# The 3d Ising model represented as random surfaces 

J. Ambjørn<br>The Niels Bohr Institute<br>Blegdamsvej 17, DK-2100 Copenhagen $\emptyset$, Denmark<br>A. Sedrakyan<br>Yerevan Phys. Institute<br>Alikhanian Brothers St. 2, 375036 Yerevan<br>Armenia<br>G. Thorleifsson<br>The Niels Bohr Institute<br>Blegdamsvej 17, DK-2100 Copenhagen $\varnothing$, Denmark


#### Abstract

We consider a random surface representation of the three-dimensional Ising model.The model exhibit scaling behaviour and a new critical index $\kappa$ which relates $\gamma_{\text {string }}$ for the bosonic string to the exponent $\alpha$ of the specific heat of the 3 d Ising model is introduced. We try to determine $\kappa$ by numerical simulations.


## 1 Introduction

The 2d Ising model formulated on the regular square lattice has a well know random walk representation. In fact we can write:

$$
\begin{gather*}
Z(\beta)=\sum_{\sigma} \exp \left(\beta \sum_{\langle n m\rangle} \sigma_{n} \sigma_{m}\right) \sim \sum_{P} \Phi(P) \frac{1}{L(P)} \exp (-\mu(\beta) L(P)),  \tag{1.1}\\
\mu(\beta)=-\ln \tanh \beta, \quad \Phi(P)=(-1)^{n(P)} . \tag{1.2}
\end{gather*}
$$

In these formulas $P$ is a closed (not necessarily connected) path without backtracking ${ }^{1}$ on the lattice, $L(P)$ the length of the path (i.e. the number of links constituting the path) and $n(P)$ the number of times the path $P$ self-intersects. The sum of paths exponentiate as a sum over connected closed paths and the free energy of the 2d Ising model:

$$
\begin{equation*}
F(\beta) \sim \ln Z(\beta) \sim \sum_{\text {connected } P} \Phi(P) \frac{1}{L(P)} \exp (-\mu(\beta) L(P)) \tag{1.3}
\end{equation*}
$$

It is known that the 2 d Ising model has a fermionic representation, a fact closely related to the sign factor $\Phi(P)$, and that the continuum field theory associated with the second order transition of the 2 d Ising model is that of a free Majorana spinor.

If we consider the 3 d Ising model defined on a regular cubic lattice it has a random surface representation[1] which can been viewed as the analogue of the random walk representation (1.1)-(1.2). It can be written as

$$
\begin{gather*}
Z(\beta)=\sum_{\sigma} \exp \left(\beta \sum_{\langle n m\rangle} \sigma_{n} \sigma_{m}\right) \sim \sum_{S} \Phi(S) \frac{1}{C(S)} \exp (-\mu(\beta) A(S)) .  \tag{1.4}\\
\mu(\beta)=-\ln \tanh \beta, \quad \Phi(S)=(-1)^{l(S)}, \tag{1.5}
\end{gather*}
$$

where $S$ denotes a closed lattice surface (without back-tracking) built from plaquettes, $C(S)$ a symmetry factor for the surface [2], $A(S)$ the lattice area of $S$ (i.e. the number of plaquettes constituting the surface $S$ ) while $l(S)$ is the number of links where the surface intersects itself. $(-1)^{l(S)}$ is obviously a surface analogy of the Kac-Ward factor $(-1)^{n(P)}$ for the random walk. Again the sum over surfaces exponentiates into a sum over connected closed surfaces[2]:

$$
\begin{equation*}
F(\beta) \sim \ln Z(\beta) \sim \sum_{\text {connected } S} \Phi(S) \frac{1}{C(S)} \exp (-\mu(\beta) A(S)) \tag{1.6}
\end{equation*}
$$

[^0]Although it is by no means clear how to take the continuum limit of this lattice surface theory, due to the oscillating sign factor $\Phi(S)$, one would nevertheless think that the second order transition of the 3d Ising model should result in continuum field theory, and that this continuum theory should allow a representation like (1.6), only with continuum surfaces instead of lattice surfaces. For a closed path $P$ in two dimensions it is possible to obtain a simple geometrical expression of the sign factor $\Phi(P)$ :

$$
\begin{equation*}
\Phi(P)=\exp \left(\frac{1}{2} i \oint e_{i} d e_{i}\right) \tag{1.7}
\end{equation*}
$$

where $e_{i}(\xi)$ is the tangent vector on the path $P$, but an analogous expression is not known for $\Phi(S)$ for a surface $S$ immersed in $R^{3}$. One can nevertheless start on the lattice, introduce a fermionic representation of $\Phi(S)$ and take the formal continuum limit of the lattice fermionic action [3, 4] (see also [5] for related considerations). In this way the continuum limit becomes a string theory where the action is sum of an area term and the induced Dirac action plus an anomaly term on the world surface immersed in $R^{3}$ and we can formally view $\Phi(S)$ as a result of first integrating over the fermions. If we accept this point of view, it makes sense to talk about a topological expansion of the surface theory, which at first glance might appear somewhat surprising since the original high temperature expansion over lattice surfaces involves crucial cancellations between surfaces of different topologies and cancellations between orientable and non-orientable surfaces. If we assume that such a topological expansion is possible close to the critical point of the Ising model ${ }^{2}$ we can discuss scaling properties of $\Phi(S)$ which are consistent with this assumption and the known critical behaviour of the Ising model.

Since the 3d Ising model has a second order phase transition at a certain value $\beta_{c}$ we have

$$
\begin{equation*}
F(\beta)=\operatorname{r.t}(\beta)+c\left(\beta_{c}-\beta\right)^{2-\alpha}+\operatorname{l.s}(\beta) \tag{1.8}
\end{equation*}
$$

where r.t $(\beta)$ stands for regular terms, l.s $(\beta)$ stands for less singular terms, while the critical exponent $\alpha$ is known from high temperature expansions and renormalization group study to be $\alpha \approx 0.11$ or $0.125[20,21]$. How can such an expression match the left hand side of (1.6)? If we ignored sign factor $\Phi(S)$ we would have an ordinary

[^1]bosonic string theory ${ }^{3}$. The entropy factor of bosonic surfaces of a fixed topology has (at least for $c<1$ ) the form
\[

$$
\begin{equation*}
Z(A) \sim A^{\gamma-3} e^{\mu_{0} A} \tag{1.9}
\end{equation*}
$$

\]

This leads to the well known singularity structure of the bosonic string:

$$
\begin{equation*}
Z(\mu)=\sum_{A} Z(A) e^{-\mu A} \sim \operatorname{r.t}(\mu)+c\left(\mu-\mu_{0}\right)^{2-\gamma}+\operatorname{l.s}(\mu) \tag{1.10}
\end{equation*}
$$

which seems unrelated to (1.8), since $\mu_{0}>\mu\left(\beta_{c}\right)$. It is however intuitively clear that $\langle\Phi(S)\rangle$ is small and it is reasonable to assume it has the following asymptotic behaviour as a function of the area $\bar{A}$ :

$$
\begin{equation*}
\Phi_{\bar{A}} \equiv \frac{\sum_{\{S \mid A(S)=\bar{A}\}} \Phi(S) e^{-\mu_{0} A(S)}}{\sum_{\{S \mid A(S)=\bar{A}\}} e^{-\mu_{0} A(S)}} \sim \bar{A}^{\kappa} e^{-\mu_{1} \bar{A}} \tag{1.11}
\end{equation*}
$$

Under this assumption the rhs of (1.6) can be written as

$$
\begin{equation*}
\sum_{A} Z(A) \Phi_{A} e^{-\mu A} \sim \operatorname{r.t}(\mu)+c\left(\mu-\left(\mu_{0}-\mu_{1}\right)\right)^{2-\gamma-\kappa}+\operatorname{l.s}(\mu) \tag{1.12}
\end{equation*}
$$

and comparing with (1.8) we reach the conclusion that

$$
\begin{equation*}
\alpha=\gamma+\kappa, \quad \mu_{0}-\mu_{1}=\mu\left(\beta_{c}\right) . \tag{1.13}
\end{equation*}
$$

At this point it should be made clear that the same arguments can be used in the case of the ordinary bosonic random walk, $\Phi(P)$ and the 2 d Ising model. It can be shown that we indeed have a relation similar to (1.11):

$$
\begin{equation*}
\Phi_{\bar{L}} \equiv \frac{\sum_{\{P \mid L(P)=\bar{L}\}} \Phi(P) e^{-\hat{\mu}_{0} L(P)}}{\sum_{\{P \mid L(P)=\bar{L}\}} e^{-\hat{\mu}_{0} L(P)}} \sim \bar{L}^{\hat{\kappa}} e^{-\hat{\mu}_{1} \bar{L}} \tag{1.14}
\end{equation*}
$$

The exact values of $\hat{\mu}_{0}$ and $\hat{\mu}_{1}$ depends on the discretized models used for the bosonic random walk. The result (1.14) is implicitly in the formalism developed for discretized fermionic walks in [6], where the phase factor (1.7) is represented as a product of Hilbert-Schmidt operators on $S^{1}$. The largest eigenvalue of that operator dominates in the scaling limit and one can prove that $\hat{\kappa}=-1$ in agreement with (1.13) since $\gamma=1$ for a closed random walk in two dimensions and $\alpha=0$ (logarithmic divergence) for the 2d Ising model.

[^2]In the case of random surfaces a formula like (1.13) immediately leads to the interesting question of topology dependence: $\mu_{0}$ is independent of topology and it is reasonable to assume that the same is true for $\mu_{1}$. This assumption is needed for $\mu_{0}-\mu_{1}=\mu\left(\beta_{c}\right)$ to make sense, since $\beta_{c}$ has no a priori relation with the topology. We know on the other hand that $\gamma$ depends on topology in a linear way. For central charge less than or equal to one we have as a function of genus $g$ of the surface:

$$
\begin{equation*}
\gamma_{g}=\gamma_{0}+K(c) g, \quad K(c)>0 \tag{1.15}
\end{equation*}
$$

$K(c)$ is not known for $c>1$. In case the surfaces for $c>1$ are dominated by socalled branched polymers we know that $K(c)<0$ for sufficiently large $c$ [22, 23]. If $K(c)<0$ it is plausible that surfaces of spherical topology will dominate the critical behaviour, but this dominance could in principle be eliminated by a $g$-dependence of $\kappa$. If that is the case the simplest scenario would be one where all genus contributions gave rise to the same $\alpha$-dependence. In both case we should be able to test $\alpha=\gamma+\kappa$ by considering spherical topology.

In sect. 2 we define the discretized model we are going to use in our attempt to verify $\alpha=\gamma+\kappa$. The results of numerical simulations are discussed in sect. 3 .

## 2 The model

Let us recall the construction of the sign factor $\Phi(S)$ for the regular lattice surface $[3,7]$. We shall cover the lattice surface by a system of closed non-self-intersecting curves, which cross each link ones. This type of covering by curves can be obtained by drawing two parallel lines connecting the midpoints of neighbouring links on each plaquette of the surface. There are two ways of drawing these lines on each plaquette (see fig. 1 in ref. [3]) and we have a class of $2^{M}$ coverings ( M is the total number of plaquettes of the surface). Each covering consists of a set of closed non-self-intersecting curves $C_{1}, \ldots, C_{m}$. Let $n(C)$ be the number of times the curve $C$ crosses the lines of self-intersection for the surface $S$ immersed in the cubic lattice. Assume we can find a function $\Phi(C)$ such that

$$
\begin{equation*}
\Phi(C)=(-1)^{n(C)} \tag{2.1}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
\Phi(S)=(-1)^{l(S)}=\prod_{k}(-1)^{n\left(C_{k}\right)}=\prod_{k} \Phi\left(C_{k}\right) \tag{2.2}
\end{equation*}
$$

where the product is over all curves of the chosen curve system. We can define a function $\Phi(C)$ with the above mentioned properties by [3] (this definition makes
sense even for a continuum surface):

$$
\begin{equation*}
\Phi(C)=\frac{1}{2} \operatorname{tr} \mathrm{P} \exp \left(\oint R^{-1} d R\right) . \tag{2.3}
\end{equation*}
$$

The matrix $R$ is defined as follows: Consider three orthonormal vectors $e_{i}^{\alpha}, \alpha=$ $1,2,3$, attached to each point of the curve $C: e_{i}^{1}$ is the tangent vector, $e_{i}^{2}$ is normal to $e_{i}^{1}$ in the tangent plan of the surface and $e_{i}^{3}$ is normal to the surface. Then $R$ rotates this orthonormal system into a given fixed orthonormal system. If the rotation matrix $R$ is in the fundamental representation and $\sigma_{i}$ denote the Pauli matrices this can be written as follows:

$$
\begin{equation*}
R \tau_{\alpha} R^{-1}=\sigma_{\alpha}, \quad \tau_{\alpha} \equiv e_{i}^{\alpha} \sigma_{i} \tag{2.4}
\end{equation*}
$$

It can be shown that $\Phi(C)=1$ if $C$ does not cross a line of self-intersection and -1 if it crosses it once. In addition $\Phi(C)$ is invariant under smooth deformations of $C$ since it takes discrete values. Only when the deformation of $C$ reaches singular points of the immersed surface, where $R$ is not defined, will it be able to change its value. The singular points are the possible endpoints of lines of self-intersection of the immersed surface and these points are so-called Whitney singularities, which are related to $\pi_{1}(S O(3))$, i.e. elements like (2.3). In the case of piecewise linear surfaces (either the cubic lattice considered above or the triangulated random surfaces immersed in $R^{3}$, to be considered below) we will assume that the curve $C$ is a straight line on the flat part of the surfaces. Then the integral (2.3) will only get contributions when the curve crosses from one flat piece to a neighbouring one. Let us denote the successive flat pieces of surface encountered by the curve by $S_{n}$, the corresponding orthonormal frames by $e^{\alpha}(n)$ and the the rotation matrices related to $e^{\alpha}(n)$ by $R(n)$. With this notation (2.3) can then be written as

$$
\begin{equation*}
\Phi(C)=\frac{1}{2} \operatorname{tr} \prod_{n} \frac{1+\tau_{\alpha}(n) \tau_{\alpha}(n+1)}{\sqrt{1+e_{i}^{\alpha}(n) e_{i}^{\alpha}(n+1)}} \tag{2.5}
\end{equation*}
$$

As was shown in ref.[3] the sign factor $\Phi(S)$ does not depend on the specific choice of one of $2^{M}$ coverings. One can choose the covering such that we have a maximum number of closed non-connected curves. These curves will then surround half of the lattice vertices separating them into two classes, say "+" and "-" vertices (see [7]). Then the sign factor $\Phi(S)$ can be said to be located at one type of the vertices, say the "+" vertices, and by construction we have

$$
\begin{equation*}
\Phi(S)=(-1)^{n_{+}} \tag{2.6}
\end{equation*}
$$

where $n_{+}$is the number of Whitney singularities, located at the "+" vertices ${ }^{4}$
Rather than applying the formula (2.5) or (2.6) to the cubic lattice surfaces we will assume that there is indeed a continuum surface theory of the 3d Ising model at the critical point, and that we are free to use any sensible approximation to the continuum surface theory, as long as it admits us to take the correct scaling limit. From this point of view it seems reasonable to use triangulated piecewise linear surfaces immersed in three dimensions as our statistical ensemble of surfaces. Their critical properties coincide with those of the bosonic string in target space dimensions $d<1$, where the theory can also be solved analytically. We have however to choose an appropriate curve system on the piecewise linear surface and for that purpose the generic triangulation of a surface is not useful. Rather we consider the class of triangulations obtained by gluing squares together as indicated in fig. 1. Each square consists of two triangles, one of the diagonals of the square being the common link of the two triangles. The four vertices are divided into two groups: "-"vertices connected by the diagonal and "+"vertices separated by the diagonal. The rule for gluing together links of different squares to form a (triangulated) surface is that "+"vertices should be glued to "+"vertices and "-"vertices to"-"vertices (see fig. 1). For this class of triangulated surfaces one can now choose a convenient curve system which separates the " + " and the "-" vertices. For each square two parts of different curves run parallel to the diagonal connecting the "-"vertices. Stated differently we can say that the closed curves on the surface surround all "+"vertices (see fig. 1). From this point of view the curve system is identical to one described above on the cubic lattice, but of course the connectivity of the triangulated surfaces are such that they cannot in general be mapped into a cubic lattice. The class of triangulations generated this way can be viewed in another way: The triangles surrounding "+"vertices form polygons, where the vertices on the boundaries are "-"vertices, and precisely one curve from the curve system runs around the "+" vertex inside this polygon. The surface is now constructed by gluing together these polygons to form a connected closed surface.

As this class of triangulations is different from the one usually used in discretized 2 d quantum gravity one could worry if they belong to the same universality class, which is clearly what we want. One can easily prove this. It is possible to write down a matrix model which precisely generates the triangulations we are considering:

$$
\begin{equation*}
Z(\lambda)=\int d \phi d \phi^{*} d A e^{-\operatorname{tr} \phi^{\dagger} \phi-\frac{1}{2} \operatorname{tr} A^{2}+\lambda \operatorname{tr} \phi^{\dagger} \phi A} \tag{2.7}
\end{equation*}
$$

[^3]In (2.7) $\phi$ denotes a complex $N \times N$ matrix, $A$ an Hermitian $N \times N$ matrix and to $\operatorname{tr} \phi^{\dagger} \phi A$ we can associate an oriented triangle with one "+"vertex and two ""vertices. The link connecting the "-"vertices corresponds to the $A$ matrix. If we first integrate over the Hermitian matrices we reproduce precisely the complex matrix model with quartic interactions which is known to have the same $\gamma_{s t r i n g}$ as the corresponding Hermitian matrix model $[8,10,9]$. The integration over the Hermitian matrices corresponds creating the squares of fig. 1 by identifying the "-" vertices of two triangle. Alternatively the integration over the complex matrices will leave us with an Hermitian matrix model with potential a $\log (1-A)$. This corresponds to gluing together polygons as mentioned above ${ }^{5}$.

An immersion in $R^{3}$ of a given (abstract) triangulation $T$ of a surface is a map from the vertices $i \in T$ into $x_{i} \in R^{3}$, such that links $\langle i j\rangle$ are mapped into straight lines connecting $x_{i}$ and $x_{j}$ and triangles $\langle i j k\rangle$ are identified with the triangles spanned by $x_{i}, x_{j}, x_{k}$. This results in a piecewise flat surface $S\left(\left\{x_{i}, T\right\}\right)$. The curve system defined above is by the same procedure mapped into piecewise linear curves on $S$ and we can use (2.2) and (2.5) to construct the sign factor $\Phi\left(S\left(\left\{x_{i}, T\right\}\right)\right.$. The definition of the model of random surfaces is as follows:

$$
\begin{equation*}
F(\mu)=\sum_{T \in \mathcal{T}} \frac{e^{-\mu N_{T}}}{C(T)} \int \prod_{i \in T} d x_{i} \delta^{3}\left(\sum_{i} x_{i}\right) e^{-\sum_{\langle i j\rangle}\left(x_{i}-x_{j}\right)^{2}} \Phi\left(S\left\{x_{i}, T\right\}\right) \tag{2.8}
\end{equation*}
$$

where $N_{T}$ denotes the number of triangles, $C(T)$ is a symmetry factor for the triangulation $T$ and $\mathcal{T}$ denotes the class of triangulations defined above. In principle we can include a summation over different topologies in the sum over triangulations, i.e. in the class $\mathcal{T}$, but as discussed above we will here try to test the conjecture that surfaces of spherical topology will be important in the scaling limit. This we do by comparing the critical exponent $\alpha$ of the 3d Ising model with $\gamma$ and $\kappa$ extracted from $F(\mu)$ defined (2.8), but with $\mathcal{T}$ restricted to spherical topology.

The free bosonic surface theory corresponding to (2.8) is given by $[11,12,13,14]$

$$
\begin{equation*}
F_{0}(\mu)=\sum_{T \in \mathcal{T}} \frac{e^{-\mu N_{T}}}{C(T)} \int \prod_{i \in T} d x_{i} \delta^{3}\left(\sum_{i} x_{i}\right) e^{-\sum_{\langle i j\rangle}\left(x_{i}-x_{j}\right)^{2}} \tag{2.9}
\end{equation*}
$$

It is in this ensemble we will try to calculate the expectation value $\langle\Phi(S)\rangle_{0}$. We expect an exponential fall off like in (1.11), only with $N_{T}$ instead of the area $A$.

[^4]
## 3 Numerical method and results

Unfortunately we have at this stage to rely on numerical methods. One such method is Monte Carlo simulations. It has been applied extensively to simulations of the bosonic string defined by (2.9). Two types of simulations have been used, a "canonical" one where the number of triangles $N_{T}$ are kept fixed, and a "grand canonical" one where $N_{T}$ is allowed to vary $[14,15,16]$. The standard canonical move, the so-called "flip" of a link, is not available for us since we consider a restricted class of triangulations $\mathcal{T}$. In fact, since we are also interested in a determination of $\gamma_{\text {string }}$, we have chosen the grand canonical updating scheme, which applies to our class of triangulations with a small modification of the standard moves. Each "+"vertex is surrounded by "-"vertices and each "-"vertex is surrounded by an alternating sequence of "+" and "-"vertices. We have shown the corresponding moves for inserting and deleting " + " and "-" vertices, respectively, in fig. 2. We have checked numerically that this algorithm indeed leads to the correct value $\gamma_{\text {string }}=-1 / 2$ in the case where the dimension of spacetime is $d=0$, but it should be mentioned that the extraction of the correct value was more difficult than if we used the standard ensemble of triangulations, i.e. all triangulations with spherical topology and length of loops of links larger than or equal three.

Let us now turn to the 3d case. There has been a number of attempts to determine $\gamma_{\text {string }}$ numerically $[16,15,17,18,19]$. The conclusion from these simulations is that $-0.2<\gamma_{\text {string }}<0.2$. for $d=3$. We will not improve these simulations, since we will be working with quite small surfaces and the only difference compared to these earlier investigations will be that our class of triangulations is different, but (at least in zero dimensions) belongs to the same universality class. Our results for $\gamma_{\text {string }}$ are compatible with the earlier measurements, indicating that also in three dimensions does our class of triangulations give the same results as the unrestricted class of triangulations.

The quantity which has our main interest is $\left\langle\Phi_{N_{T}}(S)\right\rangle_{0}$, where $N_{T}$ is the number of triangles which constitutes the surface. $\Phi_{N_{T}}(S)$ is always $\pm 1$ but will fluctuate wildly, both when we change triangulations and when we update the coordinates $x_{i}$, and in accordance which our expectations $\left\langle\Phi_{N}(S)\right\rangle_{0}$ falls off exponentially with the number of triangles. In practise this means that we have not been able to use surfaces with $N>34$. These are quite small surfaces, but hopefully large enough ${ }^{6}$ to reveal

[^5]| $N_{T}$ | $\left\langle\Phi_{N_{T}}(S)\right\rangle_{0}$ |
| :--- | :---: |
| 6 | $0.38570 \pm 0.00018$ |
| 8 | $0.29515 \pm 0.00013$ |
| 10 | $0.17897 \pm 0.00012$ |
| 12 | $0.09469 \pm 0.00012$ |
| 14 | $0.05903 \pm 0.00013$ |
| 16 | $0.03221 \pm 0.00014$ |
| 18 | $0.01882 \pm 0.00017$ |
| 20 | $0.01127 \pm 0.00011$ |
| 22 | $0.00606 \pm 0.00017$ |
| 24 | $0.00309 \pm 0.00018$ |
| 26 | $0.00190 \pm 0.00014$ |
| 28 | $0.00113 \pm 0.00017$ |
| 30 | $0.00062 \pm 0.00033$ |
| 32 | $0.00024 \pm 0.00019$ |
| 34 | $0.00040 \pm 0.00019$ |
| 36 | $0.00012 \pm 0.00020$ |

Table 1: measurement of $\left\langle\Phi_{N_{T}}(S)\right\rangle_{0}$
the qualitative behaviour of the critical exponent $\kappa$ in $\left\langle\Phi_{N}(S)\right\rangle_{0} \sim N^{\kappa} \exp \left(-\mu_{1} N\right)$.
In table 1 and fig. 3 we have shown the result of approximately $10^{9}$ sweeps over surfaces with $N$ between 6 and 40 .

The best value one can extract from the data seems to be $\kappa=0 \pm 1$. The determination of $\kappa$ is rather poor and can only be improved by going to larger surfaces, which is impossible because of the exponential fall off of the sign factor with the size of the surface. We nevertheless find the results encouraging in the sense that they are compatible which the relation $\gamma_{s t r i n g}+\kappa=\alpha$. It would obviously be very desirable to be able to find observables which have a better behaviour than $\left\langle\Phi_{N}\right\rangle_{0}$ for large $N$ and which still relate critical exponents of the Ising model to those of random surfaces. The situation is the same as for the measurements of Wilson loops in lattice gauge theories. The expectation value of the Wilson loops will always be exponentially suppressed because we make the importance sampling in the pure gauge theory which knows nothing about the static quarks. In the same way we here make the importance sampling of pure bosonic string configurations without knowledge about the fermionic string which describes the Ising model. If better observables could be found it would be possible to go to larger surfaces and thereby get more reliable critical exponents and in addition check for the contributions of different topologies and orientable versus non-orientable surfaces.

Our results indicate that surfaces of spherical topology satisfy the scaling relation $\gamma_{\text {string }}+\kappa=\alpha$ near the critical point in a random surface representation of the 3d Ising model. We are fully aware that our numerical results are such that "indicate" is the appropriate word to use. As discussed above even an improved numerical verification of this relation would not prove that only surfaces of spherical topology are important in the scaling limit, but it puts emphasis on such a possibility. In principle one could dream of settling the problem of dominance of spherical topology by numerical simulations since it is possible to perform simulations with surfaces of topology different from that of the sphere and also to include non-orientable surfaces. However, we need better observables, which do not fall off so fast with the size of the surfaces, if we want to be able to make quantitative statements.

But apart from attempts to improve the numerical simulations we hope that our point of view might simulate theoretical attempts to understand better the random surface representations of the 3d Ising model.

Acknowledgement We thank Peter Orland for reading the manuscript and for valuable comments.

## References

[1] E. Fradkin, M. Srednicki and L. Susskind, Phys.Rev.D21 (1980) 2885; V.Dotsenko, Thesis, Landau Institute 1981; A.Polyakov, Gauge Fields and Strings, Contemporary Concepts in Physics vol.3.
[2] C. Itzykson, Nucl.Phys. B210 (1982) 477.
[3] A. Kavalov and A. Sedrakyan, Nucl.Phys. B285 (1987) 264.
[4] A.G. Sedrakyan, Phys.Lett. 260B (1991) 45.
[5] P. Orland, Phys.Rev.Lett. 59 (1987) 2393; Strings in the three-dimensional Ising model, BUHEP88-2.
[6] J. Ambjørn, B. Durhuus and T. Jonsson, Nucl.Phys. B330, (1990) 509. See also J.Phys. A21 (1988) 981; Europhys.Lett. 3 (1987) 1059.
[7] N.Dolbilin, A.Sedrakyan et al.,Doklady Akad.Nauk SSSR 295 (1987) 19
[8] J. Ambjørn and Y. Makeenko, Mod.Phys.Lett. A5 (1990) 1753.
[9] J. Ambjørn, J. Jurkiewicz and Y. Makeenko, Phys.Lett. B251 (1990) 517.
[10] T. Morris, Nucl.Phys. B356 (191) 703.
[11] J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl.Phys. B257[FS14] (1985) 433.
[12] F. David, Nucl.Phys. B257[FS14] (1985)45; 543.
[13] V.A, Kazakov, I.K. Kostov and A.A. Migdal, Phys.Lett. 157B (1985) 295.
[14] J. Ambjørn, B. Durhuus, J. Fröhlich and P. Orland, Nucl.Phys. B270[FS16] (1986) 457.
[15] J. Jurkiewiecz, A. Krzywicki and B. Petersson, Phys.Lett. 168B (1986) 273; 177B (1986) 89.
[16] J. Ambjørn, B. Durhuus, J. Fröhlich, Nucl.Phys. B275[FS17] (1986) 161.
[17] F. David, J. Jurkiewiecz, A. Krzywicki and B. Petersson, Nucl. Phys. B290 [FS20] (1987) 218.
[18] J. Ambjørn, Ph. De Forcrand, F. Koukiou and D. Petritis, Phys.Lett. 197B (1987) 548.
[19] J. Ambjørn, D. Boulatov and V. Kazakov, Mod.Phys.Lett. A5 (1990) 771.
[20] J.C.Le Guillou,J.Zinn-Justin, J.Physique 48 (1987) 19 and Referenses there
[21] C.Baillie,R.Gupta,K.A.Hawick et al., Los Alamos preprint LA-UR-91-2853
[22] J. Ambjørn, B. Durhuus and T. Jonsson, Phys.Lett. B244 (1990) 403.
[23] P. Di Vecchia. M. Kato and N. Ohta, Int.J.Mod.Phys. 7A (1992) 1391.

## Figure Captions

Fig. 1 The matrix model representation of our restricted class of triangulations. Vertices with open circles represent the "-" vertices defined in the text, while vertices with black circles represent the "+" vertices. The dotted diagonals symbolize the integration over the Hermitean matrix $A_{\gamma \beta}$, while the dashed line shows the curve $C_{v}$ surrounding the " + " vertex $v$.

Fig. 2 The two classes of moves for inserting and deleting "-" vertices (open circles) and "+" vertices (black circles).

Fig. 3 A graphical representation of the data of table 1 . We expect for large $N$ :

$$
\log \left[\left\langle\Phi_{N}\right\rangle /\left\langle\Phi_{N-1}\right\rangle\right] \sim-\mu_{1}+\kappa \log [N /(N-1)]
$$


[^0]:    ${ }^{1}$ Back-tracking means in this context that two successive steps in the lattice path shares the same link. For surfaces the generalization is that two neighbouring plaquettes do no lie on top of each other.

[^1]:    ${ }^{2}$ It should be stressed that it seems difficult not to have a topological expansion, possibly involving both orientable and non-orientable surfaces, if there really exists a continuum fermionic surface representation at the critical point. Viewed in this way the assumption of a topological expansion is more or less equivalent with the assumption of a continuum surface representation. We feel there are good reasons to expect the existence of such a continuum surface representation $[3,4]$.

[^2]:    ${ }^{3}$ In the following we assume the world sheet variables are somehow regularized, either by the original cubic lattice or by dynamical triangulations.

[^3]:    ${ }^{4}$ Note that on piecewise linear surfaces the Whitney singularities have to be located on the vertices (the 0-simplexes) of the surface.

[^4]:    ${ }^{5}$ Note that we have by these arguments shown that to any order in $1 / N$ the $m=2$ Hermitian matrix model has the same critical behaviour as the $m=2$ complex matrix model, a well know result which was however not entirely trivial to prove.

[^5]:    ${ }^{6}$ At this point it is worth to recall the history of the determination of $\gamma_{\text {string }}$ by numerical methods. Although the first simulations used very small surfaces the qualitative features extracted from these simulations are essentially unchanged today, years and thousands of CPU hours later.

