# Hellinger volume and number-on-the-forehead communication complexity 

Troy Lee* ${ }^{*}$ Nikos Leonardos ${ }^{\dagger} \quad$ Michael Saks ${ }^{\ddagger} \quad$ Fengming Wang§

February 1, 2017


#### Abstract

Information-theoretic methods are a powerful tool in communication complexity, providing, in particular, elegant and tight lower bounds on disjointness in the number-in-the-hand (NIH) model. In this paper, we study the applicability of information theoretic methods to the multi-party number-on-the-forehead model (NOF). The lower bound on disjointness in the NIH model has two parts: a direct sum theorem and a lower bound on the one-bit AND function using a beautiful connection between Hellinger distance and protocols [BYJKS04]. Inspired by this connection, we introduce the notion of Hellinger volume which lower bounds the information cost of multi-party NOF protocols. We provide tools for manipulating and proving lower bounds on Hellinger volume, and use these tools to obtain a lower bound on the information complexity of the $\mathrm{AND}_{k}$ function in the NOF setting. Finally, we discuss the difficulties of proving a direct sum theorem for information cost in the NOF model.


Keywords: communication complexity, informational complexity, Hellinger volume, number-on-the-forehead.

## 1 Introduction

One of the most important research areas in communication complexity is proving lower bounds in the multi-party number-on-the-forehead (NOF) model. The NOF model was introduced in [CFL83], where it was used to prove lower bounds for branching programs. Subsequent papers revealed connections of this model to circuit complexity [BT94, HG90, Nis94, NW91] and proof complexity [BPS05]. In particular, an explicit function which requires super-polylogarithmic complexity in the NOF model with polylogarithmically many players would give an explicit function outside of the circuit complexity class $\mathrm{ACC}^{0}$.

[^0]Essentially all lower bounds on the general NOF model have been shown using the discrepancy method following [BNS92]. This method has been able to show lower bounds of $\Omega\left(n / 2^{k}\right)$ for explicit functions [BNS92, CT93, Raz00, FG05]. For the disjointness function, the plain discrepancy method shows poor bounds and a more sophisticated application of discrepancy is needed known as the generalized discrepancy method [Kla07, Raz03, LS07, She11]. The generalized discrepancy method was initially used to show lower bounds of the form $n^{1 / k} / 2^{2^{k}}$ [LS09, CA08] and $2^{\Omega(\sqrt{\log n} / \sqrt{k})-k}$ [BHN09] on the $k$-player NOF complexity of disjointness. Recent work of Sherstov in [She12] and [She13] improved the lower bounds to $\Omega\left(\left(n / 4^{k}\right)^{1 / 4}\right)$ and $\Omega\left(\sqrt{n} /\left(2^{k} k\right)\right)$, respectively. A very recent paper of Rao and Yehudayoff [RY14] gives a simplified proof of the latter lower bound and also gives a nearly tight $\Omega\left(n / 4^{k}\right)$ lower bound for deterministic protocols. An upper bound of $O\left(\log ^{2}(n)+k^{2} n / 2^{k}\right)$ for the disjointness function follows from a beautiful protocol of Grolmusz [Gro94].

In this paper we are interested in how information-theoretic methods might be applied to the NOF model. Information-theoretic methods have been very successful in the number-in-the-hand (NIH) multi-party model, in particular giving tight lower bounds on the disjointness function. The first use of information theory in communication complexity lower bounds can be traced to [Abl96]. In [CSWY01] the notions of information cost and informational complexity were defined explicitly. Building on their work, a very elegant information-theoretic framework for proving lower bounds in NIH communication complexity was established in [BYJKS04].

In [BYJKS04] a proof of the linear lower bound for two-party disjointness was given. The proof has two main stages. In the first stage, a direct-sum theorem for informational complexity is shown, which says that the informational complexity of disjointness, $\operatorname{DISJ}_{n, 2}(x, y)=\bigvee_{j=1}^{n} \operatorname{AND}_{2}\left(x_{j}, y_{j}\right)$, is lower bounded by $n$ times the informational complexity of the binary $\mathrm{AND}_{2}$ function. Although it is not known how to prove such a direct-sum theorem directly for the classical randomized complexity, Bar-Yossef et al. prove it for the informational complexity with respect to a suitable distribution. A crucial property of the distribution is that it is over the zeroes of disjointness. At this point we should point out a remarkable characteristic of the method: even though the information cost of a protocol is analyzed with respect to a distribution over zeroes only, the protocol is required to be correct over all inputs. This requirement is essential in the second stage, where a constant lower bound is proved on the informational complexity of $\mathrm{AND}_{2}$. This is achieved using properties of the Hellinger distance for distributions. Bar-Yossef et al. reveal a beautiful connection between Hellinger distance and NIH communication protocols. (More properties of Hellinger distance relative to the NIH model have been established in [Jay09].)

In this work we provide tools for accomplishing the second stage in the NOF model. We introduce the notion of Hellinger volume of $m \geq 2$ distributions and show that it can be useful for proving lower bounds on informational complexity in the NOF model, just as Hellinger distance is useful in the NIH model. However, as we point out in the last section, there are fundamental difficulties in proving a direct-sum theorem for informational complexity in the NOF model. Nevertheless, we believe that Hellinger volume and the related tools we prove, could be useful in an information-theoretic attack on NOF complexity.

Since this paper was submitted in 2011 there has been some overlapping independent work by Beame, Hopkins, Hrubeš and Rashtchian [BHHR14], including lower bounds for the information complexity of the AND function similar to those we give in Section 5 but for restricted settings, and 0 -information protocols in the "random-
ness on the forehead" model, of the type we give in Section 6 but in a more general setting.

## 2 Preliminaries and notation

Hellinger volume To transfer the information-theory framework of [BYJKS04] in the NOF model we seek a generalization of Hellinger distance that carries the same useful properties: it lower-bounds mutual information, allows one to prove lower bounds for simple functions, and has properties such as the cut-and-paste property and triangle inequality (see [BYJKS04] for these properties) of Hellinger distance that allow one to manipulate several terms of this quantity in meaningful ways with respect to the NOF model. In particular, recall the following expression for (the square of) Hellinger distance.

$$
h^{2}(p, q)=1-\sum_{\omega} \sqrt{p(\omega) q(\omega)} .
$$

It was the product structure in this expression which allowed for the cut-and-paste property. Inspired by this observation we introduce the notion of Hellinger volume of $m$ distributions. In the next section we show that it has properties similar in flavor to the ones of Hellinger distance.

Definition 1. The m-dimensional Hellinger volume of distributions $p_{1}, \ldots, p_{m}$ over $\Omega$ is

$$
h_{m}\left(p_{1}, \ldots, p_{m}\right)=1-\sum_{\omega \in \Omega} \sqrt[m]{p_{1}(\omega) \cdots p_{m}(\omega)} .
$$

Notice that $h_{2}\left(p_{1}, p_{2}\right)$ in the case $m=2$ is the square of the Hellinger distance between distributions $p_{1}$ and $p_{2}$.

The following fact follows from the arithmetic mean-geometric mean (AM-GM) inequality.

Fact 1. For any distributions $p_{1}, \ldots, p_{m}$ over $\Omega, h_{m}\left(p_{1}, \ldots, p_{m}\right) \geq 0$.
Random variables and distributions We consider discrete probability spaces $(\Omega, \zeta)$, where $\Omega$ is a finite set and $\zeta$ is a nonnegative valued function on $\Omega$ summing to 1 . If $\left(\Omega_{1}, \zeta_{1}\right), \ldots,\left(\Omega_{n}, \zeta_{n}\right)$ are such spaces, their product is the space $(\Lambda, \nu)$, where $\Lambda=\Omega_{1} \times \cdots \times \Omega_{n}$ is the Cartesian product of sets, and for $\omega=\left(\omega_{1}, \ldots, \omega_{n}\right) \in \Lambda$, $\nu(\omega)=\prod_{j=1}^{n} \zeta_{j}\left(\omega_{j}\right)$. In the case that all of the $\left(\Omega_{i}, \zeta_{i}\right)$ are equal to a common space $(\Omega, \zeta)$ we write $\Lambda=\Omega^{n}$ and $\nu=\zeta^{n}$.

We use uppercase for random variables, as in $\mathbf{Z}, D$, and write in bold those that represent vectors of random variables. For a variable $X$ with range $\mathcal{X}$ that is distributed according to a probability distribution $\mu$, i.e. $\operatorname{Pr}[X=x]=\mu(x)$, we write $X \sim \mu$. If $X$ is uniformly distributed in $\mathcal{X}$, we write $X \in_{R} \mathcal{X}$.

Information theory Let $X, Y, Z$ be random variables on a common probability space, taking on values, respectively, from finite sets $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$. Let $A$ be any event. The entropy of $X$, the conditional entropy of $X$ given $A$, and the conditional entropy
of $X$ given $Y$ are respectively (we use $\log$ for $\log _{2}$ )

$$
\begin{aligned}
H(X) & =-\sum_{x \in \mathcal{X}} \operatorname{Pr}[X=x] \cdot \log \operatorname{Pr}[X=x], \\
H(X \mid A) & =-\sum_{x \in \mathcal{X}} \operatorname{Pr}[X=x \mid A] \cdot \log \operatorname{Pr}[X=x \mid A], \\
H(X \mid Y) & =\sum_{y \in \mathcal{X}} \operatorname{Pr}[Y=y] \cdot H(X \mid Y=y) .
\end{aligned}
$$

We will need the following facts about the entropy. (See [CT06, Chapter 2], for proofs and more details.)

Proposition 2. Let $X, Y, Z$ be random variables.

1. $H(X) \geq H(X \mid Y) \geq 0$.
2. If $\mathcal{X}$ is the range of $X$, then $H(X) \leq \log |\mathcal{X}|$.
3. $H(X, Y) \leq H(X)+H(Y)$ with equality if and only if $X$ and $Y$ are independent. This holds for conditional entropy as well. $H(X, Y \mid Z) \leq H(X \mid Z)+H(Y \mid Z)$ with equality if and only if $X$ and $Y$ are independent given $Z$.

The relative entropy or divergence of distributions $P$ and $Q$ over $\Omega$ is

$$
D(P \| Q)=\sum_{x \in \Omega} P(x) \log \frac{P(x)}{Q(x)}
$$

The mutual information between $X$ and $Y$ is

$$
I(X ; Y)=H(X)-H(X \mid Y)=H(Y)-H(Y \mid X)
$$

Notation We write $[n]=\{1,2, \ldots, n\}$. For a sequence $\left(a_{1}, \ldots, a_{n}\right)$ we let, for $j \in[n], a_{<j}=\left(a_{1}, \ldots, a_{j-1}\right)$, and $a^{-j}=\left(a_{1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{n}\right)$. We will denote subsets of $\{0,1\}^{k}$ as follows: $I=\{0,1\}^{k}$; for $j \in[k], I_{j}$ is the set of points in $I$ such that the $j$-th coordinate is set to zero, i.e. $I_{j}=\left\{z \in I \mid z_{j}=0\right\} ; I_{O Z}$ (resp. $I_{E Z}$ ) is the set of points in $I$ with an odd (resp. even) number of zeros.

Communication complexity In this work we will be dealing with the multi-party private-coin randomized number-on-the-forehead communication model, introduced by [CFL83]. There are $k$ players, numbered $1, \ldots, k$, trying to compute a function $f: \mathcal{Z} \rightarrow\{0,1\}$, where $\mathcal{Z}=\mathcal{Z}_{1} \times \cdots \times \mathcal{Z}_{k}$. On input $z \in \mathcal{Z}$, player $j$ receives input $z_{j}$ (conceptually, placed on his forehead), but he has access only to $z^{-j}$. They wish to determine $f(z)$, by broadcasting messages according to a protocol $\Pi$. Let the random variable $\Pi(z)$ denote the transcript of the communication on input $z$ (where the probability is over the random coins of the players) and $\Pi_{\mathrm{out}}(z)$ the outcome of the protocol. We call $\Pi$ a $\delta$-error protocol for $f$ if, for all $z, \operatorname{Pr}\left[\Pi_{\mathrm{out}}(z)=f(z)\right] \geq 1-\delta$. The communication cost of $\Pi$ is $\max |\Pi(z)|$, where the maximum is over all inputs $z$ and over all coin tosses of the players. The $\delta$-error randomized communication complexity of $f$, denoted $R_{\delta}(f)$, is the cost of the best $\delta$-error protocol for $f$. (See [KN06] for more details.)

Communication complexity lower bounds via information theory The informational complexity paradigm, introduced by [CSWY01], and used in [SS02, BYJKS02, CKS03, BYJKS04, JKS03], provides a way to prove lower bounds on communication complexity via information theory. We are given a $k$-party function $f$ and we want to show that any $\delta$-error randomized NOF protocol $\Pi$ for $f$ requires high communication. We introduce a probability distribution over the inputs to the players. We then analyze the behavior of $\Pi$ when run on inputs chosen randomly according to the distribution. The informational complexity is the mutual information of the string of communicated bits (the transcript of $\Pi$ ) with the inputs, and provides a lower bound on the amount of communication.

More precisely, let $\Omega=(\Omega, \zeta)$ be a probability space over which are defined random variables $\mathbf{Z}=\left(Z_{1}, \ldots, Z_{k}\right)$ representing the inputs. The information cost of a protocol $\Pi$ with respect to $\zeta$ is defined to be $I(\mathbf{Z} ; \Pi(\mathbf{Z}))$, where $\Pi(\mathbf{Z})$ is a random variable following the distribution of the communication transcripts when the protocol $\Pi$ runs on input $\mathbf{Z} \sim \zeta$. The $\delta$-error informational complexity of $f$ with respect to $\zeta$, denoted $\mathrm{IC}_{\zeta, \delta}(f)$, is $\min _{\Pi} I(\mathbf{Z} ; \Pi(\mathbf{Z}))$, where the minimum is over all $\delta$-error randomized NOF protocols for $f$. The relevance of informational complexity comes from the following proposition.

Proposition 3. $R_{\delta}(f) \geq \mathrm{IC}_{\zeta, \delta}(f)$.
Proof. For any protocol $\Pi, \mathrm{IC}_{\zeta, \delta}(f) \leq I(\mathbf{X}, \mathbf{Y} ; \Pi(\mathbf{X}, \mathbf{Y}))=H(\Pi(\mathbf{X}, \mathbf{Y}))-H(\Pi(\mathbf{X}, \mathbf{Y}) \mid \mathbf{X}, \mathbf{Y})$. Applying in turn parts (1) and (2) of Proposition 2 gives $\mathrm{IC}_{\zeta, \delta}(f) \leq H(\Pi(\mathbf{X}, \mathbf{Y})) \leq$ $R_{\delta}(f)$.

For a collection of distributions $\eta=\left\{\zeta_{1}, \ldots, \zeta_{k}\right\}$, we define the $\delta$-error informational complexity of $f$ with respect to $\eta$, denoted $\mathrm{IC}_{\eta, \delta}(f)$, to be $\mathbf{E}_{j}\left[\mathrm{IC}_{\zeta_{j}, \delta}(f)\right]$, where $j$ is uniformly distributed over $[k]$.

Remark. This definition of information cost as an average, is equivalent to the (standard) conditional information cost. We choose this definition, because we think it makes the exposition cleaner.

## 3 An upper bound on the difference between the arithmetic and geometric mean.

For a nonnegative real sequence $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right)$, let $A(\alpha)$ and $G(\alpha)$ denote its arithmetic and geometric mean respectively. That is

$$
A(\alpha)=\frac{1}{m} \sum \alpha_{j} \quad \text { and } \quad G(\alpha)=\sqrt[m]{\prod \alpha_{j}}
$$

Theorem 1. For any distribution $p$ over $[m]$,

$$
A(p)-G(p) \leq \ln 2 \cdot D(p \| u),
$$

where $u$ is the uniform distribution over $[m]$.
Proof. Let $x_{j}=m p(j), x=\left(x_{1}, \ldots, x_{n}\right)$, and define

$$
f(x)=\sum x_{j} \ln x_{j}+\sqrt[m]{\prod x_{j}} .
$$

Theorem 1 is equivalent to showing that, for $x_{1}, \ldots, x_{n} \geq 0$, if $\sum x_{j}=m$, then $f(x) \geq 1$.

We proceed using Lagrange multipliers. We first need to check that $f(x) \geq 1$ when $x$ is on the boundary, i.e. $x_{j}=0$ for some $j \in[n]$. Without loss of generality, assume $x_{1}=0$. By the convexity of $t \ln t$, the minimum is attained when $x_{2}=\cdots=$ $x_{m}=m /(m-1)$. Thus,

$$
f(x) \geq(m-1) \frac{m}{m-1} \ln \frac{m}{m-1}>m\left(1-\frac{m-1}{m}\right)=1 .
$$

According to [Lue03, Theorem on page 300], it suffices to show that $f(x) \geq 1$ for any $x$ that satisfies the following system of equations.

$$
\begin{equation*}
\partial f / \partial x_{j}=1+\ln x_{j}+\sigma /\left(m x_{j}\right)=\lambda, \quad \text { for } j \in[m], \tag{L}
\end{equation*}
$$

where $\sigma=\sqrt[m]{x_{1} \cdots x_{m}} \neq 0$. Without loss of generality, since $\sum x_{j}=m$, we may assume $x_{m} \leq 1$. The system ( $L$ ) implies

$$
\begin{aligned}
& \sum_{j=1}^{m-1} x_{j}\left(\partial f / \partial x_{j}\right)=m-x_{m}+\sum_{j=1}^{m-1} x_{j} \ln x_{j}+\sigma(m-1) / m=\lambda\left(m-x_{m}\right), \\
& (m-1) x_{m}\left(\partial f / \partial x_{m}\right)=(m-1)\left(x_{m}+x_{m} \ln x_{m}+\sigma / m\right)=(m-1) \lambda x_{m} .
\end{aligned}
$$

Subtracting the second from the first we get

$$
\sum_{j=1}^{m-1} x_{j} \ln x_{j}-(m-1) x_{m} \ln x_{m}=m(\lambda-1)\left(1-x_{m}\right)
$$

We also have

$$
\sum x_{j}\left(\partial f / \partial x_{j}\right)=m+f(x)=m \lambda .
$$

Suppose $x=\left(x_{1}, \ldots, x_{m}\right)$ satisfies the system $(L)$. Since $x_{m} \leq 1$, we have $x_{m} \ln x_{m} \leq$ 0 , and using the last two equations we have

$$
f(x)=m(\lambda-1) \geq \frac{\sum_{j=1}^{m-1} x_{j} \ln x_{j}}{1-x_{m}} \geq \frac{\sum_{j=1}^{m-1} x_{j}\left(1-1 / x_{j}\right)}{1-x_{m}}=1 .
$$

This completes the proof.
Remark. The proof actually shows that the inequality of the theorem holds when the right hand side is divided by $m \ln \frac{m}{m-1}$, which is strictly bigger than 1 for any fixed $m>1$.
Corollary 2. For any nonnegative real sequence $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right)$,

$$
A(\alpha)-G(\alpha) \leq \sum \alpha_{j} \ln \frac{\alpha_{j}}{A(\alpha)}
$$

Proof. Apply Theorem 1 with $p(j)=\alpha_{j} / \sum_{j} \alpha_{j}$.
Remark. Let $\widehat{\alpha}$ to be a normalized version of $\alpha$, with $\widehat{\alpha}_{j}=\alpha_{j} / \sum \alpha_{j}$. Let also $u$ denote the uniform distribution on $[m]$. Then, the right-hand side takes the form $\sum \alpha_{j} \ln \left(m \widehat{\alpha}_{j}\right)=m A(\alpha) \sum \widehat{\alpha}_{j} \ln \left(\widehat{\alpha}_{j} / u_{j}\right)$, and the above inequality becomes

$$
\frac{A(\alpha)-G(\alpha)}{A(\alpha)} \leq m \ln 2 \cdot D(\widehat{\alpha} \| u) .
$$

## 4 Properties of Hellinger volume

Hellinger volume lower bounds mutual information The next lemma shows that Hellinger volume can be used to lower bound mutual information.

Lemma 3. Consider random variables $Z \in_{R}[m], \Phi(Z) \in \Omega$, and distributions $\Phi_{z}$, for $z \in[m]$, over $\Omega$. Suppose that given $Z=z$, the distribution of $\Phi(Z)$ is $\Phi_{z}$. Then

$$
I(Z ; \Phi(Z)) \geq \frac{h_{m}\left(\Phi_{1}, \ldots, \Phi_{m}\right)}{m \ln 2}
$$

Proof. The left-hand side can be expressed as follows (see [CT06, page 20]),

$$
\begin{aligned}
I(Z ; \Phi(Z)) & =\sum_{j, \omega} \operatorname{Pr}[Z=j] \cdot \operatorname{Pr}[\Phi(Z)=\omega \mid Z=j] \cdot \log \frac{\operatorname{Pr}[\Phi(Z)=\omega \mid Z=j]}{\operatorname{Pr}[\Phi(Z)=\omega]} \\
& =\sum_{j, \omega} \frac{1}{m} \Phi_{j}(\omega) \log \frac{\Phi_{j}(\omega)}{\frac{1}{m} \sum_{j} \Phi_{j}(\omega)},
\end{aligned}
$$

and the right-hand side

$$
h_{m}\left(\Phi_{1}, \ldots, \Phi_{m}\right)=\sum_{\omega}\left(\frac{1}{m} \sum_{j} \Phi_{j}(\omega)-\left(\prod_{j} \Phi_{j}(\omega)\right)^{\frac{1}{m}}\right) .
$$

It suffices to show that for each $\omega \in \Omega$,

$$
\sum_{j} \frac{1}{m} \Phi_{j}(\omega) \log \frac{\Phi_{j}(\omega)}{\frac{1}{m} \sum_{j} \Phi_{j}(\omega)} \geq \frac{1}{m \ln 2}\left(\frac{1}{m} \sum_{j} \Phi_{j}(\omega)-\left(\prod_{j} \Phi_{j}(\omega)\right)^{\frac{1}{m}}\right) .
$$

Let $s=\sum_{j} \Phi_{j}(\omega)$, and $\rho(j)=\Phi_{j}(\omega) / s$, for $j \in[m]$; thus, for all $j, \rho(j) \in[0,1]$, and $\sum_{j} \rho(j)=1$. Under this renaming of variables, the left-hand side becomes $\ln 2 \cdot \frac{s}{m} \sum_{j} \rho(j) \log (m \rho(j))$ and the right one $\frac{s}{m} \cdot\left(\frac{1}{m}-\sqrt[m]{\prod \rho(j)}\right)$. Thus, we need to show

$$
\ln 2 \cdot \sum_{j} \rho(j) \log (m \rho(j)) \geq \frac{1}{m}-\left(\prod_{j} \rho(j)\right)^{\frac{1}{m}}
$$

Observe that the left-hand side is $\ln 2 \cdot D(\rho \| u)$, and the inequality holds by Theorem 1 .

Symmetric-difference lemma Let $P=\left\{P_{z}\right\}_{z \in Z}$, where $Z$ is an arbitrary set of indices, be a collection of distributions over a common space $\Omega$. For $A \subseteq Z$, the Hellinger volume of $A$ with respect to $P$, denoted by $\psi(A ; P)$, is

$$
\psi(A ; P)=1-\sum_{\omega \in \Omega}\left(\prod_{z \in A} P_{z}(\omega)\right)^{1 /|A|}
$$

The collection $P$ will be understood from the context and we'll say that the Hellinger volume of $A$ is $\psi(A)$. Note that, from Fact $1, \psi(A ; P) \geq 0$.

The following lemma can be seen as an analog to the weak triangle inequality that is satisfied by the square of the Hellinger distance.
Lemma 4 (Symmetric-difference lemma). If $A, B$ satisfy $|A|=|B|=|A \Delta B|$, where $A \Delta B=(A \backslash B) \cup(B \backslash A)$. Then

$$
\psi(A)+\psi(B) \geq \frac{1}{2} \cdot \psi(A \Delta B) .
$$

Proof. By our hypothesis, it follows that $A \backslash B, B \backslash A$ and $A \cap B$ all have size $|A| / 2$. Define $u, v, w$ to be the vectors in $\mathbb{R}^{\Omega}$ defined by

$$
\begin{aligned}
u(\omega) & =\left(\prod_{z \in A \backslash B} P_{z}(\omega)\right)^{1 /|A|}, \\
v(\omega) & =\left(\prod_{z \in B \backslash A} P_{z}(\omega)\right)^{1 /|A|}, \\
w(\omega) & =\left(\prod_{z \in A \cap B} P_{z}(\omega)\right)^{1 /|A|} .
\end{aligned}
$$

By the definition of Hellinger volume,

$$
\begin{aligned}
\psi(A) & =1-u \cdot w, \\
\psi(B) & =1-v \cdot w, \\
\psi(A \Delta B) & =1-u \cdot v .
\end{aligned}
$$

Thus the desired inequality is

$$
2-(u+v) \cdot w \geq(1-u \cdot v) / 2
$$

which is equivalent to

$$
\begin{equation*}
3+u \cdot v \geq 2(u+v) \cdot w \tag{1}
\end{equation*}
$$

Since

$$
\begin{aligned}
\psi(A \backslash B) & =1-u \cdot u \\
\psi(B \backslash A) & =1-v \cdot v \\
\psi(A \cap B) & =1-w \cdot w
\end{aligned}
$$

it follows that $\|u\|,\|v\|$ and $\|w\|$ are all at most 1 . Thus $2(u+v) \cdot w \leq 2\|u+v\|$, and so (1) follows from

$$
3+u \cdot v \geq 2\|u+v\| .
$$

Squaring both sides, it suffices to show

$$
9+6 u \cdot v+(u \cdot v)^{2} \geq 4\left(\|u\|^{2}+\|v\|^{2}+2 u \cdot v\right)
$$

Using the fact that $\|u\| \leq 1$ and $\|v\| \leq 1$ this reduces to

$$
(1-u \cdot v)^{2} \geq 0
$$

which holds for all $u, v$.
Let $s_{\ell}, s_{r}$ be two disjoint subsets of $[k]$. Let $I_{s_{\ell}} \subseteq I$ (resp., $I_{s_{r}}$ ) be the set of strings with odd number of zeros in the coordinates indexed by $s_{\ell}$ (resp., $s_{r}$ ). Let $s_{p}=s_{\ell} \cup s_{r}$ and $I_{s_{p}}=I_{s_{\ell}} \Delta I_{s_{r}}$. It is not hard to see that $I_{s_{p}}$ is the set of strings with odd number of zeros in the coordinates indexed by $s_{p}$. By the symmetric-difference lemma,

$$
\begin{equation*}
\psi\left(I_{s_{\ell}}\right)+\psi\left(I_{s_{r}}\right) \geq \frac{\psi\left(I_{s_{p}}\right)}{2} \tag{2}
\end{equation*}
$$

Note that the above expression (and similarly others that follow) makes sense for any distributions indexed by subsets of $\{0,1\}^{k}$.

For each $j \in[k]$, let $I_{j} \subseteq I$ be the set of strings where the $j$-th coordinate is set to zero. Applying the above observation inductively, we can obtain the following lemma.

Lemma 5. Let $s \subseteq[k]$ be an arbitrary non-empty set and let $I_{s} \subseteq I$ be the set of strings with odd number of zeros in the coordinates indexed by s. Then,

$$
\sum_{j \in s} \psi\left(I_{j}\right) \geq \frac{\psi\left(I_{s}\right)}{2^{\lceil\log |s|\rceil}} .
$$

Proof. We prove the claim via induction on the size of $s$. If $s$ is a singleton set, it trivially holds. Otherwise, assume that for any subset of $[k]$ of size less than $|s|$, the claim is true.

Partition $s$ into two non-empty subsets $s_{\ell}, s_{r}$ with the property that $\left|s_{\ell}\right|=\lceil|s| / 2\rceil$ and $\left|s_{\ell}\right|=\lfloor|s| / 2\rfloor$. Then $\lceil\log |s|\rceil=1+\max \left\{\left\lceil\log \left|s_{\ell}\right|\right\rceil,\left\lceil\log \left|s_{r}\right|\right\rceil\right\}$. By the inductive hypothesis,

$$
\sum_{j \in s_{\ell}} \psi\left(I_{s_{\ell}}\right) \geq \frac{\psi\left(I_{s_{\ell}}\right)}{2^{\left\lceil\log \left|s_{\ell}\right|\right\rceil} \quad \text { and } \quad \sum_{j \in s_{r}} \psi\left(I_{s_{r}}\right) \geq \frac{\psi\left(I_{s_{r}}\right)}{2^{\left\lceil\log \left|s_{r}\right|\right\rceil}} . . . \text {. } . \text {. } . ~}
$$

Thus,

$$
\begin{array}{rlr}
\sum_{j \in s} \psi\left(I_{s_{\ell}}\right) & =\sum_{j \in s_{\ell}} \psi\left(I_{s_{\ell}}\right)+\sum_{j \in s_{r}} \psi\left(I_{s_{r}}\right) & \\
& \geq \frac{\psi\left(I_{s_{\ell}}\right)}{2^{\left[\log \left|s_{\ell}\right|\right.}}+\frac{\psi\left(I_{s_{r}}\right)}{2^{\left[\log \left|s_{r}\right|\right\rceil}} & \text { by the Inductive Hypothesis, } \\
& \geq \frac{1}{2^{[\log |s|\rceil-1}}\left[\psi\left(I_{s_{\ell}}\right)+\psi\left(I_{s_{r}}\right)\right] & \text { by the choice of } s_{\ell} \text { and } s_{r}, \\
& \geq \frac{1}{2^{[\log |s|\rceil} \psi\left(I_{s}\right)} \quad \text { by Equation (2). }
\end{array}
$$

Let $I_{O Z} \subseteq I$ be the set of strings which have odd number of zeros. The next corollary is an immediate consequence of Lemma 5 when $s=[k]$. In our application it will play a role analogous to that of triangle inequality for the square of Hellinger distance in the two-party setting.

## Lemma 6.

$$
\sum_{j=1}^{k} \psi\left(I_{j}\right) \geq \frac{\psi\left(I_{O Z}\right)}{2^{\lceil\log k\rceil}} .
$$

NOF communication complexity and Hellinger volume It was shown in [BYJKS04] that the distribution of transcripts of a two-party protocol on a fixed input is a product distribution. The same is true for a multi-party NOF protocol.

Lemma 7. Let $\Pi$ be a $k$-player NOF communication protocol with input set $\mathcal{Z}=$ $\mathcal{Z}_{1} \times \cdots \times \mathcal{Z}_{k}$ and let $\Omega$ be the set of possible transcripts. For each $j \in[k]$, there is a mapping $q_{j}: \Omega \times \mathcal{Z}^{-j} \rightarrow \mathbb{R}$, such that for every $z=\left(z_{1}, \ldots, z_{k}\right) \in \mathcal{Z}$ and $\omega \in \Omega$,

$$
\operatorname{Pr}[\Pi(z)=\omega]=\prod_{j=1}^{k} q_{j}\left(\omega ; z^{-j}\right)
$$

Proof. Suppose $|\Pi(z)| \leq l$. For $i=1, \ldots, l$, let $\Pi_{i}(z)$ denote the $i$-th bit sent in an execution of the protocol. Let $\sigma_{i} \in[k]$ denote the player that sent the $i$-th bit. Then

$$
\begin{aligned}
\operatorname{Pr}[\Pi(z)=\omega] & =\operatorname{Pr}\left[\Pi_{1}(z)=\omega_{1}, \ldots, \Pi_{\ell}(z)=\omega_{\ell}\right] \\
& =\prod_{i=1}^{l} \operatorname{Pr}\left[\Pi_{i}(z)=\omega_{i} \mid \Pi_{<i}(z)=\omega_{<i}\right], \\
& =\prod_{i=1}^{l} \operatorname{Pr}\left[\Pi_{i}\left(z^{-\sigma_{i}} ; \omega_{<i}\right)=\omega_{i}\right],
\end{aligned}
$$

because every bit send by player $j$ depends only on $z^{-j}$ and the transcript up to that point. We set

$$
q_{j}\left(\omega ; z^{-j}\right)=\prod_{i: \sigma_{i}=j} \operatorname{Pr}\left[\Pi_{i}\left(z^{-j} ; \omega_{<i}\right)=\omega_{i}\right]
$$

to obtain the expression of the lemma.
As a corollary, we have the following cut-and-paste property for Hellinger volume. This is analogous to the cut-and-paste property of Hellinger distance that was introduced in [BYJKS04].

Lemma 8. Let $I_{O Z} \subseteq I$ be the set of inputs which have odd number of zeros, and let $I_{E Z}=I \backslash I_{O Z}$. Then

$$
\psi\left(I_{O Z}\right)=\psi\left(I_{E Z}\right) .
$$

Proof. Using the expression of the previous lemma, we have that for any $\omega \in \Omega$,

$$
\prod_{v \in I_{O Z}} P_{v}(\omega)=\prod_{v \in I_{O Z}} \prod_{j=1}^{k} q_{j}\left(\omega ; v^{-j}\right)=\prod_{u \in I_{E Z}} \prod_{j=1}^{k} q_{j}\left(\omega ; u^{-j}\right)=\prod_{u \in I_{E Z}} P_{u}(\omega) .
$$

The middle equality holds, because for each $j \in[k]$ and $v \in I_{O Z}$ there is a unique $u \in I_{E Z}$ such that $v^{-j}=u^{-j}$.

Lower bounding Hellinger volume Eventually, we will need to provide a lower bound for the Hellinger volume of several distributions over protocol transcripts. In the two-party case, one lower bounds the Hellinger distance between the distribution of the transcripts on an accepting input and the distribution of the transcripts on a rejecting input. The following lemma will allow for similar conclusions in the multi-party case.

Lemma 9. Let $A \subseteq I$ be of size $t \geq 2$. Suppose there is an event $T \subseteq \Omega$, a constant $0 \leq \delta \leq 1$ and an element $v$ in $A$ such that $P_{v}(T) \geq 1-\delta$ and that for all $u \in A$ with $u \neq v, P_{u}(T) \leq \delta$. Then

$$
\psi(A) \geq(2-4 \sqrt{\delta(1-\delta)}) \cdot \frac{1}{t} .
$$

Proof. We need to show

$$
1-\sum_{\omega \in \Omega} \prod_{u \in A} P_{u}(\omega)^{\frac{1}{t}} \geq(2-4 \sqrt{\delta(1-\delta)}) \cdot \frac{1}{t} .
$$

Let $a=P_{v}(T)=\sum_{\omega \in T} P_{v}(\omega)$ and $b=\sum_{\omega \in T} \frac{1}{t-1} \sum_{u \neq v} P_{u}(\omega)$. Notice that by assumption $a \geq 1-\delta$ and $b \leq \delta$.

Recall Hölder's inequality: for any nonnegative $x_{k}, y_{k}, k \in m$,

$$
\sum_{k=1}^{m} x_{k} y_{k} \leq\left(\sum_{k=1}^{m} x_{k}^{t}\right)^{\frac{1}{t}}\left(\sum_{k=1}^{m} y_{k}^{\frac{t}{t-1}}\right)^{\frac{t-1}{t}}
$$

We first treat the sum over $\omega \in T$.

$$
\begin{aligned}
\sum_{\omega \in T} \prod_{u \in A} P_{u}(\omega)^{\frac{1}{t}} & =\sum_{\omega \in T} P_{v}(\omega)^{\frac{1}{t}} \prod_{u \neq v} P_{u}(\omega)^{\frac{1}{t}} \\
& \leq\left(\sum_{\omega \in T} P_{v}(\omega)\right)^{\frac{1}{t}}\left(\sum_{\omega \in T} \prod_{u \neq v} P_{u}(\omega)^{\frac{1}{t-1}}\right)^{\frac{t-1}{t}} \\
& \leq\left(\sum_{\omega \in T} P_{v}(\omega)\right)^{\frac{1}{t}}\left(\sum_{\omega \in T} \frac{1}{t-1} \sum_{u \neq v} P_{u}(\omega)\right)^{\frac{t-1}{t}} \\
& =a^{\frac{1}{t}} b^{\frac{t-1}{t}}
\end{aligned}
$$

where we first used Hölder's inequality and then the AM-GM inequality. We do the same steps for the sum over $\omega \notin T$ to find

$$
\sum_{\omega \notin T} \prod_{u \in A} P_{u}(\omega)^{\frac{1}{t}} \leq(1-a)^{\frac{1}{t}}(1-b)^{\frac{t-1}{t}} .
$$

Hence,

$$
\sum_{\omega \in \Omega} \prod_{u \in A} P_{u}(\omega)^{\frac{1}{t}} \leq a^{\frac{1}{t}} b^{\frac{t-1}{t}}+(1-a)^{\frac{1}{t}}(1-b)^{\frac{t-1}{t}} .
$$

Let $g(a, b, x)=a^{x} b^{1-x}+(1-a)^{x}(1-b)^{1-x}$. We will show that under the constraints $a \geq 1-\delta$ and $b \leq \delta$ where $\delta<1 / 2$, for any fixed $0 \leq x \leq 1 / 2, g(a, b, x)$ is maximized for $a=1-\delta$ and $b=\delta$. The partial derivatives for $g(a, b, x)$ with respect to $a$ and $b$ are

$$
\begin{gathered}
g_{a}(a, b, x)=x\left[a^{x-1} b^{1-x}-(1-a)^{x-1}(1-b)^{1-x}\right]=x\left[\left(\frac{b}{a}\right)^{1-x}-\left(\frac{1-b}{1-a}\right)^{1-x}\right] \\
g_{b}(a, b, x)=(1-x)\left[a^{x} b^{-x}-(1-a)^{x}(1-b)^{-x}\right]=(1-x)\left[\left(\frac{b}{a}\right)^{-x}-\left(\frac{1-b}{1-a}\right)^{-x}\right]
\end{gathered}
$$

Under our constraints, $\frac{b}{a}<1<\frac{1-b}{1-a}, 1-x>0$ and $-x \leq 0$, thus, $g_{a}(a, b, x)<0$ and $g_{b}(a, b, x) \geq 0$ for any such $a, b$, and $x$. This implies that for any fixed $b, g(a, b, x)$ is maximized when $a=1-\delta$ and similarly for any fixed $a, g(a, b, x)$ is maximized when $b=\delta$. Therefore, for all $a, b$, and $0 \leq x \leq 1, g(a, b, x) \leq g(1-\delta, \delta, x)$.

For $0 \leq x \leq 1 / 2$, let

$$
f(\delta, x)=1-g(1-\delta, \delta, x)=1-(1-\delta)^{x} \delta^{1-x}-\delta^{x}(1-\delta)^{1-x} .
$$

Since $f(\delta, x)$ is convex for any constant $0 \leq \delta \leq 1$,

$$
f(\delta, x) \geq \frac{f(\delta, 1 / 2)-f(\delta, 0)}{1 / 2-0} \cdot x=2(1-2 \sqrt{\delta(1-\delta)}) \cdot x .
$$

## 5 An application

In this section we show how to derive a lower bound for the informational complexity of the $\mathrm{AND}_{k}$ function. Define a collection of distributions $\eta=\left\{\zeta_{1}, \ldots, \zeta_{k}\right\}$, where, for each $j \in[k], \zeta_{j}$ is the uniform distribution over $I_{j}$. Recall that $I_{j} \subseteq I=\{0,1\}^{k}$ for $j \in[k]$ is the set of $k$-bitstrings whose $j$-th bit is 0 . We prove the following lower bound on the $\delta$-error informational complexity of $\mathrm{AND}_{k}$ with respect to $\eta$.
Remark. The choice of the collection $\eta$ is not arbitrary, but is suggested by the way the direct-sum theorem for informational complexity is proved in [BYJKS04] for the two-party setting. In particular, two properties of $\eta$ seem crucial for such a purpose. First, for each $j \in[k], \zeta_{j}$ is a distribution with support only on the zeroes of $\mathrm{AND}_{k}$. Second, under any $\zeta_{j}$, the input of each player is independent of any other input.

## Theorem 10.

$$
\mathrm{IC}_{\eta, \delta}\left(\mathrm{AND}_{k}\right) \geq \log e \cdot(1-2 \sqrt{\delta(1-\delta)}) \cdot \frac{1}{k^{2} 4^{k-1}}
$$

Proof. Let $\Pi$ be a $\delta$-error protocol for $\mathrm{AND}_{k}$. By Lemma 3 we have that,

$$
I(Z ; \Pi(Z)) \geq \frac{1}{2^{k-1} \ln 2} \cdot \psi\left(I_{j}\right)
$$

where $Z \sim \zeta_{j}$, for any $j \in[k]$, Thus, by the definition of $\mathrm{IC}_{\eta, \delta}\left(\mathrm{AND}_{k}\right)$,

$$
\mathrm{IC}_{\eta, \delta}\left(\mathrm{AND}_{k}\right) \geq \sum_{j=1}^{k} \frac{1}{k 2^{k-1} \ln 2} \cdot \psi\left(I_{j}\right)
$$

Applying in turn Lemmas 6, 8, and 9 we have

$$
\mathrm{IC}_{\eta, \delta}\left(\mathrm{AND}_{k}\right)>\frac{\psi\left(I_{O Z}\right)}{k^{2} 2^{k} \ln 2}=\frac{\psi\left(I_{E Z}\right)}{k^{2} 2^{k} \ln 2} \geq \log e \cdot(1-2 \sqrt{\delta(1-\delta)}) \cdot \frac{1}{k^{2} 4^{k-1}}
$$

where the application of Lemma 9 is with $A=I_{E Z}, t=2^{k-1}, T$ the set of transcripts that output " 1 ", and $v$ the all-one vector in $I$.

It is of interest to note, that

$$
\mathrm{IC}_{\eta, \delta}\left(\mathrm{AND}_{k}\right) \leq \frac{1}{k} \cdot H\left(1 / 2^{k-1}\right)=O\left(1 / 2^{k}\right)
$$

This is achieved by the following protocol. The players, one by one, reveal with one bit whether they see a 0 or not. The communication ends with the first player that sees a 0 . The amount of information revealed is $H\left(1 / 2^{k-1}\right)$ under $\zeta_{1}$ and 0 otherwise.

## 6 Difficulties in proving a direct-sum theorem

There seem to be fundamental difficulties in proving a direct-sum theorem on informational complexity in the NOF model. The reader familiar with the techniques of Bar-Yossef, Jayram, Kumar \& Sivakumar [BYJKS04], should recall that in the first part of the method a direct-sum for informational complexity of disjointness is proved. In particular, it is shown that with respect to suitable collections of distributions $\eta$ and $\zeta$ for DISJ $_{n, 2}$ and AND $_{2}$ respectively, the information cost of DISJ $_{n, 2}$ is at least
$n$ times the informational complexity of $\mathrm{AND}_{2}: \mathrm{IC}_{\eta, \delta}\left(\mathrm{DISJ}_{n, 2}\right) \geq n \cdot \mathrm{IC}_{\zeta, \delta}\left(\mathrm{AND}_{2}\right)$. This is achieved via a simulation argument in which the players, to decide the $\mathrm{AND}_{2}$ function, use a protocol for disjointness by substituting their inputs in a special copy of $\mathrm{AND}_{2}$ and using their random bits to generate the inputs for the rest $n-1$ copies of $\mathrm{AND}_{2}$. In the NOF model the players can no longer perform such a simulation. This is because, with private random bits, they cannot agree on what the input on the rest of the copies should be without additional communication. This problem can be overcome if we think of their random bits as being not private, but on each player's forehead, just like the input. However, In such a case, although the directsum theorem holds, it is useless. This is because $\mathrm{IC}_{\zeta, \delta}\left(\mathrm{AND}_{k}\right)=0$, as is shown by the protocol we describe in the next paragraph.

We describe a protocol that computes $\mathrm{AND}_{k}$ on every input, with one-sided error. It has the property that for any distribution over the zeroes of $\mathrm{AND}_{k}$, no player learns anything about his own input. We give the details for three players. Let $x_{1}, x_{2}, x_{3}$ denote the input. Each player has two random bits on his forehead, denoted $a_{1}, a_{2}$, $a_{3}$ and $b_{1}, b_{2}, b_{3}$. The first player does the following: if $x_{2}=x_{3}=1$, he sends $a_{2} \oplus a_{3}$, otherwise he sends $a_{2} \oplus b_{3}$. The other two players behave analogously. If the XOR of the three messages is ' 0 ', they answer ' 1 ', otherwise they know that the answer is ' 0 '. Notice that any player learns nothing from another player's message. This is because the one-bit message is XOR-ed with one of his own random bits, which he cannot see.

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[^0]:    *School of Physics and Mathematical Sciences, Nanyang Technological University and Centre for Quantum Technologies. Supported in part by the Singapore National Research Foundation under NRF RF Award No. NRF-NRFF2013-13 and by a NSF postdoctoral fellowship while at Rutgers University. Email: troyjlee@gmail.com.
    ${ }^{\dagger}$ Department of Computer Science, Rutgers University, NJ, USA Supported in part by NSF under grant CCF 0832787. Email: nikos.leonardos@gmail.com.
    ${ }^{\ddagger}$ Mathematics Department, Rutgers University, NJ, USA Supported in part by NSF under grant CCF 0832787. Email: saks@math.rutgers.edu.
    ${ }^{\S}$ Department of Computer Science, Rutgers University, NJ, USA Supported in part by NSF under grants CCF 0830133, CCF 0832787, and CCF 1064785. Email: fengming@cs.rutgers.edu.

