Hit-or-miss representation of a pattern spectrum and its optical implementation

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We demonstrate that a pattern spectrum can be decomposed into the union of hit-or-miss transforms with respect to a series of structure-element pairs. Moreover we use a Boolean-logic function to express the pattern spectrum and show that the Boolean-logic representation of a pattern spectrum is composed of hit-or-miss min terms. The optical implementation of a pattern spectrum is based on an incoherent optical correlator with a feedback operation. © 1996 Optical Society of America

1. Introduction

Mathematical morphology provides a particular approach to image processing and analysis. On the basis of morphological transforms, Matheron[1] and Serra[2] introduced the initial form of the pattern spectrum. Matheron[1] first put forward the size-distribution function. Serra[2] demonstrated its form for a continuous and binary image. Maragos[3] extended Serra's studies and set up the concept of a pattern spectrum. A pattern spectrum is a shape–size descriptor that describes shapes by a series of structure elements that increase step by step. Therefore, pattern spectra have been utilized in image analysis and feature extraction.[4] In contrast, the hit-or-miss transform detects image features or shapes from both the foreground and background. So the hit-or-miss transform can also be used to depict image shape and size distributions corresponding to structure-element pairs. From the shape–size distribution point of view, the question of interest is what is the relation between the pattern-spectrum approach and the hit-or-miss transform?

In this paper, we demonstrate that a pattern spectrum can be decomposed into a hit-or-miss representation. Maragos[5] provides the basis decomposition for morphological opening and closing transforms and also gives the basis structure elements for the opening-transform decomposition. In addition Maragos shows the constraints on the minimal basis structure elements for the closing-transform decomposition. Then, on the basis of the framework of the closing-transform decomposition by Maragos,[5] Svalbe[6] demonstrates a complex algorithm on the extraction of the minimal basis structure elements related to the closing-transform decomposition. The algorithm provides some insight into the constraints on the formation of the closing-transform basis structure elements for a two-dimensional structure element. By means of the decomposition of the opening and closing transforms, we demonstrate that a pattern spectrum can be decomposed into the union of a hit-or-miss transform.

On the other hand, from the Boolean-logic representations for two basic morphological transforms—dilation and erosion[7]—we give concrete expressions for Boolean-logic functions (BLF's) of the morphological opening and closing transforms. Then we provide the BLF's of a pattern spectrum and write them in the canonical sum-of-product form. Each min term in the canonical sum of product corresponds to a hit-or-miss transform. Therefore, from the BLF point of view, we also show that a pattern spectrum can be decomposed into hit-or-miss transforms. On this basis, a pattern-spectrum analysis can be carried out by means of optical hit-or-miss implementation architecture.[7–10] However, the decomposition of a pattern spectrum includes too many hit-or-miss min terms, so it is not convenient to implement a pattern spectrum by the use of an optical hit-or-miss architecture. Therefore we utilize an incoherent optical correlator with a feedback operation to carry out the pattern-spectrum analysis, and some proof-of-principle experimental results are given.

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2. Pattern Spectrum of a Discrete Binary Image and Its Hit-or-Miss Representation

Let $X \subseteq Z^2$ be a finite-extent discrete binary image. S is a subset of Z^2 and denotes a discrete binary structure element. The two basic morphological transforms, dilation, indicated by the direct sum symbol ($\oplus$), and erosion indicated by the symmetric difference symbol ($\ominus$), of X by S, as well as their relation, are denoted as[5,11]

$$X \oplus S = \{z: X \cap (\check{S} + z) \neq \varnothing\} = \bigcup_{s \in S} X + s,$$
$$X \ominus S = \{z: S + z \subseteq X\} = \bigcap_{s \in S} X - s,$$
$$X \oplus S = (X^c \ominus \check{S})^c, \tag{1}$$

where the symbols ($\cup$) and ($\cap$) denote the union and intersection operations, respectively, $(\cdots)^c$ represents the complement operation, and $\check{S}$ is the reflection of S. The other two important morphological transforms[5,11] of X by S are opening, denoted by the symbol ($\circ$), and closing, denoted by the symbol ($\bullet$), and are defined as

$$X \circ S = (X \ominus S) \oplus S,$$
$$X \bullet S = (X \oplus S) \ominus S. \tag{2}$$

The opening and closing transforms have the relation given by

$$X \circ S = [X^c \bullet \check{S}]^c. \tag{3}$$

The hit-or-miss transform of X by a discrete binary structure-element pair $[S_1, S_2]$ can be denoted as[5,11]

$$X \otimes [S_1, S_2] = (X \ominus S_1) \cap (X^c \ominus S_2), \tag{4}$$

where the operator ($\otimes$) denotes the hit-or-miss transform, S_1 and S_2 correspond to the X foreground and background, respectively, and $S_1 \cap S_2 = \varnothing$. Let B represent the basis structure element of the pattern spectrum PS for $B \subseteq Z^2$. Then the pattern spectrum of X with B is defined as[3]

$$\mathrm{PS}(+n, B) = A[X \circ nB \backslash X \circ (n+1)B], \qquad n \geq 0, \tag{5}$$

$$\mathrm{PS}(-n, B) = A[X \bullet nB \backslash X \bullet (n-1)B], \qquad n \geq 1, \tag{6}$$

where $nB = [B \oplus B \oplus B \oplus \cdots \oplus B]_{n \text{ times}}$, $0B = (0, 0)$, the backslash denotes the set difference, and $A[\cdots]$ is the set cardinality. According to Eqs. (5) and (6), the key operations in pattern-spectrum analysis are the opening and closing transforms.

Maragos[5] gives the basis decomposition of the opening transform corresponding to X by S, which can be described as

$$X \circ S = \bigcup_i X \ominus S_i,$$
$$\{S_i\} = \{S - s : s \in S\}, \qquad \{S_i\} \subseteq \{S \oplus \check{S}\}, \tag{7}$$

where S_i denotes the basis structure element related to the decomposition of the opening transform and S_i is the structure element S translated to the reflected location of a point inside S. Thus the maximum of i is equal to the number of points within S. Also, Maragos[5] provides the basis decomposition of the closing transform and demonstrates the following constraint for the formation of the minimal basis structure element S^k corresponding to the decomposition of the closing transform, denoted as

$$X \bullet S = \bigcup_k X \ominus S^k,$$
$$\{S^k\} = \{S^k \subseteq S \oplus \check{S} : 0 \in (S^k \bullet S)\}, \qquad S^k \text{ is minimal}, \tag{8}$$

where S^k is composed of the origin $\{0\}$ and spans the region of support $S \oplus \check{S}$. The number of S^k depends on the shape and size of S. Then, from Eqs. (3) and (5)–(8), the pattern spectrum PS can be expressed as

$$\mathrm{PS}(+n, B) = A\{\bigcup_i \bigcup_k \{[X \ominus nB_i] \cap [X^c \ominus (n+1)\check{B}^k]\}\},$$
$$\{nB_i\} = \{nB_i - b : b \in nB_i\}, \qquad \{nB_i\} \subseteq \{nB \oplus n\check{B}\},$$
$$\{(n+1)\check{B}^k\} = \{(n+1)\check{B}^k \subseteq (n+1)B \oplus (n+1)\check{B} : 0 \in [(n+1)\check{B}^k \bullet (n+1)\check{B}]\}, \qquad (n+1)\check{B}^k \text{ is minimal}, \tag{9}$$

$$\mathrm{PS}(-n, B) = A\{\bigcup_i \bigcup_k \{[X \ominus nB^k] \cap [X^c \ominus (n-1)\check{B}_i]\}\},$$
$$\{(n-1)\check{B}_i\} = \{(n-1)\check{B} - b : b \in (n-1)\check{B}\},$$
$$\times \{(n-1)\check{B}_i\} \subseteq \{(n-1)B \oplus (n-1)\check{B}\},$$
$$\{nB^k\} = \{nB^k \subseteq nB \oplus n\check{B} : 0 \in (nB^k \bullet nB)\},$$
$$nB^k \text{ is minimal}. \tag{10}$$

Note that, if the structure-element pair $[nB_i, (n+1)\check{B}^k]$ keeps $\{nB_i\} \cap \{(n+1)\check{B}^k\} \neq \varnothing$, we then have $[X \ominus nB_i] \cap [X^c \ominus (n+1)\check{B}^k] = 0$. So the intersection of nB_i with $(n+1)\check{B}^k$ has no effect on the pattern spectrum and can be eliminated. Thus we define the structure-element pair $[nB_i, (n+1)\check{B}^{kk}]$ to satisfy the following conditions: $\{(n+1)\check{B}^{kk}\} \subseteq \{(n+1)\check{B}^k\}$, $\{nB_i\} \cap \{(n+1)\check{B}^{kk}\} = \varnothing$, and $[X \ominus nB_i] \cap [X^c \ominus (n+1)\check{B}^{kk}] \neq 0$.

For the same reason, we define the structure-element pair $[nB^{kl}, (n-1)\check{B}_i]$ to keep $[X \ominus nB^{kl}] \cap [X^c \ominus (n-1)\check{B}_i] \neq 0$, where $\{nB^{kl}\} \subseteq \{nB^k\}$ and $\{nB^{kl}\} \cap \{(n-1)\check{B}_i\} = \varnothing$. The expressions $[X \ominus nB_i] \cap [X^c \ominus (n+1)\check{B}^{kk}]$ and $[X \ominus nB^{kl}] \cap [X^c \ominus (n-1)\check{B}_i]$ are just the hit-or-miss transform $\otimes$ of X by the structure-element pairs $[nB_i, (n+1)\check{B}^{kk}]$ and $[nB^{kl}, (n-1)\check{B}_i]$, respectively, according to Eq. (4). Therefore $[X \ominus nB_i] \cap [X^c \ominus (n+1)\check{B}^{kk}] = X \otimes [nB_i, (n+1)\check{B}^{kk}]$, where nB_i and $(n+1)\check{B}^{kk}$ correspond to the X foreground and background, respectively, and $[X \ominus nB^{kl}] \cap [X^c \ominus (n-1)\check{B}_i] = X \otimes [nB^{kl}, (n-1)\check{B}_i]$. Thus the pattern spectrum PS can be written in the form of the union of the hit-or-miss transforms with respect to different structure-element pairs: $[nB_i$,

$(n+1)\check{B}^{kk}]$ and $[nB^{kl}, (n-1)\check{B}_i]$, denoted as follows:

$$\mathrm{PS}(+n, B) = A\{\bigcup_i \bigcup_{kk} X \otimes [nB_i, \quad (n+1)\check{B}^{kk}]\}, \quad (11)$$

$$\{(n+1)\check{B}^{kk}\} \subseteq \{(n+1)\check{B}^{k}\}, \quad \{(n+1)\check{B}^{kk}\} \cap \{nB_i\} = \varnothing, \quad (12)$$

$$\mathrm{PS}(-n, B) = A\{\bigcup_i \bigcup_{kl} X \otimes [nB^{kl}, (n-1)\check{B}_i]\}, \quad (13)$$

$$\{nB^{kl}\} \subseteq \{nB^{k}\}, \quad \{nB^{kl}\} \cap \{(n-1)\check{B}_i\} = \varnothing, \quad (14)$$

From the basis decomposition of the opening and closing transforms we demonstrate that a pattern spectrum can be decomposed into the union of hit-or-miss transforms. The decomposition of a pattern spectrum not only demonstrates the relation of the pattern spectrum with the hit-or-miss transform but also makes the interpretation of the pattern spectrum more understandable.

3. Pattern Spectrum of a Discrete Gray-Tone Image and Its Hit-or-Miss Representation

Let $f(x, y)$, $g(x, y)$, $(x, y) \in Z^2$ be discrete gray-tone images. The erosion and dilation of $f(x, y)$ by a binary structure element S are defined as[5]

$$f(x, y) \ominus S = \min\{f(x+s) : s \in S\},$$
$$f(x, y) \oplus S = \max\{f(x-s) : s \in S\}, \quad (15)$$

respectively. The opening and closing transforms of $f(x, y)$ by S can be expressed as[5]

$$f \circ S = (f \ominus S) \oplus S,$$
$$f \bullet S = (f \oplus S) \ominus S. \quad (16)$$

Corresponding to the hit-or-miss transform of a binary image X by a binary structure-element pair $[S_1, S_2]$, the extended hit-or-miss transform definition for $f(x, y)$ by a structure-element pair $[S_1, S_2]$ can be written as

$$f(x, y) \otimes [S_1, S_2] = [f(x, y) \ominus S_1] \wedge [f^c(x, y) \ominus S_2], \quad (17)$$

where the upward wedge symbol ($\wedge$) denotes the min operation in a gray-tone image-processing system by a binary structure element. The pattern spectrum of $f(x, y)$ with a binary basis-structure element B can be denoted as[5]

$$\mathrm{PS}_f(+n, B) = A[f \circ nB - f \circ (n+1)B], \quad n \geq 0, \quad (18)$$

$$\mathrm{PS}_f(-n, B) = A[f \bullet nB - f \bullet (n-1)B], \quad n \geq 1, \quad (19)$$

where $A(f) = \Sigma_{(x,y)} f(x, y)$ and $p(x) - q(x)$ denotes the pointwise difference between the gray-tone images $p(x)$ and $q(x)$. The morphology-processing function of a binary image can be extended to that of a gray-tone image on the basis of threshold decomposition and sum superposition.[5,7,12] Let ψ_{fp} denote a parallel and cascaded operator of erosion ($\ominus$), dilation ($\oplus$), the min ($\wedge$), the max ($\vee$), and the complement $(\cdots)^c$ for processing a gray-tone image $f(x, y)$ by a binary structure element, and let ψ_{tp} be a parallel and cascaded operator to a threshold image $T_a[f(x)]$. Then the threshold decomposition of a gray-tone processing function ψ_{fp} to the threshold image-processing function ψ_{tp} can be denoted as[7]

$$Z(x, y) = \psi_{\mathrm{fp}}\{f(x, y) : \ominus, \oplus, \vee, \wedge, (\cdots)^c\}$$
$$= \sum_{a=1}^{M} Z_a(x, y),$$
$$Z_a = \psi_{\mathrm{tp}}\{T_a[f(x, y)] : \ominus, \oplus, \cap, \cup, (\cdots)^c\}, \quad (20)$$

where a is a threshold level, and the summation operation denotes a sum superposition. Let H denote the maximum value of the threshold, then $f(x, y)$ can be thresholded into a stack of binary images $f_a(x, y)$, denoted as

$$f(x, y) = \sum_{a=1}^{H} f_a(x, y),$$
$$f_a(x, y) = 1, \quad (x, y) \in T_a[f(x, y)] = \{(x, y) : f(x, y) \geq a\},$$
$$f_a(x, y) = 0, \quad (x, y) \notin T_a[f(x, y)]. \quad (21)$$

The complement of $f(x, y)$, $f^c(x, y) = H - f(x, y)$, does not obey the threshold-decomposition rule, but it can be thresholded into the sum of a binary image $f^c_{H-a+1}(x, y)$, denoted as

$$f^c(x, y) = \sum_{a=1}^{H} f^c_{H-a+1}(x, y),$$
$$f^c_a(x, y) = f^c_{H-a+1}(x, y), \quad (22)$$

Thus the dilation, erosion, max, and min of $f(x, y)$ by S, commuted with the threshold decomposition, can be expressed as

$$f(x, y) \ominus S = \sum_{a=1}^{H} [f_a(x, y) \ominus S],$$
$$f(x, y) \oplus S = \sum_{a=1}^{H} [f_a(x, y) \oplus S],$$
$$f(x, y) \vee g(x, y) = \sum_{a=1}^{H} f_a(x, y) \cup g_a(x, y),$$
$$f(x, y) \wedge g(x, y) = \sum_{a=1}^{H} f_a(x, y) \cap g_a(x, y). \quad (23)$$

The basis decompositions of the opening and closing transforms for $f_a(x, y)$ with nB can be expressed as follows:

$$f_a(x, y) \circ nB = \bigcup_i f_a(x, y) \ominus nB_i,$$
$$\{nB_i\} = \{nB - b : b \in nB\}, \quad \{nB_i\} \subseteq \{nB \oplus n\check{B}\},$$
$$f_a(x, y) \bullet nB = \bigcup_k f_a(x, y) \ominus nB^k,$$
$$\{nB^k\} = \{nB^k \subseteq nB \oplus n\check{B} : 0 \in (nB^k \bullet nB)\},$$
$$nB^k \text{ is minimal.} \quad (24)$$

Then the pattern spectrum PS of $f(x, y)$ with B can be derived as

$$
\begin{aligned}
\mathrm{PS}_f(+n, B) &= A\left\{\sum_{a=1}^{H} [f_a(x, y) \circ nB \backslash f_a(x, y) \circ (n+1)B]\right\} \\
&= A\left\{\sum_{a=1}^{H} \{\bigcup_i \bigcup_k [f_a(x, y) \ominus nB_i] \cap [f^c_{H-a+1}(x, y) \ominus (n+1)\check{B}^k]\}\right\} \\
&= A\{\bigvee_i \bigvee_k \{[f(x, y) \ominus nB_i] \wedge [f^c(x, y) \ominus (n+1)\check{B}^k]\}\},
\end{aligned}
\tag{25}
$$

To eliminate the redundant part from the structure-element pairs $[nB_i, (n+1)\check{B}^k]$ and $[nB^k, (n-1)\check{B}_i]$, which give no contributions to the pattern spectrum of $f(x, y)$ with B, we specify that the structure-element pair $[nB_i, (n+1)\check{B}^{kk}]$, $\{(n+1)\check{B}^{kk}\} \subseteq \{(n+1)\check{B}^k\}$ keeps the relations $\{nB_i\} \cap \{(n+1)\check{B}^{kk}\} = \varnothing$ and

$$
\begin{aligned}
\mathrm{PS}(-n, B) &= A\left\{\sum_{a=1}^{H} [f_a(x, y) \bullet nB \backslash_a(x, y) \bullet (n-1)B]\right\} \\
&= A\left\{\sum_{a=1}^{H} \bigcup_i \bigcup_k \{[f_a(x, y) \ominus nB^k] \cap [f^c_{H-a+1}(x, y) \ominus (n-1)\check{B}_i]\}\right\} \\
&= A\{\bigvee_i \bigvee_k \{f(x, y) \ominus nB^k] \wedge [f^c(x, y) \ominus (n-1)\check{B}_i]\}\}.
\end{aligned}
\tag{26}
$$

Here, $[f_a(x, y) \ominus nB_i] \cap [f^c_{H-a+1}(x, y) \ominus (n+1)\check{B}^{kk}] \neq 0$, and, in addition, the structure-element pair $[nB^{kl}, (n-1)\check{B}_i]$ keeps the relation $[f_a(x, y) \ominus nB^{kl}] \cap [f^c_{H-a+1}(x, y) \ominus (n-1)\check{B}_i] \neq 0$, where $\{nB^{kl}\} \subseteq \{nB^k\}$ and $\{nB^{kl}\} \cap \{(n-1)\check{B}_i\} = \varnothing$. Therefore the expressions $[f_a(x, y) \ominus nB_i] \cap [f^c_{H-a+1}(x, y) \ominus (n+1)\check{B}^{kk}]$ and $[f_a(x, y) \ominus nB^{kl}] \cap [f^c_{H-a+1}(x, y) \ominus (n-1)\check{B}_i]$ are the hit-or-miss transforms of $f_a(x, y)$ for a binary-image foreground and $f^c_{H-a+1}(x, y)$ is its background by the structure-element pairs $[nB_i, (n+1)\check{B}^{kk}]$ and $[nB^{kl}, (n-1)\check{B}_i]$, respectively. Then $[f(x, y) \ominus nB_i] \wedge [f^c\ (x, y) \ominus (n+1)\check{B}^{kk}]$ can be denoted as the hit-or-miss transform of $f(x, y)$ by the structure-element pair $[nB_i, (n+1)\check{B}^{kk}]$, and the expression $[f(x, y) \ominus nB_{kl}] \wedge [f^c(x, y) \ominus (n-1)\check{B}^i]$ can also be denoted as the hit-or-miss transform of $f(x, y)$ by the structure-element pair $[nB^{kl}, (n-1)\check{B}_i]$. Thus a pattern spectrum can be decomposed into the union of hit-or-miss transforms, denoted as follows:

$$\mathrm{PS}_f(+n, B) = A\{\bigvee_i \bigvee_{kk} f(x, y) \otimes [nB_i, (n+1)\check{B}^{kk}]\}, \tag{27}$$

$$\mathrm{PS}(-n, B) = A\{\bigvee_i \bigvee_{kl} f(x, y) \otimes [nB^{kl}, (n-1)\check{B}_i]\}, \tag{28}$$

$$
\begin{aligned}
&\{(n+1)\check{B}^{kk}\} \subseteq \{(n+1)\check{B}^k\}, \qquad \{(n+1)\check{B}^{kk}\} \cap \{nB_i\} = \varnothing, \\
&\{nB^{kl}\} \subseteq \{nB^k\}, \qquad \{nB^{kl}\} \cap \{(n-1)\check{B}_i\} = \varnothing,
\end{aligned}
\tag{29}
$$

Thus the pattern spectrum of a discrete gray-tone image $f(x, y)$ with a binary structure element B can also be represented by the gray-tone image's hit-or-miss transforms.

In the discussion in Section 4 below, we demonstrate the BLF of a pattern spectrum. We also show that a pattern spectrum can be decomposed into a hit-or-miss representation from another point of view.

4. Boolean-Logic Representation of a Discrete Binary Image's Pattern Spectrum

The BLF can be used to represent the morphological transform of a discrete binary image X with a discrete binary structure element S.[7] A morphological processing function transforms the array data of a two-dimensional binary image into new arrays. The value of each pixel in the new array is determined by the old values of the pixels in the neighborhood indicated by a structure element. So a BLF can be used to evaluate the new array of data in a neighborhood determined by a structure element. Thus BLF's at the coordinate origin in a neighborhood can be utilized to express the morphological transforms.[7] Let x_0 be a state variable at the coordinate origin $\mathbf{x}_0$, whose neighborhood is determined by a structure element S and contains $N-1$ pixels. S is the subset in the space Z^2 that connects the pixels in it to the pixel at $\mathbf{x}_0$. The state x_0 is determined by the BLF, F, on the state x_n at all $\mathbf{x}_n$ belonging to S. Thus we have the output state z_0 at the coordinate origin $\mathbf{x}_0$, denoted by[7]

$$z = F(x_n|_{x_n \in S}). \tag{30}$$

Then the BLF's of the basic morphological transforms, erosion $z_{\ominus S}$ and dilation $z_{\oplus S}$, of X to S can be denoted as[7]

$$
\begin{aligned}
z_{\ominus S} &= x_0 x_1 x_2 \cdots x_{N-1}, \\
z_{\oplus S} &= \check{x}_0 + \check{x}_1 + \check{x}_2 + \cdots + \check{x}_{N-1},
\end{aligned}
\tag{31}
$$

where $S = \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{N-1}\}$ and $\check{S} = \{\check{\mathbf{x}}_0, \check{\mathbf{x}}_1, \check{\mathbf{x}}_2, \ldots, \check{\mathbf{x}}_{N-1}\}$ denotes the neighborhoods S and $\check{S}$, respectively, $\mathbf{x}_0$. Now, the solid circle symbol ($\bullet$) represents the pointwise AND operation, and the plus symbol (+) denotes the pointwise OR operation. Therefore the BLF of erosion is the AND operation of the neighborhood states at the coordinate origin, where the neighborhood is defined by S; the BLF of dilation is the OR operation of the neighborhood states at the coordinate point, where the neighborhood is defined by the reflection of S, $\check{S}$. The morphological transforms with the set intersection, union, and complement can be represented by a similar parallel combination of the corresponding BLF's with logic AND, OR, and COMPLEMENT. A parallel combination of morphological transforms can be described by the parallel combination of morphological BLF's. So the hit-or-miss transform of X with a structure-element pair $[S_1, S_2]$ can be expressed as

$$z_{\otimes S^w} = (z_{\ominus S_1})(z^c_{\ominus S_2}) = x_0x_1\cdots x_{L-1}x_q^c x_{q+1}^c\cdots x_{q+K-1}^c, \qquad (32)$$

where $S^w = S_1 \cup S_2$, $S_1 = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{L-1}\}$, $S_2 = \{\mathbf{x}_q, \mathbf{x}_{q+1}, \ldots, \mathbf{x}_{(q+K-1)}\}$, $S_1 \cap S_2 = \varnothing$, and the neighborhoods S_1 and S_2 have $L-1$ and $K-1$ pixels, respectively. In a cascaded combination[7] of morphological transforms, if $z_{0,1}$ represents the BLF of the first morphological transform at the coordinate origin and $z_{0,2}$ denotes the BLF of the subsequent morphological transform at the coordinate origin, then we have the combined BLF, z_0, at the coordinate origin as

$$\begin{aligned} z_{0,1} &= F_1(x_k|_{x_k\in S_1}),\\ z_{0,2} &= F_2(x_l|_{x_l\in S_2}),\\ z_0 &= F_2[z_{0,1}(x_m)|_{x_m\in S_2}(x_m \in \check{S}_2)], \end{aligned} \qquad (33)$$

where $z_{0,1}(x_m)$ denotes the BLF of the first morphological transform at the coordinate point $\mathbf{x}_m$, for which the neighborhood at $\mathbf{x}_m$ is achieved by S_1 at the coordinate origin $\mathbf{x}_0$ translated by $\mathbf{x}_m$, which belongs to S_2 or to the reflection of S_2, $\check{S}_2$, according to a morphological transform. The opening and closing transforms of X to S are the cascaded combination of two basic morphological transforms: dilation and erosion. Thus the BLF's of the opening transform $z_{\circ S}$ and the closing transform $z_{\bullet S}$ can be expressed as

$$\begin{aligned} z_{\circ S} &= [z_{\ominus S}]_{\oplus S} = z_{\ominus S}(\check{x}_0) + z_{\ominus S}(\check{x}_1) + \cdots + z_{\ominus S}(\check{x}_{N-1})\\ &= (x_0x_1x_2\cdots x_{N-1})|_{\check{x}_0} + (x_0x_1x_2\cdots x_{N-1})|_{\check{x}_1} + \cdots\\ &\quad + (x_0x_1x_2\cdots x_{N-1})|_{\check{x}_{N-1}},\\ z_{\bullet S} &= [z_{\oplus S}]_{\ominus S} = z_{\oplus S}(x_0)z_{\oplus S}(x_1)\cdots z_{\oplus S}(x_{N-1}) \qquad (34)\\ &= (\check{x}_0 + \check{x}_1 + \check{x}_2 + \cdots + \check{x}_{N-1})|_{x_0}(\check{x}_0 + \check{x}_1 + \check{x}_2 + \cdots\\ &\quad + \check{x}_{N-1})|_{x_1}\cdots(\check{x}_0 + \check{x}_1 + \check{x}_2 + \cdots + \check{x}_{N-1})|_{x_{N-1}}, \end{aligned}$$

where $z_{\ominus S}(\check{\mathbf{x}}_i) = (x_0x_1x_2\cdots x_{N-1})|_{\check{x}_i}$, $(i = 0, 1, 2, \ldots, N-1)$, denotes an output state of $z_{\ominus S}$ at the coordinate point $\check{\mathbf{x}}_i$ that is the AND of the neighborhood states at the coordinate point $\check{\mathbf{x}}_i$, where the neighborhood at $\check{\mathbf{x}}_i$ can be obtained by $S = \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{N-1}\}$ at $\mathbf{x}_0$ translated by $\check{\mathbf{x}}_i$. The expression $z_{\oplus S}(\mathbf{x}_i) = (\check{x}_0 + \check{x}_1 + \check{x}_2 + \cdots + \check{x}_{N-1})|_{x_i}$ denotes an output state of $z_{\oplus S}$ at the coordinate state $\mathbf{x}_i$ that is the OR of the neighborhood states at $\mathbf{x}_i$; the neighborhood at $\mathbf{x}_i$ is achieved by the $\check{S} = \{\check{\mathbf{x}}_0, \check{\mathbf{x}}_1, \check{\mathbf{x}}_2, \ldots, \check{\mathbf{x}}_{N-1}\}$ at $\mathbf{x}_0$ translated by $\mathbf{x}_i$. Based on the BLF's of the opening and closing transforms, the BLF's of the pattern spectra $z_{\mathrm{PS}(+n,B)}$ and $z_{\mathrm{PS}(-n,B)}$ can be deduced as

$$\begin{aligned} z_{\mathrm{PS}(+n,B)} &= A\{z_{\circ n\check{B}}z^c_{\bullet(n+1)\check{B}}\} = A\{[(x_0x_1\cdots x_J)|_{\check{x}_0}\\ &\quad + (x_0x_1\cdots x_J)|_{\check{x}_1} + \cdots + (x_0x_1\cdots x_J)|_{\check{x}_J}]\\ &\quad \times[(x_0^c + x_1^c + \cdots + x_M^c)|_{\check{x}_0}(x_0^c + x_1^c + \cdots + x_M^c)\\ &\quad \times|_{\check{x}_1}\cdots(x_0^c + x_1^c + \cdots + x_M^c)|_{\check{x}_M}]\} \end{aligned} \qquad (35)$$

$$\begin{aligned} z_{\mathrm{PS}(-n,B)} &= A\{z_{\bullet nB}z^c_{\circ(n-1)\check{B}}\} = A\{[(\check{x}_0 + \check{x}_1 + \cdots + \check{x}_J)\\ &\quad \times|_{x_0}(\check{x}_0 + \check{x}_1 + \cdots + \check{x}_J)|_{x_1}\cdots\\ &\quad \times(\check{x}_0 + \check{x}_1 + \cdots + \check{x}_J)|_{x_J}]\\ &\quad \times[(\check{x}_0^c\check{x}_1^c\cdots\check{x}_{MM}^c)|_{x_0} + (\check{x}_0^c\check{x}_1^c\cdots\check{x}_{MM}^c)|_{x_1}\\ &\quad + \cdots + (\check{x}_0^c\check{x}_1^c\cdots\check{x}_{MM}^c)|_{x_{MM}}]\}, \end{aligned} \qquad (36)$$

where J is defined as the number of pixels contained in the neighborhood nB, M is the number of pixels in $(n+1)B$, and MM is the number of pixels in $(n-1)B$. All BLF's can be written in the canonical sum-of-product form. The canonical sum-of-product form of a pattern-spectrum BLF with L variables can be written into

$$\begin{aligned} z_{\mathrm{PS}(\pm n,B)} &= A\{P_1x_0x_1x_2\cdots x_{L-1} + P_2x_0^cx_1x_2\cdots x_{L-1}\\ &\quad + \cdots + P_kx_0^cx_1^cx_2^c\cdots s_{L-1}^c\}, \end{aligned} \qquad (37)$$

where $P_j \in \{0, 1\}$, and, if a min term exists, $P_j = 1$, otherwise $P_j = 0$. In Eq. (37) each min term can be divided into two groups: one group including the positive variables, and the other group containing the complement variables. Then if the coordinate points corresponding to all the positive variables in a min term are denoted by S_1, $S_1 = \{\text{all positive cells}\}$, and the coordinate points corresponding to all the complement variables are denoted by S_2, $S_2 = \{\text{all complement cells}\}$, then $S_1 \cap S_2 = \varnothing$ and $S_1 \cup S_2 = S^w$, and each min term is a hit-or-miss transform by the structure-element pair S^w. Therefore the canonical sum-of-product form of a pattern spectrum is the union of hit-or-miss min terms corresponding to different structure-element pairs S^w. Comparison of the canonical sum-of-product form of a pattern-spectrum BLF with the hit-or-miss representation for the pattern spectrum from Eqs. (11) and (13) shows that the expression $S^w = [S_1, S_2]$ corresponds to the structure-element pair $[nB_i, (n+1)\check{B}^{kk}]$ or $[nB^{kl}, (n-1)\check{B}_i]$. From the BLF of the pattern spectrum, we demonstrate that a pattern spectrum can be decomposed into the union of hit-or-miss transforms by different structure-element pairs.

On the basis of the above discussion, each of the structure-element pairs in the hit-or-miss min terms for the nth order of the pattern spectrum $\mathrm{PS}(\pm n, B)$ can be calculated with the BLF's of the pattern spectrum and will be fixed for each basis structure element B. For example, Fig. 1(a) shows a basis structure element B with three points: the origin 0, the east 1 and west 2, and $B \oplus \check{B}$, in addition to $2B \oplus 2\check{B}$. Figure 1(b) denotes the basis structure element for the opening-transform decomposition corresponding to B, and Fig. 1(c) shows the basis structure element for the closing-transform decomposition with respect to B and $2\check{B}$, deduced from the canonical sum-of-product form of the closing-transform BLF, separately. Therefore the opening-transform BLF of X with B is $z_{\circ B} = x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_4$, and of X with $0\check{B}$ is $z_{\circ(0\check{B})} = x_0$. The closing-transform BLF of X with B is $z_{\bullet B} = x_0 + x_1x_4 + x_1x_2 + x_2x_3$, and of X^c with $2\check{B}$ is $z^c_{\bullet 2\check{B}} = x_0^c + x_1^cx_2^c + x_2^cx_3^c + x_2^cx_5^c + x_2^cx_7^c +$

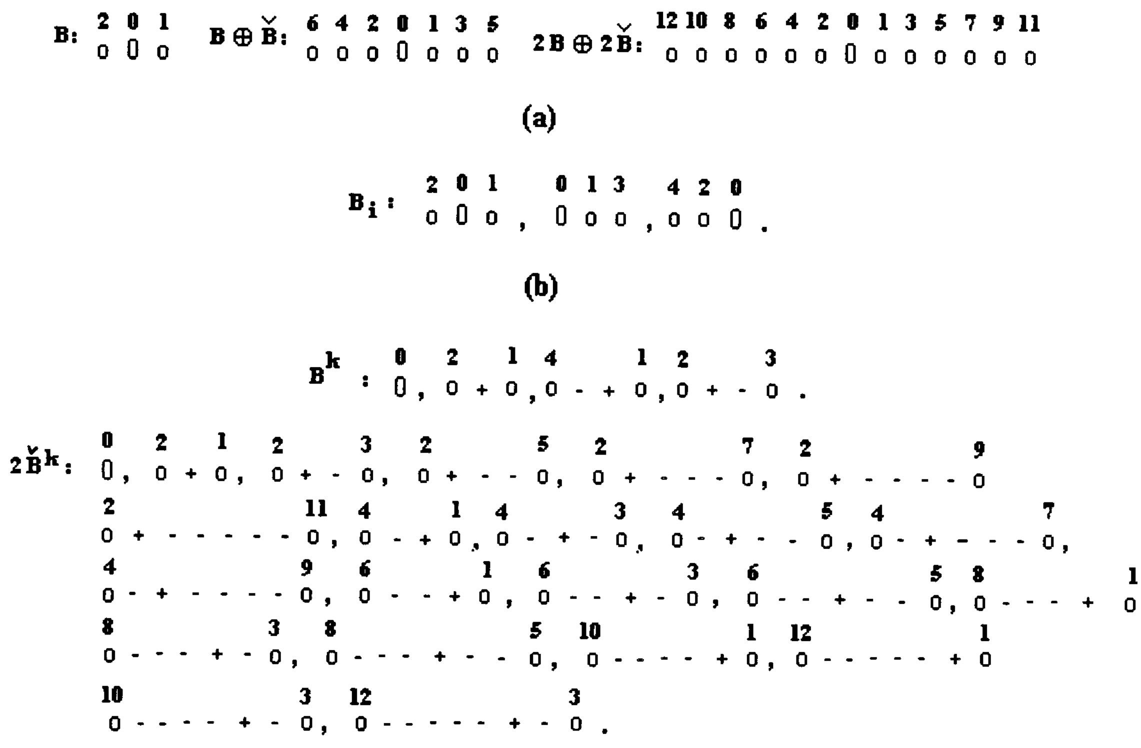

Fig. 1. (a) Basis structure element B, the dilation of B by the reflection $\check{B}$ of B, and the dilation of $2B$ by the reflection $2\check{B}$ of $2B$. (b) The basis structure elements for the opening-transform decomposition of B. (c) The minimal structure elements for the closing-transform decomposition of B and $2\check{B}$.

$x_2^c x_9^c + x_2^c x_{11}^c + x_1^c x_4^c + x_3^c x_4^c + x_4^c x_5^c + x_4^c x_7^c + x_4^c x_9^c + x_1^c x_6^c + x_3^c x_6^c + x_5^c x_6^c + x_1^c x_8^c + x_3^c x_8^c + x_5^c x_8^c + x_1^c x_{10}^c + x_1^c x_{12}^c + x_3^c x_{10}^c + x_3^c x_{12}^c$. The first-order pattern spectrum can be derived as $z_{\mathrm{PS}(1,B)} = A[x_0x_1x_2x_3^cx_4^c + x_0x_1x_2x_3^cx_6^c + x_0x_1x_2x_3^cx_8^c + x_0x_1x_2x_3^cx_{10}^c + x_0x_1x_2x_3^cx_{12}^c + x_0x_1x_2x_4^cx_5^c + x_0x_1x_2x_4^cx_7^c + x_0x_1x_2x_4^cx_9^c + x_0x_1x_2x_5^cx_6^c + x_0x_1x_2x_5^cx_8^c + x_0x_1x_3x_2^cx_5^c + x_0x_1x_3x_2^cx_9^c + x_0x_1x_3x_2^cx_{11}^c + x_0x_1x_3x_2^cx_7^c + x_0x_1x_3x_4^cx_5^c + x_0x_1x_3x_4^cx_7^c + x_0x_1x_3x_4^cx_9^c + x_0x_1x_3x_5^cx_6^c + x_0x_1x_3x_5^cx_8^c + x_0x_2x_4x_1^cx_6^c + x_0x_2x_4x_3^cx_6^c + x_0x_2x_4x_5^cx_6^c + x_0x_2x_4x_1^cx_8^c + x_0x_2x_4x_3^cx_8^c + x_0x_2x_4x_5^cx_8^c + x_0x_2x_4x_1^cx_{10}^c +$

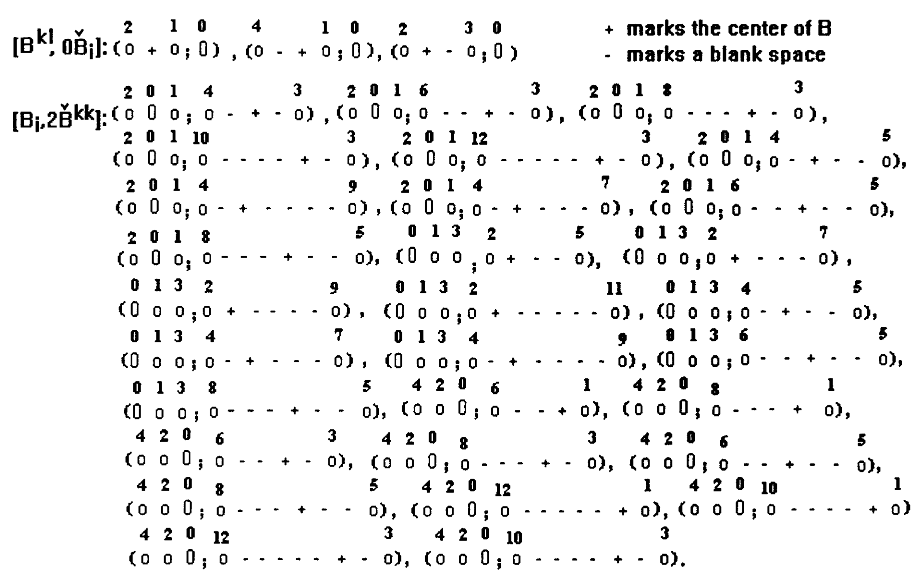

Fig. 2. Structure-element pairs $[B^{kl}, 0\check{B}_j]$ and $[B_j, 2\check{B}^{kk}]$ of hit-or-miss min terms for the decomposition of pattern spectra PS$(+1, B)$ and PS$(-1, B)$. The plus signs $(+)$ mark the center of B, and the minus signs $(-)$ mark blank spaces.

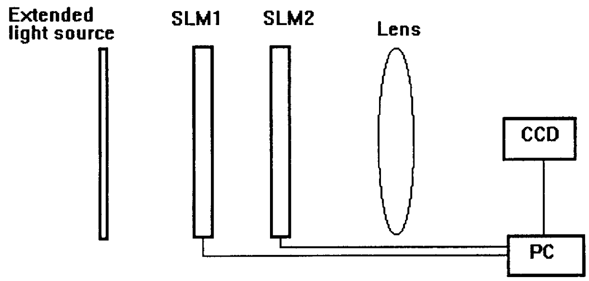

Fig. 3. Schematic diagram of the incoherent optical correlator utilized to implement the pattern spectrum.

$x_0x_2x_4x_1^cx_{12}^c + x_0x_2x_4x_3^cx_{10}^c + x_0x_2x_4x_3^cx_{12}^c]$. Similarly, $z_{\mathrm{PS}(-1,B)} = A(x_0^cx_1x_4 + x_0^cx_1x_2 + x_0^cx_2x_3)$. Thus Fig. 2 shows the structure-element pairs $[B_i, 2\check{B}^{kk}]$ and $[B^{kl}, 0\check{B}_i]$ that correspond to the hit-or-miss min terms in the $z_{\mathrm{PS}(1,B)}$ and $z_{\mathrm{PS}(-1,B)}$ pattern spectra.

Moreover through the threshold decomposition and sum superposition,[6,10] a discrete gray-tone image can be thresholded into a stack of binary images. The morphological transforms of a discrete gray-tone image from a discrete binary structure element commute with thresholding, and then the pattern spectrum of a discrete gray-tone image with a discrete binary structure element can be turned into the sum superposition of the pattern spectrum that corresponds to each of the threshold binary images with the discrete binary structure element. The pattern spectrum of a threshold binary image can be represented by the BLF shown in Eqs. (35) and (36), which can be written as the union of hit-or-miss min terms. Thus the pattern spectrum of a discrete gray-tone image with a discrete binary image can also be decomposed into the union of hit-or-miss transforms from the BLF point of view.

5. Optical Implementation of a Pattern Spectrum

From the definition of the pattern spectrum we see that the main operations in pattern-spectrum analysis are the opening and closing transforms that are the cascaded combination of dilation and erosion. Dilation and erosion can be implemented by an incoherent optical correlator.[13] Thus we utilize an incoherent optical correlator with a feedback operation to execute a pattern spectrum. Figure 3 shows a schematic diagram of the incoherent optical correlator used. The optical implementation of a pattern spectrum is outlined below: Spatial light modulator 1 (SLM1) shows a binary image, SLM2 shows a structure element, a lens implements the incoherent optical correlation, a CCD camera detects the correlation results, and then a PC, acting as a threshold and latch gate, thresholds the correlation results to obtain the dilation or erosion. The dilation or erosion pattern is next fed back to SLM1, the CCD camera detects the new correlation results, and the PC thresholds the correlation pattern, thus obtaining the opening and closing transforms. By changing the structure elements in the pattern spectrum in SLM2 sequentially, we can get the pattern spectrum of a discrete binary image. Figure 4(a) demonstrates an image X, whereas Figs. 4(b)–4(d) show the opening transform of X by B that is shown in Fig. 1(a), the

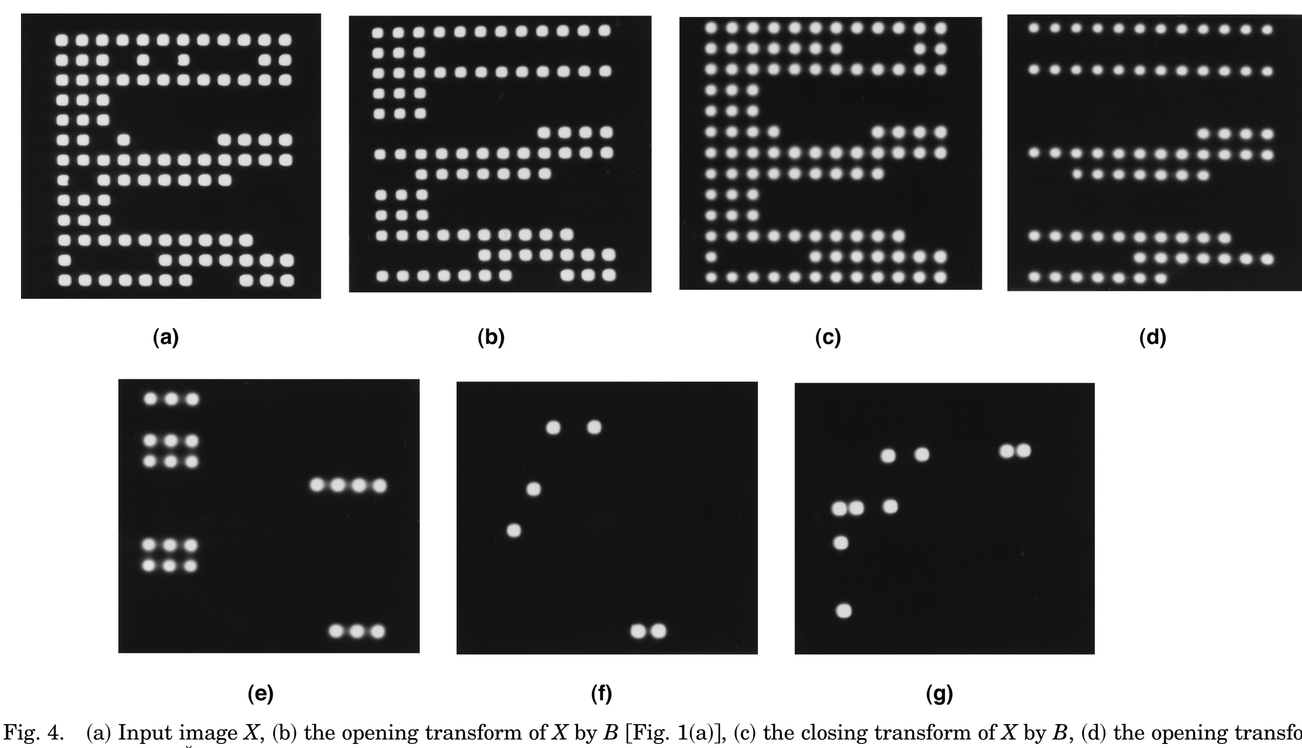

Fig. 4. (a) Input image X, (b) the opening transform of X by B [Fig. 1(a)], (c) the closing transform of X by B, (d) the opening transform of X by $2B = B \oplus \check{B}$, (e) the patterns related to pattern spectrum PS(1, B), (f) the patterns related to pattern spectrum PS(0, B), and (g) patterns related to pattern spectrum PS(-1, B).

closing transform of X by B [Fig. 1(b)], and the opening transform of X by $2B$ [Fig. 1(c)]. Figures 4(e)–4(f) correspond to the patterns of PS(1, B), PS(0, B), and PS(-1, B), respectively.

Considering an optical implementation of the pattern spectrum for a gray-tone image, we first threshold the gray-tone image into a stack of binary images[14] and then add all the pattern spectra related to each binary image to obtain the gray-tone image's pattern spectrum.

6. Conclusion

We have demonstrated that the pattern spectra of a discrete binary and a gray-tone image with a discrete structure element can be decomposed into hit-or-miss representation. First we use the basis decomposition of the opening and closing transforms to demonstrate that pattern spectra can be represented by the union of hit-or-miss transforms. We then give a concrete expression for the BLF's of the morphological opening and closing transforms; moreover we demonstrate the Boolean-logic representation for a pattern spectrum, which can be written in the canonical sum-of-product form, and each min term is shown to be a Boolean-logic representation of the hit-or-miss transform. Therefore, from the point of view of the BLF, we show that pattern spectra can be decomposed into the union of hit-or-miss transforms. We also describe an optical implementation architecture for pattern spectra that is based on an incoherent optical correlator with a feedback operation. Some proof-of-principle experimental results are also given.

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