Problem Solving

One of the primary reasons people have trouble with problem solving is that there is no single procedure that works all the time — each problem is slightly different. Also, problem solving requires practical knowledge about the specific situation. If you misunderstand either the problem or the underlying situation you may make mistakes or incorrect assumptions. One of our main goals for this semester is to become better problem solvers. To begin this task, we now discuss a framework for thinking about problem solving: Polya's four-step approach to problem solving.

Polya's four-step approach to problem solving

1. **Preparation:** *Understand the problem*
   - Learn the necessary underlying mathematical concepts
   - Consider the terminology and notation used in the problem:
     1. What sort of a problem is it?
     2. What is being asked?
     3. What do the terms mean?
     4. Is there enough information or is more information needed?
     5. What is known or unknown?
   - Rephrase the problem in your own words.
   - Write down specific examples of the conditions given in the problem.

2. **Thinking Time:** *Devise a plan*
   - You must start somewhere so try something. How are you going to attack the problem?
   - Possible strategies: (i. e. reach into your bag of tricks.)
     1. Draw pictures
     2. Use a variable and choose helpful names for variables or unknowns.
     3. Be systematic.
     4. Solve a simpler version of the problem.
     5. Guess and check. Trial and error. Guess and test. *(Guessing is OK.)*
     6. Look for a pattern or patterns.
     7. Make a list.
   - Once you understand what the problem is, if you are stumped or stuck, set the problem aside for a while. Your subconscious mind may keep working on it.
   - Moving on to think about other things may help you stay relaxed, flexible, and creative rather than becoming tense, frustrated, and forced in your efforts to solve the problem.

3. **Insight:** *Carry out the plan*
   - Once you have an idea for a new approach, jot it down immediately. When you have time, try it out and see if it leads to a solution.
   - If the plan does not seem to be working, then start over and try another approach. Often the first approach does not work. Do not worry, just because an approach does not work, it does not mean you did it wrong. You actually accomplished something, knowing a way does not work is part of the process of elimination.
   - Once you have thought about a problem or returned to it enough times, you will often have a flash of insight: a new idea to try or a new perspective on how to approach solving the problem.
   - The key is to ***keep trying until something works.***

4. **Verification:** *Look back*
   - Once you have a potential solution, check to see if it works.
     1. Did you answer the question?
     2. Is your result reasonable?
     3. Double check to make sure that all of the conditions related to the problem are satisfied.
     4. Double check any computations involved in finding your solution.
   - If you find that your solution does not work, there may only be a simple mistake. Try to fix or modify your current attempt before scrapping it. Remember what you tried—it is likely that at least part of it will end up being useful.
   - Is there another way of doing the problem which may be simpler? (You need to become flexible in your thinking. There usually is not one right way.)
   - Can the problem or method be generalized so as to be useful for future problems?

Remember, problem solving is as much an art as it is a science!!

Remember Some of the Possible Strategies Given Earlier

1. **Draw pictures**
2. **Use a variable and choose helpful names for variables or unknowns.**
3. **Be systematic.**
4. **Solve a simpler version of the problem.**
5. **Guess and check. Trial and error. Guess and test.** ***(Guessing is OK.)***
6. **Look for a pattern or patterns.**
7. **Make a list.**

Some Basic Mathematical Principles to Keep in Mind When Problem Solving:

1. **The Always Principle:**
   Unlike many other subjects, when we say a mathematical statement is true, we mean that it is true 100 percent of the time. We are not dealing with the uncertainty of statements that are "usually true" or "sometimes true".

2. **The Counterexample Principle:**
   Since a mathematical statement is true only when it is true 100% of the time, we can prove that is is **false** by finding a single example where it is not true. Such an example is called a *counterexample.*
   Of cours, when we say a mathematical statement is false, this does not mean that it is never true — it only means that it is not always true. It might be true some of the time.

3. **The Order Principle:**
   In mathematics, **order usually matters.** In a multi-step mathematical process, if we carry the steps out in a different order, we often get a different result. For example, putting your socks on first and then your shoes is quite different from putting your shoes on first and then your socks.

4. **The Splitting Hairs Principle:**
   In mathematics, details matter. Two terms or symbols that look and sound similar may have mathematical meanings that are significantly different. For example, in English, we use the term equal and equivalent interchangeably, but in mathematics, these terms do not mean the same thing. For this reason, learning and remembering the precise meaning of mathematical terms is essential.

5. **The Analogies Principle:**
   Often the formal terminology used in mathematics has been drawn from words and concepts used in everyday life. This is not a coincidence. Associating a mathematical concept with its "real world" counterpart can help you remember both the formal (precise) and intuitive meanings of a mathematical concept.

6. **The Three Way Principle:**
   When approaching a mathematical concept, it often helps to use **three** complimentary approaches:
   - Verbal – make analogies, put the problem in your own words, compare the situation to things you may have seen in other areas of mathematics.
   - Graphical – draw a graph or a diagram.
   - Examples – use specific examples to illustrate the situation.

   By combining one or more of these approaches, one can often get a better idea of how to think about and how to solve a given problem.

Problems.

Directions: Work together to solve the following problems using the problem solving strategies. Make sure at least one member of your group records the reasoning you used to arrive at your solution. You do not have to work these problems in order. Once you have found a solution to one of the problems, let me know and I can check to see if both your reasoning and you solution are correct.

1. Every person at a party of twenty-eight people said hello to each of the other people at the party exactly once. How many "hello's" were said at the party?

2. There are four volumes of Shakespeare's collected works on a shelf. The volumes are in order from left to right. The pages of each volume are exactly two inches thick. The covers are each 1/6 inch thick. A bookworm started eating at page one of Volume I and ate through to the last page of Volume IV. What is the distance the bookworm traveled?

3. Suppose that thirty-two students signed up for classes during an orientation session. If exactly twenty of them signed up for Chemistry and exactly sixteen of them signed up for English, how many of them signed up for both Chemistry and English?

4. A hunter left camp and walked five miles south and two miles east. He shot a bear and walked five miles north back to camp. What color was the bear?

5. Suppose Pat has eight shirts and four pairs of pants. How many different outfits can Pat make by combining one shirt with one pair of pants?

6. Six normal drinking glasses are standing in a row. The first three are full of water and the following three are empty. By moving only one glass, can you change the arrangement so that no full glass is next to another full glass and that no empty glass is next to an empty glass, and we still have three full and three empty glasses?

7. Suppose you work at a bowling alley. After work one day, you decide to line up bowling pins in a triangular pattern with one pin in the first row, two pins in the second, three pins in the third, and so on.

   (a) How many total pins would you need to use in order to complete four rows?

   (b) How many total pins would you need to use in order to complete ten rows?

   (c) How many total pins would you need to use in order to complete one hundred rows? How about one thousand rows?

8. Pat and Kim bought the same item. Pat said he paid 20% less than the list price. Kim said that if she would have paid 25% more for the item, she would have paid the list price. Who paid the least?

9. A family has seven children. If we list the possible genders of the children (for example *bbggbgb* where *b* is a boy and *g* is a girl), how many lists are possible?

10. In a class of 25 students, 17 lived with both parents, 21 lived with their mothers, and 20 lived with their fathers. How many lived with neither parent?

11. For each of the following statements, determine whether the statement is true or false. If the statement is true, give two specific examples that illustrate the statement. If it is false, give a specific counterexample.

    (a) If $a < b$, then $a + c < b + c$.

    (b) If $a < b$, then $ac < bc$.

    (c) If Person *X* knows Person *Y* and Person *Y* knows Person *Z*, then Person *X* knows Person *Z*.

12. How many forty passenger buses are needed for a school field trip, if 156 students and five teachers will be going on the field trip? *(They need to be legal for liability.)*

13. Use inductive reasoning to predict the next three terms in each given sequence.

    (a) 3, 6, 12, 24, …

    (b) 0.1, 0.12, 0.121, 0.1212, …

    (c) 13, 31, 15, 51, 17, 71, 19, 91, …

14. Use inductive reasoning to find the ones digit for the numeric value of 2^{50}.

15. A grocery store is having a special on cans of soup which is normally price at 45¢ per can. The advertisement reads 3 for $1.00. You buy only one can of soup. How much does it cost?

16. Bill and Sue both work a night shift. Bill has every sixth night off and Sue has every fifth night off. If they both have tonight off, how many nights will it be before they are both off at the same time again?

17. How many different rectangles with an area of twelve square units can be formed using unit squares?

18. Compare these two problems: (1) Kim made 1 out of 3 free throws on one day and 1 out of 4 free throws the next day. What fraction represents the portion of free throws Kim made over the two days? (2) Kim ate 1/3 of a pie on day and 1/4 of a pie the next day. What fraction represents the amount of the pie Kim ate over the two days?