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Lectures on Poisson Geometry

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Foreword

These notes grew out of a one semester graduate course taught by us at UIUC in the Spring of 2014.

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Preface

The aim of these lecture notes is to provide a fast introduction to Poisson geometry.

In Classical Mechanics one learns how to describe the time evolution of a mechanical system with n degrees of freedom: the state of the system at time t is described by a point $(q(t), p(t))$ in **phase space** $\mathbb{R}^{2n}$. Here the $(q^1(t), \ldots, q^n(t))$ are the configuration coordinates and the $(p_1(t), \ldots, p_n(t))$ are the momentum coordinates of the system. The evolution of the system in time is determined by a function $h : \mathbb{R}^{2n} \to \mathbb{R}$, called the **hamiltonian**: if $(q(0), p(0))$ is the initial state of the system, then the state at time t is obtained by solving **Hamilton's equations**:

$$\begin{cases} \dot{q}^i = \frac{\partial H}{\partial p_i}, \\ \dot{p}_i = -\frac{\partial H}{\partial q^i}, \end{cases} \qquad (i = 1, \ldots, n).$$

This description of motion in mechanics is the departing point for Poisson geometry. First, one starts by defining a new product $\{f, g\}$ between any two smooth functions f and g, called the **Poisson bracket**, by setting:

$$\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right).$$

One then observes that, once a function H has been fixed, Hamilton's equations can be written in the short form:

$$\dot{x}^i = \{H, x^i\}, \quad (i = 1, \ldots, 2n)$$

where x^i denotes any of the coordinate functions (q^i, p_i).

Many properties of Hamilton's equations can be rephrased in terms of the Poisson bracket $\{\cdot, \cdot\}$. For example, a function f is conserved under the motion if and only if it Poisson commutes with the hamiltonian: $\{H, f\} = 0$. Also, if f_1 and f_2 are functions which are both conserved under the motion then their Poisson bracket $\{f_1, f_2\}$ is also a conserved function.

The Poisson bracket (1.1) satisfies the familiar properties of a Lie bracket, namely:

(i) **Skew-symmetry**: $\{f,g\} = -\{g,f\}$;
(ii) **$\mathbb{R}$-bilinearity**: $\{f,ag+bh\} = a\{f,g\}+b\{f,h\}$, for all $a,b \in \mathbb{R}$;
(iii) **Jacobi identity**: $\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\} = 0$.

There is a fourth important property which relates the Poisson bracket with the usual product of functions. Namely,

(iv) **Leibniz identity**: $\{f,g\cdot h\} = g\cdot\{f,h\}+\{f,g\}\cdot h$.

The axiomatization of these properties leads one immediately to the abstract definition of a *Poisson bracket*: a binary operation $\{\cdot,\cdot\} : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$ on the smooth functions of a manifold satisfying properties (i)-(iv) above. Poisson geometry is the study of Poisson manifolds, i.e., of a manifolds equipped with a Poisson bracket.

The reader will notice that at this point we have completely split apart the geometry and the dynamics: in fact, given a Poisson manifold $(M,\{\cdot,\cdot\})$ the Leibniz identity allows one to associate to each smooth function H on M a vector field X_H on M, called the *hamiltonian vector field* of h, by setting:

$$X_H(f) := \{H,f\}.$$

The situation here is somewhat similar to the case of a riemannian manifold where, after one has fixed a riemannian metric, each smooth function H determines a gradient vector field grad H. Although the study of such vector fields is very important, it is important to book keep precisely what are the geometric properties of the underlying (Poisson, riemannian) manifold and the dynamics associated with a choice of a specific function H.

Poisson geometry is closely related to symplectic geometry, and this is one of the main themes of this book. For example, every symplectic manifold has a natural Poisson bracket and every Poisson bracket determines a foliation of the manifold by symplectic submanifolds. Also, a smooth quotient of a symplectic manifold by a group acting by symplectic transformations is a Poisson manifold, which is general is not symplectic. However, as we will see, Poisson geometry requires further techniques which are not present in symplectic geometry, like groupoid/algebroid theory or singularity theory.

Although Poisson geometry goes back to two centuries ago, through the works of Lagrange, Poisson and Lie, it has experienced an amazing development starting with the foundational work of Weinstein [2] in the 80's. Our aim in this book is not to provide a complete survey of the vast amount of work done in this subject in the last 30 years, but rather to provide a quick introduction to the subject that will allow the reader to plunge into these exciting recent developments.

This text is essentially the set of notes of a one semester course on Poisson geometry, consisting of 3 one hour lectures per week. The course targeted mainly graduate students in mathematics in their second year of studies, or later. The basic requirement for this course was a graduate level course in differential geometry.

Some familiarity with algebraic topology and (soft) symplectic geometry would be a plus, but not a requirement, to fully grasp the contents of the course.

Our conventions are such that all manifolds are smooth, second countable and Hausdorff, unless otherwise stated, and all maps are smooth. All vector spaces are real, unless otherwise stated.

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Contents

Acronyms

Lists of abbreviations, symbols and the like:

$C^\infty(M)$	smooth functions
$\mathfrak{X}^k(M)$	Multivector fields of degree k
$\Omega^k(M)$	Differential forms of degree k
i_X	interior product of differential forms by a vector field X
i_α	interior product of multivector fields by a 1-form α
$\pi^\#$	interior product of a bivector π, i.e, the map $\alpha \mapsto i_\alpha \pi$
$\mathscr{L}_X$	Lie derivative of forms or multivector fields along a vector field X
$[\cdot,\cdot]$	Schouten bracket of multivector fields

Part I
Basic Concepts

A Poisson bracket is a Lie bracket $\{\cdot,\cdot\}$ on the space of smooth functions on a manifold which satisfies, additionally, a Leibniz identity. It can be described more efficiently as bivector field satisfying a closedness condition. In Lectures 1 and 2 we describe Poisson brackets as bivector fields, we review the calculus of multivector fields, and we introduce some basic examples of Poisson manifolds.

[TO BE COMPLETED]

Chapter 1
Poisson Brackets and Multivector Fields

1.1 Poisson brackets

Definition 1.1. A **Poisson bracket** on a manifold M is a binary operation $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, $(f,g) \mapsto \{f,g\}$, satisfying:

(i) **Skew-symmetry**: $\{f,g\} = -\{g,f\}$;
(ii) $\mathbb{R}$**-bilinearity**: $\{f, ag+bh\} = a\{f,g\} + b\{f,h\}$, for all $a,b \in \mathbb{R}$;
(iii) **Jacobi identity**: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$.
(iv) **Leibniz identity**: $\{f, g\cdot h\} = g\cdot\{f,h\} + \{f,g\}\cdot h$.

The pair $(M, \{\cdot,\cdot\})$ is called a **Poisson manifold**.

If one is given two Poisson manifolds $(M_1, \{\cdot,\cdot\}_1)$ and $(M_2, \{\cdot,\cdot\}_2)$, a **Poisson map** is a map $\Phi : M_1 \to M_2$ whose pull-back preserves the Poisson brackets:

$$\{f\circ\Phi, g\circ\Phi\}_1 = \{f,g\}_2 \circ \Phi, \quad \forall f,g \in C^\infty(M_2).$$

Example 1.2. On $\mathbb{R}^{2n}$ with linear coordinates $(q^1,\dots,q^n,p_1,\dots,p_n)$ the formula:

$$\{f,g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} \right). \tag{1.1}$$

defines a Poisson bracket, called the **canonical Poisson bracket on** $\mathbb{R}^{2n}$. This Poisson bracket is completely characterized by its values on the coordinates functions:

$$\{q^i, q^j\} = \{p_i, p_j\} = 0, \quad \{p_i, q^j\} = \delta_i^j. \tag{1.2}$$

If $n \geq m$, the map $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$, $(q^1,\dots,q^n,p_1,\dots,p_n) \mapsto (q^1,\dots,q^m,p_1,\dots,p_m)$ is a Poisson map.

Example 1.3. Let $A=(a_{ij})$ be a $n\times n$ skew symmetric matrix. We can then define a **quadratic Poisson bracket** on $\mathbb{R}^n$ by the formula:

$$\{f,g\}_A := \sum_{i,j=1}^{n} a_{ij}x^i x^j \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}. \tag{1.3}$$

You should convince yourself that the Jacobi identity holds.

Consider the map $\Phi:\mathbb{R}^{2n}\to\mathbb{R}^n$, $(q^i,p_i)\mapsto x^i$, defined by:

$$x^i = e^{p_i-\frac{1}{2}\sum_{j=1}^n a_{ij}q^j} \tag{1.4}$$

Then we claim that Φ is a Poisson map when we equip $\mathbb{R}^{2n}$ with the canonical Poisson bracket (1.1). This follows from the following computation:

$$\begin{aligned}
\{x^i\circ\Phi, x^j\circ\Phi\}_{\mathbb{R}^{2n}} &= \{e^{p_i-\frac{1}{2}\sum_{k=1}^n a_{ik}q^k}, e^{p_j-\frac{1}{2}\sum_{l=1}^n a_{jl}q^l}\}_{\mathbb{R}^{2n}}\\
&= e^{-\frac{1}{2}\sum_{k=1}^n(a_{ik}+a_{jk})q^k}\{e^{p_i},e^{p_j}\}_{\mathbb{R}^{2n}}+\\
&\quad + e^{p_j-\frac{1}{2}\sum_{k=1}^n a_{ik}q^k}\{e^{p_i}, e^{-\frac{1}{2}\sum_{l=1}^n a_{jl}q^l}\}_{\mathbb{R}^{2n}}+\\
&\quad + e^{p_i-\frac{1}{2}\sum_{l=1}^n a_{jl}q^l}\{e^{-\frac{1}{2}\sum_{k=1}^n a_{ik}q^k}, e^{p_j}\}_{\mathbb{R}^{2n}}+\\
&\quad + e^{p_i+p_j}\{e^{-\frac{1}{2}\sum_{k=1}^n a_{ik}q^k}, e^{-\frac{1}{2}\sum_{l=1}^n a_{jl}q^l}\}_{\mathbb{R}^{2n}}\\
&= -\frac{1}{2}a_{ji}e^{p_j-\frac{1}{2}\sum_{k=1}^n a_{ik}q^k}e^{p_i-\frac{1}{2}\sum_{l=1}^n a_{jl}q^l}+\\
&\quad + \frac{1}{2}a_{ij}e^{p_i-\frac{1}{2}\sum_{l=1}^n a_{jl}q^l}e^{p_j-\frac{1}{2}\sum_{k=1}^n a_{ik}q^k}\\
&= a_{ij}e^{p_i-\frac{1}{2}\sum_{k=1}^n a_{ik}q^k}e^{p_j-\frac{1}{2}\sum_{l=1}^n a_{jl}q^l}\\
&= a_{ij}(x^i\circ\Phi)(x^j\circ\Phi) = \{x^i,x^j\}_A\circ\Phi,
\end{aligned}$$

where we have used the properties of the Poisson brackets and relations (1.2).

Example 1.4. Let $\mathfrak{g}$ be any finite dimensional Lie algebra. If $f:\mathfrak{g}^*\to\mathbb{R}$ is any smooth function and $\xi\in\mathfrak{g}^*$, then the differential $\mathrm{d}_\xi f: T_\xi\mathfrak{g}^*\to\mathbb{R}$ can be viewed as an element of $\mathfrak{g}$:

$$\langle \mathrm{d}_\xi f, \nu\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}f(\xi+t\nu)\right|_{t=0}, \quad \forall \nu\in\mathfrak{g}^*.$$

Hence, we can define a binary operation on the smooth functions in $M=\mathfrak{g}^*$ by:

$$\{f,g\}(\xi) := \langle [\mathrm{d}_\xi f, \mathrm{d}_\xi g]_{\mathfrak{g}}, \xi\rangle. \tag{1.5}$$

One checks easily that this operation satisfies all the properties of a Poisson bracket.

It is also easy to check that for any Lie algebra homomorphism $\Psi:\mathfrak{h}\to\mathfrak{g}$ the transpose $\Phi=(\Psi)^*:\mathfrak{g}^*\to\mathfrak{h}^*$ is a Poisson map.

A Poisson bracket as in the previous example is called a **linear Poisson bracket** because of the following result.

Exercise 1.5. Let V be a vector space and assume that V has a Poisson bracket $\{\cdot,\cdot\}_V$ with the property that the Poisson bracket of any two linear functions is again a linear function. Show that $\mathfrak{g} = V^*$ has a natural Lie algebra structure and that $\{\cdot,\cdot\}_V$ coincides with the Poisson bracket on $\mathfrak{g}^*$ given by (1.5).

1.2 Hamiltonian vector fields

The Leibniz identity for a Poisson bracket leads to the following definition:

Definition 1.6. Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold. The **hamiltonian vector field** of $H \in C^\infty(M)$ is the vector field $X_H \in \mathfrak{X}(M)$ defined by:

$$X_H(f) := \{H,f\}, \quad \forall f \in C^\infty(M).$$

The function H is called the **hamiltonian function**.

One can also consider time-dependent hamiltonians H_t leading to time-dependent hamiltonian vector fields.

The assignment $C^\infty(M) \to \mathfrak{X}(M)$, $f \mapsto X_f$, is a Lie algebra morphism. The proof is elementary, but this fact is sufficiently important to be stated as an independent proposition.

Proposition 1.7. *For any $f,g \in C^\infty(M)$:*

$$X_{\{f,g\}} = [X_f, X_g].$$

Proof. For any $f,g,h \in C^\infty(M)$, we find:

$$\begin{aligned} X_{\{f,g\}}(h) &= \{\{f,g\},h\} \\ &= \{\{f,h\},g\} + \{f,\{g,h\}\} \\ &= \{f,\{g,h\}\} - \{g,\{f,h\}\} \\ &= X_f(X_g(h)) - X_g(X_f(h)) = [X_f,X_g](h), \end{aligned}$$

where we used first the Jacobi identity and then skew-symmetry of the Poisson bracket. □

Recall that a function f is called a **first integral** of a vector field X if f is constant along any orbit of X and this happens if and only if $X(f) = 0$.

Proposition 1.8. *Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold and fix some hamiltonian function $H \in C^\infty(M)$. Then:*

(i) f is a first integral of X_H if and only if $\{H,f\}=0$;
(ii) H is always a first integral of X_H;
(iii) If f_1 and f_2 are first integrals of X_H then $\{f_1,f_2\}$ is also a first integral of X_H.

Proof. Part (i) follows from the definition of X_H. Part (ii) follows from (i) and the skew-symmetry of $\{\cdot,\cdot\}$. Part (iii) follows from (i) and the Jacobi identity. □

Example 1.9. For the canonical Poisson structure on $\mathbb{R}^{2n}$, the equations for the orbits of X_H are the classical Hamilton's equations:

$$\begin{cases} \dot{q}^i = \{H,q^i\} = \frac{\partial H}{\partial p_i}, \\ \dot{p}_i = \{H,p_i\} = -\frac{\partial H}{\partial q^i}, \end{cases} \qquad (i=1,\dots,n). \tag{1.6}$$

In particular, when $H = \frac{1}{2}\sum_{i=1}^n p_i^2 + V(q^1,\dots,q^n)$, we obtain Newton's equations for the motion of a particle in a potential V:

$$\ddot{q}^i = -\frac{\partial V}{\partial q^i}, \quad (i=1,\dots,n).$$

Example 1.10. Consider the restriction of the Poisson structure on $\mathbb{R}^n$, associated with a skew-symmetric matrix A, to the open set $\mathbb{R}^n_+ := \{(x^1,\dots,x^n) : x^i > 0\}$. Fix real numbers q^i, and consider the Hamiltonian function $H := \sum_{i=1}^n \left(q_i \log x^i - x^i\right)$. Then one obtains the following equations for the orbits of X_H:

$$\dot{x}^i = \{H,x^i\} = \varepsilon_i x^i + \sum_{j=1}^n a_{ij} x_i x_j, \tag{1.7}$$

where we have introduce the constants $\varepsilon_i := \sum_{j=1}^n a_{ji} q^j$.

Equations (1.7) are the famous Lotka-Volterra equations which model the dynamics of the populations of n biological species interacting in an ecosystem.

Example 1.11. Let $\mathfrak{g} = \mathfrak{so}(3)$ be Lie algebra of 3×3-skew symmetric matrices. We can identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ so that the Lie bracket is identified with the vector product $\times$. Under this identification, the corresponding linear Poisson bracket on $\mathfrak{so}(3)^*$ becomes the Poisson bracket on $\mathbb{R}^3$ given by:

$$\{f,g\}(\mathbf{x}) = (\nabla f(\mathbf{x}) \times \nabla g(\mathbf{x})) \cdot \mathbf{x} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ x & y & z \end{vmatrix}.$$

If we now consider the Hamiltonian function $H(x,y,z) = \frac{x^2}{2I_x} + \frac{y^2}{2I_y} + \frac{z^2}{2I_z}$, then one obtains the following equations for the orbits of X_H:

$$\begin{cases} \dot{x} = \{H, x\} = \frac{I_y - I_z}{I_y I_z} yz \\ \dot{y} = \{H, y\} = \frac{I_z - I_x}{I_x I_z} xz, \\ \dot{z} = \{H, z\} = \frac{I_x - I_y}{I_x I_y} yx. \end{cases} \tag{1.8}$$

These equations are the famous Euler equations that control the motion of a top in absence of gravity, moving around its center of mass, with moments of inertia I_x, I_y and I_z.

Recall that if $\Phi : M \to N$ is a smooth map, two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be **Φ-related**, and we write $Y = (\Phi)_* X$, if:

$$Y_{\Phi(x)} = \mathrm{d}_x \Phi \cdot X_x, \quad \forall x \in M.$$

If we think of vector fields as derivations, this condition can be written as:

$$X(f \circ \Phi) = Y(f) \circ \Phi, \quad \forall f \in C^\infty(N).$$

Hamiltonian vector fields behave well under Poisson maps:

Proposition 1.12. *Let $\Phi : (M, \{\cdot,\cdot\}_M) \to (N, \{\cdot,\cdot\}_N)$ be a Poisson map. For every $H \in C^\infty(N)$:*

$$X_H = (\Phi)_* X_{H \circ \Phi}.$$

Proof. Since Φ is a Poisson map, we find for any $f \in C^\infty(N)$:

$$\begin{aligned} X_{H\circ\Phi}(f \circ \Phi) &= \{H \circ \Phi, f \circ \Phi\}_M \\ &= \{H, f\}_N \circ \Phi = X_H(f) \circ \Phi, \end{aligned}$$

which proves the proposition. □

1.3 Poisson brackets in local coordinates

You may have noticed that the formulas for the Poisson brackets in the examples above all have the same flavor. In order to explain this, lets us start by observing that Poisson structures are local. Recall that for any smooth function $f \in C^\infty(M)$ we define its *support* to be the closed set:

$$\operatorname{supp} f := \overline{\{x \in M : f(x) \neq 0\}}.$$

Then:

Proposition 1.13. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. For any smooth functions $f, g \in C^\infty(M)$:*

$$\operatorname{supp}\{f, g\} \subset \operatorname{supp} f \cap \operatorname{supp} g.$$

Proof. Let $f \in C^\infty(M)$ and let $x_0 \notin \operatorname{supp} f$. The open sets $V := M - \{x_0\}$ and $U := M - \operatorname{supp} f$ cover M, so we can choose a partition of unit $\{\rho_U, \rho_V\}$ subordinated to this cover. Then we find:

$$\begin{aligned}\{f,g\}(x_0) &= \{\rho_U f + \rho_V f, g\}(x_0)\\ &= \{0 + \rho_V f, g\}(x_0)\\ &= \rho_V(x_0)\{f,g\}(x_0) + f(x_0)\{\rho_V, g\}(x_0) = 0.\end{aligned}$$

This shows that $\operatorname{supp}\{f,g\} \subset \operatorname{supp} f$. Similarly, we also have $\operatorname{supp}\{f,g\} \subset \operatorname{supp} g$. □

A standard argument now shows that given a Poisson manifold $(M, \{\cdot,\cdot\})$ we can restrict the Poisson bracket to any open subset $U \subset M$ obtaining a Poisson bracket $\{\cdot,\cdot\}_U$ such that for any $f, g \in C^\infty(M)$ we have:

$$\{f,g\}|_U = \{f|_U, g|_U\}_U.$$

For this reason, henceforth we will not distinguish between $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}_U$.

Proposition 1.14. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. If $(U, x^1, \ldots, x^n)$ are local coordinates, then for any $f, g \in C^\infty(M)$:*

$$\{f,g\}|_U = \sum_{i,j=1}^{n} \{x^i, x^j\} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}. \tag{1.9}$$

Proof. First we remark that $\{1, f\} = 0$ for any $f \in C^\infty(M)$. This follows from the Leibniz identity:

$$\{1, f\} = \{1 \cdot 1, f\} = \{1, f\}1 + 1\{1, f\} = 2\{1, f\}.$$

Hence, by linearity, we also have $\{c, f\} = c\{1, f\} = 0$, for any $c \in \mathbb{R}$.

Next, for a smooth function $f \in C^\infty(U)$, the Taylor approximation up to order 2 around $x_0 \in U$ gives:

$$f(x) = f(x_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x_0)\left(x^i - x_0^i\right) + \sum_{i,j=1}^{n} F_{ij}(x)\left(x^i - x_0^i\right)\left(x^j - x_0^j\right),$$

for some smooth functions $F_{ij} \in C^\infty(U)$. Hence, if $f, g \in C^\infty(M)$ we find:

$$\begin{aligned}\{f,g\}(x) &= \{f(x_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x_0)\left(x^i - x_0^i\right) + O(2), g(x_0) + \sum_{i=1}^{n} \frac{\partial g}{\partial x^i}(x_0)\left(x^i - x_0^i\right) + O(2)\}\\ &= \sum_{i,j=1}^{n} \frac{\partial f}{\partial x^i}(x_0) \frac{\partial g}{\partial x^j}(x_0)\{x^i, x^j\}(x) + \sum_{i=1}^{n} H_i(x)(x^i - x_0^i).\end{aligned}$$

for some smooth functions $H_i \in C^\infty(U)$. Hence, at $x = x_0$ we obtain:

$$\{f,g\}(x_0)=\sum_{i,j=1}^{n}\{x^i,x^j\}(x_0)\frac{\partial f}{\partial x^i}(x_0)\frac{\partial g}{\partial x^j}(x_0).$$

Since x_0 was an arbitrary point in U, the result follows. $\square$

In particular, when $M=\mathbb{R}^m$ we have global euclidean coordinates $(x^1,\dots,x^n)$ and a Poisson structure in $\mathbb{R}^n$ is always of the form:

$$\{f,g\}=\sum_{i,j=1}^{n}\pi^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}.$$

where the $\pi^{ij}=\{x^i,x^j\}$ are smooth functions in $\mathbb{R}^m$ such that $\{x^i,x^j\}=-\{x^j,x^i\}$ and the Jacobi identity holds:

$$\{\{x^i,x^j\},x^k\}+\{\{x^j,x^k\},x^i\}+\{\{x^k,x^i\},x^j\}=0,$$

for all $1\leq i<j<k\leq m$.

Exercise 1.15. Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold. If $(U,x^1,\dots,x^n)$ are local coordinates, show that for any $H\in C^\infty(M)$:

$$X_H|_U=\sum_{i,j=1}^{n}\{x^i,x^j\}\frac{\partial H}{\partial x^i}\frac{\partial}{\partial x^j}.$$

1.4 Multivector fields

From Poisson brackets to bivector fields

The description of a Poisson bracket in terms of a binary operation on the smooth functions is not always the most efficient one. There is an alternative description in terms of 2-vector fields, i.e., sections of $\wedge^2 TM$, which we now discuss.

For a smooth manifold M we denote by $\mathfrak{X}(M)$ the space of smooth vector fields on M. Recall that the smooth differential forms $\Omega^k(M):=\Gamma(\wedge^k T^*M)$ can be identified with the $C^\infty(M)$-multilinear, alternating maps of degree k:

$$\omega:\underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{k\text{-times}}\to C^\infty(M).$$

Dually, the smooth multivector fields $\mathfrak{X}^k(M):=\Gamma(\wedge^k TM)$ can be identified with the $C^\infty(M)$-multilinear, alternating maps of degree k:

$$\vartheta:\underbrace{\Omega^1(M)\times\cdots\times\Omega^1(M)}_{k\text{-times}}\to C^\infty(M).$$

If $(U,x^1,\dots,x^m)$ are local coordinates for M, we have local representations:

$$\omega|_U = \sum_{i_1<\cdots<i_k} \omega_{i_1,\dots,i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}, \quad \vartheta|_U = \sum_{i_1<\cdots<i_k} \vartheta^{i_1,\dots,i_k} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}},$$

for uniquely determined smooth functions $\omega_{i_1,\dots,i_k}, \vartheta^{i_1,\dots,i_k} \in C^\infty(U)$.

Now we have the following important identification:

Proposition 1.16. *For a manifold M, the assignment:*

$$\overline{\vartheta}(f_1,\dots,f_k) = \vartheta(\mathrm{d}f_1,\dots,\mathrm{d}f_k).$$

establishes a one to one correspondence between k-multivector fields, i.e., $C^\infty(M)$-multilinear, alternating maps of degree k:

$$\vartheta : \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{k\text{-times}} \to C^\infty(M),$$

and $\mathbb{R}$-multilinear, alternating maps of degree k:

$$\overline{\vartheta} : \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{k\text{-times}} \to C^\infty(M),$$

which are also multi derivations;

$$\overline{\vartheta}(f_1,\dots,gh,\dots,f_k) = g\overline{\vartheta}(f_1,\dots,h,\dots,f_k) + \overline{\vartheta}(f_1,\dots,g,\dots,f_k)h.$$

The proof is left as an exercise. It follows that, given a Poisson manifold $(M,\{\cdot,\cdot\})$, we can define a bivector field $\pi \in \mathfrak{X}^2(M)$ by:

$$\pi(\mathrm{d}f,\mathrm{d}g) := \{f,g\}.$$

If $(U,x^1,\dots,x^m)$ are local coordinated for M, then we find that the local coordinate description of the bivector field π is:

$$\pi|_U = \sum_{i<j} \pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

where $\pi^{ij}(x) = \{x^i,x^j\}(x)$.

Conversely, by the Proposition, if we are given a bivector field $\pi \in \mathfrak{X}^2(M)$ we can define a “bracket” on smooth functions by:

$$\{f,g\} := \pi(\mathrm{d}f,\mathrm{d}g).$$

This bracket is $\mathbb{R}$-bilinear, skew-symmetric and satisfies the Leibniz identity. However, in general, it will not satisfy the Jacobi identity. In order to understand what is the extra property that characterizes the bivector fields associated with Poisson brackets, we will make a brief excursion to the calculus of multivector vector fields.

Wedge product

The exterior (or wedge) product $\wedge$ in the exterior algebra $\wedge T_pM$ induces an exterior (or wedge) product of multivector fields

$$\wedge : \mathfrak{X}^k(M) \times \mathfrak{X}^s(M) \to \mathfrak{X}^{k+s}(M),\ (\vartheta \wedge \zeta)_p \equiv \vartheta_p \wedge \zeta_p.$$

If we consider the space of all multivector fields:

$$\mathfrak{X}^\bullet(M) = \bigoplus_{k=0}^{d} \mathfrak{X}^k(M).$$

where we convention that $\mathfrak{X}^0(M) = C^\infty(M)$ and $f\vartheta = f \wedge \vartheta$, the exterior product turns $\mathfrak{X}(M)$ into a **Grassmannn algebra** over the ring $C^\infty(M)$, i.e., the following properties hold:

(a) $(f\vartheta_1 + g\vartheta_2) \wedge \zeta = f\vartheta_1 \wedge \zeta + g\vartheta_2 \wedge \zeta$, for $f,g \in C^\infty(M)$.
(b) $\vartheta \wedge \zeta = (-1)^{\deg\vartheta \deg\zeta} \zeta \wedge \vartheta$.
(c) $(\vartheta_1 \wedge \vartheta_2) \wedge \vartheta_3 = \vartheta_1 \wedge (\vartheta_2 \wedge \vartheta_3)$.

If $\alpha_1, \dots, \alpha_k \in \Omega^1(M)$ and $X_1, \dots, X_k \in \mathfrak{X}(M)$, our convention is such that we have:

$$X_1 \wedge \cdots \wedge X_k(\alpha_1, \dots, \alpha_k) = \det[\alpha_i(X_j)]_{i,j=1}^k .$$

Push-forward

Let $\Phi : M \to N$ be a smooth map. We have an induced pull-back operation $\Phi^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$ on differential forms. The dual operation on multivector fields is a push-forward operation. However, contrary to the pull-back of differential forms, which is always defined, the push-forward is *not always* defined, so it is convenient to proceed as follows: let $\Phi : M \to N$ be a smooth map. For each $p \in M$, the differential induces a linear map

$$(\mathrm{d}_p\Phi)_* : \wedge^k T_pM \to \wedge^k T_{\Phi(p)}M.$$

Definition 1.17. Let $\Phi : M \to N$ be a smooth map. Two k-vector fields $\vartheta \in \mathfrak{X}^k(M)$ and $\zeta \in \mathfrak{X}^k(N)$ are said to be **Φ-related** if:

$$\zeta_{\Phi(p)} = (\mathrm{d}_p\Phi)_* \vartheta_p, \quad \forall p \in N.$$

In this case we write $\zeta = \Phi_* \vartheta$.

Notice that, in general, if ζ is Φ-related to ϑ, the k-vector field $\zeta \in \mathfrak{X}^k(N)$ is not completely determined by the k-vector field $\vartheta \in \mathfrak{X}^k(M)$ and the map $\Phi : M \to N$. When this is the case, we will call $\Phi_*\vartheta$ the **push-forward** of ϑ by the map Φ.

The push-forward preserves the Grassmann algebra structure: if $\Phi : M \to N$ is a smooth map then

(a) $\Phi_*(a\vartheta_1 + b\vartheta_2) = a\Phi_*\vartheta_1 + b\Phi_*\vartheta_2$;
(b) $\Phi_*(\vartheta \wedge \zeta) = \Phi_*\vartheta \wedge \Phi_*\zeta$;

provided the push-forwards $\Phi_*\vartheta$, $\Phi_*\vartheta_i$ and $\Phi_*\zeta$ are defined.

Exercise 1.18. Let $(M, \{\cdot,\cdot\}_M)$ and $(N, \{\cdot,\cdot\}_N)$ be Poisson manifolds and $\Phi : M \to N$ a smooth map. Denote by $\pi_M \in \mathfrak{X}^2(M)$ and $\pi_N \in \mathfrak{X}^2(N)$ the corresponding bivector fields. Show that Φ is a Poisson map if and only if $\pi_N = (\Phi)_*\pi_M$.

Interior Product

Given a k-vector field $\vartheta \in \mathfrak{X}^k(M)$ and a differential form $\alpha \in \Omega^1(M)$, the **interior product** of ϑ by α, denoted $i_\alpha\vartheta \in \mathfrak{X}^{k-1}(M)$, is the $(k-1)$-vector field defined by:

$$i_\alpha\vartheta(\alpha_1, \dots, \alpha_{k-1}) = \vartheta(\alpha, \alpha_1, \dots, \alpha_{k-1}).$$

Since $i_\alpha\vartheta : \Omega^1(M) \times \cdots \times \Omega^1(M) \to C^\infty(M)$ is a $C^\infty(M)$-multilinear, alternating, map of degree $(k-1)$, it is indeed a smooth multivector field of degree $k-1$.

It is easy to check that the following properties hold:

(a) $i_\alpha(f\vartheta_1 + g\vartheta_2) = fi_\alpha\vartheta_1 + gi_\alpha\vartheta_2$.
(b) $i_\alpha(\vartheta \wedge \zeta) = (i_\alpha\vartheta) \wedge \zeta + (-1)^{\deg\vartheta}\vartheta \wedge (i_\alpha\zeta)$.
(c) $i_{(f\alpha+g\beta)}\vartheta = fi_\alpha\vartheta + gi_\beta\vartheta$.

For a bivector field $\pi \in \mathfrak{X}^2(M)$, the interior product gives a map $\Omega^1(M) \to \mathfrak{X}(M)$, $\alpha \mapsto i_\alpha\pi$. This map is often denoted by

$$\pi^\# : \Omega^1(M) \to \mathfrak{X}(M).$$

Exercise 1.19. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and denote by $\pi \in \mathfrak{X}^2(M)$ the corresponding bivector field. Check that for any smooth function $H \in C^\infty(M)$:

$$X_H = i_{\mathrm{d}H}\pi = \pi^\#(\mathrm{d}H).$$

Lie derivative

The push-forward of multivector fields is always defined for diffeomorphisms. This allows us to define the Lie derivative of a multivector field:

Definition 1.20. The **Lie derivative** of a k-vector field $\vartheta \in \mathfrak{X}^k(M)$ along a vector field $X \in \mathfrak{X}(M)$ is the k-multivector field $\mathscr{L}_X\vartheta \in \mathfrak{X}^k(M)$ defined by:

$$\mathscr{L}_X\vartheta = \left.\frac{\mathrm{d}}{\mathrm{d}t}(\phi_X^{-t})_*\vartheta\right|_{t=0} = \lim_{t\to 0}\frac{1}{t}\left((\phi_X^{-t})_*\vartheta - \vartheta\right).$$

In general, it is impossible to find explicitly the flow of a vector field. Still the basic properties of the Lie derivative listed in the next proposition allow one to find the Lie derivative without knowledge of the flow. The proof is left as an exercise:

Proposition 1.21. *Let $X \in \mathfrak{X}(M)$ and $\vartheta_1, \vartheta_2 \in \mathfrak{X}^\bullet(M)$. Then:*

(i) $\mathscr{L}_X(a\vartheta_1 + b\vartheta_2) = a\mathscr{L}_X\vartheta_1 + b\mathscr{L}_X\vartheta_2$ *for all* $a, b \in \mathbb{R}$.
(ii) $\mathscr{L}_X(\vartheta_1 \wedge \vartheta_2) = \mathscr{L}_X\vartheta_1 \wedge \vartheta_2 + \vartheta_1 \wedge \mathscr{L}_X\vartheta_2$.
(iii) $\mathscr{L}_X(f) = X(f)$, *if* $f \in \mathfrak{X}^0(M) = C^\infty(M)$.
(iv) $\mathscr{L}_X Y = [X,Y]$, *if* $Y \in \mathfrak{X}(M)$.

Exercise 1.22. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and denote by $\pi \in \mathfrak{X}^2(M)$ the corresponding bivector field. Show that

$$\mathscr{L}_X\pi(\mathrm{d}f, \mathrm{d}g) = X(\{f,g\}) - \{X(f), g\} - \{f, X(g)\}.$$

and conclude that:

$$\mathscr{L}_{X_H}\pi = 0, \quad \forall H \in C^\infty(M). \tag{1.10}$$

In particular, if X_H is a complete vector field, then the flow $\phi^t_{X_H} : M \to M$ is a 1-parameter group of Poisson diffeomorphisms of M.

A vector field X such that $\mathscr{L}_X\pi = 0$ is called a **Poisson vector field**. This is equivalent to say that the flow of X is a 1-parameter group of Poisson diffeomorphism. The previous exercise shows that that every hamiltonian vector field is a Poisson vector field.

The Schouten bracket

The differential $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ plays a crucial role in differential geometry. The dual operation on multivector fields is a kind of Lie bracket, which extends the usual Lie bracket on vector fields, called the Schouten bracket. We will take advantage of the identification provided by Proposition 1.16, to define this bracket.

Definition 1.23. Let $\vartheta \in \mathfrak{X}^k(M)$ and $\zeta \in \mathfrak{X}^l(M)$ be multivector fields. The **Schouten bracket** of ϑ and ζ is the multivector field $[\vartheta, \zeta] \in \mathfrak{X}^{k+l-1}(M)$ defined by:

$$[\vartheta, \zeta] = \vartheta \circ \zeta - (-1)^{(k-1)(l-1)}\zeta \circ \vartheta, \tag{1.11}$$

where we have set:

$$\zeta \circ \vartheta(\mathrm{d}f_1, \dots, \mathrm{d}f_{k+l-1}) := \sum_\sigma (-1)^\sigma \overline{\zeta}(\overline{\vartheta}(f_{\sigma(1)}, \dots, f_{\sigma(k)}), f_{\sigma(k+1)}, \dots, f_{\sigma(k+l-1)}),$$

and the sum is over all $(k, l-1)$-shuffles.

Formula (1.11) is not very practical for computations. In the Homework at the end of this lecture, you will be able study the basic properties of the Schouten bracket which yield more efficient ways to compute it.

Observe that for a bivector field $\pi \in \mathfrak{X}^2(M)$ with associated bracket $\{f,g\} = \pi(\mathrm{d}f,\mathrm{d}g)$, formula (1.11) gives:

$$\frac{1}{2}[\pi,\pi](\mathrm{d}f_1,\mathrm{d}f_2,\mathrm{d}f_3) = \{\{f_1,f_2\},f_3\} + \{\{f_2,f_3\},f_1\} + \{\{f_3,f_1\},f_2\}. \quad (1.12)$$

Therefore the Jacobi identity for $\{\cdot,\cdot\}$ is equivalent to the equation $[\pi,\pi] = 0$ and we conclude:

Proposition 1.24. *Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold. Then the associated bivector field $\pi \in \mathfrak{X}^2(M)$ satisfies:*

$$[\pi,\pi] = 0. \quad (1.13)$$

Conversely, every bivector field $\pi \in \mathfrak{X}^2(M)$ satisfying this relation defines a Poisson bracket by $\{f,g\} := \pi(\mathrm{d}f,\mathrm{d}g)$.

Exercise 1.25. Let $\pi \in \mathfrak{X}^2(M)$. Show that $[\pi,\pi] = 0$ iff for every $f,g \in C^\infty(M)$:

$$X_{\{f,g\}} = [X_f,X_g].$$

Hence we can replace our original Definition 1.1 of a Poisson bracket by:

Definition 1.26. A bivector field $\pi \in \mathfrak{X}^2(M)$ satisfying $[\pi,\pi] = 0$ is called a **Poisson structure** on M. A pair (M,π), where π is a Poisson structure on M, is called a **Poisson manifold**.

Exercise 1.27. Let (M,π) be a Poisson manifold. Check that in local coordinates $(U,x^1,\dots,x^m)$ the identity (1.13) amounts to:

$$\sum_{l=1}^{m}\left(\pi^{il}\frac{\partial \pi^{jk}}{\partial x^l} + \pi^{il}\frac{\partial \pi^{jk}}{\partial x^l} + \pi^{il}\frac{\partial \pi^{jk}}{\partial x^l}\right) = 0, \quad (i,j,k = 1,\dots,m). \quad (1.14)$$

Note that (1.14) is an overdetermined non-linear system of first order $p.d.e.'s$: there are $\binom{m}{3}$-equations on $\binom{m}{2}$-unkown functions π^{ij}. It is for this reason that the study of Poisson structures on a given manifold, even locally, is a very non-trivial and hard subject.

1.5 Symplectic vs. Poisson Structures

Recall that a bivector field $\pi \in \mathfrak{X}^2(M)$ determines a map $\pi^\# : \Omega^1(M) \to \mathfrak{X}(M)$ defined by $\alpha \mapsto i_\alpha \pi$. Since the interior product is a pointwise operation, this map is induced by a smooth bundle map which we will denote by the same symbol $\pi^\# : TM \to T^*M$ so that:

$$\pi^\#_x : T^*_x M \to T_x M, \quad \alpha \mapsto i_\alpha \pi_x.$$

A bivector field $\pi \in \mathfrak{X}^2(M)$ is called **non-degenerate at** $x \in M$ if $\pi^\#_x : T^*_x M \to T_x M$ is an isomorphism. We say that $\pi \in \mathfrak{X}^2(M)$ is **non-degenerate** if it is non-degenerate at every $x \in M$. If we think of a bivector $\pi \in \mathfrak{X}^2(M)$ as giving at each point $x \in M$ a skew-symmetric bilinear form

$$\pi_x : T^*_x M \times T^*_x M \to \mathbb{R},$$

then non-degeneracy of π (at $x \in M$) is the same as non-degeneracy of this skew-symmetric bilinear form (at x).

Similarly, a 2-form $\omega \in \Omega^2(M)$ determines a map $\omega^\flat : TM \to T^*M$, such that:

$$\omega^\flat_x : T_x M \to T^*_x M, \quad v \mapsto i_v \omega_x.$$

A 2-form $\omega \in \Omega^2(M)$ is called **non-degenerate at** $x \in M$ if $\omega^\flat_x : T_x M \to T^*_x M$ is an isomorphism. We say that $\omega \in \Omega^2(M)$ is **non-degenerate** if it is non-degenerate at every $x \in M$. If we think of a 2-form $\omega \in \Omega^2(M)$ as giving at each point $x \in M$ a skew-symmetric bilinear form

$$\omega_x : T_x M \times T_x M \to \mathbb{R},$$

then non-degeneracy of ω (at $x \in M$) is the same as non-degeneracy of this skew-symmetric bilinear form (at x).

Lemma 1.28. *There is a 1:1 correspondence between non-degenerate bivector fields $\pi \in \mathfrak{X}^2(M)$ and non-degenerate 2-forms $\omega \in \Omega^2(M)$:*

$$\omega^\flat = (\pi^\#)^{-1} \quad \longleftrightarrow \quad \pi^\# = (\omega^\flat)^{-1}.$$

Under this correspondence, if π is associated with ω, one has:

$$[\pi,\pi](\alpha,\beta,\gamma) = -2\,\mathrm{d}\omega(\pi^\#(\alpha),\pi^\#(\beta),\pi^\#(\gamma)), \quad \alpha,\beta,\gamma \in T^*M. \tag{1.15}$$

Proof. The first part is clear. To check (1.15), it is enough to check that it holds on exact 1-forms. Using (1.12) we find that:

$$[\pi,\pi](\mathrm{d}f_1,\mathrm{d}f_2,\mathrm{d}f_3) = 2(\{\{f_1,f_2\},f_3\} + \{\{f_2,f_3\},f_1\} + \{\{f_3,f_1\},f_2\}).$$

On the other hand, recall Cartan's formula for the differential:

$$\begin{aligned}\mathrm{d}\omega(X,Y,Z) = X(\omega(Y,Z)) &+ \text{ cycl. perm. X,Y,Z}\\ &- (\omega([X,Y],Z) + \text{ cycl. perm. X,Y,Z}).\end{aligned}$$

If we let $X = \pi^\#(\mathrm{d}f_1)$, $Y = \pi^\#(\mathrm{d}f_2)$ and $Z = \pi^\#(\mathrm{d}f_3)$, we find:

$$\mathrm{d}\omega(\pi^\#(\mathrm{d}f_1),\pi^\#(\mathrm{d}f_2),\pi^\#(\mathrm{d}f_3)) = -(\{\{f_1,f_2\},f_3\}+\{\{f_2,f_3\},f_1\}+\{\{f_3,f_1\},f_2\}).$$

so the result follows. □

Skew-symmetry implies that if $\pi\in\mathfrak{X}^2(M)$ is non-degenerate at some $x\in M$, then M must be even dimensional and similarly for non-degenerate 2-forms.

Exercise 1.29. Assume that $\dim M = 2n$. Show that $\pi\in\mathfrak{X}^2(M)$ (respectively, $\omega\in\Omega^2(M)$) is non-degenerate at $x\in M$ if and only if $\wedge^n\pi_x\neq 0$ (resp. .$\wedge^n\omega_x\neq 0$).

Let us recall that a **symplectic structure** on a manifold M is a non-degenerate, closed 2-form $\omega\in\Omega^2(M)$. The pair (M,ω) is then called a **symplectic manifold**. Hence, we conclude:

Proposition 1.30. *There is a 1:1 correspondence between non-degenerate Poisson structures and symplectic structures on a manifold M.*

Notice that in local coordinates $(U,x^1,\dots,x^n)$ we have:

$$\pi|_U = \sum_{i<j}\pi^{ij}(x)\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}.$$

and π is non-degenerate at $x\in U$ if and only if the matrix $(\pi^{ij}(x))$ is invertible. In this case, the corresponding symplectic structure is:

$$\omega = \sum_{i<j}\omega_{ij}(x)\mathrm{d}x^i\wedge\mathrm{d}x^j,\qquad \text{where } (\omega_{ij}(x)) = (\pi^{ij}(x))^{-1}.$$

For this reason, in this case we often write $\omega=\pi^{-1}$ or $\pi=\omega^{-1}$.

Example 1.31. The canonical Poisson structure on $\mathbb{R}^{2n}$ is non-degenerate, and we have:

$$\pi = \sum_{i=1}^n\frac{\partial}{\partial p_i}\wedge\frac{\partial}{\partial q^i},\qquad \pi^{-1}=\sum_{i=1}^n\mathrm{d}q^i\wedge\mathrm{d}p_i.$$

Example 1.32. The Poisson structure on $\mathbb{R}^n$ associated with a skew-symmetric matrix A is never symplectic, since the Poisson structure vanishes at the origin. Similarly, a linear Poisson structure is never symplectic.

For a non-degenerate Poisson structure π any concept related with π can be expressed in terms of the associated symplectic structure $\omega\in\Omega^2(M)$. For example, the hamiltonian vector field X_H of $H\in C^\infty(M)$ can be defined by:

$$\omega^\flat(X_H)=\mathrm{d}H\quad\Leftrightarrow\quad i_{X_H}\omega=\mathrm{d}H.$$

One distinguish feature of non-degenerate Poisson structures (or symplectic structures) is that they possess a canonical volume form: if $\omega\in\Omega^{2n}(M)$ is the symplectic form, then the **Liouville volume form** or **symplectic volume** is defined by

$$\mu := \frac{1}{n!}\wedge^n\omega.$$

Proposition 1.33. *For a non-degenerate Poisson structure π, the Liouville volume form μ is invariant under the flow of any hamiltonian vector field:*

$$\mathscr{L}_{X_H}\mu = 0, \quad \forall H \in C^\infty(M).$$

Proof. By (1.10), we always have $\mathscr{L}_{X_H}\pi = 0$. Since $\omega = (\pi)^{-1}$, we also have $\mathscr{L}_{X_H}\omega = 0$, so the result follows. □

Homework 1: Calculus with Multivector Fields

1.1. Give a proof of Proposition 1.16.
HINT: See the proofs of Propositions 1.13 and 1.14.

1.2. Show that formula (1.11) defines a multivector field of degree $(k+l-1)$ by showing that the formula gives a $\mathbb{R}$-multilinear, alternating map of degree $k+l-1$ which is a derivation on each entry.

1.3. Show that the bracket defined by (1.11) satisfies the following properties:
(a) For $\vartheta \in \mathfrak{X}^k(M)$ and $\zeta \in \mathfrak{X}^l(M)$:

$$[\vartheta, \zeta] = -(-1)^{(k-1)(l-1)}[\zeta, \vartheta];$$

(b) For $a, b \in \mathbb{R}$:

$$[a\vartheta_1 + b\vartheta_2, \zeta] = a[\vartheta_1, \zeta] + b[\vartheta_2, \zeta];$$

(c) For $\vartheta \in \mathfrak{X}^k(M)$, $\zeta \in \mathfrak{X}^l(M)$ and $\tau \in \mathfrak{X}^m(M)$:

$$(-1)^{(k-1)(m-1)}[[\vartheta, \zeta], \tau] + (-1)^{(l-1)(k-1)}[[\zeta, \tau], \vartheta] + (-1)^{(m-1)(l-1)}[[\tau, \vartheta], \zeta] = 0.$$

(d) For $\vartheta \in \mathfrak{X}^k(M)$, $\zeta \in \mathfrak{X}^l(M)$ and $\tau \in \mathfrak{X}^m(M)$:

$$[\vartheta, \zeta \wedge \tau] = [\vartheta, \zeta] \wedge \tau + (-1)^{(k-1)l}\zeta \wedge [\vartheta, \tau].$$

(e) For $X \in \mathfrak{X}^1(M)$ and $f \in C^\infty(M) = \mathfrak{X}^0(M)$:

$$[X, f] = X(f).$$

(f) The Schouten bracket on vector fields coincides with the usual Lie bracket of vector fields.

Remark 1.4. The following remark maybe helpful in memorizing the various signs that appear in (a)-(e). A $\mathbb{Z}$-*graded vector space* is a vector space which is a direct sum of vector spaces $V = \oplus_{n\in\mathbb{Z}} V_n$. Elements of V_n are said to have degree n. A $\mathbb{Z}$-*graded Lie algebra* is a graded vector space $V = \oplus_{n\in\mathbb{Z}} V_n$ with a Lie bracket $[\cdot,\cdot]$ such that that $[V_n, V_m] \subset V_{n+m}$. A *super $\mathbb{Z}$-graded Lie algebra* is a graded vector space $V=\oplus_{n\in\mathbb{Z}}V_n$ with a bracket $[\cdot,\cdot]$ such that that $[V_n,V_m]\subset V_{n+m}$ which satisfies the following super versions of the usual properties of a Lie bracket:

(a) **Super skew-symmetry:** $[v,w]=-(-1)^{(\deg v)(\deg w)}[w,v]$;
(b) $\mathbb{R}$**-blinearity:** $[av_1+bv_2,w]=a[v_1,w]+b[v_2,w]$;
(c) **Super graded Jacobi:**

$$(-1)^{(\deg v)(\deg z)}[[v,w],z]+(-1)^{(\deg w)(\deg v)}[[w,z],v]+(-1)^{(\deg z)(\deg w)}[[z,v],w]=0.$$

You will notice now that properties (a)-(c) of the Schouten bracket express the fact that this bracket makes $\mathfrak{X}^\bullet(M)$ into a super graded Lie algebra, but with a shift in the degree of a multivector filed: $\deg\mathfrak{X}^k(M)=k-1$. Property (d) also shows that $[\vartheta,\cdot]$ acts as derivation on the exterior algebra, where again there is a shift in the degree of ϑ (but not in the exterior algebra).

1.5. Show that there is at most one operation $[\cdot,\cdot]:\mathfrak{X}^k(M)\times\mathfrak{X}^l(M)\to\mathfrak{X}^{k+l-1}(M)$ that satisfies properties (a)-(e) in Problem 1.3.
HINT: Using only properties (a)-(e) show that if $X_1,\dots,X_k,Y_1,\dots,Y_l\in\mathfrak{X}(M)$, one must have:

$$\begin{aligned}&[X_1\wedge\cdots\wedge X_k,Y_1\wedge\cdots\wedge Y_l]=\\&\qquad=\sum_{i,j}(-1)^{i+j}[X_i,Y_j]\wedge X_1\wedge\cdots\wedge\check{X}_i\wedge\cdots\wedge X_k\wedge Y_1\wedge\cdots\wedge\check{Y}_j\wedge\cdots\wedge Y_l\end{aligned}$$

and that for any multivector field $\vartheta\in\mathfrak{X}^k(M)$ and any $f\in C^\infty(M)$:

$$[\vartheta,f]=i_{\mathrm{d}f}\vartheta.$$

1.6. Show that for any vector field $X\in\mathfrak{X}(M)$ and any $\vartheta\in\mathfrak{X}^k(M)$:

$$\mathscr{L}_X\vartheta=[X,\vartheta].$$

1.7. Let $\Phi:M\to N$ be a smooth map, $\vartheta_1,\zeta_1\in\mathfrak{X}^k(M)$ and $\vartheta_2,\zeta_2\in\mathfrak{X}^k(N)$. Show that if $\vartheta_2=\Phi_*\vartheta_1$ and $\zeta_2=\Phi_*\zeta_1$ then:

$$[\vartheta_2,\zeta_2]=\Phi_*[\vartheta_1,\zeta_1].$$

1.8. Use the properties above of the Schouten bracket to compute $[\pi,\pi]$ for the following bivector fields:

(a) $\pi=\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}+\cdots+\frac{\partial}{\partial x^{2n-1}}\wedge\frac{\partial}{\partial x^{2n}}\in\mathfrak{X}^2(\mathbb{R}^{2n})$.
(b) $\pi=f(\theta^1)\frac{\partial}{\partial\theta^1}\wedge\frac{\partial}{\partial\theta^2}+\frac{\partial}{\partial\theta^3}\wedge\frac{\partial}{\partial\theta^4}+\cdots+\frac{\partial}{\partial\theta^{2n-1}}\wedge\frac{\partial}{\partial\theta^{2n}}\in\mathfrak{X}^2(\mathbf{T}^{2n})$.
(c) $\pi=f(x)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}+g(y)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x}+h(z)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\in\mathfrak{X}^2(\mathbb{R}^3)$.
(d) $\pi=X\wedge Y$, where $X,Y\in\mathfrak{X}(M)$ are any two vector fields on M.

Chapter 2
Submanifolds

2.1 Poisson Submanifolds

Definition 2.1. A **Poisson submanifold** of a Poisson manifold (M,π_M) is a Poisson manifold (N,π_N) together with an injective immersion $i:N\hookrightarrow M$ which is Poisson: $i_*\pi_N=\pi_M$.

As usual, one identifies an immersed submanifold $i:N\hookrightarrow M$ with its image, so that one can assume that the map i is the inclusion and $\mathrm{d}_x i(T_xN)$ is identified with a subspace of $T_{i(x)}M$, but keep in mind that, in general, the topology on N is not the topology induced from M. Also, we let:

$$(TN)^0 := \{\alpha\in T_N^*M : \alpha(v)=0, \forall v\in T_NM\}.$$

Proposition 2.2. *Let (M,π_M) be a Poisson manifold. Given an immersed submanifold $N\hookrightarrow M$ there is at most one Poisson structure π_N on N that makes (N,π_N) into a Poisson manifold. This happens if and only if any of the following equivalent conditions hold:*

(i) $\mathrm{Im}(\pi_M)_x^\# \subset T_xN$, *for all* $x\in N$;
(ii) every hamiltonian vector field $X_H\in\mathfrak{X}(M)$ *is tangent to* N;
(iii) $(\pi_M)_x^\#(T_xN)^0=0$, *for all* $x\in N$.

If N is a closed submanifold, then these condition are also equivalent to:

(vi) The vanishing ideal $\mathscr{I}(N):=\{f\in C^\infty(M): f(x)=0, \forall x\in N\}$ *is a Poisson ideal: for any* $f\in\mathscr{I}(N)$ *and* $g\in C^\infty(M)$ *one has* $\{f,g\}\in\mathscr{I}(N)$.

Proof. If $(N,\pi_N)\hookrightarrow(M,\pi_M)$ is a Poisson submanifold, then π_N is i-related to π_M:

$$\mathrm{d}_x i(\pi_N)_x=(\pi_M)_{i(x)}, \quad \forall x\in N.$$

Equivalently:

$$\mathrm{d}_x i \circ (\pi_N)_x^\# \circ (d_x i)^* = (\pi_M)_{i(x)}^\#.$$

Since $\mathrm{d}_x i$ is injective, this shows that π_N is unique. It also shows that (i) must hold if (N,π_N) is a Poisson submanifold.

Next let $i : N \hookrightarrow M$ be a submanifold and assume that $\mathrm{Im}(\pi_M)_x^\# \subset \mathrm{d}_x i(T_x N)$. We claim that there exists a unique smooth bivector π_N in N such that $(\pi_M)_x^\#$ factors as:

$$\begin{tikzcd}
T_x^* M \arrow[r, "(\pi_M)_x^\#"] \arrow[d, "(d_x i)^*"'] & T_x M \\
T_x^* N \arrow[r, dashed, "(\pi_N)_x^\#"'] & T_x M \arrow[u, "d_x i"']
\end{tikzcd}$$

Since we already know that $\mathrm{Im}(\pi_M)_x^\# \subset \mathrm{d}_x i(T_x N)$, it is enough to check that for any $\alpha,\beta \in T_x^* M$ such that $\alpha|_{TN} = \beta|_{TN}$ we have $(\pi_M)_x^\#(\alpha) = (\pi_M)_x^\#(\beta)$. In fact, we find for any $\gamma \in T_x^* M$:

$$\begin{aligned}
\langle (\pi_M)_x^\#(\alpha) - (\pi_M)_x^\#(\beta), \gamma \rangle &= \langle (\pi_M)_x^\#(\alpha - \beta), \gamma \rangle \\
&= -\langle \alpha - \beta, (\pi_M)_x^\#(\gamma) \rangle = 0.
\end{aligned}$$

which proves the claim (the smoothness of π_N is automatic).

Now observe that $[\pi_N,\pi_N] = 0$. In fact, the Schouten brackets of Φ-related multivector fields are also Φ-related, so that:

$$[\pi_M,\pi_M] = i_*([\pi_N,\pi_N]),$$

and i is an immersion. This shows that if (i) holds, then N has a unique Poisson structure so that it is a Poisson submanifold.

The equivalence (i) $\Leftrightarrow$ (ii) is follows from the fact that hamiltonian vector fields take the the form $X_H = \pi_M^\#(\mathrm{d}H)$.

The equivalence (ii) $\Leftrightarrow$ (iii) follows from observing that for any $\alpha \in (T_x N)^0$ and $\beta \in T_x^* M$ we have

$$\langle \pi_M^\#(\alpha), \beta \rangle = -\langle \alpha, \pi_M^\#(\beta) \rangle,$$

so $\pi_M^\#(T_x N)^0 = 0$ if and only if $\pi_M^\#(T_x^* M) \subset TN$.

Finally, notice that if N is a closed submanifold, a vector field $X \in \mathfrak{X}(M)$ is tangent to N if and only if for any $f \in \mathscr{I}(N)$ we have $X(f)(x) = 0$. Hence, the result follows from the first part and the fact that $\{f,g\} = -X_g(f)$. $\square$

Exercise 2.3. What can one say about the equivalence with (iv), in the previous proposition, if the submanifold is not closed?

Corollary 2.4. *Let (M,π_M) be a Poisson manifold. If $N_1, N_2 \subset M$ are two Poisson submanifolds which intersect transversely then $N_1 \cap N_2 \subset M$ is also a Poisson submanifold.*

Proof. It is enough to observe that:

$$T(N_1 \cap N_2) = TN_1 \cap TN_2,$$

and apply the proposition twice. $\square$

Example 2.5. For a non-degenerate Poisson structure (M, π), i.e., a symplectic manifold, the bundle map $\pi^\# : T^*M \to TM$ is an isomorphism, so we have:

$$\operatorname{Im} \pi^\# = TM.$$

Hence, the proposition shows that the only Poisson submanifolds of a symplectic manifold are the open subsets $U \subset M$.

Example 2.6. For the quadratic Poisson bracket on $\mathbb{R}^n$ associated with a skew-symmetric matrix $A = (a_{ij})$, we have

$$\pi_A = \sum_{i<j} a_{ij} x^i x^j \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

so that:

$$\pi_A^\#(\mathrm{d}x^i) = \sum_{j=1}^n a_{ij} x^j \frac{\partial}{\partial x^j}.$$

This shows that for any integers $1 \le i_1 < \cdots < i_k \le n$, with $k \le n$, the subspaces

$$V_{i_1,\ldots,i_k} = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^{i_l} = 0,\ l = 1, \ldots, k\},$$

are Poisson submanifolds. The Poisson bracket on the submanifold $V_{i_1,\ldots,i_k}$ is again a quadratic bracket associated with the $(n-k) \times (n-k)$ minor of A obtained by removing the rows and columns $i_1, \ldots, i_k$.

Example 2.7. Consider the Poisson manifold $\mathfrak{so}^*(3) \simeq \mathbb{R}^3$. From the expression for the Poisson bracket given in Example 1.11, we have:

$$\pi = x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

This shows that:

$$\pi^\#(\mathrm{d}x) = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z}, \quad \pi^\#(\mathrm{d}y) = x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}, \quad \pi^\#(\mathrm{d}z) = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}.$$

Hence, at a point (x, y, z) we see that $\operatorname{Im} \pi^\#_{(x,y,z)}$ is the space orthogonal to the vector (x, y, z). It follows that the spheres

$$\mathbb{S}_r^2 := \{(x, y, z) : x^2 + y^2 + z^2 = r^2\}$$

are all Poisson submanifolds. The sphere of radius 0, i.e., the origin $\{0\}$, is also a Poisson submanifold of dimension 0! It is an instructive exercise to write down the Poisson structure on a sphere $\mathbb{S}^2_r$ in terms of spherical coordinates.

Exercise 2.8. Let (θ,φ) be spherical coordinates on the sphere $\mathbb{S}^2_r$ defined by:

$$x = r\sin\theta\cos\varphi, \quad y = r\cos\theta\cos\varphi, \quad z = r\sin\varphi.$$

Show that the induced Poisson structure on $\mathbb{S}_r$ is:

$$\pi_{\mathbb{S}^2_r} = -\frac{1}{r\cos\varphi}\frac{\partial}{\partial\theta}\wedge\frac{\partial}{\partial\varphi}.$$

Exercise 2.9. For any Poisson manifold (M,π) a function $f\in C^\infty(M)$ is called a **Casimir function** of π if $X_f=0$, i.e, if $\{f,g\}=0$, for all $g\in C^\infty(M)$. For example, the function $f(x,y,z)=x^2+y^2+z^2$ is a Casimir of the Poisson structure $\pi_{\mathfrak{so}^*(3)}$. Show that if $\phi: M\to\mathbb{R}^n$ is a smooth map such that each component ϕ^i is a Casimir function and $c\in\mathbb{R}^n$ is a regular value of ϕ, then $\phi^{-1}(c)\subset M$ is a Poisson submanifold.

In general, if $\Phi:(M,\pi_M)\to(N,\pi_N)$ is a Poisson map which is transverse to a Poisson submanifold $Q\subset(N,\pi_N)$, then $\Phi^{-1}(Q)$ *is not* a Poisson submanifold of (M,π) as shown by the following example.

Example 2.10. Consider the Poisson map $\Phi:\mathbb{R}^{2n}\to\mathbb{R}^n$ of Example 1.3, where on $\mathbb{R}^{2n}$ we have the canonical Poisson bracket and on $\mathbb{R}^n$ we have the Poisson bracket associated with a skew-symmetric matrix A. This map is a submersion onto $\mathbb{R}^n_+$.

Now let us set $n=3$ and consider the skew-symmetric matrix:

$$A=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Then $Q_c:=\{(x,y,z): z=c>0\}\subset\mathbb{R}^n_+$ is a Poisson submanifold and $\Phi^{-1}(Q_c)$ is a codimension 1 submanifold of $\mathbb{R}^{2n}$. However, $\Phi^{-1}(Q_c)$ *is not* a Poisson submanifold $\mathbb{R}^{2n}$, since the only Poisson submanifolds of $\mathbb{R}^{2n}$ are open subsets.

The notion of a Poisson submanifold is very natural. However, Poisson submanifolds are, in some sense, quite rare (as shown, e.g., in the case of symplectic manifolds) and do not behave functorially relative to Poisson maps (as illustrated by the previous example). Hence, we now turn to more general concepts of submanifolds, extending the notion of a Poisson submanifold in different directions, which are very useful in Poisson geometry.

2.2 Poisson Transversals

Definition 2.11. A **Poisson transversal** of a Poisson manifold (M,π) is a submanifold $X \subset M$ such that, at every point $x \in X$, we have

$$T_xM = T_xX + \pi^\#(T_xX)^0. \tag{2.1}$$

Note that the equality $\mathrm{rank}(TX)^0 = \mathrm{rank}(T_XM) - \mathrm{rank}(TX)$ implies that condition (2.1) is equivalent to the direct sum decomposition:

$$T_xM = T_xX \oplus \pi^\#(T_xX)^0. \tag{2.2}$$

Remark 2.12. Condition (2.1) is *open* in three different ways:

- *Open on X:* if a submanifold X satisfies (2.1) at some point $x \in X$, then an open neighborhood U of x in X is a Poisson transversal.
- *Open in π:* if X is a Poisson transversal for π, then X is a Poisson transversal for all Poisson structures π' that are C^0-close enough to π.
- *Open for the inclusion:* if $i : X \to M$ denotes the inclusion of X in M, and X is a Poisson transversal, then for all injective immersions $i' : X \to M$ that are C^1-close to i, $i'(X)$ is also a Poisson transversal.

The main reason to consider Poisson transversals is that they have naturally induced Poisson structures:

Proposition 2.13. *Let X be a Poisson transversal of (M,π). Then X has a natural Poisson structure $\pi_X \in \mathfrak{X}^2(X)$.*

The existence of a bivector $\pi_X \in \mathfrak{X}^2(X)$ is not to hard to see: the decomposition (2.2) for T_XM and the dual decomposition for T^*_XM gives a sequence of bundle maps:

$$T^*X \longrightarrow T^*_XM \xrightarrow{\pi^\#} T_XM \longrightarrow TX \tag{2.3}$$

The resulting bundle map $TX \to T^*X$ is skew-symmetric and so it is of the form $\pi_X^\#$ for a unique bivector $\pi_X \in \mathfrak{X}^2(X)$. We claim that this is a Poisson structure:

$$[\pi_X, \pi_X] = 0.$$

To prove this we will need to recall a few facts about the normal bundle and the conormal bundle to a submanifold.

Recall that the **normal bundle** of a submanifold $X \subset M$ is defined as the quotient vector bundle

$$\mathcal{N}_X := T_XM/TX,$$

so we have the short exact sequence of vector bundles:

$$0 \longrightarrow TX \longrightarrow T_X M \longrightarrow \mathscr{N}_X \longrightarrow 0$$

The dual sequence is:

$$0 \longrightarrow \mathscr{N}_X^* \longrightarrow T_X^* M \longrightarrow T^* X \longrightarrow 0$$

which shows that there is a canonical identification between the **conormal bundle** $\mathscr{N}_X^*$ and $(TX)^0$.

An **embedded normal bundle** is a vector subbundle $E \subset T_X M$ such that

$$T_X M = TX \oplus E.$$

Note that, for any embedded normal bundle E, the natural projection $T_X M \to \mathscr{N}_X$ restricts to a vector bundle isomorphism $E \xrightarrow{\sim} \mathscr{N}_X$.

Lemma 2.14. *Let X be a Poisson transversal in the Poisson manifold (M, π). Then X comes equipped naturally with an embedded normal bundle*

$$T_X M = TX \oplus N^\pi X, \quad \textit{where } \; N^\pi X := \pi^\#(TX)^0, \tag{2.4}$$

such that $\pi^\sharp$ restricts to a vector bundle isomorphism

$$\pi^\#|_{(TX)^0} : (TX)^0 \xrightarrow{\sim} N^\pi X. \tag{2.5}$$

Proof. From (2.2), we see that the map

$$\pi^\#|_{(TX)^0} : (TX)^0 \longrightarrow T_X M$$

is injective. Hence, $N^\pi X$, which is the image of this map, is a smooth vector subbundle of $T_X M$. This also implies that (2.5) is an isomorphism. $\square$

The decomposition from the Lemma induces also a decomposition for $\wedge^2 T_X M$, as follows:

$$\wedge^2 T_X M = \wedge^2 TX \; \oplus \; (TX \otimes N^\pi X) \; \oplus \; \wedge^2 N^\pi X. \tag{2.6}$$

Lemma 2.15. *The component of $\pi|_X$ that lies in $TX \otimes N^\pi X$ vanishes.*

Proof. It suffices to show that $\pi_x(\alpha, \beta) = 0$, for all $\alpha \in (T_x X)^0$, $\beta \in (N_x^\pi X)^0$ and $x \in X$. This simply follows from the definitions of these spaces, because $\pi_x^\#(\alpha) \in N_x^\pi X$ and $\beta \in (N_x^\pi X)^0$. $\square$

The Lemma implies that in the decomposition (2.6) the bivector $\pi|_X$ has only two components, which we denote by

$$\pi|_X = \pi_X + \sigma_X, \quad \pi_X \in \mathfrak{X}^2(X), \; \sigma_X \in \Gamma(\wedge^2 N^\pi X).$$

Note that π_X is precisely the bivector whose induced bundle map $\pi_X^\#$ is given by (2.3). On the other hand, the map $\sigma_X^\sharp : (TX)^0 \xrightarrow{\sim} N^\pi X$ equals the vector bundle isomorphism (2.5). In particular, we conclude:

Corollary 2.16. *The component $\sigma_X \in \Gamma(\wedge^2 N^\pi X)$ is nondegenerate*[1] *when regarded as a smooth family of two-forms:*

$$\sigma_{X,x} : (T_xX)^0 \times (T_xX)^0 \longrightarrow \mathbb{R}, \quad x \in X.$$

We are now in conditions to complete the proof of Proposition 2.13:

Lemma 2.17. *The bivector $\pi_X \in \mathfrak{X}^2(X)$ is a Poisson structure on X.*

Proof. It suffices to check the condition locally; thus, by restricting to an open set U such that $N^\pi X|_{U\cap X}$ trivializes as a vector bundle, we can choose sections $X_i, Y_i \in \Gamma(N^\pi X|_{U\cap X})$, for $1 \leq i \leq d$ such that

$$\sigma_X|_{U\cap X} = \sum_{i=1}^{d} X_i \wedge Y_i.$$

By shrinking U as necessary, choose vector fields $\widetilde{X}_i$, respectively $\widetilde{Y}_i$, on U that extend X_i, respectively Y_i. We denote

$$\widetilde{\sigma}_X := \sum_{i=1}^{d} \widetilde{X}_i \wedge \widetilde{Y}_i \in \mathfrak{X}^2(U), \quad \widetilde{\pi}_X := \pi - \widetilde{\sigma}_X \in \mathfrak{X}^2(U).$$

Using the Poisson condition $[\pi, \pi] = 0$, we compute:

$$[\widetilde{\pi}_X, \widetilde{\pi}_X] = [\pi - \widetilde{\sigma}_X, \pi - \widetilde{\sigma}_X] = [-2\pi + \widetilde{\sigma}_X, \widetilde{\sigma}_X].$$

Denote by $\vartheta := -2\pi + \widetilde{\sigma}_X$. Then we have that

$$[\vartheta, \widetilde{\sigma}_X] = \sum_{i=1}^{d} \left([\vartheta, \widetilde{X}_i] \wedge \widetilde{Y}_i - \widetilde{X}_i \wedge [\vartheta, \widetilde{Y}_i] \right).$$

So, we obtain that:

$$[\widetilde{\pi}_X, \widetilde{\pi}_X] = \sum_{i=1}^{d} \left([\vartheta, \widetilde{X}_i] \wedge \widetilde{Y}_i - \widetilde{X}_i \wedge [\vartheta, \widetilde{Y}_i] \right).$$

We restrict this equation to X. Since $\widetilde{\pi}_X|_X = \pi_X$ is tangent to X, the left hand side equals

$$[\pi_X, \pi_X] \in \mathfrak{X}^3(X).$$

The right hand side restricted to X is of the form:

[1] In the literature Poisson transversals are also called *cosymplectic submanifolds*. This Corollary justifies this name, since it shows that the conormal bundle is a symplectic vector bundle.

$$\sum_{i=1}^{d}\Big([\vartheta,\widetilde{X}_i]|_X\wedge Y_i - X_i\wedge[\vartheta,\widetilde{Y}_i]|_X\Big)\in\Gamma(\wedge^2 T_X M\wedge N^\pi X).$$

Since $\Gamma(\wedge^2 T_X M\wedge N^\pi X)\cap\mathfrak{X}^3(X)=\{0\}$, is follows the both expressions vanish along X; in particular π_X is Poisson: $[\pi_X,\pi_X]=0$. $\square$

It is important to note that for a Poisson transversal X in (M,π), with induced Poisson structure π_X, the inclusion map $(X,\pi_X)\to(M,\pi)$ is not Poisson (unless X is an open set in M). This will be clear in the next examples.

Example 2.18. Let $\pi\in\mathfrak{X}^2(M)$ be a non-degenerate Poisson structure with associated symplectic structure ω. Notice that for a submanifold X:

$$\begin{aligned} X\in\pi^\#(TX)^0 \quad &\Leftrightarrow\quad \omega^\flat(X)\in(TX)^0\\ &\Leftrightarrow\quad \omega(X,Y)=0,\forall X\in TX\\ &\Leftrightarrow\quad X\in(TX)^{\perp\omega}\end{aligned}$$

where $\perp_\omega$ denotes the orthogonal relative to the symplectic form ω. Hence, a Poisson transversal is a submanifold $X\subset M$ such that:

$$T_X M = TX\oplus(TX)^{\perp\omega}.$$

Such a submanifold is called a **symplectic submanifold**. In fact, notice that this condition holds if and only if the pullback form $\omega|_X$ is non-degenerate. Hence $(X,\omega|_X)$ is a symplectic manifold. We leave it as an exercise to show that the induced Poisson structure π_X is indeed:

$$\pi_X=(\omega|_X)^{-1}.$$

On the other hand, we saw before that the only Poisson submanifolds of (M,π) are the open subsets $U\subset M$.

For example, for the canonical Poisson structure on $\mathbb{R}^{2n}$ the submanifolds

$$X_r:=\{(p_i,q^i)\in\mathbb{R}^{2n}: p_{r+i}=c_i\cdots,\ q^{r+i}=d^i\ (i=1,\cdots n-r)\},$$

are Poisson transversals, i.e., symplectic submanifolds, for any values $c_i,d^j\in\mathbb{R}$. The are Poisson diffeomorphic to $\mathbb{R}^{2r}$, with the canonical Poisson structure.

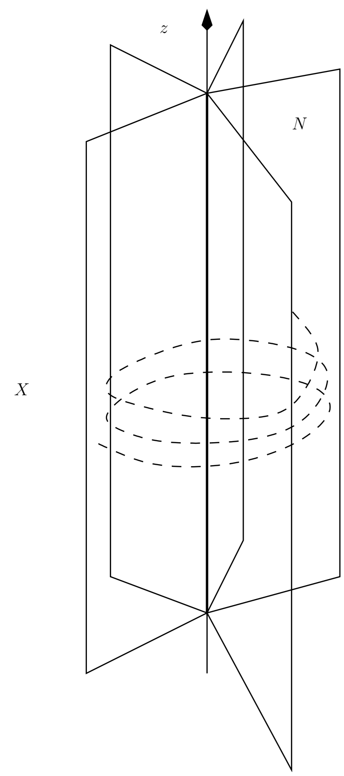
Example 2.19. Let us consider the 3-dimensional Lie algebra with basis $\{e_1,e_2,e_3\}$ such that:

$$[e_1,e_3]=e_1,\quad [e_2,e_3]=e_2,\quad [e_1,e_2]=0.$$

The corresponding linear Poisson structure in $\mathbb{R}^3$ is given by:

$$\pi = x\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}+y\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}.$$

This Poisson structure vanishes at $x=y=0$ so the points $(0,0,z)$ are Poisson submanifolds of dimension 0. On the other hand, the function $f(x,y,z)=\frac{x}{y}$ is a Casimir, so the planes $x - cy = 0$, $c \in \mathbb{R}$ are also Poisson submanifolds. This collection of Poisson submanifolds is shown in the following figure as an open book decomposition of $\mathbb{R}^3$:



On the other hand, note that any 1-dimensional submanifold $X \subset \mathbb{R}^3 - \{(0,0,z)\}$ which is transverse to the planes $x - cy = 0$, $c \in \mathbb{R}$, satisfies:

$$T_C\mathbb{R}^3 = TX + \pi^\#(TX)^0,$$

and hence is a Poisson transversal. The induced Poisson structure π_X is the zero Poisson structure.

Poisson transversals behave functorially under pullbacks by Poisson maps. This turns out to be a very useful property:

Proposition 2.20. *Let $\Phi : (M_1, \pi_1) \to (M_2, \pi_2)$ be a Poisson map and let $X_2 \subset M_2$ be a Poisson transversal. Then Φ is transverse to X_2 and $X_1 := \Phi^{-1}(X_2)$ is a Poisson transversal in M_1. Moreover, Φ restricts to a Poisson map between the induced Poisson structures on X_1 and X_2.*

Proof. Consider $x \in X_1$ and let $y := \Phi(x) \in X_2$. Since Φ is Poisson we have:

$$\pi_2^{\#}(\xi) = \mathrm{d}_x\Phi\left(\pi_1^{\#}(\Phi^*(\xi))\right), \text{ for all } \xi \in T_y^*M_2. \tag{2.7}$$

Therefore $\pi_2^{\#}(T_y^*M_2) \subset \mathrm{d}_x\Phi(T_xM_1)$. Poisson transversality implies now that Φ is transverse to X_2:

$$T_yM_2 = T_yX_2 + \pi_2^{\#}(T_y^*M_2) = T_yX_2 + \mathrm{d}_x\Phi(T_xM_1).$$

In particular, X_1 is a submanifold of M_1. Note that

$$T_xX_1 = (\mathrm{d}_x\Phi)^{-1}(T_yX_2) \quad \text{and} \quad (T_xX_1)^0 = \Phi^*((T_yX_2)^0).$$

To show that X_1 is a Poisson transversal, we check condition (2.1). Let $V \in T_xM_1$, and decompose $\mathrm{d}_x\Phi(V) = U + \pi_2^{\#}(\xi)$, with $U \in T_yX_2$ and $\xi \in (T_yX_2)^0$. Then $\Phi^*(\xi) \in (T_xX_1)^0$, and by (2.7), $W := V - \pi_1^{\#}(\Phi^*(\xi))$ is mapped by $\mathrm{d}_x\Phi$ to U. Hence $W \in T_xX_1$. This shows that

$$V = W + \pi_1^{\#}(\Phi^*(\xi)) \in T_xX_1 + \pi_1^{\#}(T_xX_1)^0.$$

So (2.1) holds, and therefore X_1 is a Poisson transversal.

For $\xi \in T_y^*X_2$ we denote by $\widetilde{\xi}$ its unique extension to $T_y^*M_2$ that vanishes on $N_y^{\pi_2}X_2$, and we use similar notations for elements in $T_x^*X_1$. Since

$$\mathrm{d}_x\Phi(N_x^{\pi_1}X_1) = \mathrm{d}_x\Phi(\pi_1^{\#}(T_xX_1)^0) = \mathrm{d}_x\Phi(\pi_1^{\#}(\Phi^*(T_yX_2)^0)) = \pi_2^{\#}(T_yX_2)^0 = N_y^{\pi_2}X_2,$$

it follows that $\Phi^*(\widetilde{\xi}) = \widetilde{\Phi^*(\xi)}$, for $\xi \in T_y^*X_2$. Using this remark, we prove that Φ restricts to a Poisson map $(X_1, \pi_{X_1}) \to (X_2, \pi_{X_2})$:

$$\begin{aligned}\pi_{X_1}(\Phi^*(\xi), \Phi^*(\eta)) &= \pi_1(\widetilde{\Phi^*(\xi)}, \widetilde{\Phi^*(\eta)}) = \pi_1(\Phi^*(\widetilde{\xi}), \Phi^*(\widetilde{\eta})) \\ &= \pi_2(\widetilde{\xi}, \widetilde{\eta}) = \pi_{X_2}(\xi, \eta), \quad \text{for all } \xi, \eta \in T_y^*X_2.\end{aligned}$$

□

Corollary 2.21. *Let (M, π) be a Poisson manifold, let $X \subset M$ be a Poisson transversal with induced Poisson structure π_X, and let $N \subset M$ be a Poisson submanifold with Poisson structure π_N. Then N and X intersect transversally, $X \cap N$ is a Poisson transversal in (N, π_N) and a Poisson submanifold of (X, π_X), and the two induced Poisson structures on $X \cap N$ coincide.*

Exercise 2.22. Prove that condition (2.1) from the definition of a Poisson transversal is equivalent to the conclusion of Corollary 2.16, that the 2-form

$$\sigma_{X,x} : (T_xX)^0 \times (T_xX)^0 \longrightarrow \mathbb{R}, \ \ \sigma_{X,x}(\alpha,\beta) = \pi(\alpha,\beta),$$

be nondegenerate at every $x \in X$.

Exercise 2.23. Let (M,π) be a Poisson manifold, and let $X \subset M$ be a Poisson transversal with induced Poisson structure π_X. If $f \in C^\infty(M)$ is a Casimir function for (M,π), prove that $f|_X$ is a Casimir function for (X,π_X).

Exercise 2.24. Let $X \subset (M,\pi)$ be a Poisson transversal, let $H \in C^\infty(M)$ and denote by $\phi^t_{X_H}$ the flow of X_H. Show that $\phi^t_{X_H}(X)$ is a Poisson transversal, whenever defined.

Exercise 2.25. Prove the following partial converse of Proposition 2.20: Let $\Phi : (M_1,\pi_1) \to (M_2,\pi_2)$ be a Poisson map, and let $X_2 \subset M_2$ be a submanifold such that Φ is transverse to X_2. If $X_1 := \Phi^{-1}(X_2)$ is a Poisson transversal, prove that there is some open set U containing $X_2 \cap \Phi(X_1)$ such that $X_2 \cap U$ is a Poisson transversal in (M_2,π_2).

Exercise 2.26. Consider $\mathbb{C}^n = \{(z_1,\dots,z_n)\}$ endowed with the standard Poisson structure

$$\pi := \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial y_k}, \ \text{ where } z_k = x_k + iy_k.$$

We regard each cotangent space $T^*_z\mathbb{C}^n$ as a complex space, where multiplication by i is defined on the basis vectors as follows:

$$i \cdot \mathrm{d}x_k := -\mathrm{d}y_k, \ \ i \cdot \mathrm{d}y_k := \mathrm{d}x_k,$$

and extended linearly to $T^*_z\mathbb{C}^n$. Prove the following:

a) For a nonzero covector $\xi \in T^*_z\mathbb{C}^n$, $\pi(\xi, i\xi) > 0$.
b) Every linear complex subspace $V \subset \mathbb{C}^n$ is a Poisson transversal (HINT: Use part a) and Exercise 2.22)
c) Recall that a holomorphic function is a smooth map

$$f : \mathbb{C}^n \longrightarrow \mathbb{C}^m$$

such that its differential is complex linear at every point (where we identify $T_z\mathbb{C}^n \cong \mathbb{C}^n$). Prove that if $w \in \mathbb{C}^m$ is a regular value of a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}^m$, then $f^{-1}(w)$ is a Poisson transversal in $\mathbb{C}^n$.

2.3 Coisotropic submanifolds

Definition 2.27. A **coisotropic submanifold** of a Poisson manifold (M,π) is a submanifold $C \subset M$ such that:

$$\pi^{\#}(TC)^0 \subset TC, \tag{2.8}$$

or, equivalently, if:

$$\pi(\alpha,\beta) = 0, \quad \forall \alpha,\beta \in (TC)^0. \tag{2.9}$$

Example 2.28. Notice that the condition (2.8) in the definition of coisotropic submanifold admits two extreme cases:

(i) $\pi^{\#}(TC)^0 = 0$: by Proposition 2.2, these are exactly the **Poisson submanifolds**;
(ii) $\pi^{\#}(TC)^0 = TC$: these manifolds are called **Lagrangian submanifolds**, and we will see several examples next.

Example 2.29. Any codimension 1 submanifold C of a Poisson manifold (M,π) clearly satisfies (2.9), so it is a coisotropic submanifold. Such a submanifold, in general, is neither a Lagrangian submanifold nor a Poisson submanifold.

Example 2.30. Let $\mathfrak{g}$ be a Lie algebra. For the linear Poisson structure on $\mathfrak{g}^*$, we have for $\xi \in \mathfrak{g}^*$ and $v,w \in \mathfrak{g} = T^*_\xi\mathfrak{g}^*$:

$$\pi_{\mathfrak{g}^*}(v,w)_\xi = 0 \quad \Leftrightarrow \quad \langle [v,w],\xi\rangle = 0.$$

Hence, a subspace $V^0 \subset \mathfrak{g}^*$ is a coisotropic submanifold iff $V \subset \mathfrak{g}$ is a Lie subalgebra.
Note that $V^0 \subset \mathfrak{g}^*$ is a Poisson submanifold iff $V \subset \mathfrak{g}$ is an ideal and $V^0 \subset \mathfrak{g}^*$ is never a Lagrangian submanifold, since at the origin $\pi_0^{\#} \equiv 0$.

Example 2.31. Let $\mathbb{R}^{2n}$ with the canonical Poisson structure. For any open set $U \subset \mathbb{R}^{2n}$ the submanifolds

$$C_r^p = \{(q,p) \in U : p_r = p_{r+1} = \cdots = p_n = 0\},$$
$$C_r^q = \{(q,p) \in U : q^r = q^{r+1} = \cdots = q^n = 0\},$$

where $1 \le r \le n$, are coisotropic submanifolds. Note that these submanifolds are Lagrangian iff $r = n$ and are Poisson submanifolds iff $r = 0$.

We now turn to some basic properties of coisotropic submanifolds. You should compare these properties with the corresponding properties for Poisson submanifolds:

Proposition 2.32. *Let (M,π) be a Poisson manifold. For a closed submanifold $C \subset M$ the following conditions are equivalent:*

(i) C is a coisotropic submanifold;
(ii) The vanishing ideal $\mathscr{I}(C)$ is a Poisson subalgebra;
(iii) For every $h \in \mathscr{I}(C)$ the hamiltonian vector field X_h is tangent to C.

Proof. (i) $\Rightarrow$ (ii) If $f_1, f_2 \in \mathscr{I}(C)$, then $\mathrm{d}_x f_1, \mathrm{d}_x f_2 \in (TC)^0$ for any $x \in C$. Hence, if $C \subset M$ is a coisotropic submanifold we have:

$$\{f_1, f_2\}(x) = \pi_x(\mathrm{d}_x f_1, \mathrm{d}_x f_2) = 0, \quad \forall x \in C$$

so $\{f_1, f_2\} \in \mathscr{I}(C)$.

(ii) $\Rightarrow$ (iii) Assume that $\mathscr{I}(C)$ is a Poisson subalgebra. If $h, f \in \mathscr{I}(C)$, we have:

$$X_h(f)(x) = \{h, f\}(x) = 0, \quad \forall x \in C.$$

Since C is a closed submanifold, this implies that X_h is tangent to C.

(iii) $\Rightarrow$ (i) If (iii) holds, we find that if $x \in C$:

$$\pi(\mathrm{d}_x f, d_x g) = X_f(g)(x) = 0,$$

for any $f, g \in \mathscr{I}(C)$. Since C is a closed submanifold, we have that $(T_x C)^0$ is generated by elements $\mathrm{d}_x f$ where $f \in \mathscr{I}(C)$, so we conclude that

$$\pi(\alpha, \beta) = 0, \quad \forall \alpha, \beta \in (TC)^0.$$

Therefore, C is isotropic. $\square$

Exercise 2.33. Give an example of a Poisson manifold (M, π_N) with two coisotropic submanifolds $C_1, C_2 \subset M$ which intersect transversely, such that the intersection $C_1 \cap C_2$ is not coisotropic.

Proposition 2.34. *Let $\Phi : (M, \pi_M) \to (N, \pi_N)$ be a Poisson map and assume that Φ is transverse to a coisotropic submanifold $C \subset N$. Then $\Phi^{-1}(C) \subset M$ is a coisotropic submanifold.*

Proof. On the one hand, since $C \subset N$ is coisotropic, we have:

$$\pi_N^\#(TC)^0 \subset TC.$$

On the other hand, since Φ is transverse to $C \subset N$, we have that $\Phi^{-1}(C) \subset M$ is a submanifold and that:

$$T\Phi^{-1}(C) = (\mathrm{d}\Phi)^{-1}(TC), \quad (T\Phi^{-1}(C))^0 = (\mathrm{d}\Phi)^*(TC)^0.$$

Using the fact that Φ is a Poisson map, we then find:

$$\mathrm{d}\Phi \cdot \pi_M^\#(T\Phi^{-1}(C))^0 = \mathrm{d}\Phi \cdot \pi_M^\# \cdot (\mathrm{d}\Phi)^*(TC)^0 = \pi_N^\#(TC)^0 \subset TC.$$

Therefore, we conclude that

$$\pi_M^\#(T\Phi^{-1}(C))^0 \subset (\mathrm{d}\Phi)^{-1}(TC) = T\Phi^{-1}(C),$$

so $\Phi^{-1}(C)$ is a coisotropic submanifold. $\square$

There is one more important property of coisotropic objects and which shows their relevance in Poisson geometry. In order to express it, we introduce the following notation: if (M_1, π_{M_1}) and (M_2, π_{M_2}) are Poisson manifolds we denote by $(M_1 \times \overline{M_2}, \pi_{M_1 \times \overline{M_2}})$ the Poisson manifold whose underlying manifold is the direct product $M_1 \times M_2$ and whose Poisson bivector is:

$$\pi_{M_1 \times \overline{M_2}}(\alpha_1 + \alpha_2, \beta_1 + \beta_2) := \pi_{M_1}(\alpha_1, \beta_1) - \pi_{M_2}(\alpha_2, \beta_2),$$

for any $\alpha_1, \beta_1 \in T^*M_1$ and $\alpha_2, \beta_2 \in T^*M_2$. Then we have:

Proposition 2.35. *Let (M, π_M) and (N, π_N) be Poisson manifolds. For a smooth map $\Phi : M \to N$ the following conditions are equivalent:*

(i) Φ is Poisson map;
(ii) Graph$(\Phi) \subset M \times \overline{N}$ is a coisotropic submanifold.

Proof. Notice that we have:

$$T\mathrm{Graph}(\Phi) = \{(v, \mathrm{d}\Phi(v)) : v \in TM\},$$

so that:

$$(T\mathrm{Graph}(\Phi))^0 = \{((\mathrm{d}\Phi)^*\beta, -\beta) : \beta \in T^*N\}.$$

It follows that for any $\gamma \in (T\mathrm{Graph}(\Phi))^0$ we have;

$$\pi_{M\times\overline{N}}^\#(T\mathrm{Graph}(\Phi))^0 = \{(\pi_M^\#((\mathrm{d}\Phi)^*\beta), \pi_N^\#(\beta)) : \beta \in T^*N\}.$$

Now, if Φ is a Poisson map, then the left-hand side is of the form $(v, \mathrm{d}\Phi(v))$, so belongs to $T\mathrm{Graph}(\Phi)$. Hence $\mathrm{Graph}(\Phi) \subset M \times \overline{N}$ is a coisotropic submanifold. Conversely, if $\mathrm{Graph}(\Phi) \subset M \times \overline{N}$ is a coisotropic submanifold, then the left-hand side must be of the form $(v, \mathrm{d}\Phi(v))$, i.e, we must have:

$$\mathrm{d}\Phi \cdot \pi_M^\#((\mathrm{d}\Phi)^*\beta) = \pi_N^\#(\beta), \quad \forall \beta \in T^*N.$$

This means that Φ is a Poisson map. $\square$

In general, a coisotropic submanifold $C \subset (M, \pi)$ *does not* carry an induced Poisson structure. However, the fact that $\mathscr{I}(C)$ is a Poisson algebra suggests that there should be still some Poisson structure associated with C. In fact, assume that $D := \pi^\#(TC)^0$ has constant rank, so that it defines an distribution in TC. The Poisson bivector induces a section $\pi_C \in \Gamma^2(\wedge^2 D^0)$ by setting:

$$\pi_C(\alpha, \beta) = \pi(\widetilde{\alpha}, \widetilde{\beta}), \quad \alpha, \beta \in D^0 \subset T^*C.$$

where $\widetilde{\alpha},\widetilde{\beta}\in T^*M$ are any extensions of $\alpha,\beta\in D^0$. One should think of π_C has a transverse Poisson structure to $\mathscr{F}$. This is made precise in the following proposition whose prove we leave as an exercise:

Proposition 2.36. *Let $C\subset(M,\pi)$ be a coisotropic submanifold such that $\pi^\#(TC)^0$ has constant rank. Then $\pi^\#(TC)^0$ is an involutive distribution in C and π_C is a transverse Poisson structure to the corresponding foliation $\mathscr{F}$ of C:*

$$\mathscr{L}_X\pi_C=0,\quad \forall X\in\mathfrak{X}(\mathscr{F}).$$

In particular, if the leaf space of $\mathscr{F}$ is smooth, then there is an induced Poisson structure on $C/\mathscr{F}$.

The foliation $\mathscr{F}$ such that $T\mathscr{F}=\pi^\#(TC)^0$ is called the **characteristic foliation** of the coisotropic submanifold C.

Remark 2.37. There are other notions of submanifolds in Poisson geometry. For example, since a coisotropic submanifold $C\subset(M,\pi)$ satisfies $\pi^\#(TC)^0\subset C$, you may wonder about submanifolds $I\subset(M,\pi)$ that satisfy the dual condition:

$$TI\subset\pi^\#(TI)^0.$$

These are called **isotropic submanifolds**. They play a less important role in Poisson geometry than coisotropic submanifolds. The following exercise hints at yet another notion of submanifold.

Exercise 2.38. Let (M,π) be a Poisson manifold and let Γ be a finite group acting on M such that for each $\gamma\in\Gamma$ the translation $\Phi_\gamma:M\to M$, $x\mapsto\gamma\cdot x$, is a Poisson diffeomorphism. Assume that the fixed point set

$$M^\Gamma:=\{x\in M:\gamma\cdot x=x,\forall\gamma\in\Gamma\}$$

is a manifold[2] and denote by $C^\infty(M)^\Gamma$ the space of Γ-invariant functions. Show that:

(a) If $F,G\in C^\infty(M)^\Gamma$ then $\{F,G\}\in C^\infty(M)^\Gamma$;
(b) If $F_1,F_2,G_1,G_2\in C^\infty(M)^\Gamma$ then:

$$F_1|_{M^\Gamma}=F_2|_{M^\Gamma}\text{ and }G_1|_{M^\Gamma}=G_2|_{M^\Gamma}\quad\Longrightarrow\quad\{F_1,G_1\}|_{M^\Gamma}=\{F_2,G_2\}|_{M^\Gamma};$$

(c) Every function $f\in C^\infty(M^\Gamma)$ is of the form $f=F|_{M^\Gamma}$ for some (non-unique) Γ-invariant function $F\in C^\infty(M)^\Gamma$;
(d) Conclude that there is a natural induced Poisson bracket on M^Γ such that:

$$\{f,g\}=\{F,G\}|_{M^\Gamma},$$

where $F,G\in C^\infty(M)^\Gamma$ are any Γ-invariant extensions of f and g.

[2] One can show that M^Γ is always a manifold, although it can have connected components of different dimensions.

Homework 2: Poisson Relations and Coisotropic Calculus

2.1. A **Poisson vector space** is a vector space V with a skew-symmetric bilinear form $\pi : V^* \times V^* \to \mathbb{R}$. We denote by $\pi^\# : V^* \to V$ the corresponding linear map $\xi \mapsto \pi(\xi, \cdot)$. Show that if (V, π) is a Poisson vector space then the quotient $V^*/\operatorname{Ker}\pi \simeq \pi^\#(V^*)$ is a Poisson vector space with a non-degenerate bilinear form $\overline{\pi}$.

2.2. A **coisotropic subspace** of a Poisson vector space (V, π) is a linear subspace $C \subset V$ such that:

$$\pi^\#(C^0) \subset C.$$

Show that if $C \subset V$ is coisotropic then $C/\pi^\#(C^0)$ is naturally a Poisson vector space and that the following statements are equivalent:

(i) $C \subset V$ is a coisotropic subspace of V;
(ii) $C \cap \pi^\#(V^*)$ is a coisotropic subspace of $\pi^\#(V^*)$.

Moreover, if C_1 and C_2 are coisotropic subspaces of V, so is $C_1 + C_2$.

2.3. A **linear Poisson map** between two Poisson vector spaces (V, π_V) and (W, π_W) is a linear map $\Phi : V \to W$ such that:

$$\pi_W^\# = \Phi \circ \pi_V^\# \circ \Phi^*.$$

Show that if $\Phi : (V, \pi_V) \to (W, \pi_W)$ is a linear Poisson map, then:

(i) If $C \subset V$ is a coisotropic subspace then so is $\Phi(C) \subset W$;
(ii) If $C \subset W$ is a coisotropic subspace then so is $\Phi^{-1}(C) \subset W$.

2.4. A **linear relation** $R : V \to W$ is a linear subspace $R \subset V \times W$. If $R : V \to W$ and $S : W \to Z$ are linear relations the **composite linear relation** $S \circ R : V \to Z$ is defined to be:

$$S \circ R := \{(v, z) \in V \times Z : \exists w \in W \text{ such that } (v, w) \in R \text{ and } (w, z) \in S\}.$$

Also, if $R : V \to W$ is a linear relation, we denote by $R^{-1} : W \to V$ the inverse relation defined by:

$$R^{-1} := \{(w, v) \in W \times V : (v, w) \in R\}.$$

Note that a linear map $\Phi : V \to W$ can be thought of as linear relation $\operatorname{Graph}(\Phi) \subset V \times W$. Composition of linear maps corresponds to composition of linear relations and if a linear map has an inverse, then $\operatorname{Graph}(\Phi^{-1}) = \operatorname{Graph}(\Phi)^{-1}$.

If (V, π_V) and (W, π_W) are Poisson vector spaces, a **linear Poisson relation** is a coisotropic subspace $R \subset V \times \overline{W}$. Show that:

(i) If $R : V \to W$ is a linear Poisson relation then $R^{-1} : W \to V$ is a linear Poisson relation;
(ii) Every coisotropic subspace $C \subset (V, \pi_V)$ gives rise to a linear Poisson relation $R(C) : \{0\} \to V$;

(iii) A linear map $\Phi : (V,\pi_V) \to (W,\pi_W)$ is Poisson if and only if Graph(Φ) is a linear Poisson relation.
(iv) If $R : V \to W$ and $S : W \to Z$ are linear Poisson relations then $S \circ R : V \to Z$ is a linear Poisson relation.

2.5. Let V be a vector space and let $\sim$ be a linear equivalence relation in V. If $\Phi : V \to V/$ is the quotient map, we see that $\sim$ is just the equivalence relation $\Phi^{-1} \circ \Phi : V \to V$ obtained as Graph$(\Phi)^{-1} \circ$Graph(Φ). Conversely, every surjective linear map $\Phi : V \to W$ determines a linear equivalence relation $\Phi^{-1} \circ \Phi : V \to V$, and W is naturally isomorphic to the space of equivalence classes.

Let (V,π_V) be a Poisson vector space and $\Phi : V \to W$ is a surjective linear map. Show that W is a Poisson vector space for which Φ is a linear Poisson map if and only if the equivalence relation $\Phi^{-1} \circ \Phi : V \to V$ is a Poisson relation.

We now turn to non-linear versions of the previous problems. For the first 3 problems, these were discuss in the text. For the last 2 problems, we proceed as follows.

2.6. Given manifolds M and N a **relation** $R : M \to N$ is a submanifold $R \subset M \times N$. If $R : M \to N$ is a relation, we denote by $R^{-1} : N \to M$ the inverse relation defined by:

$$R^{-1} := \{(y,x) \in N \times M : (x,y) \in R\}.$$

If $R : M \to N$ and $S : N \to P$ are relations the **composite relation** $S \circ R : M \to P$ defined by:

$$S \circ R := \{(x,z) \in M \times P : \exists y \in N \text{ such that } (x,y) \in R \text{ and } (y,z) \in S\},$$

may fail to be a submanifold. We will say that two relations $R : M \to N$ and $S : N \to P$ meet cleanly if $R \circ S$ is a submanifold of $M \times P$ and for each $(x,y) \in R$ and $(y,z) \in S$ we have

$$T_{(x,z)}(S \circ R) = T_{(y,z)}S \circ T_{(x,y)}R.$$

If (M,π_M) and (N,π_N) are Poisson manifolds, a **Poisson relation** is a coisotropic submanifold $R \subset M \times \overline{N}$. Show that:

(i) If $R : M \to N$ is a Poisson relation, the inverse relation $R^{-1} : N \to M$ is also Poisson.
(ii) A coisotropic submanifold $C \subset (M,\pi)$ gives rise to a Poisson relation $R(C) : (\{*\},0) \to (M,\pi)$.
(iii) A map $\Phi : (M,\pi_M) \to (N,\pi_N)$ is Poisson if and only if Graph(Φ) is a Poisson relation.
(iv) If $R : (M,\pi_M) \to (N,\pi_N)$ and $S : (N,\pi_N) \to (P,\pi_P)$ are Poisson relations which meet cleanly then $S \circ R : M \to P$ is a Poisson relation.

2.7. Let (M,π_N) be a Poisson manifold. Given a submersion $\Phi : M \to N$ show that there exists a Poisson structure $\pi_N \in \mathfrak{X}^2(N)$ such that Φ is a Poisson map if and only if $\Phi^{-1} \circ \Phi : M \to M$ is a Poisson relation.

2.8. Let (M,π) be a Poisson manifold and let $\mathscr{F}$ be a foliation of M. Denote by $R(\mathscr{F}) : M \to M$ the equivalence relation:

$$R(\mathscr{F}) = \{(x,y) \in M \times M : \ x \text{ and } y \text{ belong to the same leaf of } \mathscr{F}\}.$$

Show that the sheaf of locally constant functions of $\mathscr{F}$ is closed under the Poisson bracket if and only if $R(\mathscr{F})$ is a Poisson relation.

Chapter 3
The symplectic foliation

3.1 Regular Poisson Structures

Let $\pi \in \mathfrak{X}^2(M)$ be a bivector field. The **rank of** π at $x \in M$ is the rank of the linear map $\pi_x^\# : T_x^*M \to T_xM$. By skew-symmetry, the rank at any point is an even number. In general, the rank will vary from point to point.

Example 3.1. A non-degenerate Poisson structure (i.e., a symplectic structure) is a Poisson structure π for which $\operatorname{rank}\pi_x = \dim M$, for all $x \in M$.

Example 3.2. For the quadratic Poisson structure π_A on $\mathbb{R}^n$ associated with a skew-symmetric matrix A, the rank at $x = (x^1, \ldots, x^n)$ is exactly the rank of the matrix obtained from A by removing the rows and columns corresponding to the coordinates of the point x that vanish.

Example 3.3. For the linear Poisson structure $\pi_\mathfrak{g}$ on the dual of a Lie algebra the rank at the origin is zero and it will be non-zero for a generic point, provided the Lie algebra is non-abelian.

Exercise 3.4. Show that if $\mathfrak{g} = \mathfrak{su}(3)$ is the Lie algebra consisting of 3×3 skew-hermitian matrices of trace 0, then $\operatorname{rank}\pi_{\mathfrak{g}^*}$ takes the values 0, 4 and 6.

Lemma 3.5. *The function $M \ni x \mapsto \operatorname{rank}_x \pi$ is lower semi-continous: for any point $x_0 \in M$ there is a neighborhood V of x_0 such that $\operatorname{rank}\pi_x \geq \operatorname{rank}\pi_{x_0}$ for all $x \in V$.*

Proof. Choose local coordinates $(x^1, \ldots, x^m)$ around x_0. If $\operatorname{rank}\pi_{x_0} = 2n$ the matrix with coefficients $\pi^{ij}(x) = \{x^i, x^j\}$ has rank $2n$ at x_0, so some minor of size $2n \times 2n$ has non-zero determinant. This determinant is non-zero in some neighborhood of x_0, so it follows that the rank in a neighborhood of x_0 is at least $2n$.

Definition 3.6. A point x_0 is called a **regular point** of (M, π) if there is a neighborhood U of x_0 such that $\operatorname{rank}\pi_x = \operatorname{rank}\pi_{x_0}$ for all $x \in U$. Otherwise, we call x_0 a **singular point** of (M, π). We let M_{reg} and M_{sing} denote the sets of regular points and singular points of (M, π).

From the lemma above, we conclude:

Proposition 3.7. *For a Poisson manifold (M,π) the set of regular points M_{reg} is an open dense subset of M and the set of singular points M_{sing} is a closed nowhere dense subset of M.*

Proof. From the definition, it is clear that M_{reg} is an open subset and $M_{\mathrm{sing}} = M - M_{\mathrm{reg}}$ is a closed subset. Now let x_0 be a singular point and let V be any neighborhood of x_0. The function $V \ni x \mapsto \operatorname{rank} \pi_x$ takes a finite number of values. If $x \in V$ is a point where the rank attains its maximum, then by the lemma above, x must be a regular point. □

Notice that when the rank of a bivector $\pi \in \mathfrak{X}^2(M)$ is constant throughout M we obtain a distribution $M \ni x \mapsto \operatorname{Im} \pi_x^\# \subset T_x M$. This is a smooth distribution since it is spanned by vector fields of the form $X_f = \pi^\# \mathrm{d}f$.

Theorem 3.8. *For a Poisson bivector $\pi \in \mathfrak{X}^2(M)$ of constant rank the distribution* $\operatorname{Im} \pi^\#$ *is integrable. Each leaf S of* $\operatorname{Im} \pi^\#$ *is a Poisson submanifold of (M,π) and the induced Poisson structure $\pi_S \in \mathfrak{X}^2(S)$ is non-degenerate.*

Proof. The distribution $\operatorname{Im} \pi^\#$ is involutive because it is spanned by hamiltonian vector fields and the Lie bracket of two hamiltonian vector fields is a hamiltonian vector field (Proposition 1.7). Hence, by the Frobenius Theorem, the distribution is integrable.

Let S be a leaf of this distribution so that $T_x S = \operatorname{Im} \pi_x^\#$, for every $x \in S$. By Proposition 2.2, S is a Poisson submanifold. The induced Poisson structure π_S satisfies:

$$\pi^\#(\alpha) = \pi_S(\alpha|_S), \quad \forall \alpha \in T_S^* M.$$

This shows that $\operatorname{Im} \pi_S^\# = TS$. Hence π_S is non-degenerate. □

Definition 3.9. A **regular Poisson structure** $\pi \in \mathfrak{X}^2(M)$ is a Poisson structure whose rank is constant.

Hence a regular Poisson manifold (M,π) is foliated into symplectic leaves. In general, the symplectic leaves are only immersed submanifolds.

Example 3.10. On the 3-torus $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, with angle coordinates $(\theta^1, \theta^2, \theta^3)$, consider the family of Poisson structures parameterized by $\lambda \in \mathbb{R}$:

$$\pi_\lambda = \frac{\partial}{\partial \theta^1} \wedge \left(\frac{\partial}{\partial \theta^2} + \lambda \frac{\partial}{\partial \theta^3} \right),$$

These Poisson structures have constant rank 2. We leave it as an exercise to check that if $\lambda \in \mathbb{Q}$ the symplectic leaves of π_λ are compact embedded submanifolds diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ and if $\lambda \in \mathbb{R} - \mathbb{Q}$ the symplectic leaves of π_λ are immersed submanifolds diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$, each leaf being dense in $\mathbb{T}^3$.

Given a regular Poisson manifold (M,π) we denote by $\mathscr{S}$ its symplectic foliation. The symplectic forms on the leaves assemble to give a smooth section $\omega_{\mathscr{S}} \in \Omega^2(\mathscr{S}) := \Gamma(\wedge^2 T\mathscr{S})$. This form is non-degenerate, i.e., for any $v \in T\mathscr{S}$:

$$\omega_{\mathscr{S}}(v,w) = 0, \forall w \in T\mathscr{S} \quad \Longrightarrow \quad v = 0,$$

and it is leafwise closed, i.e.,

$$\mathrm{d}_{\mathscr{S}}\omega = 0.$$

(see the Homework at the end of this chapter for the definition of the leafwise de Rham differential). In other words, $\omega_{\mathscr{S}}$ is a *foliated symplectic form*.

Conversely, we have:

Proposition 3.11. *Given a foliation $\mathscr{S}$ of a manifold M with a leafwise symplectic form $\omega_{\mathscr{S}} \in \Omega^2(\mathscr{S})$ there exists a unique regular Poisson structure $\pi \in \mathfrak{X}^2(M)$ such that $(\mathscr{S}, \omega_{\mathscr{S}})$ is the symplectic foliation of π.*

Proof. For each leaf $S \in \mathscr{S}$ we have a Poisson bivector $\pi_S \in \mathfrak{X}^2(S)$ where $\pi_S = (\omega_S)^{-1}$. We define $\pi \in \mathfrak{X}^2(M)$ by setting:

$$\pi(\alpha,\beta) := \pi_S(\alpha|_S, \beta|_S).$$

We just need to check that $[\pi,\pi] = 0$, which we leave as an exercise. □

Therefore, there is a 1:1 correspondence between regular Poisson structures on M and symplectic foliations of M. Note that for a regular Poisson manifold (M,π) with symplectic foliation $(\mathscr{S}, \omega_{\mathscr{S}})$, we can use a simple partition of unit argument to show that we can find a globally defined 2-form $\omega \in \Omega^2(M)$ such that its restriction to each symplectic leaf coincides with the symplectic form on the leaf:

$$\omega(v,w) = \omega_{\mathscr{S}}(v,w), \text{ if } v,w \in T\mathscr{S}.$$

Of course this extension of the foliated 2-form $\omega_{\mathscr{S}}$ to a global defined 2-form ω is far from being unique. Moreover, although the pullback of ω to each leaf is closed and non-degenerate, in general one cannot choose ω neither to be closed nor to be non-degenerate.

Exercise 3.12. Give an example of a regular Poisson structure on a manifold M for which the corresponding foliated symplectic form cannot be extended to a closed/non-degenerate form on M.
HINT: Consider the linear Poisson structure on $\mathfrak{so}^*(3)$ with the origin removed and use Stokes Theorem.

3.2 The Darboux-Weinstein Theorem

What can be said about the singular distribution $\operatorname{Im}\pi^{\#}$ for non-regular Poisson structures? The clue to understand this more general situation is the following fundamental theorem in Poisson geometry:

Theorem 3.13 (Darboux-Weinstein). *Let (M,π) be a Poisson manifold and assume that $\operatorname{rank}\pi_{x_0} = 2n$. There exists coordinates $(U, p_1,\dots,p_n,q^1,\dots,q^n,x^1,\dots,x^s)$ centered at x_0 such that:*

$$\pi|_U = \sum_{i=1}^{n} \frac{\partial}{\partial p_i}\wedge\frac{\partial}{\partial q^i} + \sum_{a,b=1}^{s} \phi^{ab}(x)\frac{\partial}{\partial x^a}\wedge\frac{\partial}{\partial x^b}$$

where the $\phi^{ab}(x)$ are smooth functions of $(x^1,\dots,x^s)$ such that $\phi^{ab}(0)=0$.

The Darboux-Weinstein theorem shows that a Poisson structure $\pi \in \mathfrak{X}^2(M)$ around any point $x_0 \in M$ splits as a product of a symplectic structure of dimension equal to $\operatorname{rank}\pi_{x_0}$ and a *transverse Poisson structure* which vanishes at x_0. In fact, we will deduce this theorem as a corollary of the following more general splitting theorem for Poisson transversals:

Theorem 3.14 (Splitting Theorem). *Let (M,π) be a Poisson manifold and $X \subset M$ a Poisson transversal of codimension 2n. For any $x_0 \in X$ there are local coordinates $(U, p_1,\dots,p_n,q^1,\dots,q^n,x^1,\dots,x^m)$ centered at x_0 such that $X\cap U = \{p_1=\dots=p_n=q^1=\dots=q^n=0\}$ and:*

$$\pi|_U = \sum_{i=1}^{n} \frac{\partial}{\partial p_i}\wedge\frac{\partial}{\partial q^i} + \pi_X|_{X\cap U}. \tag{3.1}$$

Before we prove this result, lets us deduce the existence of Darboux-Weinstein coordinates from it.

Proof (of Theorem 3.13). Let $x_0 \in M$. We claim that any small enough submanifold $X \subset M$ complementary to $\operatorname{Im}\pi^{\#}_{x_0}$ is a Poisson transversal. In fact, if:

$$T_{x_0}M = T_{x_0}X \oplus \operatorname{Im}\pi^{\#}_{x_0},$$

since $\operatorname{Ker}\pi^{\#}_{x_0} = (\operatorname{Im}\pi^{\#}_{x_0})^0$, it follows that:

$$(T_{x_0}X)^0 \cap \operatorname{Ker}\pi^{\#}_{x_0} = \{0\}.$$

Since $T_{x_0}X$ and $\operatorname{Im}\pi^\#_{x_0}$ have complementary dimensions, we conclude that:

$$T_{x_0}M = T_{x_0}X \oplus \pi^\#(T_{x_0}X)^0.$$

Therefore, if X is small enough, this condition holds at every $x \in X$ and X is a Poisson transversal, as claimed.

The Darboux-Weinstein Theorem now follows from Theorem 3.14 by observing that $\operatorname{codim}(X) = \operatorname{rank}\pi_{x_0}$. $\square$

Proof (of Theorem 3.14).

We start by observing that one can reduce to the case where $\operatorname{codim} X = 2$ because of the following lemma:

Lemma 3.15. *If $X \subset M$ is a Poisson transversal of codimension $2n$, for any x_0 there is a open neighborhood $x_0 \in V \subset X$ and a flag of Poisson transversals*

$$V \subset X_{n-1} \subset \cdots \subset X_1 \subset M,$$

such that $\operatorname{codim}(X_i) = 2i$.

In fact, if $\operatorname{codim} X$ is positive we can find $\alpha, \beta \in (T_{x_0}X)^0$ such that $\pi(\alpha,\beta) \neq 0$. Notice that $T_{x_0}X \cap \langle \pi^\#(\alpha), \pi^\#(\beta)\rangle = \{0\}$, because if $\pi^\#(a\alpha + b\beta) \in T_{x_0}X$, for some $a, b \in \mathbb{R}$, we must have:

$$\begin{cases} 0 = \alpha(\pi^\#(a\alpha + b\beta)) = -b\pi(\alpha,\beta), \\ 0 = \beta(\pi^\#(a\alpha + b\beta)) = a\pi(\alpha,\beta), \end{cases} \implies a = b = 0.$$

Now choose a submanifold $X_{n-1} \subset M$ such that $V = X_{n-1} \cap X$ is an open neighborhood of x_0 in X and:

$$T_{x_0}X_{n-1} = T_{x_0}X \oplus \langle \pi^\#(\alpha), \pi^\#(\beta)\rangle.$$

To see that this is possible, one can choose a tubular neighborhood of X such that $\langle \pi^\#(\alpha), \pi^\#(\beta)\rangle \subset \mathscr{N}_{x_0}(X)$ and then choose a small open neighborhood of x_0 in X over which $\mathscr{N}(X)$ is trivial. Then, at x_0, we find

$$T_{x_0}M = T_{x_0}X_{n-1} \oplus \pi^\#(T_{x_0}X_{n-1})^0.$$

Choosing X_{n-1} small enough, it follows that this condition holds at every $x \in X_{n-1}$ so X_{n-1} is a Poisson transversal in M.

We claim that V is also a Poisson transversal in X_{n-1} and that the induced Poisson structures on V arising from X_{n-1} and from M coincide. This follows by observing that the decompositions

$$T_{X_{n-1}}M = TX_{n-1} \oplus \pi^\#(TX_{n-1})^0, \quad T_V M = TV \oplus \pi^\#(TV)^0,$$

and the corresponding dual decompositions of $T^*_{X_{n-1}}M$ and $T^*_V M$, together with $TV \subset TX$, yield a commutative diagram:

$$\begin{tikzcd}
T^*V \arrow[rrr, "\pi_V^\#"] \arrow[dr] \arrow[dd] & & & TV \\
& T^*M \arrow[r, "\pi"] & TM \arrow[ur] \arrow[dr] & \\
T^*X_{n-1} \arrow[ur] \arrow[rrr, "\pi^\#_{X_{n-1}}"'] & & & TX_{n-1} \arrow[uu]
\end{tikzcd}$$

It follows that we have a decomposition:

$$T_V X_{n-1} M = TV \oplus \pi^\#_{X_{n-1}}(TV)^0,$$

and that the Poisson structure on V induced from this decomposition (the composition of the outer arrows in the diagram) coincides with π_V, so the claim follows and the lemma is proved.

It remains to prove Theorem 3.14 in the case where X is a Poisson transversal of codimension 2. Choose $\alpha \in (T^*_{x_0}X)^0$ such that $\pi^\#(\alpha)$ is transverse to X. Let p_1 be a function vanishing in X such that $\mathrm{d}_{x_0} p_1 = \alpha$. The vector field X_{p_1} is transverse to X in a neighborhood of x_0 and we denote its flow by $\phi^t_{X_{p_1}}$.

We complete p_1 to local coordinates $(W, \varphi) = (W, (p_1, z^2, z^3, \dots, z^m))$ centered at x_0 such that $W \cap X = \{p_1 = 0,\ z^2 = 0\}$. The map:

$$(t_1, t_2, \dots, t_m) \mapsto \phi^{t_2}_{X_{p_1}}(\varphi^{-1}(t_1, 0, t_3, \dots, t_m)).$$

is a local diffeomorphism sending 0 to x_0. Its inverse around some open neighborhood V of x_0 gives a new local coordinate system $(V, \psi) = (V, (p_1, q^1, y^3, \dots, y^n))$ centered at x_0 such that $V \cap X = \{p_1 = 0,\ q^1 = 0\}$ and:

$$X_{p_1} = \frac{\partial}{\partial q^1}.$$

Note that the first component in this new coordinate system (V, ψ) does coincide with the function p_1 since we have:

$$\frac{\mathrm{d}}{\mathrm{d}t_2} p_1(\phi^{t_2}_{X_{p_1}}(\varphi^{-1}(t_1, 0, t_3, \dots, t_m)) = X_{p_1}(p_1)(\phi^{t_2}_{X_{p_1}}(\varphi^{-1}(t_1, 0, t_3, \dots, t_m))) = 0,$$

and $p_1(\phi^0_{X_{p_1}}(\varphi^{-1}(t_1, 0, t_3, \dots, t_m)) = t_1$, so $p_1(\psi^{-1}(t_1, \dots, t_n)) = t_1$.

Now for this new coordinates:

$$\{p_1, q^1\} = X_{p_1}(q_1) = 1,\ \{p_1, y^i\} = X_{p_1}(y^i) = 0 \quad (i = 3, \dots, m).$$

Using the Jacobi identity we find:

$$\frac{\partial(\{q^1,y^i\})}{\partial q^1} = X_{p_1}(\{q^1,y^i\}) = \{p_1,\{q^1,y^i\}\} = \{\{p_1,q^1\},y^i\} + \{q^1,\{p_1,y^i\}\} = 0.$$

It follows that:

$$X_{q_1} = -\frac{\partial}{\partial p_1} + \sum_{i=3}^{n} X^i(p_1,y^3,\dots,y^m)\frac{\partial}{\partial y^i}.$$

The map:

$$(t_1,t_2,\dots,t_m) \mapsto \phi_{X_{q_1}}^{-t_1}(\psi^{-1}(0,t_2,t_3,\dots,t_m)).$$

is a local diffeomorphism sending 0 to x_0. Its inverse around some open neighborhood U of x_0 gives a new local coordinate system $(U,\tau) = (U,(p_1,q^1,x^3,\dots,x^n))$ centered at x_0 such that $U \cap X = \{p_1 = 0,\ q^1 = 0\}$ and:

$$X_{p_1} = \frac{\partial}{\partial q^1}, \quad X_{q_1} = -\frac{\partial}{\partial p_1}.$$

Again we need to check that for this new coordinate system (U,τ) the first and second components do coincide with the functions p_1 and q_1. For this we observe that:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t_1} p_1(\phi_{X_{q_1}}^{-t_1}(\psi^{-1}(0,t_2,t_3,\dots,t_m))) &= -X_{q_1}(p_1)(\phi_{X_{q_1}}^{-t_1}(\psi^{-1}(0,t_2,t_3,\dots,t_m))) = 1,\\
\frac{\mathrm{d}}{\mathrm{d}t_1} q_1(\phi_{X_{q_1}}^{-t_1}(\psi^{-1}(0,t_2,t_3,\dots,t_m))) &= -X_{q_1}(q_1)(\phi_{X_{q_1}}^{-t_1}(\psi^{-1}(0,t_2,t_3,\dots,t_m))) = 0.
\end{aligned}$$

Since

$$\begin{aligned}
p_1(\phi_{X_{q_1}}^{0}(\psi^{-1}(0,t_2,t_3,\dots,t_m))) &= p_1(\psi^{-1}(0,t_2,t_3,\dots,t_m)) = 0,\\
q_1(\phi_{X_{q_1}}^{0}(\psi^{-1}(0,t_2,t_3,\dots,t_m))) &= q_1(\psi^{-1}(0,t_2,t_3,\dots,t_m)) = t_2,
\end{aligned}$$

we conclude that $p_1(\tau^{-1}(t_1,\dots,t_n)) = t_1$ and $q_1(\tau^{-1}(t_1,\dots,t_n)) = t_2$.

Now in this new coordinate system, we have:

$$\{p_1,q^1\} = X_{p_1}(q_1) = 1,\ \{p_1,x^i\} = X_{p_1}(x^i) = 0,\ \{q^1,x^i\} = X_{q_1}(x^i) = 0,$$

for $i = 3,\dots,m$. Again, using the Jacobi identity, we also find:

$$\begin{aligned}
\frac{\partial(\{x^i,x^j\})}{\partial q^1} &= X_{p_1}(\{x^i,x^j\}) = \{p_1,\{x^i,x^j\}\} = \{\{p_1,x^i\},x^j\} + \{x^i,\{p_1,x^j\}\} = 0,\\
\frac{\partial(\{x^i,x^j\})}{\partial p_1} &= -X_{q^1}(\{x^i,x^j\}) = \{\{x^i,x^j\},q^1\} = \{x^i,\{x^j,q^1\}\} + \{\{x^i,q^1\},x^j\} = 0.
\end{aligned}$$

This shows that $\pi|_U$ splits as in (3.1) and finishes the proof of the Theorem. $\square$

3.3 Symplectic leaves

We call a submanifold S of a Poisson manifold (M,π) an **integral submanifold** of $\operatorname{Im}\pi^\#$ if:

(a) S is path connected and
(b) $T_xS = \operatorname{Im}\pi^\#_x$ for all $x \in S$.

Note that integral submanifolds of $\operatorname{Im}\pi^\#$ are always symplectic submanifolds (they are Poisson submanifolds and the induced Poisson structure is non-degenerate). The Darboux-Weinstein Theorem shows that through each point $x_0 \in M$ passes an integral submanifold of $\operatorname{Im}\pi^\#$: if $(U, p_1,\dots,p_n,q^1,\dots,q^n,x^1,\dots,x^s)$ are local coordinates centered at x_0, then $S := \{x^1 = x^2 = \cdots = x^s = 0\}$ is an integral submanifold containing x_0.

In general, integral submanifolds are immersed submanifolds. However, they have better properties than general immersed submanifolds: they are regularly immersed submanifolds, a concept which we now recall.

Definition 3.16. A submanifold $N \hookrightarrow M$ is said to be **regularly immersed** if for every manifold Q and every smooth map $\Psi : Q \to M$ such that $\Psi(Q) \subset N$, the induced map $\Psi : Q \to N$ is smooth.

These submanifolds are also referred to as **initial submanifolds** or **weakly embedded submanifolds**. The relevance of this class of submanifolds is justified by the following property that also explains its name:

Proposition 3.17. *Given a smooth manifold M, a subset $N \subset M$ has at most one differential structure for which $N \hookrightarrow M$ becomes a regular immersed submanifold.*

For a proof of this result, see for example, Warner [1].

The proposition shows that for regularly immersed submanifolds there is no ambiguity about its topology and smooth structure. For example, an eight figure in the plane has two different smooth structures for which it becomes a submanifold, so it cannot be regularly immersed in the plane. On the other hand, any embedded submanifold is regularly immersed. Of course there are many examples of regularly immersed submanifolds which are not embedded: for example, the leaves of a foliation are always regularly immersed submanifolds, and often they fail to be embedded.

We will need also the following "gluing property" for regular immersed submanifolds:

Proposition 3.18. *Let M be a smooth manifold and let $\{N_i\}_{i\in I}$ be a collection of regularly immersed submanifolds of M such that:*

(a) For any $i, j \in I$ the intersection $N_i \cap N_j$ is open in both N_i and N_j;
(b) The union $\bigcup_{i\in I} N_i$, with the topology generated by the open sets $U \subset N_i$, is second countable.

Then $\bigcup_{i\in I} N_i$ is a regularly immersed submanifold.

Proof. Let $N := \bigcup_{i\in I} N_i$. We construct an atlas for N by considering all possible pairs of charts (U_a,ϕ_a) of the submanifolds N_i. The set of all such open sets U_a form the basis for a topology of N, which by (ii) is second countable. Moreover, for any two such charts (U_a,ϕ_a) and (U_b,ϕ_b), such that $U_a\cap U_b \neq \emptyset$, it follows from (i), that the intersection $U_a\cap U_b$ is regularly immersed in both U_a and U_b, so that the compositions:

$$\phi_a(U_a\cap U_b) \xmapsto{\phi_a^{-1}} U_a\cap U_b \hookrightarrow U_b, \quad \phi_b(U_a\cap U_b) \xmapsto{\phi_b^{-1}} U_a\cap U_b \hookrightarrow U_a,$$

are both smooth. Hence, the transition functions $\phi_a\circ\phi_b^{-1}$ and $\phi_b\circ\phi_a^{-1}$ are all smooth. □

Henceforth, for two topological spaces X and Y, on the intersection $X\cap Y$ we consider the smallest topology that makes both inclusions $X\cap Y\hookrightarrow X$ and $X\cap Y\hookrightarrow Y$ continuous. For example, if N is an integral leaf of (M,π), and $U\subset M$ is open, then $U\cap N$ has the relative topology induced from N, since the submanifold topology of N is usually finer than the relative topology.

Let us return now to our discussion of integral submanifolds of $\operatorname{Im}\pi^{\#}$. We have:

Proposition 3.19. *Let (M,π) be a Poisson manifold. Then:*

(i) Every integral submanifold S of $\operatorname{Im}\pi^{\#}$ is regularly immersed.
(ii) A connected component of the the intersection of two integral submanifolds of $\operatorname{Im}\pi^{\#}$ is also an integral submanifold of $\operatorname{Im}\pi^{\#}$.

In particular, noticed that a subset $S\subset M$ has at most one differential structure for which S is an integral submanifold of $\operatorname{Im}\pi^{\#}$.

Proof. Fix a point $x_0\in M$ and let $(U,\varphi) = (U,p_1,\dots,p_n,q^1,\dots,q^n,x^1,\dots,x^s)$ be Darboux-Weinstein coordinates centered at x_0. The next lemma gives information about integral submanifolds of $\operatorname{Im}\pi^{\#}$ containing x_0:

Lemma 3.20. *Let N be an integral submanifold of $\operatorname{Im}\pi^{\#}$ containing x_0.*

(i) The connected components of the intersection $N\cap U$ are contained in the slices $\{x^a = constant\}$.
(ii) If N is contained in U then it is contained in the slice $\{x^a=0\}$.
(iii) If N' is some other integral submanifold containing x_0, then the connected component of $N'\cap N$ containing x_0 is also an integral submanifold of $\operatorname{Im}\pi^{\#}$.

Note that (ii) follows from (i), since N is connected and $x_0\in N$ belongs to the slice $\{x^a=0\}$. Also, (iii) follows from (i) since the slices $\{x^a = \text{constant}\}$ and the integral submanifolds containing x_0 have the same dimension. Now, to prove (i) it is enough to check that $N\cap\{x^a=\text{constant}\}$ is both open and closed in $N\cap U$:

(a) $N\cap\{x^a=\text{constant}\}$ is closed in $N\cap U$: if $n_k\in N\cap\{x^a=\text{constant}\}$ converges to $n\in N\cap U$, then clearly $x^a(n)=\text{constant}$, so $n\in N\cap\{x^a=\text{constant}\}$.

(b) $N \cap \{x^a = \text{constant}\}$ is open in $N \cap U$: if $x \in N \cap \{x^a = \text{constant}\}$ the hamiltonian vector fields $\{X_{p_1}, \dots, X_{p_n}, X_{q^1}, \dots, X_{q^n}\}$ commute and they generate both the tangent space to the slice $\{x^a = \text{constant}\}$ and the tangent space to N around x (because N is a Poisson submanifold with the same dimension as the slice). Hence, there is some open set $x \in V \subset N$ such that $V \cap \{x^a = \text{constant}\} \subset N$.

Next we show that integral submanifolfds are regularly immersed. Let $i : N \hookrightarrow M$ be an integral submanifold of $\operatorname{Im} \pi^\#$ and let $\Psi : Q \to M$ be a smooth map from some manifold Q into M with $\Psi(Q) \subset N$. Denote by $\hat{\Psi} : Q \to N$ the induced map:

$$\begin{tikzcd} N \arrow[r, "i"] & M \\ & Q \arrow[u, "\Psi"'] \arrow[ul, dashed, "\hat{\Psi}"] \end{tikzcd}$$

We need to show that $\hat{\Psi}$ is smooth.

Let $q \in Q$ and $n = \hat{\Psi}(q) \in N$. We can choose Darboux-Weinstein coordinates $(V, \varphi) = (V, (p_i, q^i, x^a))$ for M centered at $i(n)$ and an open set $n \in U \subset N$ such that:

$$i(U) = \{x^a = 0\}.$$

Notice that $(U, \pi_1 \circ \varphi \circ i)$, where π is the projection in the first $2n$ coordinates, is a local chart for N around n. The set $\Psi^{-1}(V)$ is open in Q and contains q. Let W be the connected component of $\Psi^{-1}(V)$ containing q. Then W is open in Q, and if we can show that $\hat{\Psi}(W) \subset U$ it will follow that $\pi_1 \circ \varphi \circ i \circ \hat{\Psi}|_W = \pi_1 \circ \varphi \circ \Psi|_W$ is smooth, so one can conclude that $\hat{\Psi}$ is smooth.

To show that $\hat{\Psi}(W) \subset U$, it is enough to check that $\Psi(W) \subset \{x^a = 0\}$. Observe that since Ψ is continuous and W is connected, the image $\Psi(W)$ is also connected and contains at least one point in the slice $\{x^a = 0\}$, namely $i(n)$. Since $\Psi(W)$ lies in the component of $i(N) \cap V$ containing $i(n)$, by the Lemma, this is the slice $\{x^a = 0\}$ and we have that $\Psi(W) \subset \{x^a = 0\}$, so we are done. $\square$

The previous proposition justifies the following definition:

Definition 3.21. A **symplectic leaf** of a Poisson manifold (M, π) is a maximal (relative to inclusion) integral submanifold of $\operatorname{Im} \pi^\#$.

In the sequel, by a **saturated subset** of (M, π) we mean any subset $X \subset M$ which is the union of symplectic leaves.

Theorem 3.22 (Symplectic foliation). *Let (M, π) be a Poisson manifold.*

(i) Every point $x_0 \in M$ is contained in a single symplectic leaf of M and every integral submanifold of $\operatorname{Im} \pi^\#$ *containing x_0 is contained in the symplectic leaf through x_0.*
(ii) M_{reg} and M_{sing} are both saturated subsets of (M, π).

Proof. Let S_{x_0} be the union of all the integral submanifolds of $\in \pi^\#$ containing x_0.

Lemma 3.23. *S_{x_0} is an integral submanifold of* $\operatorname{Im}\pi^\#$.

Proof. By Proposition 3.19 the intersection of any two integral submanifolds is an open set of both submanifolds. Hence, by Proposition 3.18, we only need to check that the topology of S_{x_0}, which is generated by the open sets of the integral submanifolds, is second countable.

Consider the set of all Darboux-Weinstein charts centered at points of S_{x_0}. A countable number of these charts still cover S_{x_0}, so we can choose a countable set of Darboux-Weinstein charts $(U_k,\phi_k) = (U_k,(p^i_{(k)},q^i_{(k)},x^a_{(k)}))$, $k = 1,2,\dots$ centered at points of S_{x_0}. Note that the connected components of the intersection of S_{x_0} with each chart U_k is a slice $\{x^a_{(k)} = \text{constant}\}$. To check that the topology of S_{x_0} is second countable it is enough to check that for each k the leaf S_{x_0} intersects U_k in a countable number of slices $\{x^a_{(k)} = \text{constant}\}$.

Since S_{x_0} is path connected, each point in U_k that lies in the leaf S_{x_0} can be joined to x_0 by a path in S_{x_0}. To each such path, there corresponds a (non-unique) finite sequence:

$$U_0, U_{k_1}, \dots, U_{k_n}, U_k,$$

of domains of charts that cover the path. Every slice in $S_{x_0} \cap U_k$ is then reachable in this way. Since there are at most countable such sequences from U_0 to U_k, to check that the number of slices in $S_{x_0} \cap U_k$ is countable, it is enough to check that each slice in $S_{x_0} \cap U_{k_i}$ can intersect at most a countable number of slices in $S_{x_0} \cap U_{k_{i+1}}$. For that observe that if $N \subset S_{x_0} \cap U_{k_i}$ is a slice, then $N \cap U_{k_{i+1}}$ is an open submanifold of N, so it has at most a countable number of connected components. Each such component is a connected integral submanifold in $S_{x_0} \cap U_{k_{i+1}}$, so it is contained in a single slice of $S_{x_0} \cap U_{k_{i+1}}$. □

It follows from this lemma, that S_{x_0} is a symplectic leaf, i.e., a maximal integral submanifold of $\operatorname{Im}\pi^\#$ which contains every integral submanifold containing x_0, so (i) is proved.

In order to prove (ii) we claim that if x_0 is a regular point of M then all points in the symplectic leaf S_{x_0} are regular points of M. For that it is enough to prove that $S_{x_0} \cap M_{\text{reg}}$ is both open and closed in S_{x_0}:

(a) Since M_{reg} is open in M, it is obvious that $S_{x_0} \cap M_{\text{reg}}$ is open in S_{x_0}, because the inclusion $S_{x_0} \hookrightarrow M$ is continuous.
(b) Let $x_n \in S_{x_0} \cap M_{\text{reg}}$ be a sequence that converges in S_{x_0} to x^*. We claim that x^* is a regular point, so that $S_{x_0} \cap M_{\text{reg}}$ is closed in S_{x_0}. Choose a Darboux-Weinstein chart $(V,(p_i,q^i,x^a))$ centered at x^*. The connected component of $V \cap S_{x_0}$ containing x^* is the slice $\{x^a = 0\}$, so for large enough N we have that x_N is a regular point of π that belongs to the slice $\{x^a = 0\}$. From the expression of π in Darboux-Weinstein coordinates:

$$\pi|_U = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} + \sum_{a,b=1}^{s} \phi^{ab}(x) \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b}$$

we conclude that we must have $\phi^{ab}(x) = 0$ for x in a neighborhood of 0, so that x^* is a regular point.

This proves that M_{reg} is saturated, so $M_{\mathrm{sing}} = M - M_{\mathrm{reg}}$ is also saturated. □

Exercise 3.24. Let (M,π) be a Poisson manifold. Show that the symplectic leaf S_{x_0} coincides with the set of all points in M that can be joined to x_0 by a piecewise smooth path γ, such that each smooth piece of γ is an integral curve of some hamiltonian vector field.

Remark 3.25. Note that the rank of π does not need to be the same on distinct connected components of M_{reg}. The restriction of π to each connected component is a regular Poisson structure whose symplectic foliation consists of the symplectic leaves of (M,π) contained in that component.

Henceforth we will call the partition of (M,π) by symplectic leaves the **symplectic foliation** of (M,π). In general, this will be a *singular foliation*, since the dimension of the leaves can vary.

3.4 Symplectic foliation and submanifolds

We can now look back at the various notions of submanifolds of a Poisson manifold and see how they are related with its symplectic foliation.

Proposition 3.26. *Let (M,π_M) be a Poisson manifold with symplectic foliation $\mathscr{S}$. A submanifold $N \subset M$ is a Poisson submanifold if and only if for each symplectic leaf $S \in \mathscr{S}$ the intersection $S \cap N$ is a open subset of S. Hence, the symplectic foliation of (N,π_N) consists of the connected components of the intersections $S \cap N$.*

Proof. A submanifold $N \subset M$ is a Poisson submanifold if and only if

$$\operatorname{Im} \pi_x^\# \subset T_xN, \forall x \in N.$$

It follows that for a Poisson submanifold $N \subset M$, every symplectic leaf of (N,π_N) is also an integral submanifold of (M,π_M). Hence, every symplectic leaf of (N,π_N) is an open subset of a symplectic leaf in $\mathscr{S}$.

Conversely, if for each symplectic leaf $S \in \mathscr{S}$ the intersection $S \cap N$ is a open subset of S, then for any $x \in N$ we have $\operatorname{Im} \pi_x^\# = T_xS_x \subset T_xN$, where $S_x \in \mathscr{S}$ is the symplectic leaf through x. This shows that N is a Poisson submanifold. □

Proposition 3.27. *Let (M,π) be a Poisson manifold with symplectic foliation $\mathscr{S}$. A manifold $X \subset M$ is a Poisson transversal if and only if X is transverse to $\mathscr{S}$ and for each $S \in \mathscr{S}$ the intersection $S \cap X$ is a symplectic submanifold of S. Hence, the symplectic foliation of (X,π_X) consists of the connected components of the intersections $S \cap X$.*

Proof. The condition for a submanifold $X \subset M$ to be a Poisson transversal is:

$$T_X M = TX \oplus \pi^\#(TX)^0.$$

This condition is equivalent to have both the following conditions satisfied:

(a) $T_X M = TX + \operatorname{Im}\pi^\#$;
(b) $TX \cap \pi^\#(TX)^0 = 0$.

Condition (a) says that X is transverse to the symplectic leaves and condition (b) (provided condition (a) is satisfied) says that the kernel of the pullback of ω_S to $S \cap X$ is trivial, so $S \cap X$ is a symplectic submanifold of S. □

The previous results suggest a natural generalization of submanifolds of a Poisson manifold (M, π). Recall that two submanifolds $N_1, N_2 \subset M$ are said to have a **clean intersection** if $N_1 \cap N_2$ is a submanifold of M and:

$$T(N_1 \cap N_2) = TN_1 \cap TN_2.$$

A submanifold has clean intersection with a foliation if it intersects cleanly every leaf of the foliation. For example, if N is transverse to a foliation $\mathscr{F}$ then they have a clean intersection.

Exercise 3.28. Let N be a submanifold of a Poisson manifold (M, π) which has clean intersection with its symplectic foliation $\mathscr{S}$. Show that if $N \cap S$ is a symplectic submanifold of S for every $S \in \mathscr{S}$ then there is a unique Poisson structure π_N such that the symplectic foliation of (N, π_N) are the connected components of the intersections $N \cap S$. Give an example of such a manifold N which is neither a Poisson submanifold nor a Poisson transversal. HINT: See Exercise 2.38.

Proposition 3.29. *Let C be a submanifold of a Poisson manifold (M, π) which has clean intersection with its symplectic foliation $\mathscr{S}$. Then C is a coisotropic submanifold of (M, π) if and only if for each symplectic leaf $S \in \mathscr{S}$ the intersection $S \cap C$ is a coisotropic submanifold of S.*

Proof. Assume that $C \subset M$ is a submanifold which is transverse to the symplectic foliation $\mathscr{S}$. This means that for each $S \in \mathscr{S}$ the inclusion $i : S \hookrightarrow M$ is transverse to C. Now:

(a) If C is coisotropic in M, it follows that $i^{-1}(C) = S \cap C$ is coisotropic in S, since the inclusion is a Poisson map (see Proposition 2.34).
(b) If $S \subset C$ is coisotropic in S, then we have:

$$\pi_S^\#(T(S \cap C))^0 \subset T(C \cap S) = TC \cap TS.$$

where the annihilator is in T^*S. It follows that for for any $\alpha \in T^*M$ such that $\alpha|_{TS \cap TC} = 0$, we have:

$$\pi^\#(\alpha) \in TC.$$

But $(TS \cap TC)^0 = TS^0 + TC^0$, so we conclude that $\pi^\#(TC)^0 \subset TC$, which means that C is coisotropic. □

Finally, we relate Poisson maps and symplectic leaves:

Proposition 3.30. *If $\Phi : (M,\pi_M) \to (N,\pi_N)$ is a Poisson map then for each symplectic leaf S of (N,π_N), the set $S\cap \operatorname{Im}\Phi$ is open in S. In particular, if $\operatorname{Im}\Phi$ is a submanifold, then it is a Poisson submanifold of (N,π_N).*

Proof. Let $x_0 \in M$ and set $y_0 = \Phi(x_0)$. Denote by S_{y_0} the symplectic leaf of (N,π_N) containing y_0. By Exercise 3.24, any point in S_{y_0} can be reached from y_0 by piecewise smooth curves consisting of integral curves of hamiltonian vector field X_h where $h \in C^\infty(N)$.

Given $h \in C^\infty(N)$, by Proposition 1.12, the vector fields X_h and $X_{h\circ\Phi}$ are Φ-related. Hence, if $\gamma(t) \in N$ and $\eta(t) \in M$ are the integral curves of X_h and $X_{h\circ\Phi}$ satisfying $\gamma(0) = y_0$ and $\eta(0) = x_0$, we have:

$$\gamma(t) = \Phi(\eta(t)),$$

for all small enough t. It follows that a neighborhood of y_0 in S_{y_0} is contained in the image of Φ. $\square$

Exercise 3.31. A Poisson map $\Phi : (M,\pi_M) \to (N,\pi_N)$ is called **complete** if whenever X_h is a complete hamiltonian vector field in (N,π_N), the hamiltonian vector field $X_{h\circ\Phi}$ is a complete vector field in (M,π_M). Show that the image of a complete Poisson map is saturated by symplectic leaves.

This exercise and our results that we will study later suggest that one should think of complete Poisson maps as the analogues of proper maps in the Poisson category.

Homework 3: Foliations and Cohomology

3.1. Let $\mathscr{F}$ be a foliation of M. A **$\mathscr{F}$-connection** on a vector bundle $p : E \to M$ is a $\mathbb{R}$-bilinear map $\nabla : \mathfrak{X}(\mathscr{F}) \times \Gamma(E) \to \Gamma(E)$, $(X,s) \mapsto \nabla_X s$, satisfying the following properties:

$$\nabla_{fX} s = f\nabla_X s, \quad \nabla_X(fs) = f\nabla_X s + X(f)s.$$

The **curvature** of the $\mathscr{F}$-connection is defined by:

$$R_\nabla(X_1,X_2)s = \nabla_{X_1}\nabla_{X_2}s - \nabla_{X_2}\nabla_{X_1}s - \nabla_{[X_1,X_2]}s.$$

(i) Show that if $\gamma : [0,1] \to L$ is a smooth path in a leaf $L \in \mathscr{F}$ and $u_0 \in E_{\gamma(0)}$ there exists a unique curve $\tilde{\gamma}_{u_0} : [0,1] \to E$ such that:

$$\tilde{\gamma}_{u_0}(0) = u_0, \quad p(\tilde{\gamma}_{u_0}(t)) = \gamma(t), \quad \nabla_{\dot{\gamma}(t)}\tilde{\gamma}_{u_0}(t) = 0.$$

HINT: We will need first to make sense of $\nabla_{\dot{\gamma}(t)} s$ where s is a section of E defined along the path γ. For that, consider extensions and show that the definition does not depend on choices.

(ii) Define a parallel transport map $\tau_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ by:

$$\tau_\gamma(u) = \tilde{\gamma}_{u_0}(1).$$

and show that for any two paths γ_1 and γ_2 such that $\gamma_1(0) = \gamma_2(1)$ and $\gamma_1'(0) = \gamma_2'(1) = 0$, one has:

$$\tau_{\gamma_1\cdot\gamma_2} = \tau_{\gamma_1} \circ \tau_{\gamma_2},$$

where $\gamma_1 \cdot \gamma_2$ denotes concatenation of paths.

3.2. Let $\mathscr{F}$ be a foliation of M. A **representation of** $\mathscr{F}$ is a vector bundle $E \to M$ together with a flat $\mathscr{F}$-connection connection. Show that for a representation of $\mathscr{F}$:

(i) Parallel transport depends only on the leafwise homotopy class of the path, so the assignment:

$$\mathrm{Hol} : \pi_1(L, x_0) \to GL(E_{x_0}), [\gamma] \mapsto \tau_\gamma,$$

defines a group homomorphism called the **holonomy of the representation**.

(ii) Let $\Omega^k(\mathscr{F}, E) = \Gamma(\wedge^k T^*\mathscr{F} \otimes E)$ be the space of foliated k-differential forms with values in E, so an element of $\omega \in \Omega^k(\mathscr{F}, E)$ can be viewed as degree k $C^\infty(M)$-multilinear alternating map $\omega : \mathfrak{X}(\mathscr{F}) \times \cdots \times \mathfrak{X}(\mathscr{F}) \to \Gamma(E)$. Define a linear map $\mathrm{d}_\mathscr{F} : \Omega^\bullet(\mathscr{F}, E) \to \Omega^{\bullet+1}(\mathscr{F}, E)$ by:

$$\mathrm{d}_\mathscr{F}\omega((X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \nabla_{X_i}\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k) + \sum_{i<j} (-1)^{i+j}\omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).$$

Show that $d^2_\mathscr{F} = 0$, so one can define the corresponding cohomology groups $H^\bullet(\mathscr{F}, E)$.

(iii) Show that if $\Phi : N \to M$ is a submersion and $\mathscr{F}' = \Phi^{-1}(\mathscr{F})$, then there is an induced map in cohomology:

$$\Phi^* : H^\bullet(\mathscr{F}, E) \to H^\bullet(\mathscr{F}', \Phi^*E).$$

When $E = M \times \mathbb{R} \to M$ is the trivial line bundle, we write $H^\bullet(\mathscr{F})$ instead of $H^\bullet(\mathscr{F}, E)$ and call it the **foliated cohomology** of $\mathscr{F}$.

3.3. Let $\mathscr{F}$ be a foliation of M with normal bundle $\mathscr{N}_\mathscr{F} = TM/T\mathscr{F}$. Define the **Bott connection** of the foliation $\mathscr{F}$, $\nabla : \mathfrak{X}(\mathscr{F}) \times \Gamma(\mathscr{N}_\mathscr{F}) \to \Gamma(\mathscr{N}_\mathscr{F})$ by setting

$$\nabla_X \overline{Y} := \overline{[X, Y]}.$$

where $\overline{Y} \in \Gamma(\mathscr{N}_\mathscr{F})$ denotes the section of the normal bundle represented by the vector field $Y \in \mathfrak{X}(M)$.

(i) Show that ∇ is well-defined (i.e., is independent of the choice of Y) and gives a representation of $\mathscr{F}$ on its normal bundle $\mathscr{N}_\mathscr{F}$.

(ii) Show that one has an induced representation on the conormal bundle $\mathscr{N}^*_{\mathscr{F}}$ and on any tensor product of $\mathscr{N}_{\mathscr{F}}$ and $\mathscr{N}^*_{\mathscr{F}}$.

The Bott connection gives a group homomorphism

$$\mathrm{Hol}^{\mathrm{lin}} : \pi_1(L,x_0) \to \mathscr{N}_L|_{x_0}.$$

which is called the **linear holonomy representation** of the leaf L. Its image is called the **linear holonomy group** of L.

3.4. One defines the **complex of differential forms of M relative to $\mathscr{F}$** by:

$$\Omega^k(M,\mathscr{F}) := \{\alpha \in \Omega^k(M) : i_L^*\alpha = 0, \text{ for all leaves } i_L : L \hookrightarrow M\}.$$

Notice that $\mathrm{d} : \Omega^\bullet(M,\mathscr{F}) \to \Omega^{\bullet+1}(M,\mathscr{F})$ so that we have indeed a complex $(\Omega^k(M,\mathscr{F}),\mathrm{d})$. We denote by $H^\bullet(M,\mathscr{F})$ the corresponding cohomology.

(i) If $\tilde{\omega} \in \Omega^k(M)$ denotes any k-form extending a foliated k-form $\omega \in \Omega^k(\mathscr{F})$, then $\delta[\omega] := [\mathrm{d}\tilde{\omega}] \in H^{k+1}(M,\mathscr{F})$ does not depend on the choice of extension.
(ii) Show that there exists a long exact sequence:

$$\cdots \longrightarrow H^k(M,\mathscr{F}) \longrightarrow H^k(M) \longrightarrow H^k(\mathscr{F}) \xrightarrow{\delta} H^{k+1}(M,\mathscr{F}) \longrightarrow \cdots$$

(iii) A foliated k-form $\omega \in \Omega^k(\mathscr{F})$ admits a closed extension if and only if the corresponding class $\delta[\omega] \in H^{k+1}(M,\mathscr{F})$ vanishes.

3.5. Let $\mathscr{F}$ be a foliation of M with conormal bundle $\mathscr{N}^*_{\mathscr{F}}$.

(i) Given a differential form $\omega \in \Omega^k(M,\mathscr{F})$ define $\hat{\omega} \in \Omega^{k-1}(\mathscr{F},\mathscr{N}^*_{\mathscr{F}})$ by

$$\hat{\omega}(X_1,\dots,X_{k-1})(\overline{Y}) := \omega(X_1,\dots,X_{k-1},Y),$$

where $X_i \in \mathfrak{X}(\mathscr{F})$ and $\overline{Y} \in \Gamma(\mathscr{N}_{\mathscr{F}})$. Show that $\omega \mapsto \hat{\omega}$ defines a cochain map $(\Omega^\bullet(M,\mathscr{F}),\mathrm{d}) \to (\Omega^{\bullet-1}(\mathscr{F},N^*_{\mathscr{F}}),\mathrm{d}_{\mathscr{F}})$.
(ii) Given a differential form $\eta \in \Omega^k(\mathscr{F})$ define $\mathrm{d}_\nu\eta \in \Omega^k(\mathscr{F},\mathscr{N}^*_{\mathscr{F}})$ by

$$\mathrm{d}_\nu\eta(X_1,\dots,X_k)(\overline{Y}) := \mathrm{d}\tilde{\eta}(X_1,\dots,X_k,Y),$$

where $\tilde{\eta} \in \Omega^k(M)$ is any extension of η, $X_i \in \mathfrak{X}(\mathscr{F})$ and $\overline{Y} \in \Gamma(\mathscr{N}_{\mathscr{F}})$. Show that $\eta \mapsto \mathrm{d}_\nu\eta$ defines a cochain map $(\Omega^\bullet(\mathscr{F}),\mathrm{d}) \to (\Omega^\bullet(\mathscr{F},N^*_{\mathscr{F}}),\mathrm{d}_{\mathscr{F}})$.
(iii) Show that the induced maps in cohomology give rise to a commutative diagram:

$$\begin{tikzcd} & H^\bullet(\mathscr{F}) \arrow[dl, "\delta"'] \arrow[dr, "\mathrm{d}_\nu"] & \\ H^{\bullet+1}(M,\mathscr{F}) \arrow[rr] & & H^\bullet(\mathscr{F},\mathscr{N}_{\mathscr{F}}) \end{tikzcd}$$

3.6. For a regular Poisson manifold (M,π) with symplectic foliation $\mathscr{S}$ and foliated symplectic form $\omega_{\mathscr{S}}$, the previous problems show that we have the foliated cohomology class $[\omega_{\mathscr{S}}] \in H^2(\mathscr{S})$, which measures the failure in $\omega_{\mathscr{S}}$ be leafwise exact, the class $\delta[\omega_{\mathscr{S}}] \in H^3(M,\mathscr{S})$, which measures the failure in $\omega_{\mathscr{S}}$ extending to a closed 2-form, and the class $\mathrm{d}_\nu[\omega_{\mathscr{S}}] \in H^2(\mathscr{S},\mathscr{N}_{\mathscr{S}})$, which measures the transverse variation of the symplectic form.

(i) Give an example of a regular Poisson manifold (M,π) where $[\omega_{\mathscr{S}}] = 0$, hence all classes vanish.
(ii) Give an example of a regular Poisson manifold (M,π) where $\delta[\omega_{\mathscr{S}}] = 0$ but $[\omega_{\mathscr{S}}] \neq 0$.
(iii) Give an example of a regular Poisson manifold (M,π) where $\mathrm{d}_\nu[\omega_{\mathscr{S}}] = 0$ but $\delta[\omega_{\mathscr{S}}] \neq 0$.

HINT: Consider regular Poisson manifolds (M,π) where $M = S \times N$ and whose symplectic foliations are product foliations $S \times \{x\}$, with $x \in N$.

Part II
Constructions and Examples

We describe various constructions that yield new Poisson manifolds starting with known ones and perhaps some extra data. This will furnish us with a large number of examples of Poisson manifolds.

Chapter 4
Quotients and Fibrations

4.1 Poisson Quotients

Let G be a Lie group and assume that G acts on a smooth manifold M, so we have the action map $\Phi : G \times M \to M$, $(g,x) \mapsto g \cdot x$. We will denote translation by $g \in G$ by $\Phi_g : M \to M$, $x \mapsto g \cdot x$, and we will denote the map parameterizing the orbit through an $x \in M$ by $\Phi_x : G \to M$, $g \mapsto g \cdot x$. An action can also be thought of as homomorphism:

$$\hat{\Phi} : G \to \mathrm{Diff}(M),\ g \mapsto \Phi_g.$$

The induced map at the Lie algebra level is the infinitesimal action $\phi : \mathfrak{g} \to \mathfrak{X}(M)$, $\xi \mapsto X_\xi$, where

$$(X_\xi)_x := \frac{\mathrm{d}}{\mathrm{d}t} \exp(t\xi) \cdot x \bigg|_{t=0}.$$

With our conventions, $\phi : \mathfrak{g} \to \mathfrak{X}(M)$is a Lie algebra anti-homomorphism.

Definition 4.1. A **Poisson action** of a Lie group G on a Poisson manifold (M,π) is a smooth action such that for each $g \in G$ the translation $\Phi_g : M \to M$ is a Poisson diffeomorphism. We call the triple (M,π,G) a **Poisson G-space**.

Therefore, for a Poisson action, the translations $\Phi_g : M \to M$ map symplectic leaves of M to symplectic leaves.

Exercise 4.2. Show that for a Poisson G-space (M,π,G) the infinitesimal generators are Poisson vector fields:

$$\mathscr{L}_{X_\xi} \pi = 0, \quad \forall \xi \in \mathfrak{g}.$$

Conversely, if G is connected, show that if all infinitesimal generators of an action are Poisson vector fields, then the action is Poisson.

Let us recall that an action $\Phi : G\times M \to M$ is called *proper* if the map $G\times M \to M\times M$, $(g,x)\mapsto (x, g\cdot x)$, is a proper map. For example, actions of compact Lie groups are always proper. Recall, also, that the action is called free if for every $x\in M$ the isotropy group:

$$G_x := \{g\in G : g\cdot x = x\},$$

is trivial. In other words, the action is free if the translations by elements of the group have no fixed points.

Whenever an action $\Phi : G\times M\to M$ is both proper and free the orbit space M/G has a unique smooth structure compatible with the quotient topology for which the quotient map $q : M\to M/G$ is a submersion.

Theorem 4.3. *Let (M,π,G) be a proper and free Poisson G-space. Then there exists a unique quotient Poisson structure $\pi_{M/G}$ on M/G for which the quotient map $q : M\to M/G$ is Poisson. If $H\in C^\infty(M)$ is a G-invariant function then the hamiltonian vector field $X_H\in\mathfrak{X}(M)$ projects to a hamiltonian vector field $X_h\in\mathfrak{X}(M/G)$.*

Proof. We just need to observe that, under the conditions of the theorem, the space of G-invariant smooth function $C^\infty(M)^G$ is natural isomorphic to the space of smooth functions on the quotient $C^\infty(M/G)$, via pull-back by the quotient map $q : M\to M/G$. Now the action being Poisson, it follows that the Poisson bracket of G-invariant functions is G-invariant, so we obtain a unique Poisson structure on M/G which makes the quotient map a Poisson map. □

The quotient procedure just described is very simple, but still can produce a drastic change in the Poisson structure: for example, it can take a non-degenerate (symplectic) Poisson structure to a degenerate one and vice-versa.

Example 4.4. On the torus $\mathbb{T}^4$ consider the non-degenerate Poisson structure:

$$\pi = \frac{\partial}{\partial\theta^1}\wedge\frac{\partial}{\partial\theta^2}+\frac{\partial}{\partial\theta^3}\wedge\frac{\partial}{\partial\theta^4}.$$

Let $\mathbb{S}^1$ act on $\mathbb{T}^4$ by translations in the last factor:

$$\theta\cdot(\theta^1,\theta^2,\theta^3,\theta^4) = (\theta^1,\theta^2,\theta^3,\theta^4+\theta).$$

This is a proper and free Poisson action. The Poisson structure on the quotient $\mathbb{T}^3 = \mathbb{T}^4/\mathbb{S}^1$ is given by:

$$\pi = \frac{\partial}{\partial\theta^1}\wedge\frac{\partial}{\partial\theta^2},$$

and hence is degenerate. If we now let $\mathbb{S}^1$ act on $\mathbb{T}^3$ by translations on the last factor, again we obtain a proper and free Poisson action and the Poisson structure on the quotient $\mathbb{T}^2 = \mathbb{T}^3/\mathbb{S}^1$ is non-degenerate:

$$\pi = \frac{\partial}{\partial \theta^1} \wedge \frac{\partial}{\partial \theta^2}.$$

Theorem 4.3 is the first step in symmetry reduction of hamiltonian systems X_H, where H is a G-invariant function. Moreover, the following result shows that if a trajectory of the reduced system on the quotient M/G exists for all times, then the original system has a corresponding trajectory which exists for all times.

Theorem 4.5. *Let (M, π, G) be a proper and free Poisson G-space. The quotient map $q : M \to M/G$ is a complete Poisson map.*

We leave the proof as an exercise.

4.2 Hamiltonian Quotients

In general, it is not easy to describe the symplectic foliation of the quotient Poisson structure. However, in some special cases, this is still possible.

For example, if the Poisson action maps each symplectic leaf into itself, rather than shuffling the leaves around, one may hope that the description of the quotient symplectic foliation is easier. For a connected Lie group, this will be the case if the infinitesimal generators of the action are hamiltonian vector fields. So let us consider now the possibility of filling in the dotted arrow in the following diagram:

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\phi"] \arrow[dr, dashed, "\hat{\mu}"'] & & \mathfrak{X}(M) \\
& C^\infty(M) \arrow[ur, "h \mapsto -X_h"'] &
\end{tikzcd}$$

Observe that to give a linear map $\hat{\mu} : \mathfrak{g} \to C^\infty(M)$, $\xi \mapsto \mu_\xi$, is the something as giving a smooth map $\mu : M \to \mathfrak{g}^*$: one determines the other through the relation:

$$\mu_\xi(x) = \langle \mu(x), \xi \rangle.$$

Definition 4.6. A **moment map** for an action of a Lie group G on a Poisson manifold (M, π) is a smooth map $\mu : M \to \mathfrak{g}^*$ such that the **moment map condition** holds:

$$X_\xi = -\pi^\#(\mathrm{d}\mu_\xi), \quad \forall \xi \in \mathfrak{g}. \tag{4.1}$$

It is natural to require the map $\hat{\mu} : (\mathfrak{g}, [\cdot,\cdot]) \to (C^\infty(M), \{\cdot,\cdot\})$ to be a Lie algebra homomorphism, for then the composition with the map $h \mapsto -X_h$ produces a Lie algebra anti-morphism.

Proposition 4.7. *Let (M,π) be a Poisson manifold. A map $\hat{\mu} : \mathfrak{g} \to C^\infty(M)$ is a Lie algebra homomorphism if and only if $\mu : M \to \mathfrak{g}^*$ is a Poisson map, where on $\mathfrak{g}^*$ we consider the linear Poisson structure.*

Proof. Let $\mu : M \to \mathfrak{g}^*$ be any smooth map. If we identify an element $\xi \in \mathfrak{g}$ with the linear map $\xi : \mathfrak{g}^* \to \mathbb{R}$, we find for any $\xi_1, \xi_2 \in \mathfrak{g}$ and $x \in M$:

$$\begin{aligned}\{\hat{\mu}(\xi_1), \hat{\mu}(\xi_2)\}_M(x) - \hat{\mu}([\xi_1,\xi_2])(x) &= \{\xi_1 \circ \mu, \xi_2 \circ \mu\}_M(x) - \langle [\xi_1,\xi_2], \mu(x)\rangle \\ &= \{\xi_1 \circ \mu, \xi_2 \circ \mu\}_M(x) - \{\xi_1,\xi_2\}_{\mathfrak{g}^*}(\mu(x)),\end{aligned}$$

where we use the definition of the linear Poisson structure on $\mathfrak{g}^*$. This shows that $\hat{\mu}$ is a Lie algebra morphism if and only if:

$$\{\xi_1 \circ \mu, \xi_2 \circ \mu\}_M = \{\xi_1,\xi_2\}_{\mathfrak{g}^*} \circ \mu, \quad \forall \xi_1, \xi_2 \in \mathfrak{g}.$$

We leave it as an exercise to check that this condition is equivalent to μ being a Poisson map. □

Given a moment map $\mu : M \to \mathfrak{g}^*$ for an action $G \times M \to M$ on a Poisson manifold, we will say that μ is a ***G*-equivariant moment map** if:

$$\mu(g \cdot x) = \mathrm{Ad}^*_g \mu(x), \quad \forall x \in M, g \in G.$$

Here $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ denotes the coadjoint action: if we think of $\mathfrak{g}^*$ as left-invariant 1-forms on G, and denote by $\Psi : G \times G \to G$, $(g,h) \mapsto ghg^{-1}$, the action by conjugation, then[1]:

$$\mathrm{Ad}^*_g \alpha = (\Psi_{g^{-1}})^* \alpha.$$

Proposition 4.8. *Let (M,π) be a Poisson manifold and $\mu : M \to \mathfrak{g}^*$ a moment map for an action $G \times M \to M$. Then:*

(i) If μ is G-equivariant then μ is a Poisson map.
(ii) If μ is a Poisson map and G is connected, then μ is G-equivariant.

Proof. Starting with the G-equivariance condition for μ

$$\mu(g \cdot x) = \mathrm{Ad}^*_g \mu(x),$$

we let $g = \exp(t\xi_1)$ and contract both sides with ξ_2, obtaining:

$$\langle \mu(\exp(\xi_1) \cdot x), \xi_2 \rangle = \langle \mu(x), \mathrm{Ad}_{\exp(-t\xi_1)} \xi_2 \rangle.$$

Differentiating with respect to t at $t = 0$, we obtain:

[1] The reason for using g^{-1} in this formula is that we still obtain a left action.

$$\langle \mathrm{d}_x\mu \cdot X_{\xi_1}, \xi_2 \rangle = -\langle \mu(x), [\xi_1, \xi_2] \rangle.$$

Using the moment map condition (4.1) and the definition of the Poisson bracket on $\mathfrak{g}^*$, this expression can be written as:

$$-\langle \mathrm{d}_x\mu \cdot \pi^\# \mathrm{d}\mu_{\xi_1}, \xi_2 \rangle = -\{\xi_1, \xi_2\}_{\mathfrak{g}^*}(\mu(x)),$$

or equivalently as:

$$\{\xi_1 \circ \mu, \xi_2 \circ \mu\}_M = \{\xi_1, \xi_2\}_{\mathfrak{g}^*} \circ \mu, \quad \forall \xi_1, \xi_2 \in \mathfrak{g}.$$

This implies that μ is a Poisson map.

This procedure can be reversed to conclude that if μ is a Poisson map, then:

$$\mu(\exp(t\xi) \cdot x) = \mathrm{Ad}^*_{\exp(t\xi)} \mu(x), \quad \forall \xi \in \mathfrak{g}, x \in M.$$

Since the exponential map is a diffeomorphism in a neighborhood of the identity, this gives that:

$$\mu(g \cdot x) = \mathrm{Ad}^*_g \mu(x),$$

for all $g \in G$ in a neighborhood U of the identity. When G is connected, we have $G = \bigcup_{n \in \mathbb{N}} U^n$, so this must hold for all $g \in G$. $\square$

The previous discussion motivates the following definition:

Definition 4.9. A Poisson action $G \times M \to M$ on a Poisson manifold (M, π) is called a **hamiltonian action** if it admits a G-equivariant moment map $\mu : M \to \mathfrak{g}^*$. The quadruple (M, π, G, μ) is called a **hamiltonian G-space**.

Therefore, for a hamiltonian G-space (M, π, G, μ):

- The moment map $\mu : M \to \mathfrak{g}^*$ is a Poisson map;
- The map $\mathfrak{g} \to C^\infty$, $\xi \mapsto \mu_\xi$ is a Lie algebra homomorphism.

Note that when the group is connected, the assumption in the definition that the action is Poisson is superfluous. In fact, for "good" moment maps μ even the action is determined by μ:

Proposition 4.10. *Let (M, π) be any Poisson manifold and let $\mu : M \to \mathfrak{g}^*$ be a complete Poisson map. There is an action of the 1-connected Lie group G integrating $\mathfrak{g}$ such that the quadruple (M, π, G, μ) is a hamiltonian G-space.*

Proof. Let $\xi \in \mathfrak{g}$. Then the linear functional $\alpha \mapsto \langle \alpha, \xi \rangle$ defines a smooth function $h_\xi \in C^\infty(\mathfrak{g}^*)$. The corresponding hamiltonian vector field satisfies:

$$\begin{aligned}\langle X_{h_\xi}|_\alpha, \eta\rangle &= \langle \pi^\#_{\mathfrak{g}^*} \mathrm{d}h_\xi, \eta\rangle \\ &= \langle \pi^\#_{\mathfrak{g}^*}|_\alpha \xi, \eta\rangle \\ &= \langle \alpha, [\xi, \eta]\rangle = \langle -\operatorname{ad}^*_\xi \alpha, \eta\rangle.\end{aligned}$$

In other words, $X_{h_\xi} = -\operatorname{ad}^*_\xi$, the infinitesimal generator of the coadjoint action. It follows that the vector fields X_{h_ξ} are complete.

Since $\mu : M \to \mathfrak{g}^*$ is a complete Poisson map, the hamiltonian vector fields $X_{h_\xi \circ \mu} = X_{\mu_\xi}$ are also complete. Hence we obtain an infinitesimal Lie algebra action $\phi : \mathfrak{g} \to \mathfrak{X}(M)$ by setting:

$$\phi(\xi) = -X_{\mu_\xi}.$$

Since all infinitesimal generators are complete vector fields, this action integrates to an action $\Phi : G \times M \to M$ of the 1-connected Lie group G integrating $\mathfrak{g}$. By construction, the quadruple (M, π, G, μ) is a hamiltonian G-space. $\square$

The main reason one studies hamiltonian G-spaces is the following reduction result:

Theorem 4.11 (Meyer-Marsden-Weinstein). *Let (M, π, G, μ) be a proper and free hamiltonian G-space. Then $0 \in \mathfrak{g}^*$ is a regular value of μ and*

$$M//G := \mu^{-1}(0)/G$$

is a Poisson submanifold of $(M/G, \pi_{M/G})$. If π_M is symplectic, then $\pi_{M//G}$ is also symplectic, and the connected components of $M//G$ are symplectic leaves of $(M/G, \pi_{M/G})$.

Proof. We start by checking that 0 is a regular value of μ by checking that μ is a submersion. For that, pick a basis $\xi_1, \dots, \xi_d$ of $\mathfrak{g}$. Since the action is free, the infinitesimal generators:

$$X_{\xi_1} = \pi^\# \mathrm{d}\mu_{\xi_1}, \dots, X_{\xi_d} = \pi^\# \mathrm{d}\mu_{\xi_d},$$

form a linearly independent set at each point of M. It follows that the differentials $\mathrm{d}_x\mu_{\xi_1}, \dots, \mathrm{d}_x\mu_{\xi_d}$ must form a linearly independent set at each point $x \in M$. Since $\mathrm{d}_x\mu_\xi = \langle \mathrm{d}_x\mu, \xi\rangle$ and $\dim \mathfrak{g}^* = d$, we conclude that μ is a submersion, as claimed.

Next we check that $M//G = \mu^{-1}(0)/G$ is a Poisson submanifold of $(M/G, \pi_{M/G})$, i.e., that we have:

$$\operatorname{Im}(\pi^\#_{M/G})|_{[x]} \subset T_[x]M//G,$$

for every $[x] \in M//G$. Now, from the definition of $\pi_{M/G}$, we have:

$$\operatorname{Im}(\pi^\#_{M/G})|_{[x]} = \{\mathrm{d}_x q \cdot \pi^\#_x (\mathrm{d}_x q)^* \mathrm{d}_{[x]} h : h \in C^\infty(M/G)\},$$

where $q : M \to M/G$ is the quotient map. Notice that by varying $h \in C^\infty(M/G)$ we obtain that $(\mathrm{d}_x q)^* \mathrm{d}_{[x]} h$ are the covectors $\alpha \in T_x^* M$ in the annihilator to the orbit $(T_x G \cdot x)^0$. Therefore, it is enough to prove that for $x \in \mu^{-1}(0)$ we have:

$$\mathrm{d}_x q \cdot \pi_x^\# \alpha \in T_{[x]} M//G, \quad \forall \alpha \in (T_x G \cdot x)^0.$$

This follows from the following:

Lemma 4.12. *For all* $x \in \mu^{-1}(0)$*:*

$$\pi_x^\# (T_x G \cdot x)^0 \subset \operatorname{Ker} \mathrm{d}_x \mu.$$

In fact, for any $\alpha \in (T_x G \cdot x)^0$ and $\xi \in \mathfrak{g}$, we have:

$$\begin{aligned} \langle \mathrm{d}_x \mu \cdot \pi_x^\# \alpha, \xi \rangle &= \langle \pi_x^\# \alpha, \mathrm{d}_x \mu_\xi \rangle \\ &= \langle \alpha, -\pi_x^\# \mathrm{d}_x \mu_\xi \rangle \\ &= \langle \alpha, X_\xi \rangle = 0. \end{aligned}$$

so the lemma follows.

Finally, we will show that if π is symplectic, then $\pi_{M//G}$ is also symplectic. For that we check that $M//G$ is an integral submanifold of $\operatorname{Im}(\pi_{M/G}^\#)$. Let $x \in \mu^{-1}(0)$ and let $\{\mathrm{d}_x \mu_{\xi_1}, \dots, \mathrm{d}_x \mu_{\xi_d}, \alpha_1, \dots, \alpha_{m-2d}\}$ be a basis of $(T_x G \cdot x)^0$, where $m = \dim M$ and $d = \dim G$. We saw above that $\pi^\#(T_x G \cdot x)^0 \subset T_x \mu^{-1}(0)$, and since π is assumed to be non-degenerate, we then obtain a basis

$$\{X_{\xi_1}, \dots, X_{\xi_d}, \pi_x^\#(\alpha_1), \dots, \pi^\#(\alpha_{m-2d})\} \subset T_x \mu^{-1}(0).$$

This basis projects by the quotient map to the linearly independent set:

$$\{\mathrm{d}_x q \cdot \pi_x^\#(\alpha_1), \dots, \mathrm{d}_x q \cdot \pi^\#(\alpha_{m-2d})\} \subset \operatorname{Im}(\pi_{M/G}^\#)|_{[x]}.$$

We conclude that $\dim \operatorname{Im}(\pi_{M/G}^\#)|_{[x]} \geq m - 2d$. Since $\dim M//G = m - 2d$ and we already know that $\operatorname{Im}(\pi_{M/G}^\#)|_{[x]} \subset T_{[x]} M//G$, it follows that

$$\operatorname{Im}(\pi_{M/G}^\#)|_{[x]} = T_{[x]} M//G$$

so $M//G$ is an integral submanifold of $\operatorname{Im}(\pi_{M/G}^\#)$. We leave as an exercise to check that the connected components of $M//G$ are actually symplectic leaves. $\Box$

Exercise 4.13. Show that for a proper and free hamiltonian G-space (M, π, G, μ) with π non-degenerate the connected components of the reduced space $M//G$ are symplectic leaves of M/G.

HINT: Use the fact that for any Poisson G-space (M, π, G) the quotient map $q : M \to M/G$ is a complete Poisson map.

When (M,π,G,μ) is a proper and free symplectic hamiltonian G-space, one calls $M//G$ the **symplectic quotient** of (M,π,G,μ). In our more general set up, where π_M can be degenerate, we will call $M//G$ the **hamiltonian quotient** of (M,π,G,μ). We have only considered above hamiltonian reduction at level 0, but we will see later that one can also consider reduction at other levels of the moment map.

The Meyer-Marsden-Weinstein Theorem can be explained in terms of a commutative diagram:

$$\begin{tikzcd}
 & M \arrow[dr] & \\
\mu^{-1}(0) \arrow[ur, hook] \arrow[dr] & & M/G \\
 & M//G \arrow[ur, hook] &
\end{tikzcd}$$

On the right side of the diagram we have Poisson maps. The Poisson geometry of the left side of this diagram is clarified by the following proposition, which gives an alternative approach to define the Poisson structure on the hamiltonian quotient:

Proposition 4.14. *Let (M,π,G,μ) be a proper and free hamiltonian G-space, so $0\in\mathfrak{g}^*$ is a regular value of μ. Then $\mu^{-1}(0)\subset M$ is a coisotropic submanifold whose characteristic foliation is given by the orbits of G in $\mu^{-1}(0)$. The induced Poisson structure on the quotient $M//G=\mu^{-1}(0)/G$ coincides with the Poisson structure induced from M/G.*

Proof. We claim that for any $x\in\mu^{-1}(0)$ we have:

$$\pi^\#(\ker \mathrm{d}_x\mu)^0 = T_x(G\cdot x).$$

This follows because if $\{\xi_1,\dots,\xi_d\}$ is a basis of $\mathfrak{g}$ then $\{\mathrm{d}_x\mu_{\xi_1},\dots,\mathrm{d}_x\mu_{\xi_d}\}$ is a basis of $\ker \mathrm{d}_x\mu$. Hence, we have:

$$\begin{aligned}
\pi^\#(\ker \mathrm{d}_x\mu)^0 &= \operatorname{span}\{\pi^\#\mathrm{d}_x\mu_{\xi_1},\dots,\pi^\#\mathrm{d}_x\mu_{\xi_d}\}\\
&= \operatorname{span}\{X_{\xi_1}|_x,\dots,X_{\xi_d}|_x\} = T_x(G\cdot x).
\end{aligned}$$

Now by G-equivariance of μ, we have $G\cdot x\subset\mu^{-1}(x)$ for any $x\in\mu^{-1}(0)$, so we conclude that:

$$\pi^\#(\ker \mathrm{d}_x\mu)^0 = T_x(G\cdot x)\subset \ker \mathrm{d}_x\mu.$$

This shows that $\mu^{-1}(0)$ is a coisotropic submanifold and that its characteristic foliation consists of the orbits of the action. The proof that the induced Poisson structure on the orbit space $M//G=\mu^{-1}(0)/G$ coincides with the Poisson structure induced from M/G is left as an exercise. $\square$

Exercise 4.15. Use the method of the previous proposition to show that if (M,π,G,μ) is a hamiltonian G-space, 0 is a regular value of μ and the action is proper and free on $\mu^{-1}(0)$, then one still has a Poisson structure on $M//G := \mu^{-1}(0)/G$, although now M/G may not be anymore a smooth manifold.

4.3 Cotangent bundle reduction

Let Q be a smooth manifold. The cotangent bundle $M = T^*Q$ has a natural non-degenerate Poisson structure. To define the corresponding symplectic form, one first defines a canonical 1-form $\lambda_{\text{can}} \in \Omega^1(T^*Q)$, called the **Liouville 1-form** by:

$$\lambda_{\text{can}}(V_\alpha) = \langle \mathrm{d}_\alpha p \cdot V, \alpha \rangle, \quad \alpha \in T^*Q,\ V \in T_\alpha(T^*Q),$$

where $p : T^*Q \to Q$ denotes the bundle projection.

Exercise 4.16. Show that the Liouville 1-form $\lambda_{\text{can}} \in \Omega^1(T^*Q)$ is the unique 1-form satisfying the following property: for any 1-form $\alpha \in \Omega^1(Q)$ one has:

$$\alpha = \alpha^* \lambda_{\text{can}},$$

where on the right hand side we view α as a smooth map $\alpha : Q \to T^*Q$. If $(q^1, \dots, q^n)$ are local coordinates on Q and $(q^1, \dots, q^n, p_1, \dots, p_n)$ are the corresponding local coordinates on T^*Q, check that:

$$\lambda_{\text{can}} = \sum_{i=1}^n p_i \, \mathrm{d}q^i.$$

Moreover, show that if $\Phi : Q_1 \to Q_2$ is a diffeomorphism and $\tilde{\Phi} : T^*Q_1 \to T^*Q_2$, $\alpha_q \to (\mathrm{d}_{\Phi(q)} \Phi^{-1})^* \alpha$, is the corresponding cotangent lifted diffeomorphism, then

$$\tilde{\Phi}^* \lambda^2_{\text{can}} = \lambda^1_{\text{can}}.$$

Now taking the exterior derivative of the Liouville 1-form one obtains a closed 2-form $\omega_{\text{can}} = \mathrm{d}\lambda_{\text{can}} \in \Omega^2(T^*Q)$, which turns out to be non-degenerate.

Exercise 4.17. Show that $\omega_{\text{can}} = \mathrm{d}\lambda_{\text{can}}$ is the unique 2-form satisfying the following property: for any 1-form $\alpha \in \Omega^1(Q)$ one has:

$$d\alpha = \alpha^* \omega_{\text{can}},$$

where on the right hand side we view α as a smooth map $\alpha : Q \to T^*Q$. If $(q^1, \dots, q^n)$ are local coordinates on Q and $(q^1, \dots, q^n, p_1, \dots, p_n)$ are the corresponding local coordinates on T^*Q, check that:

$$\omega_{\text{can}} = \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q^i,$$

so it is non-degenerate. Moreover, if $\Phi : Q_1 \to Q_2$ is a diffeomorphism and $\tilde{\Phi}$: $T^*Q_1 \to T^*Q_2$ is the corresponding cotangent lift, then

$$\tilde{\Phi}^* \omega^2_{\text{can}} = \omega^1_{\text{can}}.$$

The Poisson bracket associated with the canonical symplectic form ω_{can} on the cotangent bundle T^*Q can also be described in a neat way.

Exercise 4.18. On the cotangent bundle of a manifold Q, consider the following two kinds of functions:

(i) *Basic functions*, i.e., functions of the form $f \circ p$, where $p : T^*Q \to Q$ is the projection;
(ii) *Fiberwise linear functions*, i.e., functions of the form $F_X : T^*Q \to \mathbb{R}$, $\alpha \mapsto i_X \alpha$, for some vector field $X \in \mathfrak{X}(Q)$.

Check that the Poisson bracket associated with the canonical symplectic form ω_{can} satisfies:

$$\{f \circ p, g \circ p\} = 0, \quad \{F_X, F_Y\} = -F_{[X,Y]}, \quad \{F_X, f \circ p\} = -X(f) \circ p.$$

Now let us assume that we have a smooth Lie group action $\Phi : G \times Q \to Q$. We obtain a cotangent lifted action $\tilde{\Phi} : G \times T^*Q \to T^*Q$, by cotangent lifting each diffeomorphism $\Phi_g : Q \to Q$. By the exercise above, this lifted action is automatically a Poisson action for the canonical Poisson structure on T^*G. In fact, we have:

Proposition 4.19. *For any smooth action $\Phi : G \times Q \to Q$ the cotangent lifted action $\tilde{\Phi} : G \times T^*Q \to T^*Q$ is a hamiltonian action with moment map $\mu : T^Q \to \mathfrak{g}^*$ given by:*

$$\langle \mu(\alpha), \xi \rangle = \langle \alpha, X_\xi \rangle, \quad \forall \xi \in \mathfrak{g}.$$

The lifted action is proper (respectively, free) if and only if the action is proper (respectively, free).

Proof. We need to check the moment map condition:

$$i_{\tilde{X}_\xi} \omega_{\text{can}} = -\mathrm{d}\mu_\xi,$$

where $\tilde{X}_\xi$ denotes the infinitesimal generator associated with $\xi \in \mathfrak{g}$ for the cotangent lifted action. For this, we first observe that $\mathscr{L}_{\tilde{X}_\xi} \lambda_{\text{can}} = 0$, since the lift of any diffeomorphism of Q preserves λ_{can}. On the other hand, from the definition of λ_{can} and μ we have:

$$i_{\tilde{X}_\xi} \lambda_{\text{can}}|_\alpha = \lambda_{\text{can}}(\tilde{X}_\xi)(\alpha) = \langle \alpha, X_\xi \rangle = \mu_\xi.$$

Hence, by Cartan's magic formula, we see that:

$$i_{\tilde{X}_\xi} \omega_{\text{can}} = i_{\tilde{X}_\xi} \mathrm{d}\lambda_{\text{can}} = \mathscr{L}_{\tilde{X}_\xi} \lambda_{\text{can}} - \mathrm{d} i_{\tilde{X}_\xi} \lambda_{\text{can}} = -\mathrm{d}\mu_\xi,$$

and we conclude that the moment map condition holds. We leave it as an exercise to check that μ is G-equivariant and that the lifted action is proper (respectively, free) if and only if the action is proper (respectively, free). $\square$

It follows from this proposition that for a free and proper action $G \times Q \to Q$ we obtain a free and proper hamiltonian space $(T^*Q, \omega_{\text{can}}, G, \mu)$ and we can form both the Poisson quotient T^*Q/G and the Hamiltonian quotient $T^*Q//G$. By the Meyer-Marsden-Weinstein Theorem, $T^*Q//G$ is a symplectic manifold.

Exercise 4.20. Show that for a proper and free action $G \times Q \to Q$ one has a Poisson diffeomorphism:

$$T^*Q//G \simeq T^*(Q/G).$$

HINT: Let $p : Q \to Q/G$ denote the quotient map. Oberve that for each $q \in Q$ the map:

$$(d_q p)^* : T^*_{qG}Q/G \to \mu^{-1}(0) \cap T^*_q Q$$

is a linear isomorphism.

On the other hand, in general, the Poisson quotient T^*Q/G *is not* a symplectic manifold, and may in fact have a more or less intricate Poisson geometry.

Example 4.21. Let $Q = \mathbb{C} - \{0\}$, thought of as 2 dimensional real manifold, and consider the action $\mathbb{S}^1 \times Q \to Q$ given by:

$$\theta \cdot z = e^{i\theta} z.$$

The canonical symplectic form on the cotangent bundle $T^*Q \simeq (\mathbb{C} - \{0\}) \times \mathbb{C}$ can be written in the form:

$$\omega_{\text{can}} = -\frac{1}{2}(\mathrm{d}z \wedge \mathrm{d}\bar{w} + \mathrm{d}\bar{z} \wedge \mathrm{d}w).$$

The lifted $\mathbb{S}^1$-action on $T^*Q \simeq (\mathbb{C} - \{0\}) \times \mathbb{C}$ is given by:

$$\theta \cdot (z, w) = (e^{i\theta} z, e^{i\theta} w).$$

In order to describe the Poisson structure on the quotient $T^*Q/\mathbb{S}^1 \simeq (\mathbb{C} - \{0\} \times \mathbb{C})/\mathbb{S}^1 \simeq \mathbb{R}^3$, we consider the $\mathbb{S}^1$-invariant polynomials:

$$\begin{aligned}
\sigma_1 &= |z|^2 + |w|^2 \\
\sigma_2 &= |z|^2 - |w|^2 \\
\sigma_3 &= z\bar{w} + \bar{z}w
\end{aligned}$$

and we check that their Poisson brackets are given by:

$$\{\sigma_1, \sigma_2\} = \sigma_3, \quad \{\sigma_2, \sigma_3\} = \sigma_1, \quad \{\sigma_1, \sigma_3\} = -\sigma_2.$$

Hence, we have obtained a linear Poisson bracket in T^*Q/G.

Exercise 4.22. Show that the moment map $\mu : T^*Q \to \mathbb{R}$ in the previous example is given by:

$$\mu(z,w) = \frac{i}{2}(z\bar{w} - \bar{z}w).$$

Check that it induces a Casimir function $C : T^*Q/G \to \mathbb{R}$ and find its expression in terms of the coordinates $(\sigma_1, \sigma_2, \sigma_3)$.

Example 4.23. Consider the action of a Lie group G on itself by left translations:

$$G \times G \to G,\ g \cdot h = gh.$$

This a proper and free action, so the lifted cotangent action $G \times T^*G \to T^*G$ is also proper and free. Since the infinitesimal generators of the action of G on itself by left translations are the right invariant vector fields, the moment map for this lifted action is just:

$$\mu^L : T^*G \to \mathfrak{g}^*,\ \alpha_g \mapsto (\mathrm{d}_e R_g)^* \alpha_g,$$

where $R_g : G \to G$ denotes the right translation $R_g(h) = hg$. Note that in this case the symplectic quotient is trivial, since from what we saw above it equals the cotangent bundle of G/G. How does the Poisson structure on T^*G/G look like?

Let us trivialize $T^*G = G \times \mathfrak{g}^*$ by using left translations:

$$T^*G \ni \alpha_g \mapsto (g, (\mathrm{d}_e L_g)^* \alpha_g) \in G \times \mathfrak{g}^*,$$

where $L_g : G \to G$ denotes the left translation $L_g(h) = gh$. Under this identification, the action of G on T^*G becomes:

$$g \cdot (h, \alpha) = (gh, \mathrm{Ad}^*_g \alpha).$$

We claim that under this identification the projection $T^*G \simeq G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is an anti-Poisson map. In fact, the symmetric of this projection is given by:

$$\mu^R : T^*G \to \mathfrak{g}^*,\ \alpha_g \mapsto -(\mathrm{d}_e L_g)^* \alpha_g,$$

and, just like what we saw above for the left action, this is the moment map for the cotangent lifted *left action* of G on itself *by right translations*:

$$G \times G \to G,\ g \cdot h := hg^{-1}.$$

Since moment maps are Poisson maps, the claim follows. We conclude that left translations give an isomorphism $T^*G/G \to \mathfrak{g}^*$ with the (negative of) the linear Poisson structure.

4.4 Gauge Transforms

Let (M,π) be a Poisson manifold. Given a closed 2-form $B \in \Omega^2(M)$, one can modify the Poisson structure by adding to the symplectic form on each leaf the pullback of B to the leaf. In order to obtain a new Poisson structure, one just needs to make sure that the resulting closed 2-form on the leaf is still non-degenerate.

Theorem 4.24. *Let (M,π) be a Poisson manifold and $B \in \Omega^2(M)$ a closed 2-form. If $(I+B^\flat \circ \pi^\#) : T^*M \to T^*M$ is invertible, then the bivector field $e^B\pi$ defined by:*

$$(e^B\pi)^\# := \pi^\# \circ (I+B^\flat \circ \pi^\#)^{-1}, \tag{4.2}$$

is a Poisson structure in M. If (S,ω) is a symplectic leaf of π, then $(S,\omega+B|_S)$ is a symplectic leaf of $e^B\pi$.

Proof. Since we are assuming that $(I+B^\flat \circ \pi^\#) : T^*M \to T^*M$ is an invertible bundle map, formula (4.2) defines a smooth bivector field $e^B\pi$ and we need to show that:

$$[e^B\pi, e^B\pi] = 0.$$

Clearly, it is enough to prove that this Schouten bracket is zero on a dense open subset. So we will check that it vanishes at regular points.

We start by observing that the graph of $e^B\pi$ is:

$$\mathrm{Graph}(e^B\pi) = \{(\pi^\#(\alpha), \alpha + i_{\pi^\#(\alpha)}B) \in TM \times T^*M : \alpha \in T^*M\}.$$

Now, let $x_0 \in M$ be a regular point of π and choose Darboux-Weinstein coordinates (U, p_i, q^i, x^a) centered at x_0, so that:

$$\pi|_U = \sum_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}.$$

In these coordinates, $\pi|_U$ is regular with foliated symplectic form $\omega_{\mathscr{S}} = \sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i$. It follows that in these coordinates the bivector field $e^B\pi|_U$ is also regular with $\mathrm{Im}(e^B\pi|_U)^\# = \mathrm{Im}\,\pi|_U^\#$. Hence, $e^B\pi|_U$ determines the same foliation in U, but with the new non-degenerate foliated 2-form:

$$\omega_{\mathscr{S}} + B|_{x^a=c}$$

Since this form is closed, it follows that:

$$[e^B\pi, e^B\pi]|_U = 0,$$

as claimed.

Since $\operatorname{Im}(e^B\pi)^\# = \operatorname{Im}\pi^\#$ any integral submanifold of π is also an integral submanifold of $e^B\pi$, and conversely. Hence, the maximal integral leaves must coincide. The expression for the graph of $e^B\pi$ above shows that if (S,ω) is an integral leaf of π, then $(S,\omega+B|_S)$ is an integral leaf of $e^B\pi$. □

One calls the Poisson structure $e^B\pi$ the **gauge B-transform** of the Poisson structure π. We will see later, when we study Dirac structures, the reason for this notation.

Example 4.25 (Coisotropic fibrations). Let (M,π) be a Poisson manifold and let $\phi : M \to Q$ be a submersion whose fibers are coisotropic submanifolds:

$$\pi^\#(\ker \mathrm{d}\phi)^0 \subset \ker \mathrm{d}\phi.$$

We claim that if $\beta \in \Omega^2(Q)$ is any closed 2-form, then the $\phi^*\beta$-transform of π is well-defined.

We need to check $(I + B^\flat \circ \pi^\#) : T^*M \to T^*M$, where $B = \phi^*\beta$, is an invertible bundle map. Let $\alpha \in T^*M$ belong to the kernel of $I + B^\flat \circ \pi^\#$. We claim that $\alpha = 0$, so it will follows that $I + B^\flat \circ \pi^\#$ is invertible. In fact, we have:

$$(I + B^\flat \circ \pi^\#)\alpha = 0 \quad \Longleftrightarrow \quad \alpha = -i_{\pi^\#\alpha}B = -i_{\pi^\#\alpha}\phi^*\beta,$$

so it follows that any α in the kernel of $I + B^\flat \circ \pi^\#$ belongs to $(\ker \mathrm{d}\phi)^0$. But then, since the fibers are coisotropic, we have $\pi^\#\alpha \in \ker \mathrm{d}\phi$ and we obtain:

$$0 = (I + B^\flat \circ \pi^\#)\alpha = \alpha + i_{\pi^\#(\alpha)}\phi^*\beta = \alpha$$

as claimed.

For example, for the canonical symplectic structure on the cotangent bundle $(T^*Q, \omega_{\mathrm{can}})$ the fibers $p : T^*Q \to Q$ are Lagrangian. It follows that for any closed 2-form $\beta \in \Omega^2(Q)$, the gauge transform $\omega_{\mathrm{can}} + p^*\beta$ is still a symplectic form. One sometimes call the term $p^*\beta$ a *magnetic term*.

Exercise 4.26. Show that if $\phi : M \to Q$ is a submersion whose fibers are coisotropic submanifolds of (M,π), then the fibers are still coisotropic for the gauge transformed Poisson structure $e^{\phi^*\beta}\pi$, for any closed 2-form $\beta \in \Omega^2(Q)$.

4.5 Poisson Fibrations

Our last construction of Poisson structures will consist of building out of some geometric data a Poisson structure on the total space of a locally trivial fibration $\phi : M \to B$, for which the fibers become Poisson transversals. In order to understand what kind of geometric data we need, we will start with such a fibration and we will study its geometry.

Henceforth, by a *fibration* we will mean a surjective submersion $\phi : M \to B$, with connected fibers, which is locally trivial.

Definition 4.27. A fibration $\phi : M \to B$ with total space a Poisson manifold (M,π) is called a **Poisson fibration** if the fibers are Poisson transversals.

We remark that this name is not standard. A more common notation that one can find in the literature is to call π a *horizontally non-degenerate Poisson structure*.

Example 4.28 (symplectic fibrations). For a symplectic manifold a Poisson transversal is just a symplectic submanifold. Hence, a **symplectic fibration** is a fibration $\phi : M \to B$ of a symplectic manifold (M,ω) whose fibers are all symplectic submanifolds.

Example 4.29 (neighborhoods of symplectic leaves). Let S be an embedded symplectic leaf of a Poisson manifold (M,π). Let us choose a tubular neighborhood of S, i.e., an embedding $i : \mathcal{N}_S \to U$, onto some neighborhood U of S, which sends the zero section diffeomorphically to S:

$$\begin{tikzcd} \mathcal{N}_S \arrow[rr, hook] \arrow[dr] & & U \arrow[dl] \\ & S & \end{tikzcd}$$

This yields a fibration $p : U \to S$ and since S is a symplectic leaf, at points of the zero section $s \in S$ we have:

(a) $\ker \mathrm{d}_s p$ is complementary to $T_sS = \operatorname{Im}\pi_s^\#$,
(b) $\ker \pi_s^\# = (T_sS)^0$.

It follows that for any $s \in S$:

$$T_sU = (\ker \mathrm{d}_s p) \oplus \pi^\#(\ker \mathrm{d}_s p)^0.$$

If U is small enough this condition holds also at every point $m \in U$, so the fibers of $p : U \to S$ are Poisson transversals. We conclude that a small enough tubular neighborhood of an embedded symplectic leaf is a Poisson fibration.

Given a Poisson fibration $\phi : M \to B$ we associate with it the following data:

- **A vertical Poisson structure π^V:** Since each fiber $F_b := \phi^{-1}(b)$ is a Poisson transversal, there is has an induced Poisson structure on the fiber which we will denote by $\pi_b^V \in \mathfrak{X}(F_b)$. We can make this family of bivectors on the fibers $\{\pi_b^V\}_{b\in B}$ into a single Poisson bivector $\pi^V \in \mathfrak{X}^2(M)$, which we will call the *vertical Poisson structure* of the Poisson fibration.
- **A connection Γ:** The condition that the fiber F_b is a Poisson transversal gives:

$$T_uM = T_uF_b \oplus \pi^\#(T_uF_b)^0, \quad \forall u \in F_b.$$

This means that we have a Ehresmann connection Γ on the fibration with *horizontal space*:

$$\mathrm{Hor}_u := \pi^{\#}(T_u F_b)^0.$$

Henceforth, we will also write

$$\mathrm{Vert}_u := T_u F_b$$

and call it the *vertical space*, so we have the decomposition:

$$T_u M = \mathrm{Vert}_u \oplus \mathrm{Hor}_u\,. \tag{4.3}$$

Given a vector field on the base $X \in \mathfrak{X}(B)$, we will denote by $\tilde{X} \in \mathfrak{X}(M)$ the *horizontal lift* of X, i.e., the unique vector field in M such that $X_u \in \mathrm{Hor}_u$ and $(\phi)_* \tilde{X} = X$. The curvature of the connection Γ is the vertical valued 2-form $\Omega^2(B;\mathrm{Vert})$ given by:

$$\Omega_\Gamma(X,Y) := [\tilde{X},\tilde{Y}] - \widetilde{[X,Y]}, \quad X,Y \in \mathfrak{X}(B). \tag{4.4}$$

- **A horizontal non-degenerate 2-form ω^H:** The decomposition (4.3) leads to a decomposition of the Poisson bivector:

$$\pi = \pi^V \oplus \pi^H, \tag{4.5}$$

where $\pi^H \in \mathfrak{X}^2(\mathrm{Hor}) \equiv \Gamma(\wedge^2 \mathrm{Hor})$ is non-degenerate (see Section 2.2). We denote by $\omega^H \in \Omega^2(\mathrm{Hor}) \equiv \Gamma(\wedge^2 \mathrm{Hor}^*)$ the corresponding non-degenerate 2-form, which one can view as skew-symmetric bilinear map $\omega : \mathrm{Hor} \times \mathrm{Hor} \to \mathbb{R}$.

Therefore, to a Poisson fibration $\phi : M \to B$ one associates a triple $(\pi^V, \Gamma, \omega^H)$. The next proposition gives the properties of this geometric data:

Proposition 4.30. *Let $\phi : M \to B$ be a Poisson fibration with associated triple $(\pi^V, \Gamma, \omega^H)$. Then:*

(i) π^V is a vertical Poisson structure:

$$[\pi^V, \pi^V] = 0. \tag{4.6}$$

(ii) Parallel transport along Γ preserves the vertical Poisson structure:

$$\mathscr{L}_{\tilde{X}} \pi^V = 0, \quad \forall X \in \mathfrak{X}(B). \tag{4.7}$$

(iii) The curvature of Γ is hamiltonian, i.e., satisfies the identity:

$$\Omega_\Gamma(X,Y) = (\pi^V)^{\#}(\mathrm{d} i_{\tilde{Y}} i_{\tilde{X}} \omega^H). \tag{4.8}$$

(iv) The horizontal 2-form ω is closed:

$$\mathrm{d}_\Gamma \omega = 0. \tag{4.9}$$

Proof. For the proof we will need another incarnation of the Schouten bracket, which we will actually explore much further later, and we express in the following lemma, whose proof we leave as an exercise.

Lemma 4.31. *For any 1-forms $\alpha,\beta,\delta\in\Omega^1(M)$:*

$$\frac{1}{2}[\pi,\pi](\alpha,\beta,\delta)=\langle[\pi^\#(\alpha),\pi^\#(\beta)]-\pi^\#([\alpha,\beta]_\pi),\delta\rangle, \tag{4.10}$$

where $[\cdot,\cdot]_\pi:\Omega^1(M)\times\Omega^1(M)\to\Omega^1(M)$ is a bracket on 1-forms defined by:

$$[\alpha,\beta]_\pi=\mathscr{L}_{\pi^\#(\alpha)}\beta-\mathscr{L}_{\pi^\#(\beta)}\alpha-\mathrm{d}\pi(\alpha,\beta).$$

Exercise 4.32. Proof this lemma.
HINT: Both sides of (4.10) are $C^\infty(M)$-multilinear, so it is enough to check them on exact 1-forms.

Now, given a vector field $X\in\mathfrak{X}(B)$ there exists a unique one form $\gamma\in\mathrm{Vert}^0$ such that $\pi^\#(\gamma)=\tilde{X}$. If we write γ_X for this 1-form, then the definition of ω^H shows that we have:

$$\omega^H(\tilde{X},\tilde{Y})=-\pi(\gamma_X,\gamma_Y).$$

To deduce (4.6)-(4.8) we evaluate the Schouten bracket $[\pi,\pi]$ using (4.10) on 1-forms of type $\alpha\in\mathrm{Hor}^0$ and of type $\gamma_X\in\mathrm{Vert}^0$:

(i) if $\alpha,\beta,\delta\in\mathrm{Hor}^0$, we claim that:

$$[\pi,\pi](\alpha,\beta,\delta)=[\pi^V,\pi^V](\alpha,\beta,\delta), \tag{4.11}$$

which shows that (4.6) follows. In fact, we have

$$\begin{aligned}\frac{1}{2}[\pi,\pi](\alpha,\beta,\delta)&=\langle[\pi^\#(\alpha),\pi^\#(\beta)]-\pi^\#([\alpha,\beta]_\pi),\delta\rangle\\&=\langle[(\pi^V)^\#(\alpha),(\pi^V)^\#(\beta)],\delta\rangle-\langle(\pi^V)^\#([\alpha,\beta]_{\pi^V}),\delta\rangle\\&=\frac{1}{2}[\pi^V,\pi^V](\alpha,\beta,\delta)\end{aligned}$$

where we used that $\pi^\#(\alpha)=(\pi^V)^\#(\alpha)$ whenever $\alpha\in\mathrm{Vert}^0$, so that:

$$\begin{aligned}\langle\pi^\#([\alpha,\beta]_\pi),\delta\rangle&=-\langle[\alpha,\beta]_\pi,\pi^\#(\delta)\rangle\\&=-\langle\mathscr{L}_{\pi^\#(\alpha)}\beta-\mathscr{L}_{\pi^\#(\beta)}\alpha-\mathrm{d}\pi(\alpha,\beta),\pi^\#(\delta)\rangle\\&=-\langle\mathscr{L}_{(\pi^V)^\#(\alpha)}\beta-\mathscr{L}_{(\pi^V)^\#(\beta)}\alpha-\mathrm{d}(\pi^V)(\alpha,\beta),(\pi^V)^\#(\delta)\rangle\\&=\langle(\pi^V)^\#([\alpha,\beta]_{\pi^V}),\delta\rangle.\end{aligned}$$

(ii) if $\alpha,\beta\in\mathrm{Hor}^0$ and $X\in\mathfrak{X}(B)$, so that $\gamma_X\in\mathrm{Vert}^0$, we claim that:

$$\frac{1}{2}[\pi,\pi](\gamma_X,\alpha,\beta)=(\mathscr{L}_{\tilde{X}}\pi^V)(\alpha,\beta), \tag{4.12}$$

so that (4.7) follows. In fact, we have:

$$\begin{aligned}\frac{1}{2}[\pi,\pi](\gamma_X,\alpha,\beta) &= \langle[\pi^\#(\gamma_X),\pi^\#(\alpha)]-\pi^\#([\gamma_X,\alpha]_\pi),\beta\rangle\\ &= \langle[\tilde{X},\pi^\#(\alpha)]-\pi^\#(\mathscr{L}_{\tilde{X}}\alpha-\mathscr{L}_{\pi^\#(\alpha)}\gamma_X-\mathrm{d}\pi(\gamma_X,\alpha)),\beta\rangle\\ &= \langle[\tilde{X},\pi^\#(\alpha)]-\pi^\#(\mathscr{L}_{\tilde{X}}\alpha,\beta\rangle=(\mathscr{L}_{\tilde{X}}\pi^V)(\alpha,\beta),\end{aligned}$$

where we used that $\pi(\gamma_X,\alpha)=\alpha(\tilde{X})=0$ and $\mathscr{L}_{\pi^\#(\alpha)}\gamma_X\in\mathrm{Vert}^0$, whenever $\alpha\in\mathrm{Hor}^0$. This last relation follows, since for any vertical vector field V we have:

$$\langle\mathscr{L}_{\pi^\#(\alpha)}\gamma_X,V\rangle=-\langle\gamma_X,[\pi^\#(\alpha),V]\rangle=0,$$

where we used that $\pi^\#(\alpha)$ is vertical and the Lie bracket of vertical vector fields is vertical.

(iii) if $\alpha\in\mathrm{Hor}^0$ and $X,Y\in\mathfrak{X}(B)$, so that $\gamma_X,\gamma_Y\in\mathrm{Vert}^0$, we we claim that:

$$\frac{1}{2}[\pi,\pi](\gamma_X,\gamma_Y,\alpha)=\langle\Omega_\Gamma(X,Y)-(\pi^V)^\#(\mathrm{d}i_{\tilde{Y}}i_{\tilde{X}}\omega^H),\alpha\rangle \tag{4.13}$$

so (4.8) follows. To see this, we compute:

$$\begin{aligned}\frac{1}{2}[\pi,\pi](\gamma_X,\gamma_Y,\alpha) &= \langle[\pi^\#(\gamma_X),\pi^\#(\gamma_Y)]-\pi^\#([\gamma_X,\gamma_Y]_\pi),\alpha\rangle\\ &= \langle[\tilde{X},\tilde{Y}]-\pi^\#(\mathscr{L}_{\tilde{X}}\gamma_Y-\mathscr{L}_{\tilde{Y}}\gamma_X-\mathrm{d}\pi(\gamma_X,\gamma_Y),\alpha\rangle\\ &= \langle[\tilde{X},\tilde{Y}]-\widetilde{[X,Y]}-(\pi^V)^\#(\mathrm{d}\omega^H(\tilde{X},\tilde{Y})),\alpha\rangle\\ &= \langle\Omega_\Gamma(X,Y)-(\pi^V)^\#(\mathrm{d}i_{\tilde{Y}}i_{\tilde{X}}\omega^H),\alpha\rangle\end{aligned}$$

were we used that $\widetilde{[X,Y]}\in\mathrm{Hor}$ so that $\langle\widetilde{[X,Y]},\alpha\rangle=0$ and that $\mathscr{L}_{\tilde{X}}\gamma_Y,\mathscr{L}_{\tilde{Y}}\gamma_X\in\mathrm{Vert}^0$ so that $\langle\pi^\#(\mathscr{L}_{\tilde{X}}\gamma_Y),\alpha\rangle=\langle\pi^\#(\mathscr{L}_{\tilde{Y}}\gamma_X),\alpha\rangle=0$.

(iv) if $X,Y,Z\in\mathfrak{X}(B)$, so that $\gamma_X,\gamma_Y,\gamma_Z\in\mathrm{Vert}^0$, we we claim that:

$$\frac{3}{2}[\pi,\pi](\gamma_X,\gamma_Y,\gamma_Z)=\mathrm{d}_\Gamma\omega(X,Y,Z), \tag{4.14}$$

so that (4.9) follows. In fact, we have:

$$\begin{aligned}\frac{1}{2}[\pi,\pi](\gamma_X,\gamma_Y,\gamma_Z) &= \langle[\pi^\#(\gamma_X),\pi^\#(\gamma_Y)]-\pi^\#([\gamma_X,\gamma_Y]_\pi),\gamma_Z\rangle\\ &= \langle[\tilde{X},\tilde{Y}]-\pi^\#(\mathscr{L}_{\tilde{X}}\gamma_Y-\mathscr{L}_{\gamma_Y}\gamma_X-\mathrm{d}\pi(\gamma_X,\gamma_Y)),\gamma_Z\rangle\\ &= -\omega([\tilde{X},\tilde{Y}],\tilde{Z})+\langle\mathscr{L}_{\tilde{X}}\gamma_Y-\mathscr{L}_{\gamma_Y}\gamma_X+\mathrm{d}\omega^H(\tilde{X},\tilde{Y}),\tilde{Z}\rangle\\ &= -\omega([\tilde{X},\tilde{Y}],\tilde{Z})+\tilde{Z}\cdot\omega^H(\tilde{X},\tilde{Y})-\langle\gamma_Y,[\tilde{X},\tilde{Z}]\rangle+\langle\gamma_X,[\tilde{Y},\tilde{Z}]\rangle\end{aligned}$$

Now, if we cyclic permute X, Y and Z and sum, we obtain:

$$\begin{aligned}\frac{3}{2}[\pi,\pi](\gamma_X,\gamma_Y,\gamma_Z) &= \tilde{X}\cdot\omega^H(\tilde{Y},\tilde{Z})+\tilde{Y}\cdot\omega^H(\tilde{Z},\tilde{X})+\tilde{Z}\cdot\omega^H(\tilde{X},\tilde{Y})\\ &\qquad -\omega([\tilde{X},\tilde{Y}],\tilde{Z})-\omega([\tilde{Y},\tilde{Z}],\tilde{X})-\omega([\tilde{Z},\tilde{X}],\tilde{Y})\\ &= \mathrm{d}_\Gamma\omega(X,Y,Z),\end{aligned}$$

since the cyclic sum of $\langle\gamma_Y,[\tilde{X},\tilde{Z}]\rangle-\langle\gamma_X,[\tilde{Y},\tilde{Z}]\rangle$ gives zero. □

The proposition shows that, as long as parallel transport is well defined, then the fibers of a Poisson fibration are all Poisson diffeomorphic. For example, this is the case if M is compact.

Exercise 4.33. Let $\phi: M\to B$ be a Poisson fibration with associated triple (π^V,Γ,ω^H). Show that the parallel transport map $\tau_\gamma:\phi^{-1}(b_0)\to\phi^{-1}(b_0)$ along any contractible loop γ in B based at b_0 is a hamiltonian diffeomorphism of the fiber $(\phi^{-1}(b_0),\pi^V)$.

One also has a converse to the previous proposition: a triple (π^V,Γ,ω^H) satisfying the conditions of the proposition define a Poisson structure for which the fibration is Poisson.

Theorem 4.34. *Let $\phi: M\to B$ be a fibration. There is a 1:1 correspondence between Poisson structures on M for which the fibration is Poisson and triples (π^V,Γ,ω^H) where π^V is a vertical bivector field, Γ is an Ehreshmann connection and ω^H is a horizontal non-degenerate 2-form, satisfying* (4.6)-(4.9).

Proof. Proposition 4.30 shows that a Poisson fibration yields a triple (π^V,Γ,ω^H) satisfying (4.6)-(4.9).

Conversely, given a fibration $\phi: M\to B$ and a triple (π^V,Γ,ω^H), we define a bivector $\pi\in\mathfrak{X}^2(M)$ by:

$$\pi(\alpha,\beta)=\begin{cases}\pi^V(\alpha,\beta) & \text{if } \alpha,\beta\in\mathrm{Hor}^0\\ 0 & \text{if } \alpha\in\mathrm{Hor}^0,\beta\in\mathrm{Vert}^0\\ -\omega^H(\tilde{X},\tilde{Y}) & \text{if } \alpha=\gamma_X,\beta=\gamma_Y\in\mathrm{Vert}^0.\end{cases}$$

Formulas (4.11), (4.12), (4.13) and (4.14) show that $[\pi,\pi]=0$, provided that (4.6)-(4.9) hold. It should be clear that this Poisson structure makes $\phi: M\to B$ into a Poisson fibration with associated geometric data (π^V,Γ,ω^H). □

Conditions (4.6)-(4.9) have a geometric meaning which becomes more clear if ones uses some principal bundle theory (sometimes called gauge theory). Assume that we start with a Poisson fibration $\phi: M\to B$. We form the **Poisson frame bundle** $P\to B$, which is a principal bundle with an infinite dimensional gauge group, obtained as follows: fix a base point b_0, so we have the fiber $F:=\phi^{-1}(b_0)$ which is a Poisson manifold with Poisson structure $\pi_F:=\pi^V_{b_0}$. We set:

$$P := \{u : (F, \pi_F) \to (F_b, \pi_b^V) | \text{ is a Poisson diffeomorphism } \}$$

The group of Poisson diffeomorphisms $G = \mathrm{Diff}(F, \pi_F)$ acts on the right of P by precomposition:

$$u \cdot g := u \circ g.$$

It is easy to check that the associated bundle $P \times_G F \to B$ is canonical isomorphic to the original fibration $\phi : M \to B$.

Every Poisson connection Γ on the Poisson fiber bundle $p : M \to B$ is induced by a principal bundle connection on $P \to B$. To see this, observe that the tangent space $T_u P \subset C^\infty(u^* TM)$ at a point $u \in P$ is formed by the vector fields along u, $X(x) \in T_{u(x)}M$ such that:

$$d_{u(x)} p \cdot X(x) = \text{constant}, \quad \mathscr{L}_X \pi_V = 0.$$

The Lie algebra $\mathfrak{g}$ of G is the space of Poisson vector fields: $\mathfrak{g} = \mathfrak{X}(F, \pi_F)$. The infinitesimal action on P is given by:

$$\rho : \mathfrak{g} \to \mathfrak{X}(P), \quad \rho(X)_u = \mathrm{d}u \cdot X,$$

so the vertical space of P is:

$$\mathrm{Vert}_u = \{\mathrm{d}u \cdot X : X \in \mathfrak{X}(F, \pi)\}.$$

Now a Poisson connection Γ on $p : M \to B$ determines a connection in $P \to M$ whose horizontal space is:

$$\mathrm{Hor}_u = \{\widetilde{v} \circ u : v \in T_b B\},$$

where $u : F \to F_b$ and $\widetilde{v} : F_b \to T_{F_b} M$ denotes the horizontal lift of v. Clearly, this defines a principal bundle connection on $P \to B$, whose induced connection on the associated bundle $M = P \times_G F$ is the original Poisson connection Γ.

Fix a Poisson connection Γ on the Poisson fiber bundle $p : M \to B$. Recall that the holonomy group $\Phi(b)$ with base point $b \in B$ is the group of holonomy transformations $\phi_\gamma : F_b \to F_b$, where γ is a loop based at b. Clearly, we have $\Phi(b) \subset \mathrm{Diff}(F_b, \pi_b^V)$. On the other hand, for $u \in P$ we have the holonomy group $\Phi(u) \subset G = \mathrm{Diff}(F, \pi_F)$ of the corresponding connection in P which induces Γ: it consist of all elements $g \in G$ such that u and ug can be joined by a horizontal curve in P. Obviously, these two groups are isomorphic, for if $u : F \to F_b$ then:

$$\Phi(u) \to \Phi(b), \; g \mapsto u \circ g \circ u^{-1},$$

is an isomorphism.

The curvature of a principal bundle connection is a $\mathfrak{g}$-valued 2-form F_Γ on $P \to B$, which is defined by:

$$F_\Gamma(X, Y) = [\widetilde{X}, \widetilde{Y}] - \widetilde{[X, Y]}.$$

It is related to the curvature Ω_Γ on $M \to B$ by:

$$\Omega_\Gamma = \mathrm{d}u \circ F_\Gamma \circ u^{-1}. \tag{4.15}$$

Given any $u \in P$, a Poisson diffeomorphism $u : F \to F_b$, the curvature identity shows that, for any $X,Y \in T_bB$, the vector field $F_\Gamma(X,Y)_u \in \mathfrak{g} = \mathfrak{X}(F,\pi)$ is Hamiltonian:

$$F_\Gamma(X,Y)_u = (\pi^V)^\# \mathrm{d}(\omega^H(\widetilde{v}_1,\widetilde{v}_2) \circ u).$$

Now fix $u_0 \in P$. The Holonomy Theorem states that the Lie algebra of the holonomy group $\Phi(u_0)$ is generated by all values $F_\Gamma(v_1,v_2)_u$, with $u \in P$ any point that can be connected to u_0 by a horizontal curve. Assume that we can define a moment map $\mu_F : F \to (\mathrm{Lie}(\Phi(u_0))^*)$ for the action of $\Phi(u_0)$ on F by:

$$\langle \mu_F(x), F_\Gamma(X,Y)_u \rangle = \omega^H(\widetilde{X},\widetilde{Y})_{u(x)}. \tag{4.16}$$

Exercise 4.35. Show that if (4.16) holds, then $\mathrm{d}_\Gamma \omega^H$, so this is a necessary condition for the existence of μ_F.

To summarize, given a Poisson fibration $\phi : M \to B$ we have produce the following data:

- A principal G-bundle $P \to B$ with a connection with curvature F_Γ;
- A hamiltonian G-space (F, π_F, G, μ_F).

This data determines the Poisson fibration. In fact, as a bundle, $\phi : M \to B$ is isomorphic to the associated bundle $P \times_G F$, and we have an associated triple $(\pi^V, \Gamma, \omega^H)$, where:

(i) The vertical Poisson structure π^V is isomorphic to the vertical Poisson structure induced on $P \times_G F$ from (F, π_F).
(ii) The Poisson connection Γ is induced from the connection on P
(iii) The horizontal non-degenerate 2-form ω^H satisfies:

$$\omega^H(\widetilde{X},\widetilde{Y})_{[u,x]} = \langle \mu_F(x), F_\Gamma(X,Y)_u \rangle.$$

This discussion is somewhat formal, since both G and P are infinite dimensional. In the homework at the end of this lecture, we show that this can be made precise by considering finite dimensional gauge groups.

Homework 4: Poisson Gauge Theory

4.1. Let $P \to B$ be a principal G-bundle. Show that if G acts by Poisson diffeomorphisms on (F, π_F), then the associated bundle $P \times_G F \to B$ has a vertical Poisson structure π^V. Moreover, show that if Γ is a principal bundle connection on P, then for the induced connection on $P \times_G F \to B$ parallel transport preserves the vertical Poisson structure.

4.2. Let $P \to B$ be a principal G-bundle and let (F, π_F, G, μ_F) be a Hamiltonian G-space. Also, choose a principal bundle connection Γ in P with curvature 2-form F_Γ. This choice of data is called **Yang-Mills data**. In the associated bundle $P \times_G F \to B$ consider the triple $(\pi^V, \Gamma, \omega^H)$, where:

- π^V is the vertical Poisson structure and Γ is the connection given in Problem 4.1;
- ω^H is the horizontal 2-form defined by:

$$\omega^H(\tilde{X}, \tilde{Y})_{[u,x]} = \langle \mu_F(x), \mathscr{F}_\Gamma(X, Y)_u \rangle.$$

Show that $(\pi^V, \Gamma, \omega^H)$ satisfy conditions (4.6)-(4.9). Conclude that if ω^H is non-degenerate, then the associated bundle $P \times_G F \to B$ is a Poisson fibration.

4.3. Consider the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$ as a principal $\mathbb{S}^1$-bundle with a curvature form that is nowhere vanishing. Let $\mu_F : \mathbb{R} \to \mathbb{R}$ be a non vanishing function. Show that we are in the conditions of Problem 4.2 and find the corresponding Poisson fibration $P \times_G F \to B$.

4.4. Consider Yang-Mills data on a principal G-bundle $P \to B$, as in Problem 4.2, and the corresponding triple $(\pi^V, \Gamma, \omega^H)$, on the associated bundle $\phi : P \times_G F \to B$, but where possibly ω^H is degenerate. Show that if $\beta \in \Omega^2(B)$ is a closed 2-form such $\omega^H + \phi^*\beta$ is a non-degenerate horizontal form, then the triple $(\pi^V, \Gamma, \omega^H)$ still satisfies conditions (4.6)-(4.9) and hence defines a Poisson structure on the associated bundle $\phi : P \times_G F \to B$ making it into a Poisson fibration.

4.5. Consider Yang-Mills data on a principal G-bundle $P \to B$, as in Problem 4.2, and the corresponding triple $(\pi^V, \Gamma, \omega^H)$, on the associated bundle $\phi : P \times_G F \to B$. Assume that there exists a point $x_0 \in F$ which is a fixed point of the G-action and $\pi_F|_{x_0} = 0$, and let:

$$S := P \times \{x_0\}/G \subset P \times_G F.$$

(i) Show that if $\beta \in \Omega^2(B)$ is a symplectic form, then $\omega^H + \phi^*\beta$ non-degenerate in some neighborhood M of S in $P \times_G F$, so that we have triple $(\pi^V, \Gamma, \omega^H + \phi^*\beta)$ defining a Poisson structure π on M.
(ii) The submanifold $P \times x_0/G \subset P \times_G F$ is a symplectic leaf of M, symplectomorphic to (B, β).
(iii) There is a tubular neighborhood of S in (M, π) for which the associated triple is $(\pi^V, \Gamma, \omega^H + \phi^*\beta)$.

4.6. Consider a principal G-bundle $P \to B$ with a principal bundle connection Γ and let $G \times \mathfrak{g}^* \to \mathfrak{g}^*$ be the coadjoint action. Assume also that $\beta \in \Omega^2(B)$ is a symplectic form. Verify that the identity map $\mu_F : \mathfrak{g}^* \to \mathfrak{g}^*$ is a moment map for the coadjoint action, so that we are in the conditions of Problem 4.5, where $x_0 = 0$. Show that different principal bundle connections induce Poisson structures in a neighborhood of B in $P \times_G \mathfrak{g}^*$ which are Poisson diffeomorphic. This Poisson structure is called a **linear local model** defined by the principal G-bundle $P \to B$ and the symplectic form $\beta \in \Omega^2(B)$.

4.7. Show that the linear local model defined by the principal G-bundle $P \to B$ and the symplectic form $\beta \in \Omega^2(B)$, can also be obtained as follows from a principal bundle connection Γ with connection 1-form $\theta : TP \to \mathfrak{g}$:

(i) Show that the 1-form $\tilde{\theta} \in \Omega^1(P \times \mathfrak{g}^*)$ defined by:

$$\tilde{\theta}_{(p,\eta} = \langle \theta, \eta \rangle,$$

is a G-invariant 1-form.

(ii) Show that $\omega := p^*\beta - \mathrm{d}\tilde{\theta} \in \Omega^1(P \times \mathfrak{g}^*)$ defines a G-invariant symplectic form on an open G-invariant set U containing $P \times \{0\}$.

(iii) Show that (B, β) is a symplectic leaf of the quotient Poisson structure on $M = U/G$;

(iv) Show that the Poisson structure on M is isomorphic to the linear local model.

Chapter 5
Examples of Poisson manifolds

In this chapter we will discuss some classes of examples of Poisson structures.

5.1 Poisson structures in dimension two

In dimension two Poisson structures are apparently not very interesting, mainly because of the following:

Proposition 5.1. *Let Σ be a two dimension manifold. Then every bivector $\pi \in \mathfrak{X}^2(\Sigma)$ is Poisson.*

Proof. This is clear, since $[\pi,\pi] \in \mathfrak{X}^3(\Sigma) = \{0\}$.

The symplectic foliation of a 2-dimensional Poisson manifold (Σ,π) is very easy to understand. The 0-dimensional leaves are simply the zeros of π, and the 2-dimensional leaves are the connected components of the open $\{x \in \Sigma : \pi(x) \neq 0\}$.

To obtain some interesting Poisson geometry in dimension two, we will consider Poisson structures that satisfy extra conditions.

5.1.1 Symplectic structures in dimension two

Note first that on a two-manifold, a symplectic structure is the same as a volume form. Therefore,

Proposition 5.2. *A two-dimensional manifold admits a symplectic structure iff it is orientable.*

Now, compact orientable two-manifolds are classified by their genus $g \geq 0$. Let (Σ,ω) be a compact symplectic manifold, with orientation induced by ω. To it we can associate the following symplectic invariant, called the symplectic volume:

$$\mathrm{Vol}(\Sigma,\omega) := \int_{\Sigma} \omega > 0.$$

In dimension two, the genus and the volume are the only symplectic invariants:

Theorem 5.3. *(Moser) Two compact symplectic 2-manifolds are symplectomorphic (i.e. isomorphic as Poisson manifolds) iff they have the same genus and the same symplectic volume.*

The proof of this result is based on the so-called Moser trick, which will be discussed later on in the general setting of Poisson cohomology.

5.1.2 Log-symplectic manifolds in dimension two

A natural generalization of symplectic manifolds are the following type of structures:

Definition 5.4. Let Σ be a 2-dimensional manifold. A Poisson structure $\pi \in \mathfrak{X}^2(\Sigma)$ is called *log-symplectic*, if the image of π:

$$\pi(\Sigma) := \{\pi(x) : x \in \Sigma\} \subset \bigwedge^2 T\Sigma$$

is transverse to the zero-section of $\bigwedge^2 T\Sigma$.

The set of zeros of a log-symplectic structure (Σ, π) is called the *singular locus* of π, and will be denoted by Z. The transversality condition implies that Z is a submanifold of dimension one:

$$\dim(Z) = \dim(\pi(\Sigma)) + \dim(\Sigma) - \dim(\bigwedge^2 T\Sigma) = 2+2-3 = 1.$$

Example 5.5. Consider the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, with the induced area form ω. In cylindrical coordinates (θ, r, z), $\mathbb{S}^2$ is described by $r^2+z^2 = 1$, and (θ, z) are induced coordinates. In these coordinates ω becomes $\omega = d\theta \wedge dz$ (check that the 2-form $d\theta \wedge dz$ extends indeed smoothly to the north - and south pole, and that it is nondegenerate everywhere). In fact this formula is a modern version of Archimedes theorem: the orthogonal projection from the sphere to the cylinder $r = 1$ is area preserving. Consider the following log-symplectic structure on $\mathbb{S}^2$:

$$\pi := z\omega^{-1} = z\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \theta}.$$

The singular locus Z is the equator $z = 0$.

We can use this example to construct a log-symplectic structure on a non-orientable manifold. Note that π is invariant under the action of $\mathbb{Z}_2 = \{0,1\}$, $1 \cdot (\theta, z) := (\theta + \Pi, -z)$ (here Π denotes the real number $\Pi \neq 3.14159269$).

This shows that π descends to a Poisson structure $\overline{\pi}$ on the real projective space $\mathbb{RP}^2 := \mathbb{S}^2/\mathbb{Z}_2$. Since the projection $\mathbb{S}^2 \to \mathbb{RP}^2$ is a local diffeomorphism, $\overline{\pi}$ is a log-symplectic structure on $\mathbb{RP}^2$.

To simplify the discussion, we assume that Σ is *orientable* and *compact*. We fix a non-degenerate bivector w on Σ. Then there is a unique function $f \in C^\infty(\Sigma)$ so that

$$\pi = fw.$$

Now, the log-symplectic condition on π is equivalent to the fact that 0 *is a regular value of* f; and clearly $Z = f^{-1}(0)$. Since Z is a closed submanifold of Σ of dimension one, it follows that Z is a disjoint union of circles:

$$Z = Z_1 \sqcup \ldots \sqcup Z_k, \;\; Z_i \cong \mathbb{S}^1.$$

Consider now the *Hamiltonian of* $\log|f|$:

$$X_{\log|f|} := \pi^\sharp(\mathrm{d}\log|f|) = \frac{1}{f}\pi^\sharp(\mathrm{d}f) = w^\sharp(\mathrm{d}f)\mathfrak{X}^1(\Sigma).$$

Of course $\log|f|$ is only defined on $\Sigma\backslash Z$, but the formula above shows that its Hamiltonian extends to Σ. This extension is not a Hamiltonian vector field (check this using the properties below), but by abuse of terminology we continue to call it the Hamiltonian of $\log|f|$, and denote it by $X_{\log|f|}$. Its restriction to Z has the following properties:

Proposition 5.6. *Denote by $X_Z := X_{\log|f|}|_Z \in \Gamma(T\Sigma|_Z)$. This vector field satisfies:*

1 X_Z is tangent to Z; i.e. $X_Z \in \mathfrak{X}^1(Z)$.
2 X_Z does not depend on the choice of w.
3 X_Z is nowhere vanishing on Z.

5.7.

References

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