ARITHMETICAL PARAPHRASES* (II)

BY

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IV. The fundamental series for paraphrase

1. A former paper under the same title is continued.† We first collect a few Fourier developments in a specially prepared form immediately suitable for paraphrase, and then apply the theory of the preceding paper to read off from them a few of their innumerable arithmetical consequences. The classical expansions are not in the appropriate form; we require the arithmetical developments in which the coefficients of the powers of q are given as explicit functions of the divisors of the exponents. These developments are unique. The chief elliptic theta series for paraphrase fall naturally into two sets according as they do or do not explicitly involve class numbers in their coefficients. This paper is concerned only with the latter kind and their more immediate paraphrases; but in all work like the present with the arithmetic of elliptic functions, these series of the first set appear to be fundamental, presenting themselves repeatedly in the most diverse investigations. Hence we shall give a fairly representative selection from them.

In writing down the few paraphrases of this paper we have aimed merely to show how such lists of properly prepared formulas may be used, much as a table of logarithms in other computations, for the almost immediate discovery of paraphrases broadly of the Liouville kind. We have purposely omitted all applications to specialized functions and their related theorems, the method of deriving special results being sufficiently evident from the papers of Liouville and Pepin, and from Bachmann's book. In connection with the series only brief notes on the calculations, in all cases simple, have been retained; but there are sufficient indications of the course followed for all the expansions to be quickly rechecked if desired.

2. The m, n, 2^a, d, δ, T_1, T_2, T_3 notation, explained in § 7 of Part I, is used throughout; and in the elliptic or theta series the summations refer to

* Read before the San Francisco Section of the American Mathematical Society, October, 1918. To save space, the extensive lists of theta and other expansions contained in the paper as read, have been omitted. Some of these, for the doubly periodic functions of the third kind, will appear elsewhere. Only the series necessary for illustrating the method of paraphrase in V have been retained.

† These Transactions, vol. 22 (1921), pp. 1–30.

all values > 0 of the type T_1, T_2 or T_3 indicated, the coefficients of the several powers of q being written after those powers in ( ), { } or [ ] as convenient. The ϑ_a notation is that of Jacobi (*Werke*, vol. 1, p. 501), with ϑ_0 in place of his ϑ. After glancing at the notation in §§ 3, 4 here and Part I § 7, the reader may pass at once to § 12, using §§ 3–11 only for reference. An important desideratum in regard to the series is pointed out in § 7, footnote.

3. *Numerical functions.* The constants occurring throughout this kind of work, other than class numbers, depend chiefly upon the functions now defined. Let $\zeta_r(n)$, $\zeta_r'(n)$, $\zeta_r''(n)$ denote respectively the sum of the rth powers of all, of the odd, of the even divisors of n; and let $\xi_r(n)$ denote the excess of the sum of the rth powers of all divisors $\equiv 1 \bmod 4$ of n over the sum of the rth powers of all those $\equiv -1 \bmod 4$; also let $\xi_r'(n)$ denote the excess of the sum of the rth powers of all divisors of n whose conjugates are $\equiv 1 \bmod 4$ over the sum of the rth powers of all those whose conjugates are $\equiv -1 \bmod 4$; and define $\xi_r''(n)$ by the identity $\xi_r'(n) + \xi_r''(n) = \xi_r(n)$. Write

$$\zeta_0(n),\ \zeta_0'(n),\ \zeta_0''(n),\ \xi_0(n),\ \xi_0'(n),\ \xi_0''(n) \equiv \zeta(n),\ \cdots,\ \xi''(n),$$

denoting the respective numbers of divisors pertaining to the six classes defined by the functions. For convenience we introduce six further functions

$$\alpha_r(n) = n^r\,\zeta_{-r}'(n); \qquad \lambda_r(n) = [1 + 2(-1)^n]\,\zeta_r(n);$$

$$\mu_r(n) = \zeta_r(n) + \zeta_r'(n);$$

$$\beta_r(n) = 4\xi_r'(n) - \xi_r(n); \qquad \nu_r(n) = \zeta_r''(n) - (-1)^n\,\zeta_r'(n);$$

$$\rho_r(n) = \zeta_r'(n) - \zeta_r''(n);$$

and as before write $\alpha_0(n),\ \cdots,\ \rho_0(n) \equiv \alpha(n),\ \cdots,\ \rho(n)$. Of these, $\alpha_r(n)$ is seen to be the sum of the rth powers of all those divisors of n whose conjugates are odd. The equations for the rest express frequently occurring functions of the divisors which it is unnecessary at this point to define verbally. All twelve functions $\zeta_r, \cdots, \rho_r$ will be recognized as those which first present themselves in the principal theorems concerning representations of n as a sum of 2, 4, 6 or 8 squares; and in the simplest applications of the paraphrases, such as those arising from $f(x|) = 1, x^2, x^4, \cdots, g(|x) = x, x^3, x^5, \cdots$ for all values of x, they reappear in many investigations, including those for 3, 5, 7, 9, 11 or 13 squares.* In reductions of formulas the following immediate consequences of their definitions are most frequently useful.

* The treatment of these odd numbers of squares is given in the American Journal of Mathematics, vol. 42 (1920), pp. 168–188.

$$(1)\qquad \begin{aligned} &m \equiv -1 \bmod 4, \quad \xi(m) = 0;\\ &m \equiv \pm 1 \bmod 4, \quad \xi'_r(m) = (-1)^{(m-1)/2}\xi_r(m); \end{aligned}$$

$\xi''_r(m) = 0$ or $2\xi_r(m)$ according as $m \equiv +1$ or $-1 \bmod 4$.

For $n = 2^\alpha m$, $m = d\delta$, $\alpha \geqq 0$; (2)–(7)

$$(2)\qquad \begin{aligned} &\zeta'_r(n) + \zeta''_r(n) = \zeta_r(n) = [2^{(\alpha+1)r} - 1]\zeta_r(m)/(2^r - 1);\\ &\zeta'_r(n) = \zeta_r(m); \end{aligned}$$

$$(3)\qquad (2^r - 1)\zeta''_r(n) = 2^r(2^{\alpha r} - 1)\zeta'_r(n);$$

$$(4)\qquad 2\sum 2^\alpha d = 2^{\alpha+1}\zeta'_1(n) = \zeta_1(n) + \zeta'_1(n) = \mu_1(n);$$

$$(5)\qquad \zeta'_1(2n) = \zeta'_1(n); \quad \zeta''_1(2n) = 2\zeta_1(n) = \tfrac{1}{2}\zeta''_1(2n) - \zeta'_1(2n);$$

$$(6)\qquad \zeta_1(n) + \zeta''_1(n) = \zeta''_1(2n) - \zeta'_1(2n) = -\rho_1(2n);$$

$$(7)\qquad \begin{aligned} &\sum(-1)^{(d+\delta)/2} d = (-1)^{(m+1)/2}\zeta_1(m);\\ &\sum(-1)^{(d-1)/2} = \xi(n) = \xi(m); \end{aligned}$$

and for $n = d\delta$,

$$(8)\qquad \sum(-1)^{d+\delta} d = -\lambda_1(n); \quad \sum(-1)^{d+\delta} = \zeta''(n) - \lambda(n).$$

To emphasize once more the notation, which will be followed in all subsequent lists, the Σ in (4), (7), (8) refers to all divisors d, δ of the indicated types, here T_2 for (4), (7), and T_3 for (8).

4. *Theta series and constants*; $\vartheta_\alpha(x) \equiv \vartheta_\alpha(x, q)$, $\vartheta_\alpha \equiv \vartheta_\alpha(0)$.

$$(9)\qquad \vartheta_0(x) = 1 + 2\sum(-1)^n q^{n^2}\cos 2nx, \qquad \vartheta_0 = 1 + 2\sum(-1)^n q^{n^2};$$

$$(10)\qquad \vartheta_1(x) = 2\sum(-1)^{(m-1)/2} q^{m^2/4}\sin mx, \qquad \begin{aligned}\vartheta'_1 &= \vartheta_0\vartheta_2\vartheta_3\\ &= 2\sum(-1)^{(m-1)/2} m q^{m^2/4};\end{aligned}$$

$$(11)\qquad \vartheta_2(x) = 2\sum q^{m^2/4}\cos mx, \qquad \vartheta_2 = 2\sum q^{m^2/4};$$

$$(12)\qquad \vartheta_3(x) = 1 + 2\sum q^{n^2}\cos 2nx, \qquad \vartheta_3 = 1 + 2\sum q^{n^2}.$$

With but a few exceptions which can be derived from the others by means of the transformation of the second order, all of the series for $\vartheta_\alpha^a\vartheta_\beta^b$ in which $(\alpha, \beta) = (2, 3)$, $(0, 3)$, $(0, 2)$ and $(a, b) = (1, 1)$, $(2, 2)$, $(2, 4)$, $(4, 2)$, $(3, 3)$, $(4, 4)$, $(0, 2)$, $(0, 4)$, $(0, 6)$, $(0, 8)$, can be simply found from the series for the k, k', K constants and their powers as given by Jacobi in §§ 40–42 of the *Fundamenta Nova*. For the rapid and systematic use of the method of paraphrase the series for all of these constants will be found indispensable. The coefficients of all are expressible directly in terms of the numerical functions defined in § 3. Here however we need give only the following selection:

$$\vartheta_2\,\vartheta_3 = 2\sum q^{m/4}\,\xi(m); \qquad \vartheta_0\,\vartheta_3 = 1 + 4\sum q^{2n}(-1)^n\,\xi(n);$$
$$\vartheta_0\,\vartheta_2 = 2\sum q^{m/4}(-1)^{(m-1)/4}\,\xi(m); \tag{13}$$

$$\vartheta_0^2 = 1 + 4\sum q^n(-1)^n\,\xi(n); \qquad \vartheta_2^2 = 4\sum q^{m/2}\,\xi(m);$$
$$\vartheta_3^2 = 1 + 4\sum q^n\,\xi(n); \tag{14}$$

and the following essential constants in reduced form from Jacobi, *l. c.*, § 41:

$$A = 4\sum q^n\,\mu_1(n); \qquad B = -4\sum q^n(-1)^n\,\mu_1(n);$$
$$C = 1 + 8\sum q^{2n}\,\rho_1(n); \tag{15}$$

$$B + C = 1 - 8\sum q^n(-1)^n\,\nu_1(n); \qquad A - B = 16\sum q^{2n}\,\mu_1(n);$$
$$C - A = 1 - 8\sum q^n\,\nu_1(n), \tag{16}$$

of which C is $4KE^1/\pi^2$, and all are required in the derivations of the series in § 6.

5. *Eighteen doubly periodic theta quotients.** We shall give explicitly only those of the eighteen which can not be derived from others by changes of q into $-q$ and x into $x + \pi/2$, or by the transformation of the second order.

$$\phi_1(x) = \vartheta_2\,\vartheta_3\,\vartheta_1(x)/\vartheta_0(x) = 4\sum q^{m/2}(\sum \sin dx),\ [T_1]; \tag{17}$$

$$\phi_2(x) = \vartheta_2\,\vartheta_3\,\vartheta_0(x)/\vartheta_1(x)$$
$$= \csc x + 4\sum q^n[\sum(-1)^{(d-1)/2}\cos dx],\ [T_2]; \tag{18}$$

$$\phi_9(x) = \vartheta_0\,\vartheta_3\,\vartheta_3(x)/\vartheta_0(x)$$
$$= 1 + 4\sum q^n[\sum(-1)^{(\delta-1)/2}\cos 2^{\alpha+1}dx],\ [T_2]; \tag{19}$$

$$\phi_{11}(x) = \vartheta_0\,\vartheta_3\,\vartheta_2(x)/\vartheta_1(x)$$
$$= \cot x + 4\sum q^{2n}[\sum(-1)^{\delta}\sin 2dx],\ [T_3]; \tag{20}$$

in which, *as always henceforth*, the $[T_\alpha]$ indicates the type of division for the m or n in the exponent. Note particularly that when the exponent is cn, c being a numerical constant, the type refers to the divisors of n, and not of cn, and so in all similar cases.

It will be sufficient here to indicate how the remaining fourteen functions in Hermite's list (which includes all of the doubly periodic functions considered by Jacobi, and six others), may be derived from these. In (21) the first

* Calculated (with corrections) from the equivalent set given by Hermite, *Sur les Théorèmes de M. Kronecker, etc., Oeuvres*, vol. 1, p. 243, or Journal de Mathématiques pures et appliquées, (2), vol. 9 (1864), p. 145. The suffixes correspond to the order of the functions in Hermite's list. For the connection of these with the doubly periodic functions of the second kind (§ 10), cf. Messenger of Mathematics, vol. 49 (1919–20), p. 81.

member in each triad of functions is transformed into the second by the substitution $x \sim x + \pi/2$, and into the third by $q \sim -q$; $i \equiv \sqrt{-1}$. Thus, carrying out the indicated transformations we find from (17) and the first triad the developments of

$$\phi_3(x) = \vartheta_2\,\vartheta_3\,\vartheta_2(x)/\vartheta_3(x)$$

upon changing x into $x + \pi/2$ in (17), and

$$\phi_5(x) = \vartheta_0\,\vartheta_2\,\vartheta_1(x)/\vartheta_3(x)$$

on replacing q by $-q$ in (17) and reducing. Similarly for all fourteen.

$$\begin{array}{llll} (\phi_1, \phi_3, i\phi_5), & (\phi_3, -\phi_1, i\phi_7), & (\phi_7, -\phi_5, i\phi_3); & \\ (\phi_2, \phi_4, \phi_6), & (\phi_4, -\phi_2, \phi_8), & (\phi_6, \phi_8, \phi_2), & (\phi_8, -\phi_6, \phi_4); \\ (\phi_9, \phi_{10}, \phi_{10}), & (\phi_{10}, \phi_9, \phi_9), & (\phi_{11}, -\phi_{12}, \phi_{11}), & (\phi_{12}, -\phi_{11}, \phi_{12}); \\ (\phi_{13}, -\phi_{14}, \phi_{15}), & (\phi_{14}, -\phi_{13}, \phi_{16}), & (\phi_{15}, -\phi_{16}, \phi_{13}), & (\phi_{16}, -\phi_{15}, \phi_{14}); \\ (\phi_{17}, -\phi_{17}, -\phi_{17}), & (\phi_{18}, -\phi_{18}, \phi_{18}); & & \end{array} \tag{21}$$

with which we need for the last two,

$$\phi_{17}(x, q) = 2\phi_1(2x, q^2), \qquad \phi_{18}(x, q) = 2\phi_2(2x, q^2), \tag{22}$$

which, upon reducing the functions on the right by the transformation of the second order to functions of x, q, yield Hermite's forms. It may be noted that the set yields no further quotients of the same kind under any of these transformations.

The derivation of the expansions (17)–(20) is sufficiently evident from Hermite's detailed similar calculation in the supplement to Lacroix' *Calculus*, reproduced in *Oeuvres*, vol. 1, pp. 219–220. Hermite's remarks regarding these fundamental developments are so apposite in the present connection that we quote them. "Il est impossible de ne pas être frappé du caractère arithmétique de ces expressions (Σ sin dx, etc.); elles offrent un exemple des fonctions numériques qui ont été le sujet des belles recherches de M. Liouville, et la manière simple dont elles sont amenées par la théorie des fonctions elliptiques peut aisément faire présumer le rôle de cette théorie dans l'étude des propriétés des nombres." It was from precisely this observation that the (probably) true origin of Liouville's general formulas first became evident: in fact his first is equivalent to the identity $[\phi_1(x)]^2 = \phi_1(x) \times \phi_1(x)$.. Others of his formulas (of which we shall omit discussion) follow from equally simple identities, such as

$$\phi_1(x)\phi_2(x) = \phi_3(x)\phi_4(x) = \vartheta_2^2\,\vartheta_3^2; \qquad \phi_{15}(x)\phi_{16}(x) = \vartheta_3^4;$$

$\phi_1' = \phi_7\,\phi_9$, where ϕ_1' denotes the x-derivative* of $\phi_1(x)$.

* For the complete set of these in reduced form, cf. Messenger of Matnematics, vol. 47 (1917–18), p. 55, where are also given the calculations for § 6.

6. *Squares of the eighteen* $\phi_j(x)$. All may be inferred from the following by means of (21), (22).

(23) $\phi_1^2(x) = 4\sum q^n \mu_1(n) - 8\sum q^n(\sum 2^a d \cos 2^{a+1} dx)$, $[T_2]$;

(24) $\phi_2^2(x) = 4\sum q^n \mu_1(n) + \csc^2 x - 8\sum q^{2n}(\sum d \cos 2dx)$, $[T_3]$;

(25) $\phi_9^2(x) = 1 + 8\sum q^{2n} \rho_1(n) + 8\sum q^n(\sum 2^a d \cos 2^{a+1} dx)$, $[T_2]$;

(26) $\phi_{11}^2(x) = -8\sum q^{2n} \rho_1(n) + \cot^2 x - 8\sum q^{2n}(\sum d \cos 2dx)$, $[T_3]$.

For the verification of these, (15), (16) will be found necessary.

7. In explanation of the unusual forms of these expansions which, if considered only for their use in the customary applications of elliptic functions are of slight significance, it will be well to state here what are the desirable characteristics, from the standpoint of paraphrases, for such series and identities between them to possess. Consider for example

$$2\vartheta_0^4(x)\phi_1(x)\phi_7(x)\phi_9(x) = \vartheta_1'^3\,\vartheta_1(2x),$$

which is easily seen to be true. On the left we have the product of seven known series, viz., $\vartheta_0(x)$ counted four times, by $\phi_1(x)$, $\phi_7(x)$, $\phi_9(x)$; while on the right we have the product of only four, or, if the transformation of the third order be used to give the series for $\vartheta_1'^3$, only two. Or, the left may clearly be read as the product of six in three ways, each of $\phi_1\phi_7$, $\phi_7\phi_9$, $\phi_9\phi_1$ being known from the derivative $\phi_1'(x)$. Hence, reading the identity in the first way, we shall get a paraphrase connecting L-functions of parity $(0|1)$ integrated over two separations of degrees 7 and 4 respectively, or 7 and 2; in the other ways the degrees are 6, 4 (three times) and 5, 4. Therefore since in these the separations of degree 4 or 2 are the same, we have a syzygy between integrations over separations of degrees 4 (or 2), 5, 6, 7. Thus it follows that these arithmetical facts connected with the separations of degrees 7, 6, 5 (which relate respectively to representations in quadratic forms containing 7, 6, 5 indeterminates), may be reduced to others concerning separations of degrees 4 or 2 (which relate to representations in quaternary or binary quadratic forms), and the theorems corresponding to the specializations of the L-functions, similarly reduced. This course has obviously been followed, for simpler identities, by Liouville in his 17th and 18th memoirs; also elsewhere. Our object then is to find, where possible, simple expansions for fairly complicated functions, in order to reduce complex arithmetical relations to others which are simpler. The greater the complexity of the function which is reduced to a simpler product of known series, and itself expanded in a power series in q, the greater will be the variety and interest of the paraphrases. The most desirable case is the reduction of a product of many theta factors to a single series whose coefficients are given as explicit functions of the real divisors of the exponents of q.*

8. All possible cases of the paraphrase of identities involving tangents, cotangents, secants or cosecants are covered by the sixteen formulas following. They may be verified by inspection on multiplying throughout by $\sin x$ in the cases of the cotangents and cosecants, by $\cos x$ in the others. The suffix in $u_0(x), \cdots, w_7(x)$ is even or odd according as the integer in ( ) is even or odd; and where necessary to indicate both variables, the functions will be written $u_0(2n, x), \cdots, w_7(2n, x)$.

$$(27)\quad \begin{aligned}
u_0(2n) &\equiv \sin 2nx \csc x = 2\sum_{r=1}^{n} \cos(2r-1)x,\\
u_1(m) &\equiv \sin mx \csc x = 1 + 2\sum_{r=1}^{(m-1)/2} \cos 2rx,\\
u_2(2n) &\equiv \sin 2nx \sec x = 2(-1)^n \sum_{r=1}^{n} (-1)^r \sin(2r-1)x,\\
u_3(m) &\equiv \sin mx \sec x = (-1)^{(m-1)/2}\left[\tan x + 2\sum_{r=1}^{(m-1)/2}(-1)^r \sin 2rx\right].
\end{aligned}$$

$$(28)\quad \begin{aligned}
v_0(2n) &\equiv \cos 2nx \csc x = \csc x - 2\sum_{r=1}^{n} \sin(2r-1)x,\\
v_1(m) &\equiv \cos mx \csc x = \cot x - 2\sum_{r=1}^{(m-1)/2} \sin 2rx,\\
v_2(2n) &\equiv \cos 2nx \sec x\\
&= (-1)^n\left[\sec x + 2\sum_{r=1}^{n}(-1)^r \cos(2r-1)x\right],\\
v_3(m) &\equiv \cos mx \sec x = (-1)^{(m-1)/2}\left[1 + 2\sum_{r=1}^{(m-1)/2}(-1)^r \cos 2rx\right].
\end{aligned}$$

$$(29)\quad \begin{aligned}
w_0(2n) &\equiv \sin 2nx \cot x = 1 + \cos 2nx + 2\sum_{r=1}^{n-1} \cos 2rx,\\
w_1(m) &\equiv \sin mx \cot x = \cos mx + 2\sum_{r=1}^{(m-1)/2} \cos(2r-1)x,\\
w_2(2n) &\equiv \sin 2nx \tan x\\
&= (-1)^{n-1}\left[1 + (-1)^n \cos 2nx + 2\sum_{r=1}^{n-1}(-1)^r \cos 2rx\right],\\
w_3(m) &\equiv \sin mx \tan x\\
&= (-1)^{(m-1)/2}\Bigg[\sec x + (-1)^{(m+1)/2}\cos mx\\
&\qquad + 2\sum_{r=1}^{(m-1)/2}(-1)^r \cos(2r-1)x\Bigg].
\end{aligned}$$

* In particular much light would be thrown on the arithmetic of quadratic forms in n indeterminates by the corresponding developments of

$$\vartheta_\alpha(x_1 + x_2 + \cdots + x_n)/\vartheta_\beta(x_1)\vartheta_\gamma(x_2)\cdots\vartheta_\delta(x_n),$$

$$w_4(2n) \equiv \cos 2nx \cot x = \cot x - \sin 2nx - 2\sum_{r=1}^{n-1} \sin 2rx,$$

$$w_5(m) \equiv \cos mx \cot x = \csc x - \sin mx - 2\sum_{r=1}^{(m-1)/2} \sin(2r-1)x,$$

(30)
$$w_6(2n) \equiv \cos 2nx \tan x = (-1)^n\left[\tan x + (-1)^n \sin 2nx + 2\sum_{r=1}^{n-1}(-1)^r \sin 2rx\right],$$

$$w_7(m) \equiv \cos mx \tan x = \sin mx - 2(-1)^{(m-1)/2}\sum_{r=1}^{(m-1)/2}(-1)^r \sin(2r-1)x.$$

(28)–(30) are connected by many simple and interesting group relations which, as they lie off the main course of this paper, we omit. We may notice, however, a theorem of which the proof presents no difficulty, and which often either gives additional paraphrases or affords a check on the reductions of theta series. Let $h(x)$ denote any one of $\sec x$, $\csc x$, $\tan x$, $\cot x$, and A, B quantities independent of x. Suppose that over some separation the following is an identity in x,

$$Ah(x) + B + \sum a_i t(n_i x) = 0,$$

in which t represents either sin or cos. Then this implies

$$A = 0, \qquad B + \sum a_i t(n_i x) = 0.$$

9. We frequently meet expressions involving secants, etc., of several variables to be paraphrased. An example will make clear the procedure in all such cases. Writing $z = x + y$, we have

$$\sin(mx + 2ny)\csc(x+y) = \sin\{(m-2n)x + 2nz\}\csc z$$
$$= v_0(2n, z)\sin(m-2n)x + u_0(2n, z)\cos(m-2n)x,$$

and each term is in a form suitable for paraphrase.

10. *Doubly periodic functions of the second kind.** From the standpoint of paraphrase, these functions are of the highest importance. They not only

where $\alpha, \beta, \gamma, \cdots, \delta$, are any of the numbers $0, 1, 2, 3$. The case $n = 2$ is considered below, § 11. Note that, as pointed out by Glaisher (Messenger of Mathematics, vol. 14 (1884–85), p. 162), these, the arithmetical developments, are unique, while the analytical representations are not.

* The nomenclature is that of Hermite. The doubly periodic functions of the third kind also are of great use in this subject. They are particularly valuable in the derivation of new and generalized class number relations. As the forms of these functions due to Hermite, Biehler, Appel and others can be changed to their arithmetical forms only after many reductions, we shall give the appropriate developments elsewhere.

include the doubly periodic functions as limiting cases* in one direction and give rise to a great variety of doubly periodic functions of the third kind in another, but they also afford us the first and most essential link connecting paraphrases in which the order of the functions is unity with those in which the order exceeds unity. It is of interest to note that equivalents of the series given presently can easily be derived from Jacobi's investigations on the rotation of a rigid body and were, therefore, most probably familiar to Liouville.

Consider the function

$$\phi_{\alpha\beta\gamma}(x_1, x_2) = \vartheta_1' \vartheta_\alpha(x_1 + x_2)/\vartheta_\beta(x_1)\vartheta_\gamma(x_2), \equiv \phi_{\alpha\beta\gamma};$$

and denote by $S_i\phi_{\alpha\beta\gamma}$, $P_j\phi_{\alpha\beta\gamma}$ $(i, j = 1, 2)$ the results of replacing x_i, x_j by $x_i + \pi/2$, $x_j + \pi\tau/2$ respectively, where τ has the usual meaning in terms of the half-periods; and let $\cdots P_i P_j S_i P_j$ denote the application of the substitutions P_j, S_i, P_j, P_i, $\cdots$ in the order last written. Construct the substitutions (which do not form a group) σ_i $(i = 1, 2, \cdots, 16)$:

$$\begin{array}{llll} \sigma_1 = 1, & \sigma_2 = S_2, & \sigma_3 = S_1 S_2, & \sigma_{11} = S_1; \\ \sigma_4 = P_2, & \sigma_5 = S_2 P_2, & \sigma_6 = S_1 P_2, & \sigma_7 = S_2 S_1 P_2; \\ \sigma_{12} = P_1, & \sigma_{13} = S_1 P_1, & \sigma_{14} = S_2 P_1, & \sigma_{15} = S_2 S_1 P_1; \\ \sigma_8 = P_1 P_2, & \sigma_9 = S_2 P_1 P_2, & \sigma_{10} = S_2 S_1 P_1 P_2, & \sigma_{16} = S_1 P_1 P_2; \end{array}$$

and apply them to ϕ_{100}, giving:†

$$\begin{array}{llll} \phi_1 = \phi_{100}, & \phi_2 = \phi_{203}, & \phi_3 = \phi_{133}, & \phi_{11} = \phi_{230}; \\ \phi_4 = \phi_{001}, & \phi_5 = \phi_{302}, & \phi_6 = \phi_{331}, & \phi_7 = \phi_{032}; \\ \phi_{12} = \phi_{010}, & \phi_{13} = \phi_{320}, & \phi_{14} = \phi_{313}, & \phi_{15} = \phi_{023}; \\ \phi_8 = \phi_{111}, & \phi_9 = \phi_{212}, & \phi_{10} = \phi_{122}, & \phi_{16} = \phi_{221}. \end{array}$$

It may be verified that, disregarding signs, these sixteen functions are all those that can be generated from any one of them, ϕ_k, by successive applications of S_1, S_2, P_1, P_2. Hence from the series for ϕ_k may be written down those for the remaining fifteen, and no others, by this process of transformation. The paraphrase interpretation of S_1, S_2 has been considered in Part I, § 30; that of P_1, P_2 is by no means so obvious, and need not detain us here.

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* Messenger of Mathematics, vol. 49 (1919–20), p. 81.

† The factor ± 1 being immaterial for our purpose, it is ignored. The sign is $-$ for $j = 3$, 10, and for the rest $+$. In all there are 64 possible functions $\phi_{\alpha\beta\gamma}$; the remaining 48 need not concern us here. There is an obvious advantage in deriving the functions from one of them, ϕ_{100} (any other of the 16 might have been selected), instead of from two distinct fundamental series as done by Hermite.

11. *Series for the functions of the second kind.** Applying the substitutions σ to ϕ_{100}, writing $\phi_{\alpha\beta\gamma} \equiv \phi_{\alpha\beta\gamma}(x, y)$, $x', y' = x + \pi/2, y + \pi/2$, and developing in powers of q, we find after all reductions the following four in which the type is T_1,

(31) $\phi_1(x, y) \equiv \phi_{100} = 4\sum q^{m/2}[\sum \sin(dx + \delta y)]$,

(32) $\phi_2(x, y) \equiv \phi_{203} = \phi_1(x, y') = 4\sum q^{m/2}[\sum(-1)^{(\delta-1)/2}\cos(dx + \delta y)]$,

(33) $$\begin{aligned}\phi_3(x, y) \equiv \phi_{133} &= -\phi_1(x', y')\\ &= 4\sum q^{m/2}[(-1)^{(m-1)/2}\sum \sin(dx + \delta y)],\end{aligned}$$

(34) $\phi_{11}(x, y) \equiv \phi_{230}(x, y) = \phi_2(y, x)$;

eight in which the type is T_2,

(35) $\phi_4(x, y) \equiv \phi_{001} = \csc y + 4\sum q^n[\sum \sin(2^{\alpha+1} dx + \delta y)]$,

(36) $$\begin{aligned}\phi_5(x, y) \equiv \phi_{302} &= \phi_4(x, y')\\ &= \sec y + 4\sum q^n[\sum(-1)^{(\delta-1)/2}\cos(2^{\alpha+1} dx + \delta y)],\end{aligned}$$

(37) $$\begin{aligned}\phi_6(x, y) \equiv \phi_{331} &= \phi_4(x', y)\\ &= \csc y + 4\sum q^n[(-1)^n \sum \sin(2^{\alpha+1} dx + \delta y)],\end{aligned}$$

(38) $$\begin{aligned}\phi_7(x, y) \equiv \phi_{032} &= \phi_4(x', y')\\ &= \sec y + 4\sum q^n[(-1)^n \sum(-1)^{(\delta-1)/2}\cos(2^{\alpha+1} dx + \delta y)],\end{aligned}$$

(39) $\phi_{12}(x, y) \equiv \phi_{010} = \phi_4(y, x)$,

(40) $\phi_{13}(x, y) \equiv \phi_{320} = \phi_5(y, x)$.

(41) $\phi_{14}(x, y) \equiv \phi_{313} = \phi_6(y, x)$,

(42) $\phi_{15}(x, y) \equiv \phi_{023} = \phi_7(y, x)$;

and four in which the type is T_3,

* This list being of such importance it was calculated and checked in several ways, to eliminate printers' errors prevalent in other forms in the literature. It was calculated: (i) by Hermite's method, *Sur quelques applications des fonctions elliptiques* (Comptes Rendus, vol. 85 (1877), ··· 94 (1882), *Oeuvres*, vol. 3, p. 267; (ii) by applying the σ_i to Halphen's form of $\phi_8(x, y)$ (*Traité*, vol. 1, p. 418 (15)), and comparing with the same for $\phi_1(x, y)$; (iii) by carrying out in detail the calculations in Hermite's paper *Sur une application de la théorie des fonctions doublement périodiques de seconde espèce*, (Annales de l'École Normale Supérieure, (3), vol. 2 (1885), p. 303, reprinted with corrections in *Oeuvres*, vol. 4, p. 190). Finally it was compared with Hermite's final list, *Oeuvres*, vol. 4, pp. 199–200, which still contains an error (in the expansion of $\phi_{10}(x, y)$, as may be verified upon putting $x, y = 0, \pi/2$ and comparing with the series for the elliptic functions). Equivalent forms for certain members of this list quoted by other writers with an indefinite reference to Kronecker, are unreliable, and should be recalculated before use.

(43) $\phi_8(x, y) \equiv \phi_{111} = \cot x + \cot y + 4\sum q^{2n}[\sum \sin 2(dx + \delta y)]$,

(44) $$\phi_9(x, y) \equiv \phi_{212} = \phi_8(x, y') = \cot x - \tan y + 4\sum q^{2n}[\sum(-1)^\delta \sin 2(dx + \delta y)],$$

(45) $$\phi_{10}(x, y) \equiv \phi_{122} = -\phi_8(x', y') = \tan x + \tan y - 4\sum q^{2n}[\sum(-1)^{d+\delta} \sin 2(dx + \delta y)],$$

(46) $\phi_{16}(x, y) \equiv \phi_{221} = \phi_9(y, x)$.

V. The method of paraphrase, simple illustrations *

12. As already remarked, lists of formulas in the d, δ form, such as the foregoing, are analogous in paraphrasing to tables of logarithms in common arithmetic. It will be evident that by combining the series to form identities there is implicit in the lists given an infinity of paraphrases such as those exemplified in Part I; and as a systematic derivation of all the most obvious paraphrases is out of the question in a paper of this length, we shall limit the illustrations to a few only of the paraphrases lying on the surface, choosing the examples partly for their own interest, and partly to show one or two of the simpler methods for using such lists of developments. For this purpose we may select L-functions of degrees $1, 2$, and confine our attention principally to linear separations, the other cases being treated with equal facility.

13. By the theory developed in Part I, trigonometric products are always to be written in their equivalent sum forms before proceeding to paraphrase. We shall accordingly write trigonometric identities derived from elliptic in the latter form at once, omitting the intermediate product forms, unless the separation into sums is not obvious.

As a first example, consider the eighteen identities of the form

$$\phi_j^2(x) = \phi_j(x) \times \phi_j(x).$$

From (17), (23),

$$\phi_1(x, q^2) \times \phi_1(x, q^2) = \phi_1^2(x, q^2)$$

is equivalent to

$$16\sum q^m(\sum \sin dx) \times \sum q^m(\sum \sin dx) = 4\sum q^{2n}\mu_1(n) - 8\sum q^{2n}[2^\alpha \sum d \cos 2^{\alpha+1} dx],$$

where, *as in all such cases*, the types of division being defined in the lists from which the series are transcribed, need not be written; here they are T_1 on the

* At this point it will be advantageous to glance through the Introduction to Part I, attending particularly to the bar notation for L-functions and the notation for separations there explained, also to the illustrative examples of paraphrases, as we shall not again refer to any of the notation. See also the references at the end of Part I. We shall cite Liouville's theorems by giving the numbers of the memoir, page and formula, thus: Liouville (1), 144, (A).

left, T_2 on the right. On referring to § 3 (4) for the d-form of $\mu_1(n)$, and equating coefficients of q^{2n}, we get

$$2n = m' + m''; \qquad n = 2^\alpha m; \qquad m, m', m'' = d\delta, d'\delta', d''\delta'':$$
$$\sum [\cos(d' - d'')x - \cos(d' + d'')x] = 2^\alpha \sum d[1 - \cos 2^{\alpha+1} dx];$$

and this, on writing $f(x|) \equiv f(x)$, paraphrases into

$$\text{(I)} \quad \begin{aligned} &2n = m' + m''; \qquad n = 2^\alpha m; \qquad m, m', m'' = d\delta, d'\delta', d''\delta'': \\ &\sum [f(d' - d'') - f(d' + d'')] = 2^\alpha \sum d[f(0) - f(2^{\alpha+1} d)]; \end{aligned}$$

which is Liouville's (2), 194, (a), and for $\alpha = 0$ his first formula, (1), 144, (A). It is of course not necessary in any such case as this to replace the numerical functions $\mu_1(n)$, etc., by their d-forms; but doing so increases the symmetry of the paraphrases. For simplicity in writing we shall *henceforth put*

$$f(x|) \equiv f(x).$$

In (I) each argument is even. Hence $f(x)$ may be replaced by $f(x/2)$, giving

$$\sum \left[f\left(\frac{d' - d''}{2}\right) - f\left(\frac{d' + d''}{2}\right)\right] = 2^\alpha \sum d[f(0) - f(2^\alpha d)].$$

In this case no material simplification is thus effected. But when the contrary is the case we shall make the change without notice. A similar transformation of a paraphrase involving odd arguments is not permissible, since, by the definition of an L-function, $f(m/2)$ does not necessarily exist.

14. The use of (27)–(30) will be sufficiently clear from the following, which we give in some detail as it illustrates transformations of several types which occur frequently. Any identity involving csc, tan or cot might be chosen; we take $\phi_2(x) \times \phi_2(x) = \phi_2^2(x)$, and use (18), (24), getting

$$[\csc x + 4\sum q^n (\sum \sin dx)]^2 = 4\sum q^n \mu_1(n) + \csc^2 x - 8\sum q^{2n} (\sum d \cos 2dx).$$

Equating coefficients of q^n, where $n = 2^\alpha m$, $\alpha > 0$, we have for the separations $n/2 = d_1 \delta_1$, and

$$n = n' + n''; \qquad n, n', n'' = 2^\alpha m, 2^{\alpha'} m', 2^{\alpha''} m'';$$
$$m, m', m'' = d\delta, d'\delta', d''\delta'',$$

the following

$$\sum \sin dx \csc x + \sum [\cos(d' - d'')x - \cos(d' + d'')x]$$
$$= \tfrac{1}{2}\mu_1(n) - \sum d_1 \cos 2d_1 x.$$

Since d is odd, (27) gives for $\Sigma \sin dx \csc x$ the value $\Sigma u_1(d, x)$, or

$$\sum\left[1+2\sum_{r=1}^{(d-1)/2}\cos 2rx\right],$$

$$\equiv \zeta'(n)+2\sum[\cos 2x+\cos 4x+\cos 6x+\cdots+\cos(d-1)x],$$

where we have taken ζ' from § 3. We may eliminate the separation $n/2 = d_1\delta_1$, by the following obvious identity

$$\sum d_1\cos 2d_1 x \equiv \sum[d\cos 2dx+2d\cos 2^2 dx+\cdots+2^{\alpha-1}d\cos 2^\alpha dx],$$

which results upon segregating the odd divisors d and (when $\alpha > 1$) the even divisors $2^\beta d\,(0<\beta \overline{\gtreqless}\, \alpha-1)$ among all the divisors d_1 of $2^{\alpha-1}m$. Hence, substituting the sums thus found for the respective terms of the original sin, cos, csc identity, and paraphrasing, we have

$$n=2^\alpha m=2^{\alpha'}m'+2^{\alpha''}m''; \qquad \alpha>0; \qquad m,m',m''=d\delta,d'\delta',d''\delta'':$$

$$\text{(II)}\quad \begin{aligned}\sum[f(d'-d'')-f(d'+d'')]&=[\tfrac{1}{2}\mu_1(n)-\zeta'(n)]f(0)\\&\quad-2\sum[f(2)+f(4)+f(6)+\cdots+f(d-1)]\\&\quad-\sum d[f(2d)+2f(2^2d)+\cdots+2^{\alpha-1}f(2^\alpha d)].\end{aligned}$$

The subcases of all paraphrases such as (II) in which $\alpha=0$ (although this does not directly come under (II), α being >0 therein, the paraphrases are closely related, both being consequences of the $\phi_2(x)$ identity), are of great importance in connection with the representations of primes p in the form $ax^2+br^cy^2$, where a, b are constants and r is prime, in that the specialized forms of these subcases for $f(x)=1, x^2, \cdots$ give rise to identities of the forms considered by Bouniakowsky and Liouville as the point of departure for determining the number of such representations. In the present instance, the sum $\Sigma q^{2n}(\Sigma d_1\cos 2d_1x)$ can contribute nothing to the coefficient of q^m; hence the second Σ on the right of (II) is absent. Again, for $\alpha=0$, we have $m=2^{\alpha'}m'+2^{\alpha''}m''$. Hence one and only one of $\alpha',\alpha''=0$; and therefore the value of $\Sigma[f(d'-d'')-f(d'+d'')]$ over the separation $n=2^{\alpha'}m'+2^{\alpha''}m''$ is the sum of its values over the (identical) separations

$$m=m'+2^{\alpha''}m'', \qquad m=2^{\alpha'}m'+m''.$$

Finally then we have, on referring to § 3 (4) for the numerical functions,

$$m=2^{\alpha'}m'+m''; \qquad m,m',m''=d\delta,d'\delta',d''\delta'':$$

$$\text{(III)}\quad \begin{aligned}2\sum[f(d'-d'')-f(d'+d'')]&=[\zeta_1(m)-\zeta(m)]f(0)\\&\quad-2\sum[f(2)+f(4)+\cdots+f(d-1)].\end{aligned}$$

This is Liouville's (3), 201, (D). We have not yet exhausted the obvious consequences of $\phi_2^2=\phi_2\times\phi_2$; any identity involving csc x gives at once two paraphrases, one similarly to (II), and the other by paraphrasing the result of multiplying the identity throughout by $\sin x$ to eliminate $\csc x$. In the present case we get, on equating coefficients of q^n, $n = 2^\alpha m$, $\alpha > 0$, as in proving (II):

$$\begin{aligned}\sum [\sin (1 + d' - d'')x + \sin (1 - d' + d'')x \\ + \sin (-1 + d' + d'')x - \sin (1 + d' + d'')x] \\ = \mu_1(n) \sin x - 2\sum \sin dx \\ - 2\sum \left[\sum_{r=1}^{\alpha} 2^{r-1} d\{\sin (2^r d + 1)x - \sin (2^r d - 1)x\} \right],\end{aligned}$$

which, on writing $g(x) \equiv f(|x)$, gives the paraphrase

$$n = 2^\alpha m = 2^{\alpha'} m' + 2^{\alpha''} m''; \qquad \alpha > 0; \qquad m, m', m'' = d\delta, d'\delta', d''\delta'':$$

$$\text{(IV)} \quad \begin{aligned}\sum [g(1 + d' - d'') + g(1 - d' + d'') + g(-1 + d' + d'') \\ - g(1 + d' + d'')] = \mu_1(n) g(1) - 2\sum g(d) \\ - 2\sum \left[\sum_{r=1}^{\alpha} 2^{r-1} d\{g(2^r d + 1) - g(2^r d - 1)\} \right].\end{aligned}$$

The diversity in form of (II), (IV), is the more striking in that (IV) is merely the paraphrase of the same elliptic identity as that which gives rise to (II) when it is multiplied throughout by $\sin x$. For identities involving $\csc x$ or $\cot x$ we thus get the paraphrases corresponding to (IV) on multiplying first by $\sin x$; for those containing $\sec x$ or $\tan x$ we first multiply throughout by $\cos x$. Similarly to (IV), corresponding to (III), we get,

$$m = 2^{\alpha'} m' + m''; \qquad m, m', m'' = d\delta, d'\delta', d''\delta'':$$

$$\text{(V)} \quad \begin{aligned}\sum [g(1 + d' - d'') + g(1 - d' + d'') + g(-1 + d' + d'') \\ - g(1 + d' + d'')] = \zeta_1(m) g(1) - \Sigma g(d),\end{aligned}$$

which is Liouville (3), 206, (E). He remarks that (III), (V) are ultimately the same thing, which is obvious from their origin. His (F), ibid., p. 208, is a paraphrase (among others) of Jacobi's $Z^2(u)$, *Fundamenta Nova,* § 47, equation (1); also his (4), 242, (G) is from the same source, or it follows from one of the formulas for functions $f(x, y|)$ given below in (XVII).

15. Although their arithmetical consequences are often widely different, at least in appearance, we shall regard paraphrases that may be derived from one another by means of the elementary transformations of Part I (III) as equivalent. We shall now show how the simplest properties of the elliptic or theta functions such as those in (21) are of direct use in finding all the

distinct and equivalent paraphrases implicit in a complete set of identities of a given kind, here $\phi_j(x) \times \phi_j(x) = \phi_j^2(x)$.

Obviously $\phi_{17}(x) \times \phi_{17}(x) = \phi_{17}^2(x)$ gives nothing distinct from (I), as may be seen on glancing at (22). In the same way we dispose of $j = 13, 14, 18$ when the paraphrase for $j = 11$ is known; and (21), (22) show that for the following pairs of values of j the $\phi_j^2 = \phi_j \times \phi_j$ paraphrases will be identical,

$$(1, 5),\ (2, 6),\ (3, 7),\ (4, 8),\ (9, 10),\ (13, 15),\ (14, 16),$$

since in each case the resulting elliptic identities may be transformed into one another by changing the sign of q. And from the same source it is seen that the paraphrases corresponding to the next pairs will be equivalent in the sense that either in a given pair may be transformed into the other by one of the elementary transformations considered in Part I (III),

$$(1, 3),\ (2, 4),\ (5, 7),\ (6, 8),\ (9, 10),\ (11, 12),\ (13, 14),\ (15, 16).$$

Hence we shall find all the required paraphrases by taking $j = 1, 2, 3, 4, 9, 11, 12$. The cases $j = 1, 2$ give the paraphrases (I)–(V); omitting the alternative forms that correspond to (IV), (V) we get the following in the same way for $j = 3, 4, 9, 11, 12$.

$$2n = m' + m''; \qquad n = 2^\alpha m; \qquad m, m', m'' = d\delta, d'\delta', d''\delta'':$$

$$\text{(VI)} \quad \begin{aligned} &\sum [(-1)^{(d'+d'')/2} \{f(d' - d'') + f(d' + d'')\}] \\ &\qquad = 2^\alpha \sum d[(-1)^n f(2^{\alpha+1} d) - f(0)]; \\ &\sum [(-1)^{(\delta'+\delta'')/2} \{f(d' - d'') + f(d' + d'')\}] \\ &\qquad = -2^\alpha \sum d[f(0) + f(2^{\alpha+1} d)]. \end{aligned}$$

These are identical; the first comes from $j = 3$, the second from $j = 7$. For $j = 4$ we use v_3 from (28), getting

$$n = 2^\alpha m = 2^{\alpha'} m' + 2^{\alpha''} m''; \qquad \alpha > 0;$$

$$m, m', m'' = d\delta, d'\delta', d''\delta'':$$

$$\text{(VII)} \quad \begin{aligned} &\sum [(-1)^{(d'+d'')/2} \{f(d' - d'') + f(d' + d'')\}] \\ &\quad = [\zeta'(n) - \tfrac{1}{2}\mu_1(n)] f(0) \\ &\qquad - \sum [f(2) - f(4) + f(6) - \cdots + (-1)^{(d+1)/2} f(d-1)] \\ &\qquad + \sum d[-f(2d) + 2f(2^2 d) + \cdots + 2^{\alpha-1} f(2^\alpha d)]; \end{aligned}$$

and the related form

$$m = 2^{\alpha'} m' + m''; \qquad m, m', m'' = d\delta, d'\delta', d''\delta'':$$

$$\text{(VIII)} \quad \begin{aligned} 2\sum [(-1)^{(d'+d'')/2}\{f(d'-d'')+f(d'+d'')\}] \\ = [\zeta(m) - \zeta_1(m)]f(0) \\ - 2\sum [f(2) - f(4) + \cdots + (-1)^{(d+1)/2} f(d-1)]. \end{aligned}$$

For $j = 9$ we have

$$2^{\alpha} m = 2^{\alpha'} m' + 2^{\alpha''} m''; \qquad \alpha > 0;$$

$$m, m', m'' = d\delta, d'\delta', d''\delta'':$$

$$\text{(IX)} \quad \begin{aligned} \sum [(-1)^{(\delta'+\delta'')/2}\{f(2^{\alpha'} d' - 2^{\alpha''} d'') + f(2^{\alpha'} d' + 2^{\alpha''} d'')\}] \\ = [\zeta_1''(2^{\alpha-1} m) - \zeta_1'(2^{\alpha-1} m)]f(0) \\ + \sum [(-1)^{(\delta-1)/2} - 2^{\alpha} d] f(2^{\alpha} d). \end{aligned}$$

In this the $f(x)$ which appears upon first paraphrasing has been replaced, as allowable, by $f(x/2)$.

$$m = 2^{\alpha'} m' + m''; \qquad m, m', m'' = d\delta, d'\delta', d''\delta''; \qquad \alpha' > 0:$$

$$\text{(X)} \quad \begin{aligned} 2\sum [(-1)^{(\delta'+\delta'')/2}\{f(2^{\alpha'} d' - d'') + f(2^{\alpha'} d' + d'')\}] \\ = \sum [(-1)^{(\delta-1)/2} - d] f(d). \end{aligned}$$

For $j = 11$, using w_0 from (29), we find

$$n = n' + n''; \qquad n, n', n'' = d\delta, d'\delta', d''\delta'':$$

$$\text{(XI)} \quad \begin{aligned} \sum [(-1)^{\delta'+\delta''}\{f(d'-d'') - f(d'+d'')\}] \\ = [\rho(n) - \rho_1(n)]f(0) - \sum [d + (-1)^{\delta}] f(d) \\ - 2\sum (-1)^{\delta} [f(1) + f(2) + \cdots + f(d-1)]. \end{aligned}$$

The special case in which $n = m$ is of interest:

$$\text{(XII)} \quad \begin{aligned} \sum [(-1)^{\delta'+\delta''}\{f(d'-d'') - f(d'+d'')\}] \\ = [\zeta(m) - \zeta_1(m)]f(0) - \sum (d-1) f(d) \\ + 2\sum [f(1) + f(2) + \cdots + f(d-1)]. \end{aligned}$$

The similarly derived paraphrases of the derivatives $\phi_1' = \phi_7 \phi_9$, etc., and those for the relations between pairs of functions whose products are constants, also yield simple and interesting results, but to keep this paper within reasonable limits we pass on to a very brief consideration of the more important series (31)–(46).

16. Let us first paraphrase one of the obvious identities suggested by the form of $\phi_8(x, y)$ in (43), as the resulting paraphrase is one of those which

Pepin ((1), 84–92) proved by Dirichlet's method. By the addition theorems for the ϑ-functions (Jacobi, *Werke*, vol. 1, p. 510, (C)), we have

$$\vartheta_0^2\,\vartheta_1(x+y)\,\vartheta_1(x-y) = \vartheta_1^2(x)\,\vartheta_0^2(y) - \vartheta_0^2(x)\,\vartheta_1^2(y);$$

and hence by (43), (18), on multiplying this identity throughout by

$$\vartheta_2^2\,\vartheta_3^2/\vartheta_1^2(x)\,\vartheta_1^2(x),$$

we get

$$-\phi_8(x,y)\,\phi_8(x,-y) = \phi_2^2(y) - \phi_2^2(x).$$

Substituting in this the respective series as given by (43), (24), replacing q by $\sqrt{q}$, and equating coefficients of q^n, we find

$$n = n' + n''; \qquad n, n', n'' = d\delta, d'\delta', d''\delta'':$$

$$\sum [\cot x \sin 2dx \cos 2\delta y - \cot y \cos 2dx \sin 2\delta y]$$
$$+ \sum [\cos 2\{(d'-d'')x + (\delta'+\delta'')y\}$$
$$- \cos 2\{(d'+d'')x + (\delta'-\delta'')y\}] = \sum d[\cos 2dy - \cos 2dx].$$

By (29) the first sum*

$$\equiv \sum [\cos 2dy - \cos 2dx] + 2\sum \left[\sum_{r=1}^{d-1} \{\cos 2rx \cos 2\delta y - \cos 2\delta x \cos 2ry\}\right].$$

Considering the second sum (on the left), we have†

$$\sum \sin 2(d'-d'')x \sin 2(\delta'+\delta'')y \equiv 0$$
$$\equiv \sum \sin 2(d'+d'')x \sin 2(\delta'-\delta'')y,$$

the d' being identical in reversed order with the d'', and similarly for the δ', δ''; hence the second sum reduces to

$$\sum [\cos 2(d'-d'')x \cos 2(\delta'+\delta'')y - \cos 2(d'+d'')x \cos 2(\delta'-\delta'')y].$$

Making all these reductions in the original cot, sin, cos identity, replacing then x, y by $x/2$, $y/2$ and paraphrasing, we get

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* d, δ may clearly be interchanged in either term under the Σ:

$$\Sigma[w_0(2d,x)\cos 2\delta y - w_0(2\delta,y)\cos 2dx] \equiv \Sigma[w_0(2d,x)\cos 2\delta y - w_0(2d,y)\cos 2\delta x].$$

We have made this change before writing out the Σ by (29) in the next step; henceforth it will be unnecessary to point out similar transformations.

† The reduction in this step may be obviated by paraphrasing the right of

$$-4\phi_8(x,y)\,\phi_8(x,-y) \equiv [\phi_8(x,y) - \phi_8(x,-y)]^2 - [\phi_8(x,y) + \phi_8(x,-y)]^2,$$

instead of, as above, the left. Such devices sometimes avoid complicated arithmetical reductions.

$$n = n' + n''; \qquad n, n', n'' = d\delta, d'\delta', d''\delta'';$$

$$f(x, y) \equiv f(x, y|):$$

(XIII)
$$\sum [f(d' - d'', \delta' + \delta'') - f(d' + d'', \delta' - \delta'')] = \sum [(d - 1)\{f(0, d) - f(d, 0)\}] + 2\sum \left[\sum_{r=1}^{d-1} \{f(\delta, r) - f(r, \delta)\} \right].$$

Putting $f(x, y) = f(x)$ or $f(y)$ in (XIII), we find at once

$$n = n' + n''; \qquad n, n', n'' = d\delta, d'\delta', d''\delta'':$$

(XIV)
$$\sum [f(d' - d'') - f(d' + d'')] = [\zeta_1(n) - \zeta(n)]f(0) + \sum (2\delta - d - 1)f(d) - 2\sum [f(1) + f(2) + \cdots + f(d - 1)].$$

Henceforth we shall write $f(x, y|) \equiv f(x, y)$.

17. By a simple transformation (XIII), (XIV) take more elegant forms. We remark, however, that although slightly simpler in appearance, the new forms are in reality not so simple, containing redundant terms; we give the transformation merely to show the identity of the forms above with Liouville's.

For $n = d\delta$, let $\sum' [\sum_{r=1}^{d-1} f(\delta, r)]$ denote the result of deleting from $\sum [\sum_{r=1}^{d-1} f(\delta, r)]$ every $f(\delta, r)$ for which r is a divisor of d, and similarly for $\sum' [\sum_{r=1}^{d-1} f(r, \delta)]$, $\sum' [\sum_{r=1}^{d-1} f(r)]$. Then it is easily seen that

$$\sum \left[\sum_{r=1}^{d-1} \{f(\delta, r) - f(r, \delta)\} \right] = \sum{}' \left[\sum_{r=1}^{d-1} f(\delta, r) \right] - \sum{}' \left[\sum_{r=1}^{d-1} f(r, \delta) \right],$$

$$\sum{}' [f(2) + f(3) + \cdots + f(d - 1)] = \sum [f(1) + f(2) + \cdots + f(d - 1)] - \sum [\zeta(\delta) - 1]f(d).$$

Hence (XIII), (XIV) over the same separations may be written,

(XIII′)
$$\sum [f(d' - d'', \delta' + \delta'') - f(d' + d'', \delta' - \delta'')] = \sum (d - 1)[f(0, d) - f(d, 0)] + 2\sum{}' \left[\sum_{r=1}^{d-1} \{f(\delta, r) - f(r, \delta)\} \right];$$

(XIV′)
$$\sum [f(d' - d'') - f(d' + d'')] = [\zeta_1(n) - \zeta(n)]f(0) - \sum [2\zeta(\delta) + d - 2\delta - 1]f(d) - 2\sum{}' [f(2) + f(3) + \cdots + f(d - 1)];$$

which are respectively Liouville's (5), 284, (f) and (4), 247, (H).

18. To generalize a paraphrase that arose from ϕ_j^2, we seek $\phi_k(x, y)$, $\phi_l(x, y)$ such that

$$\pm \phi_k(x, y)\phi_l(x, -y) = \phi_j^2(x) \pm \phi_j^2(y).$$

Thus for $j = 7$, we have the identity (from (31), and the application of the transformations indicated in the first triads of (21) upon (17))

$$\phi_1(x, y)\phi_1(x, -y) = \phi_7^2(x) - \phi_7^2(y),$$

where we have used

$$\vartheta_3^2 \vartheta_1(x + y)\vartheta_1(x - y) = \vartheta_0^2(x)\vartheta_2^2(y) - \vartheta_2^2(x)\vartheta_0^2(y).$$

Paraphrasing the ϕ-identity as in § 16, we find immediately

$$2n = m' + m''; \qquad n = 2^\alpha m; \qquad m, m', m'' = d\delta, d'\delta', d''\delta'':$$

$$\text{(XV)} \quad \sum [f(d' - d'', \delta' + \delta'') - f(d' + d'', \delta' - \delta'')] = 2^\alpha \sum d[f(0, 2^{\alpha+1}d) - f(2^{\alpha+1}d, 0)],$$

which is Liouville (2), 199, (b), (c), and which becomes (I) for

$$f(x, y) = f(x), f(y).$$

Since by (33), $\phi_3(x, y) = -\phi_1(x', y')$, the paraphrase of

$$\phi_3(x, y)\phi_3(x, -y) \equiv \phi_5^2(x) - \phi_5^2(y)$$

is (XV). From (32), (21) we get

$$\phi_2(x, y)\phi_2(x, -y) = \phi_7^2(x) - \phi_5^2(y),$$

whose paraphrase may be written down from (XV) by means of the elementary transformations of Part I, § 30, on observing that $\phi_2(x, y) = \phi_1(x, y')$. Over the same separation as (XV) it is

$$\text{(XVI)} \quad \sum [(-1)^{(\delta'-1)/2+(\delta''-1)/2}\{f(d' + d'', \delta' - \delta'') + f(d' - d'', \delta' + \delta'')\}] = 2^\alpha \sum d[f(2^{\alpha+1}d, 0) - (-1)^n f(0, 2^{\alpha+1}d)].$$

On putting $f(x, y) = f(x)$ in this it becomes (VI), second form; for $f(x, y) = f(y)$ it is the first form of (VI). Continuing thus with the obvious consequences of the developments in § 11, we find from

$$\vartheta_3^2 \vartheta_0(x + y)\vartheta_0(x - y) = \vartheta_0^2(x)\vartheta_3^2(y) + \vartheta_2^2(x)\vartheta_1^2(y),$$

$$-\phi_4(x, y)\phi_4(x, -y) = \phi_6^2(y) + \phi_7^2(x),$$

on using $u_1(\delta, y)$ from (27), the following paraphrase:

$$n = n' + n''; \quad n, n', n'' = 2^{\alpha} m, 2^{\alpha'} m', 2^{\alpha''} m'';$$

$$m, m', m'' = d\delta, d'\delta', d''\delta'':$$

$$\text{(XVII)} \quad \begin{aligned} &\sum [f(2^{\alpha'} d' - 2^{\alpha''} d'', \delta' + \delta'') - f(2^{\alpha'} d' + 2^{\alpha''} d'', \delta' - \delta'')] \\ &= \tfrac{1}{2}\{1 + (-1)^n\} \sum [df(0, 2d) + 2df(0, 2^2 d) + \cdots \\ &\qquad + 2^{\alpha-1} df(0, 2^{\alpha} d)] - \sum (2^{\alpha} d - 1) f(2^{\alpha} d, 0) \\ &+ 2\sum [f(2^{\alpha} d, 2) + f(2^{\alpha} d, 4) + f(2^{\alpha} d, 6) + \cdots \\ &\qquad + f(2^{\alpha} d, \delta - 1)]. \end{aligned}$$

For $f(x, y) = f(y)$ this becomes (II), (III); for $f(x, y) = f(x)$ we find after a simple reduction,

$$\text{(XVIII)} \quad \begin{aligned} &2^{\alpha} m = 2^{\alpha'} m' + 2^{\alpha''} m''; \quad m, m', m'' = d\delta, d'\delta', d''\delta''; \\ &\alpha > 0: \\ &\sum [f(2^{\alpha'} d' - 2^{\alpha''} d'') - f(2^{\alpha'} d' + 2^{\alpha''} d'')] \\ &\qquad = \sum (\delta - 2^{\alpha} d)[f(2^{\alpha} d) - f(0)]; \end{aligned}$$

and when $\alpha = 0$,

$$\text{(XIX)} \quad \begin{aligned} &m = 2^{\alpha'} m' + m''; \quad m, m', m'' = d\delta, d'\delta', d''\delta'': \\ &2\sum [f(2^{\alpha'} d' - d'') - f(2^{\alpha'} d' + d'')] = \sum (\delta - d) f(d). \end{aligned}$$

These three are Liouville's (5), 280, (e); (3), 208, (F); (4), 242, (G). Similarly from (36) we have

$$\phi_5(x, y)\phi_5(x, -y) = \phi_8^2(y) + \phi_7^2(x),$$

which gives upon reduction by $v_3(\delta, y)$ from (28),

$$n = n' + n''; \quad n, n', n'' = 2^{\alpha} m, 2^{\alpha'} m', 2^{\alpha''} m'';$$

$$m, m', m'' = d\delta, d'\delta', d'\delta'':$$

$$\text{(XX)} \quad \begin{aligned} &\sum [(-1)^{(\delta'-1)/2 + (\delta''-1)/2} \{f(2^{\alpha'} d' + 2^{\alpha''} d'', \delta' - \delta'') \\ &\qquad + f(2^{\alpha'} d' - 2^{\alpha''} d'', \delta' + \delta'')\}] \\ &= \sum (2^{\alpha} d - 1) f(2^{\alpha} d, 0) + 2\sum [f(2^{\alpha} d, 2) - f(2^{\alpha} d, 4) \\ &\qquad + \cdots + (-1)^{(\delta+1)/2} f(2^{\alpha} d, \delta - 1)] \\ &- \tfrac{1}{2}\{1 + (-1)^n\} \sum d[-f(0, 2d) + 2f(0, 2^2 d) \\ &\qquad + \cdots + 2^{\alpha-1} f(0, 2^{\alpha} d)], \end{aligned}$$

which might have been derived by the elementary transformations from (XVII); as it is the transformation affords a check of a kind which frequently is valuable. For $f(x, y) = f(x)$ this is easily seen to be

$$\sum[(-1)^{(\delta'-1)/2+(\delta''-1)/2}\{f(2^{\alpha'}d'+2^{\alpha''}d'')+f(2^{\alpha'}d'-2^{\alpha''}d'')\}]$$

(XXI)
$$=\tfrac{1}{2}\{1+(-1)^n\}(3-2^{\alpha})\zeta_1'(n)f(0)$$
$$+\sum\{2^{\alpha}d+(-1)^{(\delta+1)/2}\}f(2^{\alpha}d),$$

over the same separation as the preceding. This is essentially two theorems; the first corresponds to $\alpha = 0$, and is (X); the second is, with $\alpha > 0$,

(XXII)
$$\sum[(-1)^{(\delta'-1)/2+(\delta''-1)/2}\{f(2^{\alpha'}d'+2^{\alpha''}d'')+f(2^{\alpha'}d'-2^{\alpha''}d'')\}]$$
$$=(3-2^{\alpha})\zeta_1'(n)+\sum\{2^{\alpha}d+(-1)^{(\delta+1)/2}\}f(2^{\alpha}d).$$

This is Liouville (5), 282, (K). The examples which we have given are probably sufficient as illustrations of the simplest methods of using the tables, and the length of this paper forbids more systematic treatment here of the lists sampled or of the numerous other sets in the theory of elliptic functions which Liouville apparently did not touch. Also we must leave aside the many interesting questions that suggest themselves; e.g., the classification of the paraphrases, the inverse problem of finding paraphrases for a given separation (in which the theory of the transformation of elliptic functions finds a new application to arithmetic), and the passage from paraphrases for functions of degree 2 to degree exceeding 2. One natural starting point for the last is Jacobi's (and H. J. S. Smith's) formula for the multiplication of four theta functions, and its consequences for $\vartheta_{\alpha}(x \pm y \pm z)$, etc. This generalization is of importance, as it leads naturally to the general case of such paraphrases as the present, viz., to the paraphrase of Riemann's theta formula and the theory of the related functions, making possible a ready arithmetical interpretation of some of the most striking analytical results in the theory of abelian functions.*

In the present order of ideas the series for the $\phi_j^n(x)$, $n > 2$ furnish interesting results; e.g., $\phi_1^3(x)$ gives the principal theorems in Liouville's sixth memoir. We may conclude with two paraphrases for quadratic separations, to illustrate the fact that there is no increase in difficulty by these methods when we pass from linear separations to quadratic. This certainly is not the case in methods used hitherto.

19. From the identity

$$\frac{\vartheta_1'(x)}{\vartheta_1(x)} \times \vartheta_1(x) = \vartheta_1'(x),$$

and the following, which is easily derived from the known series,

* This generalization was carried out in some detail for the hyperelliptic functions of the first order, and in particular for the transformation theory of such functions, in 1915; the results, which lead to interesting arithmetical conclusions, will be published as soon as the papers on the elliptic case have appeared. There is a notable gain in generality when we pass beyond the elliptic case: the majority of the paraphrases refer to functions of n variables unrestricted in any way whatever; parity no longer plays an essential part.

$$\vartheta_1'(x)/\vartheta_1(x) = \cot x + 4\sum q^{2n}(\sum \sin 2dx),\ [T_3],$$

we find upon replacing q by q^4, writing $\epsilon(n) = 1$ or 0 according as n is or is not the square of an integer > 0, and paraphrasing as usual;

$$m = 8n' + m''^2; \qquad n' = \delta' d':$$

$$\text{(XXIII)} \quad \begin{aligned} &2\sum(-1)^{(m''-1)/2}[f(2d' - m'') - f(2d' + m'')] \\ &= \epsilon(m)(-1)^{(\sqrt{m}-1)/2}[(\sqrt{m} - 1)f(\sqrt{m}) - 2f(1) - 2f(3) \\ &\qquad - \cdots - 2f(\sqrt{m} - 2)]. \end{aligned}$$

Liouville gave several of the same kind, but not this. He does not seem to have used the $\vartheta_\alpha'(x)/\vartheta_4(x)$.

20. One of the simplest methods for finding quadratic paraphrases concerning functions of order > 1 is by proceeding from identities between theta functions (not their quotients) and series whose d, δ forms are known. Thus it is seen by inspection of (35) that

$$\vartheta_0(x - y)\phi_4(x - y, y) + \vartheta_0(x + y)\phi_4(x + y, -y) \equiv 0,$$

where we have not gone beyond two variables in order to keep the writing simple. Substituting the series for $\vartheta_0(x \pm y)$, $\phi_4(x \mp y, \pm y)$, and noting that $\vartheta_0(x)$ may also be written $\Sigma(-1)^{n_1} q^{n_1^2} \cos 2n_1 x$, where the range of n_1 is $-\infty$ to $+\infty$, we find after some simple reductions the following form of the ϑ, ϕ identity:

$$\sum_{n=1}^{\infty} q^{n^2}\left[(-1)^n \sin 2nx \sum_{r=1}^{n-1} \cos(2r - 1)y\right]$$
$$+ \sum_{n_1=-\infty}^{\infty} q^{n_1^2+n_2}\left[\sum(-1)^{n_1} \sin(2^{\alpha_2+1} d_2 + 2n_1)x \cos(2^{\alpha_2+1} d_2 - \delta_2 + 2n_1)y\right] \equiv 0;$$

and this being an identity in q, we have the next an identity in x, y, (the separation in both cases is the same);

$$n = n_1^2 + n_2; \qquad n_1 \gtreqless 0; \qquad 0 < n_2 = 2^{\alpha_2} m_2; \qquad m_2 = d_2 \delta_2:$$

$$\epsilon(n)(-1)^n \sin 2\sqrt{n}x \sum_{r=1}^{\sqrt{n}} \cos(2r - 1)y$$
$$+ \sum(-1)^{n_1} \sin(2^{\alpha_2+1} d_2 + 2n_1)x \cos(2^{\alpha_2+1} d_2 - \delta_2 + 2n_1)y = 0;$$

and thence, over the same separation,

$$\text{(XXIV)} \quad \begin{aligned} &\sum(-1)^{n_1} f(2^{\alpha_2+1} d_2 - \delta_2 + 2n_1 | 2^{\alpha_2} d_2 + n_1) \\ &\qquad = \epsilon(n)(-1)^{n-1} \sum_{r=1}^{\sqrt{n}} f(2r - 1 | \sqrt{n}). \end{aligned}$$

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