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Correlation Theorem for Two-Sided Quaternion Fourier Transform

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Abstract

In this paper we establish correlation theorem for the two-sided quaternion Fourier transform (QFT). A consequence of the theorem is also investigated.

Keywords: quaternion Fourier transform, correlation

I. Introduction

The quaternion Fourier transform (QFT) is a nontrivial generalization of the classical Fourier transform (FT) using quaternion algebra. A number of already known and useful properties of this extended transform are generalizations of the corresponding properties of the FT with some modifications (see, for example, [1, 2, 3, 4, 5]). *One* of the *most* powerful *properties* of the QFT is the convolution theorem. Recently, in [6] authors proposed the convolution theorem for the two-sided QFT, which describes the relationship between the two-sided QFT and convolution of two quaternion function.

Therefore, the main objective of the present paper is to establish the correlation for the two-sided QFT, which is a generalization of correlation theorem of the classical FT. We find that the correlation theorem does not work well for the right-sided quaternion Fourier transform and left-sided quaternion Fourier transform.

The quaternion algebra over $\mathbb{R}$, denoted by

$$\mathbb{H} = \{q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3\}, \qquad q_0, q_1, q_2, q_3 \in \mathbb{R}, \tag{1}$$

is an associative non-commutative four-dimensional algebra, which the quaternion units $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ obey the following multiplication rules:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1,\ \boldsymbol{ij} = -\boldsymbol{ji},\ \boldsymbol{ik} = -\boldsymbol{ki},\ \boldsymbol{jk} = -\boldsymbol{kj}, \text{and } \boldsymbol{ijk} = -1. \tag{2}$$

The quaternion conjugate is defined by

$$\bar{q} = q_0 - \boldsymbol{i}q_1 - \boldsymbol{j}q_2 - \boldsymbol{k}q_3, \tag{3}$$

which is an anti-involution, that is,

$$\bar{\bar{p}} = p, \qquad \overline{p+q} = \bar{p} + \bar{q},\ \overline{pq} = \bar{q}\bar{p}. \tag{4}$$

The norm of a quaternion is defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{{q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2}. \tag{5}$$

It is not difficult to check that

$$|qp| = |p||q|, \quad \forall\ p, q \in \mathbb{H}. \tag{6}$$

We further get the inverse

$$q^{-1} = \frac{\bar{q}}{|q|^2}.$$

This fact shows that $\mathbb{H}$ is a skew field, that means, every nonzero element has a multiplicative inverse.

For the sake of further simplicity, we will use the real vector notations

$$\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2,\ f(\boldsymbol{x}) = f(x_1, x_2), f(\boldsymbol{\omega}) = f(\omega_1, \omega_2),$$

and so on when there is confusion.

2. Main Results

In this section we introduce the definition of the two-sided QFT and establish the correlation of two quaternion-valued functions associated with the two-sided QFT.

Definition 2.1 (Two-sided QFT) Let f be in $\in L^2(\mathbb{R}^2; \mathbb{H})$. *The two-sided QFT of the quaternion function f is the transform given by the integral*

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 x_1} f(\boldsymbol{x}) e^{-\boldsymbol{j}\omega_2 x_2}\, d^2\boldsymbol{x}. \tag{7}$$

Theorem 2.1 (Inverse two-sided QFT) Suppose that $f \in L^2(\mathbb{R}^2; \mathbb{H})$ *and* $\mathcal{F}_q\{f\} \in L^1(\mathbb{R}^2; \mathbb{H})$. *Then the two-sided QFT is an invertible transform and its inverse is given by*

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 x_1} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) e^{\boldsymbol{j}\omega_2 x_2}\, d^2\boldsymbol{\omega}. \tag{8}$$

Definition 2.2 (Quaternion Correlation) *Suppose f and g are quaternion functions in $L^2(\mathbb{R}^2;\mathbb{H})$. The quaternion correlation of the functions is given by*

$$(f \otimes g)(\boldsymbol{x}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}+\boldsymbol{y})\overline{g(\boldsymbol{y})}\, d^2\boldsymbol{y}. \tag{9}$$

The following theorem is the main result of this paper, which describes how the two-sided QFT behaves under correlations.

Theorem 2.2 *Let $f, g \in L^2(\mathbb{R}^2;\, \mathbb{H})$ be two quaternion-valued functions. The two-sided QFT of correlation of f and g takes the form*

$$\begin{aligned}
&\mathcal{F}_q\{f \otimes g\}(\boldsymbol{\omega})\\
&= \left(\mathcal{F}_q\{f_0\}(\boldsymbol{\omega}) + \boldsymbol{i}\mathcal{F}_q\{f_1\}(\boldsymbol{\omega})\right)\left(\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega}) - \boldsymbol{j}\mathcal{F}\{g_2\}(-\omega_1,\omega_2)\right)\\
&\quad + \left(\mathcal{F}_q\{f_0\}(\omega_1,-\omega_2) + \boldsymbol{i}\mathcal{F}_q\{f_1\}(\omega_1,-\omega_2)\right)\left(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(-\boldsymbol{\omega}) - \boldsymbol{k}\mathcal{F}\{g_3\}(\omega_1,-\omega_2)\right)\\
&\quad + \left(\boldsymbol{j}\mathcal{F}_q\{f_2\}(-\omega_1,\omega_2) + \boldsymbol{k}\mathcal{F}_q\{f_3\}(-\omega_1,\omega_2)\right)\left(\mathcal{F}_q\{g_0\}(\omega_1,-\omega_2) - \boldsymbol{j}\mathcal{F}\{g_2\}(-\boldsymbol{\omega})\right)\\
&\quad + \left(\boldsymbol{j}\mathcal{F}_q\{f_2\}(-\boldsymbol{\omega}) + \boldsymbol{k}\mathcal{F}_q\{f_3\}(-\boldsymbol{\omega})\right)\left(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(\omega_1,-\omega_2) - \boldsymbol{k}\mathcal{F}\{g_3\}(-\boldsymbol{\omega})\right).
\end{aligned}$$

Proof. Applying the two-sided QFT definition gives

$$\begin{aligned}
&\mathcal{F}_q\{f \otimes g\}(\boldsymbol{\omega})\\
&= \int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 x_1}\left(\int_{\mathbb{R}^2} f(\boldsymbol{x}+\boldsymbol{y})\,\overline{g(\boldsymbol{y})}\, d^2\boldsymbol{y}\right) e^{-\boldsymbol{j}\omega_2 x_2}\, d^2\boldsymbol{x}.
\end{aligned}$$

By inserting the change of variables $\boldsymbol{z} = \boldsymbol{x}+\boldsymbol{y}$ to the above expression we immediately obtain

$$\begin{aligned}
&\mathcal{F}_q\{f \otimes g\}(\boldsymbol{\omega})\\
&= \int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}\left(\int_{\mathbb{R}^2} f(\boldsymbol{z})\,\overline{g(\boldsymbol{y})}\, d^2\boldsymbol{y}\right) e^{-\boldsymbol{j}\omega_2(z_2-y_2)}\, d^2\boldsymbol{z}\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}\left(\left(f_0(\boldsymbol{z}) + \boldsymbol{i}f_1(\boldsymbol{z})\right) + \left(\boldsymbol{j}f_2(\boldsymbol{z}) + \boldsymbol{k}f_3(\boldsymbol{z})\right)\right)\\
&\quad \left(\left(g_0(\boldsymbol{y}) - \boldsymbol{j}g_2(\boldsymbol{y})\right) + \left(-\boldsymbol{i}g_1(\boldsymbol{y}) - \boldsymbol{k}g_3(\boldsymbol{y})\right)\right) e^{-\boldsymbol{j}\omega_2(z_2-y_2)}\, d^2\boldsymbol{y}\, d^2\boldsymbol{z}\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}\left(f_0(\boldsymbol{z}) + \boldsymbol{i}f_1(\boldsymbol{z})\right)\left(g_0(\boldsymbol{y}) - \boldsymbol{j}g_2(\boldsymbol{y})\right) e^{-\boldsymbol{j}\omega_2(z_2-y_2)}\, d^2\boldsymbol{y}\, d^2\boldsymbol{z}\\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}\left(f_0(\boldsymbol{z}) + \boldsymbol{i}f_1(\boldsymbol{z})\right)\left(-\boldsymbol{i}g_1(\boldsymbol{y}) - \boldsymbol{k}g_3(\boldsymbol{y})\right) e^{-\boldsymbol{j}\omega_2(z_2-y_2)}\, d^2\boldsymbol{y}\, d^2\boldsymbol{z}\\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}\left(\boldsymbol{j}f_2(\boldsymbol{z}) + \boldsymbol{k}f_3(\boldsymbol{z})\right)\left(g_0(\boldsymbol{y}) - \boldsymbol{j}g_2(\boldsymbol{y})\right) e^{-\boldsymbol{j}\omega_2(z_2-y_2)}\, d^2\boldsymbol{y}\, d^2\boldsymbol{z}
\end{aligned}$$

$$+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}\Big(\big(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\big)\big(-\boldsymbol{i}g_1(\boldsymbol{y})-\boldsymbol{k}g_3(\boldsymbol{y})\big)\Big)e^{-\boldsymbol{j}\omega_2(z_2-y_2)}\ d^2\boldsymbol{y}\,d^2\boldsymbol{z}$$

$$= \int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\big(f_0(\boldsymbol{z})+\boldsymbol{i}f_1(\boldsymbol{z})\big)$$

$$\left(\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}\, g_0(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\ d^2\boldsymbol{y}-\boldsymbol{j}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 y_1}g_2(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\ d^2\boldsymbol{y}\right)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$\left(-\boldsymbol{i}\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1)}g_1(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\ d^2\boldsymbol{y}-\boldsymbol{k}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 y_1)}g_3(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\ d^2\boldsymbol{y}\right)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\big(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\big)$$

$$\left(\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 y_1)}\, g_0(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\ d^2\boldsymbol{y}-\boldsymbol{j}\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}g_2(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\ d^2\boldsymbol{y}\right)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\big(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\big)$$

$$\left(-\boldsymbol{i}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 y_1)}g_1(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\ d^2\boldsymbol{y}-\boldsymbol{k}\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1)}g_3(\boldsymbol{y})e^{\boldsymbol{j}\omega_2 y_2}\ d^2\boldsymbol{y}\right)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$= \int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\big(f_0(\boldsymbol{z})+\boldsymbol{i}f_1(\boldsymbol{z})\big)\Big(\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})-\boldsymbol{j}\mathcal{F}_q\{g_2\}(\omega_1,-\omega_2)\Big)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\big(f_0(\boldsymbol{z})+\boldsymbol{i}f_1(\boldsymbol{z})\big)\Big(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(-\boldsymbol{\omega})-\boldsymbol{k}\mathcal{F}_q\{g_3\}(\omega_1,-\omega_2)\Big)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\big(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\big)\Big(\mathcal{F}_q\{g_0\}(\omega_1,-\omega_2)-\boldsymbol{j}\mathcal{F}_q\{g_2\}(-\boldsymbol{\omega})\Big)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\big(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\big)\big(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(\omega_1,-\omega_2)+\boldsymbol{k}\mathcal{F}_q\{g_1\}(-\boldsymbol{\omega})\big)\ e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}$$

$$= \int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\big(f_0(\boldsymbol{z})+\boldsymbol{i}f_1(\boldsymbol{z})\big)e^{-\boldsymbol{j}\omega_2 z_2}d^2\boldsymbol{z}\Big(\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})-\boldsymbol{j}\mathcal{F}_q\{g_2\}(\omega_1,-\omega_2)\Big)$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\big(f_0(\boldsymbol{z})+\boldsymbol{i}f_1(\boldsymbol{z})\big)\ e^{\boldsymbol{j}\omega_2 z_2} d^2\boldsymbol{z}\Big(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(-\boldsymbol{\omega})-\boldsymbol{k}\mathcal{F}_q\{g_3\}(\omega_1,-\omega_2)\Big)$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\big(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\big)\, e^{-\boldsymbol{j}\omega_2 z_2} d^2\boldsymbol{z}\Big(\mathcal{F}_q\{g_0\}(\omega_1,-\omega_2)-\boldsymbol{j}\mathcal{F}_q\{g_2\}(-\boldsymbol{\omega})\Big)$$

$$+\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 z_1}\big(\boldsymbol{j}f_2(\boldsymbol{z})+\boldsymbol{k}f_3(\boldsymbol{z})\big)\ e^{\boldsymbol{j}\omega_2 z_2} d^2\boldsymbol{z}\ \big(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(\omega_1,-\omega_2)-\boldsymbol{k}\mathcal{F}_q\{g_3\}(-\boldsymbol{\omega})\big)$$

$$=\Big(\mathcal{F}_q\{f_0\}(\boldsymbol{\omega})+\boldsymbol{i}\mathcal{F}_q\{f_1\}(\boldsymbol{\omega})\Big)\Big(\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})-\boldsymbol{j}\mathcal{F}\{g_2\}(-\omega_1,\omega_2)\Big)$$

$$+\Big(\mathcal{F}_q\{f_0\}(\omega_1,-\omega_2)+\boldsymbol{i}\mathcal{F}_q\{f_1\}(\omega_1,-\omega_2)\Big)\big(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(-\boldsymbol{\omega})-\boldsymbol{k}\mathcal{F}\{g_3\}(\omega_1,-\omega_2)\big)$$

$$+\Big(\boldsymbol{j}\mathcal{F}_q\{f_2\}(-\omega_1,\omega_2)+\boldsymbol{k}\mathcal{F}_q\{f_3\}(-\omega_1,\omega_2)\Big)\big(\mathcal{F}_q\{g_0\}(\omega_1,-\omega_2)-\boldsymbol{j}\mathcal{F}\{g_2\}(-\boldsymbol{\omega})\big)$$

$$+\Big(\boldsymbol{j}\mathcal{F}_q\{f_2\}(-\boldsymbol{\omega})+\boldsymbol{k}\mathcal{F}_q\{f_3\}(-\boldsymbol{\omega})\Big)\big(-\boldsymbol{i}\mathcal{F}_q\{g_1\}(\omega_1,-\omega_2)-\boldsymbol{k}\mathcal{F}\{g_3\}(-\boldsymbol{\omega})\big).$$

As a consequence of the above theorem, we immediately obtain

Lemma 2.3 *Given any two quaternion-valued functions* $f,g\in L^2(\mathbb{R}^2;\ \mathbb{H})$. *If we assume that* $\mathcal{F}_q\{f\}(\boldsymbol{\omega})\in L^2(\mathbb{R}^2;\ \mathbb{R})$, *then*

$$\mathcal{F}_q\{f\otimes g\}(\boldsymbol{\omega})=\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\Big(\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})-\boldsymbol{i}\,\mathcal{F}_q\{g_1\}(-\boldsymbol{\omega})\Big)$$

$$-\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\Big(\boldsymbol{j}\mathcal{F}_q\{g_2\}(\omega_1,-\omega_2)+\boldsymbol{k}\mathcal{F}_q\{g_3\}(-\omega_1,\omega_2)\Big).$$

Proof. Straightforward computations show that

$$\mathcal{F}_q\{f\otimes g\}(\boldsymbol{\omega})$$

$$=\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 x_1}\left(\int_{\mathbb{R}^2} f(\boldsymbol{x}+\boldsymbol{y})\overline{g(\boldsymbol{y})}\ d^2\boldsymbol{y}\right)e^{-\boldsymbol{j}\omega_2 x_2}d^2\boldsymbol{x}$$

$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}\Big(f(\boldsymbol{z})\big(g_0(\boldsymbol{y})-\boldsymbol{i}g_1(\boldsymbol{y})\big)-f(\boldsymbol{z})\big(\boldsymbol{j}g_2(\boldsymbol{y})+\boldsymbol{k}g_3(\boldsymbol{y})\big)\Big)e^{-\boldsymbol{j}\omega_2(z_2-y_2)}d^2\boldsymbol{y}\,d^2\boldsymbol{z}$$

$$=\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}f(\boldsymbol{z})\int_{\mathbb{R}^2}\big(g_0(\boldsymbol{y})-\boldsymbol{i}g_1(\boldsymbol{y})\big)e^{\boldsymbol{j}\omega_2 y_2}d^2\boldsymbol{y}\ e^{-\boldsymbol{j}\omega_2 z_2}\ d^2\boldsymbol{z}$$

$$-\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1(z_1-y_1)}f(\boldsymbol{z})\int_{\mathbb{R}^2}(\boldsymbol{j}g_2(\boldsymbol{y})+\boldsymbol{k}g_3 y)e^{\boldsymbol{j}\omega_2 y_2}d^2\boldsymbol{y}\ e^{-\boldsymbol{j}\omega_2 z_2}\ d^2\boldsymbol{z}$$

$$=\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 y_1}\big(g_0(\boldsymbol{y})-\boldsymbol{i}g_1(\boldsymbol{y})\big)\ e^{-\boldsymbol{j}\omega_2 y_2}\ d^2\boldsymbol{y}$$

$$-\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\int_{\mathbb{R}^2} e^{-\boldsymbol{i}\omega_1 y_1}\big(\boldsymbol{j}g_2(\boldsymbol{y})+\boldsymbol{k}g_3(\boldsymbol{y})\big)\, e^{-\boldsymbol{j}\omega_2 y_2}\, d^2\boldsymbol{y}.$$

We further obtain

$$\begin{aligned}
&\mathcal{F}_q\{f\otimes g\}(\boldsymbol{\omega})\\
&\quad= \mathcal{F}_q\{f\}(\omega_1,-\omega_2)\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}\big(g_0(\boldsymbol{y})-\boldsymbol{i}g_1(\boldsymbol{y})\big)\, e^{\boldsymbol{j}\omega_2 y_2}\, d^2\boldsymbol{y}\\
&\qquad-\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}\big(\boldsymbol{j}g_2(\boldsymbol{y})+\boldsymbol{k}g_3(\boldsymbol{y})\big) e^{\boldsymbol{j}\omega_2 y_2}\, d^2\boldsymbol{y}\\
&\quad= \mathcal{F}_q\{f\}(\omega_1,-\omega_2)\left(\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1} g_0(\boldsymbol{y}) e^{\boldsymbol{j}\omega_2 y_2}\, d^2\boldsymbol{y}-\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}\boldsymbol{i}g_1(\boldsymbol{y}) e^{\boldsymbol{j}\omega_2 y_2}\, d^2\boldsymbol{y}\right)\\
&\qquad-\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\left(\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}\boldsymbol{j}g_2(\boldsymbol{y}) e^{\boldsymbol{j}\omega_2 y_2}\, d^2\boldsymbol{y}+\int_{\mathbb{R}^2} e^{\boldsymbol{i}\omega_1 y_1}\boldsymbol{k}g_3(\boldsymbol{y}) e^{\boldsymbol{j}\omega_2 y_2}\, d^2\boldsymbol{y}\right)\\
&\quad= \mathcal{F}_q\{f\}(\omega_1,-\omega_2)\Big(\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})-\boldsymbol{i}\,\mathcal{F}_q\{g_1\}(-\boldsymbol{\omega})\Big)\\
&\qquad-\mathcal{F}_q\{f\}(\omega_1,-\omega_2)\Big(\boldsymbol{j}\mathcal{F}_q\{g_2\}(\omega_1,-\omega_2)+\boldsymbol{k}\mathcal{F}_q\{g_3\}(-\omega_1,\omega_2)\Big).
\end{aligned}$$

which was to be proved.

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