although this hyperbolic space is infinity large it still has an "outside". The free parameter L having the dimension of length is called the "AdS radius" and its meaning for the field theory will become clear later. It is also convenient to work in the metric by changing of coordinates $z = L^2/u$,

$$ds^2 = \left(\frac{L^2}{z^2}\right)\left(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2\right). \tag{III.5}$$

We find out that the conformal field theory has somehow relationship with a non trivial Einstein space-time. We would like now to write down a gravitational theory that fulfills the Landau criteria of satisfying the symmetry requirements (in this case general covariance) with a minimal number of derivatives. Surely, the most perfect theory in this regard, that also has the AdS space time as a solotion is the Einstein-Hilbert action with an added negative cosmological constant,

$$S = \frac{1}{2\kappa^2}\int d^{D+1}x\sqrt{-g}\Big(R - 2\Lambda + \cdots\Big) \tag{III.6}$$

where $g = \det g_{\mu\nu}$, R is the Ricci scalar obtained from the metric, while $-2\Lambda = D(D-1)/L^2$ is a negative cosmological constant, parametrized in terms of the AdS radius L and '$\cdots$' means that one can add some matter terms as well as higher derivative gravity terms. Since the observation of dark energy we seem to know that our universe is de Sitter (dS), just differing from AdS by the sign of the cosmological constant. Despite a large effort [17, 18], it is still unclear how holography works out in dS.

In natural units ($\hbar = c = 1$) Newton's constant $\kappa^2 = \ell_{\rm pl}^{d-1}$ where $\ell_{\rm pl}$ is the Planck length and we learn immediately that $L \gg \ell_{\rm pl}$ in order to avoid quantum gravity. In fact, one finds invariably in the top downs that the full string theory simplifies to Einstein gravity in the appropriate classical/weak coupling limit and string theory actually tells how to add corrections to the Einstein-Hilbert action when the curvature increases. According to the top downs,

$$\frac{L^{d-1}}{2\kappa^2} \sim N^2. \tag{III.7}$$

To keep the AdS radius away from the Planck scale so that one can get away with classical gravity in the bulk one has to make the number of colors (N) in the field theory very large.

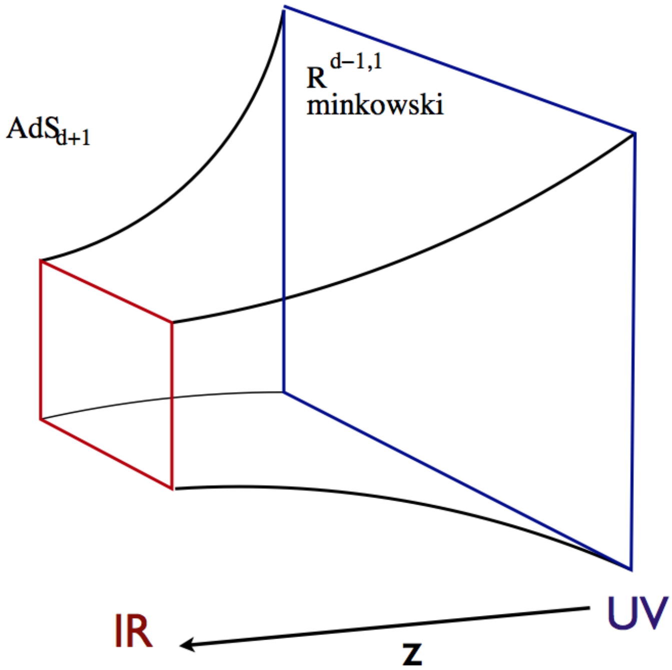

FIG. 1: The extra dimension $z = L^2/u$ in the bulk is the renormalisation group scale in the boundary field theory.

B. The minimalistic constructing of the grand unified duality

How to proceed? That one can draw a "geometrized" picture of renormalization flow is just an interesting metaphor, and it could be a mere coincidence that the AdS geometry associated with the marginal flow is also a solution of the Einstein equation. To render this serious, one does need the inputs of the string theoretical top-downs. Shortly after Maldacena's discovery, Gubser, Klebanov and Polyakov [13] and independently Witten realized a simple recipe that revolves around a single "mother" rule (the "GKPW rule") that makes the way that the physics on both sides of the duality can be translated into each other very precise. Although the rules are simple, the dictionary is also greatly counterintuitive and after all these years still seen as in many regards quite mysterious even by the professional holographists.

We will soon write down the dictionary rules. These are neat mathematical rules that have nearly the status of mathematical theorem: there is an abundance of evidence that these have to be true but the mathematical proof is lacking (the strong conjecture). In the standard texts dealing with the correspondence, one usually proceeds just posing these rules to then derive systematically bit by bit the relations between the physics in the gravitational- and field theoretical side. For the freshman holographist this might have the psychological disadvantage that this mathematics offers a hide, so that one can avoid permitting the freshman to stare from the greatly counterintuitive conceptual structure of this affair, becoming immensely beautiful when one gets used to the idea. We find it just fun to introduce the dictionary avoiding as much as possible the mathematical "shelter". Let us consider the following puzzle: it is given that the renormalization group can be "geometrized" such that a quantum critical state can be encoded in terms of classical gravity in an Anti-de-Sitter geometry. How to "string this together" with the form of the two point correlators in the conformal field theory which are known, being fixed by conformal- and Lorentz invariance? Knowing that such a relation has to exist, one gets quite far employing just common sense reasoning, although eventually one needs information from string theory to make it all work.

The point of departure is a quantity in the field theory which is known modulo a single free parameter: the two point correlation function which is completely determined by Lorentz- and Conformal invariance, with as only free parameter the scaling dimension of the operator. Consider two points apart by a distance x in Euclidean space time. Given a "bare" (UV) operator $\mathcal{O}_+(x)$ of the conformal field theory with conformal dimension Δ_+ the two point propagator will be,

$$\langle \mathcal{O}_+(x)\mathcal{O}_+(0)\rangle = \frac{2\nu\Gamma(\Delta_+)}{\pi^{D/2}\Gamma(\Delta_+ - \frac{D}{2})}\frac{1}{|x|^{2\Delta_+}}. \quad \text{(III.8)}$$

where D is the number of space-time dimensions, while $\Gamma(x)$ and ν are the Gamma function and $\nu = \Delta_+ - \frac{D}{2}$, respectively. This is likely familiar to the reader. In euclidean space-time, time is just equivalent to a space direction. mMore importantly, given scale invariance it has to be that any response becomes an algebraic function of distance. To familiarize this further, for experiments we are usually interested in the response as function of energy and momentum. After Fourier-transformation, and Wick-rotation from imaginary to real time, for time-like momentum $k^2 = -\omega^2 + \vec{k}^2 < 0$ we have

$$\langle \mathcal{O}_+(\vec{k},\omega)\mathcal{O}_+(0)\rangle_{\mathrm{R}} = 2\nu \frac{\Gamma(-\nu)}{\Gamma(\nu)}\left(\frac{ik}{2}\right)^{2\nu}. \tag{III.9}$$

These are the "branch-cut" propagators of the quantum critical state, as we know them in condensed matter physics [19]. Focussing on the spectral function $A(\vec{k},\omega)$ (the imaginary part of Eq. III.9) one finds at zero momentum $A \sim \omega^{2\nu}$ while at finite momentum there is no spectral weight until $\omega = ck$, while at higher energy the same power law tail is found as at zero momentum.

Knowing very little yet, let us focus on an ubiquitous operator in any relativistic field theory: the energy-momentum tensor $T_{\mu\nu}$. This is just associated with the fact that in a flat space-time energy and momentum are conserved, efficiently enumerated by $\partial_\mu T^{\mu\nu} = 0$. Its two point functions are at the heart of the relativistic hydrodynamics as we well see in a little while. One has to hassle with tensor indices, finding eventually,

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(y)\rangle = \frac{c_T}{s^{2d}} J_{\mu\alpha}(s)J_{\nu\beta}(s)P_{\alpha\beta,\rho\sigma} \tag{III.10}$$

where

$$s = x - y, \quad J_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{|x|^2}, \quad P_{\mu\nu,\rho\sigma} = \frac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}) - \frac{1}{D}\delta_{\mu\nu}\delta_{\rho\sigma} \tag{III.11}$$

and c_T is a constant. One notices that the conformal dimension of the energy-momentum tensor is fixed to be $\Delta_T = D$. This is just the requirement that the energy is subjected to "engineering scaling", as it is set by the volume.

The task is now to relate this propagator to things that can happen in the AdS space time. The operators in the CFT, like the energy-momentum tensor, are associated with the "bare" high energy degrees of freedom. According to the considerations of the previous section these are associated with going as far out as possible in the AdS space, and the following statement appears to be obvious

Dictionary rule 2: The spacetime of the field theory is coincident with the boundary of the gravitational AdS "bulk" space time.

Viewed from the gravity side, this is actually very special: although the AdS space is infinitely large in the sense that its radial coordinate can be send to infinite, it still has a boundary at radial infinity which we need in order to give the field theory a place to live. Our universe is de Sitter and this geometry lacks this property: there is a consensus emerging that such a space is non-holographic in the sense that it is impossible to dualize its physics in field theory on a holographic screen [20].

As a next step, let us try to use mere symmetry arguments to find out how to relate the properties of the boundary field theory in terms of the physics in the bulk. The most basic property of the boundary theory is the energy-momentum tensor. This is eventually a "Noether current" associated with *global* space- and time translations, yielding momentum- and energy conservation respectively. Similarly, the boundary field theory is also governed by *global* scale- and special conformal transformations. But in the previous section we showed with Eq. (III.4) that these global conformal transformation turn actually into isometries of the bulk geometry: these are the local coordinate transformations ("diffeomorphisms", "general covariance") that are at the heart of the structure of Einstein theory! This is a very deep feature of the correspondence:

Dictionary rule 3: AdS/CFT is a local-gobal duality. Global symmetry in the boundary field theory dualizes in gauged degrees of freedom in the bulk.

The geometry imposes in this regard also its will on internal symmetries. Later on we will see that one can address the physics associated with global internal symmetries in the boundary are encoded in gauge fields of the Yang-Mills kind in the bulk. This should be a very pleasing property for those condensed matter physicists who are familiar with the local-global dualities which appear to be ubiquitous for the simple field theories as of relevance to higher dimensional "bosonic" condensed matter physics, see for instance [21].

Turning back to the energy-momentum of the CFT, we understand that is a linear response quantity associated with infinitesimal field fluctuations on top of an interesting vacuum. Therefore, this propagator should relate to similar infinitesimal fluctuations on top of an interesting vacuum in the bulk but we have no clue yet by what theory the bulk is controlled. Surely, the theory should be invariant under general coordinate transformation, and the simplest theory of this kind we know is ... Einstein Hilbert gravity. Hesitantly, after asserting that we can get away with Einstein gravity which is a weaky coupled limit of the strongly interacting quantum gravity theory we can add a next rule,

Dictionary rule 4: AdS/CFT is a strong-weak duality, which is useful when the boundary field theory is strongly coupled and the bulk is weakly coupled.

On the field theory side there is plenty of room for large anomalous dimensions such that it is strongly coupled in the sense of the strongly interacting critical state of statistical physics which is found in between upper- and lower critical dimensions. Interpreting this in a quantum critical setting this also means that such a state is extremely quantal. The Einstein theory in the bulk is deep in the classical regime, while one should take care that one is always dealing with small gravitational forces, small curvature and so forth. We emphasize that to be sure in this regard one really needs the string theory embedding: on this level it is completely obscure that one needs also the large (matrix) N limit specifically of a Yang-Mills theory to get away with classical gravity. This aspect is encoded in the GKPW rule that is coming: this allows for the counting of thermodynamical degrees of freedom, and this gives away the role of large N.

Given the assertion that the bulk is literally governed by the Einstein Hilbert action Eq. (III.6), we are now in the position to do some real computations. Clearly, the AdS space time is the vacuum solution of the Einstein equation in the presence of a negative cosmological constant. But we also know what the infinitesimal fluctuations are of this nearly flat, classical space time: the gravitational waves! Write the full metric as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, and insist that the fluctuations $h_{\mu\nu}$ are small: the theory of linearized gravity follows where the gravitons appear as free fields. The assertion is therefore that the classical equations of motions describing the bulk gravitons somehow have to encode the information that allows to reconstruct the energy-momentum two-point propagators of the boundary field theory. As for gravity waves in flat space time, the solutions for the gravitons in AdS are quite unique so let us just compute what they are to see where this information pertaining to the boundary can reside.

In fact, the explicit computation for the gravitons is complicated by tensor-index hassles giving rise to the J and P factors in Eq. (III.10) which are usually not much liked by condensed matter physicists. Given that in other regards gravitons behave like free fields let us instead consider the propagation of free gauge fields in AdS, in a gauge fix such that they correspond with a real scalar field $\phi(x^\mu, z)$ adding a mass m as free parameter,

$$\mathcal{S} = -\frac{1}{2}\int d^{D+1}x\sqrt{-g}\Big(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2 + ...\Big) \tag{III.12}$$

Surely, gravitons are massless but let us keep m here explicit since it will turn out to play a very interesting and important role: it will turn out to encode the anomalous dimensions of the boundary operators. Vanishing m will encode for engineering dimensions.

The ϕ's have to be considered as governed by classical field theory in the curved AdS space time, with a metric $g_{\mu\nu}$ that follows directly from,

$$ds^2 = \frac{L^2}{z^2}\Big(-dt^2 + d\vec{x}^2 + dz^2\Big), \tag{III.13}$$

written here in Minkowski signature. In the remainder we will consider an Euclidean signature ($t \to i\tau$) since it is just save to Wick rotate back to real time at the end of the computation.

Let us go in some detail through the derivation: what follows is quite representative for the hard technical work associated with applying the correspondence to condensed matter physics.

The equation of motion (EOM) for ϕ is

$$(\nabla^\mu\nabla_\mu - m^2)\phi = 0. \tag{III.14}$$

where the covariant derivatives $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\sigma}V^\sigma$, in terms of the Christoffel connection $\Gamma^\nu{}_{\mu\sigma}$ which you might remember from the GR course. One now Fourier transforms to frequency-momentum space, but only with regard to the flat space-time shared by the boundary and the bulk, keeping the radial direction z (Eq. III.13) in "real space",

$$\phi(x^\mu, z) = \int \frac{d\omega d^{D-1}\vec{k}}{(2\pi)^D} f_k(z)e^{ik_\mu x^\mu} = \int \frac{d\omega d^{D-1}\vec{k}}{(2\pi)^D} f_k(z)e^{-i\omega t + i\vec{k}\cdot\vec{x}}, \tag{III.15}$$

and after some hard work one derives EOM's governing the dependence on the radial direction of the Fourier components f_k,

$$\frac{d^2 f_k}{dz^2} + (1-D)\frac{1}{z}\frac{df_k}{dz} - \big(k^2 + \frac{m^2L^2}{z^2}\big) f_k = 0 \tag{III.16}$$

where $k^2 = \omega^2 + \vec{k}^2$ (Euclidean signature).

One can now show that the general solutions are,

$$f_k(z) = a_1 z^{D/2} K_\nu(kz) + a_2 z^{D/2} I_\nu(kz) \tag{III.17}$$

where K and I are Bessel functions and $a_{1,2}$ are constants. In the interior of AdS ($z \to \infty$), the Bessel functions behave as

$$K_\nu(kz) \sim e^{-kz}, \quad I_\nu(kz) \sim e^{kz}. \tag{III.18}$$

Given that we departed from a second order differential equation we need two boundary conditions. The first one is set by the demand that ϕ is regular in the interior of AdS. It follows that $a_2 = 0$ and the solution is

$$f_k(z) = a_1 z^{D/2} K_\nu(kz). \tag{III.19}$$

The above example is for the case with Euclidean Green's function and we did the computation in the Euclidean space-time. One may also be interested to the case with retarded Green's function and one need to solve the wave function (III.14) in the real time AdS spacetime. For spacelike k, i.e. $k^2 = -\omega^2 + \vec{k}^2 < 0$, the independent solutions are the same as (III.17). Thus one need to demand that ϕ is regular in the interior of AdS. For timelike k, i.e. $k^2 = -\omega^2 + \vec{k}^2 < 0$, the independent solutions are

$$z^{D/2} K_{\pm\nu}(i\sqrt{\omega^2 - \vec{k}^2} z) \sim e^{\pm i\sqrt{\omega^2 - \vec{k}^2} z} \text{ at } z \to \infty. \tag{III.20}$$

Unlike the previous case, we can not impose regularity condition. Here we have two choices, one can impose either in-falling or out going boundary condition. They are related to time-reversal symmetry breaking and thus correspond to the calculation of real time retarded or advanced Green's function separately.

The hard work has been done. The solution can be expanded near the boundary ($z \to 0$), with the result

$$\begin{aligned} f_k(z) &\simeq \phi_0(k) z^{\Delta_-}\big(1 + \mathcal{O}(z)\big) + \phi_1(k) z^{\Delta_+}\big(1 + \mathcal{O}(z)\big), \quad (z \to 0) \\ \phi_0(k) &= a_1 2^{-1+\nu} k^{-\nu} \Gamma(\nu), \quad \phi_1(k) = a_1 2^{-1-\nu} k^{\nu} \Gamma(-\nu). \end{aligned} \tag{III.21}$$

The exponents are given by

$$\Delta_\pm = \frac{D}{2} \pm \nu, \qquad \text{where } \nu = \sqrt{(D/2)^2 + m^2 L^2}. \tag{III.22}$$

The exponents should be better be real, which implies the Breitenlohner-Freedman (BF) bound condition [22].

$$m^2 L^2 \geq -\frac{D^2}{4}. \tag{III.23}$$

This is a special property of the AdS space time. As long as this bound is satisfied the AdS space itself is stable in the presence of the massive field ϕ. When $m^2L^2 < -D^2/4$, the complex exponents signals a linear instability ("tachyon" in the string theory) of this space time; the field will acquire a finite amplitude which will in turn backreact on the geometry of the space time. In the section dealing with holographic superconductors we will learn that BF bound violations in the presence of charged black holes are the way to gravitationally encode spontaneous symmetry breaking in the boundary field theory. As will become clear in the next paragraph, in pure AdS the BF bound is encoding for the unitarity limit ($\Delta \geq (D-2)/2$ for scalar) of the CFT.

The hard gravitational work is done — the above solutions are just generic for classical free fields in the bulk, including gravitons as special cases. But where is the information as of relevance to the boundary field theory encoded in this gravitational story? We are of course prejudiced that the renormalization is governed by local differential equations: one only needs to know how to adjacent "scaling slices" talk to each other. We also observe that Einstein theory is governed by local differential equations along the scaling directions, in explicit form Eq. (III.16). Therefore, it has to be that the information is "transferred" from the bulk to the boundary field theory somehow "near" the boundary. Let us now inspect Eq. (III.21) which enumerates completely all the gravitational "knowledge" in this part of the AdS geometry. Although the field theory lives "at" the boundary ($z = 0$) the gravity theory becomes useless here because the $f_k(z)$'s all vanish.

We have arrived at a first vista exposing the splendid weirdness of the correspondence in its full glory. Although there is no "contact" at the boundary, near the boundary there is very interesting information available in the universal asymptote Eq. (III.21). Although the boundary field theory should not contain information regarding the "unphysical" scaling direction z we infer that the two z polynomials contain prefactors $\phi_0(k), \phi_1(k)$ having interesting conformal-like (k^ν) dependence on the space-time of the field theory. We know the answer already in the CFT, in the form of the propagator Eq. (III.9) and let us just try to see whether there is any way to reconstruct that using the gravitational $\phi's$. Given that the integration constant a_1 is undetermined, the simplest possibility is

$$\frac{\phi_1(k)}{\phi_0(k)} = \frac{\Gamma(-\nu)}{\Gamma(\nu)} \left(\frac{k}{2}\right)^{2\nu}. \tag{III.24}$$

Astonishingly, this works: it is coincident with the two point Euclidean conformal filed $\mathcal{O}_+$ with conformal dimension $\Delta_+ = \frac{D}{2} + \nu$ in momentum space Eq. (III.9)! Seemingly by magic we have discovered a mathematical dictionary rule

Dictionary rule 5 or the first mathematical "dictionary entry 1". The two point propagator of the field theory for conformal operators with scaling dimension $\Delta_+ = \frac{D}{2} + \nu$ can be expressed in terms of the ratio of the prefactors ϕ_0, ϕ_1 of the leading and subleading near boundary asymptotes of the AdS waves with a universal radial $z^{\Delta_\pm}$ fall off in the near boundary region with exponents $\Delta_\pm = \frac{D}{2} \pm \nu$:

$$\langle \mathcal{O}_+(x)\mathcal{O}_+(0)\rangle = 2\nu \frac{\phi_1(x)}{\phi_0(x)} \tag{III.25}$$

Surely this is still an improvisation, which is suggestive but not conclusive. Next to the stunning fact that the CFT information is encoded in the near boundary asymptotes of a classical field in the bulk, we actually did put in by hand that the bulk field has a mass, that turns out to determine the *scaling dimension* of the boundary operator. There is actually no reason for real worry because we will see in a moment that Eq. (III.25) is a first of a portfolio of dictionary entries that follow from the general GKPW rule, while the mass-scaling dimension relation is one of the general highlights of the correspondence. Before we turn to the discussion of the general structure of the dictionary we can actually push our improvisation a bit further. Is there anything more we can say based on knowing things in the boundary field theory and the bulk in general, regarding this mass-scaling dimension relation?

Let us first get back to the energy-momentum correlator $\mathcal{O} = T_{\mu\nu}$. We can assure that after repeating the above procedure for the gravitons, using Eq. (III.25) as working horse, one recovers precisely Eq. (III.10). The only essential difference is that we actually know the mass of the gravitons: by very general principle these are massless. Inserting $m = 0$ in the equation for Δ_+ we find out that the scaling dimension of energy momentum equals D, the correct answer!

A final sanity check is by inspecting the meaning of the BF bound for the field theory. In unitary conformal field theories, there is a lower bound on the conformal dimension Δ of fields, e.g. for scalar fields it is $\Delta \geq (D-2)/2$ which is the dimension of a free scalar field. For $\nu \in (0, 1]$, both modes ϕ_0 and ϕ_1 are normalizable. We can choose either ϕ_1 or ϕ_0 as the source and treat the other as the corresponding response, and the dual operators are $\mathcal{O}_-$ and $\mathcal{O}_+$ respectively which have the conformal dimension $D/2 - \nu$ and $D/2 + \nu$ (*i.e.* alternative and standard quantization). For $\nu > 1$, only ϕ_1 is normalizable. We can only take ϕ_0 as the source and treat ϕ_1 as the corresponding response. The dual operator is $\mathcal{O}_+$ with conformal dimension $D/2 + \nu$ (standard quantization). We always have the conformal dimension of the dual operator satisfying the unitary bound.

C. Getting serious: the GPKW master rule.

What is behind the miracle of the previous paragraphs, discovered by mere observation? Resting on the Maldacene discovery, Witten and independently Gubser, Klebanov and Polyakov deduced the general rule relating the bulk and the boundary in 1998. As everything else in this business it has still a conjectural status. The story in the above is surely part of the intuition behind this educated guess, but a lot more support follows from string theory. We refer the truly courageous student to some of the excellent reviews, and let us just state the "GPKW" rule as coming from a divine authority.

For condensed matter physicists, it is very familiar that the partition sum Z of a stable state in the thermodynamic limit is related to the free energy $\mathcal{F}$ by $Z = \exp\{-\beta F\}$. The essence of the GKPW rule is,

$$Z_{\mathrm{CFT}} = e^{-S^0_{\mathrm{AdS}}|_{z=0}} \tag{III.26}$$

Z_{CFT} is the full quantum partition sum or "vacuum amplitude" of the field theory. The surprise is that the free energy of the CFT is now coincident with $S^0_{\mathrm{AdS}}|_{z=0}$ the action of the gravitational theory "at the saddle-point": just the least action associated with the classical limit, evaluated at the boundary. Notice here a subtety: the bulk action is vanishing at the saddle point but the AdS space has an open boundary. One has therefore to extract a boundary term which will be finite when the bulk action is vanishing: this contribution sets the free energy of the boundary field theory. Here we will only address the equilibrium physics of the CFT and an implication of this rule is that this corresponds with *stationary* solutions of the bulk problem. Stationary "universes" are quite unnatural in GR in general but the AdS asymptotics is in this regard quite instrumental since it tends to help stationary solutions. Dealing with equilibrium, one can just rest on thermal field theory in Euclidean space-time: the familiar imaginary time axis and the Matsubara frequencies, both in the bulk and the boundary since they "share" the time direction. In the field theory finite temperature is encoded by rolling up the time dimension in a circle with radius $R_\tau = \hbar/k_B T$. But according to Eq. (III.26) this is somehow encoded in the bulk in terms of completely "mechanical" classical gravity! As we will elaborate in section V, the bottomline is that one inserts a conventional (Schwarzschild) black hole in the interior of the bulk, having the effect that the boundary theory is automatically heated to a temperature which is coincident with the Hawking temperature of the same black hole living in the quantized vacuum of the classic Hawking exercise. Eq. (III.26) turns the notion of "black hole complementarity", that the black hole can be either viewed as a lump made from space-time fabric or as a material object having temperature and entropy, into a simple calculational device!

According to linear response theory, the dynamical susceptibilities giving the information on the excitations living in the undisturbed system are part of the equilibrium agenda. When one knows how to compute partition sums, the computation of the linear response propagators becomes a variation on the theme by using the device of generating functionals. One perturbs the system with an external field ("source") $J_I(x)$'s coupling to a local operator $\mathcal{O}_I(x)$ of the field theory ("response"), by adding a term to the Langrangian of the field theory,

$$\mathcal{L}(x) \to \mathcal{L}(x) + \sum_I J_I(x)\mathcal{O}_I(x) \tag{III.27}$$

In full generality, any n-point function can then be computed by varying the log of the partition sum including the source term, taking the limit of vanishing source strength:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)...\mathcal{O}_n(x_n)\rangle = \prod_n \frac{\delta}{\delta J_n(x_n)} \ln \mathcal{Z}_{\mathrm{QFT}}|_{J=0} \tag{III.28}$$

How to incorporate this recipe on the right hand side of Eq.(III.26)? This is a greatly subtle affair. One would perhaps be inclined to just switch on the source everywhere in the bulk, coupling to some appropriate bulk field corresponding with $\mathcal{O}_I(x)$, like the graviton when the interest is in the energy-momentum operators of the CFT. But this is wrong! Gubser-Klebanov-Polyakov-Witten[13, 14] discovered the right procedure which is amazingly counterintuitive: the sources act just as boundary conditions at the AdS boundary itself as required to solve the equations of motions for the classical fields propagating in the bulk! In mathematical form:

$$\mathcal{Z}_{\mathrm{QFT}}[J] = \mathcal{Z}_{\mathrm{gravity}}[\text{boundary value } J]. \tag{III.29}$$

More explicitly,

The celebrated "GKPW rule", the working horse of AdS/CFT

$$\langle e^{\int d^dx J(x)\mathcal{O}(x)}\rangle_{\mathbf{QFT}} = e^{-S_{\mathbf{bulk}}[\phi(x,z)|_{z=0}\to J(x)]}. \tag{III.30}$$

Thinking this through one realizes that this is doing justice to the notion that the renormalization group of the CFT is "geometrized" by the AdS dynamics. The operators $\mathcal{O}_I(x)$ are the bare operators of the theory that have a meaning in the deep UV, and one wants to know their fate in the IR by tracking the energy dependence of the associated propagator. Think photoemission: this would be the literal electron injected in the interacting electron system from the outside. In the solid the electron is just the electron at very short times since there was no time yet for it to realize that there are other electrons to interact with (deep UV). But when time evolves the electron gets increasingly dressed, and after a very long time it has "fallen apart" in the true, highly collective low energy excitations associated with the strongly interacting vacuum. When these are the quasi-electrons of the Fermi liquid, these will show up as tiny but very sharp peaks in the bare electron spectral function at very low energy: the celebrated "quasiparticle peaks" seen routinely in strongly interacting Fermi-liquids by photoemission. In section IX we will find out that such quasiparticles poles can quite literally show up in holographic computations.

Paying tribute to the idea that the radial direction in AdS corresponds with the scaling direction in the CFT, it has to be that the information of the bare UV operators should be "transferred" to the bulk right *at* the boundary. But this can only be the case when it acts as mere boundary conditions for the dynamics in the bulk at the AdS boundary! This is the deep insight underlying the GKPW rule. Turning to the improvisation of the previous section aimed at the two point propagators it was quite confusing that the CFT needed the information of the bulk fields not *at* the boundary but instead in the form of the near boundary asymptote, finding that the prefactors ϕ_0, ϕ_1 of the two radial components have to be associated with the CFT propagator Eq.(III.25). This confusion is resolved by the GKPW rule as we will now demonstrate.

Correlation functions in AdS/CFT

Using the GKPW rule for correlation functions we can derive the result for the correlation function postulated earlier in Eq.(III.25). GKPW instructs us to consider the *on-shell* action with the boundary value of the field equal to the source in the dual field theory. For the simple scalar theory the action can be written in terms of a "bulk" and "boundary" contribution as,

$$S = -\frac{1}{2}\int_{\text{AdS}} d^{D+1}x\sqrt{-g}\phi(\Box + m^2)\phi - \frac{1}{2}\oint_{\partial\text{AdS}} d^Dx\sqrt{-h}\phi\partial_n\phi \tag{III.31}$$

where h is the determinant of the induced metric and the bulk part is used to derive the equations of motion. On-shell, i.e. when we substitute for ϕ a solution for to the equation of motion, the "bulk" term integrated over the whole of the AdS space vanishes. We already learned that this solution — i.e. with appropriate boundary conditions in the interior — has a universal asymptotic behavior near the boundary $z = 0$ as,

$$\phi_{\text{sol}}(\omega, k, z) = A(\omega, k)z^{\Delta_-} + B(\omega, k)z^{\Delta_+} + \dots. \tag{III.32}$$

According to GKPW the boundary value of $\phi(\omega, k, z)$ is proportional to the source J. A priori we are now facing a problem since Δ_- will be generically negative and thus the boundary value of ϕ is not well defined. However, we know what the meaning of this divergence is in the boundary field theory. Approaching the boundary is like increasing the renormalization scale to infinity and here one typically encounters UV divergences. In other words, the theory has to be regulated and this can be done in a particularly elegant way using the bulk language. One needs to subtract a counter term. GKPW proposed that one should compute at an infinitesimal distance $z = \epsilon$ away from the formal boundary and then take an appropriate limit $\epsilon \to 0$. Doing so the "regulated" on-shell action equals

$$S_{\text{on-shell}}(\epsilon) = \frac{1}{2}\oint_{z=\epsilon} \frac{d\omega d^{D-1}k}{(2\pi)^D} z^{-D+1}\left(\Delta_- A^2 z^{2\Delta_- -1} + (\Delta_- + \Delta_+)ABz^{\Delta_- + \Delta_+ -1} + \dots\right). \tag{III.33}$$

The first term is formally divergent. The pleasant circumstance is that in the bulk regularization is very easy. The key is that by adding an arbitrary boundary term to the action this never changes the equation of motion. However such an extra boundary term can be used to remove by hand remove the UV divergence. Adding a boundary counterterm of the form

$$\begin{aligned} S_{\text{counter}}(\epsilon) &= -\frac{1}{2}\Delta_- \oint_{z=\epsilon} \frac{d\omega d^{D-1}k}{(2\pi)^D}\sqrt{-h}\phi^2 \\ &= -\frac{1}{2}\Delta_- \oint_{z=\epsilon} \frac{d\omega d^{D-1}k}{(2\pi)^D} z^{-D}\left(A^2 z^{2\Delta_-} + 2ABz^{\Delta_+ + \Delta_-} + \dots\right) \end{aligned} \tag{III.34}$$

yields in combination with Eq. (X.24),

$$S_{\text{on-shell}}(\epsilon) + S_{\text{counter}}(\epsilon) = \frac{1}{2}\oint_{z=\epsilon} \frac{d\omega d^{D-1}k}{(2\pi)^D} z^{-D}\left((D - 2\Delta_-)ABz^D + \dots\right). \tag{III.35}$$

This is now all finite. We can now equate the leading behavior (A coefficient) of the ϕ_{sol} with the source J. Given that the above should coincide with the combination $iJ\langle O\rangle$ in the field theory, by taking the single derivative with respect to J this yields the expectation value $\langle O\rangle$ of the field theory operator sourced by J in AdS/CFT in the presence of the source. It is given by

$$\langle \mathcal{O}(\omega, k)\rangle_J = 2\nu B(\omega, k) \tag{III.36}$$

where we used $\Delta_\pm = \frac{D}{2} \pm \nu$. *Thus we see that equating the leading near-boundary behavior $A(\omega, k)$ of the solution ϕ_{sol} with the source implies that the subleading near-boundary behavior $B(\omega, k)$ is the corresponding response.* One can similarly obtain the two-point correlation function by taking an additional derivative and setting J to vanish. Linear response theory tells us already that $B(\omega, k)$ ought to be proportional to $A(\omega, k)$ and the proportionality is precisely the CFT Green's function,

$$\langle \mathcal{O}(-\omega, -k)\mathcal{O}(\omega, k)\rangle = 2\nu \frac{B(\omega, k)}{A(\omega, k)} \tag{III.37}$$

and we have demonstrated that the propagator rule Eq.(III.25) is indeed a consequence of the fundamental GKPW rule.

One can actually check that the GKPW rule encodes linear response theory by itself in a correct fashion. For this one has to realize that one in essence just needs to solve a simple Dirichlet boundary value problem. The equation of motion for ϕ has two independent solutions. Let us denote them ϕ_B with boundary behavior $\phi_B(z) = Bz^{\Delta_+}(1 + \sum_n c_n z^n)$. This is the appropriate Dirichlet solution with $A = 0$ that vanishes at the boundary, and $\phi_{int}(z)$ with boundary behavior determined by continuity in the interior of AdS. The Dirichlet AdS Green's function obeying $\lim_{z\to 0} \mathcal{G}^{\text{AdS}} = 0$ thus equals

$$\mathcal{G}^{\text{AdS}}(z, z') = \frac{\phi_B(z)\phi_{int}(z')\theta(z - z') + \phi_{int}(z)\phi_B(z')\theta(z' - z)}{\phi_{int}\partial\phi_B - \phi_B\partial\phi_{int}}. \tag{III.38}$$

The Wronskian in the denominator assures the correct normalization and is independent of z, i.e. it may be evaluated for any preferred z. Then for a boundary source $J(\omega, k)$ the solution to the equation of motion is

$$\begin{aligned}\phi_{\text{sol}}(\omega_1, k_1, z) &= \lim_{\epsilon\to 0} \oint_{z'=\epsilon} \frac{d\omega d^{D-1}k}{(2\pi)^D} \partial_{z'} \mathcal{G}(z, \omega_1, k_1; z', \omega, k) J(\omega, k) \\ &= \lim_{\epsilon\to 0} \int \frac{d\omega d^{D-1}k}{(2\pi)^D} \frac{\partial\phi_B(\epsilon)\phi_{int}(z)}{\phi_{int}\partial\phi_B - \phi_B\partial\phi_{int}} J(\omega, k).\end{aligned} \tag{III.39}$$

By construction this obeys $\lim_{z\to 0} \phi_{\text{sol}}(\omega, k, z) = J(\omega, k)$, which can be seen by noting that the Wronskian reduces to $\phi_{int}\partial\phi_B$ for $z \to 0$. The normal derivative of the solution follows straightforwardly

$$\begin{aligned}\partial_z\phi_{\text{sol}}(\omega_1, k_1, z) &= \lim_{\epsilon\to 0} \oint_{z'=\epsilon} \frac{d\omega d^{D-1}k}{(2\pi)^D} \partial_z\partial'_z \mathcal{G}(z, z') J(\omega, k) \\ &= \lim_{\epsilon\to 0} \int \frac{d\omega d^{D-1}k}{(2\pi)^D} \frac{\partial\phi_B(\epsilon)\partial\phi_{int}(z)}{\phi_{int}(\epsilon)\partial\phi_B(\epsilon)} J(\omega, k) = \lim_{\epsilon\to 0} \int \frac{d\omega d^{D-1}k}{(2\pi)^D} \frac{\partial\phi_{int}(z)}{\phi_{int}(\epsilon)} J(\omega, k)\end{aligned} \tag{III.40}$$

Substituting this into the action one finds

$$\begin{aligned}S_{\text{on-shell}} + S_{\text{counter}} &= \lim_{z\to 0} \left(\frac{1}{2} \int d^D x z^{-D+1} \phi_{\text{sol}}\partial_z\phi_{\text{sol}} - \frac{1}{2}\Delta_- \int d^D x z^{-D} \phi_{\text{sol}}^2 \right) \\ &= \lim_{\epsilon\to 0} \int \frac{d\omega d^{D-1}k}{(2\pi)^D} \left(\epsilon^{-D+1} \frac{1}{2} J(-\omega, -k) \frac{\partial\phi_{int}(\epsilon)}{\phi_{int}(\epsilon)} J(\omega, k) \;-\; \frac{\Delta_-}{2} \epsilon^{-D} J(-\omega, -k) J(\omega, k) \right)\end{aligned} \tag{III.41}$$

Near the boundary the solution ϕ_{int} has again the generic behavior $\phi_{int} = A_{int}(\omega, k)z^{\Delta_-} + B_{int}(\omega, k)z^{\Delta_+}$ and by taking two derivatives w.r.t. the source J one finds

$$\begin{aligned}\langle \mathcal{O}(-\omega, -k)\mathcal{O}(\omega, k)\rangle &= \lim_{\epsilon\to 0} \epsilon^{-D+1} \frac{\partial\phi_{int}(\epsilon)}{\phi_{int}(\epsilon)} - \frac{1}{2}\Delta_- \\ &= \epsilon^{-D+1} \frac{\partial_\epsilon(\epsilon^{-\Delta_-}\phi_{int}(\epsilon))}{\phi_{int}(\epsilon)} = 2\nu \frac{B(\omega, k)}{A(\omega, k)}.\end{aligned} \tag{III.42}$$

Thus one is left with the same linear response answer (III.37).

To complete this exposition of the "maximally bottom-up" zero temperature CFT dictionary, we still need to address the fate of other quantities than the energy-momentum tensor that might be part of the physics of the CFT. One is dealing according to the top downs with a Yang-Mills style gauge theory, and the only observables that make sense in the CFT are gauge singlets. These are associated with *global* internal symmetries ("flavor symmetries" in the high energy language) which are quite familiar to condensed matter physicists. The most primitive one is a "global $U(1)$" associated with stuff that has a conserved overall density or "charge" (like the electrical charge). By applying a chemical potential this can be forced to a finite density and this is the main theme of the condensed matter applications discussed in section VII. One can generalize this to non-abelian conserved charges, roughly corresponding with the study of spin-densities and currents in the condensed matter context. One should consider as well currents associated with Cooper pairs that can drive superfluid transitions (holographic superconductivity), while even the single fermion propagators of photoemission have a natural habitat in conformal field theories. We learned already that the energy momentum tensor associated with the *global* space-time and conformal symmetries in the boundary field theory translate via the symmetry-isometry principle in the gravitons of the gauged space time of the bulk. This local-global duality turns out to be generic also for the internal symmetries: the global degrees of freedom in the boundary are invariably dual to gauge fields in the bulk. The current response associated with the global $U(1)$ charge in the boundary corresponds with gauged $U(1)$ fields in the bulk. Operationally, to compute the behavior of the (optical) conductivity of conserved $U(1)$ stuff in the boundary one has to evaluate literally the propagation of Maxwell electromagnetic fields in the AdS bulk. In fact, the symmetry correspondence global $U(1)$ in the boundary – local $U(1)$ in the bulk is generic. Due to the same strong-weak duality principle that gave us the right to consider the simplest classical field theory in the bulk (GR), one can get away with the minimal classical gauge theory for the internal symmetries: just classical Einstein-Maxwell and its non-abelian (classical Yang-Mills) generalizations. As we will see in section VIII and IX, at finite density one has to deal with "material" fields like the pair field associated with holographic superconductivity and the fermions that give away the existence of Fermi-liquids. These turn out to be represented in the bulk by *gauged* Higgs fields and gauged fermion fields, respectively.

One can actually directly infer from the GKPW rule that this local-global duality principle has to be at work. Starting from the operator $\mathcal{O}_\mu(x)$ representing a gauge singlet, conserved $U(1)$ current in the boundary, this enters the bulk description as a condition at the boundary $A_\mu = J_\mu$ where J_μ is the source for $\mathcal{O}_\mu(x)$, acting on a classical field A_μ in the bulk. One can now demonstrate that this condition is invariant under $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(x)$ with Λ being an arbitrary function of x: this is just another manifestation of the freedom to add arbitrary boundary terms to the bulk action! This implies that the bulk field has to be a $U(1)$ gauge field. Surely in the top down setting the reasoning becomes even more beautiful. The internal symmetries arise from the Kaluza-Klein compactification of five of the ten dimensions of the string theory on the S^5, with the other five dimensions form the AdS_5 space time. The isometries act out on in the same way on the five compact dimensions as they do in AdS itself, implying that the Kaluza-Klein fields have to be gauge fields.

We are now in the position to present the dictionary in the form of a table, II. Anticipating the outcomes for finite temperatures as discussed in section V, we already include the most important entries in this regard as well. The amateur holographist should always have this "dictionary table" at hand since it is too easy to get some of the "words" of the bulk language wrong ...

boundary: gauge (operator)	**bulk: gravity (field)**
energy momentum tensor $T^{\mu\nu}$	metric field g_{ab}
global current J^μ	Maxwell field A_a
scalar operator $\mathcal{O}_\mathrm{b}$	scalar field ϕ
fermionic operator $\mathcal{O}_\mathrm{f}$	Dirac field ψ
global symmetry	local isometry
spin/charge of the operator	spin/charge of the field
conformal dimension of the operator	mass of the field
source of the operator	boundary value of the field
VEV of the operator	boundary value of radial momentum of the field
temperature	Hawking temperature
chemical potential/charge density	boundary values of the gauge potential
phase transition	Instability of black holes

TABLE II: The mini-dictionary for AdS/CFT correspondance.

IV. THE HOLOGRAPHIST' NIGHTMARE: THE LARGE N DELUSION

In order to appreciate the remainder of this exposition, the moment has arrived to discuss the big headache associated with applying the correspondence to real life condensed matter problems. This is called the large N limit. It is at present not clear how damaging it is. There is a hope that it should be possible to filter out its malignant effects, ending up with universal wisdoms that do have relevance to the strongly interacting electrons of condensed matter. In the string theory jargon this is referred to as “UV independence”. This is pretty much the same thing as Landau’s concept of phenomenological theory: although the numbers cannot be computed, the structure of the theory describing the physics of the thermodynamic limit is fixed by universal emergence principle, while the specifics of the microscopic degrees of freedom do not matter.

There is quite some evidence for UV independence at work: in the next section you will see that the great theories of finite temperature matter (thermo- and hydrodynamics) are perfectly encoded in generic AdS settings. Even the numbers are correct by order of magnitude for quantum-critical states: the only UV dependence enters through the very general property of scale invariance in the quantum UV. The issue is that this is quite unsettled in the context of the recent developments studying finite density, zero temperature “quantum matter” states. The correspondence throws up new field-theoretical phenomena like the AdS_2 metals and holographic superconductors that cannot be checked directly in the field theory since apparently the Fermion-sign problem is taking the role of the proverbial brick wall. Are these exciting insights actually artefacts of the large N UV, or are they giving away new universal emergence principles for quantum matter, so desired by condensed matter physicists? The jury is still out, and let us therefore give the readership some idea what this horrid (for condensed matter physics) large N limit is about.

The symmetry considerations behind AdS/CFT which we emphasized much in the previous section are believed to be very general. However, to do computations we assumed that we could get away with the simplest weakly coupled theory to describe the bulk: classical Einstein-Maxwell theory and extensions there-off. But here is the sore spot: from the string theoretical embedding one learns that the general dictionary rule should be like:

The nightmare is that AdS/CFT only works for very special ”large N” theories that have no relation of any kind to physical reality: $\mathcal{Z}$ (CFT in d dimensions for arbitrary N and ’t Hooft coupling) = $\mathcal{Z}$ (stringy quantum gravity in d+1 dimensions at arbitrary “$\hbar$”)

reducing to the useful GKPW dictionary rule Eq. (III.26) only when one takes the limit of large number of colors (N) and strong ’t Hooft coupling on the left hand side, because that corresponds with the weak coupling ($\hbar \to 0$) limit on the right hand side, with the effect that the intractable quantum gravity in the bulk reduces to the familiar Einstein theory.

Why should a condensed matter physicist dislike this large N? One finds surely plenty of occasions in condensed matter physics where large N theories are quite useful, but these are different. They are “vector large N”, of a typical form

$$S = \int d^d x \left(|(\partial_\mu - iA_\mu)\vec{\Phi}(x)|^2 + m^2 \vec{\Phi}(x)\cdot\vec{\Phi}(x) + w(\vec{\Phi}(x)\cdot\vec{\Phi}(x))^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \cdots \right) \tag{IV.1}$$

where $\vec{\Phi}(x)$ is a vector field with N entries, like the $O(3)$ vector describing a Heisenberg quantum antiferromagnet. In the context of the slave theories we have learned that these can be gauged involving compact gauge fields leaving room for (de)confinement physics. However, these gauge fields are in the limit of few “colors” (generically $U(1)$, at best $SU(2)$). The boundary field theories as of relevance to the classical bulk have to be however of the *matrix* large N variety. Instead of the vectors these fields are of the kind Φ_{ab}, stored in an $N \times N$ matrix with N^2 entries. The specific way that this is realized in the top-down construction is in the form of gauged matrix theory. This is just the familiar $SU(N_c)$ non-abelian Yang-Mills theory where the number of degrees of freedom per space-time point are counted according to $N_c^2 - 1$. For instance, in QCD where $N_c = 3$ one has to deal with 8 gluons.

In the case of the Maldacena top-down it is quasi-proven that the classical gravity in the bulk is indeed dual to such a $\mathcal{N} = 4$ supersymmetric large N_cYang-Mills theory. There are a variety of elegant ways of understanding this more deeply using the underlying string theory. These arguments also make clear that one has to be at a large ’t Hooft coupling $\lambda = g^2 N \to \infty$. This is a pleasant circumstance for condensed matter physics since strong ’t Hooft coupling coincides in these particular theories with the way one deals with strong coupling in condensed matter physics. ’t Hooft pointed out already in the 1970’s that in this limit the Feynman diagrams span up dense “nets” which look like triangulated forms of string worldsheets. However, these are forced by large N to be “planar” meaning that the worldsheets have no “holes”: these are genus zero. This connects directly to the wisdoms of perturbative string theory: such “no holes” closed worldsheets are known to eventually describe gravity. Upon lowering N one should allow for increasing numbers of topologically non trivial worldsheets and this is supposed to represent the quantization of the geometry but this “expansion in the string coupling g_s” is hell for the string technicians. The second parameter in string theory is the string tension α', parametrizing how easily a string for a fixed genus is fluctuating. It turns out that by reducing the ’t Hooft coupling α' is increasing. The effect of α' on the effective long wavelength theory is much better understood: at least at first it amounts to switching on higher gradient terms and higher spin gauge fields in the Einstein theory.

A final motive that is also alien to condensed matter systems is supersymmetry. It turns out that supersymmetry appears to be not at all that essential. It is just a greatly simplifying circumstance for the mathematically minded, but there is plenty of evidence that it is not a prerequisite for holography. There are a number of explicit non-supersymmetric top-downs in existence and for the condensed matter applications supersymmetry has invariably disappeared: both finite temperature and finite density just break supersymmetry explicitly. One should be aware of one feature that is perhaps more of a convenience than a hazard for condensed matter physics applications. In higher than 1+1D it has proven to be very difficult to identify "lines" of unstable fixed points: quantum critical/conformal states are invariably associated with localized (in parameter space) quantum critical points, where one has to fine tune to a critical coupling. Supersymmetric systems are special in the regard that the bosonic and fermionic quantum fluctuations tend to cancel each other and this leads to the "non-renormalization theorems". The effect is that it is way easier to find marginal operators in the renormalization group flow, and this has in turn the effect that one finds critical states with continuously varying anomalous dimensions in *whole planes* in coupling constant space. The conformal field theories associated with pure AdS are of this kind. But this is no problem: AdS/CFT is directly associated with the "quantum critical UV's" of condensed matter and in this regard the superconformal CFT's are just a convenience since one does not have to do any work to create such starting points.

A first inconvenience associated with this alien UV physics met by condensed matter physics is that one has to alert that "Yang-Mills stamp collection" is around the corner. We will later find out how the QCD-type confinement mechanism shows up in holography. The effect of it is that one has to know about the big zoo of confined particles coming with the theory. These are typically scores of mesons "living on Regge trajectories" of increasing mass, but in the supersymmetric theories these come also with the "fermions in the adjoint" like the "mesino's" (also called "mesonino's"), gravitino's and so forth. You then also need to know that in the large N limit these gauge singlets no longer interact: this is actually quite beautifully holographically encoded in the bulk as "hard wall electron stars", describing the confined mesino Fermi-gas (section IX). On this level, this is largely a hassle, baroque stuff that one needs to filter out to infer the underlying beauty of the emergence principles given away by holography (like the Fermi-gas per se in this example).

The real concerns with the large N versus condensed matter applications are elsewhere. AdS/CFT is in the first place a very non-perturbative renormalization group machine, where the renormalization flow is vividly "imaged" by the "journey" along the radial direction from a featureless Anti-de-Sitter space near the boundary to an interesting black hole world in the deep interior coding for the emergence in the infrared. There are here two issues: (a) Starting from the large N superconformal theory in the UV, can all generic infrareds be reached or is this very un-condensed matter physics UV? Or does it lead to a particular subset of IR fixed points? The other side of the coin: could it be that specifics of the condensed matter UV (like the "Mottness") are such that they imply IR's that will not be found when these are not wired in all along? This is the "UV independence" question. (b) It is very clear that for the UV a classical gravitational dual implies very special theories (large N) that have no relevance to condensed matter physics. Is it so that for the purpose of chasing down the *structure* of the theories describing the deep IR that the dual classical gravity description is still representative? This is the hundred thousand dollar question. As we will see in the next sections, the case is compelling. It is by now well established that the theories of classical physics (thermodynamics, hydrodynamics) in the regime where temperature is the largest scale are miraculously encoded in classical black hole physics in the bulk. The only way that large N exerts qualitative influences in this regime is by rendering thermal phase transitions to be very mean field. It appears that even for this "flaw" the remedy is already available in the form of mild re-quantization of the bulk physics. The case is less strong in the quantum regime of zero temperature and finite density. In the cases that we know what to expect on the field theory side (superconductivity, Fermi liquid) the classical gravity infrared encoding again produces miracles, but the great question is whether the "strange metals" which are lying on the surface on the gravity side can be trusted as generic IR physics in the boundary as well.

V. THE FINITE TEMPERATURE MIRACLE: HOLOGRAPHIC THERMODYNAMICS

About a month after Einstein proposed the famous Einstein's theory of general relativity, Schwarzschild found the first exact solution for Einstein's equation. The four dimensional Schwarzschild metric solution is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{V.1}$$

with the redshift factor $f(r)$

$$f(r) = 1 - \frac{2GM}{r} \tag{V.2}$$

where G is Newton constant and M is a parameter which can be interpreted as the mass. When $M = 0$, the above geometry (V.1) becomes Minkowski spacetime. When $r \to \infty$, the geometry (V.1) becomes a Minkowski flat spacetime. This property is known as asymptotic flatness. When $r = 2GM$, we have $g_{tt} = 0$. This is the horizon, and when $r < 2GM$, the time and radial direction will change with each other. The horizon is a coordinate singularity. Nothing can escape from this horizon, thus we call it a "black hole".

The notion of black hole complementarity appeals equally to the popular mind as to the expert in quantum gravity. It started with Hawking's discovery that the space time of a Schwarzschild black hole "tears apart" the coherent vacuum of free quantum fields, rendering the virtual vacuum fluctuations to become the real radiation of a black body at a Hawking temperature,

$$T_{\rm H} = \frac{1}{8\pi M} = \frac{\hbar c^3}{8\pi G M k_B} \tag{V.3}$$

either expressed in natural units where M is the mass of the black hole, or either in explicit units involving the surface area. Hawking's celebrated exercise departs from a space time which is gravitationally classical while the fields are quantized. Actually, before Hawking's discovery Bekenstein had realized that in essence via a relation of the kind Eq. (III.26) the purely mechanical Hilbert-Einstein action and the partition sum of a finite temperature field theory system showed strange coincidences. This suggested that a "Bekenstein" entropy could be associated with the black hole,

$$S_{\rm BH} = \frac{A}{4} = 4\pi M^2 = \frac{k_B G}{c\hbar} 4\pi M^2 \tag{V.4}$$

again in natural and explicit units. This turned out to be the same as the entropy expected from Hawking's considerations. The salient feature of this Bekenstein-Hawking (or black hole) entropy is its proportionality to the horizon area: it is coincident with the thermal entropy associated with a surface having the area of the horizon, divided in cells with a size set by the Planck length, where every cell contains one bit of information. This is the origin of the holographic principle stating that the number of degrees of freedom of a gravitational system in d space time dimensions can be counted in a non-gravitational field theoretical system living in $d-1$ dimensions.

The Bekenstein moral was taken a step further by Thorn *et al.* [23], finding out that the GR equations associated with small, time dependent fluctuations of the black hole metric were strangely coincidental with the Navier-Stokes hydrodynamical equations one would get when the horizon would be viewed as a fluid: the "membrane paradigm". These ideas that emerged in the relativist community in the 1970's and 1980s' have been quite instrumental in guiding the discovery of the AdS/CFT correspondence. But this sector of thermal physics is also the great established success of the correspondence: it has taken it to a higher level, of a mathematical automatism that yields a rather complete description of this physics. What matters most in the present context is that it demonstrates the overwhelming power of the correspondence as "generating functional" of emergence principle.

How to address finite temperature in the correspondence? The interest is of course in the finiteness of temperature in the boundary field theory, since there is no meaning to the notion of temperature in the bulk since this is in the large N limit supposed to be governed by "mechanical" classical field theory. Especially in the equilibrium applications it is assuringly unambiguous how to deal with finite temperature. Also the condensed matter readership should be familiar with the notion of imaginary times and Matsubara frequencies when one is dealing with quantum problems in finite temperature. The way this works is most obvious in the path integral language. One departs from the real time (Minkowski space time (it, x_1, x_2, x_3)), to analytically continue to imaginary time $t \to i\tau$. One now insists that at finite temperature the imaginary time is "compactified" in a circle with radius $R_\tau = \hbar/(k_B T)$. In this "Euclidean space-time" one automatically computes thermodynamic quantities, and when the interest is in dynamical linear response properties (dynamical susceptibilities or propagators) one just Wick rotates back to real frequencies at the end of the computation. This simple recipe reproduces all of quantum statistical physics with the caviat that it only works at equilibrium. For non-equilibrium problems one has to work with the Schwinger-Keldysh formalism; this turns out to be also perfectly compatible with AdS/CFT and we refer the reader to the literature.

Perhaps the non-gravitational inclined reader might have wondered whether it is an accident that the encoding of temperature has some geometrical root. After all, the "curling up" of the imaginary time direction is quite like the Kaluza-Klein compactification which is natural in geometrical theories: by turning one *spatial* direction of a D dimensional Einstein space time in a small circle one finds at scales large compared to its radius that one is dealing with a D-1 dimensional gravitational universe, with in addition an electromagnetic field governed by a U(1) gauge degree of freedom associated with the diffeomorphic small circle. Among relativists this is known as the "problem of signature". Einstein theory does not care whether the signature is positive (imaginary time, Euclidean space-time) or negative (real time, Minkowski space time). It is just empirical observation that the universe preferse a negative signature.

Throughout relativist' history debates have been raging regarding the significance of Euclidean signature physics. So much is clear that in case of stationary universes it is save to Wick rotate back and forth. As realized, predictably, by Witten [24] this is of great help in the deduction of how to wire in the finite temperature of the boundary field theory in the *geometry* of the bulk. In the Euclidean boundary field theory at equilibrium finite temperature is already "geometrized" in the form of the imaginary time circle with finite radius. But the boundary field theory and the bulk share the same space-time. In the bulk one just has the extra radial direction, while the boundary field theory "feels" the bulk imaginary time of the bulk at the radial infinity. How to continue this into the bulk, in terms of a solution of Euclidean Einstein gravity? The interesting circumstance is that the solution of this exercise is a famous result which is familiar to any student of gravity: the Gibbons-Hawking construction for the Schwarzschild black hole. This is just the quick and dirty way to deduce the Hawking temperature. One departs from the usual Schwarzschild black hole metric in Minkowski signature, to analytically continue to imaginary time: the result is the black hole solution of Euclidean gravity. How does this metric look like? The Euclidean space-time just disappears behind the horizon, and right at the horizon imaginary time is compactified in a circle with vanishing radius. Upon moving away radially from the black hole, this time circle is expanding while the compactification perimeter asymptotes at radial infinity at a value off $R_\tau = \hbar/(k_B T_H)$ where T_H is the Hawking temperature. One immediately concludes that whatever quantum field theory is at work, it will feel a temperature equal to the Hawking temperature when you observe sufficiently far away from the black hole!

This all works the same in an AdS space time. Therefore, given an imaginary time axis that is rolled up with $R_\tau = \hbar/(k_B T)$ at the radial asymptote where the field theory lives on the AdS boundary, it has to be that in the precise middle of AdS (the "deep interior") a black hole is present with properties such that it encodes for the Hawking temperature desired by the experimentalist who wants to measure the temperature dependence of the field theory physics!

Dictionary rule for finite temperature: finite temperature in the field theory is encoded by a black hole in the deep interior of the AdS bulk, while the temperature is equal to the Hawking temperature of this black hole.

There is more wonderful stuff to appreciate. Let us first review some standard wisdoms regarding the finite temperature physics of quantum critical states. The notion of "energy-temperature" scaling and the associated concept of "Planckian dissipation" should ring a bell for those condensed matter physicists who are familiar with the basics of the quantum critical state – it is a main theme in for instance the book by Sachdev. This also rests on Euclidean field theory. Dealing with a zero temperature quantum critical state/CFT, space and time are just the same thing in Euclidean signature, and time like correlations are as algebraic/scale invariant as space like correlations; after Wick rotation back to real time one finds the "branch cut" dynamical susceptibilities of the form Eq. (III.9). However, at finite temperature this scale invariance is actually lifted because the imaginary time axis acquires a finite radius. One can now just mobilize the wisdoms of finite size scaling. These insist that any dynamical response should be described by a scaling function which has to be a function of the ratio of energy and temperature: the "energy-temperature scaling" that has been used with much effect to find out whether laboratory systems are actually quantum critical. The energy $\hbar\omega$ where the response is measured is inversely proportional to a real "measurement time". It is obvious that when this time is shorter than R_τ the experiment cannot know that the system is at a finite temperature and in this "coherent regime" $\hbar\omega > k_B T$ one just finds the zero temperature branch cuts. But what to expect in the "hydrodynamical regime" $\hbar\omega \ll k_B T$? Here the measurement time is much larger than the "size" of Euclidean space time! The Wick rotation back to real time is causing here an interesting miracle. As illustrated explicitly in Sachdev's book, one has to compute the Euclidean correlators with very high precision, and upon Wick rotation one finds that the procedure describes *dissipative*, entropy producing classical dynamics. At least when one departs from simple quantum critical fields such dynamics is set by a single time scale: the energy relaxation time which tells how much time it takes to convert work into heat, i.e. the characteristic time for entropy production. Again resting on finite size scaling: since the only scale in the system is the imaginary time radius, it has to be that in the quantum critical system the relaxation time is just of order of the time-radius. This is the idea of Planckian dissipation: the finite temperature quantum critical system has a hydrodynamical (long time) classical regime governed by an energy relaxation $\tau_\hbar = A R_\tau = A\hbar/(k_B T)$ where A is a constant of order unity which is characteristic for the universality class of the CFT. Perhaps paradoxically, this entropy production time as set by Planck's constant ("Planckian") turns out to be extremely short. As we will see in a little while, this underlies the famous AdS/CFT "minimal viscosity" as well.

AdS/CFT describes correctly the "Planckian dissipation" property of the finite temperature fluids formed from zero temperature quantum critical states: these fluids are maximally dissipative with a time scale for entropy production $\tau_\hbar = A_\hbar \hbar/(k_B T)$.

Let us now take up again the way that the black hole in AdS deals with these matters. The classic Schwarzschild solution (V.1) is valid as an asymptotically flat space-time, associated with a vanishing cosmological constant. There is a twist when one deals with an AdS asymptotics: the bottomline is that in contrast the the flat case where the Hawking temperature is inversely proportional to the horizon area, in AdS it becomes directly proportional to the "size" of the black hole.

To demonstrate how this works let us have a look at some of the metrics. The Einstein equation for a $D+1$ dimensional Minkowski-signature space time with a negative cosmological constant is,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{D(D-1)}{2L^2} g_{\mu\nu} = 0. \tag{V.5}$$

This is solved by a Schwarzschild black hole background with the metric,

$$ds^2 = \frac{r^2}{L^2}(-f(r)dt^2 + dx_i^2) + \frac{L^2}{r^2 f(r)} dr^2, \quad i = 1, ..., D-1 \tag{V.6}$$

where L is the AdS radius, r is the radial direction, and r_0 is the horizon and the redshift factor

$$f(r) = 1 - r_0^D / r^D. \tag{V.7}$$

To see how the Gibbons-Hawking trick works, consider a general static black hole metric

$$ds^2 = -g_{tt}(r)dt^2 + \frac{dr^2}{g^{rr}(r)} + g_{xx}(r) d\vec{x}^2,$$

and Wick rotate to the Euclidean signature $\tau = it$ with the result

$$ds_E^2 = g_{tt}(r) d\tau^2 + \frac{dr^2}{g^{rr}(r)} + g_{xx}(r)(d\vec{x})^2. \tag{V.8}$$

Although our main interest is in the asymptotic regime, we first have to inspect how this works near the horizon, since in this regime we can find out that the imaginary time axis has to be compact. At the horizon g_{tt} and g^{rr} are vanishing, and we can therefore near the horizon expand as $g_{tt}(r) = g_{tt}'(r_0)(r - r_0) + ...$, $g^{rr}(r) = g^{rr\prime}(r_0)(r - r_0) + ...$ and $g_{xx}(r) = g_{xx}(r_0) + ...$ where the prefactors g''s are just numbers. It is now just convenient to re-parametrize the radial direction in $R = 2\sqrt{r - r_0}/\sqrt{g^{rr\prime}}$. All what matters is the plane spanned by this R and the imaginary time direction τ and here the metric becomes $ds_E^2 = dR^2 + \frac{1}{4} R^2 g_{tt}' g^{rr\prime} d\tau^2$. This is just like the metric of a plane in polar coordinates with τ being the compact direction. Upon approaching the horizon $R \to 0$ one sees that the prefactor of $d\tau^2$ is vanishing: this means that the Euclidean time direction shrinks to a point! The Euclidean Einstein equation insists that the geometry is smooth at the horizon and this implies that τ is periodic with period $\frac{4\pi}{\sqrt{g_{tt}'(r_0) g^{rr\prime}(r_0)}}$. It is now easy to find out how the time circle evolves as function of the radial direction for the particular case of a Schwarzchild black hole in AdS . Consider the Euclidian version of Eq. (V.6), the perimeter of the time circle is (See Fig. 2)

$$\ell_\tau(r) = \sqrt{g_{tt}(r)} \frac{4\pi}{\sqrt{g_{tt}'(r_0) g^{rr\prime}(r_0)}}. \tag{V.9}$$

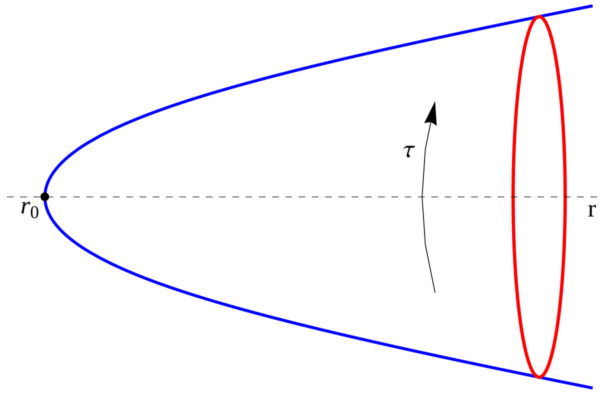

FIG. 2: The Euclidean geometry $ds^2 = g_{tt}(r)d\tau^2 + g_{rr}(r)dr^2$. The period of τ is $\frac{4\pi}{\sqrt{g'_{tt}(r_0)g^{rr\prime}(r_0)}}$ such that the geometry is smooth at $r = r_0$. The proper length of the perimeter of the circle at r is $\sqrt{g_{tt}(r)}\frac{4\pi}{\sqrt{g'_{tt}(r_0)g^{rr\prime}(r_0)}}$. In the asymptotical flat case, it approaches a constant and the above geometry looks like a cigar. In the asymptotical AdS case near the boundary it is diverging as r since the boundary of AdS is conformal to the flat spacetime with a conformal factor r/L.

At the boundary $r \to \infty$ this asymptotes to $\ell_\tau L/r = \frac{4\pi L^2}{D r_0}$, implying that the temperature felt by the boundary field theory becomes,

$$T = \frac{D r_0}{4\pi L^2} = \frac{\hbar c}{k_B}\frac{D r_0}{4\pi L^2}. \tag{V.10}$$

in terms of the horizon radius r_0, the number of space-time dimensions D, and the AdS radius L.

The bottom line is that according to Eq. (V.10) this temperature is increasing when the black hole radius is growing along the radial direction. This is elegantly in harmony with the notion that the radial direction is coincident with the scaling direction of the field theory. The perfect scale invariance in the field theory is embodied by the AdS metric. However, at finite temperature this geometry is disturbed by the black hole horizon at increasingly large scales when temperature rises. This encodes the finite size scaling effect of temperature in the field theory in a geometrical language. The black hole horizon takes the role of gravitational dual of the origin of dissipation and entropy production. In fact, as we will find out a little later, the modes of the classical fields propagating in the bulk will generically fall through the horizon meaning that these have a finite life time (the "quasinormal modes"). Generically, this disappearance behind the horizon encodes for the relaxational nature of the responses in the field theory in the hydrodynamical regime. Before we turn to these matters, let us first find out how thermodynamics rolls out of this finite temperature AdS/CFT machinery.

The starting point is the relation Eq.(III.26) relating the partition sum of the CFT to the on-shell action of the bulk gravity. For the reasons discussed in the above, this works equally well for the field theory at a finite temperature invoking Euclidean Einstein theory for the black hole in the deep interior of AdS. The free energy of the field theory is thereby given by the dictionary entry,

$$F = -k_B T \ln \mathcal{Z} = k_B T S_E[g_E] \tag{V.11}$$

where $S_E[g_E]$ is the on-shell Euclidean Einstein-Hilbert action: notice that temperature is now entirely encoded in the bulk through the Euclidean black hole geometry. To compute the right hand side, one would be tempted to just insert the Euclidean metric (g_E) of the AdS-Schwarzschild black hole in the Euclideanized action. However, one runs here into a GR subtlety which is associated to a well-defined variation problem. The Einstein-Hilbert action turns out to be only appropriate for a closed spacetime. AdS geometry is special in the sense that such a spacetime has still a boundary, and it was discovered in the 1970's [25, 26] by Gibbons, Hawking and York that one has to supplement the action with a "Gibbons-Hawking-York" boundary term to assure that the variational principle is well defined. This is similar as to for instance the requirement to add a Wess-Zumino-Witten (WZW) boundary term (edge states) to a Chern-Simons topological bulk theory in the quantum Hall context. The result is as follows,

$$S_E = -\frac{1}{2\kappa^2}\int d^{D+1}x\sqrt{g}(R + \frac{D(D-1)}{L^2}) + \frac{1}{2\kappa^2}\int_{r\to\infty} d^D x\sqrt{\gamma}\Big(-2K + \frac{2(D-1)}{L}\Big). \tag{V.12}$$

The second term is the boundary term, determined by the induced metric γ_{ab} on the boundary at $r \to \infty$, while $K = \gamma^{ab}\nabla_a n_b$ is the trace of the extrinsic curvature of the induced metric with n^ν an outward unit vector normal to the boundary. By inserting the Euclidean version of the metric Eq. (V.6) one finds after a lengthy but straightforward calculation a remarkably simple result for the free energy,

$$F = -k_B T \ln \mathcal{Z} = k_B T S_E[g_E] = -\frac{(4\pi)^D L^{D-1}}{2\kappa^2 d^D} V_{D-1} T^D = -\frac{k_B^D c}{\hbar^D}\frac{(4\pi)^D L^{D-1}}{2\kappa^2 d^D} V_{D-1} T^D \tag{V.13}$$

where V_{D-1} is the volume of the $D-1$ spatial dimensional volume of the boundary field theory while $\kappa^2 = 8\pi G$ where G is Newton's constant and L is the AdS radius. For the time being we express this in the natural units of the gravitational bulk theory .

From the free energy (V.13) and the temperature (V.10) the entropy for the dual field theory follows immediately as,

$$S = -\frac{\partial F}{\partial T} = \frac{(4\pi)^D L^{D-1}}{2\kappa^2 D^{D-1}} V_{D-1} T^{D-1} = \frac{k_B^D c}{\hbar^D}\frac{(4\pi)^D L^{D-1}}{2\kappa^2 d^{D-1}} V_{D-1} T^{D-1} = \frac{k_B c^D}{\hbar}\frac{1}{4G}\frac{r_0^{D-1}}{L^{D-1}} V_{D-1} \tag{V.14}$$

while the entropy density is, obviously,

$$s_{\lambda=\infty} = \frac{S}{V_{D-1}} = \frac{k_B^D c}{\hbar^D}\frac{(4\pi)^D L^{D-1}}{2\kappa^2 d^{D-1}} V_{D-1} T^{D-1} = \frac{1}{4G}\frac{r_0^{D-1}}{L^{D-1}}\frac{k_B c^D}{\hbar}. \tag{V.15}$$

Although not immediately obvious, it turns out that this is equal to the Bekenstein-Hawking entropy that is calculated from the area of the horizon area A_{horizon} (Eq. (V.4)), i.e. $S = A_{\text{horizon}}/4G$. To a degree this is not surprising since the Bekenstein derivation is along similar lines as in the above, although this dealt with a flat asymptotics. It is yet quite different from the view underlying the Hawking derivation that the entropy produced by the black hole geometry "tearing apart" the quantized free field vacuum in the gravitional bulk: in this AdS/CFT perspective, the bulk is still completely classical and the entropy is associated with the field theory "hologram" living on the boundary. This shift in interpretation becomes even more manifest focussing in on the actual magnitude of the field theoretical entropy. This is still expressed in the Newton's constant κ and the AdS radius L specifying the gravitataional bulk. How to translate this into the counting of the degrees of freedom of the boundary field theory? Although there are ways to a posteriori rationalize matters, eventually one has to rest on the underlying string theory to get this counting right. The weakly coupled, classical gravity in the bulk is invariably associated with the large N limit of a matrix field theory characterized by N^2 degrees of freedom at the UV cut-off. The following relation between the gravitational quantities and those off the field theory is believed to be exact (for $D = 4$)

$$\frac{c^4}{\hbar}\frac{L^3}{\kappa^2} = \frac{N^2}{4\pi^2} \tag{V.16}$$

In order that one can get away with classical gravity the curvature in the bulk has to be small ($L^3/\kappa^2 \gg 1$) in the gravitational unit. This implies that the classical gravity limit of the bulk implies that $N^2 \gg 1$.

Combining Eq.'s (V.15, V.16) we arrive at the answer for the entropy of the large N CFT. Explicitly in 3+1D dimensions,

$$s = \frac{\pi^2}{2} N^2 T^3. \tag{V.17}$$

The dependence on temperature is in fact devoid of any information: it is just like the T^{D-1} behavior of the Debye entropy associated with e.g. acoustic phonons (D is the number of space and time dimensions). This follows from a simple scaling consideration that pertains to any scale invariant quantum theory: the free energy, as the entropy, is just set by the dimensions of space D and the effective number of time dimensions (the dynamical critical exponent z, equal to unity because of Lorentz invariance): $s \sim T^{D/z}$. The physics is therefore merely in the prefactor, but this reveals a very interesting surprise that is regarded as one of the early highlights of the correspondence. As we discussed in section IV, besides the large N limit the top-down also reveals that one is dealing with the physics of the maximally supersymmetric Yang-Mills theory for the strong t' Hooft coupling $\lambda = g^2 N$ when one evaluates matters with Einstein theory in the bulk. One can now compare the entropy of the strongly coupled theory Eq. (V.17) with the result for the free theory. The latter is trivial since one just to count the number of massless modes, and the result is $s_{\lambda=0} = \frac{4}{3}\frac{\pi^2}{2}N^2T^3$. Remarkably, these are nearly the same: $s_{\lambda=0}/s_{\lambda=\infty} = 4/3$. Apparently, the thermodynamic potentials of non-Abelian gauge theory plasma's vary very slowly upon changing the coupling strength as illustrated in Fig. 3.

An early surprise was the discovery that the "conformal" entropy of the "maximally" strongly interacting Yang-Mills theory of Maldacena is barely different from the limit of free conformal fields.

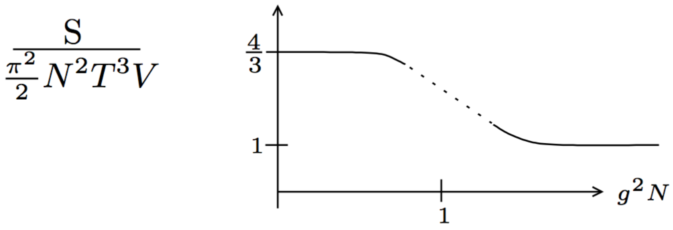

FIG. 3: The thermodynamic potentials of non-Abelian gauge-theory plasmas as a function of the 't Hooft coupling in the large N limit. The plot is taken from [27].

A. Holographic thermodynamics: the Hawking-Page transition

Conformal invariance is an extremely powerful symmetry, having as perhaps undesired consequence that it by itself largely determines the qualitative features of the physics. All what remains to be determined is numbers, like anomalous scaling dimensions, universal amplitudes like the prefactor of the entropy of the previous paragraph and so forth. AdS/CFT is however about a lot more than studying the CFT's itself: as of central interest to the condensed matter applications, it also tells intriguing stories regarding the way that stable states of matter emerge from the quantum critical state. Much of the later sections are entirely devoted to this theme (holographic superconductivity, the emergent Fermi-liquids). To give a first taste, let us discuss here a quite elementary but famous example: the discovery by Witten [24] that the Hawking-Page transition in AdS is describing a thermal confinement-deconfinement phase transition in the boundary field theory.

This is about a thermodynamics that is in fact a bit remote from what one encounters in condensed matter systems. In fact, it is a vivid example of the rather odd nature of large N limit super-conformal field theories. Besides illustrating the way that thermodynamics gets a gravitational encoding, it might also serve as a warning for the subjects discussed in the next sections. In the large N limit common wisdoms regarding thermal (and quantal) fluctuations do not always apply. The most striking effect is that the large N limit "overrules" the rules associated with the behavior of fluctuations pending the number of dimensions. The issue at stake here is that the CFT of Maldacena's correspondence behaves in a way similar as to a system ruled by e.g. discrete global symmetries in one space dimension. The following properties have been established by direct calculations in the field theory for weak 't Hooft coupling [28]. This system is "ordered" precisely at zero temperature, but this order is destroyed at any finite temperature in all space dimensions equal and larger than 2. Oddly, this $T \to 0$ transition is of first order (!), and it can be studied by putting the system in a finite volume maintaining the thermodynamic limit. The finite size splittings in the excitation spectrum have the effect of stabilizing the ordered state with the consequence that in the finite volume this first order transition now happens at a finite temperature. It is actually a confinement/deconfinement transition, from a low temperature confining- to a high temperature deconfining state, while due to the protection by supersymmetry both phases are conformal, at least at distances shorter than the finite size cut-off. The physics is that of confinement without dynamical mass generation: see fig. 4.

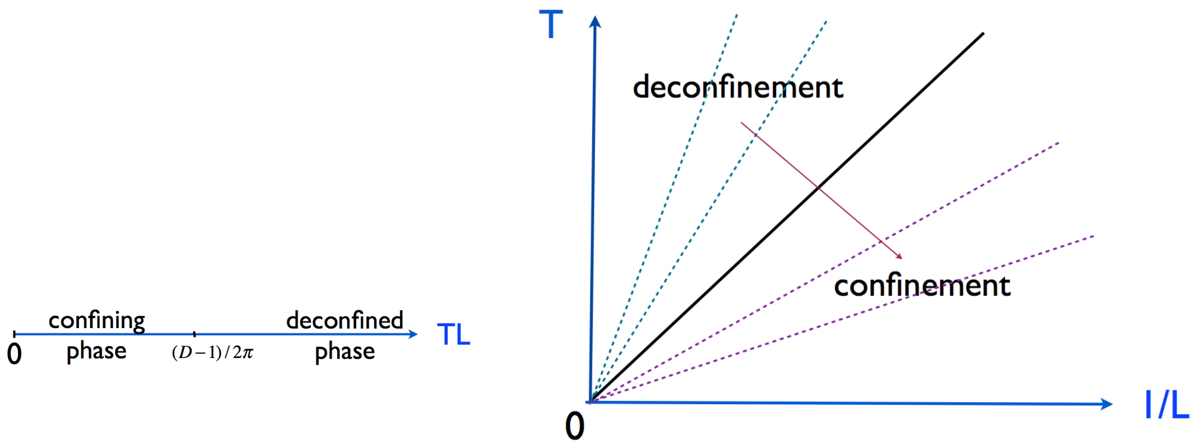

FIG. 4: Hawking Page phase transition in the bulk is the holographic dual of the confinement-deconfinement phase transtion in the boundary field theory.

The question arises what happens at the large 't Hooft coupling limit. In the hands of Witten this turned into an early highlight of the correspondence. He showed that the finite size/temperature behavior of the CFT precisely matches a famous relativist's story: the Hawking-Page transition of a black hole in a compact AdS space! First, how to recognize the confinement phenomenon of the Yang-Mills theory in the gravitational bulk? The entropy associated with the black hole scales with N^2 and this counts the deconfined degrees of freedom. However, precisely in the zero temperature limit of the field theory the bulk black hole vanishes and one has a pristine AdS geometry with a vanishing entropy. This just signals that the entropy is of order N^0 and this is expected for the confining state where N^2 "quarks" combine in a single gauge singlet (baryon). This can be made more precise by considering a Wilson loop, a closed loop in the boundary space. This dualizes in a minimal surface in the bulk, bounded by the loop in the boundary space and penetrating the bulk in the radial direction. For reasons that are not well understood this is very similar to the way that the entanglement entropy is computed (see section IX). This surface is "ruptured" when it touches the black hole horizon, while is stays intact all the way to the deep interior of the pristine AdS space, and this translates into the perimeter and area laws of the Wilson loop being the measures of (de)confinement. In the infinite volume limit one recovers therefore the $T \to 0$ confinement transition with this simple gravitational story. But how to understand the finite T_c in the finite volume?

The answer to this question goes back to the relativist's hay days when Hawking together with Page [29] studied the GR of black holes in AdS space times. Their discovery is related to a subtle aspect of this physics: the topology of the horizon, going hand in hand with the topology of the asymptotic space, plays a crucial role. Up to this point we have been arguing about black "holes" but in this regard we have been sloppy. Given that the AdS space has a boundary, one is forced to specify the precise topology of this boundary before one can name the precise nature of the black "thing" at the origin. The time direction is easy: it is the "Euclidean circle" S^1. However, regarding the space directions of the boundary one has still to make a choice. The more intuitive choice is choose a compact space with boundary conditions at infinity such that the boundary forms a sphere, say S^{D-1} in $D-1$ space dimensions. At infinite Hawking temperature the black hole horizon can be made coincident with this boundary, and upon lowering the temperature this horizon shrinks along the radial direction to eventually approach to a point when the temperature goes to zero: this is the usual black hole. The topology of the boundary is thereby $S^1 \times S^{D-1}$. However, instead of this compact space one can as well take a non-compact space that extends in all directions to infinity with a boundary topology $S^1 \times R^{D-1}$. One immediately infers that in this case the black hole horizon cannot be a closed surface! Instead the horizon is now a surface that covers all of the space directions such that "one cannot fly by" in these directions. When temperature decreases, this horizon recedes along the radial direction but it is there in all space directions. This is called is "black brane".

Most of the condensed matter applications depart from these black branes, for the reason that the compact boundary space automatically leads to physics that one not necessarily wants to study in this particular context, although it is fascinating by itself. This is the Hawking-Page transition that works as follows. One just looks for black-hole type solutions in the Euclidean-signature Einstein equations for a negative cosmological constant and the compact boundary geometry. Remarkably, Schwarzschild type solutions are possible and thermaldynamical stable but these require a special condition: the radius of the imaginary time circle has to be larger than the radius of the compact spatial directions of the boundary! The consequence is that only at high temperature the filed theory can have a thermal state that is encoded by the Schwarzschlld black hole in the bulk. When the time circle is larger than the spatial radius one finds instead a very simple gravitational solution: the black hole has disappeared one finds a pristine AdS geometry extending all the way to the deep interior. Since the time circle is still finite, this represents a *confining* thermal state in the field theory which is called "thermal AdS". This state is a-priori and is still conformal, except that the conformality is explicitly broken not only by the finiteness of the Euclidean time circle but also by the sphere with finite radius in the spatial directions. This implies an energy scale and one finds out that this acts as the gap stabilizing the confining state in the finite volume. The conclusion is that the gravitational Hawking-Page transition in the bulk describes a thermal transition between a high temperature deconfined quark-gluon plasma to a low temperature confining phase of the gauge theory when the volume is finite.

Maldacena's superconformal Yang-Mills theory is in a confining state at exactly temperature, while it is deconfining at any finite temperature, with the deconfined ("fractionalized") degrees of freedom encoded by the presence of the black hole horizon. By putting this stuff in a finite spatial volume a first order deconfinement transition happens at a finite temperature and this is encoded holographically by the black hole that "evaporates" when the AdS space acquires a finite volume: the celebrated "Hawking-Page" transition from black hole physics.

Let us illustrate the above with the most important equations for the bulk. The Euclidean black hole solution has the same generic form as Eqn. (V.1)

$$ds_2^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-1}^2, \tag{V.18}$$

where

$$f(r) = 1 + \frac{r^2}{L^2} - \omega_D \frac{M}{r^{D-2}}, \qquad \omega_D = \frac{2\kappa^2}{(D-1)\mathrm{Vol}(S^{D-1})}. \tag{V.19}$$

Considering the scaling limit

$$t \to \lambda t, r \to \lambda^{-1} r, d\Omega_{D-1}^2 \to \lambda^2 d\vec{x}^2 \tag{V.20}$$

and

$$\lambda \to 0, \tag{V.21}$$

the metric (V.18) changes to the vacuum planar AdS_{D+1}. If we scale $M \to \lambda^{-D} M$ at the same time, one gets the planar AdS Schawartzshild black hole (V.6).

The outer horizon r_+ is the larger solution of the equation

$$1 + \frac{r^2}{L^2} - \omega_D \frac{M}{r^{D-2}} = 0. \tag{V.22}$$

In fact, in the previous section we deduced the Euclidean time circle Eq.(V.9) for the black brane, but matters are for the real black hole in the compact space

$$\beta = \frac{4\pi L^2 r_+}{D r_+^2 + (D-2)L^2}. \tag{V.23}$$

in units of $c = k_B = \hbar = 1$ and its inverse gives the temperature of the black hole $T = 1/\beta$.

The "competing" thermal AdS space has just the pristine AdS metric

$$ds_1^2 = \left(\frac{r^2}{L^2} + 1\right) d\tau^2 + \frac{dr^2}{\frac{r^2}{L^2} + 1} + r^2 d\Omega_{D-1}^2. \tag{V.24}$$

Note that the AdS radius L does not set a scale for the boundary field theory. In the field theory, we only have the scales related to the size of time circle and the spherical radius. Using the algorithm of the previous section one can now compute the free energies of the boundary corresponding with these two different bulk geometries. A crucial extra ingredient is that in order to compare these free energies, one has to insist that the bulk geometries describe precisely the same boundary space-time. Choosing a cut-off in the radial direction at $r = R$, one can express the imaginary periodicity of the thermal AdS β' in terms of the black hole inverse temperature β such that both systems live in the same boundary space, sharing the same boundary time circle. At the boundary these temperatures are the same but when one moves the cut-off R into the bulk the temperatures at a given R starts to deviate due to the different geometry. Henceforth, thermal AdS feels an effective different temperature β' at finite R that can be computed by,

$$\beta' \big(1 + \frac{R^2}{L^2}\big)^{1/2} = \beta \big(1 + \frac{R^2}{L^2} - \frac{\omega_D M}{R^{D-2}}\big)^{1/2}. \tag{V.25}$$

One now computes the difference in the boundary field theory free energy straightforwardly and it is related to the Euclidean action difference

$$\beta(F_2 - F_1) = \frac{d}{\kappa^2 L^2} \lim_{R\to\infty} \left(\int_0^\beta dt \int_{r_+}^R dr \int_{S^{D-1}} d\Omega r^{D-1} - \int_0^{\beta'} dt \int_0^R dr \int_{S^{D-1}} d\Omega r^{D-1} \right)$$
$$= \frac{4\pi \mathrm{Vol}(S^{D-1}) r_+^{D-1}(L^2 - r_+^2)}{2\kappa^2(Dr_+^2 + (D-2)L^2)}. \tag{V.26}$$

where F_1 and F_2 are the free energies for the black hole- and thermal AdS solutions. This is expressed in units $c = k_B = \hbar = 1$: the usual coupling constant $\kappa^2 = 8\pi G$, the volume of unit $D-1$ dimensional sphere $\mathrm{Vol}(S^{D-1})$, the AdS radius L, the radial coordinate of the black horizon r_+ and the time-periodicity β. With the units restored, we have

$$\beta(F_2 - F_1) = \frac{k_B^D c}{\hbar^D} \frac{4\pi \mathrm{Vol}(S^{D-1}) r_+^{D-1}(L^2 - r_+^2)}{2\kappa^2(Dr_+^2 + (D-2)L^2)}. \tag{V.27}$$

As noted first by Hawking and Page [29], a sign change can happen in this free energy difference as function of the black hole radius r_+. At low temperature in the field theory the black hole has a radius smaller than the AdS radius ($r_+ < L$) and the thermal AdS has a lower free energy than the black hole solution. When temperature rises a sign change occurs because r_+ becomes larger than L, signaling a first order transition to a high temperature associated with the the black hole geometry in the bulk: the Hawking Page transition, that takes place at a critical temperature $T_c = (D-1)/2\pi L$. This is Witten's [24] explanation for the finite temperature/finite volume confinement transition in the field theory.

In summary, this Hawking-Page transition gives away the important moral that Einstein-theory has a generic attitude to switch suddenly between qualitatively different stationary space times, pending specific conditions. This specific example has a special appeal because it is just involving space time itself and nothing else. In the next sections we will find that we have to add all kinds of material "stuffs" to the AdS physics to describe the behavior of the boundary field theory. In this context a plethora of such holographic transitions have been identified and this remarkable richness of the gravitational theory forms a main reason for the appeal of AdS/CMT. Another important moral is that one has to be constantly aware that the physics of Yang-Mills theory is hard wired in the correspondence. In the presence of horizons in the bulk one is somehow dealing with deconfined states involving "fractionalized" degrees of freedom (quarks, gluons).

In Witten's example [24], the boundary on which the finite temperature field theory lives is a compact space $S^1 \times S^{D-1}$. The radius of the $D-1$ dimensional sphere breaks the conformal symmetry of the field theory, which makes the phase transition possible. For the case with a non-compact boundary $S^1 \times R^{D-1}$, because of the conformal invariance, no Hawking-Page phase transition exists and on the SYM side only the deconfinement phase is present even in a finite temperature case [30–32]. However, the authors of Ref. [33] are able to realize confinement in certain supersymmetric theories by removing a small radius part of the AdS geometry when the boundary is noncompact. In the framework of gauge/gravity correspondence, the radial coordinate on the gravity side corresponds to the energy scale on the field theory side. Thus the small radius cutoff on the gravity side implies introducing an IR cutoff in the field theory. The so-called hard wall AdS/QCD model has been extensively employed in discussing various properties of low energy QCD [34]. There is one point to remind here that for supersymmetric field theories which live on a non-compact space, introducing an IR cutoff is an effective way to realize a confinement-deconfinement phase transition, while for those which live on a flat but at least one dimension compact space, ie. $S^1 \times S^1 \times R^{D-2}$ or so, there is a kind of AdS soliton [35] which can be used to realize confinement. Hawking-Page phase transitions can occur between Ricci flat AdS black holes and AdS solitons both with at least one dimension compact, see *e.g.* [36]. The application to AdS/CMT can be found in [37, 38].

We will later encounter an interesting version of the confinement transition which goes hand in hand with the dynamical mass generation as is familiar from QCD and so forth. This is generically encoded in the bulk in the form of the geometry "coming to an end" in the deep interior using the bottom up soft- and hard wall constructions. One open question which is right now at the center of attention in the condensed matter applications is whether this bears any direct relevance to the "electron-fractionalization" ideas associated with the slave-particle theories developed for the strongly interacting electron systems of condensed matter physics.

VI. HOLOGRAPHIC HYDRODYNAMICS

As we already highlighted, next to thermodynamics one can also study the linear response of the system by using the GKPW formula. Considering the finite temperature states of the field theoretical system as described by Schwarzschild black holes in the bulk one can ask a very deep question. Regardless the nature of the system at short times, when temperature is finite and one measures at sufficiently long times it has to be that one finds out that the system behaves like a classical, dissipative medium. Moreover, when the system lives in a Galilean continuum and it does not break any symmetry it has to behave like a classical fluid. Already in the 19-th century the universal theory describing such a fluid in the "deep IR" was written down: Navier-Stokes hydrodynamics. One should appreciate that the theory of hydrodynamics has a completely different structure than any quantum field theory: it is even impossible to derive it from an action principle. For this reason it has been proven to be very difficult to derive it directly from a truly microscopic description. As demonstrated by our 19-th century forefathers, it is a "triumph of emergence" in the sense that it can be derived without much knowledge of the microscopic degrees of freedom: it just rests on the way conservation laws act out at macroscopic scales. The microscopic information enters through the numbers in the theory, like the viscosity. One can attempt to derive these numbers via the quantum Boltzmann equations but there is a glass ceiling. In order for Boltzmann theory to work one has to be close to a gaseous limit so that one can speak about particles having a mean free path and so forth. In fact, physicists have been largely unsuccessful in computing these numbers from first principles even in dense, strongly correlated classical liquids like water. Dealing with strongly interacting quantum critical states this problem looks at first sight even more intractable. In order to use Boltzmann equations one needs the ingredient of particles that scatter against other particles. A particle is a collection of quantum numbers which has a well defined location in space time in the form of its worldlines, that one might want to sum over or not pending whether $\hbar$ is important. In a quantum critical state scale invariance rules: one might want to try to localize a quantum number in some restricted space-time location. However, the scale invariance insists that nothing should change when one blows up that region such that it fills all of space-time! Fact is that in condensed matter physics we have learned to think qualitatively about these matters, with a special credit to the work of Sachdev during the last 20 years. However, it has proven to be quite difficult to arrive at truly quantitative answers.

In all these regards, AdS/CFT has delivered a series of miracles. The only short coming is that anything which is quantitative (universal amplitudes, anomalous dimensions; in general, cross over functions) is tied to special large N and large λ UV's. However, it appears that it has delivered a complete description of the *qualitative* physics. This includes the generic renormalization flow from the high energy quantum coherent regime to the dissipative regime at energies less than temperature. The most stunning result is however that the *structure* of the Navier-Stokes equations has been shown to be precisely encoded in the gravitational physics near the horizon of the bulk black hole. This works so well that it is at present used to debug the hydrodynamical equations in particularly complicated situations. The point in case is the recent demonstration of an error in the famous Landau-Lifshitz hydrodynamical theory for finite temperature superfluids!

Let us follow here more or less the historical order in which this art developed. We start out on the simple linear response level, with the discovery of the "minimal viscosity" that impacted first on the experimental study of the quark-gluon plasma, and later on the unitary fermion gas of the cold atom physicists (section VI A, VI A). We then turn to the holographic derivation of Navier-Stokes in its simplest setting; this is doing justice to the non-linear and non-equilibrium aspects of hydrodynamics and one obtains a first glimpse of the use of the correspondence to deal with real time evolution (section VI B). Finally, we will return to the linear response level to introduce the correspondence at one higher level of sophistication: we leave the purely gravitational bulk to address the fate of the finite temperature AC transport properties associated with a conserved charge in the boundary theory employing Einstein-Maxwell theory in the bulk (section VI C).

A. Quantum criticality and the minimal viscosity

This will be the first example of the use of the GKPW formula to compute a field theoretical linear response quantity using the propagation of the classical fields in the bulk. There is a price to pay for the condensed matter physicists: one is dealing here with the hydrodynamics of a relativistic fluid which looks a bit different from the non-relativistic Navier-Stokes theory that is taught at introductory courses. As usual, Lorentz invariance is a simplifying circumstance and the structure of relativistic hydrodynamics is actually simpler than the non-relativistic limit. Recall the non-relativistic Navier-Stokes equation,

$$\rho \frac{D\vec{v}}{Dt} = -\nabla p + \nabla \cdot \vec{T} + \vec{f} \tag{VI.1}$$

where $\vec{v}$ is the fluid velocity and

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \tag{VI.2}$$

is the "material" derivative; ρ is the mass density and this equation just expresses that the mass density times the acceleration of the fluid is equal to the forces exerted on the fluid, as collected on the right hand side. This included the gradient of the pressure p, the external forces $\vec{f}$, and the gradient of the stress tensor $\vec{T}$. The latter describes the viscous stresses which are in turn proportional to the gradients of the velocity field, $\vec{T} \sim \nabla \vec{v}$. The constants of proportionality in this equation are the viscosities; the shear- (η) and volume (ζ) viscosities are associated with the components transversal- and parallel to the flow. For incompressible flows (typical velocities small compared to the sound velocity) only the shear viscosity plays a role. In fact, in conformal systems ζ has to vanish always.

Newton's second law is of course a consequence of the conservation of linear momentum in the Galilean continuum. Together with the conservation of energy, this implies that the Navier-Stokes equation can be written in a relativistic system in a very compact form,

$$\partial_\mu T^{\mu\nu} = 0 \tag{VI.3}$$

just expressing the conservation of the energy momentum tensor $T^{\mu\nu}$.[1] In the next section we will dwell further in the structure of the relativistic hydrodynamics. These basic observations might already suffice to let the reader accept that a very convenient linear response Kubo relation exists, expressing the shear viscosity in terms of the absorptive part of the spatially transversal (T_{xy}) energy momentum propagator through,

$$\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im} G^R_{xy,xy}(\omega, 0), \tag{VI.4}$$

where $G^R_{xy,xy}(\omega, 0)$ is the retarded Green's function of the xy component of the energy momentum tensor, defined as

$$G^R_{\mu\nu,\alpha\beta}(\omega, \vec{k}) = -i \int dt d\vec{x} e^{i\omega t - i\vec{k}\cdot\vec{x}} \theta(t) \langle [T_{\mu\nu}(t, \vec{x}), T_{\alpha\beta}(0, 0)] \rangle. \tag{VI.5}$$

In the holographic context this is a quite interesting, and elementary property: at the very start (section III) we learned that the energy-momentum in the boundary dualizes in metric perturbations in the bulk. In fact, the transversal components T_{xy} dualize in literal gravitational waves in the bulk. The difference with section III is that we have now to consider what happens in the presence of the black hole in the deep interior. One has now to pay more care with the causality structure of the propagators. As usual we are interested in the retarded propagator in the real time formalism which we discussed shortly in section III. It is perhaps not surprising that to get its absorptive (imaginary) part right in the boundary, one has to impose *infalling* boundary conditions at the horizon in the bulk. The stuff that disappears behind the horizon in the bulk encodes for the losses in the boundary field theory.

Underneath we will attempt to give the reader an idea of how the full calculation works. The bottom line is however quite simple to understand. Eq. (VI.4) expresses the obvious fact that the viscosity emerges in the long time limit. In the bulk, this corresponds with the black hole horizon where the radial direction ends. The absorptive part of this response function dualizes in the rate that the gravitons fall through the horizon: the shear viscosity of the finite temperature conformal field theory is therefore expected to be proportional to the absorption cross section of gravitons by the black hole! The outcome of the full computation will be [39],

$$\eta = \frac{\sigma_{\mathrm{abs}}(0)}{16\pi G} \tag{VI.6}$$

where the graviton absorption cross section of the black hole in the $\omega \to 0$ limit turns out to be

$$\sigma_{\mathrm{abs}}(0) = A_{\mathrm{hor}} \tag{VI.7}$$

where A_{hor} is the horizon area of the black hole. Since the black hole entropy density Eq. (V.15) is set by the same quantities A and G, the ratio of the viscosity and the entropy density is now completely independent of anything other than geometrical factors. Restoring the explicit units one arrives at a very famous result [40],

$$\frac{\eta}{s} = \frac{1}{4\pi} \frac{\hbar}{k_B} \tag{VI.8}$$

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[1] Note that we will not distinguish the bulk and boundary indexes and label them all by μ, ν *etc.* as it is easy to know whether they are bulk or boundary quantities.

which turns out to be generic for any field theoretical plasme with an Einstein-gravitational dual [39–42], while it is even unchanged at a finite chemical potential [43, 44]. It is surely tied to the large N limit: when finite N corrections are taken into account, the prefactor will become larger than $1/4\pi$ [45].

The viscosity to entropy density ratio η/s is a natural measure for the "Planckian dissipation". According to AdS/CFT the viscosity of the field is dual the absorption cross section of gravitons by the black hole, yielding $\eta/s = (1/4\pi)\hbar/k_B$ in the case of Maldacena's large N superconformal Yang-Mills. It is believed that the factor $1/4\pi$ represents an absolute lower bound for this ration ("minimal viscosity") or upper bound for the dissipative power of this hydrodynamical liquid.

This ratio of viscosity and entropy turns out to be a natural observable quantity in experiments on relativistic plasma's. Compared to typical values found in normal fluids, Eq. (VI.8) is numerically extremely small [39] and it is therefore called the "minimal viscosity" (or KSS bound [41] by high energy community) representing the closest approach to the perfect, non-viscous fluid. This result caused a big splash when experimental results became available for this quantity at the relativistic heavy ion collider (RIHC) in Brookhaven. By smashing heavy nuclei together the purpose of this machine was to create and study the quark gluon plasma (QGP): nuclear matter is supposed to fall apart at sufficiently high temperatures in this deconfined plasma. Although still controversial, it was claimed around 2005 that the QGP was created during a short instance at the fire balls caused by the colliding nuclei and by studying the so-called "elipticity" of the collisions it is relatively easy to determine the viscosity-entropy density ratio of the QGP. In fact, the only way one can address this quantity using other means is by weak coupling perturbation theory departing from the UV asymptotic free regime. The numerical lattice QCD is nowadays quite reliable for the thermodynamics in the low density-high temperature regime as of relevance to the RIHC experiments. However, due to the extreme accuracy that is required for the Wick rotation to real time in the low frequency regime transport quantities like the viscosity are still out of reach of the brute force numerics. According to the perturbative computations the ratio should be (see e.g. [46]),

$$\frac{\eta}{s}\Big|_{\lambda\to 0} = \frac{A}{\lambda^2 \ln(B/\sqrt{\lambda})} \tag{VI.9}$$

where A, B are of order 1 constants and theory dependent. In the weak-coupling case, it has a strong dependence on the 't Hooft coupling λ, and diverge when λ flows to 0 in the deep UV. This just reflects the somewhat confusing fact that the viscosity of a near-ideal gas is very high because the momentum is carried around by the microscopic degrees of freedom over long distances between the collisions. It was therefore a big surprise when the RIHC experimentalists found that the viscosity-entropy ratio is within the error bar consistent with the AdS/CFT result!

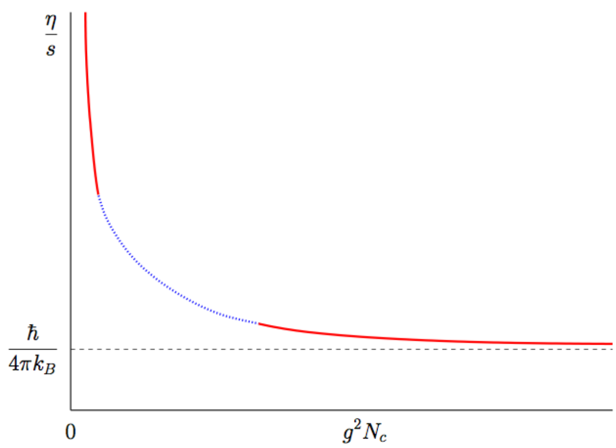

FIG. 5: The dependance of η/s on the 't Hooft coupling $\lambda = g^2 N_c^2$. The plot is taken from [39].

Although it took a little while before it was recognized, it was immediately understood by the condensed matter physicists that had been studying the finite temperature physics of their quantum critical states. In section V, we already discussed the notion of Planckian dissipation: the simple observation further substantiated by especially Sachdev that inverse temperature acts as a finite scale in the otherwise scale invariant quantum dynamics, showing up in observables as a “Planckian dissipation time” $\tau_\hbar = A\hbar/k_B T$ where A is a constant number. This is also the secret of the minimal viscosity. In the presence of a single energy relaxation time τ it should be on dimensional grounds that the viscosity of a relativistic fluid $\eta = (\epsilon + p)\tau$ where ϵ and p are the energy density and pressure. The entropy density is $s = (\epsilon+p)/T$, and therefore $\eta/s = T\tau$. In a Planckian dissipator $\eta/s = T\tau_\hbar = A\hbar/k_B$! What really matters in Eq. (VI.8) is that the magnitude of the purely dissipative (entropy producing) quantity viscosity is set by Planck's constant: this is the reason the ratio is so small compared to regular fluids that do not emerge from the quantum conformal zero temperature vacuum. In stable states the scales present in Euclidean space time will completely wipe out the knowledge regarding the shrinking Euclidean time circle associated with the rising temperature. The prefactor A is a universal amplitude associated with the universality class of the quantum critical state. In the general case it is very hard to compute, all one knows is that it has to be of order one. In this regard AdS/CFT has an interesting message: the large N- large λ conformal theory is perhaps in absolute terms the most strongly interacting CFT that can be imagined. It might well be that therefore the smallest value that can be attained by the factor A is $1/(4\pi)$.

To end this physics story, it appears that the quark gluon plasma near the deconfinement transition approaches rather closely a quantum critical state: lattice calculations that the “non-conformality” that can be measured easily in Euclidean signature becomes quite small in this regime, with the effect that the plasma mimics a real strongly interacting conformal state. In the mean time this same theme has becoming quite fashionable in the cold atom community. Tuning the Feshbach resonances it appears possible to create the “unitary fermi gas” that is strongly interacting but scale free due to the diverging scattering length. Various claims have appeared that the minimal viscosity has been observed in these systems.

Computing the viscosity explicitly: linearized gravity in the bulk

Let us now sketch the explicit computation for the shear viscosity [48]. In principle this is among the simplest calculations for a linear response property one can imagine in the AdS/CFT context. The bulk is just pure Einstein gravity, while the geometry (an AdS Schwarzschild black hole eq. (V.6)) is quite simple as compared to the situations we will encounter later. Given that we are interested in linear response in the boundary, it just suffices to consider the infinitesimal fluctuations of the metric around the AdS Schwarzschild background – we need to know how the gravity waves of linearized gravity after being “excited” at the boundary (the GKPW formula), travel along the radial direction to get absorbed by the black hole. There is now an issue in how one wants to deal with the Wick rotation. In principle one could compute the energy momentum propagator Eq. (VI.5) first in Euclidean signature, to Wick rotate to real time at the end of the calculation. However, with AdS/CFT it is remarkably straightforward to compute directly in real time: this is encoded in AdS by standard classical field theory in a Minkowski signature spacetime and the horizon takes care of the dissipation at long times.
As usual in linearized gravity, one writes the metric as $g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$ where $g^0_{\mu\nu}$ is the background metric that solved for the AdS Schwarzschild black hole while $h_{\mu\nu}$ is the infinitesimal metric fluctuation, *i.e.* the gravitational wave/graviton. The counting of the number of DOF for the massless graviton is a bit of a tricky affair:

We focus on $4+1$ dimensional bulk. We first classify the graviton modes as follows. We take the spatial momentum to be in the x_3 direction $\vec{k} = (0,0,k)$ and this means that the perturbation $h_{\mu\nu} = h_{\mu\nu}(t,r,x_3)$. The system has an $SO(2)$ symmetry in the x_1x_2-plane and according to the behavior of the graviton modes under this symmetry we have three decoupled sets of graviton modes: the tensor mode (transverse mode) $h_{x_1x_2}$, the vector mode (shear mode), a linear combination of h_{tx_1}, $h_{x_3x_1}, h_{rx_1}$ together with h_{tx_2}, $h_{x_3x_2}, h_{rx_2}$ and two scalar modes (sound mode) from two linearly independent combinations of h_{tt}, h_{tx_3}, $h_{x_3x_3}$, $h_{x_1x_1} + h_{x_2x_2}$, h_{rr}, h_{tr} and h_{rx_3}. It directly follows from the linearized coordinate transformations on $h_{\mu\nu}$

$$\delta h_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \tag{VI.10}$$

that $h_{x_1x_2}$ in this momentum configuration is a gauge-invariant, i.e. physical mode.

We now perturb the Einstein equation with a negative cosmological constant, eqn (V.5), around the solution of the AdS-Schwarzchild black hole geometry (V.6). Denoting the perturbation $h^{x_1}_{x_2}(t,r,x_3)$ with [47]

$$\delta g^{x_1}_{x_2} = h^{x_1}_{x_2}(t,r,x_3) = \int \frac{d\omega dk}{(2\pi)^2} \phi(r;\omega,k) e^{-i\omega t + ikx_3}, \tag{VI.11}$$

one obtains linearized equation of motion for $\phi(r;\omega,k)$. In accordance with the tensorial analysis, it equals the equation for a massless scalar

$$\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \phi(r;t,x_3) = 0 \tag{VI.12}$$

Substituting the background metric eq. (V.6) and Fourier transforming we find

$$\phi(r;\omega,k)'' + \left(\frac{D+1}{r} + \frac{f'}{f}\right)\phi'(r;\omega,k) + \frac{(\omega^2 - k^2 f)L^4}{r^4 f^2}\phi = 0 \tag{VI.13}$$

with $f = (1 - \frac{r_0^D}{r^D})$.

To obtain the retarded Green's function in the CFT we must now solve this equation with infalling boundary conditions at the horizon. The retarded Green's function is then given by the ratio of the subleading to the leading coefficient of the solution at the boundary:

$$\phi_{\rm sol}(u,\omega,k) = A(\omega,k)r^{-\Delta_-} + B(\omega,k)r^{\Delta_- - D} + \ldots \tag{VI.14}$$

with $\Delta_- = 0$ in this case, and

$$G^{CFT}_{R,xy,xy}(\omega,k) \sim \frac{B(\omega,k)}{A(\omega,k)} \tag{VI.15}$$

To obtain $A(\omega,k)$ and $B(\omega,k)$ is a relatively straightforward numerical exercise; in some special cases the answer is analytically known. Rather than using this brute force short-cut, we will give an alternate way [53, 54] to compute the solution which captures some of the essential physics. We first write the Green's function directly in terms of the solution $\phi_{\rm sol}$ following the GKPW construction

$$G^{CFT}_R(\omega,k) = \frac{1}{2\kappa^2}\lim_{r\to\infty} r^{-2\Delta_-}\sqrt{-g}g^{rr}\frac{\partial_r\phi_{\rm sol}(r)}{\phi_{\rm sol}(r)} \tag{VI.16}$$

The overall factor $1/2\kappa^2$ follows from the normalization of the Einstein-Hilbert action.

The key step is that the viscosity (and all other transport coefficients) follow from the imaginary part of the retarded Green's function. Also inserting $1 = \phi^*(r)/\phi^*(r)$ inside the limit one obtains for this imaginary part the expression

$$\text{Im}G^{CFT}_R(\omega,k) = \lim_{r\to\infty} r^{-2\Delta_-}\sqrt{-g}g^{rr}\frac{\phi^*_{\rm sol}(r)\partial_r\phi_{\rm sol}(r) - \phi_{\rm sol}(r)\partial_r\phi^*_{\rm sol}(r)}{2i\phi^*_{\rm sol}(r)\phi_{\rm sol}(r)} \tag{VI.17}$$

The numerator is readily recognized as the Wronskian which measures the flux density through a surface at fixed r. The physics is that for a field obeying a second order equation of the type

$$\phi'' + P(r)\phi' + Q(r)\phi = 0 \tag{VI.18}$$

with $P(r)$ and $Q(r)$ real, the generalized Wronskian

$$W(r) = e^{\int^r P(r)}(\phi^*_{\rm sol}(r)\partial_r\phi_{\rm sol}(r) - \phi_{\rm sol}(r)\partial_r\phi^*_{\rm sol}(r)) \tag{VI.19}$$

is conserved: $\partial_r W = 0$. Note that the factors $\sqrt{-g}g^{rr}$ is precisely this necessary prefactor: this is readily seen by acting with ∂_r including these metric factors and using the equation of motion (VI.12). Rewriting the imaginary part of the retarded Green's function directly in terms of the conserved Wronskian we find

$$\text{Im}G^{CFT}_R(\omega,k) = \lim_{r\to\infty} r^{-2\Delta_-}\frac{W(r)}{2i\phi^*_{\rm sol}(r)\phi_{\rm sol}(r)} \tag{VI.20}$$

We will discuss the physics of this rewriting in a moment. Mathematically its computational power for frequency-independent transport coefficients is immediate. We can use the conservation of the Wronskian to evaluate the numerator at any point r. The most convenient is the horizon itself where the infalling boundary conditions are set. Near the horizon, near the single zero of $f(r)$, the equation of motion (VI.13) reduces to

$$\phi'' + \frac{f'}{f}\phi' + \frac{L^4\omega^2}{r^4 f^2}\phi + \ldots = 0 \tag{VI.21}$$

Writing $f(r) = (r - r_0)\frac{D}{r_0} + \ldots$,

$$\phi'' + \frac{1}{(r-r_0)}\phi' + \frac{L^4\omega^2}{D^2 r_0^2(r-r_0)^2}\phi + \ldots = 0 \tag{VI.22}$$

we can deduce the powerlaw dependence of the solution near the horizon by substituting the ansatz

$$\phi_{sol}(r;\omega,k) = (r-r_0)^\alpha(1+\ldots) \tag{VI.23}$$

One finds

$$\alpha(\alpha-1)+\alpha+\frac{L^4\omega^2}{D^2r_0^2}=0 \tag{VI.24}$$

with solutions $\alpha=\pm\frac{i\omega L^2}{Dr_0}=\pm\frac{i\omega}{4\pi T}$. In the last step we used the relation between the horizon location and the black hole temperature derived in eqn. (V.10). The choice $\alpha=-i\omega/4\pi T$ corresponds to the infalling solution. Thus near the horizon we may parametrize

$$\phi_{\rm sol}(z;\omega,k)=(r-r_0)^{-i\omega/4\pi T}F(r;\omega,k), \tag{VI.25}$$

where $F(r;\omega,k)$ is regular at the horizon $r=r_0$. Evaluating now the conserved Wronskian near the horizon one finds

$$\begin{aligned}
W(r_0) &= \lim_{r\to r_0}\sqrt{-g}g^{rr}\phi^*\overleftrightarrow{\partial}\phi \\
&= \lim_{r\to r_0}\frac{r^{D+1}}{L^{D+1}}(1-\frac{r_0^D}{r^D}\left(\frac{-2i\omega}{4\pi T}(r-r_0)^{-1}\right)F^*(1)F(1)+\dots \\
&= \frac{r_0^{D+1}}{L^{D+1}}\frac{D}{r_0}\frac{-2i\omega}{4\pi T}F^*(1)F(1) \\
&= (\frac{4}{D}\pi TL)^{D-1}(-2i\omega)F^*(1)F(1)
\end{aligned} \tag{VI.26}$$

In the last line, we have again used the definition of the temperature $4\pi TL=Dr_0/L$. Substituting, one sees that the remaining unknown in the Green's function

$$\mathrm{Im}G_R^{CFT}(\omega)=\frac{1}{2\kappa^2}\lim_{r\to\infty}(\frac{4}{D}\pi TL)^{D-1}(-\omega)\frac{F^*(1)F(1)}{F^*_{\rm sol}(r)F_{\rm sol}(r)} \tag{VI.27}$$

is the ratio of the absolute value of $F(r)$ at the horizon to $|F(r)|$ at the boundary. Formally one still needs to solve for $F(r)$ to determine this. However, in the limit $\omega\to 0, k\to 0$ the leading contribution will be the ω-independent solution for the remaining function F. From equation (VI.13) this is readily seen to be the trivial constant function. This is the leading solution $\phi\sim Ar^{-\Delta_-}$ near $r\to\infty$ with our special case of $\Delta_-=0$. Hence one obtains

$$\mathrm{Im}G_R^{CFT}(\omega,k)=\frac{1}{2\kappa^2}(\frac{4}{D}\pi TL)^{D-1}(-\omega) \tag{VI.28}$$

From the Kubo relation

$$\eta=-\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\langle T_{x_1x_2}(-\omega)T_{x_1x_2}(\omega)\rangle=-\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G_R^{CFT}(\omega,0) \tag{VI.29}$$

the shear viscosity therefore equals

$$\eta=\frac{1}{2\kappa^2}(\frac{4}{D}\pi TL)^{D-1} \tag{VI.30}$$

Recalling that $2\kappa^2\equiv 16\pi G$. Comparing this to the entropy density (V.15) $s=\frac{4\pi}{2\kappa^2}(\frac{4}{D}\pi TL)^{D-1}$ for a $D+1$-dimensional AdS Schwarzschild black hole, we find the famous ratio

$$\frac{\eta}{s}=\frac{1}{4\pi}\frac{\hbar}{k_B}. \tag{VI.31}$$

In 2001, Policastro, Son and Starinets pioneered the calculation of transport coefficients in strongly coupled field theories which are dual to Anti-de-Sitter gravities with the computation of the now famous ratio of $\eta/s=1/4\pi$ [40]. During the following several years, they and others calculated this ratio for some other dual gravity systems. They found that in any dimension, pure Einstein gravity always gives $\eta/s=1/4\pi$ [39–42] and when there is a nonzero chemical potential, the ratio is still $\eta/s=1/4\pi$ [43, 44] . When large N corrections are taken into account, the ratio will get positive corrections [45]. With the fact that all known fluids in nature have $\eta/s>1/4\pi$, KSS gave a conjecture that $1/4\pi$ is the lower bound of this ratio in nature [41].

However, in 2007, H.Liu *et al.* found that this ratio for AdS Gauss-Bonnet gravity can be smaller than $1/4\pi$ [47, 51]. After considering the constrains from causality they proposed a new lower bound $4/25\pi$.

B. The Navier-Stokes fluid as the near horizon hologram

The capacity of AdS/CFT to deal with classical hydrodynamical properties was taken a step further by the explicit demonstration by Bhattacharya, Minwalla and Wadia [58] that the coarse grained near horizon *dynamical* gravity is in a one-to-one relation with the Navier-Stokes equations in the boundary, which form the fundament of classical fluid dynamics. This is quite an accomplishment given that classical hydrodynamics is a theory that is completely detached in its structure from quantum field theory. It does describe the non-linear and non-equilibrium dissipative dynamics of the fluid just under the assumption of local equilibrium, and that is surely a physics that is completely different from unitary quantum field theory. The only non-generic feature is that one derives the hydrodynamics for special parameters associated with the zero temperature underlying CFT, like the minimal viscosity of the previous section. The first indications that there is this deep connection between black hole physics and hydrodynamics date back to the 1980's when Wheeler and Thorne identified it as a mathematical coincidence (the 'membrane paradigm' [23]). In the context of quantum gravity it is still a conundrum whether this connection might extend beyond the "practical" realms of AdS/CFT: for instance, Strominger *et al.* [59] demonstrated the connection also for an asymptotically flat space-time.

A highlight of AdS/CFT is the demonstration that the Navier-Stokes equations describing the macroscopic classical fluid realized in the finite temperature boundary field theory is in its full non-linear and non-equilibrium sense encoded in the space-time jitters near the horizon of the black hole in the bulk, as described by Einstein gravity.

Let us sketch here the bare bones of the derivation. As we already pointed out, the dynamical side of fluid dynamics becomes quite elegant for a relativistic fluid. It is just captured in the simple conservation law of the energy momentum tensor, $\partial_\mu T^{\mu\nu} = 0$ (Eq. VI.3). This part one gets for free in the correspondence. At the very beginning we departed from the observation that the global space-and time translations which imply the conservation of energy-momentum in the boundary are encoded in the form of the diffeomorphisms of the bulk theory. This part cannot fail. However, matters become very different from field theory when one specifies the actual form of the energy-momentum tensor associated with a classical relativistic fluid. The derivation underneath is much helped by the fact that one can organize matters on both sides in terms of a gradient expansion [49]. On the hydrodynamical side the zero-th order is the "perfect fluid" as characterized by vanishing viscosity. We already learned that the viscosity of the AdS/CFT fluid is very small, and this surely helps. According to the textbook,

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}, \tag{VI.32}$$

where the u^μ's are the local relativistic "four"-velocities of the fluid associated with a Minkowski signature ($u^\mu u_\mu = -1$); $g^{\mu\nu}$ is the background metric of the space (here flat, $g^{\mu\nu} = \eta^{\mu\nu}$), while ϵ and P are the free-energy and pressure, respectively, associated with the assumption of local equilibrium characterized by a local temperature $T(x)$. In addition one has the thermodynamical relations $d\epsilon = TdS$, $dP = sdT$, and $\epsilon + P = Ts$, where s is the entropy density. Resting on symmetry arguments one arrives at the form of the leading order gradient corrections

$$\sigma^{\mu\nu} = P^{\mu\kappa} P^{\nu\lambda}\left[\eta(\partial_\kappa u_\lambda + \partial_\lambda u_\kappa - \frac{2}{3} g_{\kappa\lambda}\partial_\alpha u^\alpha) + \zeta g_{\kappa\lambda}\partial_\alpha u^\alpha\right], \tag{VI.33}$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection operator onto directions perpendicular to u^ν. In this order one meets the shear- and bulk viscosities η and ζ , with the latter vanishing in conformal systems. This defines the leading order (in gradients) relativistic Navier-Stokes theory. The usual non-relativistic Navier-Stokes is just a special limit, obtained by taking the non-relativistic limit $\varepsilon \to 0$,

$$\partial_i \to \varepsilon\partial_i, \quad \partial_\tau \to \varepsilon^2\partial_\tau, \quad v_i \to \varepsilon v_i, \quad P \to \bar{p} + \varepsilon^2 P. \tag{VI.34}$$

It has been obvious all along that the stationary black-hole geometries in the bulk encode for global thermodynamic equilibrium in the field theory [24]. Given that the boundary- and bulk theories share their spatial- and time directions, and since one considers in the hydrodynamics long wavelength- and long time deviations from this global equilibrium one has to study the gravitational of the corresponding metric fluctuations in the bulk. The procedure of Bhattacharyya *et al.* [56] is as follows (see also [55] and the review ref. [60]). Matters are computed in terms of units such that the AdS radius $L = 1$. Departing from the Schwarzschild geometry Eq. (V.6) one performs a coordinate transformation to ingoing static coordinates u_μ which are subsequently promoted to slowly varying functions $u_\mu(x)$ of space-time x. In addition, one also allows for slow fluctuations of the local horizon area which are parametrized directly in the Hawking temperature $T(x)$ of the boundary

$$ds^2 = -2u_\mu(x)dx^\mu dr - r^2 f\left(\frac{r}{T(x)}\right) u_\mu(x)u_\nu(x)dx^\mu dx^\nu + r^2\left(u_\mu(x)u_\nu(x) + \eta_{\mu\nu}\right)dx^\mu dx^\nu, \ \ \mu,\nu = 0,...D-1. \tag{VI.35}$$

where $f(\frac{r}{T(x)}) = 1 - (\frac{4\pi T(x)}{dr})^D$.

Although the metric $g^{(0)}_{\mu\nu}(T(x), u(x))$ solves the Einstein equations for constant u^μ, T it is surely not a solution of Eq. (V.5) for generic functions of space-time. However, one can now expand the u, T's in a gradient expansion with regard to x, and obtain perturbative solutions order by order, just as in the derivation of fluid dynamics. Identifying the orders in this expansion in terms of the small parameter ϵ as $\partial_\mu \sim \varepsilon, \partial_r \sim \varepsilon^0, u_\mu \sim \varepsilon^0, p \sim \varepsilon^0$, this gradient expansion relative to the stationary solution takes the form,

$$g_{\mu\nu} = \sum_{n=0}^{\infty} \varepsilon^n g^{(n)}_{\mu\nu}(T(\varepsilon x), u(\varepsilon x)), \quad u^\mu = \sum_{n=0}^{\infty} \varepsilon^n u^{\mu(n)}(\varepsilon x), \quad T = \sum_{n=0}^{\infty} \varepsilon^n T^{(n)}(\varepsilon x), \tag{VI.36}$$

and this can be used to solve iteratively the bulk Einstein equations (V.5) in principle up to arbitrary order. To reproduce the standard hydrodynamics at the beginning of this section one can get away with the first order in ε. In fact, this graviational procedure has proven to be quite useful to derive the higher derivative generalizations of Navier Stokes which are hard to get at using the 19-th century "by hand" methods. Using GR this exercise just turns into straightforward algebra! Up to first order in ε one finds the gravitational solution,

$$\begin{aligned} ds^2 &= ds_0^2 + ds_1^2 + \dots \\ ds_0^2 &= -2u_\mu dx^\mu dr - r^2 \tilde{f}(\beta r) u_\mu u_\nu dx^\mu dx^\nu + r^2 \big(u_\mu u_\nu + \eta_{\mu\nu}\big) dx^\mu dx^\nu, \\ ds_1^2 &= 2r^2 \beta F(\beta r) \sigma_{\mu\nu} dx^\mu dx^\nu + \frac{2}{D-1} r u_\mu u_\nu \partial_\alpha u^\alpha dx^\mu dx^\nu - r u^\alpha \partial_\alpha (u_\mu u_\nu) dx^\mu dx^\nu, \end{aligned} \tag{VI.37}$$

where

$$\tilde{f}(r) = 1 - \frac{1}{r^D}, \quad \beta = \frac{D}{4\pi T}, \quad F(r) = \int_r^\infty dy \left(\frac{y^{D-1} - 1}{y(y^D - 1)} \right). \tag{VI.38}$$

This time dependent solution describes the metric throughout the bulk, including the region near the boundary. However, there is still an issue of how to dualize this information to the boundary field theory. To make this work, one has to reformulate the dictionary entry for the expectation value of the energy-momentum tensor of the boundary theory. The boundary stress tensor associated with the bulk gravity has been found by Brown and York to be

$$T_{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{grav}}}{\delta \gamma^{\mu\nu}} = \lim_{r\to\infty} \frac{-r^D}{\kappa^2} \left[K_{\mu\nu} - K\gamma_{\mu\nu} + (D-1)\gamma_{\mu\nu} - \frac{1}{d-2} \big({}^\gamma R_{\mu\nu} - \frac{1}{2} {}^\gamma R \gamma_{\mu\nu} \big) \right], \tag{VI.39}$$

where $\gamma_{\mu\nu}$ is the induced metric near the boundary surface $r \to \infty$, ${}^\gamma R_{\mu\nu}$ and ${}^\gamma R$ are the corresponding Ricci tensor and scalar, $K_{\mu\nu}$ and K are the extrinsic curvature and its trace. According the dictionary, the Brown York stress tensor (VI.39) can be interpreted as the expectation value of the stress tensor of the dual fluid

$$\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{-\gamma}} \frac{\delta \ln \langle e^{\int d^D x \sqrt{-\gamma} \gamma^{\mu\nu} T_{\mu\nu}} \rangle}{\delta \gamma^{\mu\nu}} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{grav}}}{\delta \gamma^{\mu\nu}}. \tag{VI.40}$$

As usual, the dictionary causes the miracles. Evaluating this dictionary item for the bulk metric Eq. (VI.37, VI.38), the outcome is

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P \eta^{\mu\nu} - 2\eta \big(\frac{1}{2} P^{\mu\alpha} P^{\nu\beta} [\partial_\alpha u_\beta + \partial_\beta u_\alpha] - \frac{1}{3} P^{\mu\nu} \partial_\alpha u^\alpha \big) \tag{VI.41}$$

which has exactly the form of the energy momentum tensor of the relativistic Navier-Stokes fluid! Surely, the parameters are special: in 3+1 dimensions, one finds the standard equation of state for a conformal fluid $\epsilon = 3P$. The pressure $P = \frac{\pi^2}{8} N^2 T^4$ which can be easily understood as the temperature remains the only scale in a CFT. Finally, the shear viscosity is $\eta = \frac{\pi}{8} N^2 T^3$. Given the usual expression for the entropy this turns into the "minimal viscosity ratio" $\eta/s = 1/4\pi$, as anticipated since this is just a more general way to compute the viscosity than by using the linear response Kubo equation of the previous section.

This is the point in case. It is perhaps the greatest success of the AdS/CFT correspondence that Einstein gravity can be used as "generating functional" to determine hydrodynamical equations. The above is of course a particular basic example but at present this "method" is used with significant practical consequence, by re-assesing particularly complicated forms of hydrodynamics, as e.g. higher order gradient Navier-Stokes, hydrodynamics in the absence of parity, and last but not least to "debug" the Landau hydrodynamics for the finite temperature superfluid.

C. Conserved currents as photons in the bulk

In writing a text like this it is a bit of an unpleasant circumstance that one has to start with a subject that is most remote from the daily life of condensed matter physics: the pure gravity of the bulk. As we will see in the next chapters, this is quite literally a fundament on which the "material" aspects as of direct interest to the condensed matter physicst are erected. Via the bulk isometry-boundary gobal conformal symmetry connection, the pure gravity in the bulk takes care of everything associated with the global space-time symmetries in the boundary field theory. This governs the (free) energy/thermodynamics which is invariably encoded in the bulk geometry, but also the hydrodynamics of the previous sections that is rooted in energy and momentum conservation.

Matter is more: one also needs a "conserved charge" in the jargon of high energy physics. This is just the usual stuff of condensed matter physics: in water we know that water molecules do not disappear by themselves and the number of electrons in a piece of solid is not changing in time. The number of conserved electrons, or either their total electrical charge, is governed by a global $U(1)$ symmetry: familiar to condensed matter physicists as the symmetry which is spontaneously broken in a superfluid. In the AdS/CMT context this is the most important global symmetry. One can also address larger, non-abelian "flavor" symmetries (again, high energy jargon) like the $SU(2)$ associated with the spin currents of condensed matter. A complicating aspect is that these are only covariantly conserved and not truly conserved but this does not matter when wants to address the non-abelian triplet superconductors and so forth [62].

As we already discussed in section III, through the general global-local duality structure of the AdS/CFT correspondence, it has to be that a global internal symmetry in the boundary is dual to its gauged version in the bulk. The global $U(1)$ "counting the matter" in the boundary should therefore be represented by a local $U(1)$ symmetry in the bulk. In the bottom-up setting, the next rule is the weak-strong duality which insists that the bulk theory should have a minimal number of gradients, while it should be considered in the classical limit associated with the large N limit of the "color" degrees of freedom in the boundary. This bulk theory is of course nothing else than classical Maxwell electrodynamics. This lives together with the Einstein theory in the bulk and the bottom line is that the physics of material systems characterized by a conserved number are described by the classical Einstein-Maxwell system in the bulk,[2]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(R + \frac{6}{L^2}\right) - \frac{1}{4g_4^2}F_{\mu\nu}F^{\mu\nu}\right] \tag{VI.42}$$

This simple system is the point of departure for the recent AdS/CMT discoveries: by sourcing the bulk Maxwell fields with an electrical charge residing in the black hole one can address the physics of the field theory at finite density with the effect that all hell breaks loose: the remainder of this narrative. However, as a first introduction to the way that these global internal symmetries are handled by the correspondence, let us here focus in on its hydrodynamical aspects in the zero density but finite temperature conformal state. We will keep this sketchy: when you want to know more details, we refer to the very accessible papers [64, 65] by Sachdev and his collaborators discussing these matters in a much greater detail.

Dictionary entry: "material stuff" as encoded by a conserved current associated with a global symmetry in the boundary dualizes in a gauge field with the same symmetry in the bulk. Dealing with a simple conserved number (global $U(1)$) in the boundary, this implies that the theory in the bulk has to be extended with a $U(1)$ gauge field. The minimal implementation is in terms of Einstein-Maxwell theory in the bulk.

This has a direct relevancy a particular type of quantum critical states that are in principle of relevance to condensed matter physics. The key is zero density, and where do we find such states? One needs special circumstances and the two most appealing examples are the "zero chemical potential" superfluid-Bose Mott insulator transition and the "interacting graphene" model. The very first requirement is the presence of continuous quantum phase transitions. The quantum critical states realized at these transitions should however also be strongly interacting in order to rely on universality. The conformal states of the correspondence are automatically of this kind but dealing with the condensed matter systems one has to assure that these are below the upper critical dimension: the effective dimensionality $d+z$ should be less than 4 for the examples that are quoted here. Since these are effectively relativistic ($z=1$), while one dimensional systems are special, only the systems with two space dimensions are of interest in this specific condensed matter context.

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[2] We focus on 3+1 dimensional gravity, *i.e.* 2+1 dimensional field theory.

The graphene model is perhaps most intuitive. One starts out with the tight binding electrons hopping around on the hexagonal (graphene) lattice giving rise to the Nobel-prize winning effective Dirac fermions near the Fermi energy. Keeping the chemical points at the Dirac nodes ("zero density") one imagines a local "Hubbard U" repulsive interaction active on the "carbon" sites. For some critical value of this interaction an instability will occur to an ordered antiferromagnet, which will gap the fermion spectrum. This quantum phase transition is continuous: the quantum dynamics of the bosonic order parameter field of the antiferromagnet will be governed by an emergent conformal invariance. But it can be demonstrated that the "Yukawa" coupling between this order parameter field and the Dirac fermions is finite at the transition, such that the latter "take part" in the critical regime [63]. The question is: how does the temperature dependent optical conductivity associated with the carbon electrons look like at this transition?

From the point of view of field theory the "Bose-Hubbard" model in 2+1D is even more primitive. The easiest example is the Bose-Mott insulator as prepared by cold atom physicists: one departs from an optical lattice, filling it with an integer number of bosonic atoms per site. One tunes up the local repulsions U using the Feshbach resonance, until this interaction overwhelms the hopping t. The result is the Bose Mott insulator where all atoms are localized on the sites since it costs the large energy U to delocalize anything. The massive excitations are the "holons" (one less boson on a site) and "doublons" (one extra boson). Upon decreasing U at some point the holons and doublons will again delocalize forming directly a superfluid. This maps precisely on the theory of a zero density relativistic superfluid undergoing a condensation in a dual relativistic superconductor: the Mott insulator can be shown to be the same as a dual Higgs phase formed from the vortices of the superfluid, which is now gauged because of the long range interactions between the vortices. This is nothing else than an XY problem in d+1 dimensions, among the best studied critical states. Again the question arises, what is the temperature dependent optical conductivity at the transition? The two point correlation functions are fixed by kinematics and one finds by straightforward scaling arguments that the conductivity is set by engineering dimensions: at zero temperature $\sigma(\omega, T=0) \propto \omega^{(2-d)/z}$ while at zero frequency and finite temperature $\sigma(\omega, T=0) \propto T^{(2-d)/z}$. The dimensionless two dimensionless case is particularly simple since one finds both a temperature independent DC conductivity, and a frequency independent high frequency behavior: $\sigma(\omega=0, T) = \sigma_0$ and $\sigma(\omega, T=0) = \sigma_\infty$. However, the two universal amplitudes σ_0, σ_∞ are rooted in qualitatively very different physics: the Planckian dissipation regime and the high frequency quantum coherent regime. There is therefore no a-priori reason for these to be related and one expects the crossover function to be an interesting function of ω/T. Remarkably, even for this simple system this is not precisely known. The reason is that for the usual Wick rotation reasons even numerics fails: one needs to know the Euclidean correlators so accurately to address the low energy regime that quantum Monte Carlo fails even in this very simple system.

All what remains is to compute matters using the quantum Boltzmann approach, but this fails qualitatively since it gives rise to an elegant paradox as realized first by Sachdev. Here one departs from a language of scattering particles and invariably one obtains a relaxational Drude like response associated with the "high" temperature, low frequency regime, translating into a peak in the optical conductivity at low frequency. However, the Abelian-Higgs duality principle insists that the conductivity of the particles should be identical to the resistivity associated with the dual system of vortices. The latter are again like scattering particles, and their conductivity peak translates in a resistivity dip: one obtains the opposite behavior pending whether one computes the conductivity in a particle or vortex language!

Let us now turn to the holographic computation of the optical conductivity in the 2+1D zero density system. Surely, the result should not to be taken as the literal solution to the XY problem in the above. Instead, as for the minimal viscosity the perspective should be: what is the generic outcome for the conductivity in the case of the most strongly interacting quantum critical state that humanity has identified up to now? From this point of view, the result will be most satisfactory!

Let us walk through this elementary holographic computation slowly, since it is a useful template to see how the more complicated calculations of the next sections work. The starting point is the Einstein-Maxwell action Eq. (VI.42) and we are interested in the "probe limit of the photons": the Maxwell sector is not affecting the AdS/Schwarzschild bulk in any way (zero density) and we just want to infer how infinitesimal electromagnetic pertubations propagate through this fixed space-time. Let us first inspect the dictionary to find out the quantity in the bulk that codes for the conductivity in the boundary. This follows the pattern of the graviton calculation Eq. (III.14-III.24), employing the "ratio rule", Eq. (III.25).

The optical conductivity can be computed from the two point retarded current-current correlation function

$$\sigma(\omega) = -\frac{i}{\omega} G^R_{xx}(\omega, \vec{k}=0) \tag{VI.43}$$

where

$$G^R_{xx}(\omega, \vec{k}) = -i \int dt d\vec{x} e^{i\omega t - i\vec{k}\cdot\vec{x}} \theta(t) \langle [J_x(t, \vec{x}), J_x(0,0)] \rangle. \tag{VI.44}$$

Let us just take the dictionary rule that the $U(1)$ currents J_μ in the boundary dualize in $U(1)$ gauge fields in the bulk. The matter in the boundary is neutral and the sources/applied fields are therefore like chemical potential differences. However, it is a habit to imagine that one switches on the electrical charge in the boundary perturbatively at the end of the calculation for the purpose of determining the linear response of this "weakly charged matter". This should be kept in mind when one reads electrical field in the boundary, or when in the next chapter holographic "superconductivity" is discussed which is in fact about a holographic superfluid. This is just language.

We are interested in the specific AdS_4/CFT_3 case, and we are supposedly dealing with standard Maxwell theory in the bulk, although in the AdS geometry. The first step is to choose a gauge: given that the components in the radial direction have no physical meaning in the boundary a convenient choice is the radial gauge $A_r = 0$. One ends up with the spatial $A_{x,y}$ and temporal A_t components as the physical fields in the bulk. The spatial components decouple. For the full momentum dependent conductivity one finds that at finite $\vec{k}$ the bulk fluctuations A_x and A_t will couple [64]. However, the $\vec{k} = 0$ conductivity as of relevance to experiment is easier to compute since the fluctuation $A_x(\omega, \vec{k} = 0)$ decouples completely from the other fluctuations of the gauge field. We therefore only have to consider the spatially uniform fluctuation $A_x = a_x(r)e^{-i\omega t}$ in the Schwartzschild background (V.6).

Let us first exercise what happens at zero temperature: perhaps the most elementary holographic computation. We are now dealing with the pure AdS_4 geometry, and one obtains straightforwardly from the Einstein-Maxwell the equation of motion for the radial dependence of the coefficients $a_x(r)$, in terms of the radial derivatives,

$$a_x'' + \frac{2}{r}a_x' + \frac{L^4\omega^2}{r^4}a_x = 0. \tag{VI.45}$$

The solution is very simple in this case,

$$a_x = c_0 e^{i\frac{\omega L^2}{r} - i\omega t}, \tag{VI.46}$$

where c_0 is a constant. Near the boundary ($r \to \infty$) it follows immediately from this simple expression that,

$$A_x = A_x^{(0)} + \frac{A_x^{(1)}}{r} + \dots \tag{VI.47}$$

where $A_x^{(0)} = c_0 e^{-i\omega t}$ and $A_x^{(1)} = ic_0\omega L^2$.

In fact, this is a Taylor expansion where $A_x^{(0)}$ is the tail of the gauge potential itself, while $A_x^{(1)} = -r^2\partial_r A_x$: it is in fact the near boundary asymptote of the bulk electrical *field strength* in this radial gauge. This explains the dictionary entries in the table II of section III: according to the ratio rule the leading term $A_x^{(0)}$ is the source/applied field in the boundary and the sub-leading term $A_x^{(1)}$ is the expectation value of the response, the induced current which are dual to the gauge field itself and its field strength in the bulk, respectively.

Using the ratio rule Eq. (III.25) of section III, we immediately read off that that $A_x^{(0)}$ is the source and $A_x^{(1)}$ is the corresponding response. Consistent with the scaling dimension of the $T = 0$ conductivity ($(2-d)/z$) we recover that the conformal dimension for the conserved current J_x is 1. Surely, AdS/CFT gets this right! The conductivity itself follows also from the ratio rule Eq. (III.25),

$$\sigma(\omega) = -\frac{i}{\omega}G^R_{xx}(\omega, \vec{k} = 0) = \frac{1}{L^2 g_4^2}\frac{-i}{\omega}\frac{A_x^{(1)}}{A_x^{(0)}} = \frac{1}{g_4^2} \tag{VI.48}$$

where the overall factor in front of the ratio is determined from the GKPW rule (III.25).

For the connaisseurs of linear response theory, an other way of understanding this result is as follows. The leading order contribution of the gauge field near the boundary gives the electric field on the boundary: $E_x = -\dot{A}_x^{(0)}$, and the VEV of the induced current (response) is the corresponding bulk field strength: $J_x = \frac{1}{L^2 g_4^2}A_x^{(1)}$. From Ohm's law, we have

$$\sigma = \frac{J_x}{E_x} = \frac{1}{L^2 g_4^2}\frac{A_x^{(1)}}{-\dot{A}_x^{(0)}} = \frac{1}{L^2 g_4^2}\frac{-iA_x^{(1)}}{\omega A_x^{(0)}}. \tag{VI.49}$$

and filling in the results of the previous paragraph yields immediately the conductivity.

Let us now compute the optical conductivity for the CFT at finite temperature: it repeats the above but now the Schwarzschild black hole is in the background. This changes the radial equation of motion Eq. (VI.45) into,

$$a_x'' + \left(\frac{2}{r} + \frac{f'}{f}\right)a_x' + \frac{L^4\omega^2}{r^4 f^2}a_x = 0. \tag{VI.50}$$

now involving the black hole red shift factor $f(r)$ of Eq. (V.7) and its radial derivatives: $f(r) = 1$ corresponds with the absence of the black hole and this result reduces to the zero temperature case. As explained in section III, to obtain the retarded Green's function in this real time formalism we need to impose infalling boundary conditions for the photons at the horizon. From section V we know that near the horizon the red shift factor becomes

$$f \simeq \frac{4\pi T L^2}{r_0^2}(r - r_0) + \mathcal{O}((r - r_0)^2).$$

Inserting this into Eq. (VI.50) to determine the solution near the black hole horizon one finds two branches of solutions near the horizon

$$a_x \propto e^{\pm \frac{i\omega}{4\pi T}\ln(r - r_0)}\big(1 + \mathcal{O}(r - r_0)\big). \tag{VI.51}$$

Since $A_x = a_x(r)e^{-i\omega t}$, near the horizon, the '+' and '-' branches correspond with the outgoing- and infalling waves. We need the infalling one, such that the final solution behaves as $a_x \propto (r - r_0)^{-i\omega/(4\pi T)}$ when $r \to r_0$.

In this case it is still possible to derive a closed solution for the radial differential equation for this boundary condition —— for anything more complicated it appears that one has to rely on numerics. This explicit solution is,

$$A_x = c_0\left(\frac{4\pi T L^2 - 3r}{\sqrt{9r^2 + 12\pi T L^2 r + (4\pi T)^2 L^4}}\right)^{-\frac{i\omega}{4\pi T}} e^{-\frac{i\omega}{4\pi T}\left(\sqrt{3}\arctan(\frac{1}{\sqrt{3}}[1+\frac{6r}{4\pi T L^2}]) - \frac{\sqrt{3}}{2}\pi - i\pi\right)} e^{-i\omega t} \tag{VI.52}$$

where c_0 is an integration constant. It is easy to check that this satisfies the infalling near horizon boundary condition. The near boundary ($r \to \infty$) now becomes,

$$A_x^{(0)} = c_0, \quad A_x^{(1)} = i c_0 \omega L^2. \tag{VI.53}$$

Inserting this into Eq. (VI.49) we find, surprisingly,

$$\sigma(\omega/T) = \frac{1}{g_4^2}. \tag{VI.54}$$

The result at zero temperature, Eq. (VI.48), is unchanged at finite temperature. The conductivity is just a constant, regardless the energy and the temperature. In a way this is the most perfect solution of Sachdev's paradox: when the conductivity is completely featureless as function of energy and temperature, it is automatically coincident with its dual vortex conductivity! In fact, it can be shown that this holographic result is rooted in the electromagnetic self-duality property of the bulk theory that forces the conductivity to be a constant [64]: this is in turn holographically related to the particle-vortex duality in the boundary theory. This in turn has the remarkable consequence that the σ_0 and σ_∞ constants are the same, showing an unanticipated connection between the coherent and dissipative regimes of quantum critical transport.

The explicit holographic computation of the optical conductivity associated with the conserved current in the zero density, strongly coupled large N in 2+1 dimensions shows that it is completely independent of both frequency and temperature. This revolves Sachdev's "particle-vortex" paradox that one encounters when the problem is addressed with the quantum Boltzmann approach.

In moving away from the strong 't Hooft coupling limit one expects that this relation has to be severed: for the usual phase dynamics it is very well understood that the particle-vortex duality is anything but a self-duality. It is the well known 2+1D local-global abelian Higgs duality. The self duality has to be special to the strong 't Hooft coupling limit. From string theory one has some clue of how to get away from this limit in terms of the α' expansion that has the effect that higher order gradients terms have to be added to the Einstein-Maxwell theory. The culprit is the 4D electromagnetic self-duality, and the lowest order term spoiling this duality in the bulk is [65]

$$S = S_{\text{EM}} + \int d^4x\sqrt{-g}\frac{\gamma L^2}{g_4^2}C_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \tag{VI.55}$$

where γ is a dimensionless parameter, L is the AdS radius, g_4 is the gauge coupling constant and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma}) + \frac{1}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R$, while stability and causality constraints restrict $|\gamma| < 1/12$. This should give the condensed matter reader a taste of the fancifulness of the GR one encounters in moving away from the Maldacena limit. When $\gamma = 0$ this reduces to Eq. (VI.42) while for $\gamma \neq 0$, the electromagnetic self-duality is lost. Repeating the procedure in the above but now using numerics one finds the results indicated in Fig.6. The conductivity is no longer a constant, and instead one can demonstrate that the relation between the σ_0 and σ_∞ transport coefficients become,

$$\frac{\sigma(\omega = 0)}{\sigma(\omega = \infty)} = 1 + 4\gamma, \qquad \sigma(\omega = \infty) = \frac{1}{g_4^2}. \tag{VI.56}$$

The dissipative part σ_0 is now no longer identical with the (unaltered) high frequency σ_∞. It is now waiting for experiments using cold atom systems as "analogue quantum computers" to find out how closely this result will match the laboratory phase dynamics system. This is the first instance in this narrative where we meet an un-trivial, high quality prediction from string theory that can be tested by a difficult but doable condensed matter laboratory experiment!

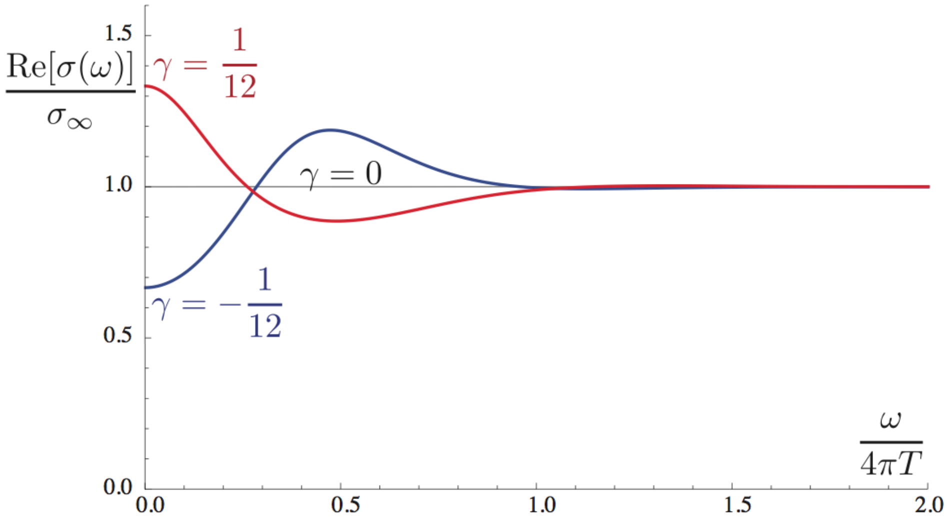

FIG. 6: The optical conductivity as a function of ω/T for the field theory at zero density. This plot is taken from [65].

VII. FINITE DENSITY AND THE REMARKABLE REISSNER-NORDSTROM BLACK HOLE.

We have arrived at the state of the art of holography as it existed around 2007 when the first contact was established between the string theorists and condensed matter physics. Back then the focus was entirely on the miracle exposed in the above that the generic behaviors of matter at finite temperature appears to be fully encoded in the bulk gravity. In fact, the communication lines between condensed matter and string theory where established in 2007 by Sachdev getting on speaking terms with a team of holographists, exploiting the ease of computing hydrodynamics using gravity to address the quite hairy nature of thermal transport phenomena near quantum critical points. The computation of, among others, the Nernst effect just boils down to computing photon-graviton propagators in the bulk [64]. Directly after this hand shake, a very obvious question inspired on condensed matter reality spread around in this network: what has holography to say about "quantum matter", the stuffs that exists at zero temperature but where the density is finite? As we already explained in the previous paragraphs, the zero density quantum critical states which were the main stay of AdS/CFT before 2007 are rare phenomena in the laboratory, which need a lot of fine tuning. The generic stuffs that are studied in condensed matter laboratories are made from electrons (cold atoms, whatever) that form the truly material systems having an existence because their density is finite.

This turned out to be a most inspiring question: in a very rapid development it became clear that a rich world of finite density quantum matter phenomena was waiting to be explored with the correspondence. At present this exploration is still in full swing. Different from the thermal physics of the previous section, even a basic understanding of what it all means is still lacking. Quantum liquid ground states are found of a completely new kind. Are these pathologies of esotheric supersymmetric Yang-Mills theories in the large N limit, only of interest as a mathematical excecise? Or are these revealing new emergence principles governing quantum matter, of a similar quality as the hydro- and thermodynamics of the previous section? Is it so that these have been overlooked up to now because these states are fundamentally out of reach of conventional field-theoretical methods because of the destructive action of the fermion signs?

At present, the center of this quest is the object called the “Reissner-Nordstrom (RN) strange metal”. Viewed from the boundary perspective, this is some form of quantum liquid which is completely different from anything that is known in the traditional condensed matter folklore. We will make a start in this section exploring some of its properties, but it will come back prominently when we discuss the holographic superconductors and the fermions. It tends to heat up the emotions: it appears that at present the community is split in two groups. It seems that one either worships the RN metal as a revelation that is the key to the secrets of the high T_c bad metals and so forth, or it is dismissed as an unphysical pathology of the large N limit, which is just an irrelevancy for the life in laboratories. Fact is that it has a very strong appeal to the mind of the theoretical holographist. The reason is that its gravitational dual is maximally simple, natural and elegant.

In this section we will give a first introduction, focussing on the gravitational aspects embodied by the interesting properties of the so-called Reissner-Nordstrom black hole that at finite density takes over the “IR control” from the Schwarzschild black hole of the zero density regime. We will discuss what it means for the thermodynamics and elementary transport properties. This in turn forms a fundament for the physics discussed in the sections that follow.

The Reissner-Nordstr ́om black hole is just the most primitive gravitational structure one encounters when one wants to address a finite density in the boundary field theory. Let us depart from the boundary field theory. In the last section it was already explained that one can deal with a global $U(1)$ symmetry in the boundary that just encodes for a conserved quantity like the number of electrons. In the absence of electromagnetic sources in the bulk, we found out that the field theory is at zero density, with the chemical potential right at the Dirac node departing from the relativistic theory. In order to lift this to a finite density one has to expose the field theory to a chemical potential that is conjugate to the density. What is the bulk dual of such a uniform potential? We learned in the previous section that an electrical field strength in a spatial direction of the boundary corresponds with a similar field strength in the bulk, Eq. (VI.49). To encode for the chemical potential, one therefore needs electrical field lines that are aligned along the radial direction in the bulk, “piercing” through the boundary in a spatially uniform way. The response associated with this source in the boundary field theory then corresponds with a density associated with the conserved $U(1)$ charge switching on. This can be seen from the fact that the charge density is $\rho = \frac{1}{g_F^2}\sqrt{-g}F^{tr}(r \to \infty)$ and from the maxwell equation it follow that $\partial_r\left(\sqrt{-g}F^{tr}\right) = 0$.

How to accomplish this with the Einstein-Maxwell bulk? In the case of the compact boundary space (section V A) we have somehow to store an electrical charge “somehow” in the deep interior; for the non-compact case one needs a condenser plate (compare with the black brane). In later chapters we will find out that this can be accomplished in various way at zero temperature. However, departing from finite temperature one has in one or the other way merry this charge with the requirement that a black hole (or black brane) has to be present as well. The simplest way out is to just consider the electrically charged black hole: this is the Reissner-Nordstrom black hole/brane.

This black hole has a long history. Directly after Einstein’s seminal GR work,and Schwarzschild’s discovery of the metric that is named after him, Reissner and Nordstrom independently analyzed the Einstein-Maxwell system discovering in the period 1916-1918 that it allowed for a new solution: the “RN” back hole that is composed of EM field energy embodied by a net monopole charge Q and a space-time, having an overall mass M. In AdS space, the solution of the Einstein-Maxwell equation including the cosmological constant Eq. (VI.42) can be written in the form of a metric and gauge potential A_t as,

$$ds^2 = \frac{r^2}{L^2}(-f dt^2 + d\vec{x}^2) + \frac{L^2}{r^2}\frac{dr^2}{f}, \tag{VII.1}$$

$$f(r) = 1 + \frac{Q^2}{r^{2D-2}} - \frac{M}{r^d}, \quad A_t = \mu\left(1 - \frac{r_0^{D-2}}{r^{D-2}}\right) \tag{VII.2}$$

where

$$\mu = \frac{\sqrt{2} g_F Q}{c_d \kappa L r_0^{D-2}}, \text{ with } c_d = \sqrt{\frac{2(D-2)}{D-1}}. \tag{VII.3}$$

The differences in the geometry as compared to the now familiar AdS-Schwarzschild solution Eq. (V.6) is encapsulated by the altered redshift factor $f(r)$. The horizon is determined by the zero of this function, $f(r_0) = 0$. One infers directly that this factor allows for two solutions, representing an *inner* and *outer* horizon. In the remainder of the present story the inner horizon does not play any role, and one just needs to know that the outer horizon is taking over the role of the horizon of the Schwarzschild black hole. The larger root of the redshift factor sets the radial position r_0 of this outer horizon. This can be used in turn to conveniently parametrize the gauge potential which is according to Eq. (VII.2) just vanishing at the horizon. The mass of the black hole can also be expressed in terms of the outer horizon position as,

$$M = r_0^D + \frac{Q^2}{r_0^{D-2}}. \tag{VII.4}$$

There is surely a Hawking temperature and entropy associated with this black hole, which is recovered using the holographic recipe for the thermodynamics of section V. When the mass M of the Reissner-Nordstr'om black hole is large compared to its charge Q the black hole forgets that it is charged and one recovers the results for the Schwarzschild black hole. This is not particularly interesting, since it just means that the temperature in the boundary is large compared to the chemical potential μ.

Obviously one expects to find the physics associated with finite density to show up when T becomes less than μ. The RN black hole is in this regard most interesting. Upon lowering the mass M (temperature) for a fixed charge Q (chemical potential) the horizon will recede towards the deep interior. However, although the energy stored in the space-time fabric decreases when the black hole shrinks, the energy stored in the electromagnetic field is fixed given the fixed charge. In order to avoid a naked singularity, the solution Eq. (VII.2) can only exist under the condition,

$$M^{\frac{2D-2}{D-2}} \geq \left(\frac{2D-2}{D-2}\right)\left(\frac{2D-2}{D}\right)^{\frac{D}{D-2}} Q^{\frac{2D}{D-2}}. \tag{VII.5}$$

The interesting case is when this inequality gets exhausted. This is the celebrated "extremal black hole" characterized by a mass that is entirely due to its electromagnetic charge! It was early on recognized that in the context of black hole complementarity such extremal black holes are fascinating objects, taking the Schwarzschild story a step further. The Hawking type calculation shows that its temperature is zero. Giving it a little thought, this makes sense. The Schwarzschild black hole in an asymptotically flat space-time is unstable since it emits Hawking radiation such that its mass is decreasing in time. However, when its mass is entirely due to its conserved electrical charge it is just impossible to reduce this mass further, impeding the emission of radiation: its temperature has to be zero. Although this behavior is reproduced by the explicit calculation, it is far from obvious why this fits so neatly. This is generally regarded as a mystery by itself.

At extremality the RN black hole therefore codes for a zero temperature state. But now it comes: when the bound (VII.5) is saturated the black hole is still in existence with a horizon radius being still finite,

$$r_{\rm h} = \left(\frac{D-2}{D}\right)^{1/(2-D)} Q^{1/(D-1)}. \tag{VII.6}$$

This implies in turn that the extremal black hole carries a Bekenstein-Hawking entropy $S \propto A$ which is finite since the horizon area A is still finite. The bottom line is that this gravitational structure that is supposedly the most elementary way to encode finite density in the field theory predicts a ground state that is highly degenerate: this is a state characterized by zero temperature entropy!

The "Reissner-Nordstrom" strange metal is characterized by a finite entropy at zero temperature. The ground state degeneracy is of a new kind: different from the frustrated systems of condensed matter physics it arises in the "$\hbar \to \infty$" limit, while it is not obviously related to spatial configurations having the same energy.

Such states are abundant in *classical* matter. In condensed matter they are called frustrated systems. An elementary example are the antiferromagnet Ising spins on a triangular lattice as characterized by an extensive number of different spin configurations that all have the same minimal energy. A more fanciful and timely example is "spin-ice", again formed by Ising-like spins but now living on a more intricate lattice geometry where it has been demonstrated that "non-gauge" analogues of Dirac monopoles are formed. However, the RN state is different from anything that has been identified explicitly in statistical physics and/or explicit quantum field theory. To address this in more detail we need the mathematical precision of the AdS/CFT correspondence. As we will show in the next paragraphs, the AdS/CFT correspondence tracks precisely the qualitative story in the above, with the difference that we now get much more information regarding what is happening on the field theory side. We learn from the correspondence:

a. The phenomenon is associated with "doping" the strongly interacting quantum critical state at zero density, think metaphorically that one increases the charge density of the "graphene critical state" of the previous section lifting the chemical potential away from the Dirac points.

b. Most importantly, the nature of the "frustration" is entirely different from what is encountered in the spin systems of condensed matter. In so far results are available one finds out that upon switching on $\hbar$ one invariably runs into the quantum order-out-of-disorder ("Coleman-Weinberg") mechanism, lifting the degeneracy of the classical limit. We learn from the correspondence that the ground state entropy of the Reissner-Nordstrom metal in the boundary arises at strong 't Hooft coupling. This is literally a $\hbar \to \infty$ limit and there is not a single instance in conventional quantum field theory hinting at such a ground state degeneracy emerging in the quantum limit. The only example that bears some similarity (although in a contrived way) is the critical multichannel Kondo problem where one finds a $\log 2$ ground state entropy associated with a single impurity. We will find out later that the degeneracy is not lifted by explicit breaking of the translational invariance: it is not in some simple way related to degenerate spatial field configurations. Most alarming, the von Neuman entropy calculations as discussed in section VII A indicate that the RN metal is characterized by a *volume* entanglement entropy: perhaps the best way to express the extreme quantum nature of this "frustrated" state.

c. Particularly fascinating for the condensed matter physicists that have been paying attention to the "strange metals" found in cuprate superconductors, heavy fermion systems and so forth: an intriguing renormalization flow is revealed, showing an *emergent* novel conformal state in the deep IR controlled by the RN black hole. As discussed in detail in section VII B, this is implied by a generic $AdS_2 \times R^{D-1}$ geometry associated with the black hole extremality. This implies for the boundary field theory that one is dealing with an emergent *local quantum criticality*: one is dealing with a purely temporal quantum criticality while matters are "stable" in spatial directions. This looks quite like the dynamical exponent "$z = \infty$" behavior which is regularly seen in experiments on the strange metals.

d. On the negative side for condensed matter applications, we will see underneath that the ground state entropy scales with N^2, just as was the case for the Schwarzschil black hole. In section V A dealing with the Hawking-Page transition, we explained that this signals that the state is deconfined and that one is somehow dealing with a soup of quarks and gluons associated with the specific physics of the large N Yang-Mills theory. We also learned in the previous sections that the *qualitative* features of the thermal physics are not critically on large N. However, this is much less clear in this zero temperature-finite density setting. For instance, there are general arguments indicating that upon re-quantizing the bulk geometry ("serious" large N corrections) the ground state degeneracy of the field theory is lifted.

e. States with ground state entropy are of course sick: this fact was already expressed by the 19-th century thermodynamicist Nernst with his "third law". Such states are just infinitely unstable to anything breaking this degeneracy. However, this is not a real concern: the third law is beautifully dealt with by the correspondence, and the next chapters are devoted to the plethora of instabilities of the RN metal.

With the above in mind, let us spell out the big question faced at present by the AdS/CMT pursuit. Although more forms of "strange" quantum matter have been identified using the correspondence, the RN metal is king of the hill in this regard. On the gravitational side it is just the simplest structure encoding for finite density zero temperature matter, while in the field theory it is just championing the weird behaviors which are unrecognizable viewed from the established wisdoms of zero-density field theory and the fermiology of condensed matter physics. Undoubtedly, this signals the physics behind the "fermion-sign brick wall" because otherwise we would know how to interpret it. But is it generic IR physics associated with strongly interacting "sign infested" matter? We have no guidance from explicit field theory and also the status of the RN metal in top-down approaches is not well understood. We already alluded to the issue that it might be quite specific to large N Yang-Mills. But there is more: the headache is in the workings of the third law in the "string landscape". It might well be that in the absence of special protection mechanisms, the system will flow away from the RN fixed point long before the system starts to discover the RN geometry. The best reasons to be more optimistic are actually given in by empiricisms. As Hong Liu *et al.* emphasized, it is very suggestive to identify the RN metal with the idea of an "intermediate state" [66] which is at work at higher temperatures in cuprate superconductors and so forth. Instead of postulating that the strange metal is *caused* by a quantum phase transition at zero temperature (in turn, associated with some form of competing order), Anderson [67] suggested that at the "intermediate" temperatures where one observes the laboratory strange metals one is dealing with a conformal metallic phase that needs the intermediate temperatures to exist. This state is inherently extremely unstable and as "quantum frustrated" entity it can flow in many different directions. The instability of the intermediate phase has the meaning that it can be the birth place of many competing phases: the zoo of pseudogap orders, the superconductivity itself, the Fermi-liquid in the overdoped regime. Pending the "chemistry" specifics (doping, strains, number of layers) one instability is preferred over the other.

This postulate is particularly attractive since it does explain the most salient features of experiment: (a) The strange metals in the laboratory systems seem to show invariably the local quantum criticality. This is a very strong prediction associated with the AdS metal. There is clearly no other truly mathematical theory available to explain it. (b) There is potentially a very interesting answer to a very important question. Why is T_c high? Because there is superglue? Wrong, T_c is high not because the superconducting particularly stable but because the competing metal state is so extremely unstable!

At present the significance of the RN metal is just a conundrum. We believe that the most promising alley to make here progress is to mobilize condensed matter experiment as "analogue quantum computer" to shed further light on this matter, and in the next chapters we will return repeatedly to this subject, hoping to inspire the reader to come up with something better. Let us in the remainder of this section give some first details of the RN metal. In the next subsection we present the detailed equations of its holographic thermodynamics, to continue with the remarkable near horizon geometry of the RN black hole revealing the local quantum criticality, to end with its quite reasonable conductivity properties.

A. Reissner-Nordstrom thermodynamics: the equations

The recipe for computing the thermodynamics as discussed in section V works as well for the Reissner-Nordstrom black hole. In terms of the units $k_B = c = \hbar = 1$ one obtains for the temperature

$$T = \frac{Dr_0}{4\pi L^2}\left(1 - \frac{(D-2)Q^2}{Dr_0^{2D-2}}\right). \tag{VII.7}$$

In the field theory, turning on a finite chemical potential μ for the global $U(1)$ current J^μ means perturbing the field theory with

$$\delta S_{\mathrm{FT}} = \mu \int d^D x J^t. \tag{VII.8}$$

According to the dictionary, the boundary value of A_μ is the source for the corresponding operator J^μ, thus $A_t(r \to \infty) = \mu$. Similar to sec. V, we will work in the grand canonical ensemble and the thermal potential for the dual field theory can be computed from the Euclidean on-shell action

$$\Omega = -T \ln \mathcal{Z} = TS_E[g_E] = -V_{D-1}\left(\frac{2(D-2)Q\mu}{\sqrt{2}\kappa c_D g_F L^D} - \frac{2\pi}{\kappa^2}(\frac{r_0}{L})^{D-1} T\right) \tag{VII.9}$$

where $V_{D-1} = \int d^{D-2}x$. Thus the charge density ρ and entropy density s and the energy density ϵ for the boundary field theory are

$$\begin{aligned} \rho &= -\frac{1}{V_{D-1}}\frac{\partial \Omega}{\partial \mu} = \frac{2(D-2)}{c_D}\frac{Q}{\sqrt{2}\kappa L^D g_F}, \\ s &= -\frac{1}{V_{D-1}}\frac{\partial \Omega}{\partial T} = \frac{2\pi}{\kappa^2}(\frac{r_0}{L})^{D-1}, \\ \epsilon &= \frac{\Omega}{V_{D-1}} - \mu\rho = \frac{D-1}{2\kappa^2}\frac{M}{L^{D+1}}. \end{aligned} \tag{VII.10}$$

As $L^{D-1}/2\kappa^2 \sim N^2$, and $g_F \sim \kappa/L$, all these thermodynamical quantities are of order N^2. One can check that the first law of thermodynamics for a finite density system,

$$d\epsilon = Tds + \mu d\rho$$

is indeed satisfied.

It is convenient to introduce a length scale r_* to parameterize Q as

$$Q = \sqrt{\frac{D}{D-2}} r_*^{D-1}. \tag{VII.11}$$

In terms of r_*, all the physical quantities can be expressed as functions of r_0 and r_* and we have

$$\rho = \frac{1}{\kappa^2}\left(\frac{r_*}{L}\right)^{D-1}\frac{1}{e_D}, \quad \mu = \frac{D(D-1)r_*}{(D-2)L^2}\left(\frac{r_*}{r_0}\right)^{D-2} e_D, \quad T = \frac{Dr_0}{4\pi L^2}\left(1 - \frac{r_*^{2D-2}}{r_0^{2D-2}}\right) \tag{VII.12}$$

where $e_D = \frac{Lg_F}{\kappa\sqrt{D(D-1)}}$ is a dimensionless number and we are in units $k_B = \hbar = c = 1$.

At zero temperature $T = 0$, we have $r_0 = r_*$ and $M = \frac{2(D-1)}{D-2} r_*^D$. Note that the horizon area is nonzero. Thus we have non vanishing ground state entropy. The entropy density is

$$s = (2\pi e_D)\rho. \tag{VII.13}$$

At low temperature $T/\mu \ll 1$ we have

$$s = \frac{2\pi}{\kappa^2}\left(\frac{r_0}{L}\right)^{D-1} = (2\pi e_D)\rho + \frac{4\pi r_*^{D-2}}{D\kappa^2 L^{D-3}} T + \mathcal{O}(T^2). \tag{VII.14}$$

One can easily find that the specific heat $c_v = T\partial s/\partial T \propto T$. When the temperature is very high, we know that RN AdS is quite similar to the one containing a Schwartzschild black hole and we have $s \propto r_0^{D-1}$ and $T \propto r_0$, and thereby $c_v = T\partial s/\partial T \propto T^{D-1}$ as expected from the D dimensional CFT at finite temperature.

B. AdS_2 near horizon geometry and the emergent local quantum criticality

Let us now turn to the near horizon geometry of the zero temperature, extremal black hole. We learned already that the horizon is finite, having a radial coordinate r_*. From the general expression of the metric Eq. (VII.2) it is far from obvious that there is anything interesting going on with the geometry very close to this horizon, let alone that this signals a spectacular emergent quantum criticality in the field theory. This theme of the black hole changing the geometry close to its horizon will recur a number of times in the remainder. The RN black hole is a most interesting example of this phenomenon and let us therefore step slowly through the analysis, as a very useful exercise especially for those who are not too familiar with GR.

It is all about the redshift factor f in Eq. (VII.2). In general, the horizon r_h is determined from $f(r_h) = 0$ and we can make a Taylor expansion of $f(r)$ near $r - r_h$ as

$$f(r) = f'(r_h)(r - r_h) + f''(r_h)(r - r_h)^2 + \ldots$$

where the '...' are higher order terms in $r - r_h$. As will become clear later, one finds a rather abrupt transition from this near horizon geometry to the UV *AdS* geometry along the radial direction. This happens more or less at the same radial coordinate where one has to start taking the higher orders into account, and for this reason the leading order in this expansion are quite representative for the IR physics in the field theory. From the discussions around eqn.(V.9) from the previous section we learned that it has to be that $f'(r_h) \propto T$. The finite temperature near horizon geometry is therefore quite different from the zero temperature near horizon geometry, which is necessarily governed by the next order coefficient $f''(r_h)$. Let us now consider the redshift factor explicitly. In terms of the $T = 0$ horizon coordinate r_* (Eq. VII.11) this becomes quite transparant,

$$\begin{aligned} f(r) &= 1 - \frac{2D-2}{D-2}\left(\frac{r_*}{r}\right)^D + \frac{D}{D-2}\left(\frac{r_*}{r}\right)^{2D-2} \\ &= D(D-1)\frac{(r-r_*)^2}{r_*^2} + \ldots, \quad (r \to r_*) \end{aligned} \tag{VII.15}$$

The redshift factor has a double zero at the horizon: the inner- and outer horizon of the finite temperature merge in a single double horizon for the extremal zero temperature case, which is a key ingredient in the remainder.

Inserting the near-horizon redshift factor in the full metric yields the metric describing the near horizon geometry in the regime $\frac{r-r_*}{r_*} \ll 1$,

$$ds^2 = -\frac{D(D-1)(r-r_*)^2 dt^2}{L^2} + \frac{L^2 dr^2}{D(D-1)(r-r_*)^2} + \frac{r_*^2}{L^2} dx^2, \tag{VII.16}$$

while the gauge potential becomes, straightforwardly,

$$A_t = \frac{D(D-1)e_D}{L^2}(r-r_*). \tag{VII.17}$$

Although the manipulations are simple, a miracle actually has occurred! One infers that in terms of the near horizon coordinate $r - r_*$, the metric factors multiplying dt^2 and dr^2 acquire a similar structure as the bare AdS_{D+1} metric Eq. (III.3): this is an effective Anti de Sitter geometry, except that the space directions (dx^2) are multiplied by the constant $(r_*)^2$! Therefore, the space directions form just a flat space, while the effective anti de Sitter geometry is realized in the time-radial direction plane! To render this more explicit, let us re-parametrize in terms of a radial coordinate ζ which is the inverse of the distance from the horizon, and a radius L_2

$$\zeta = \frac{L_2^2}{r - r_*}, \quad L_2 = \frac{L}{\sqrt{D(D-1)}}, \tag{VII.18}$$

and one infers directly that this is a space time with an $\mathrm{AdS}_2 \times R^{D-1}$ geometry, where the AdS_2 part refers to the radial- and time direction with an effective radius L_2 while in the spatial directions just a flat space is realized,

$$ds^2 = \frac{L_2^2}{\zeta^2}(-dt^2 + d\zeta^2) + \frac{r_*^2}{L^2} d\vec{x}^2, \quad A_t = \frac{e_d}{\zeta}. \tag{VII.19}$$

We started out this story with the "central dogma" of holography, that the isometries in the bulk code for the global space-time and scaling symmetries in the boundary field theory (section III). We therefore just have to inspect the scaling isometry of the metric Eq. (VII.19),

$$t \to \lambda t, \quad \zeta \to \lambda \zeta, \quad \vec{x} \to \vec{x}. \tag{VII.20}$$

Comparing this with the scaling isometry of the pure AdS_{D+1} geometry (eq. III.5) (z is the radial direction),

$$t \to \lambda t, \quad z \to \lambda z, \quad \vec{x} \to \lambda \vec{x} \tag{VII.21}$$

we directly infer that we are dealing with a (quasi) local quantum critical state in the boundary field theory: in spatial directions there is no sense of scale invariance, while the dynamics of the fields is scale invariant merely in temporal regards! Such an emergence scale invariance seems to be ubiquitous in the strange metals, according to a variety of experimental properties in various laboratory systems. It is just fascinating that the most primitive gravitational set up for a finite density system (the RN black hole) gives rise to this very peculiar behavior. Both the zero temperature entropy as this local quantum criticality are phenomena which are alien in any known bosonic field theory. The conclusion seems inescapable that these somehow require the action of finite density fermions, where the fermion signs make possible a completely novel type of emergence principle.

In fact, a first glimpse of such new principle was seen in condensed matter physics quite some time ago when the limit of infinite dimensions was explored. Although the spatial fluctuations in the interacting fermion system are frozen out in infinite dimensions, it was found that in temporal regards one is still dealing with a non trivial dynamics that can be viewed as a quantum impurity problem living in an effective medium. This "single site" DMFT is nowadays used on a large scale as a pragmatic band structure tool (DMFT) to address physical systems, but it was initially haunted by the uncontrolled, ad hoc character of the construction. However, in recent times the method was improved in order to address the spatial aspects of the dynamics by implementing it for finite clusters handled by quantum Monte Carlo that are subjected to the DMFT effective medium boundary conditions. In terms of the convergence of the finite scaling it performed above all expectations, further helped by a beneficial suppression of the sign problem. One might want to interpret as a further evidence for the notion that the fermion signs promote an effective spatial locality of the quantum dynamics, perhaps governed by the same general principle that underlies the RN local quantum criticality.

Even within conventional realms one can attempt to guess how this works. Describing the Fermi gas in a first quantized, spatial worldline path integral one infers immediately that the fermion signs amount to a destructive interference. In the so-called Ceperley representation where one stores this sign information in the form of the nodal surface of the density matrix, it is immediately seen that the signs cause effectively a spatial confinement of the now sign free bosonic walkers. In this representation, the Fermi energy is nothing else than the energy penalty associated with confining the individual fermions in a small volume (the "nodal pocket"). Although the physical consequences are quite different, one meets even in the Fermi gas also this "fermion locality" when one chooses a representation that is as bosonic as possible. Last but not least, very recently it was demonstrated that the remnant sign structure one encounters dealing with one hole in the t-J model (Zeng-Yu Weng's "phase strings") are capable of Anderson localizing the carrier in the absence of any quenched disorder!

in physical dimensions, attempts to demonstrate that such a local dynamics can survive in lower dimensions didn't go anywhere. Even in infinite dimensions it proved very difficult to turn this into critical theory, given the tendency of the effective impurity problems to favor first order quantum phase transitions. However,

We will find out later that via the near-far matching method (Eq. VIII.15) one obtains rather direct information regarding this deep IR dynamics through the boundary propagators, as encoded by the bulk fields propagating in the RN near horizon geometry. As we will see, this amounts to a procedure which is similar as to the way one derives the field theory propagators from the bulk using the GKPW rule, except that one has now to put an effective boundary at the radial position where the near horizon geometry starts. As for the case of the standard propagators one can now ask the question how the field theory propagators associated with the deep IR behave, given the scaling properties hardwired in the bulk geometry. Even knowing little about the specifics, the general form of these deep IR two-point propagators is easy to deduce. One can first exploit that in the bulk one is dealing with a simple Galilean invariant flat space, with the ramification that the Fourier components indexed by spatial momenta $\vec{k}$ decouple. For every $\vec{k}$ one has now an independent problem associated with the AdS_2 time-radial direction plane in the bulk: according to the dictionary this translates into a CFT_1, a 0+1 dimensional critical "quantum mechanical" theory. Given the temporal conformal invariance the "curly" propagators have to look like, for a given $\vec{k}$

$$\mathcal{G}(\omega, k) = c_k e^{i\phi_k} \omega^{2\nu_k} \tag{VII.22}$$

in real frequency, where the k dependence enters in the phase factor, and further more merely in the scaling exponent ν_k! The "effective mass" of the field in the bulk that corresponds with this scaling dimension in the boundary will quite generally pick up k dependence. We will see later regarding the spatial dimensions: the reason that it should be called "quasi-local quantum criticality". We will see later that these curly G's play a big role in various spectroscopic properties. More than anything else these are the real "signals emitted by the finite density black holes" and perhaps the most interesting challenge is for the condensed matter physicists is to find out whether these signals can be digged out from the big pile of experimental information.

At low energy the propagators of the field theory are eventually dominated by the local quantum criticality, where they become algebraic functions of energy, while the exponents associated with the temporal behavior become function of momentum. The RN metals are characterized by "algebraic pseudogap behavior".

To finish this section let us address a final issue which is still not quite settled. We have arrived at a stage where we discern a quite attractive feature of holography with regard to condensed matter: the local quantum criticality. However, employing the RN black hole to get there one has to pay a prize. Its finite ground state entropy signals that this metallic state is inherently unstable. Are there gravitational settings which show an AdS_2 type IR geometry which are yet not involving an extremal black hole? As explained at the beginning of this section, to induce a finite charge density one has to deal with a conserved flux in the bulk because of Gauss's law. The source of this flux is hidden behind the horizon of the RN black hole, but to get rid of the zero temperature entropy this horizon has to disappear. There are two ways to realize this. One can either introduce charged sources to "discharge" the horizon: we will see this at work in the scalar hair and the electron stars of the next two sections, but there we will see that we will also loose the AdS_2 geometry. The second way is to attempt to violate Gauss's law. This can be realized in a simple way by employing the so called "Einstein-Maxwell Dilaton" (EMD) gravity [68–70]. Dilaton fields are quite unfamiliar to condensed matter physicists but these turn out to be quite ubiquitous in string theory where they naturally appear in the low energy spectrum, and thereby also in top-down theories. A dilaton field ϕ is an object having its own potential while it typically shows up as well as determining dynamically the gauge coupling $Z(\phi)$ multiplying *e.g.* the

Maxwell term. A typical explicit form for such an EMD gravity is

$$\mathcal{L} = R - \frac{1}{2}(\partial_\mu \phi)^2 - \frac{Z(\phi)}{4}F^2 + V(\phi). \tag{VII.23}$$

Such dilaton actions typically produce the "soft walls" that code for the confinement scale, giving rise to quite appealing phenomenological holography models in the QCD context. In EMD gravity one also finds black hole solutions but these have the beneficial property of showing a vanishing ground state entropy, A particular interesting example of five dimensional EMD gravity theory [68] departs from the particular choice for the potentials,

$$Z(\phi) = e^{2\phi/\sqrt{6}}, \quad V(\phi) = \frac{1}{L^2}(8e^{\phi/\sqrt{6}} + 4e^{-2\phi/\sqrt{6}}) \tag{VII.24}$$

which can be demonstrated to be part of a top-down construction. One finds the black hole solution,

$$ds^2 \;=\; e^{2A}(-hdt^2 + d\vec{x}^2) + \frac{e^{2B}}{h}dr^2, \quad A_t = \frac{Q\sqrt{2\mu}}{r^2+Q^2} - \frac{Q\sqrt{2\mu}}{r_H^2+Q^2}, \quad \phi = \frac{2}{\sqrt{6}}\ln\left(1+\frac{Q^2}{r^2}\right) \tag{VII.25}$$

where

$$A = \ln\frac{r}{L} + \frac{1}{3}\ln\left(1+\frac{Q^2}{r^2}\right), \quad B = -\ln\frac{r}{L} - \frac{2}{3}\ln\left(1+\frac{Q^2}{r^2}\right), \quad h = 1 - \frac{\mu L^2}{(r^2+Q^2)^2} \tag{VII.26}$$

The finite temperature its thermodynamics is quite different from the RN AdS black hole [68]. The black hole becomes extremal when $\mu L^2 = Q^4$, but now the horizon has disappeared because this solution describes a naked singularity at $r = 0$! The immediate reaction could be that this is bad since naked singularities are not supposed to occur (*e.g.*, the "cosmic censorship hypothesis"). However, we are dealing here with very modern GR and there appears to be a loop hole. This particular singularity is a so-called good singularity. It is obtained from a consistent truncation departing from 10 dimensional type II string theory. One constructs a healthy Reissner Nordstrom black hole in ten dimensional space time, and by compactifying this to 5 dimensions the "projection" of this higher dimensional black hole, which is effectively encoded by the dilaton, turns it into the good singularity. The near horizon geometry of this solution turns out to be,

$$ds^2 = \left(\frac{L_2^2}{Q\zeta}\right)^{2/3}\left[\frac{L_2^2}{\zeta^2}(-dt^2 + d\zeta^2) + \frac{Q^2}{L^2}d\vec{x}^2\right], \quad A_t = \frac{L_2^2}{Q\zeta^2} \tag{VII.27}$$

where

$$\zeta = \frac{L_2^2}{r}, \quad L_2 = \frac{L}{\sqrt{2}}. \tag{VII.28}$$

It is easy to see that under the scaling transformation (VII.20) $t \to \lambda t, \zeta \to \lambda\zeta, \vec{x} \to \vec{x}$ we have $ds^2 \to \lambda^{-2/3}ds^2$. Therefore, the near horizon geometry is conformal to $\mathrm{AdS}_2 \times \mathrm{R}^3$, while the zero ground state entropy indicates that the geometry (VII.25) is also stable. Such geometries are at present barely explored and the future will tell whether these might be as genuine top-down solutions of greater relevancy to this pursuit than the simple RN black holes.

C. The conductivity of the RN metal: with and without Umklapp

Let us finish this introductory chapter on the strange RN metals by addressing its optical conductivity. Above all this highlights the hairy aspects of trying to learn about finite density critical states involving quantities that have to submit to hydrodynamical conservation laws. The bottom line is that when both momentum and charge are conserved one learns in facts very little. However, there is a very exciting recent result: upon incorporating translational symmetry breaking that lifts the momentum conservation one finds an answer bearing some intriguing resembles with a rather mysterious experimental result.

We save the readers the details of the holographic calculation of the conductivity. As in the subsection VI C, one probes the bulk by a photon but it turns out that this couples also to the metric component g_{tx}. One has now to solve the coupled equations in the bulk containing the RN black hole and this can only be done numerically [71]. The result for the real part and imaginary part of the conductivity as function of ω/T is shown in Fig. 7. Compared to the ω/T independent conductivity of the zero density case there is now quite a bit more structure. One finds a depletion of spectral weight at energy scales less than μ. Holography surely knows about the f-sum rule, and this missing weight accumulates in the Drude peak at zero frequency: as usual, when momentum is conserved this is a perfect delta function at zero frequency with a pole strength (Drude weight) that measures the number of "carriers" that contribute to the perfect metallic conduction. This result makes a perfect sense, since it is completely dictated by hydrodynamical principle. At zero density one can just have finite spectral weight at finite frequencies since in the relativistic system momentum conservation is not putting constraints: the particles and antiparticles move in opposite direction, carrying an opposite charges. However, in the finite density system momentum conservation takes control, insisting that the weight has to accumulate in the Drude delta function, and the f-sum rule together with the scale of the chemical potential takes care of the remainder.

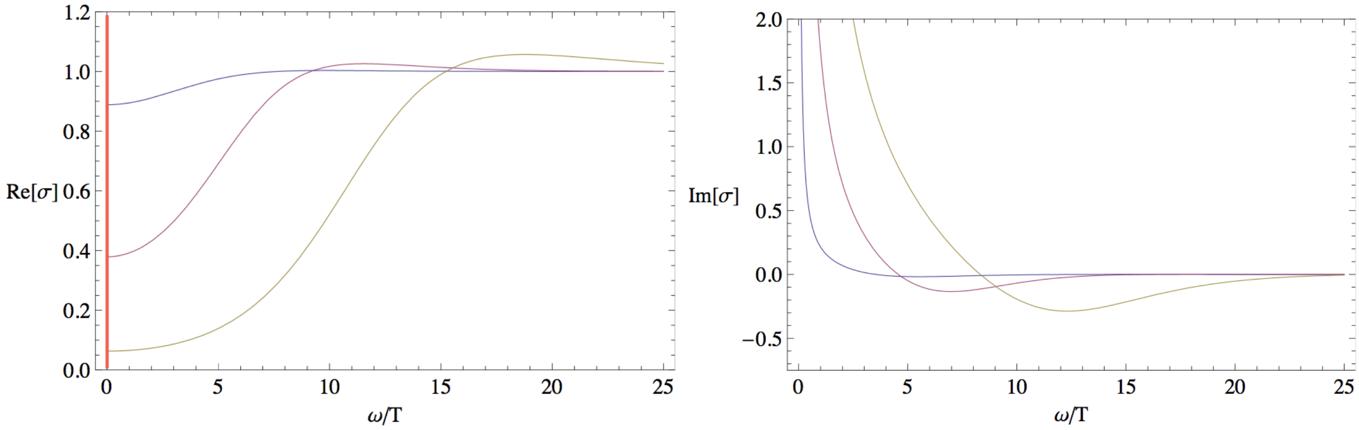

FIG. 7: The conductivity as a function of ω/T. When $\omega \to 0$, the imaginary of the optical conductivity behaves as $1/\omega$. This reflects the existence of a delta-function $\delta(\omega)$ in Reσ. In these plots, from top down, the chemical potential grows larger. These plots are taken from [1]

At the same time, this is the conductivity response of the very strange local quantum critical RN metal, but due to the hydrodynamics there is no signature of any kind of its strange IR in this particular reponse! As we will find out in section IX, this is surely different when conservation laws are not in the way. For instance, we will find later than this IR is very visible in photoemission measurements but also in (non-conserved) order parameter susceptibilities as we discuss in the next section. This just highlights an important caveat dealing with optical data, which is surely realized by many in condensed matter physics: in principle, it is an instance where much excitement can be hidden behind the "hydrodynamical curtain". But fortunately this is not the whole story because in real electron systems as realized in solids there is always a background potential coming from the ions that breaks the galilean symmetry, with this "Umklapp scattering" lifting the momentum conservation. It is technically quite hard work to incorporate this holographically [1] but the first results are just appearing. The simplest way to incorporate a periodic lattice is by just giving the electrical potential on the black brane a periodic oscillation, corresponding with a modulated chemical potential in the bulk. A more fanciful set up is used by Horowitz, Santos and Tong in the form of an auxiliary field that encodes an eventually quite similar modulation of the electrical fields in the bulk. In a tour de force, these authors compute the optical conductivity for a nominally rather weak periodic Umklapp potential finding numerically a great surprise [72].

Because of straightforward hydrodynamical reasons, the optical conductivity in the galilean continuum reveals no information whatever regarding the finite density strange metal. The energy independent conductivity of zero density gets depleted at energies below the chemical potential, and because of the f-sum rule this weight is all stored in the diamagnetic contribution at zero frequency. However, first results obtained in the presence of Umklapp scattering give hope that there is much to be learned when momentum is no longer conserved.

This result is indicated in Fig. 8. The Drude delta function has now spread out in a peak with a finite width at low energy. (See Fig. 9) This low energy peak is nearly indistinguishable from a Lorentzian, indicative of a "normal" Drude response characterized by a single momentum relaxation time ("scattering time"). The cross over to the zero-density like high energy regime is now smeared and surprisingly a large dynamical regime has developed at $\omega > T$ which is well fitted by a conformal form $\sigma(\omega) \sim 1/(i\omega)^{2/3}$. See Fig. 10. Why is this striking?

This goes back a while, to an experimental endeavor where one of the authors was involved as theoretical advisor. As it turns out, very high precision measurements in the best, optimally doped high T_c superconductors yield in the strange metal normal state results that have a striking resemblance with the above [73]. On the one hand one finds in the hydrodynamical regime a standard Drude response, be it with a Planckian momentum relaxation time $\tau_\hbar = 0.7\hbar/(k_B T)$ – perhaps the best reason available to leave no doubt for the conformal nature of this strange metal. On the other hand, for $\omega > T$ it was found that the conductivity is very well described by $\sigma(\omega) \sim 1/(i\omega)^{2/3}$. Both the very fact that the high energy branch cut goes hand in hand with the low energy Drude response, but also the high energy exponent were perceived as plainly mysterious (see e.g. [74]). These features are reproduced by the holographic calculation for reasons that are at present still unclear. A feature where the holographic computation fails is in the behavior of the momentum relaxation time. This follows the behavior argued on general grounds by Hartnoll and Hofman [75] that due to Umklapp $1/\tau \sim T^{2\nu_Q}$ where the AdS_2 exponent associated with the currents should be taken at the Umklapp momentum Q. There is no a-priori reason that this should take the value $\nu_Q = 1/2$.

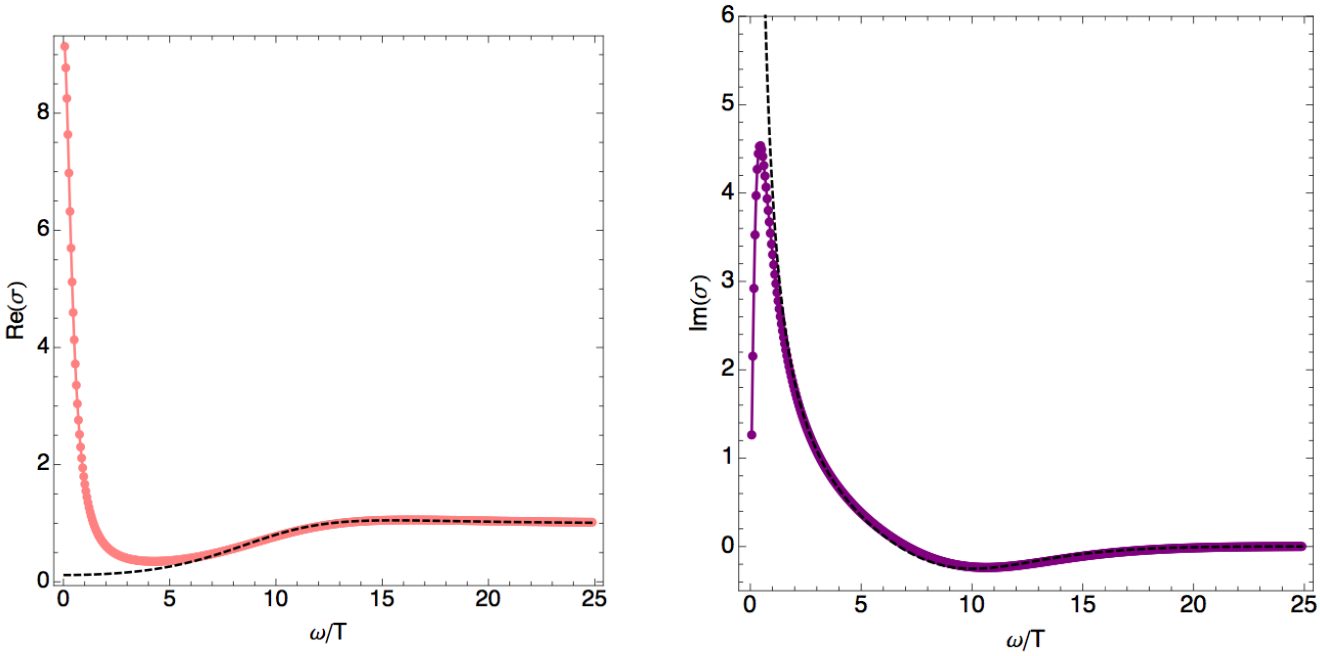

FIG. 8: The optical conductivity with (solid line) and without lattice (dashed line). When $\omega \to 0$, the imaginary part of the optical conductivity becomes finite in the presence of lattice. This means that there is no delta function peak for $\mathrm{Re}\sigma$. These plots are taken from [72]

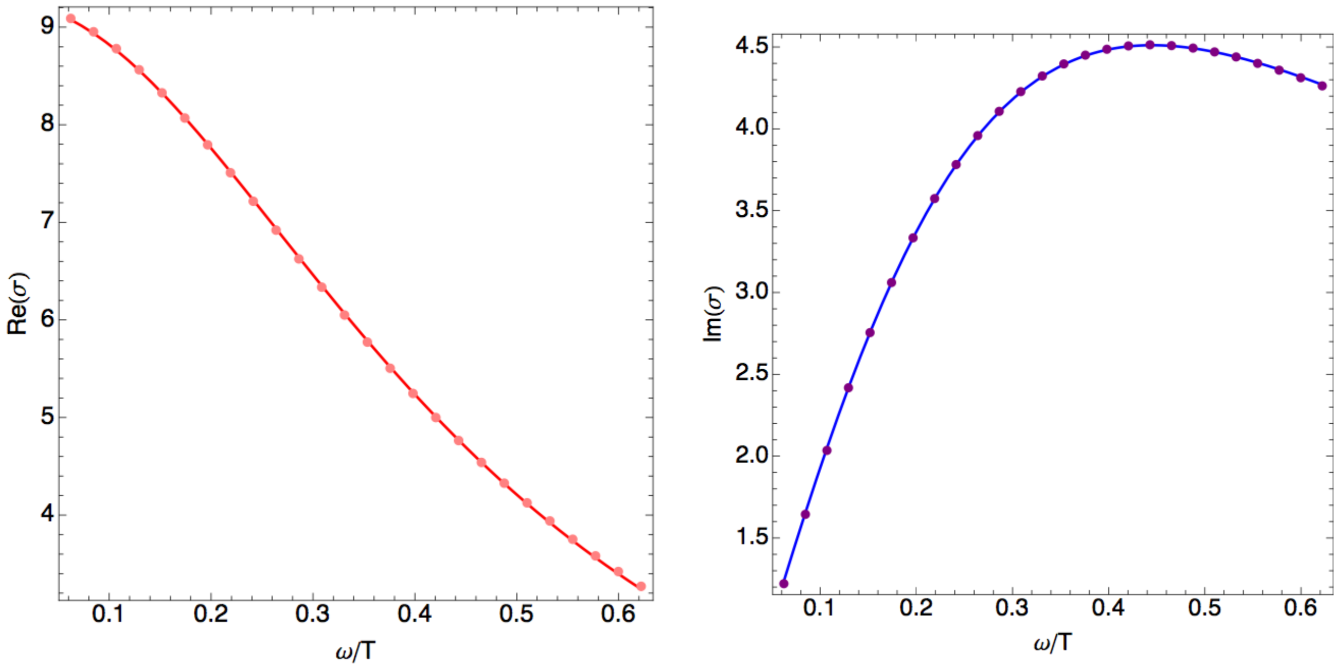

FIG. 9: The optical conductivity lattice at low frequency. The data can be fit pretty well by the Drude form $\sigma(\omega) = K\tau/(1-i\omega\tau)$. These plots are taken from [72]

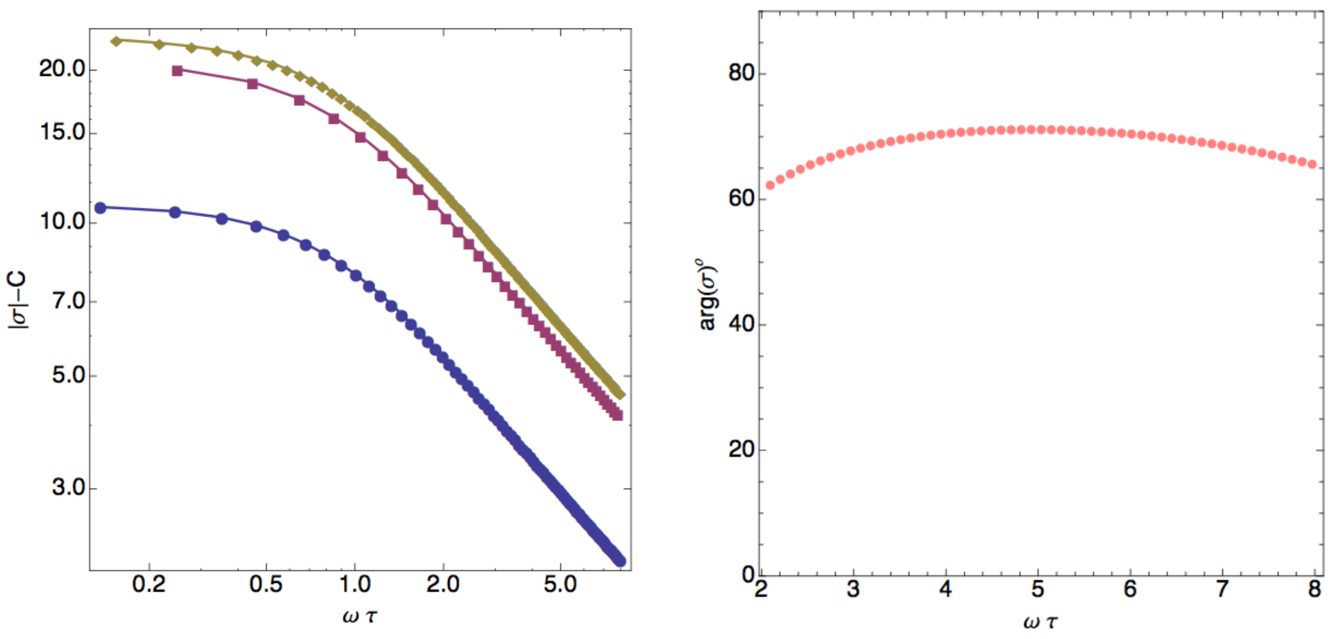

FIG. 10: Left: power law fitting: $|\sigma(\omega)| = \frac{B}{\omega^{2/3}} + C$. Right: the phase angle of the conductivity $\arg(\sigma)$. These plots are taken from [72]

VIII. HOLOGRAPHIC SUPERCONDUCTORS

The AdS/CMT pursuit aimed at addressing the physics of finite density matter started seriously in 2008 with the discovery of the holographic superconductivity, first suggested by Gubser [76] and subsequently further implemented in an explicit minimal bottom-up construction by Hartnoll, Herzog and Horowitz (H^3) [77, 78]. This triggered subsequently a large research effort in the string theory community and at present this phenomenon is quite well settled theoretically, much more so than for instance fermion physics that we address in the next section.

It is quite an achievement. From the condensed matter perspective it should be viewed as the first truly mathematical theory for the mechanism of superconductivity that goes beyond the celebrated Bardeen-Cooper-Schrieffer (BCS) theory discovered in 1957. The debate regarding the nature of the pairing mechanism has been raging in condensed matter physics since the discovery of high T_c superconductivity in 1986. Despite the large (and confusing) literature addressing the mechanism [79], these all depart from two fundamental mathematical theories. Either one departs from some notion of "pre-formed pairs", be it the naive bipolarons or the more fanciful resonating valence bond pairs, or it is assumed that the Cooper mechanism is at work. At least in the best high Tc superconductors, it looks superficially as if the BCS mechanism is at work. There is no sign of a (pseudo) gap in the normal state. The gap turns on at Tc to grow roughy like one would expect from BCS, be it that the gap to T_c ratio is much larger than in weak coupling BCS. In the course of time the BCS view has been slowly overtaking while the debate has increasingly focussed on the nature of the "superglue" that is required in this mind set to explain why T_c is so high.

However, there is a hidden assumption involved. The Cooper instability underlying the BCS mechanism is a quite deep phenomenon associated with Fermion physics. However, for it to work the normal state should be a well developed Fermi liquid, since one needs the sharp discontinuity associated with the quasiparticle Fermi gas. However, in the high T_c family the normal states appear to be non-Fermi-liquid strange metals. What gives then the right to rely on the BCS mechanism?

According to holography the formation of a superconducting state at low temperatures is quite natural. At first glance this is superconducting state is quite similar to BCS: fermion pairs are formed at T_c and a gap develops in the spectrum upon lowering temperature. The big difference is however that this pairing instability arises in a non-Fermi liquid where the Cooper mechanism is not operative.

The holographic superconductivity mechanism can be viewed as a generalization of the BCS mechanism to such strange metals. It shares with BCS that a gap opens up at the transition, growing as function of decreasing temperature, while it is associated with the binding energy of fermion pairs. In fact, as we will discuss towards the end of this section the way that the instability develops in the normal state is governed by the same time dependent mean-field/RPA equations as the BCS instability. The difference is however that the normal state itself is instead the strange metal controlled by the Reissner-Nordström black hole, showing the local quantum criticality and so forth. The bottom line is that holography is telling that such strongly interacting non-Fermi-liquid states of matter at least in the appropriate limit (large N and so forth) are subjected to an instability that has some crucial traits in common with BCS superconductors. As we will explain at the end of the section, the differences are subtle but these are in principle observable although this requires an unconventional experiment that has yet to be carried out.

The construction is also a marvel viewed from the string theory side. It interferes directly with the long standing relativist's wisdom as formulated first by Wheeler: "black holes do not have hair" [80]. This refers to the "no hair theorems" formulated in the 1960's, demonstrating that in an asymptotically flat spacetime the Einstein equations only allow for black hole solutions that are characterized by an overall mass, charge, (angular) momentum but nothing else. Black holes behave in this regard like elementary particles and do not carry any other specific properties (the "hair"). Motivated by holography this wisdom was challenged in recent years, and the holographic superconductor is the case in point. The subtlety is that the no-hair theorems do not apply to the asymptotically AdS spacetime and the moral of the story is that the symmetry breaking associated with the holographic superconductor is dual to the Reissner-Nordström black hole acquiring hair in the form of a Higgs field forming an "atmosphere" surrounding the black hole! For the relativist this is quite a revelation since it demonstrates the existence of an unanticipated, new kind of black holes. This is in turn rooted in the fundamental *gravitational instability* of the Reissner-Nordström black hole as we will explain in more detail underneath. The beauty of this holographic mechanism is that this dualizes into the field theory in the form of an instability that looks quite like a BCS instability. However, it has no longer anything to do with free fermions that cannot cope with attractive interactions, but instead it is eventually about a very non-Fermi liquid metal that has to turn into a superconductor to avoid its exceedingly unstable finite entropy ground state.

Holographic superconductivity is gravitationally encoded in the form of a "discharging" instability of the Reissner-Nordstrom black hole. When temperature is lowered at some point the black hole develops an "atmosphere" in the form of a Higgs field that acquires a finite amplitude peaking near the black hole horizon ("scalar hair"). Eventually, at zero temperature the black hole uncollapses and what remains is a "Higgs lump" in AdS, which is lacking a horizon and therefore corresponds with a zero entropy $T = 0$ state.

Let us now turn to the explicit holographic construction. The first question is, how does the dictionary entry look like encoding for the "superconductivity" (in fact, superfluidity, see previous section) in the boundary field theory. We are dealing with the $U(1)$ conserved current associated with the Maxwell fields in the bulk and we are after a way to break this symmetry spontaneously. Elegantly, the symmetry rules underlying the correspondence are so tight that they do not leave any room for any ambiguity. The full gauge invariance of the Maxwell fields in the bulk encode automatically for the conserved current. To break this symmetry boundary one has to "break" the gauge symmetry in the bulk, and this can happen in only one way: in the bulk a Higgs condensate/superconductor has to form. To make this possible, one has to add at a minimum a complex scalar (Ginzburg-Landau) field Ψ to the bulk action in $d+1$ dimensions which is minimally coupled to the Maxwell gauge fields, [3]

$$\mathcal{L} = \frac{1}{2\kappa^2}\left(R + \frac{d(d-1)}{L^2}\right) - \frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} - |\nabla\Psi - iqA\Psi|^2 - V(\Psi), \tag{VIII.1}$$

where $2\kappa^2$ denotes the Newton's constant, q is the charge of the scalar field and e is the Maxwell coupling constant. This is all fixed by gauge invariance and minimal number of derivatives and the only quantity that needs more information is the potential of the scalar field $V(\Psi)$.

We observe that there is now the extra scalar field Ψ in the bulk that is dual to an operator $\mathcal{O}$ in the boundary. Given the action Eq. (VIII.1) one can compute the near boundary asymptote of Ψ and as usual the leading- and subleading components are associated with the source and the expectation value of the response. Now we have arrived at the key question: how is the spontaneous symmetry breaking in the boundary encoded in this information? This is very beautiful: spontaneous symmetry breaking means that one has a vacuum expectation value of an operator, which stays non-zero even when the external field breaking the symmetry explicitly is turned off. This means that this is signaled in the bulk by a classical field configuration which is such that near the boundary its leading piece vanishes (the explicit symmetry breaking field) while the subleading piece (the VEV) stays finite! This general holographic wisdom was realized early on but it came alive properly with the discovery of the hairy black hole.

Before delving in the details of the "hair-cut", we still have to address the physical identity of the operator "$\mathcal{O}$" in the boundary field theory. From the above bottom-up construction we just learn that such an operator has to exist in order to have a chance to break the $U(1)$ but in other regards it does not give a clue what it means in terms of the bare (UV) degrees of freedom of the explicit boundary field theory. Similarly, we cannot be sure that such an operator and the associated bulk scalar field make any sense in a real theory. But now the good news is that help is available from top-down constructions: there is much confidence in the reality of the holographic superconductor because it turns out to be a relative ubiquitous phenomenon when the full string theory exerts control. Although there is no room for the $U(1)$ charge in the "minimal" Maldacena $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, there are several more fanciful top-down extensions available that have a richer field content, including the global $U(1)$ "flavor" symmetry in the boundary. This includes the "consistent truncations" of 11 dimensional supergravity [85, 86] and of type IIB string theory [87] employing Sasaki-Einstein compactification where well behaved holographic superconductor have been identified. Perhaps most useful in the present context is a so-called Dp/Dq brane construction by the Erdmenger group [88]. We will discuss these "brane intersections" in the final section in somewhat more detail. All one needs to know is that these wire in "fundamental fermions" carrying explicit flavor degrees of freedom in the Yang-Mills theory. In the Yangs-Mills theory of Maldacena the fermions are in the "adjoint", meaning that they are just the superpartners of the gluons.

Although the Erdmenger top-down holographic superconductor is more complicated (among others, it is a non-abelian triplet) one can read off from their top-down dictionary the precise nature of the symmetry breaking field

$$\mathcal{O} = \mathrm{Tr}[\psi\psi] + \cdots \tag{VIII.2}$$

where the ψ's are the fundamental fermion of the UV theory, that occur as Cooper pairs in a gauge invariant combination (the trace is over the color degrees of freedom). We therefore know for sure that the symmetry breaking

[3] One can set $\kappa^2 = 1/2$, $e = 1$ and $L = 1$ as in [78].

field is a pair field of fermions like in the BCS theory. The '$\cdots$' in this equation signal an important but general caveat: in a strongly interacting IR there is no a-priori reason to expect that only pair correlations exist, and in principle one has to also include multi-point correlators in Eq. (VIII.2). However, these all need to have the same symmetry as the lowest order pairs, and this is just the appropriate way to think about Cooper pairs in such a strongly interacting soup. The conclusion is that the Higgs field in the bulk is dual to a Cooper pair current in the boundary field theory, which is sourced by an external pair field.

With regard to the dictionary there is one remaining subtlety. The alert superconductivity expert might have already detected an oddity in this construction. Dealing with a single component simple superfluid as formed by e.g. ^{4}He, it is of course the case that the zero temperature state can be described in terms of a single order parameter field $\Psi = |\Psi| \exp(i\phi)$. The conserved $U(1)$ current in the condensate is then $J_\mu = \partial_\mu \phi$ and the specialty is that this current is now irrotational since vorticity is gapped. In the holographic superconductor we meet however two fields: next to the current J_μ we also need the pair field VEV $\langle \mathcal{O} \rangle$ which is supposedly equivalent to Ψ. What is going on here? This was initially overlooked by the string theorists and it is actually right now sorted out. Although it is not yet proven it appears to be simple. In the UV theories there are lots of (fermionic, bosonic) degrees of freedom and not all of them talk with the condensate. Therefore, the actual ground is still a two fluid system where the superfluid coexists with some soup of neutral degrees of freedom. The nature of these remnant low energy degrees of freedom coexisting with the superconductor is a very interesting subject by itself that will be discussed in terms of the back-reacted deep interior geometries of the (near) zero temperature superconductor in subsection VIII C.

A. The black hole with scalar hair

Let us now discuss how to the bulk Einstein-Maxwell-scalar field system is solved. One is invariably dealing in the bulk with the equations of motions for the various classical fields which, forming a system of coupled one dimensional non linear differential equations in terms of the radial coordinate r. Up to this point closed solutions were available but upon adding the scalar field hell breaks loose. All one can do is to use numerical shooting methods: it turns out that it is even very hard work to find the numerical solutions for the fully back-reacted holographic superconductor system. At present, an effort is still under way to fill in the last blanks.

The novelty is the scalar field and we still have to specify its potential. In bottom up approaches one just departs from the simple Ginzburg-Landau form,

$$V(\Psi) = m^2 \Psi \Psi^* + \frac{\lambda}{2} (\Psi \Psi^*)^2 + \cdots \tag{VIII.3}$$

Perhaps surprising, the essence of the mechanism is already contained in the so-called "minimal holographic superconductor" where one ignores the self-interaction of the scalar field choosing just a quadratic potential,

$$V(\Psi) = m^2 |\Psi|^2. \tag{VIII.4}$$

The point of departure is to add a complex scalar field to the Maxwell-Einstein theory in the bulk, which is minimally coupled to the Maxwell gauge fields. Although the mass of this field is positive, due to the strong curvature near the RN horizon this field nevertheless condenses, with its amplitude peaking near the horizon. Via the GKPW rule, this finite field amplitude in the bulk translates in a finite VEV in the absence of a source in the boundary, coding for the spontaneous breaking of the global $U(1)$ symmetry of the boundary field theory.

In the absence of gauge couplings and gravitational back-reaction, this is just like the free scalar field that we considered in the very beginning where we introduced the dictionary rules for propagators, Eq. 's (III.14)-(III.24). We learned there that a mass m in the bulk corresponds with a scaling dimension in the UV of the boundary theory $\Delta_+ = d/2 + \nu$ where $\nu = \sqrt{(d/2)^2 + m^2 L^2}$. We also learned that in the AdS_{d+1} spacetime one has to satisfy the Breitenlohner-Freedman (BF) bound $m^2 L^2 \geq -d^2/4$, Eq. (III.23). When this is not satisfied ν would become complex signaling an instability, in the form of the field Ψ acquiring a finite amplitude, which would be unphysical in the zero density limit. One already infers that in the curved AdS geometry the logic of Bose condensation works differently from the flat space time where one needs a finite w while the condensate is driven by a change of sign in m^2.

Let us now consider finite density case at zero temperature. We discussed the RN black hole already at length, and one can now derive a BF bound associated with the AdS_2 near horizon regime. This turns out to be different from the UV AdS$_{d+1}$ BF bound, and it is violated when,

$$(mL)^2 - \frac{q^2 e^2 L^2}{\kappa^2} < -\frac{d(d-1)}{4}. \tag{VIII.5}$$

The consequence is that a window opens up where the UV theory is above its BF bound, while in the deep IR the field has to acquire a finite amplitude since its violates the BF bound in the near horizon region of the RN black hole. Dimensionality does not play a critical role and for simplicity we consider from now the 2+1D field theory. In this case, the UV BF bound requires $(mL)^2 > -\frac{9}{4}$ while in the IR condensation occurs whenever $(mL)^2 - q^2e^2L^2/\kappa^2 < -\frac{3}{2}$. The effect is that near the black hole horizon an "atmosphere" of scalar "hair" develops. When one now computes the boundary asymptote of the scalar field which has turned into the hair in the deep interior one finds that the sub-leading contribution is finite while the leading contribution vanishes. As we discussed, the dictionary spells out that this signals the development of an order parameter in the boundary system. This is the elegant essence of the "black hole super-radiance" mechanism as the holographic dual of the superconductivity of the field theory. Physically it amounts to the effect that near the horizon of the RN black hole the space-time curvature becomes so strong that one runs into spontaneous "pair production" associated with the Higgs field, despite the fact that the field still is massive.

This scalar condensate carries stress-energy and will therefore back react on the geometry. As we will discuss in section VIII C, this has eventually the effect that the black hole completely un-collapses in the zero temperature limit. Instead one ends up with a "scalar hair" (perhaps better, "Higgs-lump") where all the charge is carried by the lump of condensate residing in the deep interior. This lump extends over all of the AdS space-time, and since it has no edge it is awkward to call it a star. As we will discuss in section VIII C, one encounters here a new "Lifshitz" geometry in the deep interior that encodes for massless degrees of freedom in the field theory which form in the presence of the condensate.

To give some idea regarding the problem which should be solved in the bulk, let us just list here the system of radial equations of motion. Specializing to 4 dimensional gravity case, corresponding to the 2+1 dimensional field theory, the Einstein equation of motion is,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{3}{L^2}g_{\mu\nu} = \kappa^2\Big[\frac{1}{e^2}(F_{\mu\rho}F_\mu{}^\rho - \frac{1}{4}F^2g_{\mu\nu}) + (\partial_\mu + iqA_\mu)\Psi^*(\partial_\nu - iqA_\nu)\Psi$$
$$+(\partial_\mu - iqA_\mu)\Psi(\partial_\nu + iqA_\nu)\Psi^* - g_{\mu\nu}\big(|(\partial_\alpha - iqA_\alpha)\Psi|^2 + m^2|\Psi|^2\big)\Big], \tag{VIII.6}$$

the equation of motion for the scalar is

$$(\nabla^\nu - iqA^\nu)(\nabla_\nu - iqA_\nu)\Psi - m^2\Psi = 0, \tag{VIII.7}$$

and the Maxwell equation

$$\nabla_\mu F^{\mu\nu} = iqe^2[\Psi^*(\partial^\nu - iqA^\nu)\Psi - \Psi(\partial^\nu + iqA^\nu)\Psi^*]. \tag{VIII.8}$$

One has now to look for solutions with an asymptotic AdS_4 geometry at the boundary. One can consider various simplifying circumstances. In the normal state above the superconducting transition, the probe limit suffices. One computes the infinitesimal fluctuations of both the gauge fields (dual of the conductivity) and the scalar field in the RN finite temperature black hole. This is straightforward and the pair field propagators one obtains in this way reveal very interesting information regarding the way that the instability above the transition temperature as we will discuss in detail in the next subsection. The next level of approximation is to just switch of the back reaction: the changes in the geometry driven by the stress-energy of the scalar hair. As discussed in section VIII C, these gravitational back reaction effects become only significant at quite low temperatures, while at temperatures close to T_c the bulk physics is dominated by the interplay between the scalar and Maxwell fields. This has the effect that down to quite low temperatures one maintains the RN geometry. Even numerically it is quite hard work to solve the system in full, including the gravitational back reaction.

To give an idea of how the bulk solutions look like, we show in Fig. 11 some typical profiles of the scalar hair along the radial direction. These are computed for the minimal set up, and we show the hair for different temperatures for a fixed ρ. As the temperature goes down, the location of the horizon $r_+/\sqrt{\rho} \to 0$ and at the zero temperature, the black hole horizon disappears.

Let us now zoom in what this all means for the boundary field theory. A first interest is to find out how the conductivity reacts to the presence of the finite order parameter. This is just computed as before in terms of the fluctuations of the Maxwell gauge fields in the bulk, but now these acquire a mass in the deep interior due to the finite Higgs field. A priori it is not clear what this means for the boundary – mass in the bulk is supposed to alter the scaling dimension in the boundary. However, the exciting finding is that an extra depletion of spectral weight occurs

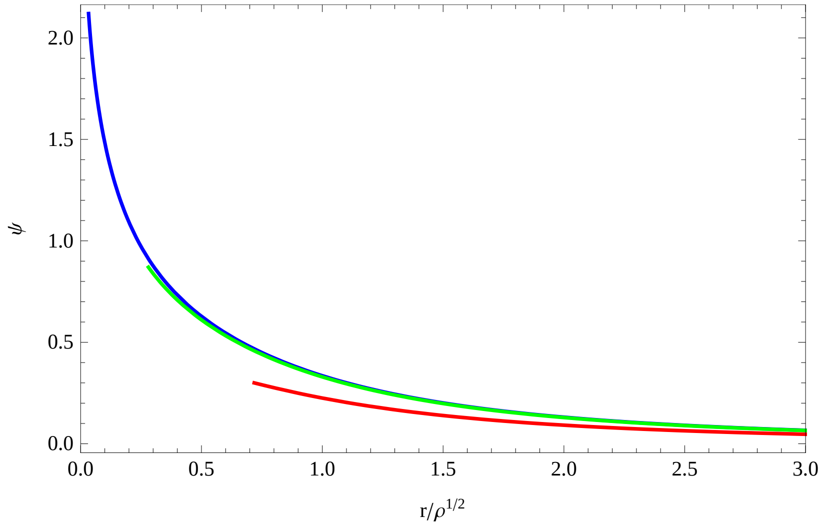

FIG. 11: The scalar hair along the radial direction for different temperature $T \simeq 0.84T_c$ (red), $0.37T_c$ (green), $0.06T_c$(blue). Here $m^2 = -2, q = 3$ and we focus on the case with the condensate $\mathcal{O}_2$.

at low energy, as illustrated in Fig. 12. This signals the formation of a gap that grows with the order parameter. In a way it looks like the optical conductivity of a dirty-limit BCS superconductor, but one should be aware of the qualitatively different physics at work. In the absence of momentum relaxation (no Umklapp- and elastic disorder scattering) one would not see anything happening in the optical conductivity of a BCS superconductor when the order parameter develops. The reason is of course that in the Galilean continuum the metal is as perfect as a conductor as the superconductor and the f-sum rule would be exhausted by the Drude delta function at zero frequency. One has to inspect the magnetic/Meissner response to measure the difference. The BCS gap becomes a big signal in the optical conductivity only in the "dirty limit" where the momentum relaxation exceeds the gap scale. The holographic superconductor behind Fig. 12 is also living in the Galilean continuum and the finite conductivity at finite frequency is just the "remnant" of the zero density CFT conductivity which we discussed at the end of section VII. It is this weight that is removed from the optical conductivity of the superconductor.

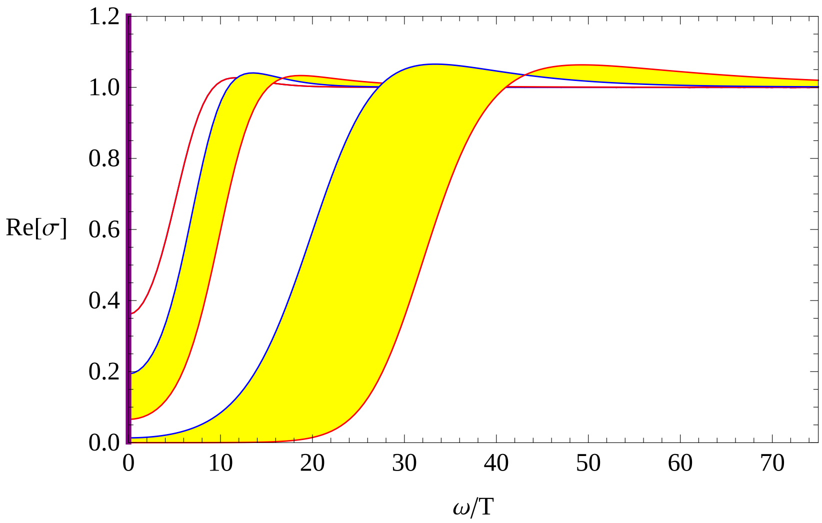

FIG. 12: Electric conductivity for holographic superconductors of $\mathcal{O}_2$ with $q = 3, m^2 = -2$ (red line) and RN black holes (blue line): $T/T_c = 1, 0.775, 0.274$ from top to bottom.The curves at $T = T_c$ coincide. There is a delta function at the origin in all cases.

In fact, the case that the holographic superconductor closely parallels a BCS superconductor in the regard that a gap opens up that is just proportional to the order parameter is most persuasive when one computes the fermion spectra [81]. The fermionic "sector" is the subject of the next chapter. The casual reader might mistake this for a typical fermiology computation where one takes care of the self energy of the quasiparticles invoking fluctuations of electronic origin. At low temperatures one sees very sharp Bogoliubov quasiparticles submitting to the spectral weight rules, separated by a s-wave gap. Upon heating, the gap shrinks like a BCS gap while the quasiparticle damping increases rapidly. A quantitative difference is that the gap to T_c ratio is large ($\simeq 8$). However, the striking qualitative difference is that these sharp Bogoliubov develop from a high temperature state that is a non Fermi-liquid strange metal with overdamped fermions. For high T_c connoisseurs this is perhaps the greatest danger posed by the gravitational pull of the hairy black hole!

With regard to its thermodynamics as well as its transport and photoemission properties, the holographic superconductor mimics the BCS superconductor quite closely. Like in BCS the phase transition is mean field, but for the holographic superconductor this is caused by the large N limit.

Let us finally turn to the most basic aspect of holographic superconductivity, its thermodynamics: the thermal history of the superconducting order parameter $\langle \mathcal{O} \rangle$. As already emphasized, one finds from the boundary behavior of the scalar fields in the bulk that when the hair has switched on it results in a "VEV without source". Equivalently one can compute the free energy and its behavior is fully consistent with the order parameter itself. The free energy difference between the metal and the superconductor obtained by switching on the coupling with the scalar field behaves like $(T_c - T)^2$ near the phase transition, and accordingly the order parameter switches on with an order exponent 1/2, see Fig. 13 and 14. It is a thermal second order transition of the mean field kind. This result holds regardless the spatial dimensionality, including dimensions less than 4. It seems to violate the basic rules of statistical physics. A global $U(1)$ symmetry is broken and below the upper critical dimension one should pick up anomalous dimensions. Although it mimics weak coupling BCS, this is a mere coincidence. In BCS the mean field nature is enforced by the large coherence length acting as a cutoff for the critical fluctuations, but this gradient length can be checked to be quite microscopic in generic holographic superconductors. The real reason for this mean-field attitude has been actually known since the 1970's. It is the "invisible hand" of the large N limit. By computing the diagrams associated with the thermal fluctuations in the flavor sector of a large N Yang-Mills theory, one finds out that these are $1/N$ suppressed. The order parameter therefore behaves effectively as a purely classical, non-fluctuating entity: infinite N is similar to infinite dimensions. Regarding the issue whether holography resting on a classical bulk is representative for general emergence principle, this "deep mean-field attitude" is the best documented flaw.

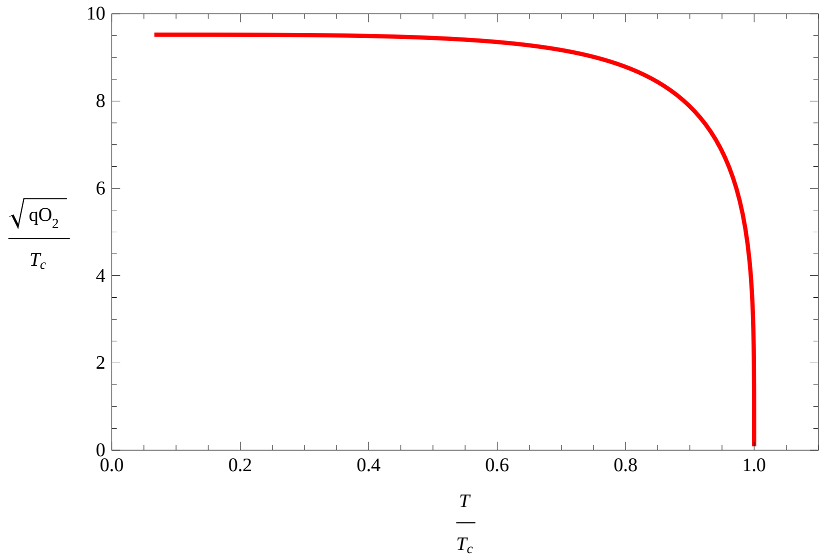

FIG. 13: The condensate $\mathcal{O}_2$ as a function of temperature for $q = 3, m^2 = -2$. Near the critical temperature, $\mathcal{O} \propto (T_c - T)^{1/2}$.

However, this flaw appears to be not lethal and it appears that it is already fairly well understood how to restore the order parameter fluctuations. The case in point is the demonstration by Anninos, Hartnoll and Iqbal [91] that by semi-classically re-quantizing the bulk hair one picks up the order $1/N$ thermal fluctuations. These have the effect of turning the infinite N pathological true long range order of the 2+1D finite temperature $U(1)$ superconductor into

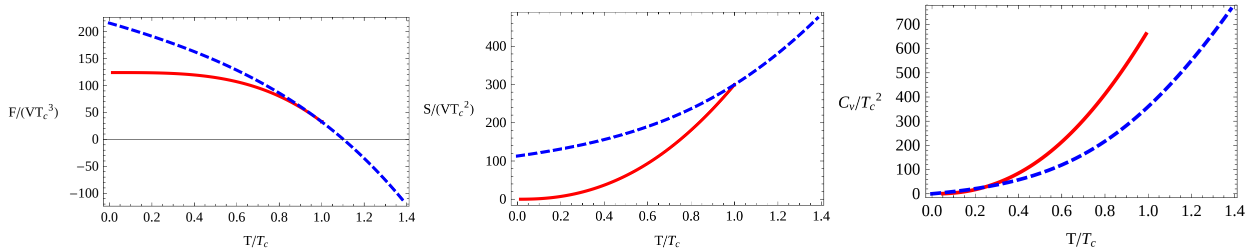

FIG. 14: The free enengy, entropy and specific heat for the RN black hole (blue dashed line) and hairy black hole (red solid line) of $\mathcal{O}_2$ as a function of temperature for $q = 3, m^2 = -2$. It is easy to see that below T_c the hairy black hole is a more stable state.

the correct algebraic long range order. On the bright side, the mean-field behavior can also be an asset since it makes possible to compare apples with apples and pears with pears, as we shall find out in the next sub-section, .

To finish this exposition, it is well understood how to generalize the above s-wave holographic superconductor to "unconventional" p-wave [82, 83] and d-wave superconductors (e.g. [84]). The difference is that now has to employ non-Abelian SU(2) Higgs-gauge theory to encode this superconductivity instead of the Abelian-Higgs bulk theory of the above. Of course there is now a richer order parameter theory in the boundary but the gross features are shared with the s-wave case. Further emphasizing the case that holographic superconductivity is quite like the laboratory version is the construction of holographic Josephson junctions by Horowitz, Santos and Way [92]. In the simplest setting (see also [93]) this just involves a chemical potential that varies in spatial directions, similar as the periodic potential discussed in the context of the RN metal conductivity (section VII C). It is relatively easy to compute the effects of strong variation of the chemical potential on length scales that are large compared to the intrinsic length scale set by the chemical potential itself. By varying in this way the density from zero- to deep in the holographic regime one can mimic the barriers separating the Josephson islands , and these "devices" show all the right Josephson properties. In addition, the hydrodynamical properties are precisely recovered. At zero temperature one finds at finite momenta the superfluid zero sound mode while we already alluded to the stunning success of using the gravitational dual to debug the finite temperature hydrodynamical theory of the superfluid by Tisza and Landau [94, 95].

B. Measuring holographic superconductivity: the pair susceptibility

These black hole stories are fascinating but do they have anything to do with the mundane matter found in earthly laboratories? Perhaps the grandest question of condensed matter during the last quarter century has been regarding the mechanism of superconductivity at a high temperature. Could it be that the holographic superconductor contains the key to unlock this secret? Although the story that now follows is due to our Leiden collaboration [96], we do not hesitate to advertise it at this occasion since we find it at least a quite appropriate metaphor for the ultimate goal of this AdS/CMT pursuit.

The string theorists should be aware of the culture differences. The black hole stories in the above are of course stunning examples of the beauty of mathematics at work in physical realms. However, this sentiment eventually does not count in the condensed matter culture. The healthy reason is that condensed matter is ruled by the primacy of reality over human imagination, and reality is probed through experiment and not by thought alone. Einstein counts because of Eddington, but how does the incarnation of the famous solar eclipse expedition look like for AdS/CMT? At the least one should come up with a ploy that has the capacity to push the experimentalists into taking up an endeavor which would be otherwise beyond their own imagination. Such an experiment should have a smoking gun status: the outcome of the experiment should directly proof the theory right or wrong. Eddington's observation of the shift of star light by the sun is the benchmark.

We claim to have such a ploy in the offering. Although the outcome is not (dis)proving the validity of holography per se, it is a critical test of the renormalization group flow that underlies the pairing instability. As we will demonstrate in the below, this is for the holographic superconductor qualitatively different from any of the established "mechanism" theories of condensed matter. This scaling behavior can actually be measured *directly* with a non-standard condensed matter experiment. At present, we are waiting impatiently for experimentalists having the guts to take up this risky pursuit.

As we already highlighted, the holographic superconductor behaves quite like a perturbatively dressed BCS superconductor. This pertains to its thermodynamical- and hydrodynamical properties, but also to the spectroscopic properties (photoemission, optical conductivity) probing microscopic features of the physics. The high T_c expert will have noticed that on basis of this information it is very hard, if not impossible, to judge whether holographic superconductivity makes sense or not. It surely looks like the real data, but so does the mainstream folklore in condensed matter that takes the BCS fundaments for granted.

One should now realize that the focus on photoemission, conductivity and so forth is rooted in the availability for a long time of laboratory equipment that allow for routine measurements of these properties. However, do they represent the best experimental probes of the origin of the pair instability? Of course, they do not: the case in point is that this repertoire of established experiments is apparently even failing to discriminate in a sharp way between the vast differences between holographic and conventional superconductivity. These experiments just yield quite indirect information and the only way out of this log jam is to perform an experiment that yields direct information regarding the origin of the pairing.

There is only one way to distinguish sharply in experiment between the mechanism of BCS and holographic superconductivity: one has to measure the dynamical pair susceptibility in the normal state. Away from the transition this pair susceptibility should be subjected to "energy-temperature" scaling in the case of the holographic superconductor, since this emerges from a truly conformal metal. On the other hand, at least away from the weak coupling limit a BCS type superconductor will not be characterized by a conformal pair susceptibility since the perturbative corrections pick up the Fermi-energy, the scale of the Fermi-liquid, that wrecks the scale invariance.

In fact, the adjacent field of magnetism is in this regard in a much better shape. The reason is inelastic neutron scattering that makes possible to measure directly the propagator associated with the order parameter field itself: the dynamical magnetic susceptibility, that can be measured as function of energy, momentum and temperature. It is well established that by following the evolution of this dynamical susceptibility in the normal state one obtains a wealth of information regarding the origin of the magnetic instability. it is even the case that linear response theory seems to insist that the evolution of the dynamical magnetic susceptibility in the normal state embodies the *maximal* knowledge that can be obtained regarding the origin of the order. What is the corresponding propagator when one is interested in superconductivity? For the holographist it is immediately obvious: the order parameter propagator $\langle \mathcal{O}(x)\mathcal{O}(x')\rangle$ (x is space and time). This is also familiar to the superconductivity expert as the quantity which is at the center of the BCS formalism: the pair susceptibility,

$$\chi_p(\mathbf{q},\omega) = -i\int_0^\infty dt e^{i\omega t - 0^+ t}\langle\left[b^\dagger(\mathbf{q},0), b(\mathbf{q},t)\right]\rangle, \qquad \text{(VIII.9)}$$

where $b^\dagger(\mathbf{q},t) = \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}+\mathbf{q}/2,\uparrow}(t) c^\dagger_{-\mathbf{k}+\mathbf{q}/2,\downarrow}(t)$, and $c^{(\dagger)}_{\mathbf{k},\sigma}$ the usual annihilation (creation) operators for electrons with momentum $\mathbf{k}$ and spin σ, while this can be decomposed into the various s, p, d, $\cdots$ pairing channels. Its imaginary part at finite frequency is most revealing and this is also the quantity that is accessible by experiment as we will see later.

Holographically this is very easy to compute in the normal state at temperatures above the superconducting transition. Using the usual dictionary rules one just computes the "probe" infinitesimal fluctuations of the scalar field in the bulk in the fixed finite temperature RN background. With the help of the GKPW rule one then directly obtains the desired answer for the pair propagator Eq. (VIII.9). Before we discuss the outcomes let us first lay down a template for the interpretation of this information by considering how this works out for a conventional, weakly coupled BCS superconductor. One computes first the "bare" pair susceptibility of the non-interacting Fermi-gas which is just the particle-particle loop. Let us inspect its imaginary part at $q = 0$ where the instability will develop,

$$\mathrm{Im}\chi^0_{\mathrm{pair}}(\omega, T) = \frac{1}{2E_F}\tanh\left(\frac{\hbar\omega}{4k_B t}\right) \qquad \text{(VIII.10)}$$

One now injects a weak attractive interaction V between the free fermions and according to the conventional formalism one "re-sums the bubbles", resulting in the "Random Phase Approximation" (RPA) expression for the full dynamical susceptibility of the normal state,

$$\chi_{\mathrm{pair}}(\omega, T) = \frac{\chi^0_{\mathrm{pair}}(\omega, T)}{1 - V\chi^0_{\mathrm{pair}}(\omega, T)}. \qquad \text{(VIII.11)}$$

Coming from high temperature, the superconducting instability is signalled by the condition that a pole develops at zero frequency in the full susceptibilty. This requires,

$$1 - V\mathrm{Re}\chi^0_{\mathrm{pair}}(\omega = 0, T) = 0. \qquad \text{(VIII.12)}$$

By Kramers-Kronig transformation,

$$\chi'(\omega=0) = \int d\omega' \chi''(\omega',T)/\omega' \simeq \int_{k_B T}^{\hbar\omega_B} 1/(2E_F) = N_0 \ln(\hbar\omega_B)/(k_B T),$$

where N_0 is the density state, while we smuggled in a retardation scale/phonon frequency ω_B. Plugging this into Eq. (VIII.12), one obtains immediately the famous BCS expression,

$$k_B T_c \simeq \hbar\omega_B \exp(-1/(N_0 V)). \tag{VIII.13}$$

It is now interesting to follow the evolution of the absorptive part of χ_{pair} in the ω, T plane in the normal state, approaching the transition (Fig. 16a). One infers at high temperature the bare Fermi gas response Eq. (VIII.10), but upon approaching T_c the denominator switches on with the result that a peak develops that moves down in energy when temperature decreases, while it sharpens up, to turn into the pole at $\omega = 0$ at T_c. This is the "relaxational peak", which can as well be understood in terms of the effective order parameter theory,

$$\mathcal{L} = \frac{1}{\tau_r}\Psi\partial_t\Psi + |\nabla\Psi|^2 + i\frac{1}{\tau_\mu}\Psi\partial_t\Psi + \alpha_0(T-T_c)|\Psi|^2 + w|\Psi|^4 + \cdots \tag{VIII.14}$$

This describes and "Ornstein-Zernike" mean field relaxational order parameter dynamics, with the ramification that at low frequencies close to T$_c$

$$\chi_{\text{pair}}(\omega,T) = \frac{\chi'_{\text{pair}}(\omega=0,T)}{1 - i\omega\tau_r - \omega\tau_\mu}.$$

This just describes the overdamped, relaxational response of the order parameter, persisting during a time τ_r in the normal state before it relaxes away. The time τ_μ measures the breaking of the charge conjugation symmetry at the transition: this is perhaps less familiar since it only plays a role in strongly coupled superconductors where one cannot get away with a constant density of states on the scale of the gap. The bottom line is that the RPA form Eq. (VIII.11) for the dynamical pair susceptibility is controlled by the mean field nature of the order parameter: RPA is well understood as representing time dependent mean field theory.

This mean-field attitude is actually a very helpful circumstance when we want to compare BCS with the outcomes of holographic superconductivity. Although the mean field is rooted in very different circumstances (large N), the evolution in the normal state of the holographic pair susceptibility is actually also implicitly governed by the RPA logic! As we already announced, the computation of the pair susceptibility in terms of the bulk scalar field fluctuations is straightforward but general solutions are not available in closed form. A typical outcome is shown in Fig 16. Different from the BCS case one discerns now a peak at high temperatures and frequencies that morphs continuously into the relaxational peak that dominates the spectrum close to T_c. How to understand this in more detail?

In fact, this is governed literally by RPA, but how is this encoded in the explicit physics of the bulk? This becomes manifest using the "matching technique" which was developed in the context of the fermion propagators [103] adressed in the next section. One finds that RPA has in fact an elegant encoding in the space-time geometry of the bulk! One has to find out how the classical waves propagate in the bulk, starting near the boundary and falling towards the RN black hole in the deep interior. Near the boundary the geometry is AdS$_{d+1}$ while in the deep interior one encounters the near-horizon geometry of the black hole . As it turns out, this switch of geometry happens rather suddenly: one can picture this as if along the radial direction a sudden change occurs at a "goemetrical domain wall" connecting the UV AdS$_{d+1}$ and the near horizon geometry. In both regimes it is easy to find solutions for the scalar "waves" but these have be matched into a single solution at the geometrical wall. For $\omega \to 0$ the solution of this matching problem can be obtained analytically with as result,

$$G_R(\omega,T,e) \sim \frac{b_+^{(0)} + b_+^{(1)}\omega + \mathcal{O}(\omega^2) + \mathcal{G}(\omega,T)\big(b_-^{(0)} + b_-^{(1)}\omega + \mathcal{O}(\omega^2)\big)}{a_+^{(0)} + a_+^{(1)}\omega + \mathcal{O}(\omega^2) + \mathcal{G}(\omega,T)\big(a_-^{(0)} + a_-^{(1)}\omega + \mathcal{O}(\omega^2)\big)}, \tag{VIII.15}$$

where $\mathcal{G}(\omega,T)$ is the near-horizon "IR-CFT" Green's function defined in a similar way as the full "AdS-CFT" Green's function. This has a direct meaning in the field theory and it is obtained by taking the ratio of the leading and the subleading coefficients, but now at the "domain wall" forming the boundary of the near-horizon region. The coefficients $a_\pm^{(n)}(e,T), b_\pm^{(n)}(e,T)$ are determined by matching these IR-solutions to the "UV"-solutions associated with the zero density "waves" propagating from the boundary to the domain wall. These coefficients can only be obtained numerically. Near T_c this simplifies further,

$$G_R(\omega, T, e) \sim \frac{\gamma_0}{\beta_0(T - T_c) + i\omega\beta_1 + \omega\beta_2}, \tag{VIII.16}$$

where $\gamma_0 = b_+^{(0)}(e, T_c),\ \beta_0 = \partial_T a_+^{(0)}(e, T_c),\ \beta_1 = \lim_{\omega\to 0} \frac{1}{i\omega}\mathcal{G}(\omega, T_c) a_-^{(0)}(e, T_c)$ and $\beta_2 = a_+^{(1)}(e, T_c)$

$$\chi(\omega, T) = \frac{\chi'(\omega = 0, T)}{1 - i\omega\tau_r - \omega\tau_\mu} \tag{VIII.17}$$

where one recognizes the Curie-Weiss like "bulk pair susceptibility" ,

$$\chi'(\omega = 0, T) = \frac{\gamma_0}{\beta_0(T - T_c)}.$$

One can choose the parameters such that T_c is very high, of order of the chemical potential. It is in a way like a "local pair" superconductor where the particle-hole asymmetry is manifest. It follows generally that this implies the presence of the term $\omega\tau_\mu$ given in terms of the the particle-hole asymmetry parameter $\tau_\mu = -\frac{\beta_2}{\beta_0(T-T_c)}$. In other regards this susceptibility has the ubiquitous Ornstein-Zernike form, characterized by the order parameter relaxation time,

$$\tau_r = \lim_{\omega\to 0} \frac{i}{\omega\beta_0(T - T_c)}\mathcal{G}(\omega, T_c) a_-^{(0)}(e, T_c) \tag{VIII.18}$$

given in terms of the IR Green's function $\mathcal{G}(\omega, T)$. The net result is qualitatively strikingly similar as to what one finds in the BCS case. For $\omega \ll T$, the near-horizon IR Green's function takes the universal form $\mathcal{G}(\omega, T) = -i\omega/4\pi T$, such that $\tau_r = \alpha_0/(T - T_c)$. This is the gravitational ibcarnation of the universal hydrodynamical requirement that in the regime $\omega \ll T$ χ'' should always be linear in ω.

The similarity with BCS/RPA is in fact much deeper. Around the same time that holographic superconductivity was discovered, one of the authors developed together with She a very simple, entirely phenomenological scaling theory generalizing BCS to apply to a general "conformal metal" normal state [98]. They called this construction quantum critical BCS (QCBCS). This departs from the assumption that the order parameter is ruled by mean field and therefore the pairing instability is governed by the RPA expression Eq. (VIII.11). However, this metal does not need to be a Fermi-liquid and instead it is postulated that the "background" pair susceptibility χ^0_{pair} is some conformal propagator, with an "energy-temperature" scaing form,

$$\chi^0_{\text{pair}}(\omega, T) = \frac{1}{T^{\Delta_p}}\mathcal{F}\left(\frac{\omega}{T}\right) \tag{VIII.19}$$

such that in the regime $\omega > T$, $\chi^0_{\text{pair}} \sim 1/(i\omega)^{\Delta_p}$ and one can use various models for the cross-over function $\mathcal{F}$. One recognizes that the standard weak coupling BCS involves actually the special case for such a χ^0, characterized by the "marginal" scaling dimension $\Delta_p = 0$. This is an interesting accident happening only in the weak coupling limit. The Fermi-liquid is not conformal at all since the scale invariance is broken by the Fermi energy, but the Femi-gas pair response is effectively conformal. As we will see in a moment, this is just an oddity of the non-interacting limit: at the moment one dressed up the fermion propagators with perturbative self-energy corrections the Fermi energy is remembered and one looses the conformality of the pair response.

One now assumes that the attractive interaction 'V' is due to some outside source (like phonons) and amusing ramifications follow from this simple construction. For a *relevant* $\Delta_p > 0$ pair susceptibility one obtains a strongly modified, algebraic gap equation that predicts much higher T_c's than the regular BCS expression with its famous exponential. However, here we are interested what QCBCS has to reveal regarding holographic superconductivity. This becomes manifest when one compares the outcomes for the imaginary parts of the pair susceptibility. In Fig. 16e the result for quantum critical BCS is shown taking similar scaling dimensions as for the minimal holographic case (panel c) and one discerns immediately that these look very similar. This similarity becomes even more obvious by the "scaling collapses" shown in Fig. 17. By multiplying χ'' with T^{Δ_p} and plotting the results as function of ω/T, these should become temperature independent when these are conformal (compare with Eq. (VIII.19). At high temperature this is the case because there is no knowledge of the instability yet. Upon descending to T_c, the breaking of the scale invariance becomes manifest but entirely so in the form of the relaxational peak developing near the origin.

Perhaps the most convincing way to demonstrate the RPA nature of this holographic affair is by involving the so-called "double trace deformation". This is yet another contribution to the dictionary by Witten [97]. Its meaning is best understood in the context of the zero density conformal field theory. One can in principle include a "double trace" operator in the CFT of the form,

$$S_{\rm FT} \to S_{\rm FT} - \int d^3x \tilde{\kappa} \mathcal{O}^\dagger \mathcal{O}, \tag{VIII.20}$$

where $\tilde{\kappa} = 2(3 - 2\,\Delta_-)\kappa$, which will be generically a relevant perturbation in the usual sense that its coupling constant $\tilde{\kappa}$ will grow in the renormalization flow to the IR. However, one can now exploit the mean-field character of the CFT in the large N limit, which implies that the expectation value of the four point propagator factorizes in the product of the two point propagators,

$$G_R \sim \frac{\psi_-}{\kappa\psi_- - \psi_+}. \tag{VIII.21}$$

Since $\chi_0 = -\psi_-/\psi_+$, this in turn implies that the two point propagator is modified by the double trace deformation into a literal RPA form,

$$\chi_\kappa = \frac{\chi_0}{1 + \kappa\chi_0}. \tag{VIII.22}$$

The moral is that at least in the normal state this is like a condensed matter system that in its "pristine" state already wants to superconduct: the holographic χ_0. But now one can add from the outside a repulsive interaction (e.g., Coulomb potential) or an attractive interaction (electron-phonon glue, whatever) and these influences enter the equation in a literal RPA form as they would in a BCS superconductor!

There are now quite a number of parameters to play with in this bottom up holographic superconductivity affair, and in the next subsection we will sketch what this implies for the zero-temperature phase diagram. All one needs to know at this point is that one can depart from a "strongly coupled" holographic superconductor with a T_c of order of the chemical potential as the "AdS$_4$" one in Fig.'s 16 and 17. It is called this way because the conformal metal above T_c is in this case like the zero density CFT in 2+1 dimensions, since at the moment that the system discovers that the density is finite it falls prey to the pairing instability. This is perhaps of less interest to condensed matter applications since the zero density CFT's as realized in condensed matter physics (e.g. "quantum critical" graphene) do not seem to have such an appetite for pairing at finite density. However, departing from such a holographic superconductor one can now switch on a "repulsive" double trace deformation κ. This acts like a Coulomb pseudo-potential in suppressing the pair instability, and in this way one can suppress the superconductivity such that $T_c \ll \mu$, see Fig. 15. The effect is that the system stays normal down to such "low" temperatures that the system starts to feel the AdS$_2$ geometry of the low temperature RN black hole. This codes for the "strange" holographic metal that we already discussed in the previous section, showing the local quantum criticality an so forth. This might have bearing on the strange metals in the laboratory systems and one can now study how the pairing instability builds up in such a strange metal. The outcomes are the "AdS$_2$" panels Fig.'s 16d 17d. These look qualitatively much like the "AdS_4" and quantum critical BCS outcomes. Compared to BCS there is again a "conformal peak" at high temperatures, at an energy that is set by temperature. One again finds the "energy-temperature" scaling collapse (Fig. 17d): this is no wonder since the metal at "higher" temperatures is now controlled by the emergent "local quantum critical" AdS_2 geometry of the RN black hole, and in the context of the "matching" expression Eq. (VIII.15) we already explained that the IR behavior of the pair propagator is determined by the near horizon geometry imposing the scaling behavior on the pair propagator.

This "Energy-temperature" scaling collapse of the dynamical pair susceptibility in the normal state is the smoking gun prediction of holographic superconductivity. It would prove the existence of the "conformal metal" as the birth place of superconductivity and such a metal is entirely different from a Fermi-liquid. We will see in the next section that its single fermion spectral functions ("photoemission") can mimic "marginal Fermi-liquid" like behavior, in the sense that still a Fermi surface might be present. The crucial difference that this AdS_2 metal is not derived from a stable Fermi-liquid that is perturbatively dressed. One big difference is that the scaling behavior of the collective pair response should be completely detached from the single fermion properties. In a perturbative setting one expects that the pair susceptibility is associated with the particle-particle loop where one has to insert the dressed lines. When one computes these loops involving these "marginal Fermi-liquid" like single fermion self-energy one obtains a pair susceptibility which is not at all conformal: it seems even impossible to obtain a "conformal peak" associated with a relevant scaling of the pair operator [98].

A most interesting, sharp contrast is obtained by comparing the holographic pair susceptibility with the state of the art computations departing from the popular "Hertz-Millis" scenario for quantum phase transitions in metals. For the string theorists, this is an industry standard in condensed matter physics. The belief is widespread that

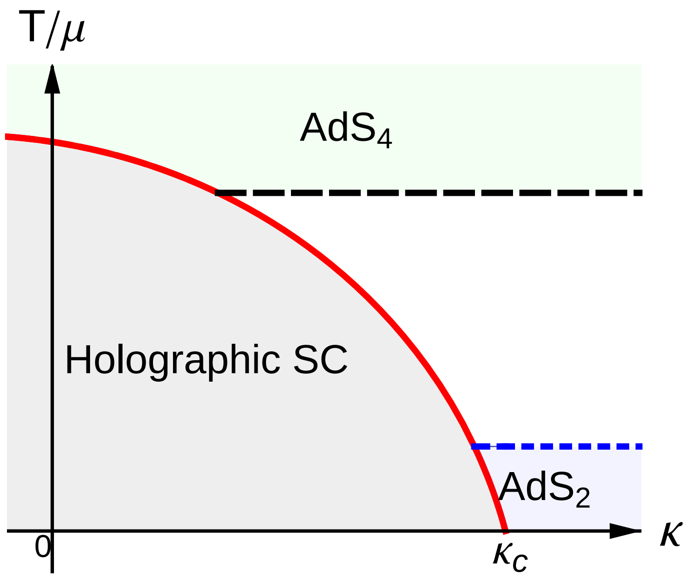

FIG. 15: A phase diagram of holographic superconductor including a double-trace deformation with strength κ. For $\kappa = 0$ one has the minimal holographic superconductor, case D, where $T_c \sim \mu$. Increasing the value of κ can decrease the critical temperature all the way to $T_c = 0$ if one includes a non-minimal coupling to the AdS-gauge field (see text). The shaded regions indicate which region of the geometry primarily determines the susceptibility. It shows that one must turn on a double-trace coupling to describe superconductors whose susceptibility is determined by AdS_2-type physics. This is of interest as AdS_2-type physics contains fermion spectral functions that are close to what is found experimentally.

it is literally at work in some heavy fermion quantum critical superconductors (the "good actors") and in the iron superconductors, while it might also catch the essence of the "bad actor" heavy fermions and the cuprates, although some modifications are needed. This departs from a *postulate* [104] which is neither supported by mathematics or experimental evidence which is nevertheless taken as self-evident by the main stream. This assumes that at low energies the strongly interacting electrons from the UV (Hubbard-, Anderson lattice models) renormalize in a tranquil fermionic quasiparticle gas that coexists with a order parameter field associated with ordered magnetism, which is undergoing a conventional (bosonic) quantum phase transition. The fermions and the bosonic field are weakly coupled and one then just evaluates perturbatively how these systems influence each other near the quantum phase transition. The bottom line is that the fermions Landau damp the critical bosonic fluctuations, raising the dynamical exponent in the bosonic critical theory. However, the critical fluctuations also "back react" on the fermions and the most dominant singularity one picks up in the perturbation theory resides in the pair channel [105]: the critical fluctuations just act as a "pairing glue" driving a BCS superconductivity "surrounding" the quantum critical point, explaining the superconducting "domes" surrounding the quantum phase transitions in the experimental systems.

Right at the critical point this translates into a "glue function", the frequency dependent attractive interaction felt by the fermions, of the form $\lambda(\omega) \sim (1/|\omega|)^\gamma$. This glue is therefore strongly retarded and one can exploit this "Migdal parameter" to argue that the standard Eliashberg formalism is applicable [106]. This amounts to dressing up all lines in the diagrams into fully dressed double lines, while the vertex corrections are suppressed by the strong retardation, Given the "critical glue", one ends up with a renormalized perturbation theory which is pushed to its extreme. One can even demonstrate that when the metallic state would be stable to the lowest temperatures this theory would eventually describe an emergent conformal metal. However, long before one enters this regime the superconductivity takes over. The self-energy effects on the "high temperature" metal above the superconducting temperature are very strong. In a tour de force, the Leiden postdoc Overbosch managed to compute the real frequency pair susceptibility in this regime and the results are shown in the panels Fig. 16a and 16b. Also in this case a peak develops in the high temperature normal state pair susceptibility. However, this pair susceptibility is not at all conformal: as seen in Fig. 17, it is impossible to collapse it by energy-temperature scaling. This is of course not an accident. Although one finds an accidental conformal invariance in the free Fermi gas pair susceptibility of the weak coupling BCS limit, this is no longer true when the interaction effects become sizable. The perturbative corrections are computed around the Fermi-gas and these diagrams all know about the Fermi energy. Once again, the Fermi energy is a very muscular scale which is generically imprinting on any perturbative fermiology calculation, with the effect that conformal invariance is broken by default.

The figures Fig. 16, 17 embody the best question one can ask at present to the high temperature superconductors. It could well be that the "Fermi-gas shaken from below" postulate underlying the Hertz-Millis main stream is actually correct. Implicitly it rests on the belief that upon dealing with fermions at a finite density the Fermi-liquid cannot be avoided as IR fixed point, and that it is eventually the organizing principle controlling everything else. However, one can as well argue that this is a tunnel vision, given in by the fact that with established technology it was impossible to describe mathematically any other state of fermionic matter. From this angle, the AdS_2 metal could be viewed as an eye opener: even when it is not literally realized it at least demonstrates that fermionic matter exists that is completely different from the Fermi liquid and its descendants like the BCS superconductor. As stressed by Hong Liu and coworkers [66], with the AdS_2 metal at hand one can have a very different view on the very basics of the physics of any of the quantum critical superconductors, including the cuprates, pnictides and heavy fermions. This was actually foreseen by Philip Anderson [67] a long time ago, based on empirical intuition. Instead of assuming that the zero temperature quantum phase transition is the cause and the strange metal is the effect, one should depart from seeking the cause in the high temperature bad metal. Departing from the chemistry at very high energy, one envisions entering an intermediate temperature regime where the system discovers an emergent conformal invariance. This is of an unusual kind in the sense that it does not require fine tuning to an isolated quantum critical point, while it has strange conformal properties like the local quantum criticality. The RN black hole is a perfect metaphor: this state is in turn intrinsically extremely unstable when temperature is further reduced. This instability is of the kind reminiscent of classically frustrated systems in the sense that are many competing ground states (the zero temperature entropy of the RN black hole). Although these also include quite classical states like the stripes and so forth, the frustration resides in the $\hbar \to \infty$ limit such that the most natural instabilities lead to quantum liquids. This surely included holographic superconductivity, but we shall see that also the Fermi-liquid is part of this black hole agenda. In the next section we will see that Fermi-liquids can emerge from the conformal metal, by a mechanism that is surprisingly similar to the RPA logic underlying the superconductivity. This is perhaps the most alarming message signaled by the black holes to the condensed matter physicist: the Fermi-liquid is not necessarily the unavoidable destiny of finite density fermion matter which is therefore the cause of everything, but instead it is just one of the many "ordering" phenomenon that can be easily fluctuated into disorderly pieces.

The only way to shed light on these very fundamental questions is by employing experiment. Can the dynamical pair susceptibility be measured? Right now, there is surely not a community in condensed matter physics that is routinely producing pair susceptibility data. It should be quite obvious that this is not only the best way to get at the origin of the superconductivity, but that it is in fact the *only way* to directly interrogate the pairing mechanism. It is a bit of an embarrassment for the experimentalists: the reasons for this to be not a standard activity are rather accidental. In fact, in the 1970's Allen Goldman [107, 108] already demonstrated that the imaginary part of the dynamical pair susceptibility can be directly measured in a straightforward Josephson junction set up. This involves a junction between a "probe" superconductor with a high Tc separated by a thin insulating barrier from a metal with a much lower superconducting T_c which one wants to investigate. Based on a precise linear response consideration, Ferrell and Scalapino [109, 110] predicted already in the 1960's that in the temperature regime where the probe superconducts while the system that is investigated is in its normal state, the current thought the junction is due to the second order Josephson effect with the ramification that this current is proportional to the imaginary part of the pair susceptibility of the metal, measured at an energy equal to the bias over the junction.

The dynamical pair susceptibility can be actually directly measured using the special "Ferrell-Scalapino" type of Josephson-junction device: Goldman delivered in the 1970's already proof of principle by measuring the relaxational BCS peak in aluminum in this way. Although this is quite a challenge for materials science, realizing such an experiment involving "strange metal" superconductors should become the highest priority item in experimental superconductivity research.

Exploiting a lead-aluminum junction Goldman demonstrated the measurement of the temperature evolution of the BCS relaxational peak in the normal state of the lower T_c aluminum metal. For accidental reasons this experiment was forgotten: the only follow up was the recent demonstration [111] of the vanishing of the particle hole asymmetry parameter τ_μ of Eq. (VIII.17) in an underdoped 123 cuprate superconductor probed by an optimally doped 123. It is in principle straightforward to directly measure the pair susceptibility in the metallic phase of a strange metal superconductor. As a lucky circumstance, one finds in the heavy fermion family such superconductors having a T_c that is quite low, just because of the chemistry conspiring to push the UV cut-off down. All what needs to be done is to build a junction involving such a heavy fermion metal from a superconductor with a much higher T_c which are also available in many forms. One is just dealing with the practical problem that it is not easy to fabricate a thin insulating barrier without pinholes between materials which are chemically so different.

Perhaps the condensed matter reader should take the message home from this subsection that it is quite unfair, if not silly, to put away the activity of string theorists as "stylized gibberish". More than anything else in AdS/CMT, the

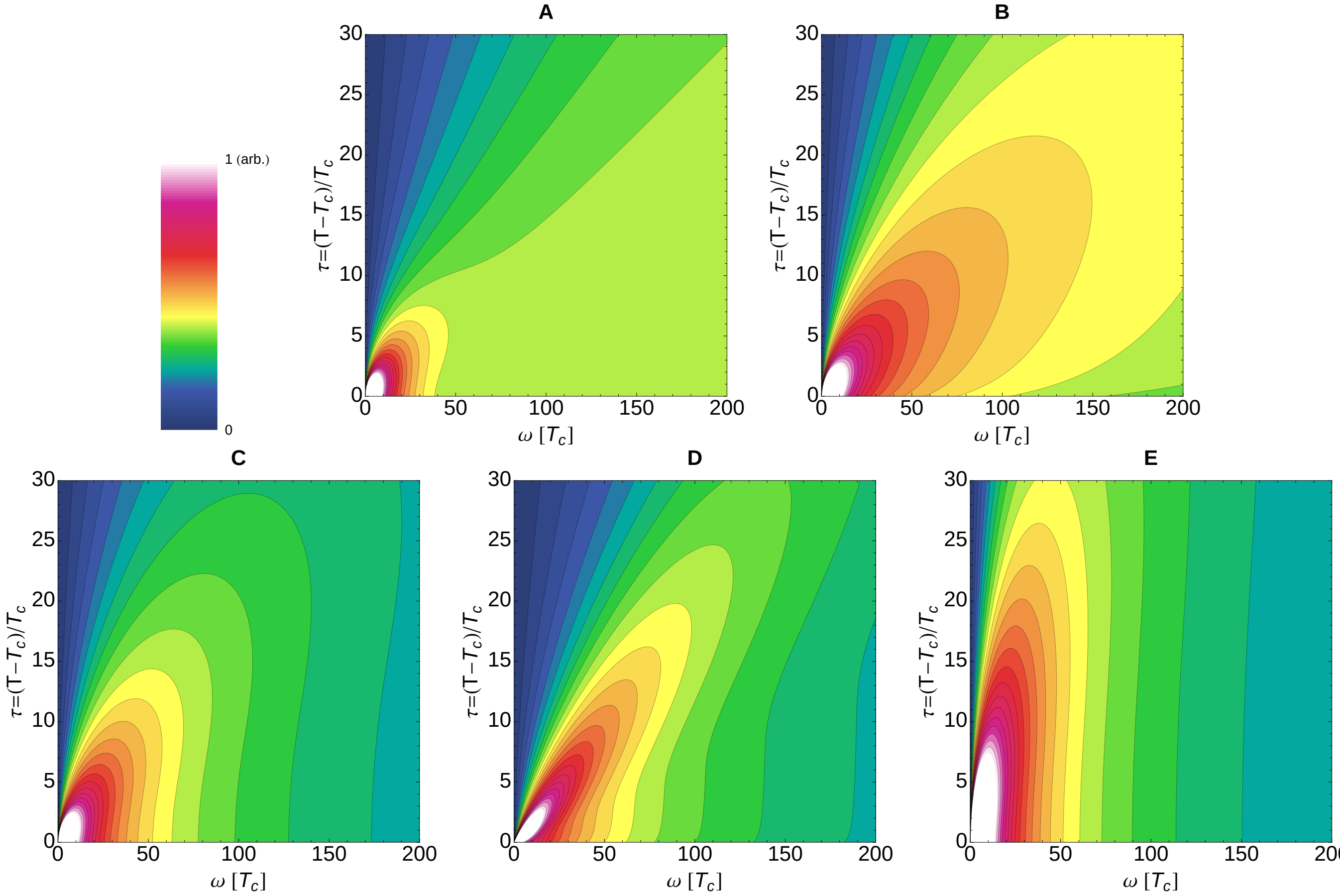

FIG. 16: **Imaginary part of the pair susceptibility. A-E**, False-color plot of the imaginary part of the pair susceptibility $\chi''(\omega, T)$ in arbitrary units as function of ω (in units of T_c) and reduced temperature $\tau = (T - T_c)/T_c$, for five different cases: case A represents the traditional Fermi liquid BCS theory), case B is the Hertz-Millis type model with a critical glue , case C is the phenomenological "quantum critical BCS" theory , case D corresponds to the "large charge" holographic superconductor with AdS_4 type scaling and case E is the "small charge" holographic superconductor with an emergent AdS_2 type scaling . $\chi''(\omega, T)$ should be directly proportional to the measured second order Josephson current (experiment discussed in the text). In the bottom left of each plot is the relaxational peak that diverges (white colored regions are off-scale) as T approaches T_c. This relaxational peak looks qualitatively quite similar for all five cases, while only at larger temperatures and frequencies qualitative differences between the five cases become manifest.

theory of holographic superconductivity is of such a quality that it is capable of producing smoking gun predictions for real life experiments. That the above narrative is not yet at the center of attention in the condensed matter community, let alone that millions of euro's, dollars, yens or yuans are invested in trying to realize the "Goldman" experiment, is just testimony of tribal conservatisms which come natural to any human activity.

C. The zero temperature states of holographic superconductors

It might be that this last part of this chapter is of less direct relevance to condensed matter physics, It deals with what we know about the zero temperature phases of holographic superconductors. This is still under production: although the crude outlines seem clear there are still instances where the problems are not quite solved yet. One aspect are the bottom-up zero temperature phase diagrams. In the phenomenological approach there are quite a number of free parameters and it is of interest to find out how these span up such phase diagrams. One caveat is that in the controlled top-down approaches one is dealing with explicit field theories corresponding with very particular choices of the parameters, and it is not at all clear whether these "fill up" all of the parameter space of the bottom-up models. The other caveat is that it is rather obscure how these bottom up parameters relate to anything one can vary in real condensed matter systems. A final aspect that we will discuss is the occurrence of "massless stuff" which is still around in the zero temperature holographic superconductors. In the single component electron superconductors of condensed

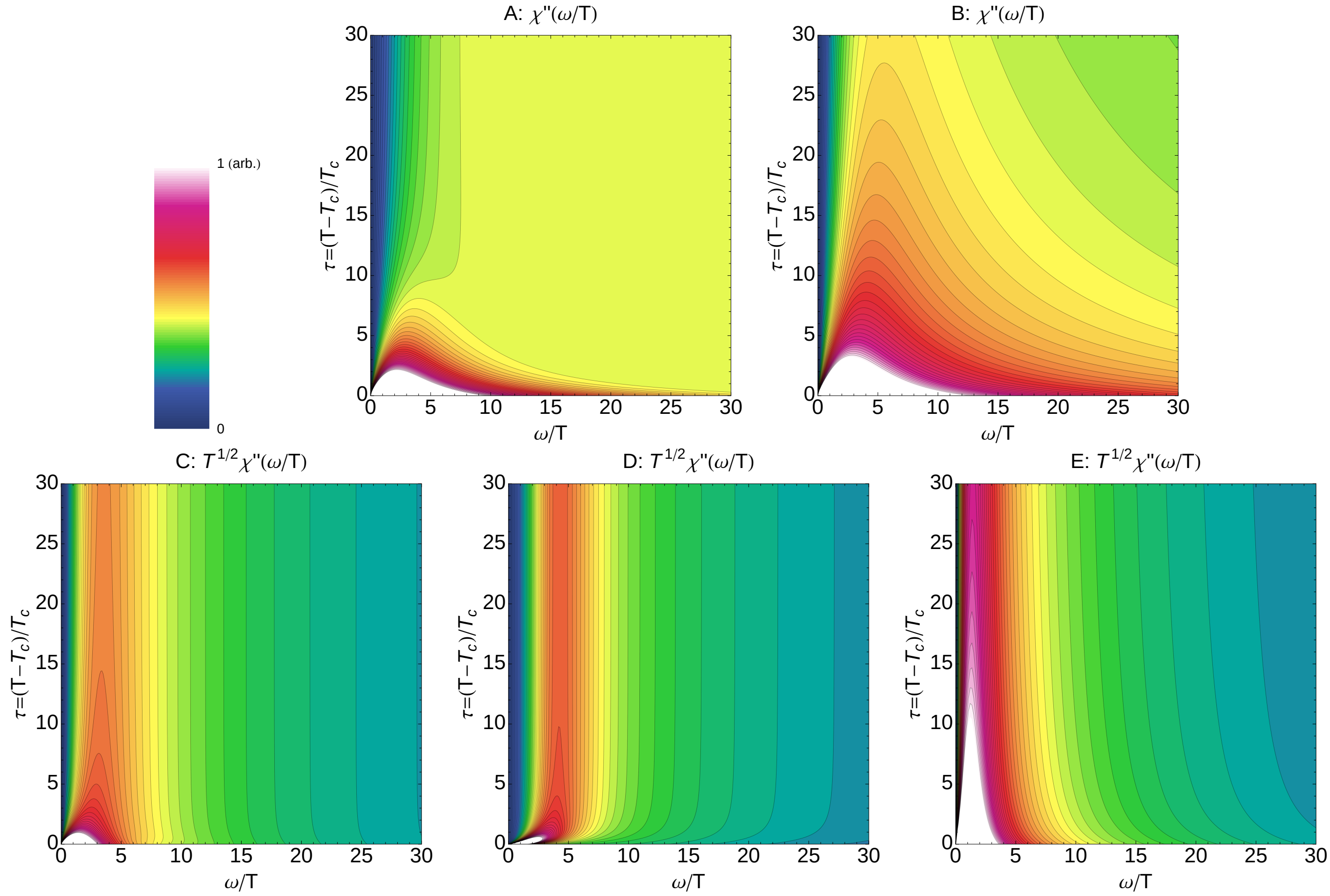

FIG. 17: **Energy-temperature scaling of the pair susceptibility.** **A-E**, False-color plots of the imaginary part of the pair susceptibility, like in Fig. 16, but now the horizontal axis is rescaled by temperature while the magnitude is rescaled by temperature to a certain power: we are plotting $T^{\delta}\chi''(\omega/T, \tau)$, in order to show energy-temperature scaling at high temperatures. For quantum critical BCS (case C), AdS_4 (case D) and AdS_2 (case E), with a suitable choice of the exponent $\delta > 0$, the contour lines run vertically at high temperatures, meaning that the imaginary part of the pair susceptibility acquires a universal form $\chi''(\omega, T) = T^{\delta}\mathcal{F}(\omega/T)$, with $\mathcal{F}$ a generic scaling function, the exact form of which depends on the choice of different models. Here we choose in cases C-D-E $\delta = 1/2$, by construction. The weak coupling Fermi liquid BCS case A also shows scaling collapse at high temperatures, but with a marginal exponent $\Delta = 0$. In the quantum critical glue model (case B) energy-temperature scaling fails: for any choice of δ, at most a small fraction of the contour lines can be made vertical at high temperatures (here $\delta = 0$ is displayed).

matter physics the low energy world is very empty: modulo the nodal fermions associated with unconventional order parameters all electronic degrees of freedom are gapped, and all what remains are the phonons associated with the ionic subsystem in the solids. This is quite different in the field theories as of relevance to the AdS/CFT correspondence. These descent from large N supersymmetric Yang-Mills theories, and these have a bewildering variety of different degrees of freedom. The order parameter of the holographic superconductors gaps out a number of these degrees of freedom but apparently there is quite some stuff which is not talking directly to the order parameter. The precise identity of this stuff in the field theory is unclear but the gravitational side of the duality is telling us that something quite interesting is going on. At very low temperatures the effective geometry in the deep IR reconfigures due to the effects of back reaction associated with the stress energy of the scalar- and Maxwell fields. These geometries encode for the massless character of the "left over" degrees of freedom, but these fall into a quite different category of emergent conformal fields than the one implied by RN near horizon geometry: the most ubiquitous geometries are of the Lifshitz kind. The type of boundary scale invariance which is encoded by such geometries is quite familiar to condensed matter physicists: they correspond with zero density like CFT's characterized now by a dynamical critical exponent $z \neq 1$. Although the conformal symmetry is restored, Lorentz invariance (implying $z = 1$) is seriously broken.

A rather mysterious feature of the zero temperature holographic superconductors is the appearance of new mass-less (conformal) degrees of freedom deep inside the superconducting gap. These are governed by a "Lifshitz geometry", which is just the gravitational encoding for a scaling behavior governed by a dynamical critical exponent z that can vary between 1 (Lorentz invariance) and ∞, pending the details of the system.

Let us first address the rough features of the zero temperature phase [112, 113]. We have various parameters to play with. Let us first consider the "bare bone" holographic superconductor defined in the bulk through Einstein-Maxwell with an added scalar Higgs field. The latter needs a potential and the simplest choice is the Ginzburg-Landau "ϕ^4" form, Eq. (VIII.3). The first free parameter is the mass of the scalar field m and this will be the number one tuning parameter because it directly translates into the scaling dimension of the scalar field at high energy via $\Delta_+ = D/2 + \sqrt{D^2/4 + m^2 L^2}$. The next parameter is the scalar field self-interaction strength λ. In the presentation in the above we focussed on the minimal holographic superconductor where this self-interaction is just ignored. This is a simplifying circumstance at higher temperature, but it turns into a nuisance at zero temperature. As we already announced, the fully back reacted solutions have in common that much is going on in the bulk upon lowering temperature. Close to T_c the deep infrared is governed by the RN black hole with a "crew cut" scalar hair, having a near horizon geometry that is quite like a simple Schwarzschild thermal state. Upon lowering temperature, the black hole horizon recedes along the radial direction and the hair "grows". The result is that the near horizon regime becomes increasingly extremal RN-like. However, at a quite low temperature estimated underneath the hair starts to take over and at zero temperature the black hole has completely disappeared. Instead one finds a "lump" of scalar field that has "absorbed" the black hole charge (the system is still at finite density) which is stabilized by a fine balance between gravitational attraction, Coulomb repulsion and the "box like" nature of the overall AdS space time. This is therefore no longer a hairy black hole, and since the scalar field has no edge where it ends the string theorists do not like to call it a "star" either. By lack of a better word let's call it the "Higgs lump". The beauty is surely in the fact that the Higgs lump defeats by gravitational means the extremal black hole: the ramification for the field theory is that the absence of a horizon means that the ground state of the holographic superconductor has removed the zero temperature entropy. This thing is stable.

The gravitational computation of the Higgs lump is hard work. One finds a typical solution that we will call "Lifshtiz" for reasons that will become clear soon. For other kinds of zero temperature solutions we refer to [113]. The Lifshitz solutions are characterized by a particular dimensionless number z. One can show that z and the parameters in the action m, q, λ satisfy the relations

$$2q^2(m^2 - 2q^2)z^3 + (m^4 + 2m^2q^2 - 4q^4 + 12\lambda)z^2 + 4q^2(2m^2 - q^2)z + 12q^4 = 0. \tag{VIII.23}$$

For large self-interaction λ and the parameters which satisfy the relation (VIII.5), we have

$$z = \frac{6}{q^2(-m^2 + 2q^2)}\lambda + \big(-1 + \frac{m^4}{2q^2(-m^2 + 2q^2)}\big) + \mathcal{O}(\frac{1}{\lambda}). \tag{VIII.24}$$

For small or vanishing λ the potential becomes very flat, with all the difficulties associated with stability one expects intuitively. For increasing m ,while fixing q and λ, at some point a quantum phase transition occurs at zero temperature from the holographic superconductor back to the RN zero temperature metal. Departing from the metallic (RN) state, this is easy to understand from the bulk physics. When the mass is increasing one looses the violation of the BF bound even in the AdS_2 geometry of the near horizon area of the RN black hole. On the field theory side it also makes perfect sense. The mass of the scalar field in the bulk is dual to the scaling dimension of the pair operator in the boundary, *e.g.* in $2+1$ dimensions $\Delta_+ = 3/2 + \sqrt{9/4 + m^2L^2}$. Upon approaching the UV BF bound the pair propagator of the effectively zero density (UV) theory is as relevant as it can be, peaking towards low energy. Although holography tells us that the field theory wants to undergo a pairing instability all by itself, without "help" from the outside, the same logic is at work as in quantum critical BCS. The divergence of the pair susceptibility of the normal state in the zero energy, zero temperature limit has the effect via the implicit RPA logic explained in the previous section that the superconductivity sets in at a very high temperature of order of the chemical potential. Upon increasing the mass the UV scaling dimension turns first marginal (like BCS) and eventually irrelevant (decreasing towards lower energy) with the effect that the superconducting instability gets at some point completely suppressed. A subtlety is here of course that for increasing mass one first enters the AdS_2 like geometry of the RN black hole, and close to the boundary with the normal state the instability criterium becomes associated with the scaling dimension of the scalar field as set by the near horizon AdS_2 geometry (compare with fig.'s 16 c,d, 17 c,d).

Although for small λ other fully back reacted solutions have been found, the generic solution is the one with "Lifshitz geometry". The "Higgs lump" extends all the way to the deep interior where is back reacts on the geometry via its stress-energy associated with its mass and the charge it carries. This special gravitational bound state defines a quite interesting zero temperature geometry as indicated in the right plot in Fig.18 for the 2+1D field theory case. Starting near the boundary one of course has the pristine AdS_4 geometry. Descending along the radial axis, at a scale associated with the chemical potential the charge of the Higgs lump becomes dominant and the geometry turns into the quasi-local quantum critical AdS_2. However, descending further along the radial direction something has to happen since the RN black hole has uncollapsed, and the system crosses over to the Lifshitz geometry, at a scale

$$r = r_* - L_2 e^{-z} \tag{VIII.25}$$

where r_* is the horizon of RN black hole, z is the Lifshitz scaling dimension and L_2 is the AdS_2 radius (VII.18). Interestingly, one still finds a "hair" solution when the charge of the Higgs field q is set to zero: see the left plot in Fig. 18. The lump is now literally scalar hair again, kept together entirely by gravity, since the charge associated with the finite chemical potential in the boundary has to be stored in the extremal RN black hole. Accordingly, in this zero charge case one still has an AdS_2 geometry in the deep interior which is yet different from the "intermediate scale" geometry which is similar as to the Lifshitz case (Fig. 18). Potential applications of this zero charge case to condensed matter are further discussed in ref. [114].

What is this deep interior Lifshitz geometry? Similarly as to the extremal RN case discussed in section VII, one can derive an effective metric in this regime,

$$ds^2 = -r^{2z} dt^2 + g_0 \frac{dr^2}{r^2} + r^2(dx^2 + dy^2) \tag{VIII.26}$$

with

$$g_0 = \frac{2m^2 z + q^2(3 + 2z + z^2)}{6q^2} \tag{VIII.27}$$

while both the gauge potential and the scalar field become independent of the radial coordinate

$$A_t = \sqrt{\frac{z-1}{z}}, \quad \Psi = \sqrt{\frac{12z}{2m^2 z + q^2(z^2 + 2z + 3)}}. \tag{VIII.28}$$

The z is in turn given by Eq. (VIII.24). This quantity is actually the working horse. Consider the scaling of the effective metric Eq. (VIII.26),

$$r \to b^{-1} r, \qquad t \to b^z t, \quad \{x, y\} \to b\{x, y\} \tag{VIII.29}$$

This is quite like the scaling of the pure AdS geometry: when $z = 1$ it would be the same but for $z > t$ time scales faster by b^z. Using the isometry-conformal symmetry bulk-boundary correspondence where this whole story started, this means that in the deep infrared realized inside the holographic superconductor a field theory emerges characterized by a dynamical critical exponent $z \neq 1$. This is called "Lifshitz scaling" by the string theorists, but such field theories are overly familiar to the condensed matter physicists. After all, $z = 1$ signals Lorentz-invariance and this is in condensed matter systems always badly broken in the UV: although it is relatively ubiquitous in the IR it needs to be "generated dynamically". This emergent Lorentz invariance is fragile as well: in the presence of a heat bath coupling to the order parameter, like in the Hertz-Millis theories, z is always larger than 1.

The scaling behavior of the field theory is modified in ways that are quite familiar to the condensed matter physicist. For instance, for $z = 1$ the entropy is for a Lorentz invariant CFT scaling like $S \sim T^{D-1}$ (D is the number of space-time dimensions). This is in fact associated with the finite size scaling associated with the time direction in the Euclidean field theory, and when there are "z time directions" this turns into $S \sim T^{(D-1)/z}$; for $z \to \infty$ this becomes a constant, the extremal RN zero temperature entropy. Fundamentally, it is here the same story. Due to the finite density Lorentz-invariance is lost and z can take a-priori any value. In the condensed matter context one is used to small z's (like 2 or 3 for antiferromagnetic or ferromagnetic Hertz-Millis systems) but in this bottom up setting z can be anything between 1 and ∞ according to Eq. (VIII.24). The upper limit is just set by the AdS_2 geometry as one get check easily by inserting $z \to \infty$ in the metric Eq. (VIII.26): this is just the meaning of quasi-local quantum criticality which is so difficult (f not impossible) to realize in explicit field theoretical systems.

Interestingly, there is top-down evidence for a $z = 1$ deep infrared as well. This is found in the holographic superconductor constructed using the consistent truncation of 11 dimensional supergravity employing Sasaki-Einstein compactification [86]. At face value, this is quite absurd: one starts out with a pristine CFT_3, to break the Lorentz and conformal symmetry by the chemical potential, the system reacts vigorously by forming the holographic superconductor, to rediscover in its deep infrared again a not-so-pristine CFT_3! What are these low energy degrees of freedom in an explicit field theory language? The same theme will re-occur in the next chapter dealing with the electron stars describing something that looks much like a Fermi-liquid, coexisting however with similar deep IR degrees of freedom. The answer is that right now it is a conundrum. From the discussion of the Hawking-page transition we learned to appreciate that the presence of a horizon signals deconfined degrees of freedom. However, for the present case of the Higgs lumps the horizon has disappeared and in analogy with the thermal AdS one would perhaps be tempted to identify the "Lifshitz" sector as some kind of confined degrees of freedom. However, these are still massless given that the geometry extends all the way to the deep interior. One can now in addition wire in a confinement scale by "capping off" the geometry using a hard wall or the so-called AdS soliton. This has the (predictable) effect that the "Lifshitz" degrees of freedom are gapped out, while the holographic superconductivity is in first instance not affected. Such a "confined" holographic superconductor is quite like a condensed matter superconductor and it is now possible to mimic a superfluid-bose Mott insulator transition [38]: the "normal" state is now a simple confining state that can be identified with a featureless condensed matter insulator.

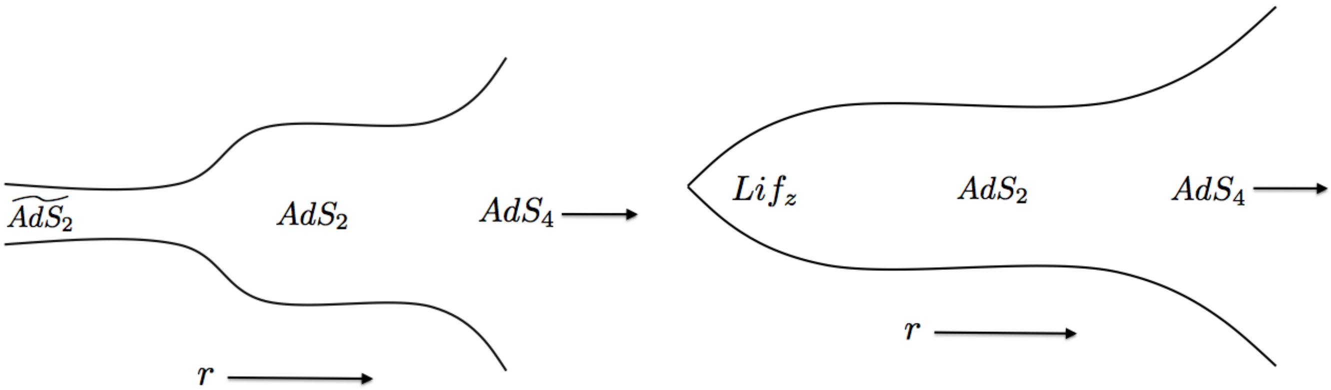

FIG. 18: The geometry after considering the backreaction of the boson. Left: $q = 0$, Right: $q \neq 0$. The crossover scale is $r \simeq r_* - L_2 e^{-z}$. Plot is taken from [115].

The presence of these additional, rather mysterious massless degrees of freedom has also interesting consequences for the nature of the zero temperature quantum phase transitions (QPT's). Different from the straightforward, standard mean-field transitions one finds at finite temperature, one finds novel types of universality classes at zero temperature which are unfamiliar in the condensed matter context. By just tuning the scaling dimension of the UV scalar field one finds a "holographic BKT transition". Such transitions are quite ubiquitous dealing with zero temperature transitions departing from RN strange metals. These behave quite literally like the infinite order Kosterlitz-Thouless transitions familiar from two dimensional XY systems. However, they appear to have no relation with the standard BKT logic associated with the unbinding of vortices governed by logarithmic vortex-vortex interactions. These holographic BKT transitions are rather insensitive to dimensionality and these are not rooted in unbinding of topological excitations. The large N limit is still exerting control and eventually even the holographic BKT's are mean-field transitions of a new kind, perhaps again associated with the influence of fermion signs.

Different from the finite temperature transitions, the quantum phase transitions from the RN metal to the holographic superconductor are of a new kind. These occur either as "hybridized" transitions or as large N mean field versions of the Berezinskii-Kosterlitz-Thouless transitions where the mean-field order parameter acquires extra damping by its decay in the deep infrared AdS$_2$ quasi-local quantum critical degrees of freedom associated with the RN extremal horizon.

The other way of forcing a zero temperature transition is by suppressing the holographic superconductivity by a "repulsive" double trace deformation, as discussed in the previous subsection. Here one finds yet another type of quantum phase transition transiiton, where now the mysterious "deconfined" or "behind the horizon" degrees of freedom play an active role. The order parameter itself is now subjected to a normal mean-field behavior but now it mode couples or "hybridizes" with the AdS$_2$ degrees of freedom associated withe the extremal black hole. In the spirit of Hertz-Millis theory, this hybridization with the massless strange metal degrees of freedom adds an extra self energy to the order parameter dynamics, as if the order parameter is talking to an effective heat bath.

Let us now consider these zero temperature transitions in more detail. To get a full view it is useful to employ besides λ, q also the double trace deformation introduced in the previous subsection, Eq's (VIII.20, VIII.21). Here one needs another subtlety associated with the double trace deformation, called "alternative quantization". Note in the Sec. III when we introduced GKPW rule, we treated A as the source and B as the corresponding response: this is the so called "standard quantization". In the region $\nu \leq 1$, one can alternatively treat B as the source while A as the response and we call it the "alternative quantization". The following discussion applies for both these two quantizations. As an example we consider the alternative quantization case. We use κ_- which means we add the double trace deformation of the operator $\mathcal{O}_-$ which duals to ϕ (*i.e.* alternative quantization) and we use κ_+ to parametrize the coefficient of the double trace deformation of the operator $\mathcal{O}_+$ which duals to ϕ (*i.e.* standard quantization) [115]. The other natural parameter to vary at zero temperature is u which combines the scalar field mass m and charge q in terms of a single quantity as of relevance to the stability of the holographic superconductor (Fig. 22):

$$u = m^2 L_2^2 + 1/4 - q^2 e_d^2. \tag{VIII.30}$$

One finds now the two different types of quantum phase transitions we already announced[115] between the zero temperature superconducting ("condensed", C) and RN metal ("uncondensed", U) phases. These are indicated in Fig. 19 where one finds the BKT transition (called "bifurcating") upon varying u, while the double trace deformation is found to cause the "hybridized" transition. When these transitions meet at $u = 0$ and $\kappa_- = 1.5$ one finds yet another behavior called "marginal".

The system has a vacuum IR instability for $\kappa_- < 0$ as a pole shows up in the upper-half ω plane for the double trace deformed Green's function [115]. For $u < 0$, the system is unstable in the AdS_2 region as it violates the BF bound with $u = 0$ the system develops an IR instability for $\kappa_- > \kappa_c$ giving a the critical line for a bifurcating QCP. For $u > 0$, the critical line is pushed into the region $\kappa_- > 0$ by finite density effect.

Let us first discuss the hybridized type transition. The procedure to depart from the scalar field propagator/pair susceptibility from the previous subsection, to reconstruct from this an effective action that governs the transition, in the spirit of the relaxational GL action deduced for the thermal transition. For the hybridized case one departs again from the matching procedure. According to Eq. (VIII.21), we have $\psi_+ = b_+(\omega, k) + b_-(\omega, k)\mathcal{G}_k(\omega)$ and $\psi_- = a_+(\omega, k) + a_-(\omega, k)\mathcal{G}_k(\omega)$ by the near far matching method [103]. As for the thermal transitions, the pair propagators close to the QPT's are governed by the matching/RPA-like expressions,

$$G_R(\omega, \vec{k}) \simeq \frac{a_+(\omega, k) + a_-(\omega, k)\mathcal{G}_k(\omega)}{\tilde{b}_+(\omega, k) + \tilde{b}_-(\omega, k)\mathcal{G}_k(\omega)} \tag{VIII.31}$$

where

$$\tilde{b}_\pm(\omega, k) = b_\pm(\omega, k) - \kappa_- a_\pm(\omega, k) = \tilde{b}_\pm^{(0)}(k) + \mathcal{O}(\omega). \tag{VIII.32}$$

The hybrized type of phase transition is realized when the following condition is full filled,

$$0 < \kappa_- \leq \kappa_c = \frac{b_+^{(0)}(0)}{a_+^{(0)}(0)} \tag{VIII.33}$$

noticing that at $\kappa_- = \kappa_c$ we have $\tilde{b}_\pm(0, 0)|_{\kappa_c} = 0.$, such that at larger κ_- values the matching expressions acquire an RPA form. As will become clear in the next chapter, this type of quantum phase transition is very closely related to the way that the Fermi-liquid emerges from the RN metal. The fermions have still their mysteries but the present case of the bosonic order parameter dynamics is quite transparent, although the consequences are less familiar from the condensed matter context.

Starting from uncondensed side and expanding $\tilde{a}_\pm(k, \omega)$ around $\omega = 0, \vec{k} = 0$ and $\kappa_- = \kappa_c$, Eq. (VIII.31) can be written as

$$G_R(\omega, \vec{k}) \simeq \frac{Z}{\kappa_c - \kappa_- + h_k \vec{k}^2 - h_\omega \omega^2 + h\mathcal{G}_k(\omega)}. \tag{VIII.34}$$

This looks quite like the finite temperature case where the relaxational peak of the thermal transition turns into a propagating boson corresponding with the free quantum field associated with the holographic superconductor at the transition. This has a mass $\sim \kappa_c - \kappa_-$, and the vanishing of this mass signals the transition. The order parameter itself is still governed by the gaussian/large N mean field dynamics but different from the finite temperature case, at zero temperature one has now to take serious the presence of the $h\mathcal{G}_k(\omega)$ factor in the propagator. This indicates that the order parameter acquires an additional damping due to the presence of the massless degrees of freedom associated with the AdS_2 deep infrared of the strange metal. We learned in chapter VIII that this has the propagator of the unusual form $\mathcal{G}_k(\omega) = c_k e^{i\phi_k} \omega^{\nu_k}$ where the exponents $\nu_k \sim \sqrt{u + k^2}$ where u is defined in (VIII.30) and is set by the specifics of the AdS_2 geometry. This is the same u as in the phase diagram Fig. 22, and this shows that at least when $\kappa_- > 0.5$ the onset of the holographic superconductivty is coincident with the violation of the AdS_2 BF bound at zero momentum, where the scaling dimension ν_k turns imaginary. For bosons this is coincident with the statement that for negative u's one runs into "tachyons" (linear instability) such that the vacuum has to change.

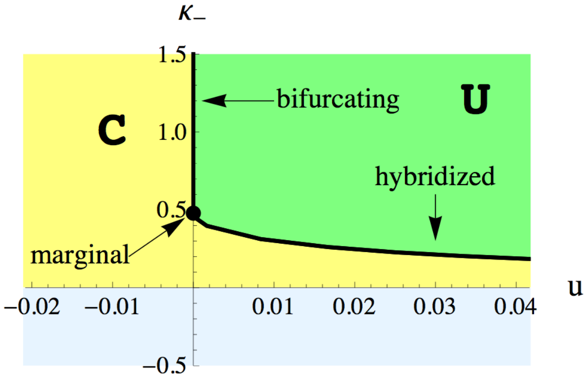

FIG. 19: Phase diagram for the alternative quantization with double trace deformation. Plot is taken from [115].

The bottom line is that although the phase transition is in first instance governed by a purely gaussian field theory, the order parameter field is still communicating with a "heat-bath" (the AdS_2 sector) in a simple, linear coupling way as if one can get away with second order perturbation theory/Fermi's golden rule. Another way of making the same statement is in terms of an effective field theory of the form,

$$S_{\text{eff}} = S_{\text{AdS}_2}[\Phi] + \int \lambda(k,\omega)\Phi_{-\vec{k}}\Psi_{\vec{k}} + S_{\text{LG}}[\Psi] \tag{VIII.35}$$

where S_{LG} is the usual Ginzburg-Landau theory to be literally interpreted in the mean-field sense,

$$S_{\text{LG}} = -\frac{1}{2}\int \Psi_{-\vec{k}}(\kappa_c - \kappa_- + h_k k^2)\Psi_{\vec{k}} + h_t \int (\partial_t \Psi)^2 + \ldots \tag{VIII.36}$$

and $S_{\text{AdS}_2}[\Phi]$ is a strongly coupled sector in terms of unknown fields Φ. In the next section we will encounter a very close sibbling of this structure, which has dealings with the holographic quasi-Fermi-liquids. This appears to be explained in terms of the "semi-holography" language of Faulkner and Polchinski [116] where this odd "second order perturbation theory" behavior is explained as being deeply rooted in the mean field structure of the large N limit.

These hybridized transitions are found in a regime where the BF bound is not yet violated in the AdS_2 geometry: one just uses a "strongly attractive" double trace deformation to force in the instability while the RN metal of the bulk is still stable. The "bifurcating" transition is comparable to the mean-field thermal transitions we discussed in the beginning, except that at zero temperature there is more action: one finds out that the transition changes its nature. Instead of simple field, it turns into one that has the same scaling properties as the familiar Berezinskii-Kosterlitz-Thouless (BKT) transition familiar from the thermal 2+1 D XY system. The mechanism for BKT phase transition follows when an IR fixed point of the system merges with a UV fixed point and its characteristic is the exponential behavior. Among others, one finds that the susceptibility does not diverge approaching the transition, but instead it develops a branch cut singularity ("bifurcation") at the critical coupling with the effect that it jumps as at a Kosterlitz-Thouless transition [117] (see Fig. 20) and one finds that the condensate depends on the tuning parameter as $\langle\mathcal{O}\rangle \sim \exp\big(-\pi/2\sqrt{-u}\big)$. Interestingly, the correlation length shows still a mean-field divergence $\xi \simeq 1/\sqrt{u}$ for $u > 0$ where $u = 0$ is the critical coupling.

We can also tune κ_- and u at the same time to have the critical point at $u = 0, \kappa_- = \kappa_c$, where the susceptibility both diverges and bifurcates. This is so called the marginal quantum critical point. Interestingly, around this point the susceptibility has the similar form as the bosonic fluctuations underlying the "Marginal Fermi Liquid" proposed by Varma et al. [118] for describing the strange metal region of the high Tc cuprates. For details for this case please refer to [115].

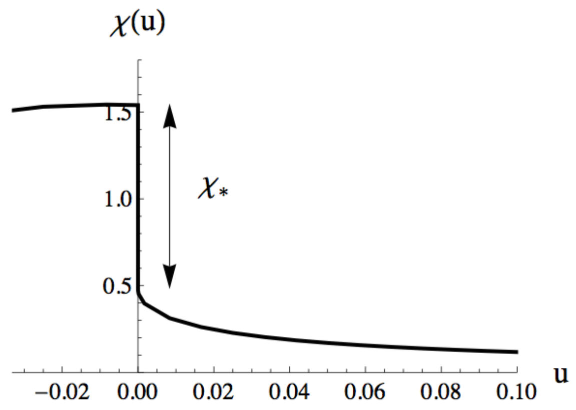

FIG. 20: The static susceptibility has a jump across the critical point. Plot is taken from [115].

IX. HOLOGRAPHIC STRANGE METALS

A. The quest for non-Fermi liquids

We now turn to one of the most successful achievements of AdS/CMT holography. From the experiments on high Tc superconductors and heavy fermion systems, we know that finite density electron systems can display remarkably novel behavior. Qualitatively they are brutally distinct from regular metals or insulators described by perturbative Landau Fermi liquid theory. Experimental probes of these "strange metals" indicate that the system is not only strongly coupled, but also displays critical behavior. Most notably, response functions exhibit energy/temperature scaling. The combination of strong coupling and criticality make strange metals an ideal arena to apply AdS holography. As we shall show, the results have thoroughly vindicated this idea. Not one, but whole classes of quantum critical strange metals have been found. As a proof of principle this has already given an explosively positive answer to the question whether interacting finite density fermi systems exist that are different from the Landau Fermi liquid or a BCS-paired condensed phase. The practical task has therefore been turned on its head and instead of finding a single strange metal theory, we must now all of sudden dissect this embarrassment of riches to decipher whether and how it connects to observed strange metals in experiment.

B. A modern view on the Landau Fermi liquid

Before we do so, we present for reference a modern view of Landau Fermi liquid theory. The Landau Fermi liquid starts from a zero temperature free Fermi gas with a characteristic Fermi momentum k_F associated with the last occupied state.The dynamics are described by expanding the free (non-relativistic spinless) electron Lagrangian at finite chemical potential μ

$$\mathcal{L} = \int d^d x \Psi^\dagger(x)(i\partial_t + \mu + \frac{1}{2m^2}\nabla^2)\Psi(x) \tag{IX.1}$$

around the Fermi momentum $k_F = \sqrt{2m\mu}$.

$$\mathcal{L}_{\text{Fermi gas}} = \int \frac{d^d k}{(2\pi)^d}\Psi^\dagger(k)(i\partial_t - v_F(k-k_F))\Psi(k) + \ldots \tag{IX.2}$$

with $v_F = k_F/m$. Conventionally one invokes the Pauli principle that turning on small interactions cannot in any way change this occupation barrier and the existence of a momentum scale k_F, and postulates that the relevant degrees of freedom are still the fermions above. In the modern view this insight directly follows from the renormalization group [89, 90]. Assuming that the interactions between the fermions are controlled by a scale M, one can integrate out the interactions to arrive at an effective theory for the low-energy fermions. In perturbation theory this results in *analytic* higher order corrections to the dispersion relation plus induced interactions.

$$\begin{aligned}\mathcal{L}_{\text{Fermi liquid}} &= 1\int \frac{d^dk}{(2\pi)^d}\Psi^\dagger(k)(i\partial_t - v_F(k-k_F) + \sum_{n,m=1,2,\ldots} c_{n,m}\frac{\omega^{n+1}(k-k_F)^{m+1}}{M^{n+m}} + \ldots)\Psi(k) \\ &\quad + \sum_{k=1}\frac{1}{M^{dk}}(\Psi^\dagger\Psi)^{k+1} + \ldots \\ &= \int \frac{d^dk}{(2\pi)^d}\Psi^\dagger(k)(i\partial_t - v_F(k-k_F) + \frac{\alpha}{M}\omega^2 + \frac{\beta}{M}(k-k_F)^2 + \ldots)\Psi(k) + \frac{\gamma}{M^d}(\Psi^\dagger\Psi)(\Psi^\dagger\Psi) + \ldots\end{aligned} \tag{IX.3}$$

Straightforward dimensional analysis shows that all interactions between the electrons are irrelevant in any dimension $d > 1$. The generic fixed point of an interacting finite density fermi system is thus a field theory of fermionic *quasiparticles*. This generic effective low-energy theory is the Fermi liquid. Its tell-tale sign is that the two-point correlation function of two such quasiparticles

$$G(\omega, k) = \frac{1}{\omega - v_F(k-k_F) + \Sigma(\omega, k)} \tag{IX.4}$$

has a self-energy $\Sigma(\omega, k)$ whose imaginary part $Im(\Sigma) \equiv \Sigma''$ behaves as $\Sigma'' \sim \frac{1}{M}\omega^2$ (for $k = k_F$). This is the reflection of the analyticity of the gradient expansion. The spectral density $a = -\frac{1}{\pi}G_R(\omega, k)$ therefore has a beautiful Lorentzian line shape around $\omega = 0$ for $k = k_F$.

The modern view is thus that the Landau Fermi Liquid is the generic low energy effective action near an IR free fixed point of an theory of interacting fermions. This is why Landau Fermi liquid theory explains essentially all qualitative macroscopic properties of metals. For instance the low temperature T^{-2} dependence of the DC conductivity of metals directly follows from the Landau Fermi Liquid properties. Its main property is that the low energy physics is controlled by the $d-1$ dimensional Fermi surface in momentum space. Due to the Pauli principle the quasiparticles have only the normal direction $k_\perp$ to this surface to explore. This quasi 1D behavior is reflected in all low T thermodynamics. For a generic system entropy density, and specific heat behave as

$$s \sim \frac{1}{L^d} \sim T^d v^d \ ; \quad c_V \sim T\frac{dS}{dT} \sim T^d \tag{IX.5}$$

Substituting the low energy effective $d = 1$ dimensionality one finds the Fermi liquid entropy and specific heat scaling linear in T. This naive scaling argument can be explicitly verified by exact calculations.

The DC conductivity in Landau Fermi Liquid Theory

Because the effective Lagrangian is essentially free at low energies/temperatures the DC conductivity is to leading order just the lowest order current-current correlator and one readily computes

$$\begin{aligned}\sigma_{DC} &\sim \lim_{\omega\to 0}\frac{1}{i\omega}\text{Im}\bar{\Psi}\Psi\bar{\Psi}\Psi \\ &\sim \lim_{\omega\to 0}\frac{1}{i\omega}\int d^dk\frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\frac{f(\omega_1)-f(\omega_2)}{\omega_1-\omega-\omega_2-i\epsilon}A(\omega_1,k)\Lambda(\omega_1,\omega_2,\omega,k)A(\omega_2,k)\Lambda(\omega_2,\omega_1,\omega,k)\end{aligned} \tag{IX.6}$$

Here $A(\omega, k) \equiv -\frac{1}{\pi}\text{Im}G_R(\omega, k)$ is the spectral function of the fermionic quasiparticles with correlation function (IX.4). The interaction vertex $\Lambda(\omega_1, \omega_2, \omega, k)$ must be analytic in all its arguments and to leading order in ω is simply a constant. Thus

$$\begin{aligned}\sigma_{DC} &\sim \lim_{\omega\to 0}\frac{1}{i\omega}\int d^dk\frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\frac{f(\omega_1)-f(\omega_2)}{\omega_1-\omega-\omega_2-i\epsilon}A(\omega_1,k)A(\omega_2,k) \\ &\sim \int d^dk d\omega_1\frac{df}{d\omega_1}|A(\omega_1,k)|^2\end{aligned} \tag{IX.7}$$

One can get the dominant T-dependence in the spectral function by shifting the the imaginary part of the self-energy $\Sigma''(\omega) \to \Sigma''(\omega + T)$. This lifts the pole, but nevertheless the spectral function $A(\omega, k)$ remains dominated by the Fermi-surface $\omega \sim 0, k \sim k_F$. We can thus estimate

$$G = \frac{1}{\omega - v_F k_\perp + i\frac{(\omega+T)^2}{M}} . \tag{IX.8}$$

The key observation is that the Fermi surface is $d-1$-dimensional surface in momomentum space with a single normal direction $k_\perp$. This means that $d-1$ of the momentum integrals computing the DC conductivity localize on the Fermi surface. Doing so

$$\sigma_{DC} \sim k_F^{d-1} \int dk_\perp d\omega \frac{df}{d\omega} |A(\omega, k_\perp)|^2 \tag{IX.9}$$

we see that the system has reduced to an effective 1D system. The derivative of the Fermi-Dirac distribution is essentially a delta-function at $\omega = 0$. A redefinition $k_\perp \to T^2 k_\perp$ scales the temperature out of the spectral function and the full integral and we deduce characteristic temperature-squared dependence of the conductivity of a Fermi-liquid.

$$\sigma_{DC} \sim \frac{T^2}{T^4} \sim \frac{1}{T^2} \tag{IX.10}$$

C. A Holographic Fermi liquid

Since AdS/CFT essentially states that every quantum field theory has a gravitational dual, one could directly ask what the gravitational description of a Fermi liquid is. At face value there is a conflict. As we have just explained the Fermi liquid is intrinsically weakly coupled (at low energies) with a small number of degrees of freedom whereas the *useful* classical gravitational regime is dual to a strongly coupled theory with a large number of microscopic degrees of freedom. The careful adjectives in the previous sentence shows how this is nevertheless possible. AdS/CFT is dual to a microscopic system of a large number of ddegrees of freedom. Furtermore the extra radial direction encodes the RG flow of the theory. If this theory flows in the IR to a generic Fermi liquid, then AdS/CFT should be able to capture this. We have already *seen* this explicitly in the holographic superconductor. There the IR, the weakly coupled Goldstone boson, is as generic as the Fermi liquid. In practical terms to initiate this RG flow in AdS/CFT language one must have some non-trivial field configuration in the bulk which backreacts on the metric. Clearly for an intrinsically fermionic system this should be a fermionic configuration. Doing so, one does not get the Fermi liquid, however. Below we will see what very interesting physics arises in this case, but to obtain the actual Landau Fermi Liquid, one further ingredient needed. One has to ensure that the low-energy degrees of freedom are only the fermionic quasiparticles and none other [127]. Since one starts with a CFT there is no obvious scale in the theory that could explain a gap. By dimensional transmutation such a gap can arise after the RG flow of course, but here we will follow a more direct way and simply build in this scale. Since a gap removes degrees of freedom from the IR, a brute force way to do so, is to simply cut-off the corresponding geometric region in the AdS bulk. I.e. we insert a "hard wall" in the AdS geometry

$$ds^2 = r^2 \left(-dt^2 + dx_1^2 + dx_2^2\right) + \frac{dr^2}{r^2} \tag{IX.11}$$

by limiting the range of $\infty > r > r_c$. The prototype of such engineering arose in ways to model QCD confinement in AdS/CFT and despite its incredibly coarse and brute nature, such hard wall models are remarkably successful. Following them we will do so here.

The effect of this hard wall tremendously simplifies the problem. Since in AdS the gravitational curvature results in a confining potential at $r \to \infty$, one just ends up with particles in a box. To add a fermionic field configuration is now a conventional Hartree exercise. For a given set of background bosonic fields (the gravitational metric and electrostatic potential) we construct the fermion wavefunctions, occupy them up to the chemical potential (i.e. weigh them with the Fermi-Dirac distribution), compute their combined energy and charge, recompute the gravitational metric and potential for this charge and iterate until convergence.

The exact equations follow from the AdS Lagrangian for a minimally coupled fermion of charge q. We will see that the Dirac field is dual to a fermionic operator $\mathcal{O}_f$ with charge q and conformal dimension $\Delta = \frac{D}{2} + mL$.

$$S = \int d^4x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\Psi}\left[e^\mu_a\Gamma^a(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\Gamma^{ab} - iqA_\mu) - m\right]\Psi\right) \tag{IX.12}$$

Note that in a gravitional background the Dirac action gets modified and contains both a vielbein, defined through $e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}$, and a spin connection $\omega_{\mu ab}$ defined through the fact that the vielbein should be covariantly constant $D_\mu e^a_\nu = 0 = \partial_\mu e^a_\nu - \Gamma^\tau_{\mu\nu} e^a_\tau + \omega^a_{\mu\ \ b} e^b_\nu$. Just as for the holographic superconductor, the essentials for the backreaction of a fermionic field configuration are already present in the large fermion charge limit where we can ignore gravitational backreaction. Moreover for metrics of the type (IX.11) the spin connection $\omega_{\mu ab}$ can be removed from the Dirac equation

$$\left(e^\mu_a(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\Gamma^{ab} - iqA_\mu) - m\right)\Psi = 0 \tag{IX.13}$$

by a redefinition

$$\Psi = (-gg^{rr})^{-1/4}\begin{pmatrix}\chi_+ \\ \chi_-\end{pmatrix} \tag{IX.14}$$

In flat spacetime we would now Fourier transform and project the 4-component Dirac spinor onto 2-component spin-eigenstates. Because the radial direction of anti-de-Sitter space breaks four-dimensional Lorentz invariance, this cannot be done here. However, a similar projection exists onto *transverse helicities*, where the spin is always orthogonal to both the direction of the boundary momentum and the radial direction (see box). Using rotational invariance to choose the boundary momentum along the x_1-direction, $\vec{k} = (k, 0)$ and projecting onto the t-helicities $\chi_\pm$ eigenstates of $\Gamma^5\Gamma^{x_1}$, the Dirac equation reduces to

$$\sqrt{\frac{g_{ii}}{g_{rr}}}\,(i\sigma_2\partial_r - \sqrt{g_{rr}}\sigma_1 m)\,\chi_\pm(r;\omega,k) = -(\pm k - \sqrt{\frac{g_{ii}}{-g_{tt}}}(\omega + q\Phi)\chi_\pm \tag{IX.15}$$

It suffices to consider only χ_+ from here on, as the results for χ_- simply follow by changing $k \to -k$.

Similar to scalar case (see e.g. [99]) near the AdS boundary the 2-component spinor χ_+ has the asymptotic behavior

$$\chi(r) = a\begin{pmatrix}0\\1\end{pmatrix}r^m + b\begin{pmatrix}1\\0\end{pmatrix}r^{-m} + \ldots. \tag{IX.16}$$

For a generic mass the first component is not-normalizable, but the second one is. To build a fermionic field configuration in the bulk, we must therefore seek solutions with $a = 0$.

Before we continue to do so, however, recall that the AdS/CFT dictionary identifies the leading non-normalizable component as the source and the subleading normalizable component as the response. Eq. (IX.16) illustrates this same concept, once we recall that for fermions obeying the first order Dirac equation the second component is the conjugate momentum, i.e. the response. Indeed the CFT Green's function for the dual fermionic operator is precisely

$$G(\omega, k) = \frac{b}{a}. \tag{IX.17}$$

We will use this later on, but let us now return to building a holographic Fermi liquid with a normalizable field configuration in the bulk. A very important aspect is that AdS fermions are *chiral*. For a fixed k_x only one of the t-helicities $\chi_\pm$ will have normalizable solutions we $a_\pm = 0$. To construct an isotropic fermionic field configuration out of solutions of the Dirac equation there is therefore *no* spin degeneracy. For $k_x > 0$ one has only χ_+ solutions, whereas for $k_x < 0$ only χ_- solutions exist. The exact normalization condition follows from the conserved norm for the Dirac Equation. It states that the wavefunction for each mode must be scaled to obey

$$\int_{r_c}^\infty dr\sqrt{-g}\bar{\Psi}(\omega,k)e^r_{\hat{r}}\Gamma^{\hat{r}}\Psi(\omega,k) = 1 \to \int_{r_c}^\infty dr\left(\chi_+^\dagger\chi_+ + \chi_-^\dagger\chi_-\right) = 1. \tag{IX.18}$$

On the right hand side there will always only be one of the two t-helicities that contributes and the chirality therefore implies the simple normalization condition for the two-component spinor

$$\int_{r_c}^\infty dr\chi_+^\dagger\chi_+ = 1. \tag{IX.19}$$

To deduce the boundary conditions at the hard wall, we insert the Dirac operator in the normalization condition, take the complex conjugate and integrate by parts. This shows that we must demand

$$\chi^\dagger(r_c)\sigma^y\chi(r_c) = 0. \tag{IX.20}$$

to ensure real eigenvalues, i.e. on such solutions the Dirac operator is self-adjoint.

These equations and boundary conditions are to be combined with the Maxwell equation. Since the system is isotropic only $A_0 = \Phi$ and A_r are relevant degrees of freedom and the latter can be set to vanish by a gauge choice.

$$\partial_r^2\Phi = -q\langle\langle\bar{\Psi}^\dagger(k)\Psi(k)\rangle\rangle. \tag{IX.21}$$

The double brackets on the right-hand-side denotes the expectation value of the number operator. For fermions at zero temperature this is just the sum over all "negative" energy states with normalizable wave functions.

$$\begin{aligned}\partial_r g^{rr}\partial_r\Phi &= -q\int\frac{d\omega d^2k}{(2\pi)^3}\theta(-\omega)\bar{\Psi}^\dagger_{nsol}(k)\Psi(k)_{nsol} \\ &= -q\int\frac{d\omega d^2k}{(2\pi)^3}\theta(-\omega)\chi_+^\dagger\chi_+. \end{aligned}\tag{IX.22}$$

Following the by now familiar dictionary rules the AdS boundary values of the electrostatic potential

$$\Phi = \mu - \frac{\rho}{r} + \ldots \tag{IX.23}$$

encode the chemical potential and charge density of the dual CFT. In the interior at the hard wall we will demand that the electric field vanishes, $\partial_r\Phi(r_c) = 0$ i.e. there are no sources of charge emanating from behind the wall. This is as it should be if the system is gapped, then there are no low-energy charges carriers remaining.

We can now initiate our Hartree procedure. We compute the spectrum of normalizable solutions for a given background potential $\Phi(r)$ dual to a CFT chemical potential μ, sum over all negative energy wavefunctions, find the corrected potential and iterate. Let's start at $\mu = 0$. There the Dirac equation can be solved exactly by Bessel functions [127]

$$\chi(r;\omega,k) = \frac{1}{\sqrt{r}}\begin{pmatrix}\frac{-M_\ell}{k+\omega(\ell,k))}J_{m+1/2}(\frac{M_\ell}{r}) \\ J_{m-1/2}(\frac{M_\ell}{r})\end{pmatrix} \tag{IX.24}$$

Here $M_\ell = j_{m-1/2,\ell} * r_c$ with $j_{m-1/2,\ell}$ the ℓ'th zero of the Bessel function $J_{m-1/2}(x)$. This ensures the correct boundary condition (IX.20) at the hard wall. The normalizability condition below Eq. (IX.16), $a = 0$, picks out distinct energy-eigenvalues

$$\omega(k,\ell) = \pm\sqrt{k^2 + M_\ell^2}. \tag{IX.25}$$

This spectrum is illustrated in Fig. 21.

Following [127] we will ignore all the bare $\mu = 0$ negative energy solutions in the Dirac Sea, and only consider the solutions $\omega(k,\ell) > 0$. In the absence of occupied fermion states a constant value of $\Phi = \mu$ is a solution to the Maxwell equations of motion with the right boundary conditions. Increasing the chemical potential this way thus shifts the eigenvalues $\omega(k,\ell)$ to $\omega(k,\ell;\mu) = \omega(k,\ell) - q\mu$. Nothing happens until one reaches the critical value where $\omega(k,\ell;\mu) = 0$. At this moment one can occupy the state and through Maxwell's equations this will subtly alter the AdS electrostatic potential. This effect increases as one increases μ. The resulting spectrum is also displayed in Figure 21.

The claim is that this state is precisely the holographic dual of the regular Landau Fermi liquid. The way to see this is to recall from (IX.17) that the solutions to the Dirac equation we construct to occupy the states in the bulk can also be used to construct the Green's function in the CFT. As it should be, precisely for a normalizable solution $a = 0$ one finds a pole in the Green's function. In our system this pole is really a branch cut along the full dispersion curve in Fig 2. but this is just an artifact of the free Fermi-gas approximation in the bulk. Once we include loop-effects the conventional Landau Fermi Liquid argument for interacting fermions will give rise to a self-energy $\Sigma \sim i\omega^2$. The true pole will therefore lie at the zero of energy corresponding to a distinct momentum value k_F.

A Holographic Luttinger Theorem— Since this k_F in the strongly coupled boundary Green's function is the very same k_F corresponding to the bulk weakly coupled Fermi gas, one can immediately draw an important conclusion

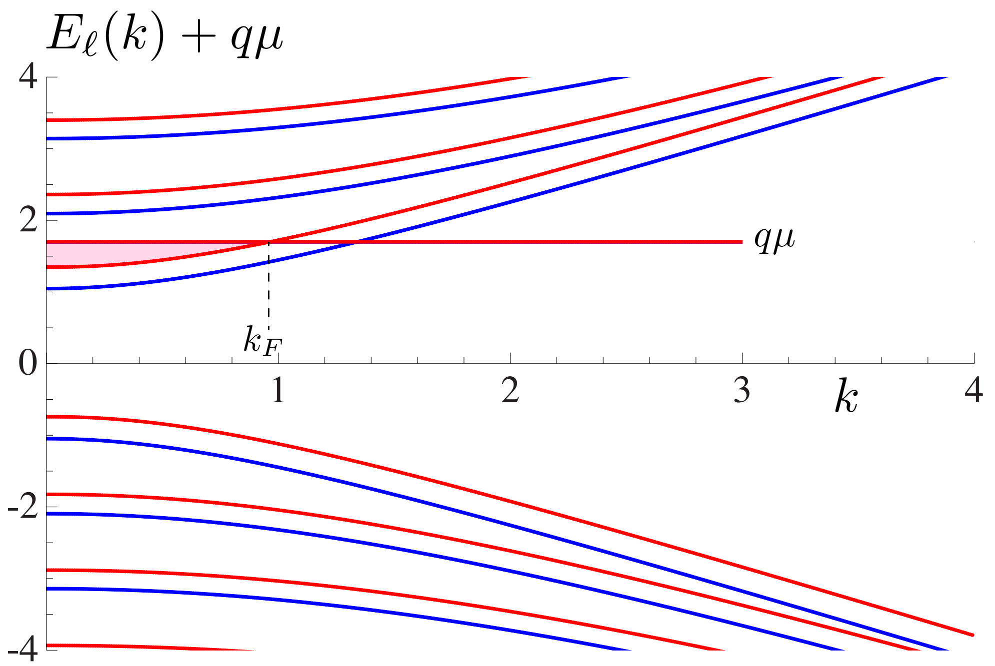

FIG. 21: Spectrum of normalizable fermionic modes in AdS. The blue lines reflect the pure AdS spectrum for $m = 1$, $r_c = 1/3$. Increasing the chemical potential μ there is a critical value at which the energies become negative. Occupying these states changes the electrostatic potential which modifies the subsequent fermion spectrum. The red lines show this adjusted spectrum for $\mu = 1$ and $q = \sqrt{3}$. The horizontal red line shows the zero of energy at the effective chemical potential $q\mu = \sqrt{3}$. The shaded region shows the occupied states. Figure is from [127].

[127]. For holographic fermionic systems a Luttinger theorem is satisfied iff all the charge is carried by the dual bulk fermions. The argument is straightforward. The macroscopic charge density in a system is by definition the differential of the free energy w.r.t. the chemical potential

$$\langle Q \rangle = -\frac{\partial F}{\partial \mu}. \tag{IX.26}$$

In AdS/CFT the chemical potential is encoded in the asymptotic behavior of the electrostatic potential

$$A_t = \mu + \mathcal{O}(\frac{1}{r}) \tag{IX.27}$$

and the free energy is equal to the on-shell Euclidean AdS action. In an isotropic system the relevant part of the action is (we have used gauge freedom to set $A_r = 0$)

$$\begin{aligned} S_{Eucl} &= \int dr\sqrt{g}\left(g^{rr}g^{tt}(\partial_r A_t)^2 - A_t J\right) \\ &= \int dr\sqrt{g}\left(A_t(-\partial_r g^{rr}g^{tt}\partial_r A_t) - J\right) + \oint_{r=\infty} \sqrt{g}g^{rr}g^{tt}\mu\partial_r A_t. \end{aligned} \tag{IX.28}$$

The first term, the equation of motion, vanishes on-shell and (in isotropic systems) the free energy and charge density in AdS/CFT are thus given by

$$\begin{aligned} F = S^{on-shell}_{Eucl} &= \oint_{r=\infty} \sqrt{g}g^{rr}g^{tt}\mu\partial_r A_t|_{on-shell} \\ \langle Q \rangle &= -\oint_{r=\infty} \sqrt{g}g^{rr}g^{tt}\partial_r A_t|_{on-shell}. \end{aligned} \tag{IX.29}$$

Integrating the equation of Motion

$$\partial_r\sqrt{g}g^{rr}g^{tt}\partial_r A_t = -J \tag{IX.30}$$

we can determine that the bulk charge density equals

$$\left[\sqrt{g}g^{rr}g^{tt}\partial_r A_t\right]_{r_{hor}}^{r=\infty} = -\int_{r_{hor}}^{\infty} J$$

$$\Leftrightarrow \qquad \sqrt{g}g^{rr}g^{tt}\partial_r A_t|_{r=\infty} - \sqrt{g}g^{rr}g^{tt}\partial_r A_t|_{r=r_{hor}} = -Q_{bulk}. \tag{IX.31}$$

Thus

$$\langle Q\rangle = Q_{bulk} - \underbrace{\sqrt{g}g^{rr}g^{tt}\partial_r A_t\big|_{r_{hor}}}_{\text{flux from horizon}}. \tag{IX.32}$$

The Luttinger theorem for the weakly coupled bulk fermions reads $Q_{bulk} \sim k_F^2$. If all charge is carried by the bulk fermions and there is no flux from the horizon also $\langle Q\rangle = k_F^2$. We just argued that the bulk Fermi momentum equals the boundary Fermi momentum and a boundary Luttinger theorem follows.

Fermionic correlation functions from holography

Due to the first order nature of the Dirac action, there is some technical changes to the prescription of applying the AdS/CFT dictionary. That such changes are necessary is most readily apparent simply by counting components of fermions. In dimensions $d = 2n, 2n+1$ a spinor has 2^n components. Which means that if one has even dimensional bulk the spinor has double the components than one would naively expect based on the boundary. Due to the first order nature of the action this counting is wrong. The first order action simultaneously describes the fluctuation — half of the components — and its conjugate momentum — the other half. Clearly only the former should correspond to a boundary degree of freedom. The most ready way to do so is to use the extra direction to project the fermion into two distinct eigenstates $\Psi_\pm$ of Γ^r and call one the fluctuation, say Ψ_+ Under this projection the Dirac action reduces to

$$S = \int d^4x\sqrt{-g}\left(-\bar{\Psi}_+ \not{D}\Psi_+ - \bar{\Psi}_- \not{D}\Psi_- - m\Psi_+\Psi_- - m\bar{\Psi}_-\Psi_+\right). \tag{IX.33}$$

The second issue is that the AdS/CFT correspondence instructs us to derive the CFT correlation functions from the on-shell action. The Dirac action, however, is proportional to its equation of motion. This reflects the inherent quantum nature of fermions that they never influence the saddle point. This is not quite true as we shall see. Having chosen Ψ_+ as the fundamental degree of freedom, we will choose a boundary source $\Psi_+^0 = \lim_{r\to\infty}\Psi_+(r)$. Then the boundary value Ψ_-^0 is not independent but related to that of Ψ_+^0 by the Dirac equation. We should therefore not include it as an independent degree of freedom when taking functional derivatives with respect to the source Ψ_+^0. Instead it should be varied to minimize the action To ensure a well-defined variational system for Ψ_- we add add a boundary action,

$$S_{bdy} = \int_{z=z_0} d^3x\,\sqrt{-h}\,\bar{\Psi}_+\Psi_- \tag{IX.34}$$

with $h_{\mu\nu}$ the induced metric, similar the earlier encountered Gibbons-Hawking term [141]. The variation of $\delta\Psi_-$ from the boundary action,

$$\delta S_{bdy} = \int_{z=z_0} d^3x\sqrt{-h}\,\bar{\Psi}_+\delta\Psi_-\Big|_{\Psi_+^0\,\text{fixed}}, \tag{IX.35}$$

now cancels the boundary term from variation of the bulk Dirac action

$$\begin{aligned}\delta S_{bulk} = &\int \sqrt{-g}\left(-\delta\bar{\Psi}_+(\not{D})\Psi_+ - \overline{((\not{D})\Psi_+)}\delta\Psi_+ - \delta\bar{\Psi}_-(\not{D})\Psi_- - \overline{((\not{D})\Psi_-)}\delta\Psi_-\right) - \\ &+ \int_{z=z_0}\sqrt{-h}\left(-\bar{\Psi}_+\delta\Psi_- - \bar{\Psi}_-\delta\Psi_+\right)\Big|_{\Psi_+^0\,\text{fixed}}.\end{aligned} \tag{IX.36}$$

The next complication is that the Fermionic correlation function is in general a matrix between the various spin components. A completely covariant formulation exists [102, 141], but with some insight we can diagonalize this matrix beforehand [103]. One does by undoing the projection on Γ^r, rotating to a fixed momentum frame $\vec{k} = (k_x, 0)$ and projecting on the t-helicity eigenstates $\psi_{1,2}$ of $\Gamma^r\Gamma^t\Gamma^x$. As we explained earlier this projector commutes with the Dirac operator and one recovers an independent action for each of the two t-helicities. Using the basis for Dirac matrices

$$\Gamma^z = \begin{pmatrix} -\sigma^3 1\!\!1 & 0 \\ 0 & -\sigma_3 1\!\!1 \end{pmatrix} \; \Gamma^t = \begin{pmatrix} i\sigma^1 1\!\!1 & 0 \\ 0 & i\sigma_1 1\!\!1 \end{pmatrix} \; \Gamma^1 = \begin{pmatrix} -\sigma^2 1\!\!1 & 0 \\ 0 & \sigma_2 1\!\!1 \end{pmatrix} \ldots \tag{IX.37}$$

the Dirac equation reduces to Eq. (IX.15) with asymptotic solution Eq.(IX.16). Given these homogenous solutions to the Dirac equation, the Green's function for one of the two t-helicities, and therefore still a two by two matrix, equals

$$\mathcal{G}(\omega, k, r_1, r_2) = \frac{\psi_b(r) \otimes \bar{\psi}_{int}(r')\theta(r - r') - \psi_{int}(r) \otimes \bar{\psi}_b(r')\theta(r' - r)}{\frac{1}{2}\left(\bar{\psi}_{int}(r)\gamma^r\psi_b - \bar{\psi}_b(r)\gamma^r\psi_{int}\right)} \tag{IX.38}$$

where $\psi_b(r)$ is the normalizable solution with leading coefficient $a = 0$. and $\psi_{int}(r)$ is determined by the appropriate boundary conditions in the interior. Note that in this basis a boundary source only has a lower component

$$\psi_+^0 = \begin{pmatrix} 0 \\ J \end{pmatrix}.$$

Substituting this Green's function into the boundary action one obtains

$$S = \lim_{r\to\infty} \int \sqrt{-h}\bar{\psi}_+^0 \frac{1}{2}(1 + \sigma_3) \frac{\psi_{int}(r) \otimes \bar{\psi}_b(\infty)}{\frac{1}{2}\left(\bar{\psi}_{int}(r)\gamma^r\psi_b - \bar{\psi}_b(r)\sigma^3\psi_{int}\right)} \psi_+^0. \tag{IX.39}$$

Writing also $\psi_{int}(r) = \begin{pmatrix} b_{int}r^{-m} + \ldots \\ a_{int}r^m + \ldots \end{pmatrix}$, $\psi_b(r) = \begin{pmatrix} br^{-m} + \ldots \\ 0 + \ldots \end{pmatrix}$, one finds

$$S = \lim_{r\to\infty} \int \sqrt{-h} r^{-2m} \frac{(J^\dagger b_{int})(b^\dagger J)}{b^\dagger a_{int} + a_{int}^\dagger b}. \tag{IX.40}$$

The final step is that an inspection of the Dirac equation reveals that one can always choose b and a_{int} real (but not b_{int}).

$$S = \lim_{r\to\infty} \int \sqrt{-h} r^{-2m} J^\dagger \frac{b}{a} J. \tag{IX.41}$$

Differentiating the on-shell action w.r.t J and $J^\dagger$, and dropping th overall r^{-2m} term gives the expression for the fermionic CFT correlation function

$$G_{\text{fermions}} = \frac{b}{a}. \tag{IX.42}$$

This beautiful story of the emergent Fermi liquid only resulted after two years of hard work. The difficulty experienced in arriving here is related to two glaring open questions that follow.

- *Why in fact do we need to introduce an explicit hard wall?.* In a true Fermi liquid all other modes are automatically gapped. We should therefore also recover the Fermi liquid if we study an unmanipulated AdS set-up with now the chemical potential itself acting as an infrared cut-off.

- *What is the role of the other discrete set of states visible in Fig 2?* A closer look at the spectrum reveals that we have a states for each radial harmonic. This implies that the system does not describe a single Fermi surface, but a multitude of Fermi surfaces with more and more visible as one increases μ.

We will address this second question later, let us now turn to the first. What happens if we study the same system but now without the hard wall? To explain the remarkable success-story we shall find, we first need to recall the definition of a non-Fermi liquid.

D. Non-Fermi liquids

Non-Fermi liquids or strange metals are by definition characterized by the presence of a Fermi surface but a self-energy scaling that is *not* quadratic in the frequency at $k = k_F$. Thus one finds a single fermion spectral function (note that due to the Pauli principle this can still be non-perturbatively defined)

$$G = \frac{1}{\omega - v_F(k - k_F) + \Sigma(\omega, k)} \tag{IX.43}$$

but $\Sigma'' \sim \omega^a$ with $a \neq 2$. Due to the same special fermion physics as in the Landau Fermi liquid, this microscopic effect is immediately reflected in the macroscopic properties of the system. E.g. the temperature dependence of the conductivity now scales as

$$\sigma_{DC} \sim T^{-a} \tag{IX.44}$$

A famous case of a non-Fermi liquid is the so-called marginal Fermi liquid $a = 1$ with $\Sigma \sim \omega \ln \omega$ [118]. The observed strange metals in high T_c superconductors and heavy fermions all exhibit a characteristic linear-in-temperature resitivity. The marginal Fermi liquid is designed to match this.

Theoretically the essential point is that the self-energy in all these cases is a non-analytic function in ω. From a Wilsonian RG point of view, the only way one can obtain such behavior if one sits precisely at a non-trivial interacting fixed point. Any generic effective field theory will always have analytic funcions of frequencies and momenta. Moreover, this non-trivial fixed point must exist at zero-temperature, because a Fermi surface is a groundstate property. This leads us directly to consider Quantum Critical Points at Quantum Phase Transitions and Quantum criticality as the set-up to find non-Fermi liquids. The fermionic nature of the relevant excitations makes it incredibly hard, if not practically impossible to build Quantum Critical Points from the ground up. The only way is to tune large sets irrelevant operators and hoping one hits a dangerous point where the operator nevertheless survives in the infrared. But this is like searching for a needle in the haystack.

As is familiar by now, this is where AdS/CFT is revolutionary. It is a generating mechanism of interacting critical theories. There is a large chance therefore that AdS/CFT with fermions is able to encode non-Fermi liquids. In a breakthrough 2009 computation it was shown that this is so. Here is how.

E. Holographic non Fermi liquids

The first step is to set the starting point correct. We wish to obtain a critical theory rather than a new or generic infrared. The simplest way to do is is to *not* excite any non-trivial fields in the bulk. At the same time we do wish to study the system at finite density. We need a finite chemical potential. The simplest system is therefore the previously encountered charged AdS-Reissner-Nordstrom black-hole (VII.2). To study the system we shall use the fermionic spectral function $A(\omega, k) = -\frac{1}{\pi}\Im G_R(\omega, k)$. This mimics real life experiments where the fermionic spectral function is directly measured by Angle-Resolved Photon Emission Spectroscopy (ARPES).

The basic ingredients needed to contruct the boundary retarded Green's function we have already seen in our construction of the holographic Fermi liquid. The difference is that the background is now fixed. We simply need to compute the solution to the Dirac Equation for infalling boundary conditions at the horizon. The boundary spinor Green's function is given by the ratio of the subleading to the leading term

$$G(\omega, k) = \frac{b}{a}. \tag{IX.45}$$

For small frequency $\omega \ll \mu$, the retarded Green's function can be solved semi analytically by the near far matching method explained in section VIII. The final result is

$$G_R(\omega, k) = \frac{b_+^{(0)} + \omega b_+^{(1)} + \mathcal{O}(\omega^2) + \mathcal{G}_k(\omega)\big(b_-^{(0)} + \omega b_-^{(1)} + \mathcal{O}(\omega^2)\big)}{a_+^{(0)} + \omega a_+^{(1)} + \mathcal{O}(\omega^2) + \mathcal{G}_k(\omega)\big(a_-^{(0)} + \omega a_-^{(1)} + \mathcal{O}(\omega^2)\big)} \tag{IX.46}$$

where a, b are UV data and functions of k. Here $\mathcal{G}_k(\omega)$ is the IR AdS_2 Green's function for the operator $\mathcal{O}_f(k)$ in the IR CFT

$$\mathcal{G}_k(\omega) = c_k e^{i\phi_k} \omega^{2\nu_k} \quad \text{with} \quad \nu_k = \sqrt{\frac{2k^2}{\mu^2} + \frac{m^2}{6} - \frac{q^2}{3}} \tag{IX.47}$$

and c_k an analytic real function in k.

What we seek to know whether the Green's function gives rise to a pole at $\omega = 0$ at a finite k_F, i.e. a Fermi surface. This happens whenever $a_+^{(0)}(k_F) = 0$.[4] One finds that for $m \lesssim \frac{2}{\sqrt{3}}q$, isolated poles appear in the Green's function [103]. Rewriting the Green's function in an expansion around this Fermi surface, we have that

$$G_R(\omega, k) \simeq \frac{h_1}{k_\perp - \frac{1}{v_F}\omega - h_2 c_{k_F} e^{i\phi_{k_F}} \omega^{2\nu_{k_F}}}, \tag{IX.48}$$

with $k_\perp = k - k_F$ and h_1, v_F, h_2 real. What jumps out immediately is that the self-energy of the quasi-particles is *non-analytic. We have found that the computational framework of AdS-CFT naturally gives rise to non-Fermi liquids.*

The implausibility of non-Fermi liquids has not only been spectularly resolved. There is an embarrassment of riches in that we have whole families of non-Fermi liquids whose detailed properties depend on the non-analytic self-energy scaling $\nu_{k_F}(m, q)$. This parameter in itself depends on the fundamental properties of the dual fermion: its mass/conformal dimension $mL = \Delta + D/2$ and charge q (Table III). For $2\nu_{k_F} > 1$ the quasiparticle is stable (its width spreads much slower than it disperses) but simply has irregular width, outside the accidental value $2\nu_{k_F} = 2$. For $2\nu_{k_F}$, however the quasiparticle is unstable, as it will decay before its had chance to appreciable disperse. At the border $2\nu_{k_F} = 1$ a detailed reanalyis reveals that the self-energy becomes logarithmic. *We have computationally recovered the phenomenologically inspired marginal Fermi liquid.*

$\Sigma \sim \omega^{2\nu_{k_F}}$	**Fermi-system Phase**	**Quasiparticle Properties: (dispersion, peak)**
$2\nu_{k_F} > 1$	regular FL	$\omega_*(k) = v_F k_\perp, \ \ \frac{\Gamma(k)}{\omega_*(k)} \propto k_\perp^{2\nu_{k_F}-1} \to 0, \ \ Z = \text{constant}$
$2\nu_{k_F} = 1$	marginal FL	$G_R = \frac{h_1}{c_2\omega - v_F k_\perp + c_R \omega \ln \omega}, \ \ Z \sim \frac{1}{\ln \omega_*} \to 0$
$2\nu_{k_F} < 1$	singular FL	$\omega_*(k) \sim k_\perp^{1/2\nu_k} \ \ \frac{\Gamma(k)}{\omega_*(k)} \to \text{const}, \ \ Z \propto k_\perp^{\frac{1-2\nu_{k_F}}{2\nu_{k_F}}} \to 0$

TABLE III: The zoo of non-Fermi liquids one can find in AdS/CFT. c_2 is complex while c_R is real.

1. *Semi-holography, AdS_2 instabilities and Semi local quantum liquid (SLQL)*

The above near far matching method between the near horizon geometry $AdS_2 \times R^{d-1}$ of extremal AdS RN black hole and the asymptotic AdS_4 space of the UV CFT illustrates in a very intuitive way the effective dynamics of a non-Fermi liquid fixed point. The AdS RN solution is in fact a domain-wall interpolating between the exact 1+1 dim IR CFT described by the dual $AdS_2 \times R^{D-1}$ geometry and the 3+1 dim CFT in the UV described by the asymptotic AdS_4 space. Tracking the wave-function of the fermionic excitations near the pole, one notes that they are localized halfway down the bulk: they are intrinsic parts of the domain wall solution rather than the strict UV or the strict IR.

This insight is directly evident in the Green's function. Just as in the case of the holographic superconductor the IR AdS_2 Green's function affects RPA-like the quasiparticles descending from the UV. In essence the full Fermi dynamics is captured by a hybridization between free fermion and the IR CFT.

$$S = \int d\omega d^2k \Big(\bar{\psi}(\omega - v_F k_\perp) + g\bar{\psi}\chi + g^*\bar{\chi}\psi + \bar{\chi}\mathcal{G}^{-1}\chi \Big) \tag{IX.49}$$

where χ is an operator in the IR CFT and $\mathcal{G} = c_k e^{i\phi_k}\omega^{2\nu_k}$. In this "semi-holography" [116], the holographic spectral function (IX.48) is directly reproduced by a Dyson resummation

$$\langle \bar{\psi}\psi \rangle = \frac{1}{\omega - v_F k_\perp - |g|^2 \mathcal{G}} \tag{IX.50}$$

---

[4] For the special case $m = 0$ one can actually solve the full Green's function analytically [119]. For generic m one has to do a numerical evaluation.

Naively this effective semi-holographic theory seems to be too primitive to be consistent, but its AdS/CFT origins give it solidity. Even if it is essentially turns out to be a beefed up Dyson resummation, it can hardly be emphasized how this first construction of a theory with an emergent non-Fermi liquid, is a grand success. There is, however, another *as* important feature in this Green's function. When $m_{eff}^2 < 2q^2$, there is a range of momenta k, k_o where the AdS$_2$ scaling dimension are *imaginary*. Due to this the Green's function displays log-oscillatory behavior in this range

$$G \sim \frac{1}{a_+^{(0)} + \omega a_+^{(1)} + e^{i|2\nu_k|\ln\omega}}. \tag{IX.51}$$

Imaginary scaling dimensions always signal an instability. On the gravity side this instability is well known, it is the threshold for Schwinger pair production of charged particles near the horizon [120]. We have also encoutered this instability for bosons in section VIII. It is the onset of the "bifurcating" tachyonic instability holographic superconductor. It raises the immediate question what this log-oscillatory instability means for fermions?

Before we address this question let us note that phenomenologically as well as analytically the Fermi system thus vary analogous to the detailed structure of holographic superconductors. Both systems have RPA-like spectral functions where either bare UV can cause singular behavior (an isolated Fermi surface/a hybridized critical point) or the "self-energy" can become complex signalling an instability in the field theory. As suggested in [66], the physics described by the IR fixed pointhas been called a *semi-local quantum liquid* (SLQL). As it has a finite entropy density, it is only a universal intermediate-energy phase and it might not be the final stable ground state. Indeed from the gravity side we know that a charged black hole suffers various bosonic and fermionic instabilities. The final state can be a superconducting/superfluid state which was studied in the last section, an antiferromagnetic state or a Fermi liquid state. Back to the question at hand, what instability might the log-oscillatory behavior signal in the AdS/CFT description of a strongly coupled critical Fermi system? It strongly suggests there should be a ground-state with occupied fermions in the bulk. This is also indicated by entropy. The confusing part is that *this instability always happens before one finds singular non-Fermi-liquid Fermi surfaces in the bulk* (see Fig. 22). Normally one needs a pole in the Green's function to be able to occupy a state: this pole perturbatively directly corresponds to a normalizable mode. We have seen this already in detail in the construction of the holographic Fermi liquid. What made the system tractable there was the ad-hoc inclusion of a hard wall. But by cutting off the IR geometry this hard wall completely removed the SLQL AdS$_2$ region which accounts for the non-trivial self-energy and log-oscillatory behavior. As we presaged at the end of the holographic Fermi liquid system, we therefore have to readdress the question of a system of occupied Fermi states but now without the presence of a hard wall.

As before this finite density of bulk fermions will generically backreact on both the gauge-field and the metric. In particular the instability signals that the IR of the system will now be different and that the corresponding interior region of the AdS metric will change. Due to this the system is much more intricate system to solve than before. There are, however, two obvious limits where we can do so: the Thomas Fermi approximation where we can treat the occupied fermi states as a fluid (the limit where the number of fermions is infinite, but the total charge finite). With our knowledge of the holographic Fermi liquid we can already state before hand that this will be a strange system with a formally infinite number of Fermi surfaces. Or we can be guided by our condensed matter knowledge and consider only single fermionic radial mode corresponding to a single Fermi surface. Clearly to arrive at the same charge density such a single fermion will need a macroscopic charge. We will now treat both in turn. It is worth emphasizing at this point that in both cases we are only constructing the putative groundstate that might arise as a consequence of the log-oscillatory instability in the system. Unlike the case for bosons at this time we do not yet know precisely what dynamics of the log-oscillatory behavior in actuality induces.

X. HOLOGRAPHIC STRANGER METALS

A. Fluid approximation to finite density of bulk fermions

In the fluid approximation one considers starts with abstract charged matter coupled to Einstein-Maxwell theory

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2\kappa^2}(R + \frac{6}{L^2}) - \frac{1}{4q^2}F^2 + \mathcal{L}_{matter}\Big], \tag{X.1}$$

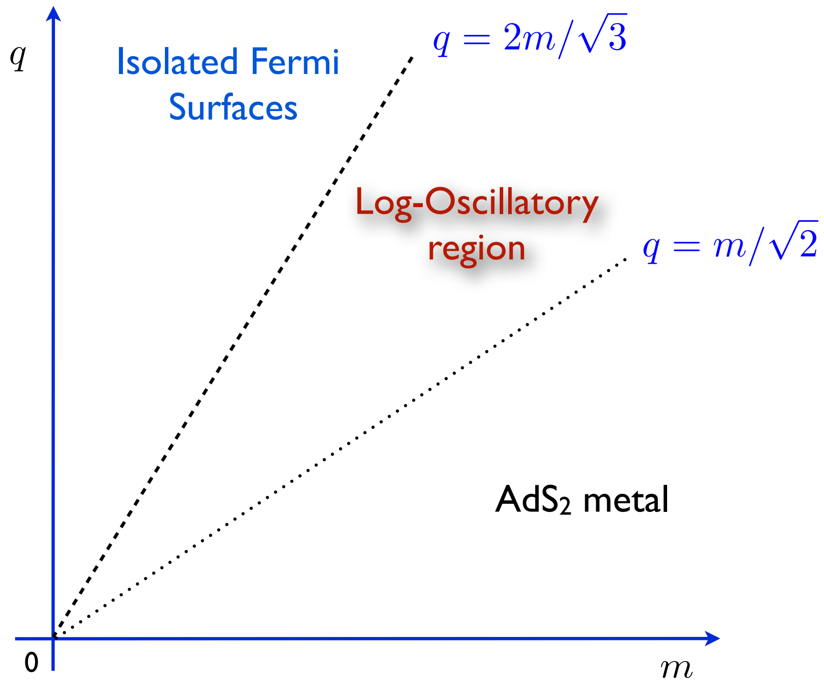

FIG. 22: Phase diagram from the retarded Green's function of probe fermion.

whose relevant properties are encoded only in the stress-energy and charge density that source the Einstein and Maxwell equations

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R+6) &= F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + T_{\mu\nu}^{matter} \\
D_\mu F^{\mu\nu} &= -\frac{qL}{\kappa}J^{\nu}_{\text{matter}}
\end{aligned}
\tag{X.2}
$$

Here we have rescaled $A_\mu \to \frac{L}{\kappa}A_\mu$ to make it easy to compare gravitational and electromagnetic effects Intuitively this fluid approximation is valid when the number of constituents is macroscopic. For a charged system this is easy to trace, it is when the microscopic charge is nearly vanishing, but the total charge density is not.

$$
q \ll Q. \tag{X.3}
$$

In the fluid approximation the contribution of fermions can be described by

$$
\begin{aligned}
T_{\mu\nu}^{\text{matter}} &= T_{\mu\nu}^{\text{fluid}} = (\rho+p)u_\mu u_\nu + p g_{\mu\nu} \\
J_\mu^{\text{matter}} &= J_\mu^{\text{fluid}} = n u_\mu
\end{aligned}
\tag{X.4}
$$

with n the number density. In trying to understand neutron stars Tolman, Oppenheimer and Volkov had the deep insight that for fermions one can determine the equation of state needed to solve the system almost independent of the gravitational background. When the number of constituents is macroscopic, the Fermi momentum is appreciable. This means that the fermion wavefunctions near the Fermi-surface are very highly localized. In turn this means that they are essentially unaffected by the background metric. Since simple volume growth also states that it is these wavefunctions that predominantly determine the macroscopic properties of the system, we may simply use the flat space expressions for the energy, pressure etc of a macroscopic Fermi liquid.

$$
\begin{aligned}
\rho &= \frac{1}{\pi^2}\frac{\kappa^2}{L^2}\int_{mL}^{\mu_{\text{loc}}} dE E^2\sqrt{E^2-(mL)^2}, \\
n &= \frac{1}{\pi^2}\frac{\kappa^2}{L^2}\int_{mL}^{\mu_{\text{loc}}} dE E\sqrt{E^2-(mL)^2}, \\
-p &= \rho - \mu_{\text{loc}} n
\end{aligned}
\tag{X.5}
$$

where the local chemical potential is given by

$$\mu_{\rm loc}(r) = \frac{qL}{\kappa} e^t_{\underline{0}}(r) A_t(r). \tag{X.6}$$

Note the inclusion of the vielbein $e^t_{\underline{0}}(r)$. This accounts for the change between the coordinates we are using and the natural coordinate system of a local observer. The two notable difference with the well-known Tolman-Oppenheimer-Volkov equations for neutron stars is that our asymptotic geometry is AdS and not flat Minkowski spacetime, and that we are considering charged fermions. In our universe these electron-stars as opposed to neutron stars do not exist as the relative strength of the electromagnetic vs the gravitational force means that cosmologically all matter is neutral. In our virtual AdS description of strongly coupled field theories we are not limited by this restriction, however.

From fermion microscopics to fluids One can derive the fluid description of the fermions directly from the fundamenal Lagrangian of Einstein-Maxwell theory coupled to charged fermions

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}(R+\frac{6}{L^2}) - \frac{1}{4q^2}F^2 + \mathcal{L}_{\tt matter}(e^A_\mu, A_\mu)\right], \tag{X.7}$$

where L is the AdS radius, q is the electric charge and κ is the gravitational coupling constant. It is useful to scale the electromagnetic interaction to be of the same order as the gravitational interaction and measure all lengths in terms of the AdS radius L:

$$g_{\mu\nu} \to L^2 g_{\mu\nu}\ , \quad A_\mu \to \frac{qL}{\kappa}A_\mu. \tag{X.8}$$

The system then becomes

$$S = \int d^4x\sqrt{-g}\left[\frac{L^2}{2\kappa^2}\left(R+6-\frac{1}{2}F^2\right) + L^4\mathcal{L}_{\tt matter}(Le^A_\mu, \frac{qL}{\kappa}A_\mu)\right]. \tag{X.9}$$

Note that in the rescaled variables the effective charge of charged matter now depends on the ratio of the electromagnetic to gravitational coupling constant: $q_{\tt eff} = qL/\kappa$. For the case of interest, charged fermions, the Lagrangian in these variables is

$$L^4\mathcal{L}_{\tt fermions}(Le^A_\mu, \frac{qL}{\kappa}A_\mu) = -\frac{L^2}{\kappa^2}\bar{\Psi}\left[e^\mu_A\Gamma^A(\partial_\mu + \frac{1}{4}\omega_\mu^{BC}\Gamma_{BC} - i\frac{qL}{\kappa}A_\mu) - mL\right]\Psi, \tag{X.10}$$

where $\bar{\Psi}$ is defined as $\bar{\Psi} = i\Psi^\dagger\Gamma^0$. Compared to the conventional normalization the Dirac field has been made dimensionless $\Psi = \kappa\sqrt{L}\psi_{\rm conventional}$. With this normalization all terms in the action have a factor L^2/κ^2 and it will therefore scale out of the equations of motion

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - 3g_{\mu\nu} &= \left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + T^{\tt fermions}_{\mu\nu}\right), \\ D_\mu F^{\mu\nu} &= -q_{\tt eff}J^\nu_{\tt fermions} \end{aligned} \tag{X.11}$$

with

$$T^{\tt fermions}_{\mu\nu} = \frac{1}{2}\bar{\Psi}e_{A(\mu}\Gamma^A\left[\partial_{\nu)} + \frac{1}{4}\omega_{\nu)}^{BC}\Gamma_{BC} - i\frac{qL}{\kappa}A_{\nu)}\right]\Psi - \frac{\kappa^2L^2}{2}g_{\mu\nu}\mathcal{L}_{\tt fermions}, \tag{X.12}$$

$$J^\nu_{\tt fermions} = i\bar{\Psi}e^\nu_A\Gamma^A\Psi, \tag{X.13}$$

where the symmetrization is defined as $B_{(\mu}C_{\nu)} = B_\mu C_\nu + B_\nu C_\mu$ and the Dirac equation

$$\left[e^\mu_A\Gamma^A(\partial_\mu + \frac{1}{4}\omega_\mu^{BC}\Gamma_{BC} - i\frac{qL}{\kappa}A_\mu) - mL\right]\Psi = 0. \tag{X.14}$$

The stress-tensor and current are to be evaluated in the specific state of the system. For multiple occupied fermion states, even without backreaction due to gravity, adding the contributions of each separate solution to (X.14) rapidly becomes very involved. In such a many-body-system, the collective effect of the multiple occupied fermion states is better captured in a "fluid" approximation

$$T_{\mu\nu}^{\tt fluid} = (\rho+p)u_\mu u_\nu + pg_{\mu\nu}, \qquad N_\mu^{\tt fluid} = nu_\mu \tag{X.15}$$

with

$$\rho = \langle u^\mu T_{\mu\nu} u^\nu \rangle_{\texttt{matter only}} \ , \quad n = -\langle u_\mu J^\mu \rangle_{\texttt{matter only}}. \tag{X.16}$$

In the center-of-mass rest frame of the multiple fermion system ($u_\mu = (e_{t\underline{0}}, 0, 0, 0)$), the expressions for the stress-tensor and charge density are given by the one-loop equal-time expectation values (as opposed to time-ordered correlation functions)

$$\rho = \langle \bar{\Psi}(t) e_{\underline{0}}^t \Gamma^{\underline{0}} (\partial_t + \frac{1}{4}\omega_t^{AB}\Gamma_{AB} - iq_{\texttt{eff}} A_t)\Psi(t)\rangle. \tag{X.17}$$

By the optical theorem the expectation value is equal to twice imaginary part of the Feynman propagator[5]

$$\rho = \lim_{t\to t'} 2\mathrm{ImTr}\left[e_{\underline{0}}^t \Gamma^{\underline{0}} (\partial_t + \frac{1}{4}\omega_t^{AB}\Gamma_{AB} - iq_{\texttt{eff}} A_t) G_F^{AdS}(t', t)\right]. \tag{X.18}$$

In all situations of interest, all background fields will only have dependence on the radial AdS direction; in that case the spin connection can be absorbed in the normalization of the spinor wavefunction.[6] In an adiabatic approximation for the radial dependence of $e_{t\underline{0}}$ and A_t — where $\mu_{\texttt{loc}}(r) = q_{\texttt{eff}} e_{\underline{0}}^t(r) A_t(r)$ and $\omega(r) = -ie_{\underline{0}}^t(r)\partial_t$; — this yields the known expression for a many-body-fermion system at finite chemical potential

$$\begin{aligned}
\rho(r) &= \lim_{\beta\to\infty} 2\int \frac{d^3k d\omega}{(2\pi)^4} [\omega(r) - \mu_{\texttt{loc}}(r)]\, \mathrm{ImTr}\, i\Gamma^{\underline{0}} G_F^\beta(\omega, k) \\
&= \lim_{\beta\to\infty} \int \frac{dk d\omega}{4\pi^3} [k^2(\omega - \mu)] \left[\frac{1}{2} - \frac{1}{2}\tanh(\frac{\beta}{2}(\omega-\mu))\right] \mathrm{Tr}(i\Gamma^0)^2 \frac{\kappa^2}{L^2}\pi\delta((\omega-\mu) - \sqrt{k^2 + (mL)^2}) \\
&= \lim_{\beta\to\infty} \frac{\kappa^2}{\pi^2 L^2} \int d\omega f_{FD}(\beta(\omega-\mu)) \left[(\omega-\mu)^2 - (mL)^2\right] [\omega - \mu] \frac{(\omega-\mu)\theta(\omega-\mu-mL)}{\sqrt{(\omega-\mu)^2 - (mL)^2}} \\
&= \frac{1}{\pi^2}\frac{\kappa^2}{L^2} \int_{mL}^{\mu_{\text{loc}}} dE E^2 \sqrt{E^2 - (mL)^2}\ .
\end{aligned} \tag{X.19}$$

The normalization κ^2/L^2 follows from the unconventional normalization of the Dirac field in eq. (X.10).[7] Similarly

$$n = \frac{1}{\pi^2}\frac{\kappa^2}{L^2}\int_{mL}^{\mu_{\text{loc}}} dE E\sqrt{E^2 - (mL)^2} = \frac{1}{3\pi^2}\frac{\kappa^2}{L^2}({\mu_{\text{loc}}}^2 - (mL)^2)^{3/2}. \tag{X.20}$$

The adiabatic approximation is valid for highly localized wavefunctions, i.e. the expression must be dominated by high momenta (especially in the radial direction). The exact expression on the other hand will not have a continuum of solutions to the harmonic condition $-\Gamma^0\omega + \Gamma^i k_i + \Gamma^z k_z - \Gamma^0\mu_{\text{loc}} - imL = 0$. Normalizable solutions to the AdS Dirac equations only occur at discrete momenta — one can think of the gravitational background as a potential well. The adiabatic approximation is therefore equivalent to the Thomas-Fermi approximation for a Fermi-gas in a box.

To get an estimate for the parameter range where the adiabatic approximation holds, consider the adiabatic bound $\partial_r\mu_{\texttt{loc}}(r) \ll \mu_{\texttt{loc}}(r)^2$. Using the field equation for $A_{\underline{0}} = \mu_{\texttt{loc}}/q_{\texttt{eff}}$:

$$\partial_r^2 \mu_{\texttt{loc}} \sim q_{\texttt{eff}}^2 n, \tag{X.21}$$

this bound is equivalent to requiring

$$\partial_r^2\mu_{\texttt{loc}} \ll \partial_r\mu_{\texttt{loc}}^2 \quad \Rightarrow \quad (\frac{qL}{\kappa})^2 n \ll 2\mu_{\texttt{loc}}\partial_r\mu_{\texttt{loc}} \quad \Rightarrow \quad (\frac{qL}{\kappa})^2 n \ll \mu_{\texttt{loc}}^3 \tag{X.22}$$

where in the last line we used the original bound again. If the chemical potential scale is considerably higher than the mass of the fermion, we may use (X.35) to approximate $n \sim \frac{\kappa^2}{L^2}\mu_{\texttt{loc}}^3$. Thus the adiabatic bound is equivalent to,

$$q = \frac{q_{\text{eff}}\kappa}{L} \ll 1 \tag{X.23}$$

the statement that the constituent charge of the fermions is infinitesimal. Note that in the rescaled action (X.9, X.10), L/κ plays the role of $1/\hbar$, and eq. (X.23) is thus equivalent to the semiclassical limit $\hbar \to 0$ with q_{eff} fixed. Since AdS/CFT relates $L/\kappa \sim N_c$ this acquires the meaning in the context of holography that there is a large N_c scaling limit [66, 124] of the CFT with fermionic operators where the RG-flow is "adiabatic". Returning to the gravitational description the additional assumption that the chemical potential is much larger than the mass is equivalent to

$$\frac{Q^{\text{total}}_{\text{phys}}}{V_{\text{spatial AdS}}} = \frac{LQ^{\text{total}}_{\text{eff}}}{\kappa V_{\text{spatial AdS}}} \equiv \frac{L}{\kappa V_{\text{spatial AdS}}}\int dr\sqrt{-g_{\text{induced}}}\,(q_{\text{eff}}n) \simeq \frac{1}{V_{\text{spatial AdS}}}\int dr\sqrt{-g}\frac{q_{\text{eff}}\kappa}{L}\mu^3_{\text{loc}}(r) \quad \gg \quad q(mL)^3 \ . \tag{X.24}$$

This implies that the total charge density in AdS is much larger than that of a single charged particle (as long as $mL \sim 1$). The adiabatic limit is therefore equivalent to a thermodynamic limit where the Fermi gas consists of an infinite number of constituents, $n \to \infty$, $q \to 0$ such that the total charge $Q \sim nq$ remains finite.

We can now solve this system to find whether a state with a fermionic field configuration exists and whether it is favored over pure AdS-RN. Since the fluid is homogeneous and isotropic, the background metric and electrostatic potential will respect these symmetries and will be of the form

$$ds^2 = L^2\Big(- f(r)dt^2 + g(r)dr^2 + r^2(dx^2+dy^2)\Big), \quad A_t = h(r). \tag{X.25}$$

The tricky part as before is to figure out the deep IR $r \to 0$ behavior of the resulting geometry. Considering a scaling ansatz, we have just as in the case of the holographic superconductor an emergent Lifshitz scaling behavior

$$f(r) = r^{2z} + \dots \ , \quad g(r) = \frac{g_\infty}{r^2} + \dots \ , \quad h(r) = h_\infty r^z + \dots \tag{X.26}$$

and dynamical exponent z and invariance under the Lifshitz scaling

$$r \to \lambda^{-1}r, \quad t \to \lambda^z t, \quad (x,y) \to \lambda(x,y). \tag{X.27}$$

One can now take this IR-geometry and integrate numerically up the RG-flow to complete the full background. For technical details we refer to [122]. Fig.23 is an example illustrating the resulting geometry. The full electron star geometry flows from the Lifshitz near horizon to the AdS_4 boundary. We see exactly what we predicted and what we encoutered before several times. Due the presence of the fermionic field configuration in the bulk, the deep IR geometry in the interior changes from $AdS_2 \times R^2$ to a Lifshitz spacetime. The dual field theory thus has an IR Lifshitz fixed point. Moreover, the Lifshitz horizon has zero entropy supporting that it is true groundstate. Finally, it is straightforward to see that there is no charge density on the horizon. Therefore the Fermi fluid carries all the charge of the system, and a Luttinger theorem should hold. This can be explicitly verified [66, 124].

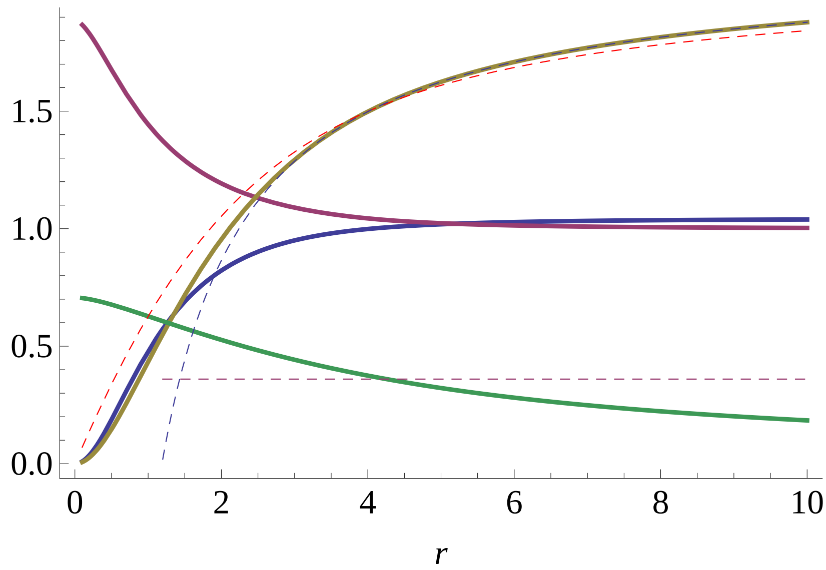

FIG. 23: Electron star metric for $z = 2, \hat{m} = 0.36$, $c \simeq 1.021, \hat{M} \simeq 3.601, \hat{Q} \simeq 2.534, \hat{\mu} \simeq 2.132$ compared to pure AdS. Shown are $f(r)/r^2$ (Blue), $r^2 g(r)$ (Red) and $h(r)$ (Orange). The asymptotic AdS-RN value of $h(r)$ is the dashed blue line. For future use we have also given $\mu_{\rm loc} = h/\sqrt{f}$ (Green) and $\mu_{q_{\rm eff}} = \sqrt{g^{ii}}h/\sqrt{f}$ (Red Dashed) At the edge of the star $r_s \simeq 4.253$ (the intersection of the purple dashed line setting the value of $m_{\rm eff}$ with $\mu_{\rm loc}$) one sees the convergence to pure AdS in the constant asymptotes of $f(r)/r^2$ and $r^2 g(r)$.

Now, the vanishing ground state entropy and its implied stability plus the validity of the Luttinger theorem suggests we can easily guess what field theory state this "electron star" describes. We expect it to be the generic Fermi liquid ground state. But that is not true. The Lifshitz scaling in the geometry immediately infers that the entropy density and specific heat scale as

$$s \sim T^{d/z} \;\; ; \;\; c_V \sim T^{d/z} \tag{X.28}$$

which is certainly not the characteristic linear Fermi liquid scaling (IX.5). So what is this state, that the electron star describes?

We already have an inkling from our construction of the holographic Fermi liquid. Each radial mode corresponds to Fermi surface in the dual field theory and in the fluid limit we should have a very large number — formally infinite — of such surfaces. To detect these and further analyze the nature of the field theory dual to the electron star state we will study its fermionic response in the spectral function. There is a very important technical detail in doing so. In the fluid approximation the charge of the probe fermion is formally scaled infinitesimally small. To actually probe the system, we have to resolve the discreteness of charge again. As a consequence the spectral function is not unique determined by the electron star background, but also by how well the underlying system given by the microscopic values of the charge q and the gravitational coupling κ approximate the fluid picture. A detailed analysis (see box) reveals that the parameter can controls the fluid approximation is κ/L. We show this dependence in Fig.24 where we plot the momentum-distribution-function (MDF) (the spectral function as a function of k) for fixed $\omega = 10^5, z = 2, \hat{m} = 0.36$ while changing the value of κ.

There are several predicted features that are immediately apparent:

- As expected, there is a multitude of Fermi surfaces. They have very narrow width and their spectral weight decreases rapidly for each higher Fermi-momentum k_F. This agrees with the exponential width $\Gamma \sim \exp(-\left(\frac{k^z}{\omega}\right)^{1/(z-1)})$ predicted by [116] for gravitational backgrounds that are Lifshitz in the deep interior, which is the case for the electron star. This prediction is confirmed in [66, 123, 124] and the latter two articles also show that the weight decreases in a corresponding exponential fashion. This exponential reduction of both the width and the weight as k_F increases explains why we only see a finite number of peaks, though we expect a very large number. We will confirm this in a moment with a WKB counting.
- Increasing the parameter κ/L the peak location k_F decreases orderly and peaks disappear at various threshold values of momentum. This is the support for our argument that changing κ changes the number of microscopic constituents in the electron star.

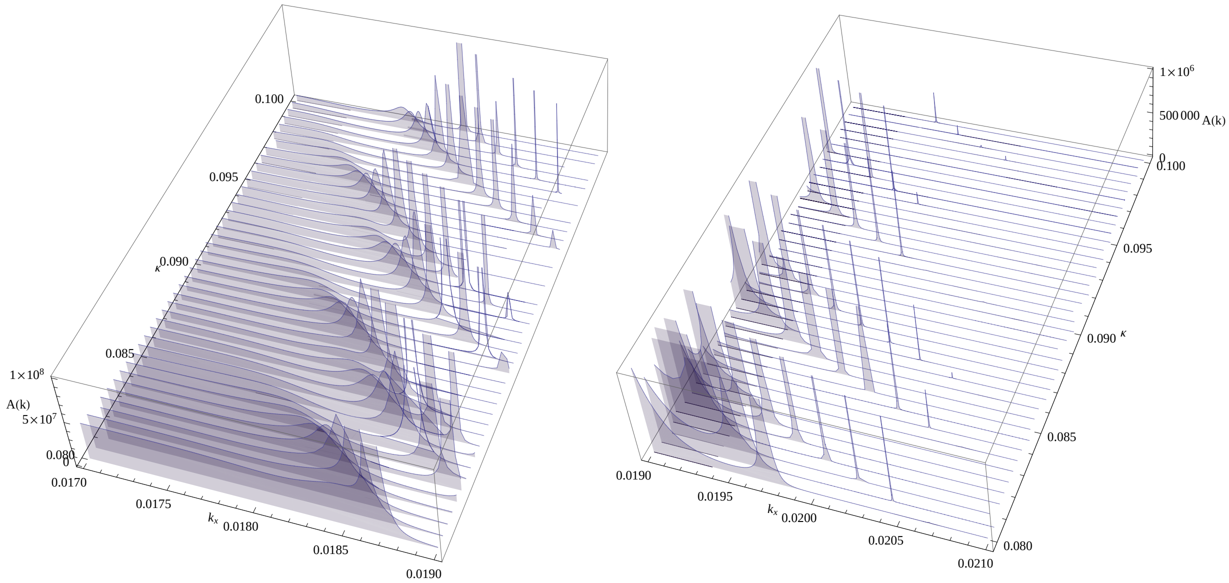

FIG. 24: Electron-star MDF spectral functions as a function of κ for $z = 2, \hat{m} = 0.36, \omega = 10^{-5}$. Because the peak height and weights decrease exponentially, we present the adjacent ranges $k \in [0.017, 0.019]$ and $k \in [0.019, 0.021]$ in two different plots with different vertical scale.

- The generic value of k_F of the peaks with visible spectral weight is *much* smaller than the effective chemical potential μ in the boundary field theory. This is quite different from the RN-AdS case where the Fermi momentum and chemical potential are of the same order. One can guess in view of the verification of the Luttinger count, why this has to be so. Each Fermi surface must carry only a fraction of the total charge, whereas the total charge density is set by μ.
- Consistent with the boundary value analysis, there is no evidence of an oscillatory region. It suggests the electron star is stable.

We can confirm the discrete countable nature of the number of Fermi surfaces for finite κ by a WKB count. One can reformulated the spectral function calculation as a Schrodinger problem. Solving this in the WKB approximation, we can get the number of Fermi surfaces by studying the normalizable modes in Schrodinger potential. (See Fig 25.)

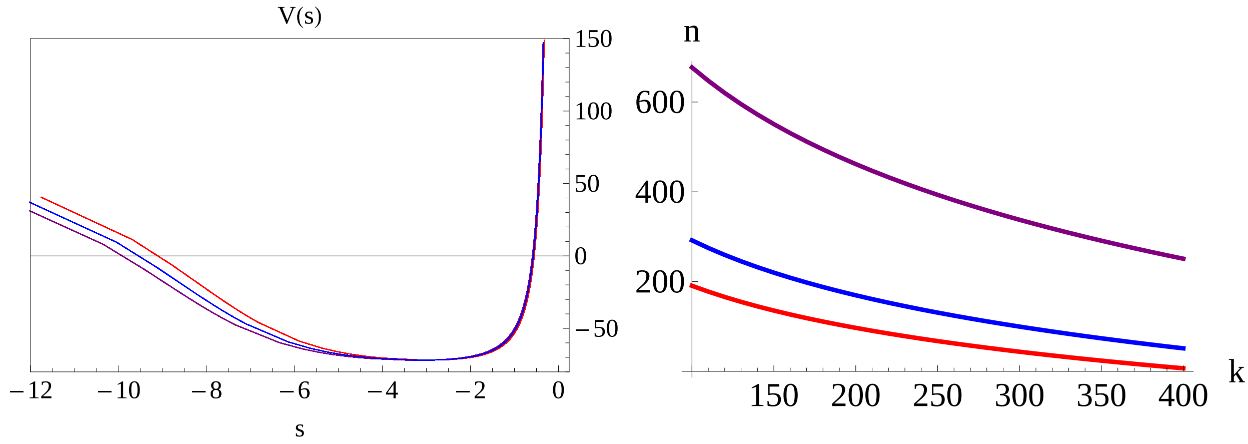

FIG. 25: The Schŕodinger potential $V(s)$ for the fermion component z_+ of in the ES background $\hat{m} = 0.36, z = 2, c_0 = 0.1$. The left plot shows the dependence on $\kappa = 0.086$ (Purple), $\kappa = 0.092$ (Blue), $\kappa = 0.1$ (Red) for $k = 0.0185$. Recall that $s = 0$ is the AdS boundary and $s = -\infty$ is the near-horizon region. The right plot is the bound states counting. The WKB estimate of the number of bound states n as a function of the momentum k for $\kappa = 10^{-5}$(Purple), 3×10^{-5}(Blue), 5×10^{-5}(Red). Note the parametric increase in number of states as the adiabaticity of the background improves for smaller κ.

A multiple Fermi surface system is difficult to understand from the condensed matter point view. It is far more natural from that perspective to consider the case where there is only a single Fermi surface. This would the opposite the fluid approximation and is a configuration we have called Dirac hair [126]. We shall turn to this next. There is one further aspect that is puzzling. Even for a large number of Fermi surfaces one would have expected the thermodynamics to reflect the generic Fermi liquid nature. The Lifshitz scaling contradicts this. Even though the number of fermionic degrees of freedom is very much macroscopic, it appears that the thermodynamics is still set by the geometry. What is going on, should have been realized from the beginning in AdS/CFT. Nevertheless it wasn't until 2011 that the solution was found. Taking the canonical AdS/CFT correspondence with $N = 4$ SYM, what happens in AdS/CFT is that it is the very strongly coupled gauge dynamics gets geometrized. If this geometry extends infinitely deep into the interior than through the extra dimension/RG correspondence this means that the strongly coupled gauge dynamics contains modes that contribute all the way to the deep IR. The exponentially suppressed self-energy precisely captures the weak but still present interactions between the geometrized gauge sector and the Fermions. Essentialy this is the same physics that happens when one simply couples a charged Fermi liquid to electro-magnetism. One of the two polarizations is gapped, but the other survives in the IR. The thermodynamics then reveals is that the macroscopics is still controlled by the dualized strongly coupled degrees of freedom, rather than the bulk star. What has not happened therefore is that the introduction of a chemical potential has confined the remaining degrees of freedom. The confirmation of the Luttinger count does say all the charged degrees of freedom have done so, but a large neutral sector remains in the IR that does not decouple. The details of this neutral sector is currently under active investigation (see e.g. [128]).

B. Dirac Hair: An electron the size of the universe

We now turn part of our philosophy around. So far we have let for the most part let the gravitational dynamics prescribe what happens in our system. Just as for the holographic Fermi liquid, however, we will now enforce that we have only a single Fermi surface in the system. We must therefore find a dual gravity state with only a single radial mode occupied, but without a hard wall. Because a fermionic mode is inherently quantum generically the effect of a single fermionic mode will barely affect the background. Compared to the electron star, it appears therefore to be even less likely to affect the IR in such a way that either a generic Landau Fermi Liquid emerges. It does affect the IR, however, as we shall so and one can emphasize it by scaling up the microscopic charge of the fermion such that its corresponding electric field is macroscopic. Qualitatively the geometry will be very similar to the electron star.

We therefore look at occupying individual fermionic wavefunctions. As before the demand that the solutions be normalizable means that the construction of the AdS black hole solution with a finite single fermion wavefunction is analogous to the construction of a holographic superconductor [77] with the role of the scalar field now taken by the Dirac field. What we shall do here is address a different question about the instability. The nature of its transition at finite temperature. The holographic superconductor famously is a prototypical second order transition. However, the ill-understood aspect of the fermionic instabilities means that we do not know the nature of the phase transition for the fermion system. A strong argument can be made that it is first order, as anything of second order or weaker should be tractable in the instabilities of the system. We will now verify this.

The starting point therefore is the charged AdS_4 black-hole background (VII.2) and we should show that at low temperatures this AdS Reissner-Nordström black hole is unstable towards a solution with a finite Dirac profile. A fortunate effect of the large charge limit that is the opposite limit to the fluid approximation of the electron star, allows to simplify the system by ignoring the gravitational dynamics. We already saw in holographic superconductor studies (see e.g. [1, 77, 78]) and in the construction of the holographic Fermi liquid that this limit already captures much of the essential physics.

In this large charge non-gravitational limit the equations of motion for the action (IX.26) again reduce to electrodynamics coupled to a fermion with charge q but now in the background of the black hole rather than the hard wall:

$$\begin{aligned} D_M F^{MN} &= iqe^N_A \bar{\Psi}\Gamma^A\Psi \ , \\ 0 &= e^M_A\Gamma^A(D_M + ieqA_M)\Psi + m\Psi \ . \end{aligned} \tag{X.29}$$

To simplify the analysis we shall use a somewhat coarser approach to the system than the Hartree summation employed in the construction of the holographic Fermi liquid. We will first rewrite this coupled non-trivial set of equations of motion in terms of the currents while at the same time use symmetries to reduce the complexity. Although a system at finite fermion density need not be homogeneous, the Fermi liquid groundstate is. As the Dirac field transforms non-trivially under rotations and boosts, we cannot make this ansatz in the strictest sense. However, in some average sense which we will make precise, the solution should be static and translationally invariant. Then time- translation and rotational invariance allow us to set $A_i = 0$, $A_z = 0$, as before. Again denoting $A_0 = \Phi$, the equations reduce to the following after the projection onto $\Psi_\pm = \frac{1}{2}(1+\Gamma^r)\Psi$.

$$\begin{aligned} \partial_r g^{rr}\partial_r\Phi &= -q\sqrt{g^{tt}}\left(\bar{\Psi}_+ i\gamma^0\Psi_+ + \bar{\Psi}_- i\gamma^0\Psi_-\right) \ , \\ (\partial_r + \mathcal{A}_\pm)\,\Psi_\pm &= \mp\mathcal{T}\Psi_\mp \end{aligned} \tag{X.30}$$

with

$$\begin{aligned} \mathcal{A}_\pm &= -\frac{r}{2}\left(3 - \frac{rf'}{2f}\right) \pm \frac{rmL}{\sqrt{f}} \ , \\ \mathcal{T} &= \frac{i(-\omega + q\Phi)}{\alpha f}\gamma^0 + \frac{i}{\alpha\sqrt{f}}k_i\gamma^i \ . \end{aligned} \tag{X.31}$$

One cannot "impose" staticity and rotational invariance for the non-invariant spinor, but by rephrasing the dynamics in terms of fermion current bilinears, rather than the fermions themselves, we can require that these are static and rotational invariant. This may seem unfamiliar, but the theoretical justification to do so is that the equations obtained this way are in fact the flow equations for the Green's functions and composites $\mathcal{J}^I(r)$, $G^I(r)$ [126]. In terms of the local vector currents[8]

$$J^\mu_+(x,r) = \bar{\Psi}_+(x,r)i\gamma^\mu\Psi_+(x,r) \ , \quad J^\mu_-(x,r) = \bar{\Psi}_-(x,r)i\gamma^\mu\Psi_-(x,r) \ , \tag{X.32}$$

or equivalently

$$J^\mu_+(p,r) = \int d^3k\bar{\Psi}_+(-k,r)i\gamma^\mu\Psi_+(p+k,r) \ , \quad J^\mu_-(p,r) = \int d^3k\bar{\Psi}_-(-k,r)i\gamma^\mu\Psi_-(p+k,r) \ . \tag{X.33}$$

rotational invariance means that spatial components $J^i_\pm$ should vanish on the solution and the equations can be rewritten in terms of $J^0_\pm$ only. Staticity and rotational invariance in addition demand that the bilinear momentum p_μ vanish. In other words, we are only considering temporally and spatially averaged densities: $J^\mu_\pm(r) = \int dtd^2x\bar{\Psi}(t,x,r)i\gamma^\mu\Psi(t,x,r)$. When acting with the Dirac operator on the currents to obtain an effective equation of motion, this averaging over the relative frequencies ω and momenta k_i also sets all terms with explicit k_i-dependence

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[8] In our conventions $\bar{\Psi} = \Psi^\dagger i\gamma^0$.

to zero.[9] Restricting to such averaged currents and absorbing a factor of q/α in Φ and a factor of $q\sqrt{L^3}$ in $\Psi_\pm$, we obtain effective equations of motion for the bilinears directly

$$\begin{aligned}
\left(\partial_r + 2\mathcal{A}_\pm\right) J^0_\pm &= \mp\frac{\Phi}{f} I \ , \\
\left(\partial_r + \mathcal{A}_+ + \mathcal{A}_-\right) I &= \frac{2\Phi}{f}(J^0_+ - J^0_-) \ , \\
\partial_r^2 \Phi &= -\frac{r^3}{\sqrt{f}}(J^0_+ + J^0_-) \ ,
\end{aligned} \tag{X.36}$$

with $I = \bar{\Psi}_-\Psi_+ + \bar{\Psi}_+\Psi_-$, and all fields are real. The remaining constraint from the A_z equation of motion decouples. It demands $\mathrm{Im}(\bar{\Psi}_+\Psi_-) = \frac{i}{2}(\bar{\Psi}_-\Psi_+ - \bar{\Psi}_+\Psi_-) = 0$. What the equations (X.36) tell us is that for nonzero $J^0_\pm$ there is a charged electrostatic source for the vector potential Φ in the bulk.

The justification of using (X.36) follows from a high-temperature expansion of a Hartree sum. One can show that at asymptotic values of the radial coordinate r at finite $T \sim \mu$ the dominant contribution is complete captured by J_-. Its value in turn is related to the zero temperature number density discontinuity Δn.

C. At the horizon: Entropy collapse to a Lifshitz solution

Before we show the solution of non-trivial Dirac hair solutions to Eqs. (X.36), we briefly consider the boundary conditions at horizon necessary to solve the system. Insisting that the right-hand-side of the dynamical equations (X.36) is subleading at the horizon, the near horizon behavior of $J^0_\pm,\ I,\ \Phi$ is:

$$\begin{aligned}
J^0_\pm &= J_{hor,\pm}(1-\frac{1}{r})^{-1/2} + \ldots, \\
I &= I_{hor}(1-\frac{1}{r})^{-1/2} + \ldots, \\
\Phi &= \Phi^{(1)}_{hor}(1-\frac{1}{r})\ln(1-\frac{1}{r}) + (\Phi^{(2)}_{hor} - \Phi^{(1)}_{hor})(1-\frac{1}{r}) + \ldots.
\end{aligned} \tag{X.37}$$

If we insist that Φ is regular at the horizon $r = 1$, i.e. $\Phi^{(1)}_{hor} = 0$, so that the electric field is finite, the leading term in $J^0_\pm$ must vanish as well, i.e. $J_{hor,\pm} = 0$, and the system reduces to a free Maxwell field in the presence of an AdS black hole and there is no fermion density profile in the bulk. Thus in order to achieve a fermion-density profile in the bulk, we must have an explicit source for the electric-field on the horizon. This does not indicate a manifest violation of Luttinger's theorem. Rather it shows the breakdown of our our neglect of backreaction as the electric field and its energy density at the location of the source will be infinite. As we argued, this backreaction is expected to geometrically encode the new IR groundstate of the theory. Nevertheless as the divergence in the electric field only increases logarithmically as we approach the horizon, and our results shall hinge on the properties of the equations at the opposite end near the boundary, we can ignore it here.

One can now solve these equations numerically (Fig. 26). What the results show explicitly is that the transition at finite temperature is first order. This is satisfyingly consistent as a resolution of the log-oscillatory instability in the system, but we should warn the reader again that we have only shown that there is a new groundstate. Whether it is truly the log-oscillatory dynamics that cause the emergence of a new IR or something else is still open question.

As foreseen also with a single Fermi surface the equations foretell that a fully gravitationally backreacted solution will still be of the Lifshitz type. This again means that this gravity system is *not* the generic Fermi liquid, but has

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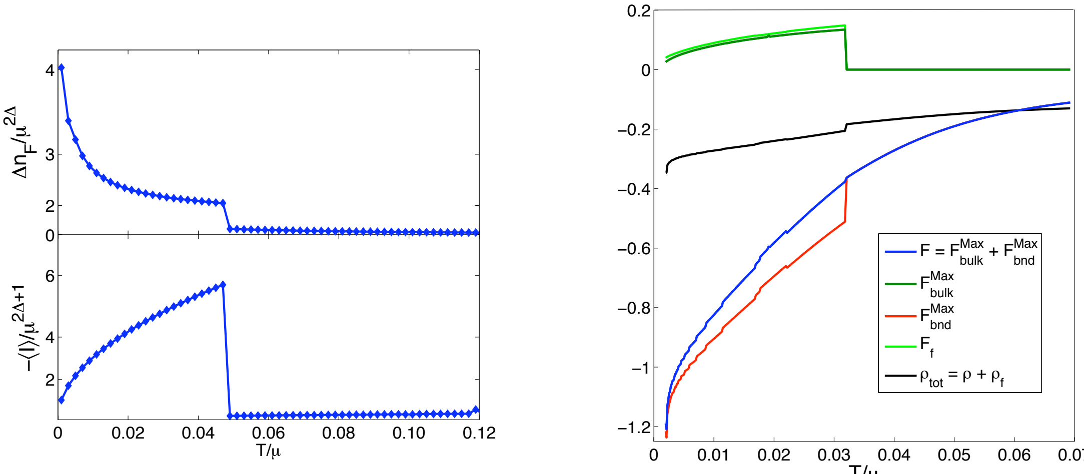
[9] To see this consider

$$(\partial + 2a_\pm)\Psi^\dagger_\pm(-k)\Psi_\pm(k) = \mp\frac{\Phi}{f}\left(\Psi^\dagger_- i\gamma^0\Psi_+ + \Psi^\dagger_+ i\gamma^0\Psi_-\right) + \frac{ik_i}{\sqrt{f}}\left(\Psi^\dagger_-\gamma^i\Psi_+ - \Psi^\dagger_+\gamma^i\Psi_-\right) \ . \tag{X.34}$$

The term proportional to Φ is relevant for the solution. The dynamics of the term proportional to k_i is

$$(\partial + a_+ + a_-)(\Psi^\dagger_-\gamma^i\Psi_+ - \Psi^\dagger_+\gamma^i\Psi_-) = -2i\frac{k^i}{\sqrt{f}}(\Psi^\dagger_+\gamma^0\Psi_+ + \Psi^\dagger_-\gamma^0\Psi_-) \ . \tag{X.35}$$

The integral of the RHS over k^i vanishes by the assumption of translational and rotational invariance. Therefore the LHS of (X.35) and thus the second term in eq. (X.34) does so as well.



(A) (B)

FIG. 26: (A) Temperature dependence of the Fermi liquid occupation number discontinuity Δn_F and operator I for a fermionic field of mass $m = -1/4$ dual to an operator of dimension $\Delta = 5/4$. We see a large density for T/μ small and discontinuously drop to zero at $T \approx 0.05\mu$. At this same temperature, the proxy free energy contribution per particle (the negative of I) vanishes. (B) The free energy $F = F^{fermion} + F^{Maxwell}$ (Eq. (X.16)) as a function of T/μ ignoring the contribution from the gravitational sector. The blue curve shows the total free energy $F = F^{Maxwell}$, which is the sum of a bulk and a boundary term. The explicit fermion contribution $F_{fermion}$ vanishes, but the effect of a non-zero fermion density is directly encoded in a non-zero $F^{Maxwell}_{bulk}$. The figure also shows this bulk $F^{Maxwell}_{bulk}$ and the boundary contribution $F^{Maxwell}_{bulk}$ separately and how they sum to a continuous F_{total}. Although formally the explicit fermion contribution $F_f \sim I$ in equation (X.17) vanishes, the bulk Maxwell contribution is captured remarkably well by its value when the cut-off is kept finite. The light-green curve in the figure shows F_f for a finite $z_0 \sim 10^{-6}$. For completeness we also show the total charge density, Eq. (X.22). The dimension of the fermionic operator used in this figure is $\Delta = 1.1$.

a large strongly coupled sector surviving in the IR together with the Fermi system. We now have a clearer picture of why to get the true Fermi-liquid one needed to introduce the hard wall to get rid of this IR surviving geometrized gauge degrees of freedom.

D. Lattice potentials and fermions in holographic non Fermi-liquids

We now wish to ask the question: how do both these non-Fermi-liquid metals react to the presence of a static periodic potential characterized by a wave-vector of order of the Fermi-momentum? This is complementary to recent similar studies of lattice effects in holographic duals of strongly coupled theories: Hartnoll and Hofman consider the lattice effect on momentum relaxation and show that it equals $\Gamma \sim T^{2\Delta_{k_L}}$ with Δ_{k_L} the scaling dimension of the charge density operator $J(\omega, k)$ in the locally critical theory at the lattice momentum k_L [75]. From the momentum relaxation one can directly extract the DC conductivity. The broadening of $\omega = 0$ delta-function into a Drude peak in the AC conductivity due to lattice effects was numerically studied by Horowitz, Santos and Tong with the remarkable observation that the holographic computation has a lot in common with experimental results in cuprates [72]. Similar to the latter study, we shall use a simple and natural encoding of a spatially varying static "lattice potential". One just adds a spatial variation of the electric field on the black hole horizon, dualizing into a spatial variation of the chemical potential in the boundary field theory.

We will assume that this is a simple unidirectional potential with only a one-dimensional periodicity. We do so by simply modifying the chemical potential *alone*:

$$ds^2 = -r^2 f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2(dx^2 + dy^2), \tag{X.38}$$

$$A_t = \left[\mu_0 + \mu_1(x)\right]\left(1 - \frac{1}{r}\right), \tag{X.39}$$

$$f(r) = 1 - \frac{1+Q^2}{r^3} + \frac{Q^2}{r^4}, \tag{X.40}$$

$$\mu_0 = 2Q, \quad \mu_1 = 2\epsilon \cos Kx \tag{X.41}$$

corresponding with an Umklapp wavevector (lattice momentum) in momentum space $K = 1/a$.

This corresponds to a bulk problem which is conceptually simple, but technically quite hazardous [121]. In the present approximation, the free Dirac fermions in the bulk experience this electrical field modulation just as a diffracting potential in a spatial direction. The curved geometry along the radial direction and the boundary conditions associated with the dictionary translate this to a very complicated "band structure" problem in the bulk that we did not manage to tackle in general. However, as in the case of elementary band structure, it is expected that the weak potential scattering limit is representative for its most salient effects. Using ϵ/μ_0 as the small parameter, we performed a weak potential scattering perturbation theory for the fermions in the bulk.

The outcomes are quite unexpected and to emphasize the physics over the technical steps we summarize them here. The predictable part is associated with the domain wall fermions responsible for the background free-fermion dispersion in Eq. (IX.49). These react to the presence of the periodic potential just as free fermions. For a unidirectional potential along the x-direction we find that a bandgap opens in the "background" dispersion right at the scattering wavevector (K, k_y). For small ϵ the size of the gap behaves as

$$\Delta(K, k_y) \simeq \epsilon \sqrt{1 - \frac{1}{\sqrt{1 + (2\frac{k_y}{K})^2}}}. \tag{X.42}$$

The band gap is vanishing for this unidirectional potential when the transversal momentum $k_y = 0$. This might look unfamiliar but holographic fermions have a chiral property that causes them to react to potentials in the same way as the helical surface states of three dimensional topological insulators: at $k_y = 0$ the gap disappears since such fermions do not scatter in a backward direction. The new quasi-Fermi-surface in the presence of the weak potential can now be constructed as usual (Fig. 27). The Umklapp surfaces are straight lines in the k_y direction, centered at the Umklapp momenta. The gaps are centered at these Umklapp surfaces and when these intersect the Fermi-surface, the latter split in Fermi-surface "pockets".

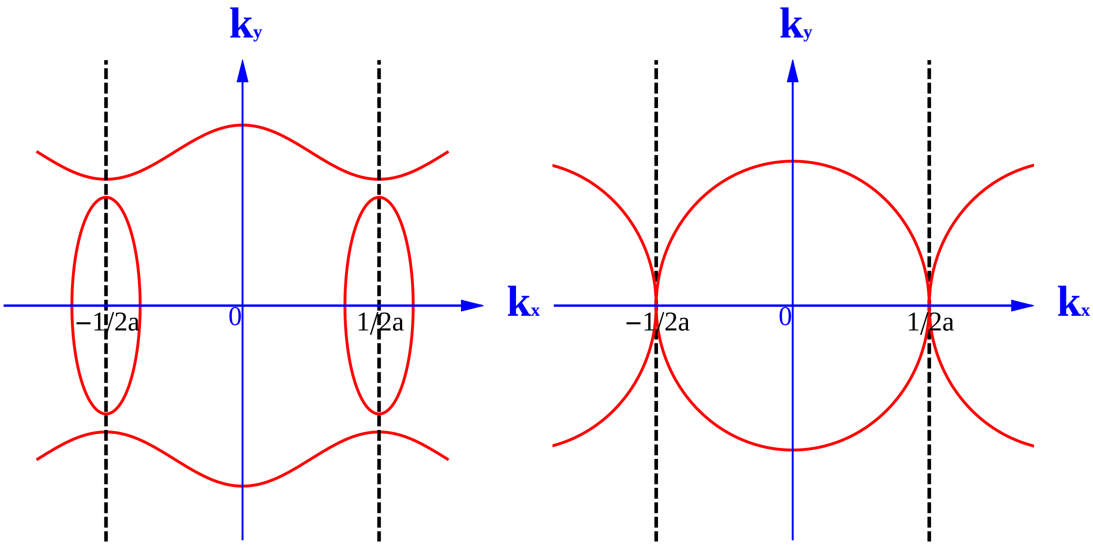


FIG. 27: A cartoon of our results of the band structure for different k_F. The system under consideration has a lattice structure only in x direction. The red line curve is the Fermi surface ($\omega = 0$) and the black dashed line is the first BZ boundary. The left plot is for $k_F > \frac{K}{2}$ and the right plot is for $k_F = \frac{K}{2}$. We have a band gap at the first Brillouin Zone boundary $k_x = \pm\frac{K}{2}$ (the black dashed line) for generic $k_F > \frac{K}{2}$ (i.e. $k_y \neq 0$) which will close when $k_F = \frac{K}{2}$ (i.e. $k_y = 0$). At generic k_x, the self-energy receives a second order correction related to the lattice effect, i.e. $\Sigma = \alpha_{\vec{k}}\,\omega^{2\nu_{\vec{k}}} + \beta^{(-)}_{\vec{k}}\omega^{2\nu_{\vec{k}-\vec{K}}} + \beta^{(0)}_{\vec{k}}\omega^{2\nu_{\vec{k}}}\ln\omega + \beta^{(+)}_{\vec{k}}\omega^{2\nu_{\vec{k}+\vec{K}}} + \ldots$. Note that this picture is a result in the extended Brillouin zone scheme.

However, the deep infrared of the field theory as encoded by the near horizon AdS_2 geometry in the bulk reacts in a surprising way to the weak potential. The hybridization of the fermions bulk dualizes into a linear combination of CFT_1 "local quantum critical" propagators in the bulk, characterized by momentum dependent exponents displaced by lattice Umklapp vectors. This has the consequence that the metals showing quasi-Fermi surfaces cannot be localized in band insulators. In the AdS_2 metal regime, where the conformal dimension of the fermionic operator is large and no Fermi surfaces are present at low T/μ, the lattice gives rise to a characteristic dependence of the energy scaling as a function of momentum. We predict crossovers from a high energy standard momentum AdS_2 scaling to a low energy regime where exponents found associated with momenta "backscattered" to a lower Brillioun zone in the extended zone scheme.

XI. OTHER ASPECTS

A. Spatially modulated phase

Now let us come to another groundstate candidate, the so called "spatially modulated phase". In condensed matter systems, spatially modulated order widely exists, for example, spin density waves, charge density waves, FFLO phase, stripe phase of underdoped cuprate superconductors etc. It is interesting to study the gravity dual for this phase. The five dimentional case was first studied by [129, 130] and the four dimensional case was considered by [131, 132].

The action for Maxwell-Chern-Simons system in the five dimensional flat spacetime $R^{1,4}$ is

$$S=\int d^5x\Big(-\frac{1}{4}F_{IJ}F^{IJ}+\frac{\alpha}{3!}\epsilon^{IJKLM}A_IF_{JK}F_{LM}\Big), \tag{XI.1}$$

where α the Chern-Simons coupling. The system has a constant electric field solution

$$F_{\mu\nu}=E\epsilon_{\mu\nu},\quad F_{\mu i}=F_{ij}=0,\quad \mu=0,1,\ i=2,3,4. \tag{XI.2}$$

However, the system is unstable around this background [129] because the gauge field fluctuation around the solution is tachyonic in the parameter region $0<k<4|\alpha E|$. In contrast, a constant magnetic field solution is stable.

In the 5D Einstein-Maxwell theory with Chern-Simons term,

$$S=\frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\Big(R+\frac{12}{L^2}-\frac{L^2}{4}F_{IJ}F^{IJ}\Big)+\frac{\alpha L^3}{3!}\epsilon^{IJKLM}A_IF_{JK}F_{LM}, \tag{XI.3}$$

the near-horizon geometry of $U(1)$ charged RN AdS black hole of is $AdS_2\times R^3$ with an electric field proportional to the volume form of AdS_2. In this case, the fluctuation of the gauge field will couple to the fluctuation of the gravity field. If the effective mass of the modes below the BF bound, we have the instability which is quite similar to the flat spacetime case. The instability happens when $\alpha>0.2896...$ and this instability happens at non-vanishing momenta which indicates a spatially modulated phase in the holographically dual field theory with spontaneous current generation in a helical configuration. The end-point of this phase transition was studied in [130]. In general there is a second order phase transition between the normal phase and spatially modulated phase.

A similar instability can occur in four dimensional Maxwell-axion theory

$$S=\int d^4x\Big(-\frac{1}{4}F_{IJ}F^{IJ}+\frac{1}{2}(\partial_I\Phi)^2-\frac{1}{2}m^2\Phi^2+\frac{\alpha}{2}\epsilon^{IJKL}F_{IJ}F_{KL}\Big). \tag{XI.4}$$

It is easy to show it has a constant electric field solution and an instability for the fluctuation when $0<k<\sqrt{16\alpha^2E^2-m^2}$. When coupled to the gravity, one can similarly realize the holographic spatially modulated phase in four dimensional gravity system [131, 132]. There are also attempts to combine the holographic spatically modulated phase and holographic superconducting phase to study the stable phase [133].

B. Holographic entanglement entropy

Entanglement entropy is an important nonlocal observable of quantum field theory. For zero temperature zero density D-dimensional field theory, there is a famous *area law* for the entanglement entropy S_Σ: the leading divergent term is proportional to the area of the boundary of Σ

$$S_\Sigma=\frac{A_{\partial\Sigma}}{\delta^{D-2}}+\mathcal{O}\Big(\frac{1}{\delta^{D-3}}\Big) \tag{XI.5}$$

where δ is a UV cutoff for the field theory. It was discovered that in the presence of Fermi surface the area law would be violated logrithmically [134]

$$S_\Sigma=cA_{\rm FS}A_{\partial\Sigma}\ln(A_{\rm FS}A_{\partial\Sigma}) \tag{XI.6}$$

where c is a constant and $A_{\rm FS}$ is the area of the Fermi surface $A_{\rm FS} \propto k_F^{D-2}$. It is expected that the log behavior is also true for both Fermi liquid and non Fermi liquid.

Entanglement entropy can be computed holographicly [135]. In particular the entanglement entropy S_Σ of a spatial region Σ in a D-dimensional field theory which has a dual description in terms of Einstein-Hilbert gravity with matter is given by

$$S_\Sigma = \frac{A(\gamma_\Sigma)}{4G_N} \tag{XI.7}$$

where γ_Σ is a codimension two minimal area surface in the bulk whose boundary $\partial\gamma_\Sigma$ coincides with $\partial\Sigma$. This is so called "holographic entanglement entropy".

In the previous section, we studied the finite density system holographically and found evidences for Fermi surface. It is natural to ask what we can learn from the entanglement entropy. A large class of geometry was studied [136] and it turns out the most interesting geometry is [137–139]

$$ds^2 = r^{\frac{2\theta}{D-1}}\Big(-\frac{dt^2}{r^{2z}} + \frac{dr^2 + dx_i^2}{r^2}\Big). \tag{XI.8}$$

Under the transformation $t \to \lambda^z t, x_i \to \lambda x_i, r \to \lambda r$, the metric is not scale invariant, but it transforms as

$$ds \to \lambda^{\theta/(D-1)} ds. \tag{XI.9}$$

Here z is the dynamical critical exponent and θ is the violation of hyperscaling exponent.

At finite temperature we can construct the black hole solution and the thermal entropy density scales as

$$S \sim T^{(D-1-\theta)/z} \tag{XI.10}$$

A positive specific heat imposes the constraint $\theta < D-1$ for positive z. In this region, the null energy condition (NEC) implies $z \geq 1 + \frac{\theta}{D-1}$.

The entanglement entropy S_Σ scales as [137]

$$S_\Sigma \sim \begin{cases} A_{\partial\Sigma} & \text{if } \theta < D-2 \\ A_{\rm FS} A_{\partial\Sigma} \ln(A_{\rm FS} A_{\partial\Sigma}) & \text{if } \theta = D-2 \\ A_{\partial\Sigma}^{\theta/(D-2)} & \text{if } \theta > D-2 \end{cases} \tag{XI.11}$$

In the parameter region $D-2 < \theta < D-1$ we have new violations of area law. The most interesting case is when $\theta = D-2$, there is a logarithmic violation of the area law which indicates the appearance of Fermi surfaces in the dual theory.

Acknowledgments

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