Worksheet: Proofs involving Functions

The structure of proofs involving functions and their properties can be derived from the formal definitions of these properties. For example, here are the standard ways to prove injectivity or surjectivity of a function, using the definitions of these concepts.

- **Proving injectivity:** To prove that a function $f : A \to B$ is injective, it is best to use the definition of injectivity in the first form: (The second form is more cumbersome to use for proofs because it involves the negated statements $x \neq y$ and $f(x) \neq f(y)$.)

  $(\forall x, y \in A)[f(x) = f(y) \Longrightarrow x = y]$.

  This translates into a proof of the following structure

  > Let $x, y \in A$ be given.
  > Assume $f(x) = f(y)$.
  > ... [Logical deductions] ...
  > Therefore $x = y$.
  > Hence f is injective.

- **Proving surjectivity:** The definition of surjectivity of a function $f : A \to B$ is:

  $(\forall b \in B)(\exists a \in A)[f(a) = b]$

  This translates into a proof of the following form:

  > Let $b \in B$ be given.
  > ... [Find an a that maps into the given element b.] ...
  > ... [Show that $f(a) = b$.] ...
  > Hence f is surjective.

Sample proof: Composition of injective functions.

Claim: Let $f : A \to B$ and $g : B \to C$ be functions, and let $h = g \circ f$ be the composition of g and f. Prove that if f and g are injective, then so is h.
Proof:

- Suppose f and g are injective.
- Let $x, y \in A$ be given, and assume $h(x) = h(y)$.
- Since $h = g \circ f$, this means that $g(f(x)) = g(f(y))$, by the definition of the composition of functions.
- Since g is injective, this implies $f(x) = f(y)$.
- Since f is injective, it follows that $x = y$.
- Summarizing, we have shown that, for any $x, y \in A$, $h(x) = h(y)$ implies $x = y$.
- Thus, h is injective, by the definition of injectivity.

Practice Problems: Proofs and Counterexamples involving Functions

The following problems serve two goals: (1) to practice proof writing skills in the context of abstract function properties; and (2) to develop an intuition, and "feel" for properties like injective, increasing, bounded, etc., so that you can easily come up with a "guess" whether a statement is likely true, and find counterexamples for false statements. None of the problems is particularly difficult: The proofs for true statements are all quite routine, and counterexamples for false statements are not hard to discover once you have a good intuitive understanding of the definitions. **Try to master them all!**

Most of the problems below are homework problems from HW 5 and HW 6, and are variations or generalizations of problems in the text (see 4:12, 4:33, 4:34)

1. **Proofs involving surjective and injective properties of general functions:** Let $f : A \to B$ and $g : B \to C$ be functions, and let $h = g \circ f$ be the composition of g and f. For each of the following statements, either give a formal proof or counterexample. (A counterexample means a *specific* example of sets A, B, C and functions $f : A \to B$, and $g : B \to C$, for which the statement is false.)
   (a) If f and g are injective, then h is injective.
   (b) If f and g are surjective, then h is surjective.
   (c) If h is injective, then f is injective.
   (d) If h is injective, then g is injective.
   (e) If h is surjective, then f is surjective.
   (f) If h is surjective, then g is surjective.

2. **Proofs involving bounded functions:** Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. For each of the following statements, either give a formal proof or a counterexample. (Here $f + g$ is the function defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in \mathbb{R}$, fg is defined analogously, and e^f is the function defined by $e^f(x) = e^{f(x)}$ for all $x \in \mathbb{R}$.)
   (a) If f and g are bounded, then so is $f + g$.
   (b) If f and g are bounded, then so is fg.
   (c) If $f + g$ is bounded, then so are f and g.
   (d) If fg is bounded, then so are f and g.
   (e) If f is bounded, then so is the function e^f.
   (f) If e^f is bounded, then so is f.

3. **Relations between various properties:** Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. For each of the following statements, either give a formal proof or a counterexample.
   (a) If f is surjective, then f is unbounded.
   (b) If f is unbounded, then f is surjective.
   (c) If f is increasing, then f is injective.
   (d) If f is increasing, then f has an inverse.