

QUANTUM AND CLASSICAL GASES IN CONFINED GEOMETRY

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The solution of kinetic equation for the quantum and classical gases with diffusion boundary conditions has been obtained for an arbitrary capillary width d -to-mean free path l ratio. This solution allowed us to obtain analytical expressions describing both the stationary states of the gas filling in two containers connected by a capillary and the thermal slippage effect. The relations obtained have made it possible to observe the transition from the Knudsen regime ($d \ll l$) to the hydrodynamic one ($d \gg l$). This results take into account the influence of statistics and geometry of the capillary as well.

Non-equilibrium stationary states of the gas [1] filling in the two vessels connected by the channels with a small ratio of its width d to the mean free path l_T (Knudsen effect) is used for the investigation of the classical kinetics of the high rarefied gas. As far as we know the Knudsen effect has been investigated for the classical gases experimentally and theoretically in the limit $l_T \gg d$ only [1].

The present paper continues the investigations started in [2] and here we shall consider the Knudsen effect observed in the quantum gas of the impuritons of the superfluid mixtures of helium isotopes. Non-equilibrium stationary states of the classical and quantum gases with the arbitrary ratio of l_T/d and various geometry is investigated theoretically in this paper.

The explicit expressions are obtained for the distribution function of the particles without any restrictions in the statistics both for the cylindrical and plane-parallel capillaries of various cross-section. Proceeding from the expression for the distribution function one can calculate all significant values for the gas flow in the capillary. In this way the relations between gradients of the temperature and concentration were obtained. This results made it possible to obtain both the stationarity condition for the gas flow analytically and a explicit expression for the thermal slippage coefficient. The above expressions being essentially different for the classical and quantum gases.

A conventional device for the quasiparticles osmotic pressure measurements consists of two vessels filled with superfluid ^3He - ^4He mixtures. This vessels are connected by the Vicor glass superleak [3]. The values of concentration n_3 and temperature T may be different in these containers. The state with the constant value of the ^4He chemical potential ($\mu_4 = \text{const}$) is reached by the overflowing of the superfluid component through the superleak relatively fast. As we know this state was studied both theoretically

and experimentally. Otherwise this state is not stationary one and it should be considered since a quasistationary one as the gas of impuritons can flow through the superleak representing a set of channels with the finite cross-section. In consequence of the impuritons overflow a real stationary state is established. In this state the flow velocity of impuritons [3] must be equal to zero.

The condition of stationarity can be written as follows [1]:

$$\int dS_{\Pi} \int d\Gamma \frac{d\epsilon_3}{dp_z} g = 0. \quad (1)$$

Here dS_{Π} is the cross-section element, $d\Gamma$ is the momentum space element, axis z lies along the flow, p_z is z -th component of momentum, ϵ_3 is the energy of impuriton; $g = f - f_0$ is the nonequilibrium term adding to the local-equilibrium function f_0 . It should be noted that the left-hand part of equation (1) represents the macroscopic velocity of impuriton gas

$$\vartheta = \int d\Gamma \frac{d\epsilon_3}{dp_z} g. \quad (2)$$

which is averaged over the cross-section of a capillary.

Relation (1) makes it possible to determine the values of the temperature and concentration in each of the vessels connected by a capillary in the stationary state. This values must be different essentially in the quasistationary state. The situation when the inequality $l_T \gg d$ is not fulfilled can be observed in a number of experiments (see for example [4]). In this connection the theoretical investigations of the Knudsen effect with an arbitrary ratio of l_T/d are of great interest.

To investigate the relation of stationarity (1) which is valid for the capillary of an arbitrary cross-section, one has to solve the kinetic equation for the gas of the quasiparticles filling in this capillary with respective boundary condition. The length of the capillary is supposed to be much greater than all typical lengths (mean free paths, width of the capillary). This assumption allows us to consider only slow changes of pressure and temperature along the z -axis of the capillary and to linearize the kinetic equation with respect to the small deviation g from the local equilibrium distribution function f_0 .

The kinetic equation should be completed with the boundary conditions. The latter describes the type of interaction between the quasiparticles and the capillary walls. In the case of diffusive reflection this conditions may be written as

$$g[(x; y) \in S; (\vec{p}; \vec{n}) \geq 0] = 0, \quad (3)$$

where S are the points of the capillary surface, and \vec{n} is positive normal to S .

Let us consider the flow of quasiparticles gas in a cylindrical capillary of radius R by using the BGK -approximation for the collision integral [5]. In this case the kinetic equation gives

$$s_\rho \frac{dg}{d\rho} \cos(\varphi - \psi) + s_z \frac{df_0}{dz} = -g + \frac{p_z f_0}{m P_F} \int d\Gamma p_z g. \quad (4)$$

Here we use the cylindrical coordinate system. The location and the momentum of the particles are defined by the vectors $\vec{r} = \vec{r}(\rho, \varphi, z)$ and $\vec{p} = \vec{p}(p_\rho, \psi, p_z)$ respectively; $\vec{s} = \tau \vec{p}/m$; and P_F is the pressure of the ideal

Fermi-gas. The general solution of equation (4) with diffusion boundary condition (3) may be found as the sum

$$g = \tilde{g} + g_H. \quad (5)$$

The first term in (5) defines the kinetics of the quasiparticles gas in the vicinity of the capillary walls inside the layer of typical width of mean free path. This term represents the exact solution of the Cauchy-problem.

$$s_\rho \frac{d\tilde{g}}{d\rho} \cos(\varphi - \psi) + s_z \frac{d\tilde{f}_0}{dz} + \tilde{g} = 0, \quad (6)$$

$$\tilde{g}(\rho = R; (\vec{p}, \vec{n}) \geq 0) = 0. \quad (7)$$

It respects to the usual τ -approximation and diffusive reflection on the capillary walls.

The second term in (5) is the solution of equation

$$s_\rho \frac{dg_H}{d\rho} \cos(\varphi - \psi) + g_H = \frac{p_z f_0}{m P_F} \int d\Gamma p_z (\tilde{g} + g_H) \quad (8)$$

with the boundary condition (3). As it will be shown later this term should be considered in hydrodynamic flow of quasiparticles.

The problem (6)-(7) has the only solution

$$\tilde{g} = -s_z \frac{df_0}{dz} \left[1 - \exp \left(-\frac{\rho}{s_\rho} \left[\cos(\varphi - \psi) + \sqrt{\left(\frac{R}{\rho}\right)^2 - \sin^2(\varphi - \psi)} \right] \right) \right]. \quad (9)$$

Here the term with the exponent defines the interaction of quasiparticles with the walls. It differs from zero in the vicinity of the capillary walls.

The expression for the deviation g_H is obtained from the equation (8) which describes the motion of the quasiparticles gas at a distant from the capillary walls (hydrodynamical flow). The asymptotic expression for the deviation of g_H with respect to the small parameter l_T/ρ ($l_T = \tau\sqrt{2T/m}$) can be written as

$$g_H = \frac{5F_{3/2}}{4F_{5/2}} \frac{\tau p_z \nabla P_F}{m P_F l_T^2} [\rho^2 - R^2 - 4\Omega s_\rho^2] + \frac{p_z f_0}{m P_F} (1 - e^{-\Omega}) \mu \nabla T, \quad (10)$$

$$\text{where } \Omega = \frac{\rho \cos(\varphi - \psi)}{s_\rho} + \frac{\sqrt{R^2 - \rho^2 \sin^2(\varphi - \psi)}}{s_\rho},$$

$$\mu = \frac{5}{2} \frac{\tau}{m} \frac{F_{3/2}}{F_{5/2}} \left[\frac{7F_{5/2}}{15F_{1/2}} - \frac{5}{9} \left(\frac{F_{3/2}}{F_{1/2}} \right)^2 + \frac{3}{8} \frac{F_2}{F_{3/2}} \left(\frac{3}{4} \frac{F_2}{F_{1/2}} - \frac{5}{6} \frac{F_{3/2} F_1}{F_{1/2}^2} \right) \right] \quad (11)$$

$$\text{and Fermi-function } F_\nu = \frac{1}{\pi^2 \hbar^3} \int_0^\infty \frac{d\bar{\varepsilon} \bar{\varepsilon}^\nu}{1 + \exp(\bar{\varepsilon} - \frac{\mu_F}{T})}.$$

The coefficient μ (11) is the coefficient of thermal slippage. It was introduced phenomenologically for the classical gases [1].

From (11) we have

$$\mu = \frac{\tau}{2m} \left(1 + \frac{1}{\pi}\right) \xi, \quad \mu = \frac{3\tau}{m} \frac{T}{T_F}, \quad (12)$$

for the cases of classical ($T \geq T_F$) and degenerate ($T \leq T_F$) gases, respectively.

The same calculations for the plane-parallel capillary give

$$g = \tilde{g}_\Pi + g_{\Pi H}, \quad (13)$$

$$\tilde{g}_\Pi = -s_z \frac{df_0}{dz} \left[1 - \exp \left(-\frac{d}{|s_x|} - \frac{x}{s_x} \right) \right]. \quad (14)$$

The x -axis is perpendicular to the walls of the capillary and z -axis coincides with the direction of the gradient

$$g_{\Pi H} = \frac{5F_{3/2}}{2F_{5/2}} \frac{\tau p_z}{m P_F l_T^2} [x^2 - d^2 - 2\Lambda s_x] \nabla P_F + \frac{p_z f_0}{m P_F} (1 - e^{-\Lambda}) \mu \nabla T, \quad (15)$$

where $\Lambda = d/|s_x| + x/s_x$.

Relations (9), (10), (14), (15) completely define the flow of the gas without any restriction on statistics of the gas in both cylindrical and plane-parallel capillaries with an arbitrary cross-section. The substitution of this relations in (1) determines, in general case, the relation between the gradients of temperature and concentration that makes the state stationary but non-equilibrium.

The general relations will not be written here since they are too cumbersome. The expressions for the limiting cases of both relatively wide and narrow capillary will be presented.

Proceeding from both the expressions for nonequilibrium term g (8)-(10), (13)-(15) and stationarity condition (1) we have for the wide plane-parallel capillary ($d/l_T \geq 1$)

$$\nabla P_F = \frac{3F_{5/2}}{5F_{3/2}} \frac{l_T^2}{d^2} \frac{m n_3 \mu}{\tau} \left(1 - \frac{9l_T F_2}{8d F_{3/2}} \right) \nabla T \quad (16)$$

and for the cylindrical one ($R_T/l_T \geq 1$):

$$\nabla P_F = \frac{8F_{5/2}}{5F_{3/2}} \frac{l_T^2}{R^2} \frac{m n_3 \mu}{\tau} \left(1 - \frac{3l_T F_2}{R F_{3/2}} \right) \nabla T. \quad (17)$$

In classical region of temperatures ($T \geq T_F$) relations (16) and (17) give

$$\nabla(n_3 T) = \frac{3l_T^2}{4d^2} \left(1 + \frac{1}{\pi} \right) \left(1 - \frac{3l_T}{d\sqrt{\pi}} \right) n_3 \nabla T \quad (18)$$

and

$$\nabla(n_3 T) = \frac{2l_T^2}{R^2} \left(1 + \frac{1}{\pi} \right) \left(1 - \frac{8l_T}{R\sqrt{\pi}} \right) n_3 \nabla T \quad (19)$$

respectively.

If the gas of impuritons becomes degenerate (for the temperatures $T \leq T_F$), relations (16), (17) may be written as

$$\nabla P_F = \frac{9}{7} \frac{l_F^2}{d^2} \left(1 - \frac{15l_F}{16d}\right) \frac{T}{T_F} n_3 \nabla T, \quad (20)$$

$$\nabla P_F = \frac{24}{7} \frac{l_F^2}{R^2} \left(1 - \frac{5l_F}{2R}\right) \frac{T}{T_F} n_3 \nabla T. \quad (21)$$

Here $l_F = \tau \sqrt{2T_F/m}$. Expressions (16)-(21) show that the constant difference of temperatures $\delta T = T_2 - T_1$ at the ends of a wide capillary is, in a long run, compensated by the constant difference of pressures $\delta P = P_2 - P_1$, so

$$\frac{\delta P}{P_1} \sim \frac{l_T^2 \delta T}{d^2 T_1}.$$

In this case the gas of impuritons flows along the temperature gradient direction near the axis of the capillary and it flows in the opposite direction in the vicinity of the capillary walls making the complete mass flow be equal to zero. This result coincide with the result of [1].

Now let us consider the stationary non-equilibrium state of ^3He - ^4He mixture filling in the vessels connected by a narrow capillary ($d/l_T \leq 1$, $R/l_T \leq 1$) with constant values of temperature at its ends [2]. This situation is similar to the stationary state of the system of the gas filling in the volumes separated by means of the thin walls with the pores of size much less than mean free path of the particle of the gas (Knudsen effect [1]).

Proceeding from condition (1) and expressions (8)-(10), (13)-(15) with reference to the limit considered one can find that the stationary state is determined by the fact that the function Ψ must be constant along the capillary axis

$$\nabla \Psi_{\Theta;\Pi} = 0. \quad (22)$$

In classical range of temperatures ($T \geq T_F$) the asymptotic expression for the function Ψ with respect to the small parameter $a = 2d/l_T \leq 1$ may be written as

$$\Psi_{\Pi}(a) = n_3 T \left[\frac{3}{\sqrt{\pi}} a \left(1 - \Re - \frac{2}{3}\right) + \frac{a^2}{3} \right] \quad (23)$$

for the plane-parallel capillary ($\Re = 0.577\dots$ – is the Euler's constant), and

$$\Psi_{\Theta}(a) = n_3 \sqrt{\frac{mT}{2\pi}} \frac{8R}{3\tau} \quad (24)$$

for the cylindrical one.

In the region of the degeneracy $T \leq T_F$ one can obtain

$$\Psi_{\Pi} = -\frac{15}{16} a_F P_F \left[\ln a_F + \Re - \frac{3}{4} \right] \left[1 - \frac{\pi^2}{4} \frac{T^2}{T_F^2} \left(1 + \frac{1}{6 \ln a_F} \right) \right], \quad (25)$$

for the plane-parallel capillary and

$$\Psi_{\Theta} = P_F \left[\frac{5}{4} a_F - \frac{5}{6} a_F^2 \ln a_F \right] \left[1 - \frac{\pi^2}{4} \frac{T^2}{T_F^2} \left(1 - \frac{4}{9} a_F \ln a_F \right) \right]; a_F = \frac{2d}{\tau} \sqrt{\frac{m}{2T_F}} \quad (26)$$

for the cylindrical one.

It should be noted that there is essential difference between asymptotic relations (23), (24) for the plane-parallel capillary and for the cylindrical one, respectively. The main term in the asymptotic derivation (23) of the function Ψ for the case of the plane-parallel capillary is $a \cdot \ln a$. As to the case with the cylindrical capillary the main term in expression (24) is linear $\sim a$. This result corresponds to the well-known formula of Knudsen describing the establishment of the equilibrium state in the above mentioned classical system of the gas filling in two volumes separated by the porous wall. In this case the mechanical equilibrium is determined by the following equality between temperatures and pressures of the gas filling in the vessels

$$P_1/\sqrt{T_1} = P_2/\sqrt{T_2}. \quad (27)$$

This relations may be considered as a requirement for the function Ψ (24) to be constant along the capillary axis. For the plane-parallel capillary the stationarity requires the fulfillment of the another condition

$$\frac{P_1}{\sqrt{T_1}} \left(1 - \Re - \frac{2}{3} \ln \left(\frac{2d}{l_{T_1}} \right) \right) = \frac{P_2}{\sqrt{T_2}} \left(1 - \Re - \frac{2}{3} \ln \left(\frac{2d}{l_{T_2}} \right) \right) \quad (28)$$

This difference is caused by the fact that when the walls of the plane-parallel capillary approach each another we have hydrodynamical regime in each plane parallel to walls. When the radius R of the cylindrical capillary tends to zero the hydrodynamical regime is retained along the direction of the capillary axis only.

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ТЕПЛОПРОВІДНІСТЬ ТА ДИФУЗІЯ У БАГАТОКОМПОНЕНТНИХ КЛАСИЧНИХ ТА КВАНТОВИХ СИСТЕМАХ

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Одержано розв'язок кінетичного рівняння з дифузними умовами на межі для квантових і класичних газів при довільних відношеннях між шириною капіляра d і середньою довжиною вільного пробігу l . Цей розв'язок дозволяє дістати вирази для теплового ковзання і досліджувати стаціонарні стани газу, що перебуває у двох контейнерах, які з'єднані між собою капіляром. Отримані співвідношення дозволяють простежити перехід від режиму Кнудсена ($d \ll l$) до гідродинамічного режиму ($d \gg l$) і визначити вплив статистики і геометрії капіляру.