Many-Valued Logics

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Abstract

The paper considers the fundamental notions of many- valued logic together with some of the main trends of the recent development of infinite valued systems, often called mathematical fuzzy logics.

Besides this logical approach also a more algebraic approach is discussed. And the paper ends with some hints toward applications which are based upon actual theoretical considerations about infinite valued logics.

Key words: mathematical fuzzy logic, algebraic semantics, continuous t-norms, left-continuous t-norms, Pavelka-style fuzzy logic, fuzzy set theory, non-monotonic fuzzy reasoning

1 Basic ideas

1.1 From classical to many-valued logic

Logical systems in general are based on some formalized language which includes a notion of well formed formula, and then are determined either semantically or syntactically.

That a logical system is semantically determined means that one has a notion of *interpretation* or *model*[1] in the sense that w.r.t. each such interpretation every well formed formula has some *(truth) value* or represents a function into

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[1] We prefer to use the word interpretation in general and to restrict the use of the word model to particular interpretations which are tied in a specific way with sets of well formed formulas.

the set of (truth) values. It furthermore means that one has a notion of validity for well formed formulas and, based upon it, also a natural *entailment relation* between sets of well formed formulas and single formulas (or sometimes also whole sets of formulas).

That a logical system is syntactically determined means that one has a notion of *proof* and of provable formula, i.e. of (formal) *theorem*, as well as a notion of *derivation from* a set of *premisses*.

From a philosophical, especially epistemological point of view the semantic aspect of (classical) logic is more basic than the syntactic one, because it are mainly the semantic ideas which determine what are suitable syntactic versions of the corresponding (system of) logic.

The most basic (semantic) assumptions of classical, i.e. two-valued – propositional as well as first order – logic are the principles of bivalence and of compositionality. Here the *principle of bivalence* is the assumption that each sentence[2] is either true or false under any one of the interpretations, i.e. has exactly one of the truth values $\top$ and $\bot$, usually numerically coded by 1 and 0. And the *principle of compositionality*[3] is the assumption that the value of each compound well formed formula is a function of the values of its (immediate) subformulas.

The most essential consequence of the principle of compositionality is the fact that each one of the propositional connectives as well as each one of the (first order) quantifiers is semantically determined by a function (of suitable arity) from the set of (truth) values into itself, or by a function from its powerset into the set themselves.

Disregarding the quantifiers for a moment, i.e. restricting the considerations to the propositional case, the most essential (semantical) point is the determination of the truth value functions, i.e. of the operations in the truth value set which characterize the connectives. From an algebraic point of view, hence, the crucial point is to consider not only the *set* of truth values, but a whole *algebraic structure* with the truth value set as its support. And having in mind that all the classical connectives are definable from the connectives for conjunction, disjunction, and negation, this means to consider the set $\{0, 1\}$ of truth values together with the truth functions $\min, \max$ and $1 - \dots$ of these connectives – and this is a particular Boolean algebra. The semantical notion of validity of a well formed formula φ w.r.t. some interpretation now means that φ has truth value 1 at this particular interpretation. (And universal va-

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[2] By a sentence one means either any well formed formula of the corresponding formalized propositional language, or any well formed formula of the corresponding formalized first order language which does not contain any free individual variable.
[3] Sometimes this principle is also named principle of extensionality.

lidity of course means being valid for every interpretation.)

Generalizing the notions of truth value, of interpretation and of validity in such a way that the truth value structure may be any (nontrivial) Boolean algebra $\mathcal{B} = \langle B, \sqcap, \sqcup, ^*, \mathbf{0}, \mathbf{1} \rangle$, that an interpretation is any mapping from the set of all propositional variables into B, the truth value functions for conjunction, disjunction, negation are chosen as $\sqcap, \sqcup, ^*$, respectively, and validity of a well formed formula φ w.r.t. a given interpretation means that φ has the Boolean value $\mathbf{1}$ at this interpretation.

This generalized type of interpretations can easily become extended to the first order case: one then has to consider only complete Boolean algebras and has to consider the operations of taking the infimum or supremum as the operations corresponding to the universal or existential quantifier.

It is well known that the class of universally valid formulas of classical logic is just the class of all formulas valid for all (nontrivial) – and complete (in the first order case) – Boolean algebras as truth value structures. This fact is referred to by saying that the class of (complete) Boolean algebras is characteristic for classical logic.

Many-valued logic deviates from the two basic principles of bivalence and of compositionality only in that it neglects the principle of bivalence. Therefore, any system S of many-valued logic is characterized (i) by a suitable *formalized language* $\mathcal{L}_{\mathsf{S}}$ which comprises

- its (nonempty) family $\mathcal{J}^{\mathsf{S}}$ of (basic) *propositional connectives*,
- its (possibly empty) family of *truth degree constants*,
- its set of *quantifiers*,[4]

and adopts the usual way of defining the class of well formed formulas w.r.t. these syntactic primitives, and parallel to these syntactic data (ii) by the corresponding semantic data, i.e. by

- a (nonempty) set $\mathcal{W}^{\mathsf{S}}$ of *truth degrees*,[5]

[4] Each quantifier for simplicity is supposed here to be a unary one. This means that we allow to have some kinds of generalized quantifiers in the sense of Mostowski [117] but we do not consider the possibility to have quantifiers with more than one scope, as is allowed e.g. in [129,68].

[5] For systems of many-valued logic we prefer to call their semantic values truth *degrees* to emphasize the difference to the truth values of classical logic. Moreover this term appears to be a bit more neutral concerning ontological commitments. And this is important because one does not have any preferred ontological reading for the truth degrees of many-valued logics. The intuitive understanding of these degrees completely depends upon the particular applications under consideration. Because this is a completely different situation compared with classical logic, this

- a family of *truth degree functions* together with a correspondence between these truth degree functions and the propositional connectives of the (formal) language,
- a (possibly empty) family of nullary operations, i.e. of elements of the truth degree set together with a one-one correspondence between the members of this family and the truth degree constants of the (formal) language,
- a set of *quantifier interpreting functions* from the power set $\mathbb{P}(W^{\mathsf{S}})$ of $\mathcal{W}^{\mathsf{S}}$ into $\mathcal{W}^{\mathsf{S}}$ together with a one-one correspondence between these functions and the quantifiers of the (formal) language.

Based upon these data which constitute any particular system S of many-valued logic, it is a routine matter to combine with each truth degree evaluation e of all atomic formulas an extension $\mathsf{Val}^{\mathsf{S}}$ which gives for each well formed formula φ of the language of S its truth degree $\mathsf{Val}^{\mathsf{S}}(\varphi, e)$ under e.

As usual, $\mathsf{Val}^{\mathsf{S}}(\varphi, e)$ could be defined by recursion on the complexity of φ.

And, again as usual, in the propositional case such an evaluation e is given immediately, and in the first-order case $e = (\mathfrak{A}, v)$ is determined by a suitable interpretation $\mathfrak{A}$ together with an assignment v of objects from $\mathfrak{A}$ to the (individual) variables of the language.

1.2 Particular truth degree sets

Usually one additionally assumes that the classical truth values (or some "isomorphic" copies of them, also coded by 1 and 0) appear among the truth degrees of any suitable system S of many-valued logic:

$$\{0, 1\} \subseteq \mathcal{W}^{\mathsf{S}} . \tag{1}$$

Formally, for the systems S of many-valued logic there is essentially no restriction concerning the set $\mathcal{W}^{\mathsf{S}}$ of truth degrees of S besides (1). Nevertheless the choice of $\mathcal{W}^{\mathsf{S}}$ as a set of numbers (either integers or rationals or even reals) is widely accepted use. At least as long as one is not interested to have an ordering of the truth degrees which allows for incomparable truth degrees[6]. The existence of incomparable truth degrees, however, may be crucial for certain particular applications, e.g. in a situation where the truth degrees are intended to code parallel evaluations of different points of view. To imagine such a situation assume to be interested, in image processing, to evaluate whether a

change in the terminology is intended to underline just this difference.

[6] Of course, even in such a situation one can take a set of numbers for the set of truth degrees – and adjoin another ordering relation to these numbers than their natural ordering. But this is rather unusual.

certain point P belongs to a certain figure $\mathcal{F}$, i.e. to determine the truth value of the sentence "P is a point of $\mathcal{F}$". For pure black and white pictures, the evaluation yields one of the truth values $\top, \bot$. Being instead confronted with a graytone picture, the evaluation of this sentence may yield as truth degrees the values of some scale which characterizes the different gray levels. And being, finally, confronted with a colored picture, e.g. on the screen of some monitor, which consists of (colored) pixels which themselves are generated by superposing pixels of the three basic colors, then it may be reasonable to evaluate the above mentioned sentence by a truth degree which is a triple of the levels of intensity of the basic colors which give point P.

Another widely accepted kind of approach is to assume that among the truth degrees there is a smallest one, usually interpreted as an equivalent for the truth value $\bot$, and a biggest one, usually interpreted as an equivalent for $\top$.

Based on these common assumptions it is usually at most a simple matter of isomorphic exchange of the structure of truth degrees to assume (1) together with

$$\mathcal{W}^{\mathsf{S}} \subseteq [0,1] \subseteq \mathbb{R}. \tag{2}$$

And this choice of the truth degree set shall be the standard one in the following discussions.

For the case of infinitely many truth degrees it is common usage to consider either countably many or uncountably many truth degrees and furthermore to choose either one of the truth degree sets

$$\mathcal{W}_0 =_{def} \{x \in \mathbb{Q} \,|\, 0 \leq x \leq 1\} \quad \text{or} \quad \mathcal{W}_\infty =_{\text{def}} \{x \in \mathbb{R} \,|\, 0 \leq x \leq 1\}. \tag{3}$$

For the case of finite sets of truth degrees usually one additionally assumes that these truth degrees form a set of equidistant points of the real unit interval $[0,1]$, i.e. are of the kind

$$\mathcal{W}_m =_{\text{def}} \Big\{ \frac{k}{m-1} \mid 0 \leqq k \leqq m-1 \Big\} \tag{4}$$

for some integer $m \geqq 2$.

1.3 Designated truth degrees

In classical logic there is a kind of superiority of the truth value $\top$ over the other one $\bot$: given some well formed formula (or some set of formulas) one mainly is interested in those interpretations for the system of many-valued logic one is working in which make the given formula(s) true.

With the set $\mathcal{W}^{\mathsf{S}}$ of truth degrees each system S of many-valued logic has its equivalent to the truth value set $\{\top, \bot\}$. However, even admitting condition (1), this does not mean that the truth degree 1 has to be *the* equivalent of $\top$. The rather vague philosophical idea that in many-valued logic one considers some kind of "splitting" of the classical truth values does not by itself determine which truth degrees "correspond" to $\top$. Therefore, to determine some system S of many-valued logic does not only mean to fix its set of truth degrees and its formal language, i.e. its connectives, quantifiers, predicate symbols and individual constants, together with their semantic interpretations, it also means to fix which truth degrees "correspond" to $\top$.

Formally this means, that with each system S of many-valued one connects not only its set $\mathcal{W}^{\mathsf{S}}$ of truth degrees but also some set $\mathcal{D}^{\mathsf{S}}$ of *designated* truth degrees. Of course, usually one supposes

$$1 \in \mathcal{D}^{\mathsf{S}} \subseteq \mathcal{W}^{\mathsf{S}} \quad \text{together with} \quad 0 \notin \mathcal{D}^{\mathsf{S}}. \tag{5}$$

Such a choice of designated truth degrees is of fundamental importance for the generalization of the notions of logical validity and of logical consequence.

1.4 Logical validity and logical consequence

Based on the notion of designated truth degrees it is essentially a routine matter to generalize the notions of logical validity and of logical consequence to any system S of many-valued logic.

Adopting the standard usage to mean by a *sentence* either a well formed formula in the case of a propositional language $\mathcal{L}_{\mathsf{S}}$ or a well formed formula without free individual variables of a first order language $\mathcal{L}_{\mathsf{S}}$, one calls a sentence φ *valid* w.r.t. an evaluation e of the atomic formulas iff it has a designated truth degree for that evaluation, and one calls a sentence φ *logically valid* iff it is valid for every suitable evaluation.

As usual, furthermore, a *formula* φ of a first order language $\mathcal{L}_{\mathsf{S}}$ is called *valid in a* given *interpretation* $\mathfrak{A}$, iff it has a designated truth degree w.r.t. any assignment of objects of $\mathfrak{A}$ to the individual variables of the language $\mathcal{L}_{\mathsf{S}}$.

On the other hand, an evaluation e is called a *model* of a formula φ iff this formula φ is valid under this evaluation e. And e is called a *model* of a set Σ of well formed formulas iff it is a model of any formula $H \in \Sigma$. The fact that e is a model of φ, or of Σ, as usual is denoted by $e \models \varphi$ or $e \models \Sigma$. And, as usual, one considers the (crisp) *model classes*

$$\mathsf{Mod}^{\mathsf{S}}(\Sigma) =_{\text{def}} \{e \mid e \models \Sigma\} = \{e \mid \mathsf{Val}^{\mathsf{S}}(\varphi, e) \in \mathcal{D}^{\mathsf{S}} \text{ for all } \varphi \in \Sigma\}. \tag{6}$$

As usually, hence, one has that the logical validities are just those formulas φ whose model class is the class of all evaluations.

Sometimes, the notion of model is generalized a bit further in many-valued logic. Given some truth degree α and some formula φ, an evaluation e is called an *α-model* of φ iff the truth degree of φ under the evaluation e equals α or [7] iff it is greater or equal to α. We prefer here, to speak in the last mentioned case of a $(\geq \alpha)$-model.

Correspondingly the notion of α-model of a set of sentences is used. In this case, the most suitable way is to call an evaluation an $(\geq \alpha)$-model of a set Σ of sentences iff this interpretation is an $(\geq \alpha)$-model of each sentence $\varphi \in \Sigma$.

Based on these preliminaries, the notion of logical consequence is defined almost in the standard way, but again with mainly two slightly different basic intuitions. For simplicity, we restrict again the considerations to sentences only. The extension to all well formed formulas happens in many-valued logic exactly as in classical logic.

One defines that a sentence φ is a *logical consequence* of a set Σ of sentences, usually also written $\Sigma \models \varphi$, iff

(*Version 1*): each model of Σ is also a model of φ, or equivalently

$$\Sigma \models \varphi \iff \mathsf{Mod}^{\mathsf{S}}(\Sigma) \subseteq \mathsf{Mod}^{\mathsf{S}}(\varphi)\,. \tag{7}$$

(*Version 2*): each $(\geq \alpha)$-model of Σ is also a $(\geq \alpha)$-model of φ.

With $\models \varphi$ as shorthand for $\emptyset \models \varphi$ in any case $\models \varphi$ means that φ is logically valid.

The checking of well formed formulas of any propositional system of many-valued logic for being logically valid can be done in the same way as for classical logic by determining complete truth degree tables – and this is effective provided the set of truth degrees is finite.

Theorem 1 *For each finitely many-valued system S of propositional logic the property of being a logically valid sentence is decidable, and for each finite set Σ of sentences of S also the property of being a logical consequence of Σ is decidable.*

[7] Both these variants are in use. One has to check the use of the term α-model in the particular case to see which version applies.

2 Outline of the history

Many-valued logic as a separate part of logic was created by the works of J. Łukasiewicz [105] and E.L. Post [124] in the beginning 1920th. Admittedly, both authors have not been the first ones which did not assume the principle of bivalence, but earlier attempts to do logic without this principle of bivalence did not prove to be influential [8].

The prehistory of many-valued logic, however, may be traced back up to Aristotle [9] who e.g. in his *De Interpretatione*, chap. 9, discussed the problem of future contingencies, i.e. the problem which truth value a proposition should have today which asserts some future event. This problem was stimulating even for J. Łukasiewicz [106], and it is closely tied with the philosophical problem of determinism. The link is provided by the interpretation that the classification of some future event as (actually) "possible" or "undetermined" may well be seen as the acceptance of a third "truth value" besides $\top$ and $\bot$. Surely, this reading is not the necessary one. Nevertheless, the ancient philosophical school of Epicureans which tended toward indeterminism refused the principle of bivalence, whereas the school of the Stoics did accept it—and strongly advocated determinism.

The same problem of *contingentia futura* was also the source for several extended discussions during the Middle Ages, cf. e.g. [107,114,7,129], without getting resolved. And in the phase of the general revival of investigations into the field of logic during the second half of the 19th century the idea of neglecting the principle of bivalence appeared (partly without clear mentioning of this fact) to H. McColl [112], cf. also [103], and Ch.S. Peirce, cf. [121, vol. 4] as well as [55,145].

The real starting phase of many-valued logic was the time interval from about 1920 till about 1930, and the main force of development was the Polish school of logic under J. Łukasiewicz. The papers [110,106] as well as the influential textbook [102], all published in 1930, explain the core ideas as well as the background of philosophical ideas and the main technical results proven up to this time. [10] They also stimulated further research into the topic.

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[8] Even the previous paper [104] of J. Łukasiewicz which also admitted generalized truth "degrees" besides the traditional truth values $\top, \bot$ did not influence the development of logic toward many-valued logic in any perceivable manner.

[9] The interested reader may consult e.g. [106,109,129,119].

[10] As a side remark it has to be mentioned that P. Bernays [12] used more than the usual two truth values of classical logic to study independence problems for systems of axioms for systems of classical propositional calculus. But in his case these multiple values were only formal tools for his unprovability results.

In [106] Łukasiewicz intends to give a modal reading to his many-valued propositional logic, claiming that only the 3-valued and the infinite valued case (with the set of all rationals between 0 and 1 as truth degree set) are really of interest for applications. In [110] however all finitely many-valued propositional systems and the just mentioned infinitely many-valued one are discussed, always based on a negation and an implication connective as primitive ones characterized semantically by their truth degree functions.

Parallel with this "Polish" development the US-American mathematician E.L. Post [124] designed a family of finitely valued systems. However, it seems that his main aim was not a philosophical one, but "only" a technical: he was interested in the problem of functional completeness, i.e. in the representation of *all* truth degree functions (say from $\mathcal{W}_m$ into $\mathcal{W}_m$) with the help of only a few of them.

Basic theoretical results for systems of many-valued logics which followed the initial phase of "Polish" many-valued logic have e.g. been

- M. Wajsberg's [148] axiomatization of the three valued (propositional) system L_3 of Łukasiewicz, i.e. of that one propositional system with Łukasiewicz's implication and negation connectives as primitive connectives,
- the extension of Łukasiewicz's system L_3 to a functionally complete one and its axiomatization by J. Słupecki [140],
- the work of K. Gödel [64] and S. Jaśkowski [89] which clarified the mutual relations of intuitionistic and many-valued logic in the sense that it was proven that there does not exist a single (propositional) many-valued system whose set of logically valid formulas coincides with the set of logically valid formulas of intuitionistic (propositional) logic,
- the application of systems of three valued logic to the problems of logical antinomies by Bočvar [17,18] with the third truth value read as "senseless",
- the application of systems of three valued logic to problems of partially defined function by S. Kleene [95,96] with the third truth value read as "undefined".

Furthermore, during the 1940th basic approaches have been generalized and essential results were proven by B. Rosser and A.R. Turquette in a series of papers and later on most of this material collected in their monograph [135], which besides the Łukasiewicz papers of 1930 was the standard reference for years. At least up to 1969 in which year the nice monograph [129] appeared which became one of the standard references to many-valued logics up to the 1990s.

In the 1950s and 1960s then there was some decline in the interest in many-valued logics – and at the same time some shift in the focus of what were considered as the more interesting problems. The decline as well as the shift may have been caused by the same situation: the fact – mentioned in the Rosser/Turquette monograph [135] quite open – that up to this time no really convincing applications had been approached by the methods of many-valued logic. The shift in interest inside many-valued logic thus happened toward problems of definability of operations in $\{1, 2, \ldots, n\}$ from particular sets of such operations, i.e. toward problems connected with the functional incompleteness of most of the then "usual" systems of connectives of (finitely) many-valued systems of (propositional) logic. In the background, however, there was not only the theoretical problem of functional completeness, there was (and is) also the related problem in switching theory of sets of suitable elementary circuits which allow to generate all the (finitary) operations in $\{1, 2, \ldots, n\}$ by being combined into suitable circuits – or the related problem to determine the class of all operations which can be generated by some given ones.

3 Basic Systems of Many-Valued Logics

If one looks systematically for many-valued logics which have been designed for quite different applications, one finds four main types of systems:

- the Łukasiewicz logics L_κ as explained in [106];
- the Gödel logics G_κ from [64];
- the product logic Π studied in [80];
- the Post logics P_m for $2 \leq m \in \mathbb{N}$ from [124].

The first two types of many-valued logics each offer a uniformly defined family of systems which differ in their sets of truth degrees and comprise finitely valued logics for each one of the truth degree sets (4) together with an infinite valued system with truth degree set (3), which formally is indicated by choosing $\kappa \in \{n \in \mathbb{N} \mid n \geq 2\} \cup \{\infty\}$. For the fourth type an infinite valued version is lacking.

In their original presentations, these logics look rather different, regarding their propositional parts. For the first order extensions, however, there is a unique strategy: one adds a universal and an existential quantifier such that quantified formulas get, respectively, as their truth degrees the infimum and the supremum of all the particular cases in the range of the quantifiers.

As a reference for these and also other many-valued logics in general, the reader may consult [68].

3.1 The Gödel logics

The simplest ones of these logics are the *Gödel logics* G_κ which have a conjunction $\wedge$ and a disjunction $\vee$ defined by the minimum and the maximum, respectively, of the truth degrees of the constituents:

$$u \wedge v = \min\{u, v\}\,, \qquad u \vee v = \max\{u, v\}\,. \tag{8}$$

For simplicity we denote here and later on the connectives and the corresponding truth degree functions by the same symbol.

These Gödel logics have also a negation $\sim$ and an implication $\to_{\mathsf{G}}$ defined by the truth degree functions

$$\sim u = \begin{cases} 1\,, \text{ if } \; u = 0; \\ 0\,, \text{ if } \; u > 0. \end{cases} \qquad u \to_{\mathsf{G}} v = \begin{cases} 1\,, \text{ if } \; u \le v; \\ v\,, \text{ if } \; u > v. \end{cases} \tag{9}$$

The systems differ in their truth degree sets: for each $2 \le \kappa \le \infty$ the truth degree set of G_κ is $\mathcal{W}_\kappa$.

3.2 The Łukasiewicz logics

The *Łukasiewicz logics* L_κ, again with $2 \le \kappa \le \infty$, have originally been designed in [106] with only two primitive connectives, an implication $\to_{\mathsf{L}}$ and a negation $\neg$ characterized by the truth degree functions

$$\neg u = 1 - u\,, \qquad u \to_{\mathsf{L}} v = \min\{1, 1 - u + v\}\,. \tag{10}$$

The systems differ in their truth degree sets: for each $2 \le \kappa \le \infty$ the truth degree set of G_κ is $\mathcal{W}_\kappa$.

However, it is possible to define further connectives from these primitive ones. With

$$\varphi \,\&\, \psi =_{\mathrm{df}} \neg(\varphi \to_{\mathsf{L}} \neg\psi)\,, \qquad \varphi \,\veebar\, \psi =_{\mathrm{df}} \neg\varphi \to_{\mathsf{L}} \psi \tag{11}$$

one gets a (strong) conjunction and a (strong) disjunction with truth degree functions

$$u \,\&\, v = \max\{u + v - 1, 0\}\,, \qquad u \veebar v = \min\{u + v, 1\}\,, \tag{12}$$

usually called the *Łukasiewicz* (arithmetical) *conjunction* and the *Łukasiewicz* (arithmetical) *disjunction*. It should be mentioned that these connectives are linked together via a De Morgan law using the standard negation of this system:

$$\neg(u \,\&\, v) = \neg u \veebar \neg v\,. \tag{13}$$

With the additional definitions

$$\varphi \wedge \psi =_{\mathrm{df}} \varphi \,\&\, (\varphi \to_{\mathsf{L}} \psi) \qquad \varphi \vee \psi =_{\mathrm{df}} (\varphi \to_{\mathsf{L}} \psi) \to_{\mathsf{L}} \psi \tag{14}$$

one gets another (weak) conjunction $\wedge$ with truth degree function min, and a further (weak) disjunction $\vee$ with max as truth degree function, i.e. one has the conjunction and the disjunction of the Gödel logics also available.

3.3 The Product logic

The *product logic* Π, in detail explained in [80], has a fundamental conjunction $\odot$ with the ordinary product of reals as its truth degree function, as well as an implication $\to_\Pi$ with truth degree function

$$u \to_\Pi v = \begin{cases} 1\,, \text{ if } \; u \le v; \\ \frac{u}{v}\,, \text{ if } \; u < v. \end{cases} \tag{15}$$

Additionally it has a truth degree constant $\overline{0}$ to denote the truth degree zero.

In this context, a negation and a further conjunction are defined as

$$\sim \varphi =_{\mathrm{df}} \varphi \to_\Pi \overline{0}\,, \qquad \varphi \wedge \psi =_{\mathrm{df}} \varphi \odot (\varphi \to_\Pi \psi)\,. \tag{16}$$

Routine calculations show that both connectives coincide with the corresponding ones of the infinite valued Gödel logic G_∞. And also the disjunction $\vee$ of this Gödel logic becomes available, now via the definition

$$\varphi \vee \psi =_{\mathrm{df}} ((\varphi \to_\Pi \psi) \to_\Pi \psi) \wedge ((\psi \to_\Pi \varphi) \to_\Pi \varphi)\,. \tag{17}$$

There is, however, no natural way to combine with this (infinite valued) product logic a whole family of finite valued systems by simply restricting the set of truth degrees to some $\mathcal{W}_m$ as in the previous two cases: besides $\mathcal{W}_2$ no such set is closed under the ordinary product, and for $\mathcal{W}_2$ the product coincides e.g. with the minimum operation.

3.4 The Post logics

The Post system P_m for $m \ge 2$ has truth degree set $\mathcal{W}_m$. These propositional systems have been originally formulated uniformly in negation and disjunction as basic connectives with the following truth degree functions:

$$\sim u = \begin{cases} 1\,, & \text{for } u = 0\,, \\ u - \frac{1}{m-1}\,, & \text{for } u \ne 0\,, \end{cases} \qquad u \vee v = \max\{u, v\}\,.$$

Contrary to the previous systems, the definition of negation here does not seem to be given in a uniform way independent of the number of truth degrees. However, it is always just a cyclic permutation of all the truth degrees (in their natural order).

For the sets of designated truth degrees a canonical choice does not exist; already Post [124] has discussed the possibility that there may be chosen truth degrees different from 1 as designated ones. Nevertheless, $\mathcal{D}^P = \{1\}$ is a kind of standard choice.

The set of basic connectives of each one of the Post systems P_m is functionally complete, i.e. allows to represent every possible truth degree function (over $\mathcal{W}_m$). Therefore each one of the Post systems P_m, with $\mathcal{D}^P = \{1\}$ as the set of designated truth degrees, covers its corresponding Łukasiewicz system with the same set of truth degrees—in the sense that the set of L_m-tautologies is a subset of the set of P_m-tautologies, and that this set of P_m-tautologies does not contain any formula φ whose Łukasiewicz negation $\neg H$ is L_m-satisfiable, of course always via a suitable reading of the Łukasiewicz connectives in the Post systems. And the same holds true for the corresponding m-valued Gödel system G_m.

If one enriches all the finitely many-valued (propositional) Łukasiewicz systems L_m with truth degree constants for all their truth degrees, then these enriched systems L_m^* become functionally complete. And this means that the extended m-valued Łukasiewicz systems L_m^* and the m-valued Post logics become interdefinable (for each fixed number m of truth degrees). Hence there is in principle no essential difference between both types of (finitely valued) systems: all what can be expressed in the "Post world" can also be expressed in the (extended) "Łukasiewicz world", and vice versa.

4 Standard and Algebraic Semantics

The fundamental many-valued logics have their *standard semantics* as explained: the sets $\mathcal{W}_m$ or the whole real unit interval $[0, 1]$ as truth degree sets, and the connectives (and quantifiers) as mentioned.

And derived from these basic choices one has the notions of validity under some evaluation, of logical validity, and of model as explained.

Besides these standard semantics, all these many-valued logics have also *algebraic semantics* determined by suitable classes $\mathcal{K}$ of truth degree structures. The situation is similar here to the case of classical logic: the logically valid formulas in classical logic are also just all those formulas which are valid in all

Boolean algebras.

Of course, these structures have to have the same signature as the language $\mathcal{L}$ of the corresponding logic. This means that these structures provide for each connective of the language $\mathcal{L}$ an operation of the same arity, and they have to have—in the case that one discusses the corresponding first order logics—suprema and infima for all those subsets which may appear as value sets of formulas. Particularly, hence, they have to be (partially) ordered, or at least pre-ordered.

For each formula φ of the language $\mathcal{L}$ of the corresponding logic, for each such (generalized truth degree) structure $\mathbf{A}$, and for each evaluation e which maps the set of atomic formulas of $\mathcal{L}$ into the carrier of $\mathbf{A}$, one has to define a value $\mathsf{Val}(\varphi, e)$, and finally one has to define what it means that such a formula φ is *valid in* $\mathbf{A}$. Then a formula φ is *logically valid* w.r.t. this class $\mathcal{K}$ iff φ is valid in all structures from $\mathcal{K}$.

The standard way to arrive at such classes of structures is to start from the Lindenbaum algebra of the corresponding logic, i.e. its algebra of formulas modulo the congruence relation of logical equivalence. For this Lindenbaum algebra one then has to determine a class of similar algebraic structures which—ideally—forms a variety.

A *variety* is a class $\mathcal{K}$ of algebraic structures which is equationally definable, i.e. for which there exists a set $\mathcal{E}$ of equations between terms of the language of these structures such that an algebraic structure $\mathbf{A}$ belongs to $\mathcal{K}$ iff $\mathbf{A}$ is a model of $\mathcal{E}$. Besides this characterization in logical terms there is also a characterization in purely algebraic terms: a variety is a class $\mathcal{K}$ of algebraic structures which is closed under the formations of subalgebras, of homomorphic images, and of direct products. For the algebraic details the interested reader may e.g. consult [23,34,71].

4.1 Gödel and Łukasiewicz logics

It is remarkable that for both these types of many-valued logics corresponding algebraic semantics have mainly been developed for the infinite valued systems, and have been considered in the context of completeness proofs.

For the infinite valued Gödel logic G_∞ such a class of structures is, according to the completeness proof given by Dummett [36], the class of all *Heyting algebras*, i.e. of all relatively pseudo-complemented lattices, which satisfy the pre-linearity condition

$$(u \rightarrowtail v) \sqcup (v \rightarrowtail u) = \mathbf{1}\,. \tag{18}$$

Here $\sqcup$ is the lattice join and $\rightarrowtail$ the relative pseudo-complement.

For the infinite valued Łukasiewicz logic L_∞ the corresponding class of structures is the class of all *MV-algebras*, first introduced again within a completeness proof by Chang [25], and more recently extensively studied in [28].

And for the product logic the authors of [80] introduce a class of lattice ordered semigroups which they call *product algebras*.

It is interesting to recognize that all these structures—prelinear Heyting algebras, MV-algebras, and product algebras—are abelian lattice ordered semigroups with an additional "residuation" operation.

For the Łukasiewicz as well as for the Gödel infinite valued logics these algebraic semantics have long been considered as a mathematically nice, but logically not really important tool.

The recent development of infinite valued logics, which shall be discussed later on, beginning with Section 6, has completely modified this point of view: this development got very important stimulations from suitable algebraic semantics.

For the finite valued logics from both families, separately developed algebraic semantics did not yet find considerable interest. There was, again in the context of a completeness proof, an approach by R.S. Grigolia [72] toward MV-algebras for m-valued Łukasiewicz logics, called MV_m-algebras. [11] But these structures play only a marginal rôle in algebraic investigations toward many-valued logics.

Besides a certain minor interest in this topic, this situation may mainly be caused by the fact that the finite valued Łukasiewicz logics L_m become functionally complete after enriching their language either with truth degree constants for all truth degrees, or with suitable unary (and binary) connectives which are sufficient to characterize each one of the truth degrees. And for these enriched systems G.C. Moisil [115,116], and later on R. Cignoli [26], offered algebraic semantics. These *Lukasiewicz algebras*, however, are quite difficult structures which do not have a primitive counterpart for the Łukasiewicz implication. And therefore there is a tendency to consider them more as variants of Post algebras, as explained in Section 4.3, than as "natural" counterparts to MV-algebras.

[11] It seems, to the best of this authors knowledge, that for the finite valued Gödel logics there are no separate algebraic studies. In principle, however, Heyting-algebras with m elements should do the job for G_m.

4.2 Product logic

The product logic, as introduced in [80], was from the very beginning designed as a logic which had, in parallel, a standard semantics—provided by the real unit interval and by a product based conjunction as a fundamental connective—as well as an algebraic semantics, formed by the class of all product algebras—introduced in [80] again within a completeness proof.

We shall not explain more details here because this whole approach proved to become paradigmatic for the development of t-norm based infinite valued logics, a topic which shall be discussed later on, starting with Section 6.

4.3 Post logics

Contrary to the situation for the Łukasiewicz and the Gödel systems, for the Post systems in their original form there exist only very few syntactically oriented studies toward constituting or investigating logical calculi for these systems. Instead, for the Post systems one mainly was interested in corresponding algebraic structures, which were suitable to form an algebraic semantics, and investigated such structures earlier, and in more detail, as similar structures for the Łukasiewicz and the Gödel systems. Rosenbloom in a paper [134] of 1942 was the first one to do this. His algebraic structures shall here be called P-algebras for short—but not be considered in detail: the interested reader may e.g. consult [68].

One of the main reasons for the difficulty and complexity of the defining conditions of P-algebras is the fact that the Post systems as well as the P-algebras have only two primitive notions, their connectives resp. their basic operations, but have maximal expressive power in the sense of being functionally complete. That this choice of the primitive notions really is the main obstacle toward a simplification became clear as Epstein [42] in 1960 changed these basic operations and found a much simpler class of "essentially" these P-algebras, now called Post algebras. The reservation "essentially" here comes from the fact that formally the choice of other basic operations creates another type of algebraic structures, and it means–in more technical terms– that Post algebras and P-algebras are definitionally equivalent. We shall not go into details and refer e.g. again to [68] for details.

What are not covered by these basic considerations are possible infinite valued generalizations of these logical calculi, or of these Post algebras. Approaches toward this problem started e.g. with papers on generalizations of the notion of Post algebras like [24,39,40,144]. The most influential paper, however, which also discussed the corresponding logical systems was the paper [128] of Rasiowa in which Post algebras of order $\omega^{+}1$ and corresponding systems of infinitely many-valued (first-order) logic have been introduced. The algebraic theory of these Post algebras of order $\omega^{+}1$ is partly given in [128].

Another such infinitely many-valued generalization of the standard Post systems is discussed e.g. in [44,45], Post algebras of order $\omega + \omega^{*}$.

The Post algebras of finite or infinite order and the systems of many-valued logic related with them seem to be of particular importance for investigations in computer science, which rely on many-valued logic as a toolbox, because these Post systems are functionally complete and well suited to study the representability of truth degree functions on the basis of some predetermined set of basic truth degree functions, as determined e.g. by available electronic components, cf. [133] for a good introduction.

5 Particular Three- and Four-Valued Systems

Each system of many-valued logic which is not only intended to be some particular kind of formalism, but supposed to express some meaning, is confronted with the problem to offer a meaning for its truth degrees, or at least for its "additional" truth degrees different from the degrees 1 for "true" and 0 for "false". In the light of this problem, three-valued as well as four-valued systems get particular importance because for them only one or two "additional" truth degrees exist and need an interpretation. Hence it should not appear as a surprise that from the viewpoint of philosophically-oriented applications three- and four-valued systems assumed a more prominent role than other systems, at least as other finitely many-valued systems.

5.1 Three-Valued Systems

Here we shall be interested only in some such 3-valued systems and mainly restrict the attention to two strongly related systems introduced by Bočvar [17] and Kleene [95], and a system designed as the "true" logic of the natural language by Blau [16]. The intentions of these authors, connected with their systems, have been quite different despite some strong similarities of the systems.

The main problem of Bočvar has been the philosophical and logical analysis of logical and semantical antinomies as they appear in first-order and higher-order logic, often in connection with some lack of care e.g. in the use of the (set theoretic) comprehension principle or of metatheoretical notions, cf. e.g.

[13]. Therefore his preferred interpretation of the additional truth degree $\frac{1}{2}$ was its reading as "meaningless", "paradoxical", or "senseless". The starting point of Kleene, on the other hand, was a mathematical one and related to his research on partial recursive relations. Such relations sometimes may be undefined. Therefore in his case the intended reading of the additional truth degree $\frac{1}{2}$ was "undefined" or "undetermined". Both systems coincide in their approach to consider the truth degrees $1, 0$ just as the counterparts of the classical truth-values $\top, \bot$. Accordingly for both systems the degree 1 is the only designated one.

Bočvar subdivides his truth degree functions and thus also his connectives into *internal* and *external* ones. The characteristic property of the internal truth degree functions is that they have always a (truly: the) function value different from $0, 1$ if a argument value differs from $0, 1$. The external truth degree functions, on the other hand, map into $\{0, 1\}$. Hence this subdivision is not a (complete) classification because there obviously exist truth degree functions which are neither external nor internal ones. The the 3-valued system $\boldsymbol{B}_3$ of Bočvar has four basic connectives $\neg$, $\wedge_+$, J_0, J_1 for internal negation, internal conjunction, external negation, and external assertion. The internal negation $\neg$ is nothing but the negation of the Łukasiewicz system L_3. The internal conjunction may, again in terms of L_3, be defined as

$$\varphi \wedge_+ \psi =_{\text{def}} (\varphi \wedge \psi) \vee (\varphi \wedge \neg\varphi) \vee (\psi \wedge \neg\psi) \tag{19}$$

or by the formula

$$u \wedge_+ v =_{\text{def}} \begin{cases} \min(u, v)\,, & \text{if } u, v \in \{0, 1\}\,, \\ \frac{1}{2}\,, & \text{if } u = \frac{1}{2} \text{ or } v = \frac{1}{2}\,. \end{cases}$$

The truth degree functions of J_0, J_1 are given by

$$\mathsf{J}_0(u) = \begin{cases} 1\,, & \text{if } u = 0 \\ 0\,, & \text{otherwise} \end{cases} \qquad \mathsf{J}_1(u) = \begin{cases} 1\,, & \text{if } u = 1 \\ 0\,, & \text{otherwise} \end{cases}$$

Thus $\boldsymbol{B}_3$ is a subsystem of L_3. It is a proper subsystem because it was shown in [53] that the Łukasiewicz implication $\rightarrow_{\mathsf{L}}$ is not $\boldsymbol{B}_3$-definable.

Further internal versions of disjunction, implication and biimplication can be defined as

$$\varphi \vee_+ \psi =_{\text{def}} \neg(\neg\varphi \wedge_+ \neg\psi)\,, \tag{20}$$
$$\varphi \rightarrow_+ \psi =_{\text{def}} (\varphi \wedge_+ \neg\psi), \tag{21}$$
$$\varphi \leftrightarrow_+ \psi =_{\text{def}} (\varphi \rightarrow_+ \psi) \wedge_+ (\psi \rightarrow_+ \varphi)\,, \tag{22}$$

and get truth degree functions which coincide over $\{0, 1\}$ with their classical counterparts and are internal truth degree functions. Corresponding external versions of these connectives result by a uniform approach which e.g. for external conjunction and external disjunction reads as

$$\varphi \Cap \psi =_{\mathrm{def}} \mathsf{J}_1(\varphi) \wedge_+ \mathsf{J}_1(\psi)\,,$$
$$\varphi \Cup \psi =_{\mathrm{def}} \mathsf{J}_1(\varphi) \vee_+ \mathsf{J}_1(\psi)\,.$$

The axiomatizability problem for $\boldsymbol{B}_3$ has been discussed and solved in [52,54]. We shall not treat it here.

The 3-valued system $\boldsymbol{K}_3$ of Kleene has the so-called *strong* connectives

$$\neg, \wedge, \vee, \rightarrow_{\mathrm{K}}, \leftrightarrow_{\mathrm{K}}\ . \tag{23}$$

The first three of them are again the (equally denoted) connectives of L_3. The last two of them are characterized by the truth degree tables:

$\rightarrow_{\mathrm{K}}$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	0	$\frac{1}{2}$	1

$\leftrightarrow_{\mathrm{K}}$	0	$\frac{1}{2}$	1
0	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

Also for this system $\boldsymbol{K}_3$ the connectives $\neg, \wedge, \vee$ are just the same as in the Łukasiewicz system L_3.

This system $\boldsymbol{K}_3$ furthermore has *weak* connectives $\wedge_+, \vee_+, \rightarrow_+$ which coincide with the equally denoted connectives of the Bočvar system $\boldsymbol{B}_3$, i.e. which yield formulas which have truth degree $\frac{1}{2}$ iff one of their constituents has truth degree $\frac{1}{2}$. These weak connectives are definable from the connectives of the list (23) because of (19), (20), and (21).

Because the implication $\rightarrow_{\mathrm{K}}$ of the Kleene system is L_3-definable, e.g. by

$$\varphi \rightarrow_{\mathrm{K}} \psi =_{\mathrm{def}} (\varphi \rightarrow_{\mathsf{L}} \psi) \wedge (\varphi \vee \neg\varphi \vee \psi \vee \neg\psi)\,, \tag{24}$$

also the 3-valued Kleene system $\boldsymbol{K}_3$ is a subsystem of the Łukasiewicz system L_3. Again, however, the Łukasiewicz implication $\rightarrow_{\mathsf{L}}$ is not definable in the Kleene system and this system therefore is a proper subsystem of L_3. This undefinability follows from the fact that each (binary) connective $\vartriangle$ which is definable from the basic connectives in the list (23) has a corresponding truth degree function which assumes the value $\frac{1}{2}$ if all its arguments have this value, and hence cannot be the truth degree function of the Łukasiewicz implication.

If one combines the basic connectives of the Bočvar system with the basic connectives of the Kleene system, then $\rightarrow_{\mathsf{L}}$ becomes definable as mentioned by Šestakov [137]. Therefore neither are all the connectives of the Kleene system definable in the Bočvar system, nor conversely are all the connectives of the Bočvar system definable in the Kleene system.

Systems of many-valued propositional logic with an intended reading of the third truth degree as "meaningless", or as "undefined", or as something like, as in the Bočvar and Kleene systems, and with basic connectives which essentially can be defined in L_3, have also been investigated e.g. in [3,41,83,122,136]. In [19] a longer, but concise survey of such and other 3-valued systems is given. Additionally the interested reader should also consult [63].

It is interesting to notice that the intended reading of the third truth degree as "undefined" in the Kleene system, and the connection with the similar approach via truth value gaps which we mentioned earlier, was a substantial fact that this system was more recently considered in connection with partial, i.e. sometimes undefined, truth predicates e.g. in [73,99,111,113].

Another approach toward 3-valued systems comes from the consideration of vague predicates like "hot water", i.e. of predicates which in some cases neither really apply nor really do not apply to some objects. This effect can be modeled in different ways. A usual one is via fuzzy sets, which supported the recent investigations into infinite valued systems, as explained later on. But also 3-valued systems provide a (rough) possibility—with the third truth degree read as "neither completely applies nor fully does not apply". From this point of view three-valued logic was studied with a philosophical attitude quite early in [14,15] and more recently e.g. in [98] and also in [16,33,92,93].

It should additionally be mentioned that the phenomenon of presuppositions was discussed within the realm of truth value gaps, as well as in the realm of 3-valued – but also of 4-valued – systems. And also systems of paraconsistent logic have been discussed which can be based on finitely many truth degrees, cf. e.g. [43,138,139].

Both of these aspects, i.e. vague predicates as well as presuppositions, are covered in the use of three-valued logic for the analysis of natural language given by Blau [16]. His approach is essentially based on discussions in the realm of the philosophy of language. From these discussions he gets the fundamental motivations for the intuitive understanding of the three truth degrees. He identifies the classical truth value $\top$ with the truth degree 1, and he splits the classical truth value $\bot$ into the two degrees 0 and $\frac{1}{2}$. He takes the truth degree 0 as a modified version of the (usual) truth value $\bot$, and his intended reading for the truth degree $u = \frac{1}{2}$ is "undetermined", combined with the understanding that the appearance of this degree is caused either by the use of vague predicates, or by the use of non-denoting names, i.e. by reference to unsatisfied presuppositions.

On the propositional level his considerations are based on three primitive connectives, two kinds of negation connectives $\neg, \approx$ and a conjunction connective $\wedge$. Again $\neg$ is the Łukasiewicz negation of L_3, and $\wedge$ the weak conjunction of that system. The additional negation $\approx$ is characterized by the truth degree function

$$\approx(u) = \begin{cases} 0\,, & \text{if } u = 1\,, \\ 1\,, & \text{if } u \neq 1\,. \end{cases} \tag{25}$$

All these connectives satisfy the normal condition, hence this system is not functionally complete. However, these connectives suffice to introduce some further, interesting connectives, e.g. the Gödel negation $\sim$ as

$$\sim \varphi =_{\text{def}} \neg{\approx}\neg\varphi\,,$$

and to introduce an implication connective $\rightarrow_{\mathsf{Bl}}$ by the formula

$$\varphi \rightarrow_{\mathsf{Bl}} \psi =_{\text{def}} {\approx}\varphi \vee \psi\,.$$

The main background idea behind the choice of this implication connective is the authors claim that only this implication connective is suitable for a three-valued modeling of (two-valued) sentences of the form "All A are B" in natural language, of course read as short form for: all objects which have property A also have property B. And this claim is essentially based upon the idea that for the truth of a sentence of this form "all objects which have property A also have property B" it is completely out of any rational interest to allow the antecedent "a particular object has property A" to be undetermined. Formally this means that inside a universally quantified sentence the case that the antecedent has truth degree $\frac{1}{2}$ should not be a reason that the whole sentence may become not true.

A core intuitive point of the whole approach is the idea that natural language uses only such connectives which satisfy the normal condition, because the appearance of the truth degree $u = \frac{1}{2}$ for some sentenceφ(in some particular situation) is simply an unintended mistake caused either by the use of vague predicates, or by the use of non-denoting names.

5.2 Four-Valued Systems

In contrast to the situation with three-valued systems, where there are a lot of approaches and interpretations, only a few approaches concern four-valued systems and give particular interpretations to the four truth degrees. One of these rare exceptions is Łukasiewicz [108] who, in his later years, preferred a four-valued approach via his system L_4 toward a modal reading of the truth degrees over his original three-valued one via L_3 in [105].

However, instead of this reference to the system L_4 with its linearly ordered truth degree set $\mathcal{W}_4$, an approach toward four-valued systems has become prominent more recently which makes essential use of the natural partial ordering of the truth degree set $\mathcal{W}_4^* = \{0,1\}^2 = \{(0,0),(0,1),(1,0),(1,1)\}$, and which connects a very natural interpretation with these degrees.

This approach was inspired by theoretical work on systems of relevance logic and later on also applied to considerations on how to treat – possibly inconsistent – information in computers, e.g. in data bases or knowledge bases, like in [10,11,37,38], but also to discussions related to the liar paradox, cf. [147].

In data and knowledge bases information is stored, e.g. in the form of "facts", i.e. sentences which are marked as true or false. This information may have been collected from different sources, and at different times. One kind of use which can be made from such information is that one asks such data or knowledge bases, or the computers they are stored in, to answer some suitable questions, e.g. for (the confirmation or refutation of) simple statements concerning facts. The crucial point is that this information usually is incomplete – and often even inconsistent. Therefore one should allow a computer to answer not only "true" or "false", but also "I don't know" – and even "true and false". Of course, the answer "I don't know" indicates incompleteness of the information stored in the data or knowledge base, and the answer "true and false" indicates that this information is inconsistent – in the simplest case because contradictory facts have been stored.

For the computer these answers are just "marks" which it has to connect with sentences (which e.g. formulate the questions it is asked to answer). And with these four "values" the computer should also be able to "reason" internally, because one likes (within a bit more sophisticated applications) that the computer is not only able to repeat something that he was told before, he should also be able to connect different facts by a kind of (internal) reasoning mechanism.

Of course, as this explanation shows, this is an epistemic understanding of these four "values", and not an ontological one: the "real world" is, of course, treated here as well covered by the basic ideas of two-valued logic. This remark does not say, however, anything about the suitability of a four-valued approach based on this (epistemic) understanding of the truth degrees.

It is interesting to notice that some (preliminary) form of an ontological understanding of these four degrees appears in ancient Indian logic, e.g. in the work of Sanjaya who worked prior to the sixth century B.C. There the principle of bivalence was rejected in favor of an idea of so-called “four corners”, which correspond just to an ontological reading of these four degrees, cf. [88,127].

For simpler reading we write $\mathbf{T} = (1,0)$, $\mathbf{F} = (0,1)$, $\mathbf{N} = (0,0)$, $\mathbf{B} = (1,1)$, and understand the truth degree $\mathbf{N}$ (for “none”) as indicating “underdetermination” or a *gap*, i.e. the lack of information on a truth value, and the truth degree $\mathbf{B}$ (for “both”) as “overdetermination” or a *glut*, i.e. the presence of contradictory information on some truth value.

One has even more than only the truth degree set $\mathcal{W}_4^*$ in this case: one has a natural (partial) ordering of these degrees, having in mind that they evaluate the (computers) knowledge about the truth value of some sentence φ. Then surely the degree $\mathbf{T} = (1,0)$ is ranked in top position, and the degree $\mathbf{F} = (0,1)$ is ranked in lowest position—because it is most preferable to have a (definitely) true sentence φ, and worse to have a (definitely) false one. The two other degrees $\mathbf{N} = (0,0)$, $\mathbf{B} = (1,1)$ are ranked somehow “between” the degrees $\mathbf{T}, \mathbf{F}$ because they, in some suitable sense, if assigned to φ, leave open both possibilities that φ may “really” be true or be false. As a result, this provides the truth degree set $\mathcal{W}_4^*$ with a lattice structure as indicated in Fig. 1.

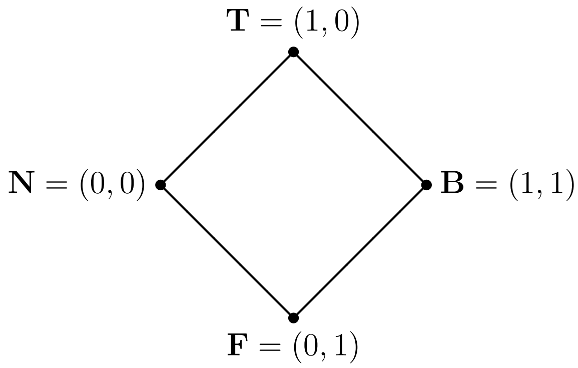

Fig. 1. Four-valued truth degree lattice $\mathcal{W}_4^*$

In this truth degree lattice $\mathcal{W}_4^*$ the lattice ordering $\leqq$ goes “bottom-up”, i.e. lattice elements which are on a lower level position are smaller ones.

With the corresponding lattice operations $\sqcap, \sqcup$ one has natural candidates for truth degree functions for a conjunction and a disjunction connective $\curlywedge, \curlyvee$ of a (propositional) system D_4 of four-valued logic which is to be based on the intuitions discussed up to now. And this choice fits even well into the intuitive picture mentioned previously.

This intuitive picture provides also the basis for the introduction of a negation connective $\neg$ into D_4, with a similar relationship to classical negation, as $\curlywedge$ and $\curlyvee$ have to classical conjunction and disjunction, respectively. This essentially means that to a formula $\neg H$ the truth value $\bot$ (or: $\top$) should be assigned if to φ the truth value $\top$ (or: $\bot$) is assigned. So one immediately has for $\neg$ the truth degree table given in Fig. 2.

φ	**F**	**N**	**B**	**T**
$\neg H$	**T**	**N**	**B**	**F**

Fig. 2. Truth table characterization of the D_4-negation

These connectives $\curlywedge, \curlyvee, \neg$ form the basic vocabulary of D_4.

One can also prove that the two three-valued subsystems of D_4, which are constituted by the restrictions of the truth degrees to $\{\mathbf{T}, \mathbf{F}, \mathbf{N}\}$ or $\{\mathbf{T}, \mathbf{F}, \mathbf{B}\}$, coincide as three-valued systems, and are subsystems of L_3.

The crucial point now is to define a suitable entailment relation $\models_{\mathsf{D}}$ which fits well into this intuitive realm.

A very natural approach seems to be, to say again that a set of formulas Σ entails a formula φ iff each model of Σ is also a model of φ. However, what shall we understand by a model of a set Σ of formulas? Well, nothing but some valuation which gives to all the formulas of Σ a designated truth degree.

So the problem arises what the designated truth degrees should be. Up to now there is no agreement on this point. One of the possible approaches is to take only $\mathbf{T}$ as a designated truth degree, i.e. to put $\mathcal{D}^{\mathsf{D}} = \{\mathbf{T}\}$.

In this case the resulting notion of model fits well into the background intuition: for our computer a model of a set of formulas should be any (partial and non-functional) two-valued valuation which makes all the formulas of Σ definitely true – with "definitely true" understood here as meaning *true but not false or valueless*. And this represents the standard notion of a model. Thus we get in this case according to (7):

$$\Sigma \models_{\mathsf{D}} H \;\Leftrightarrow\; \mathsf{Mod}^{\mathsf{D}}(\Sigma) \subseteq \mathsf{Mod}^{\mathsf{D}}(H)\,, \tag{26}$$

with the notion of model class defined as in (6).

With applications to relevance logic in mind, Dunn [37] considers instead both truth degrees $\mathbf{T}, \mathbf{B}$ as designated. The intuition behind this choice is that a formula which has such a designated truth degree is considered as "at least true".[12] Problems of definability of truth degree functions, i.e. of connectives,

[12] This type of four-valued semantics may be used to give adequate semantical interpretations for different systems of relevance logic. We will not discuss details here. The interested reader may e.g. consult [125,130].

and of relations between truth degrees for this choice of designated truth degrees have been discussed e.g. in [4,126].

The models in the sense of [37] for this situation are just the ($\geqq \mathbf{B}$)-models, with $\leqq$ for the lattice ordering of the truth degree lattice $\mathcal{W}_4^*$. Let us call them *weak models* for the moment, and denote the class of all weak models of some set Σ of formulas by $\mathsf{Mod}_0^{\mathsf{D}}(\Sigma)$.

Then Dunn discusses the following notion $\models_{\mathsf{D}}^0$ of *weak entailment*:

$$\Sigma \models_{\mathsf{D}}^0 H \;\Leftrightarrow\; \mathsf{Mod}_0^{\mathsf{D}}(\Sigma) \subseteq \mathsf{Mod}_0^{\mathsf{D}}(H)\,,$$

which is more general than the previous notion from (26), as the following result shows.

Proposition 2 *For each set Σ of formulas of D_4 and each formula φ one has*

$$\Sigma \models_{\mathsf{D}}^0 H \quad \Rightarrow \quad \Sigma \models_{\mathsf{D}} H\,.$$

Still another notion of entailment $\models_{\mathsf{D}}^*$ is taken into account in [10,11]. There Belnap considers however only entailment relationships of the form $\varphi \models_{\mathsf{D}}^* H$. This corresponds for the two former cases to a restriction to finite sets Σ of formulas. The definition of $\models_{\mathsf{D}}^*$ is the following:

$$\varphi \models_{\mathsf{D}}^* \psi \quad =_{\mathrm{def}} \quad \mathsf{Val}^{\mathsf{D}}(\varphi, \beta) \leqq \mathsf{Val}^{\mathsf{D}}(\psi, \beta) \text{ for all valuations } \beta.$$

It is immediately clear that one has for all formulas φ, ψ of D_4:

$$\varphi \models_{\mathsf{D}}^* \psi \quad \Rightarrow \quad \varphi \models_{\mathsf{D}}^0 \psi\,.$$

For this notion of entailment Belnap gives in [10,11] a finite list of principles claiming that it is a complete list to infer all the valid entailments. We shall not give this list here, the interested reader may consult [10] or [11].

Instead we take into consideration another aspect which in a natural way is connected with the epistemic understanding of the truth degree set $\mathcal{W}_4^* = \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$. The lattice ordering of $\mathcal{W}_4^*$, symbolized in Fig. 1, was based on the intuition that the "larger" elements are the better ones in the sense that the best possible case is definite truth, and the worst possible case is definite falsehood. Hence this lattice ordering $\leqq$ is a kind of *truth ordering*.

This truth ordering can be contrasted with a *knowledge ordering* which counts "complete" lack of knowledge of the truth value of some sentenceφas the worst case, and which prefers "more complete" knowledge. In some sense, this point of view transforms $\mathcal{W}_4^*$ into a set of knowledge degrees. From this point of view, another lattice ordering $\sqsubseteq$, and hence another lattice structure, becomes interesting which may be symbolized by the diagram of Fig. 3.

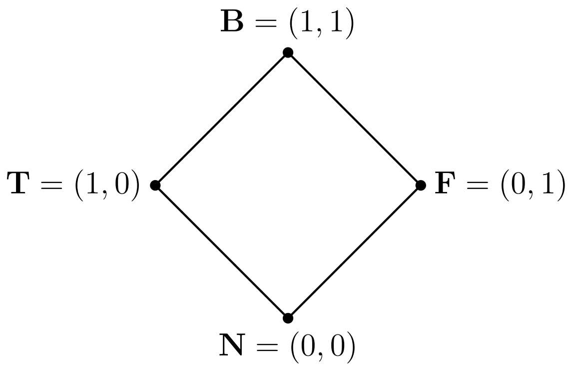

Fig. 3. Four-valued knowledge degree lattice $\mathcal{W}_4^*$

Again here, the lattice ordering $\sqsubseteq$ goes "bottom-up". Of course, the diagram of Fig. 3 is just the diagram of Fig. 1 "turned around", which also means that the knowledge ordering $\sqsubseteq$ goes "left-to-right" in the diagram of Fig. 1.

From an algebraic point of view the truth degree set $\mathcal{W}_4^*$ with these two lattice structures becomes a particular case of a *bilattice*. This is a type of structure which more recently has been introduced in lattice theory, and which seems to be of particular interest for applications in logic and computer science, cf. e.g. [6,56,57,61,62].

6 Logics with T-Norm Based Connectives

From now on we shall restrict our considerations to the case of infinite valued logics, even to logics with the real unit interval (3) as truth degree set. The main reason for this decision is that the recent development of many-valued logics has its focus on the development of such infinite valued systems. The core motivation for the emphasis on this type of systems is that they count as the background logics for the theory of fuzzy sets, as introduced by Zadeh [150]. And fuzzy sets have become an important tool in knowledge engineering as well as in artificial intelligence, hence are highly application relevant.

The fundamental infinite valued logics from Section 3 look quite different if one has in mind the form in which they first were presented.

Fortunately, however, there is a common generalization which allows to present all these three logics in a uniform way. In this uniform presentation one of the conjunction connectives becomes a core role: $\wedge$ in the system G_∞, & in the system L_∞, and $\odot$ in the system Π.

But this uniform generalization covers a much larger class of infinite valued logics over $[0,1]$: the core conjunction connective—which shall now in general be denoted &—has only to have a truth degree function $\otimes$ which, as a binary operation in the real unit interval, should be an associative, commutative, and isotonic operation which has 1 as a neutral element, i.e. should satisfy for arbitrary $x, y, z \in [0,1]$:

(T1) $x \otimes (y \otimes z) = (x \otimes y) \otimes z$,
(T2) $x \otimes y = y \otimes x$,
(T3) if $x \leq y$ then $x \otimes z \leq y \otimes z$,
(T4) $x \otimes 1 = x$.

Such binary operations are known as *t-norms* and have been used in the context of probabilistic metric spaces, cf. e.g. [97]. At the same time they are considered as natural candidates for truth degree functions of conjunction connectives. And from such a t-norm one is able to derive (essentially) all the other truth degree functions for further connectives.

The minimum operation $u \wedge v$ from (8), the Łukasiewicz arithmetic conjunction $u \,\&\, v$ from (12), and the ordinary product are the best known examples of t-norms.

In algebraic terms, such a t-norm $\otimes$ makes the real unit interval into an *ordered monoid*, i.e. into an abelian semigroup with unit element. And this ordered monoid is even *integral*, i.e. its unit element is at the same time the universal upper bound of the ordering. Additionally this monoid has because of

$$0 \otimes x \leq 0 \otimes 1 = 0 \tag{27}$$

the number 0 as an *annihilator*.

Starting from a t-norm $\otimes$ one finds a truth degree function $\rightarrowtail$ for an implication connective via the *adjointness condition*

$$x \otimes z \leq y \quad \Longleftrightarrow \quad z \leq (x \rightarrowtail y)\,. \tag{28}$$

However, to guarantee that this adjointness condition (28) determines the operation $\rightarrowtail$ uniquely, one has to suppose that the t-norm $\otimes$ is a *left continuous* function in both arguments. Indeed, the adjointness condition (28) is equivalent to the condition that $\otimes$ is left continuous in both arguments, cf. [68].

Instead of this adjointness condition (28) one could equivalently either give the direct definition

$$x \rightarrowtail y = \sup\{z \mid x \otimes z \leq y\} \tag{29}$$

of the *residuation operation* $\rightarrowtail$, or one could force the t-norm $\otimes$ to have the *sup-preservation property*

$$\sup_{i\to\infty}(x_i \otimes y) = (\sup_{i\to\infty} x_i) \otimes y \tag{30}$$

for each $y \in [0,1]$ and each non-decreasing sequence $(x_i)_{i\to\infty}$ from the real unit interval.

In this framework one additionally introduces a further unary operation $-$ by

$$-x =_{\mathrm{df}} x \rightarrowtail 0\,, \tag{31}$$

and considers this as the truth degree function of a negation connective. That this works also in the formalized language of the corresponding system of logic forces to introduce into this language a truth degree constant $\overline{0}$ to denote the truth degree zero.

And finally one likes to have the weak conjunction and disjunction connectives $\wedge, \vee$ available. These connectives should also be added to the vocabulary. However, it suffices to add only the min-conjunction $\wedge$, because then for each left continuous t-norm $\otimes$ and its residuated implication $\rightarrowtail$ one has, completely similar to the situation (17) in the product logic,

$$u \vee v = ((u \rightarrowtail v) \rightarrowtail v) \wedge ((v \rightarrowtail u) \rightarrowtail u)\,. \tag{32}$$

All these considerations lead in a natural way to algebraic structures which, starting from the unit interval, consider a left continuous t-norm $\otimes$ together with its residuation operation $\rightarrowtail$, with the minimum-operation $\wedge$, and the maximum operation $\vee$ as basic operations of such an algebraic structure, and with the particular truth degrees $0,1$ as fixed objects (i.e. as nullary operations) of the structure. Such an algebraic structure

$$\langle [0,1], \wedge, \vee, \otimes, \rightarrowtail, 0, 1\rangle \tag{33}$$

shall be coined to be a *t-norm algebra*.

7 Residuated Implications versus S-Implications

With the basic properties of classical logic in mind, particularly because of the logical equivalence of the formulas

$$\varphi \to \psi \qquad \text{and} \qquad \neg\varphi \vee \psi \qquad \text{and} \qquad \neg(\varphi \wedge \neg\psi) \tag{34}$$

in classical logic, the introduction of the implication connective directly as residuation via (29) or via the adjointness condition (28) seems to be quite sophisticated, and perhaps unnecessarily complicated.

If one has in mind an arrow-free approach similar to (34) for the present case, one would have to start for a definition of an implication connective either with a generalized disjunction or with a generalized conjunction, and had to add in any case a generalized negation.

In this case the implication, defined according to one of the equivalences in (34), is often coined "S-implication".

This has been done e.g. in [22,86], but this approach does not really become simpler as the former one because one needs to fix either, besides the basic t-norm, an additional negation, or one has to fix a negation together with a disjunction.

However, the main disadvantage of such a modification is that one looses a quite natural strong soundness property one has for the rule of detachment in the former t-norm based approach. From the adjointness condition (28) one has always

$$u \otimes (u \rightarrowtail v) \le v \tag{35}$$

simply because of

$$u \otimes (u \rightarrowtail v) \le v \quad \text{iff} \quad (u \rightarrowtail v) \otimes u \le v \quad \text{iff} \quad u \rightarrowtail v \le u \rightarrowtail v\,.$$

Therefore one has, in the t-norm based approach with implication defined as residuation, a natural lower bound for the truth degree of a formula ψ which has been derived from $\varphi \to \psi$ and φ via the rule of detachment: the truth degree of the formula $\varphi \,\&\, (\varphi \to \psi)$.

A similar property is lacking in general for the approach via S-implications. In this case, say starting from a t-norm $\otimes$ and a negation $\neg\!\!\!\rightharpoondown$, the corresponding property to (35) would be

$$u \otimes \neg\!\!\!\rightharpoondown (u \otimes \neg\!\!\!\rightharpoondown v) \le v\,. \tag{36}$$

But this fails already in the case that, independent of the choice of the t-norm $\otimes$, the negation $\neg\!\!\!\rightharpoondown$ is the Gödel negation $\sim$ of (9), i.e. the common negation of the systems G and Π. For this negation and any $v > 0$ one has $\sim v = 0$, hence $u \otimes \sim v = 0$, which means $\sim (u \otimes \sim v) = 1$ and $u \otimes \sim (u \otimes \sim v) = u$. Now one may choose $u > v$ to see that (36) fails.

8 Continuous T-Norms

Among the large class of all t-norms the continuous ones are the best understood. A t-norm is continuous iff it is continuous as a real function of two variables, or equivalently, iff it is continuous in each argument (with the other one as a parameter), cf. [68,97].

Furthermore, all continuous t-norms are ordinal sums of only three of them: the Łukasiewicz arithmetic t-norm $u \,\&\, v$ from (12), the ordinary product t-norm, and the minimum operation $u \wedge v$. The definition of an ordinal sum of t-norms is the following one.

Definition 3 *Suppose that* $([a_i, b_i])_{i \in I}$ *is a countable family of non-overlapping proper subintervals of the unit interval* $[0,1]$*, let* $(\boldsymbol{t}_i)_{i \in I}$ *be a family of t-norms, and let* $(\varphi_i)_{i \in I}$ *be a family of mappings such that each* φ_i *is an order isomorphism from* $[a_i, b_i]$ *onto* $[0,1]$*. Then the (generalized)* ordinal sum *of the combined family* $(([a_i, b_i], \boldsymbol{t}_i, \varphi_i))_{i \in I}$ *is the binary function* $T : [0,1]^2 \to [0,1]$ *characterized by*

$$T(u,v) = \begin{cases} {\varphi_k}^{-1}(\boldsymbol{t}_k(\varphi_k(u), \varphi_k(v)), & \textit{if } u, v \in [a_k, b_k] \\ \min\{u, v\} & \textit{otherwise.} \end{cases} \tag{37}$$

Often it is helpful to visualize the construction of an ordinal sum. For a simple case which shows some of the interval summands this is done in Fig. 4.

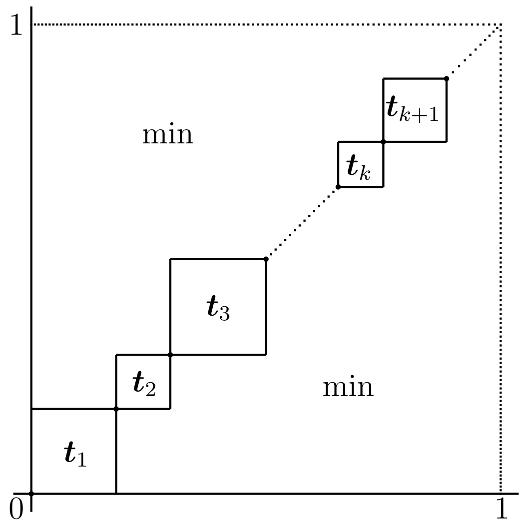

Fig. 4. The basic construction of an ordinal sum

It is easy to see that an order isomorphic copy of the minimum t-norm is again the minimum operation. Thus the whole construction of ordinal sums of t-norms even allows to assume that the summands are formed from t-norms different from the minimum t-norm. This detail, however, shall be inessential for the present considerations.

But it should be mentioned that all the endpoints a_i, b_i of the interval family $([a_i, b_i])_{i\in I}$ give *idempotents* of the resulting ordinal sum t-norm T:

$$T(a_i, a_i) = a_i\,, \quad T(b_i, b_i) = b_i \quad \text{for all } i \in I.$$

Conversely, if one knows all the idempotents of a given continuous t-norm $\boldsymbol{t}$, i.e. all $u \in [0,1]$ with $\boldsymbol{t}(u,u) = u$, then one is able to give a representation of $\boldsymbol{t}$ as an ordinal sum, as explained again in [97].

The general result, given e.g. in [68,97], reads as follows.

Theorem 4 *Each continuous t-norm* $\boldsymbol{t}$ *is the* (*generalized*) *ordinal sum of* (*isomorphic*) *copies of the Łukasiewicz t-norm, the product t-norm, and the minimum t-norm.*

As was mentioned in Section 3, the t-norm based logics which are determined by these three t-norms are well known and adequately axiomatized.

Therefore one is interested to find adequate axiomatizations also for further continuous t-norms. A global solution of this problem, i.e. a solution which did not only cover some few particular cases, appeared as quite difficult. Therefore, instead, one first has been interested to find all those formulas of the language of t-norm based systems which are logically valid in each one of these logics.

There seems to be a natural way to get an algebraic semantics for these considerations: the class of all t-norm algebras with a continuous t-norm should either form such an algebraic semantics, or should be a constitutive part—preferably a generating set—of a variety of algebraic structures which form such an algebraic semantics.

However, there seems to be an inadequacy in the description of this algebraic semantics: on the one hand the notion of t-norm algebra is a purely algebraic notion, the notion of continuity of a t-norm on the other hand is an analytical one. Fortunately, there is a possibility to give an algebraic characterization for the continuity of t-norms. It needs a further notion.

Definition 5 *A t-norm algebra* $\langle [0,1], \wedge, \vee, \otimes, \rightarrowtail, 0, 1\rangle$ *is* divisible *iff one has for all* $a, b \in L$:

$$a \wedge b = a \otimes (a \rightarrowtail b)\,. \tag{38}$$

And this notion gives the algebraic counterpart for the continuity, as shown e.g. in [68,97].

Proposition 6 *A t-norm algebra* $\langle [0,1], \wedge, \vee, \otimes, \rightarrowtail, 0, 1\rangle$ *is divisible iff the t-norm* $\otimes$ *is continuous.*

9 The Logic of Continuous T-Norms

The class of t-norm algebras (with a continuous t-norm or not) is not a variety: it is not closed under direct products because each t-norm algebra is linearly ordered, but the direct products of linearly ordered structures are not linearly ordered, in general. Hence one may expect that it would be helpful for the development of a logic of continuous t-norms to extend the class of all divisible t-norm algebras in a moderate way to get a variety. And indeed this idea works: it was developed by P. Hájek and in detail explained in [75].

The core points are that one considers instead of the divisible t-norm algebras, which are linearly ordered integral monoids as mentioned previously, now lattice ordered integral monoids which are divisible, which have an additional residuation operation connected with the semigroup operation via an adjointness condition like (28), and which satisfy a pre-linearity condition like (18). These structures have been called BL-algebras; they are completely defined in the following way.

Definition 7 *A* BL-algebra *is an algebraic structure*

$$\mathbf{L} = \langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1}\rangle \tag{39}$$

with four binary operations and two constants such that

(*i*) $(L, \vee, \wedge, \mathbf{0}, \mathbf{1})$ *is a bounded lattice, i.e. has* $\mathbf{0}$ *and* $\mathbf{1}$ *as the universal lower and upper bounds w.r.t. the lattice ordering* $\leq$,

(*ii*) $(L, *, \mathbf{1})$ *is an abelian monoid, i.e. a commutative semigroup with unit* $\mathbf{1}$ *such that the multiplication* $*$ *is associative, commutative and satisfies* $\mathbf{1} * x = x$ *for all* $x \in L$,

(*iii*) *the binary operations* $*$ *and* $\rightarrow$ *form an adjoint pair, i.e. satisfy for all* $x, y, z \in L$ *the* adjointness *condition*

$$z \leq (x \rightarrow y) \iff x * z \leq y, \tag{40}$$

(*iv*) *and moreover, for all* $x, y \in L$ *one has satisfied the* pre-linearity *condition*

$$(x \rightarrow y) \vee (y \rightarrow x) = \mathbf{1} \tag{41}$$

as well as the divisibility *condition*

$$x * (x \rightarrow y) = x \wedge y\,. \tag{42}$$

The axiomatization of Hájek [75] for the *basic t-norm logic* BL (in [68] denoted BTL), i.e. for the class of all well-formed formulas which are valid in all BL-algebras, is given in a language $\mathcal{L}_T$ which has as basic vocabulary the connectives $\rightarrow, \&$ and the truth degree constant $\overline{0}$, taken in each BL-algebra $\langle L, \cap, \cup, *, \rightarrowtail, 0, 1\rangle$ as the operations $\rightarrowtail, *$ and the element 0. Then this t-norm based logic has as axiom system $\mathsf{Ax}_{\mathsf{BL}}$ the following schemata:

$(\mathsf{Ax}_{\mathsf{BL}}1)$ $\quad (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))\,,$
$(\mathsf{Ax}_{\mathsf{BL}}2)$ $\quad \varphi \,\&\, \psi \rightarrow \varphi\,,$
$(\mathsf{Ax}_{\mathsf{BL}}3)$ $\quad \varphi \,\&\, \psi \rightarrow \psi \,\&\, \varphi\,,$
$(\mathsf{Ax}_{\mathsf{BL}}4)$ $\quad (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \,\&\, \psi \rightarrow \chi)\,,$
$(\mathsf{Ax}_{\mathsf{BL}}5)$ $\quad (\varphi \,\&\, \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))\,,$
$(\mathsf{Ax}_{\mathsf{BL}}6)$ $\quad \varphi \,\&\, (\varphi \rightarrow \psi) \rightarrow \psi \,\&\, (\psi \rightarrow \varphi)\,,$
$(\mathsf{Ax}_{\mathsf{BL}}7)$ $\quad ((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)\,,$
$(\mathsf{Ax}_{\mathsf{BL}}8)$ $\quad \overline{0} \rightarrow \varphi\,,$

and has as its (only) inference rule the rule of detachment, or: modus ponens (w.r.t. the implication connective $\rightarrow$).

The logical calculus which is constituted by this axiom system and its inference rule, and which has the standard notion of derivation, shall be denoted by $\mathbb{K}_{\mathsf{BL}}$ or just by BL. (Similarly in other cases.)

Starting from the primitive connectives $\rightarrow, \&$ and the truth degree constant $\overline{0}$, the language $\mathcal{L}_T$ of BL is extended by definitions of additional connectives $\wedge, \vee, \neg$:

$$\varphi \wedge \psi =_{\mathrm{df}} \varphi \,\&\, (\varphi \rightarrow \psi)\,, \tag{43}$$
$$\varphi \vee \psi =_{\mathrm{df}} ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)\,, \tag{44}$$
$$\neg\varphi =_{\mathrm{df}} \varphi \rightarrow \overline{0}\,, \tag{45}$$

where φ, ψ are formulas of the language of that system.

Calculations (in BL-algebras) show that the additional connectives $\wedge, \vee$ just have the lattice operations $\cap, \cup$ as their truth degree functions.

It is a routine matter, but a bit tedious, to check that this logical calculus $\mathbb{K}_{\mathsf{BL}}$, usually called the axiomatic system BL, is sound, i.e. derives only such formulas which are valid in all BL-algebras. A proof is given in [75], together with a proof of a corresponding completeness theorem.

Corollary 8 *The Lindenbaum algebra of the axiomatic system* BL *is a BL-algebra.*

Theorem 9 (General Completeness) *A formula* φ *of the language* $\mathcal{L}_T$ *is* *derivable within the axiomatic system* BL *iff* φ *is valid in all BL-algebras.*

The proof method yields that each BL-algebra is (isomorphic to) a subdirect product of linearly ordered BL-algebras, i.e. of BL-chains. Thus it allows a nice modification of the previous result.

Corollary 10 (General Completeness; Version 2) *A formula* φ *of* $\mathcal{L}_T$ *is derivable within the axiomatic system* BL *iff* φ *is valid in all BL-chains.*

But even more is provable and leads back to the starting point of the whole approach: the logical calculus $\mathbb{K}_{\mathsf{BL}}$ characterizes just those formulas which hold true w.r.t. all divisible t-norm algebras. This was proved in [27].

Theorem 11 (Standard Completeness) *The class of all formula which are provable in the system* BL *coincides with the class of all formulas which are logically valid in all t-norm algebras with a continuous t-norm.*

The main steps in the proof are to show (i) that each BL-algebra is a subdirect product of subdirectly irreducible BL-chains, i.e. of linearly ordered BL-algebras which are not subdirect products of other BL-chains, and (ii) that each subdirectly irreducible BL-chain can be embedded into the ordinal sum of some BL-chains which are either trivial one-element BL-chains, or linearly ordered MV-algebras, or linearly ordered product algebras, such that (iii) each such ordinal summand is locally embedable into a t-norm based residuated lattice with a continuous t-norm, cf. [27,74] and again [68].

This is a lot more of algebraic machinery as necessary for the proof of the General Completeness Theorem 9 and thus offers a further indication that the extension of the class of divisible t-norm algebras to the class of BL-algebras made the development of the intended logical system easier. But even more can be seen from this proof: *the class of BL-algebras is the smallest variety which contains all the divisible t-norm algebras.* And the algebraic reason for this is that each variety may be generated from its subdirectly irreducible elements, cf. again [23,34,71].

And another generalization of Theorem 9 deserves to be mentioned. To state it, let us call *schematic extension* of BL every extension which consists in an addition of finitely many axiom schemata to the axiom schemata of BL. And let us denote such an extension by $\mathsf{BL}(\mathcal{C})$. And call BL($\mathcal{C}$)-algebra each BL-algebra $\mathbf{A}$ which makes $\mathbf{A}$-valid all formulas of $\mathcal{C}$.

Then one can prove, as done in [75], an even more general completeness result.

Theorem 12 (Extended General Completeness) *For each finite set* $\mathcal{C}$ *of axiom schemata and any formula* φ *of* $\mathcal{L}_T$ *there are equivalent:*

(i) φ *is derivable within* $\mathsf{BL}(\mathcal{C})$*;*
(ii) φ *is valid in all* $\mathsf{BL}(\mathcal{C})$*-algebras;*
(iii) φ *is valid in all* $\mathsf{BL}(\mathcal{C})$*-chains.*

The extension of these considerations to the first-order case is also given in [75], but shall not be discussed here.

But the algebraic machinery allows even deeper insights. After some particular results e.g. in [84,85], the study of such subvarieties of the variety of all BL-algebras which are generated by single t-norm algebras $\langle [0,1], \wedge, \vee, \otimes, \rightarrowtail, 0, 1\rangle$ with a continuous t-norm $\otimes$ led to (finite) axiomatizations of those t-norm based logics which have a standard semantics determined just by this continuous t-norm algebra. These results have recently been presented in [51].

10 The Logic of Left Continuous T-Norms

The guess of Esteva/Godo [47] has been that one should arrive at the logic of left continuous t-norms if one starts from the logic of continuous t-norms and deletes the continuity condition, i.e. the divisibility condition (38).

The algebraic approach needs only a small modification: in Definition 7 of BL-algebras one has simply to delete the divisibility condition (42). The resulting algebraic structures have been called *MTL-algebras*. They again form a variety.

Following this idea, one has to modify the previous axiom system in a suitable way. And one has to delete the definition (43) of the connective $\wedge$, because this definition (together with suitable axioms) essentially codes the divisibility condition. The definition (44) of the connective $\vee$ remains unchanged.

As a result one now considers a new system MTL of mathematical fuzzy logic, characterized semantically by the class of all MTL-algebras. It is connected with the axiom system

$(\mathsf{Ax}_{\mathsf{MTL}}1)$ $\quad (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))\,,$
$(\mathsf{Ax}_{\mathsf{MTL}}2)$ $\quad \varphi \,\&\, \psi \to \varphi\,,$
$(\mathsf{Ax}_{\mathsf{MTL}}3)$ $\quad \varphi \,\&\, \psi \to \psi \,\&\, \varphi\,,$
$(\mathsf{Ax}_{\mathsf{MTL}}4)$ $\quad (\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)\,,$
$(\mathsf{Ax}_{\mathsf{MTL}}5)$ $\quad (\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))\,,$
$(\mathsf{Ax}_{\mathsf{MTL}}6)$ $\quad \varphi \wedge \psi \to \varphi\,,$
$(\mathsf{Ax}_{\mathsf{MTL}}7)$ $\quad \varphi \wedge \psi \to \psi \wedge \varphi\,,$
$(\mathsf{Ax}_{\mathsf{MTL}}8)$ $\quad \varphi \,\&\, (\varphi \to \psi) \to \varphi \wedge \psi\,,$
$(\mathsf{Ax}_{\mathsf{MTL}}9)$ $\quad \overline{0} \to \varphi\,,$
$(\mathsf{Ax}_{\mathsf{MTL}}10)$ $\quad ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)\,,$

together with the rule of detachment (w.r.t. the implication connective $\rightarrow$) as (the only) inference rule.

It is a routine matter, but again tedious, to check that this logical calculus $\mathbb{K}_{\mathsf{MTL}}$ is sound, i.e. derives only such formulas which are valid in all MTL-algebras.

Corollary 13 *The Lindenbaum algebra of the logical calculus $\mathbb{K}_{\mathsf{MTL}}$ is an MTL-algebra.*

Proofs of this result and also of the following completeness theorem are given in [47].

Theorem 14 (General Completeness) *A formula φ of the language $\mathcal{L}_T$ is derivable within the logical calculus $\mathbb{K}_{\mathsf{MTL}}$ iff φ is valid in all MTL-algebras.*

Again the proof method yields that each MTL-algebra is (isomorphic to) a subdirect product of linearly ordered MTL-algebras, i.e. of MTL-chains.

Corollary 15 (General Completeness; Version 2) *A formula φ of $\mathcal{L}_T$ is derivable within the axiomatic system MTL iff φ is valid in all MTL-chains.*

And again, similar as for the BL-case, even more is provable: the logical calculus $\mathbb{K}_{\mathsf{MTL}}$ characterizes just these formulas which hold true w.r.t. all those t-norm based logics which are determined by a left continuous t-norm. A proof is given in [90].

Theorem 16 (Standard Completeness) *The class of all formulas which are provable in the logical calculus $\mathbb{K}_{\mathsf{MTL}}$ coincides with the class of all formulas which are logically valid in all t-norm algebras with a left continuous t-norm.*

This result again means, as the similar one for the logic of continuous t-norms, that the variety of all MTL-algebras is the smallest variety which contains all t-norm algebras with a left continuous t-norm.

Because of the fact that the BL-algebras are the divisible MTL-algebras, one gets another adequate axiomatization of the basic t-norm logic BL if one extends the axiom system $\mathbb{K}_{\mathsf{MTL}}$ with the additional axiom schema

$$\varphi \wedge \psi \rightarrow \varphi \,\&\, (\varphi \rightarrow \psi)\,. \tag{46}$$

The simplest way to prove that this implication is sufficient is to show that the inequality $x * (x \rightarrowtail y) \leq x \cap y$, which corresponds to the converse implication, holds true in each MTL-algebra. Similar remarks apply to further extensions of MTL we are going to mention.

Also for MTL an extended completeness theorem similar to Theorem 12 remains true.

Theorem 17 (Extended General Completeness) *For each finite set $\mathcal{C}$ of axiom schemata and any formula φ of $\mathcal{L}_T$ the following are equivalent:*

(i) φ is derivable within the logical calculus $\mathbb{K}_{\mathsf{MTL}} + \mathcal{C}$;
(ii) φ is valid in all MTL($\mathcal{C}$)-algebras;
(iii) φ is valid in all MTL($\mathcal{C}$)-chains.

Again the extension to the first-order case is similar to the treatment in [75] for BL and shall not be discussed here.

11 Some Generalizations

The standard approach toward t-norm based logics, as explained in Sections 9 and 10, has been modified in various ways. The main background ideas are the extension or the modification of the expressive power of these logical systems.

A first, quite fundamental addition to the standard vocabulary of the languages of t-norm based systems was proposed in [5]: a unary propositional operator $\triangle$ which has for t-norm algebras the semantics

$$\triangle(x) = 1 \quad \text{for } x = 1\,, \qquad \triangle(x) = 0 \quad \text{for } x \neq 1\,. \tag{47}$$

This unary connective can be added to the systems BL and MTL via the additional axioms

(Δ1)	$\triangle\varphi \vee \neg\triangle\varphi\,,$
(Δ2)	$\triangle(\varphi \vee \varphi) \rightarrow (\triangle\varphi \vee \triangle\psi)\,,$
(Δ3)	$\triangle\varphi \rightarrow \varphi\,,$
(Δ4)	$\triangle\varphi \rightarrow \triangle\triangle\varphi\,,$
(Δ5)	$\triangle(\varphi \rightarrow \psi) \rightarrow (\triangle\varphi \rightarrow \triangle\psi)\,.$

This addition leaves all the essential theoretical results, like correctness and completeness theorems, valid: of course w.r.t. suitably expanded algebraic structures.

A second stream of papers discusses the addition of an idempotent negation, i.e. a negation which satisfies the double negation law, for those cases where the standard negation of the t-norm based system is not idempotent. This is e.g. the case for the product logic which, as explained at the end of Subsection 3.3, has the Gödel negation (9) as its standard negation. By the way, it should be noticed that (routine calculations show that) this non-idempotent Gödel negation is the standard negation of all those t-norm algebras with a t-norm $\otimes$ which does not have zero-divisors.[13] A very general approach is given in [48], and a more particular axiomatization problem discussed in [69].

Another stream of papers, partly related to the previously mentioned one, is devoted to the problem of a unified treatment of different, usually two, t-norms and their related connectives within one logical system. Here the focus is on the join of the systems based upon the Łukasiewicz t-norm and upon the product t-norm. The great advantage of this unification is that the Łukasiewicz t-norm essentially allows to treat the addition, as may be seen from the truth degree function (12) of the Łukasiewicz (arithmetical) disjunction, and that the product t-norm adds the treatment of the usual product: and this means that the elementary arithmetic (in the unit interval) can be discussed in this combined system. This combined system has been considered in two strongly related forms, denoted $\text{Ł}\Pi$ and $\text{Ł}\Pi\frac{1}{2}$. The distinction between both systems is that $\text{Ł}\Pi$ has both t-norms $\&$ and $\odot$ and their related (residual) implications and negations among their basic connectives, and that $\text{Ł}\Pi\frac{1}{2}$ adds a truth degree constant for the truth degree $\frac{1}{2}$. These two systems are discussed in detail in [29–31,46,50].

A fourth stream of papers intends to weaken the systems BL and MTL in such a way that one deletes the explicit reference to the truth degree constant $\overline{0}$ and considers the *falsity free* fragments of the previous systems. From the algebraic point of view their characteristic structures become the *hoops* which in general are defined as algebraic structures $H = \langle H, *, \Rightarrow, \mathbf{1}\rangle$ such that $\langle H, *, \mathbf{1}\rangle$ is an abelian monoid and that the further binary operation $\Rightarrow$ satisfies the equations

$$\begin{aligned} x \Rightarrow x \;&=\; \mathbf{1}\,, \\ x * (x \Rightarrow y) \;&=\; y * (y \Rightarrow x)\,, \\ (x * y) \Rightarrow z \;&=\; x \Rightarrow (y \Rightarrow z)\,. \end{aligned}$$

The definition

$$x \sqsubseteq y \;=_{\text{def}}\; x \Rightarrow x = \mathbf{1}$$

provides an ordering $\sqsubseteq$ with universal upper bound $\mathbf{1}$ which makes $\langle H, *, \mathbf{1}\rangle$ an ordered monoid, and which has the additional property that the operations $*, \Rightarrow$ become an adjoint pair w.r.t. this ordering.

In particular, hoops with the additional property

$$x \Rightarrow (y \Rightarrow z) \;\sqsubseteq\; (y \Rightarrow (x \Rightarrow z)) \Rightarrow z$$

can in a natural way be generated from t-norm algebras with continuous t-norms, as has been shown in [1]. So one has a kind of competing generalization of t-norm algebras. And for this kind of algebraic semantics one can find

[13] Zero-divisors of a t-norm $\boldsymbol{t}$ are such reals $0 < u, v < 1$ for which $\boldsymbol{t}(u, v) = 0$ holds.

adequate axiomatizations for corresponding hoop logics quite similar to the approaches of Sections 9 and 10. The details have been developed in [49].

And a fifth stream discusses the generalization of the algebraic semantics from the case of abelian lattice ordered monoids with residuation to the case of non-commutative lattice ordered semigroups. In this context one tries to define non-commutative BL-algebras or non-commutative MTL-algebras, and similarly defines non-commutative t-norms, also called pseudo-t-norms. And these considerations become combined with the design of an adequate axiomatization, with similar results as in Sections 9 and 10. The most important ones of these papers are [35,58,76–78,91,100].

And finally it should be mentioned that HÁJEK [79] even gives a common generalization of all of these generalized fuzzy logics, thus giving up divisibility, the falsity constant, and commutativity. The corresponding algebras are called *fleas* (or *flea algebras*), and the logic is the *flea logic* FIL. There are examples of fleas on $(0, 1]$ not satisfying divisibility, nor commutativity, and having no least element.

It shall be sufficient to mention these generalizations here. The interested reader can find more details in the survey paper [70] and in the original publications.

12 Pavelka Style Extensions

Having in mind that fuzzy logics, also in their form as formalized logical systems, should be a (mathematical) tool for approximative reasoning makes it desirable that they should be able to deal with graded inferences.

The systems of t-norm based logics discussed up to now have been designed to formalize the logical background for fuzzy sets, and they allow themselves for degrees of truth of their formulas. But they all have crisp notions of consequence, i.e. of entailment and of provability.

It is natural to ask whether it is possible to generalize these considerations to the case that one starts from *fuzzy sets of formulas*, and that one gets from them as consequence hulls again fuzzy sets of formulas. This problem was first treated by PAVELKA [120]. The basic monograph elaborating this approach is [118]. We discuss in the present section this kind of approach, because it uses graded relations of entailment and of provability.

However, it should be mentioned that there is also another, more algebraically oriented approach toward consequence operations for the classical case, originating from Tarski [143] and presented e.g. in [149]. This approach treats consequence operations as closure operations. And this type of approach has been generalized to closure operations in classes of fuzzy sets of formulas by Gerla [59]. It shall be discussed in Section 13.

The Pavelka-style approach has to deal with fuzzy sets $\Sigma^{\sim}$ of formulas, i.e. besides formulas φ also their membership degrees $\Sigma^{\sim}(\varphi)$ in $\Sigma^{\sim}$. And these membership degrees are just the truth degrees. We may assume that these degrees again form a residuated lattice $\mathbf{L} = \langle L, \cap, \cup, *, \rightarrowtail, 0, 1\rangle$. Thus we (slightly) generalize the standard notion of fuzzy set (with membership degrees from the real unit interval). Therefore the appropriate language has the same logical connectives as in the previous considerations.

The Pavelka-style approach is an easy matter as long as the entailment relationship is considered. An evaluation e is a *model* of a fuzzy set $\Sigma^{\sim}$ of formulas iff

$$\Sigma^{\sim}(\varphi) \leqslant \mathsf{Val}(\varphi, e) \tag{48}$$

holds for each formula φ. This immediately yields as definition of the entailment relation that the semantic consequence hull of $\Sigma^{\sim}$ should be characterized by the membership degrees

$$\mathcal{C}^{\mathsf{sem}}(\Sigma^{\sim})(\psi) = \bigwedge\{\mathsf{Val}(\psi, e) \mid e \text{ model of } \Sigma^{\sim}\} \tag{49}$$

for each formula ψ.

For a syntactic characterization of this entailment relation it is necessary to have some calculus $\mathbb{K}$ which treats formulas of the language together with truth degrees. So the language of this calculus has to extend the language of the basic logical system by having also symbols for the truth degrees. Depending upon the truth degree structure, this may mean that the language of this calculus becomes an uncountable one.

Further on we indicate these symbols by overlined letters like $\overline{a}, \overline{c}$. And we realize the common treatment of formulas and truth degrees by considering *evaluated formulas*, i.e. ordered pairs $(\overline{a}, \varphi)$ consisting of a truth degree symbol and a formula. This trick transforms in a natural way each fuzzy set $\Sigma^{\sim}$ of formulas into a (crisp) set of evaluated formulas, again denoted by $\Sigma^{\sim}$.

So $\mathbb{K}$ has to allow to derive evaluated formulas out of sets of evaluated formulas, of course using suitable axioms and rules of inference. These axioms are usually only formulas φ which, however, are used in the derivations as the corresponding evaluated formulas $(\overline{1}, \varphi)$. Derivations in $\mathbb{K}$ out of some set $\Sigma^{\sim}$ of evaluated formulas are finite sequences of evaluated formulas which either are axioms, or elements of (the support of) $\Sigma^{\sim}$, or result from former evaluated formulas by application of one of the inference rules.

Each $\mathbb{K}$-derivation of an evaluated formula $(\overline{a}, \varphi)$ counts as a derivation of φ *to the degree* $a \in L$. The *provability degree* of φ from $\Sigma^{\sim}$ in $\mathbb{K}$ is the supremum over all these degrees. This now yields that the syntactic consequence hull of $\Sigma^{\sim}$ should be the fuzzy set $\mathcal{C}^{\mathsf{syn}}_{\mathbb{K}}$ of formulas characterized by the membership function

$$\mathcal{C}^{\mathsf{syn}}_{\mathbb{K}}(\Sigma^{\sim})(\psi) = \bigvee\{a \in L \mid \mathbb{K} \text{ derives } (\overline{a}, \psi) \text{ out of } \Sigma^{\sim}\} \tag{50}$$

for each formula ψ.

Despite the fact that $\mathbb{K}$ is a standard calculus, this is an *infinitary* notion of provability.

For the infinite-valued Łukasiewicz logic L this machinery works particularly well because it needs in an essential way the continuity of the residuation operation. In this case we can form a calculus $\mathbb{K}_{\mathsf{L}}$ which gives an adequate axiomatization for the graded notion of entailment in the sense that one has suitable soundness and completeness results.

This calculus $\mathbb{K}_{\mathsf{L}}$ has as axioms any axiom system of the infinite-valued Łukasiewicz logic L which provides together with the rule of detachment an adequate axiomatization of L, but $\mathbb{K}_{\mathsf{L}}$ replaces this standard rule of detachment by the generalized form

$$\frac{(\overline{a}, \varphi) \qquad (\overline{c}, \varphi \to \psi)}{(\overline{a * c}, \psi)} \tag{51}$$

for evaluated formulas.

The soundness result for this calculus $\mathbb{K}_{\mathsf{L}}$ yields the fact that the $\mathbb{K}_{\mathsf{L}}$-provability of an evaluated formula $(\overline{a}, \varphi)$ says that $a \leq \mathsf{Val}(\varphi, e)$ holds for every valuation e, i.e. that the formula $\overline{a} \to \varphi$ is valid—however as a formula of an *extended* propositional language which has all the truth degree constants among its vocabulary. Of course, now the evaluations e have also to satisfy $e(\overline{a}) = a$ for each $a \in [0, 1]$.

And the soundness and completeness results for $\mathbb{K}_{\mathsf{L}}$ say that a *strong completeness theorem* holds true giving

$$\mathcal{C}^{\mathsf{sem}}(\Sigma^{\sim})(\psi) \;=\; \mathcal{C}^{\mathsf{syn}}_{\mathbb{K}_{\mathsf{L}}}(\Sigma^{\sim})(\psi) \tag{52}$$

for each formula ψ and each fuzzy set $\Sigma^{\sim}$ of formulas.

If one takes the previously mentioned turn and extends the standard language of propositional L by truth degree constants for all degrees $a \in [0, 1]$, and if one reads each evaluated formula $(\overline{a}, \varphi)$ as the formula $\overline{a} \to \varphi$, then a slight modification $\mathbb{K}^{+}_{\mathsf{L}}$ of the former calculus $\mathbb{K}_{\mathsf{L}}$ again provides an adequate axiomatization: one has to add the *bookkeeping axioms*

$$\begin{aligned}(\overline{a} \mathbin{\&} \overline{c}) &\equiv \overline{a * c}\,,\\ (\overline{a} \to \overline{c}) &\equiv \overline{a \rightarrowtail_{\mathsf{L}} c}\,,\end{aligned}$$

as explained e.g. in [118]. And if one is interested to have evaluated formulas together with the extension of the language by truth degree constants, one has also to add the *degree introduction rule*

$$\frac{(\overline{a}, \varphi)}{\overline{a} \to \varphi}\,.$$

However, even a stronger result is available which refers only to a notion of derivability over a countable language. The completeness result (52), for $\mathbb{K}_{\mathsf{L}}^{+}$ instead of $\mathbb{K}_{\mathsf{L}}$, becomes already provable if one adds truth degree constants only for all the *rationals* in $[0,1]$, as was shown in [75]. And this extension of L is even only a conservative one, cf. [82], i.e. $\mathbb{K}_{\mathsf{L}}^{+}$ proves only such *constant-free* formulas of the language with rational constants which are already provable in the standard infinite-valued Łukasiewicz logic L.

For more details the reader may also consult e.g. [75,118,146].

13 Gerla's General Approach

For completeness we mention also a much more abstract approach toward fuzzy logics with graded notions of entailment as the previously explained one for the t-norm based fuzzy logics is.

The background for this generalization by G. Gerla, in detail explained in [59], is that (already) in systems of classical logic the syntactic as well as the semantic consequence relations, i.e. the provability as well as the entailment relations, are closure operators within the set of formulas. This is a fundamental observation made by Tarski [143] already in 1930. And the same holds true for the Pavelka style extensions of Section 12 and the operators $\mathcal{C}^{\mathsf{sem}}$ and $\mathcal{C}^{\mathsf{syn}}$ introduced in (49) and (50), respectively: they are generalized closure operators.

The context, chosen in [59], is that of L-fuzzy sets, with $\mathbf{L} = \langle L, \leqslant \rangle$ an arbitrary complete lattice. A *closure operator in* $\mathbf{L}$ is a mapping $J : L \to L$

satisfying for arbitrary $x, y \in L$ the well known conditions

$$
\begin{array}{ll}
x \leqslant J(x)\,, & \text{(increasingness)} \\
x \leqslant y \Rightarrow J(x) \leqslant J(y)\,, & \text{(isotonicity)} \\
J(J(x) = J(x)\,. & \text{(idempotency)}
\end{array}
$$

And a *closure system* in $\mathbf{L}$ is a subclass $C \subseteq L$ which is closed under arbitrary lattice meets.

For fuzzy logic such closure operators and closure systems are considered in the lattice $\mathcal{F}_L(\mathbb{F})$ of all fuzzy subsets of the set $\mathbb{F}$ of formulas of some suitable formalized language.

An *abstract fuzzy deduction system* now is an ordered pair $\mathcal{D} = (\mathcal{F}_L(\mathbb{F}), D)$ determined by a closure operator D in the lattice $\mathcal{F}_L(\mathbb{F})$. And the *fuzzy theories* T of such an abstract fuzzy deduction system, also called $\mathcal{D}$-theories, are the fixed points of D: $T = D(T)$, i.e. the deductively closed fuzzy sets of formulas.

A rather abstract setting is also chosen for the semantics of such an abstract fuzzy deduction system: an *abstract fuzzy semantics* $\mathcal{M}$ is nothing but a class of elements of the lattice $\mathcal{F}_L(\mathbb{F})$, i.e. a class of fuzzy sets of formulas. These fuzzy sets of formulas are called *models*. The only restriction is that the universal set over $\mathbb{F}$, i.e. the fuzzy subset of $\mathbb{F}$ which has always membership degree one, is not allowed as a model. The background idea here is that, for each standard interpretation $\mathfrak{A}$ (in the sense of many-valued logic – including an evaluation of the individual variables) for the formulas of $\mathbb{F}$, a model M is determined as the fuzzy set which has for each formula $\varphi \in \mathbb{F}$ the truth degree of φ in $\mathfrak{A}$ as membership degree. Accordingly the satisfaction relation $\models_{\mathcal{M}}$ coincides with inclusion: for models $M \in \mathcal{M}$ and fuzzy sets Σ of formulas one has:

$$M \ \models_{\mathcal{M}} \ \Sigma \ \ \Leftrightarrow \ \ \Sigma \subseteq M\,. \tag{53}$$

In this setting, one has a semantic and a syntactic consequence operator, both being closure operators, i.e. one has for each fuzzy set Σ of formulas from $\mathbb{F}$ a semantic as well as a syntactic consequence hull, given by

$$\mathcal{C}^{\mathsf{sem}}(\Sigma) = \bigcap\{M \in \mathcal{M} \mid M \models_{\mathcal{M}} \Sigma\}\,, \qquad \mathcal{C}^{\mathsf{syn}}(\Sigma) = D(\Sigma)\,. \tag{54}$$

Similar to the classical case one has $\mathcal{C}^{\mathsf{sem}}(M) = M$ for each model $M \in \mathcal{M}$, i.e. each such model provides a $\mathcal{C}^{\mathsf{sem}}$-theory.

However, a general completeness theorem is not available. What one needs instead, in search for a completeness result, that are specifications which restrict the full generality of this approach, and lead mainly back to situations which have been discussed in the previous sections.

14 Some Recent Applications

14.1 Fuzzy sets theory

It is an old approach, dating back to the early days of fuzzy set theory, to identify the membership degrees of fuzzy sets with truth degrees of a suitable many-valued logic. In different forms, this idea has been offered and explained e.g. in [60,65–67,94]. And it has since been the topic of occasional investigations like in [141,142].

This point of view toward fuzzy set theory has been one of the motivations behind the development of mathematical fuzzy logics. Therefore one may expect that the recent results in this field of mathematical fuzzy logics give rise to a return to this starting point to use the new insights e.g. for a coherent development of a (formalized) fuzzy set theory.

Indeed, the paper [81] and the subsequent Ph.D. Thesis [85] use the (first-order) logic BL of continuous t-norms, extended with the $\triangle$-operator mentioned in (47), to develop a ZF-like axiomatization for a formalized fuzzy set theory together with a kind of standard model constructed in the style of Boolean valued models for (standard) set theory, as explained e.g. in [9].

The axioms are suitable versions of the axioms of extensionality, pairing, union, powerset, $\in$-induction (i.e. foundation), separation, collection (i.e. comprehension), and infinity, together with an axiom stating the existence of the *support* of each fuzzy set.

The standard model for this theory is formed w.r.t. some complete BL-chain $\mathbf{L} = \langle L, \wedge, \vee, *, \rightarrow, 0, 1\rangle$ and given by the transfinite hierarchy

$$V_0^{\mathbf{L}} = \emptyset\,, \qquad V_{\alpha+1}^{\mathbf{L}} = \left\{ f \in {}^{\mathsf{dom}(u)}L \mid \mathsf{dom}(u) \subseteq V_\alpha^{\mathbf{L}} \right\}, \tag{55}$$

with unions at limit stages. [14]

The primitive predicates $\in, \subseteqq, =$ are interpreted using the following definitions for their truth degrees $[\![\ldots]\!]$:

[14] Here, as usual, by ${}^{B}A$ one denotes—for crisp sets A, B—the class of all functions from B into A.

$$
\begin{aligned}
[\![x \in y]\!] &= \bigcup_{u \in \mathsf{dom}(y)} ([\![u = x]\!] * y(u)) \,, \\
[\![x \subseteqq y]\!] &= \bigcap_{u \in \mathsf{dom}(x)} (x(u) \Rightarrow [\![u \in y]\!]) \,, \\
[\![x = y]\!] &= \triangle [\![x \subseteqq y]\!] * \triangle [\![y \subseteqq x]\!] \,.
\end{aligned}
$$

The last condition forces the equality to be crisp.

Besides this "global" approach toward a generalization of the idea of the cumulative set universe for fuzzy sets, there is also a recent more "local" one [8] which only aims to give a unified treatment of a theory of fuzzy subsets of a given universe of discourse, i.e. which–in a suitable sense–restricts the considerations to the first level of the transfinite hierarchy (55).

The authors of [8] use the (first-order) fuzzy logic ŁΠ, extended again with the $\triangle$-operator, as the background logical system. They take it as a two-sorted language with one sort of variables for objects of the universe of discourse and the other sort for fuzzy sets. The advantage of this choice is that (i) this logic is well understood, cf. e.g. [87], and that (ii) it has sufficiently high expressive power such that former approaches, like [67], which used a mixture of object and metalanguage considerations, can be unified and given in a uniform way within the language of ŁΠ, again with the primitive predicates $\in, =$. So one can e.g. express the fundamental *comprehension axiom* by the schema

$$\exists X \triangle \forall x (x \in X \leftrightarrow \varphi(x))$$

which has $\varphi(x)$ as an arbitrary formula of the language (not containing the set variable X free). And one can express the *axiom of extensionality* by

$$\forall x \triangle (x \in X \leftrightarrow x \in Y) \rightarrow X = Y \,.$$

This allows to denote fuzzy sets by class terms as in [67], with

$$a \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(a) \tag{56}$$

as the guiding principle.

To guarantee the existence of fuzzy sets which are not crisp ones, one may either start with the logic $Ł\Pi\frac{1}{2}$ or add a specific *axiom of fuzziness* reading

$$\exists X \exists x (x \in X \leftrightarrow \neg_{\mathsf{L}} (x \in X)) \,.$$

Some examples shall illustrate the expressive power of this language:

$\{x \mid \neg_\Pi \neg_{\mathsf{L}} (x \in X)\}$	defines	the kernel of X ,
$\{x \mid \neg_\Pi \neg_\Pi (x \in X)\}$	defines	the support of X ,
$\{x \mid \triangle(\overline{\alpha} \rightarrow x \in X)\}$	defines	the (closed) α-cut of X ,

of course with $\overline{\alpha}$ as truth degree constant to denote the truth degree α.

A further expansion of the language with additional sorts of variables allows the authors to develop a machinery to discuss also fuzzy sets of higher level, and finally also a kind of fuzzy type theory and higher order fuzzy logics. Actually this is work in progress, partly contained in [32].

14.2 Non-monotonic fuzzy reasoning

One of the core areas for the application of logic in computer science is artificial intelligence. And inside AI, non-monotonic reasoning has a prominent position.

So it is natural to ask how the basic ideas of non-monotonic inference can be generalized from the crisp case to the fuzzy case, i.e. to the case in which either the knowledge comes e.g. with degrees of vagueness, or of confidence, or in which e.g. the defaults are accepted only to some degrees.

A first idea was offered in [101]. This paper generalizes the circumscription approach in a straightforward way from classical logic to the infinite-valued Łukasiewicz logic L, and gives some basic properties of the non-monotonic inference operator defined via minimal models.

Interesting new ideas, based upon the abstract approach toward fuzzy logic discussed in Section 13, have quite recently been offered in [21,132].

It is possible to define, quite similar to the standard case (6), for abstract fuzzy semantics $\mathcal{M}$ the *model class of* a fuzzy set Σ of formulas as

$$\mathbf{mod}_{\mathcal{M}}(\Sigma) = \{M \in \mathcal{M} \mid M \models_{\mathcal{M}} \Sigma\}\,, \tag{57}$$

and to define the *theory of* a class $\mathbb{K} \subseteq \mathcal{M}$ of models as

$$\mathbf{th}(\mathbb{K}) = \bigcup \{u \in \mathcal{F}_L(\mathbb{F}) \mid M \models u \text{ for all } M \in \mathbb{K}\}\,, \tag{58}$$

which means, in accordance with (53), that one has

$$\mathbf{th}(\mathbb{K}) = \bigcup \Big\{u \in \mathcal{F}_L(\mathbb{F}) \mid u \subseteq \bigcap \mathbb{K}\Big\} = \bigcap \mathbb{K}\,.$$

It is a routine matter to prove that for each fuzzy set $\Sigma \in \mathcal{F}_L(\mathbb{F})$ of formulas one obtains

$$\mathcal{C}^{\mathsf{sem}}(\Sigma) = \mathbf{th}(\mathbf{mod}_{\mathcal{M}}(\Sigma))\,,$$

i.e. $\mathbf{th}(\mathbf{mod}_{\mathcal{M}}(\Sigma))$ is a $\mathcal{C}^{\mathsf{sem}}$-theory.

Therefore it is possible to adapt within this abstract setting the model theoretic method of non-monotonic inference which connects with each set Σ of formulas as its non-monotonic inference hull $\mathbf{C}_{\sim}(\Sigma)$ the theory of a subclass $\Phi(\mathbf{mod}(\Sigma))$ of the class $\mathbf{mod}(\Sigma)$ of all models of Σ:

$$\mathbf{C}_{\sim}(\Sigma) = \mathbf{th}(\Phi(\mathbf{mod}(\Sigma)))\,,$$

e.g. the subclass of all normal or of all minimal models. In this generalized setting one can prove quite similar theoretical results as in the crisp case, as can be seen from [131].

Also another tool from non-monotonic reasoning has a natural generalization to a fuzzy setting: *Poole systems* as introduced in [123]. Such a crisp Poole system P is determined by a pair (D, C) of sets of sentences understood as the relevant *defaults* and *constraints*. For each set Σ of formulas and a suitably chosen closure operator $\mathbf{C}$ it defines a class $\mathbf{E}_P$ of extensions by

$$\mathbf{E}_P(\Sigma) = \{\mathbf{C}(\Sigma \cup D_m) \mid D_m \subseteq D \text{ maximal w.r.t. consistency of } \Sigma \cup C \cup D_m\}\,,$$

and an inference operator $\mathcal{C}_P$ by

$$\mathcal{C}_P(\Sigma) = \bigcap \mathbf{E}_P(\Sigma)\,.$$

All these definitions allow, in the abstract setting of Section 13, a natural extension to the case of fuzzy sets of defaults and constraints. Details again may be found in [131,132].

However, even a more practical application becomes available: toward fuzzy belief revision.

A fuzzy belief base B is just a fuzzy set of formulas $B \in \mathcal{F}_L(\mathbb{F})$. The revision information (φ/a), understood as the fuzzy singleton of φ with membership degree a, tells that a "new" formula φ should be integrated with degree a. As in the AGM framework [2] for the crisp case this may happen in the following steps, cf. [20,21]:

(1) Form the family $B \perp (\varphi/a)$ of all maximal $X \in \mathcal{F}_L(B)$ consistent with (φ/a).
(2) Select a subset $\gamma(B \perp (\varphi/a)) \subseteq B \perp (\varphi/a)$ and form its meet.
(3) Add the revision information to get the revised belief base

$$B \star (\varphi/a) = \bigcap \gamma(B \perp (\varphi/a)) \cup (\varphi/a)\,.$$

The adaptation of this procedure to the case of the revision of fuzzy theories is not as straightforward as in the crisp case, but can also be handled sufficiently well with some extra care regarding the moment for taking (deductive) closures. Details are in [132].

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