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Lectures on pseudo-differential operators: Regularity Theorems and applications to non-elliptic problems, by Alexander Nagel and E. M. Stein, Mathematical Notes, Princeton Univ. Press, Princeton, N. J., 1979, 159 pp., $6.75.

Pseudodifferential operators may be considered from the ontological, the teleological, or the archeological standpoint: what are they, what do they do, where do they come from? Quick answers are that they are linear operators expressed via the Fourier transform as (formal) integral operators, that they are used extensively in the study of partial differential equations, and that the direct line of descent is through singular integrals. We shall consider each point in more detail and detect as well a thread of Hegelian dialectic.

If $P = \Sigma a_\alpha(x)D^\alpha$ is a differential operator in R^n and e_ξ denotes the exponential $e_\xi(x) = \exp(ix \cdot \xi)$, then $Pe_\xi(x) = p(x, \xi)e_\xi(x)$, where $p(x, \xi) = \Sigma a_\alpha(x)\xi^\alpha$. Expressing a test function as a sum of exponentials by the Fourier inversion formula, one obtains

$$Pu(x) = \int e^{ix\cdot\xi}p(x, \xi)(\xi)\, d'\xi = \int\int e^{ix\cdot\xi}p(x, \xi)u(y)\, dy\, d'\xi, \qquad (1)$$

where $d'\xi = (2\pi)^{-n}\, d\xi$. Thus the differential operator P is expressed formally as an integral operator defined by a conditionally convergent "oscillatory" integral by means of the "symbol" p. Here $p = p_r + p_{r-1} + \cdots$, where p_j is homogeneous of degree j in ξ. Thus at least locally in x one has estimates for the derivatives

$$\left|D_x^\beta D_\xi^\alpha\left(p - \sum_{k>j} p_k\right)\right| < C_{\alpha\beta j}|\xi|^{j-|\alpha|} \quad \text{if } |\xi| > 1. \qquad (2)$$

These estimates may be used in conjunction with integration by parts in (1) to convert (1) into a convergent integral and verify directly that P maps $C_c^\infty(R^n)$ to $C^\infty(R^n)$. Now if Q is a second differential operator, with symbol $q = q_s + q_{s-1} + \cdots$, then the composition QP has symbol which can be calculated by Leibniz' rule:

$$q \circ p = \sum (\alpha!)^{-1} i^{-|\alpha|} D_\xi^\alpha q D_x^\alpha p. \qquad (3)$$

In particular the highest order part is just the product $p_r q_s$. Note in passing that the identity operator has symbol 1.

Several points might be noted here. First, (1) makes sense for symbols p which are not polynomials in ξ, for example symbols with asymptotic expansions as infinite sums of terms homogeneous in ξ of decreasing order: $p \sim p_r + p_{r-1} + \cdots$, understood in the sense of (2). Second, such a symbol will again map $C_c^\infty(R^n)$ to $C^\infty(R^n)$. Third, the composition formula (3) may be expected to hold, in the same asymptotic sense. Fourth, if p is *elliptic*, i.e. if $p_r(x, \xi) \neq 0$ when $\xi \neq 0$, $\xi \in R^n$, then one may take $q_{-r} = p_r^{-1}$ and use (3) to solve $q \circ p \sim 1$ asymptotically with $q \sim q_{-r} + q_{-r-1} \cdots$. Thus one would have an algebra of operators which includes the differential operators and also the parametrices (approximate inverses) of elliptic operators, and the regularity theory should be a consequence. Fifth, one may equally well consider systems by allowing matrix-valued symbols. Sixth, if these notions are intrinsic then the class of operators should be invariant under diffeomorphism, the effect on symbols should be calculable, and thus there should be a theory on manifolds.

The theory just described in outline might appear to be little more than an elegant way of rederiving elliptic regularity results by a modern version of the parametrix method of Levi (1909), but introducing nonpolynomial symbols allows a surprising amount of flexibility. The increased freedom in dividing and factoring symbols allowed Calderón [7] to obtain the first very general result on local existence of solutions to equations with variable coefficients and was instrumental in the work of Nirenberg and Treves on local solvability [26]; see also [5]. Having enough symbols to account for the operations occurring in K-theory was necessary for the index theorem [2]. Also, as observed by Calderón, general boundary value problems for elliptic operators may be reduced to pseudodifferential equations on the boundary. A simple example is the oblique derivative problem for the Laplacian: find u harmonic in the half plane $y > 0$ such that

$$a(x)\frac{\partial}{\partial x}u(x, 0) + b(x)\frac{\partial}{\partial y}u(x, 0) = g(x). \tag{4}$$

Let $v(x) = u(x, 0)$ and take the Fourier transform in x; then the problem becomes

$$\int e^{ix\cdot\xi}\{ia(x)\xi - b(x)|\xi|\}\hat{v}(\xi)d'\xi = g(x). \tag{5}$$

Various weakenings of the assumptions on the symbols are possible. One does not need an asymptotic expansion in homogeneous parts since the composition, boundedness, and invariance results use only the estimates on the derivatives. These estimates may be allowed to depend nontrivially on x and be nonisotropic:

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| < C_{\alpha\beta}w_{\alpha\beta}(x, \xi). \tag{6}$$

For various conditions of this type, see [17], [3], [5], [18], [34]. The correspondingly more general classes of operators provide still greater flexibility in attacking fundamental questions of linear theory, and contain parametrices of nonelliptic operators.

The discussion so far makes it appear that pseudodifferential operators came as a natural outgrowth of general linear PDE theory, specifically of the

Fourier transform-L^2-C^∞-methods of Schwartz, Gårding and others. This was almost the case **[15]**, but paradoxically the actual development came through an L^p theory less directly related to PDE. This part of the story begins with a boundary value problem even simpler than (4). A function harmonic on the upper half plane and suitably small at infinity is determined by its boundary value f on R and determines the boundary value g of its conjugate harmonic function. Thus f determines g and in fact g is the Hilbert transform of f:

$$g(t) = Hf(t) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|t-s|>\varepsilon} (t-s)^{-1} f(s)\, ds. \tag{7}$$

Now H is a generalized convolution operator whose (distribution) kernel has Fourier transform $i \operatorname{sgn} \xi$. By Plancherel one has the L^2 equality $\|Hf\|_2 = \|f\|_2$. M. Riesz proved in 1927 that $\|Hf\|_p \leq A_p \|f\|_p$, $1 < p < \infty$. Various clever proofs of this result are known, exploiting the complex variable aspects of the situation. In 1952 Calderón and Zygmund gave a real variable proof of such L^p inequalities which demonstrated that the essential features of the kernel $k(t, s) = (t - s)^{-1}$ are first that it have enough cancellation for an L^2 estimate to hold, and second that although k is singular on the diagonal the relationship between singularity and smoothness is such that

$$\int_{|x-y| \geq 2|x-y'|} |k(x,y) - k(x,y')|\, dx \leq C, \quad \text{all } y, y'. \tag{8}$$

This inequality is satisfied by smooth convolution kernels in R^n which are homogeneous of degree $-n$. The Calderón-Zygmund proof uses a general interpolation argument and a specific decomposition of an L^1 function which exploits the measure-theoretic covering properties of cubes or balls in R^n.

The relationship between singular integral operators and partial differential operators is that our operator P can be factored as $P_0(I - \Delta)^{r/2}$ where P_0 is a singular integral operator of the above type (plus a multiple of the identity). Various important estimates follow from this factorization **[8]**, **[1]**, for example the basic L^p estimate

$$\sum_{|\alpha| \leq m} \|D^\alpha u\|_p \leq C_K(\|Pu\|_p + \|u\|_p), \qquad u \in C_K^\infty, \tag{9}$$

where P is elliptic of order m and K is compact. Moreover, the commutator of a singular integral operator and a differential operator of order m is an operator of order $m - 1$, while the commutator of two singular integral operators has order -1. This theory was transplanted to manifolds by Seeley **[27]** and thus came to play its role in the index theorem. The approximate commutativity properties just mentioned were essential in the various applications and suggested that a more complete calculus of such operators should be possible; it finally blossomed in **[22]**, **[28]**, **[35]**, **[16]** as the theory of pseudodifferential operators in the first version sketched above.

Roughly speaking, singular integral operators are pseudodifferential operators of degree zero and conversely, at least before generalization sets in. For example the Hilbert transform is the pseudodifferential operator with symbol $i \operatorname{sgn} \xi$, and in general the symbol is a partial Fourier transform of the kernel.

Thus in principle the symbol and kernel may be used interchangeably. In practice the viewpoints have diverged. As we have seen, looking at symbols and the symbol calculus is natural for the L^2 theory and its C^∞ applications, and leads to close analysis of the estimates (6). One generally tries to preserve L^2 boundedness of zero order operators, $\|Qu\|_2 < C\|u\|_2$, but the usual generalizations have rarely had the corresponding L^p boundedness property.

For the L^p theory one seems forced to examine the kernel, and the Calderón-Zygmund method suggest analysis of the geometry of distinguished sets to replace the euclidean balls or cubes used in (8) and in the decomposition of L^1 functions. For example the original "elliptic" homogeneity associated to the form $\xi_1^2 + \cdots + \xi_n^2$ may be replaced by the "parabolic" homogeneity associated to $|\xi_1| + \xi_2^2 + \cdots + \xi_n^2$, [20], [13], or by other obvious generalizations, or by more complicated twisted homogeneities associated to Lie groups or to the boundaries of strictly pseudoconvex complex domains [23], [14], [11].

The reader interested in a fuller account of the history, the applications, or the state of the art in singular integral and pseudodifferential operators should see [3], [4], [8], [17], [18], [25], [29], [30]–[33], [36], and the references therein, or the forthcoming books by M. Taylor and by F. Treves. We have slighted various developments and viewpoints in this brief survey, including the Banach algebra approach [12], the analytic case [6], and the Heisenberg view [19].

Nagel and Stein [24] achieved a significant synthesis of the kernel-geometric and the symbol-estimate viewpoints. They considered families of second order hypoelliptic differential operators which include the following (in R^2):

$$D_1^2 + D_2^2, \quad iD_1 + D_2^2, \quad iD_1 + x_1 D_2^2, \quad D_1^2 + x_1^2 D_2^2 \tag{10}$$

or any Hörmander operator $\Sigma X_j^2 + Y$ where the X_j and Y are vector fields and they and the commutators $[X_j, X_k]$ span the tangent space at each point. To each such operator P they associate a class of pseudodifferential operators defined by estimates like (6), which amounts to a geometry on the cotangent bundle of R^n, and a dual geometry in R^n itself defined by a pseudometric and therefore having a family of distinguished neighborhoods. The latter geometry is suited to a version of the singular integral theory [11]; the pseudodifferential operators of order zero are admissible singular integral operators, while the parametrix of the operator P is an admissible pseudodifferential operator. Thus one obtains not only the known L^2 estimates for P but the corresponding L^p estimates analogous to (9), and in the process one achieves a grasp of the singularities of the corresponding kernels and the naturally associated geometry. (A related but somewhat different synthesis is given in [4].)

The monograph which is the excuse for this long discussion is a clear and unhurried presentation of the authors' viewpoint and methodology in the work just described. It is not a comprehensive treatise or general introduction to pseudodifferential operators, but a guided tour which starts from basic knowledge of real analysis and Fourier analysis, and spirals upward to a view of the theory at work on a new frontier. The introduction maps the way fully in advance. A discussion of homogeneous distributions in R^n and on homogeneous (nilpotent Lie) groups follows. There is a brief general discussion of pseudodifferential operators, followed by the introduction of the author's new classes and a sketch of the basic L^p boundedness result. General Sobolev and Lipschitz spaces are introduced and corresponding mapping theorems obtained. The class of operators is shown to be closed under composition and to behave as expected under diffeomorphisms. Finally, the theory is applied to the boundary Laplacian $\square_b$, to the general oblique derivative problem, and to general versions of the examples (10). Throughout, the exposition is as pleasant as the sometimes very technical nature of the material will allow; only the repeated idiosyncratic spelling of "parametrices" is an irritant.

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Bedienungsprozesse, by Gennadi P. Klimow, translated from the Russian by V. Schmidt, Birkhäuser Mathematische Reihe 68, Basel and Stuttgart, 1979, xi + 244 pp., $38.00.

Queues or waiting lines are among the richest sources of problems and examples in the theory of stochastic processes. The simplest queues serve well to illustrate the basic theory of Markov chains. The service systems, which are of research interest today, involve complex interactions of queues and are described in terms of typically multi-dimensional stochastic processes. Such descriptions hold, only if we are able to formalize the processes at all. Realistic queueing systems are formidable dynamic-stochastic systems indeed.

From isolated but distinguished contributions by Felix Pollaczek and A. Ya. Khinchin in the thirties and forties, the theory of queues emerged as a subdiscipline of probability theory during the years 1950–1960. Its growth since then has truly been astounding. I usually place the number of journal articles on queues at 7500 and know of some fifty books fully devoted to this subject. Both numbers probably underestimate the actual size of the literature. By those studying the job flows in telecommunication and manufacturing systems or inside computers or computer networks, the relevance of understanding the behavior of queues is taken for granted. In spite of this,