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Chapter 13

Qualitative Spatial Representation and Reasoning

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13.1 Introduction

The need for spatial representations and spatial reasoning is ubiquitous in AI—from robot planning and navigation, to interpreting visual inputs, to understanding natural language—in all these cases the need to represent and reason about spatial aspects of the world is of key importance. Related fields of research, such as geographic information science (GIScience) [70], have also driven the spatial representation and reasoning community to produce efficient, expressive and useful calculi.

Whereas there has been considerable research in spatial representations which are based on metric measurements, in particular within the vision (e.g., [62, 137]) and robotics communities (e.g., [198]), and also on raster and vector representations in GIScience (e.g., [214]), in this chapter we concentrate on symbolic, and in particular *qualitative* representations. Chapter 9 is devoted to qualitative reasoning (QR) more generally, whereas here we limit our attention specifically to qualitative spatial, and spatio-temporal reasoning (henceforth QSR).

13.1.1 What is Qualitative Spatial Reasoning?

Chapter 9 concentrates on linear quantities; in some cases this suffices to reason about space in a qualitative way, for example, when reasoning about the position of a sliding block, or the level of a tank. However, space is multidimensional, and is not in general adequately represented by a single scalar quantity. Consider using Allen's interval calculus, briefly mentioned in Chapter 12, which distinguishes 13 jointly exhaustive and pairwise disjoint relations that may hold between a pair of convex (one-piece) intervals, see Fig. 13.1(a). Now we consider using this representation to model two-dimensional regions, by projecting 2D space onto two separate linear dimensions; in

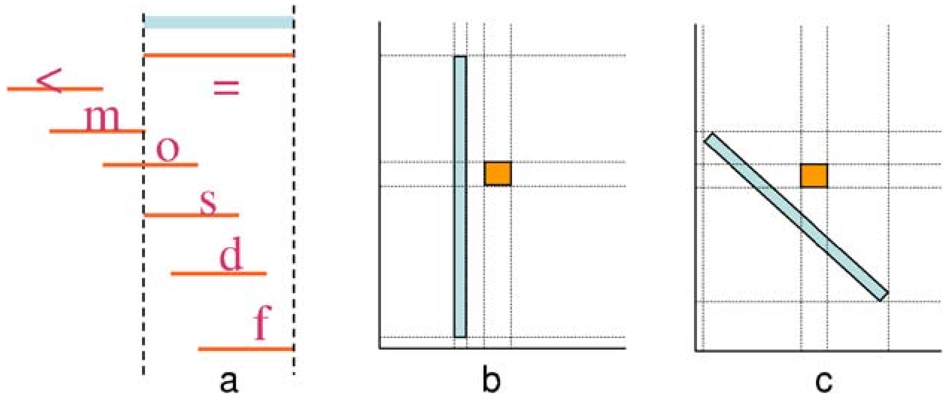

Figure 13.1: (a) The 13 jointly exhaustive and pairwise disjoint Allen interval relations between a pair of convex intervals (the top thick line and each of the thinner lines below)—only seven are displayed—the last six are asymmetric and have inverses. Projecting regions onto axes and using Allen's interval calculus can give misleading results: in (b) the small region is discrete from the larger along the x-axis, whilst in (c) it is contained in the larger region along both axes.

Fig. 13.1(b) this works well, but in Fig. 13.1(c) it is not so satisfactory—the smaller region appears to be contained in the larger.[1]

Early attempts at qualitative spatial reasoning within the QR community led to the 'poverty conjecture' [84]. Although purely qualitative representations were quite successful in reasoning about many physical systems [209], there was much less success in developing purely qualitative reasoners about spatial and kinematic mechanisms and the poverty conjecture is that this is in fact impossible—there is no purely qualitative spatial reasoning mechanism. Forbus et al. correctly identify transitivity of values as a key feature of qualitative quantity spaces but doubt that this can be exploited much in higher dimensions and conclude that the space of representations in higher dimensions is sparse and for spatial reasoning nothing weaker than numbers will do.

The challenge of QSR then is to provide calculi which allow a machine to represent and reason with spatial entities without resort to the traditional quantitative techniques prevalent in, for, e.g., the computer graphics or computer vision communities.

There has been an increasing amount of research in recent years which tends to refute, or at least weaken the 'poverty conjecture'. Qualitative spatial representations addressing many different aspects of space including topology, orientation, shape, size and distance have been put forward. There is a rich diversity of these representations and they exploit the 'transitivity' as demonstrated by the relatively sparse *composition tables* (cf. the well known table for Allen's interval temporal logic [209]) which have been built for these representations.

This chapter is an overview of some of the major qualitative spatial representation and reasoning techniques. We focus on the main ideas that have emerged from research in the area; there is not sufficient space here to be comprehensive and some

[1] In certain domains, containing rectangular objects which are uniformly aligned, this can still be a useful representation, see, for example, [208] where the layout of text blocks on envelopes is learned. A theoretical analysis into the n-dimensional generalisation of the Allen calculus can be found in [9].

interesting approaches have had to be omitted though we give some pointers to the wider literature.[2]

In Section 13.1.2 we will mention some possible applications of qualitative spatial reasoning. Thereafter, in Section 13.2 we survey the main aspects of the representation of qualitative spatial knowledge including ontological aspects, topology, distance, orientation and shape. Section 13.3 discusses qualitative spatial reasoning and Section 13.4 reasoning about spatial change. The chapter concludes with some remarks on cognitive validity in Section 13.5 and a glimpse at future work in Section 13.6. This chapter is based on a number of earlier papers, in particular [47].

13.1.2 Applications of Qualitative Spatial Reasoning

Research in QSR is motivated by a wide variety of possible application areas including Geographic Information System (GIS), robotic navigation, high level vision, spatial propositional semantics of natural languages, engineering design, common-sense reasoning about physical systems and specifying visual language syntax and semantics. There are numerous other application areas including qualitative document-structure recognition [208], biology (e.g., [191, 42]) and domains where space is used as a metaphor (e.g., [127, 160]).

Even though GIS are now a commonplace, the major problem is that of interaction. With gigabytes of information stored either in vector or raster format, present-day GISs do not sufficiently support intuitive or common-sense oriented human–computer interaction. Users may wish to abstract away from the mass of numerical data and specify a query in a way which is essentially, or at least largely, qualitative. Arguably, the next generation GIS will be built on concepts arising from *Naive Geography* [70], wherein QSR techniques are fundamental. Examples of research employing qualitative spatial techniques in geography include reasoning about shape in a qualitative way such as [32].

Although robotic navigation ultimately requires numerically specified directions to the robot to move or turn, hierarchical planning with detailed decisions (e.g., how or exactly where to move) being delayed until a high level plan have been achieved has been shown to be effective [196]. Further, the robot's model of its environment may be imperfect, leading to an inability to use standard robot navigation techniques. Under such circumstances, a qualitative model of space may facilitate planning. One such approach is the development of a robust qualitative method for robot exploration, mapping and navigation in large-scale spatial environments described in [125]; another is the work of Liu and Daneshmend [133] on spatial planning for robotic motion and path planning using qualitative spatial representation and reasoning. Another example of using QSR for robotic navigation is [207]. A qualitative solution to the well known 'piano mover's problem' is [78]. Some work in cognitive robotics has addressed the issue of building topological maps of the robot's environment (rather than metrical ones), e.g., [165, 123].

[2] Much relevant material is published in the proceedings of COSIT (the Conference on Spatial Information Theory), GIScience (the International Conference on Geographical Information Science), the journal Spatial Cognition and Computation, as well as regular AI outlets such as the AI journal, the Journal of Artificial Intelligence Research (JAIR) the International Journal of Geographical Information Science, and the proceedings of such conferences as KR, AAAI, IJCAI, PRICAI and ECAI.

QSR has been used in computer vision for visual object recognition at a higher level which includes the interpretation and integration of visual information. QSR techniques have been used to interpret the results of low-level computations as higher level descriptions of the scene or video input [80, 121]. The use of qualitative predicates helps to ensure that scenes which are semantically close have identical or at least very similar descriptions. Work in this area from a cognitive robotics viewpoint includes that of Santos [181, 180].

In natural language, the use and interpretation of spatial propositions tend to be ambiguous. There are multiple ways in which natural language spatial prepositions can be used (e.g., [114] cites many different meanings of "in"); this motivates the use of qualitative spatial representation for finding some formal way of describing these prepositions (e.g., [5, 178, 24]).

Engineering design, like robotic navigation, ultimately normally requires a fully metric description. However, at the early stages of the design process, a reasonable qualitative description would suffice. The field of qualitative kinematics (e.g., [77]) is largely concerned with supporting this type of activity.

Finally, visual languages, either visual programming languages or some kind of representation language, lack a formal specification of the kind that is normally expected of a textual programming or representation language. Although some of these languages make metric distinctions, the bulk of it is often predominantly qualitative in the sense that the exact shape, size, length, etc. of the various components of the diagram or picture is unimportant—rather, what is important is the topological relationship between these components [98, 107]. In a similar vein, research continues on the application of qualitative spatial reasoning for sketch interpretation, e.g., [83, 79, 66, 183, 107, 85].

13.2 Aspects of Qualitative Spatial Representation

Representing space has a rich history in the physical sciences—and serves to locate objects in a quantitative framework. At the other extreme, spatial expressions in natural languages tend to operate on a loose partitioning of the domain. Representation for this less precise description of space proliferated, more or less on an *ad hoc* basis until the emergence of qualitative spatial reasoning; thereafter the partitioning was done more systematically [142].

There are many different aspects to space and therefore to its representation. Not only do we have to decide on what kind of spatial entity we will admit (i.e., commit to a particular ontology of space), but also we can consider developing different kinds of ways of describing the relationship between these kinds of spatial entities; for example, we may consider just their topology, or their sizes or the distance between them, their relative orientation or their shape. In the following sections we will overview the principal techniques which have emerged to represent these different aspects of qualitative spatial knowledge.

13.2.1 Ontology

In this chapter we concentrate on what might be termed "pure space", i.e., purely spatial entities such as points, lines and regions, rather than entities which have spatial extensions, such as physical objects or geographic regions.

Traditionally, in mathematical theories of space, points are considered as the primary primitive spatial entities (or perhaps points and lines), and extended spatial entities such as regions are defined, if necessary, as sets of points. A minority tradition ('mereology' or 'calculus of individuals'—Section 13.2.3) regards this as a philosophical error.[3] Within the QSR community, there is a strong tendency to take regions of space as the primitive spatial entity—see [206]. Even though this ontological shift means building new theories for most spatial and geometrical concepts, there are strong reasons for taking regions as the ontological primitive. If one is interested in using the spatial theory for reasoning about physical objects, then one might argue that the spatial extension of any physical object must be region-like rather than a lower dimension entity. Further, one can always define points, if required, in terms of regions [18]. However, it needs to be admitted that at times it is advantageous to view a 3D physical entity as a 2D or even a 1D entity. Of course, once entities of various dimensions are permitted, a pertinent question would be whether mixed dimension entities are allowed. Further discussion of this issue can be found in [43, 44, 100] and also in [155, 157] who argues that in a first order 2D planar *mereotopology*,[4] a region based ontology is not as parsimonious as a point based one, from a model theoretic viewpoint. Whether points or regions are taken as primitive, it is clear that regions nevertheless are conceptually important in modelling physical and geographic objects.

However, even once one has committed to an ontology which includes regions as primitive spatial entities, there are still several choices facing the modeller. For example, in most mereotopologies, the null region is excluded (since no physical object can have the null region as its extension) though technically it may be simpler to include it [13, 193]. It is fairly standard to insist that regions are all *regular*, though this choice becomes harder to enforce once one allows regions of differing dimensionalities (e.g., 2D and 3D, or even 4D) since the sum of two regions of differing dimensions will not be regular. One can also distinguish between regular-open and regular-closed alternatives. Some calculi [21, 65] insist that regions are connected (i.e. one-piece). A yet stronger condition would be that they are *interior connected*—e.g., a 2D region which pinches to a point is not interior connected. In practice, a reasonable constraint to impose would be that regions are all rational polygons [156].

Another ontological question is what is the nature of the embedding space, i.e., the universal spatial entity? Conventionally, one might take this to be R^n for some n, but one can imagine applications where discrete (e.g., [71]), finite (e.g., [99]), or non-convex (e.g., non-connected) universes might be useful. There is a tension between the continuous-space models favoured by high-level approaches to handling spatial information and discrete, digital representations used at the lower level. An attempt to bridge this gap by developing a high-level qualitative spatial theory based on a discrete model of space is [91]. For another investigation into discrete vs continuous space, see [139].

Once one has decided on these ontological questions, there are further issues: in particular, what primitive "computations" should be allowed? In a logical theory, this amounts to deciding what primitive non-logical symbols one will admit without definition, only being constrained by some set of axioms. One could argue that this set of

[3] Simons [189] says: "No one has ever perceived a point, or ever will do so, whereas people have perceived individuals of finite extent".

[4] Mereotopology is defined and discussed in detail in Section 13.2.4 below.

primitives should be small, not only for mathematical elegance and to make it easier to assess the consistency of the theory, but also because this will simplify the interface of the symbolic system to a perceptual component because fewer primitives have to be implemented. The converse argument might be that the resulting symbolic inferences may be more complicated or that it is more natural to have a large and rich set of concepts which are given meaning by many axioms which connect them in many different ways [108]. As we shall see below, in a full first order theory one can define perhaps surprisingly many concepts from just a few primitives; however sometimes it is desirable to restrict the language used to a less expressive language for computational reasons—in this case one will typically need to increase the number of primitives. The next section considers the most common class of such primitives, relations between spatial entities.

13.2.2 Spatial Relations

It is one of the basic assumptions of qualitative representation and reasoning that situations are represented by specifying the relationships between the considered entities. Hence it is natural to represent qualitative information using relations, and in this chapter spatial relations. Formally, a *relation* R is a set of tuples $(d_1, \ldots, d_k)$ of the same arity k, where d_i is a member of a corresponding *domain* $\mathcal{D}_i$. In other words, a relation R of arity k is a subset of the cross-product of k domains, i.e., $R \subseteq \mathcal{D}_1 \times \cdots \times \mathcal{D}_k$.

Very often, *spatial relations* are *binary relations* and very often the considered domains are identical, namely, the set of all spatial entities of a particular space. In these cases spatial relations are of the form $R = \{(a, b) \mid a, b \in \mathcal{D}\}$. The considered domain is usually an infinite domain and the spatial relations contain infinitely many tuples.

Given a set of relations $\mathcal{R} = \{R_1, \ldots, R_n\}$ we can use algebraic operators such as union, intersection, complement, converse, or composition of relations and in this way obtain an *algebra of relations*.[5] Since the relations contain an infinite number of tuples, applying these operators might not be feasible. It is therefore a common assumption in qualitative representation and reasoning to select a (small) finite set of relations which are *jointly exhaustive and pairwise disjoint* (*JEPD*), i.e., each tuple $(a, b) \in \mathcal{D} \times \mathcal{D}$ is a member of exactly one relation. JEPD relations are also called *atomic, base*, or *basic relations*. Given a set of JEPD relations, the relationship between any two spatial entities of the considered domain must be exactly one of the JEPD relations. Indefinite information can be expressed by taking the union of those base relations that can possibly hold (representing the disjunction of the base relations). If no information is known and all possible base relations can hold, we use the *universal relation* which is the union of all base relations. The set of all possible relations is then the powerset of the set of base relations, i.e., all possible unions of the JEPD relations.

In the following sections we discuss various sets of spatial relations, and in particular some different sets of JEPD relations that have been studied in the literature. These are usually restricted to one particular aspect of space such as topology, orientation, shape, etc. How to reason about these relations and more about the consequences of having infinite domains is covered in Section 13.3, while more about general considerations of defining a qualitative calculus can be found in [132].

[5] See [59] for a review of the use of relation algebras in spatial and temporal reasoning.

13.2.3 Mereology

Mereology is concerned with the theory of *parthood*, deriving from the Greek $\mu\epsilon\rho o\varsigma$ (*part*), and forms a fundamental aspect of spatial representation, with practical applications in many fields, e.g., [187]. The books by Simons [189], and more recently by Casati and Varzi [27] are excellent reference works for mereology. Simons proposes a number of different mereological theories, depending on what properties one wishes to ascribe to. Perhaps the most widely used theory is his *minimal extensional mereology* [189, pp. 25–30]. The proper part relation is taken as primitive, symbolised PP.[6] The logical basis of the system is:

(SA0) Any axiom set sufficient for first-order predicate calculus with identity.

(SA1) $\forall x, y[\mathsf{PP}(x, y) \rightarrow \neg\mathsf{PP}(y, x)]$.

(SA2) $\forall x, y, z[[(\mathsf{PP}(x, y) \wedge \mathsf{PP}(y, z)] \rightarrow \mathsf{PP}(x, z)]$.

(SA1) and (SA2) simply assert that the system's basic relation is a strict partial ordering. Simons goes on to define part (symbolised 'P'). The next step is to require that an individual cannot have a *single* proper part. After defining overlapping ('O', having a common part), Simons gives the 3rd axiom:

(SA3) $\forall x, y[\mathsf{PP}(x, y) \rightarrow \exists z[\mathsf{PP}(z, y) \wedge \neg\mathsf{O}(z, x)]]$.

This axiom he refers to as the *Weak Supplementation Principle* (WSP), asserting that any individual with a proper part has another that is disjoint with the first. The axiom set (SA0)–(SA3) still permits various models Simons regards as unsatisfactory, in which overlapping individuals do not have a unique product or intersection. Such models are ruled out by adding:

(SA6) $\forall x, y[\mathsf{O}(x, y) \rightarrow \exists z\forall w[\mathsf{P}(w, z) \equiv \mathsf{P}(w, x) \wedge \mathsf{P}(w, y)]]$,

which ensures the existence of such a unique product. This system of four axioms defines the system known as minimal extensional mereology. We do not have space here to present the many other variations of mereology, but refer the reader to the literature, in particular [189, 27].

13.2.4 Mereotopology

It is clear that topology must form a fundamental aspect of qualitative spatial reasoning since topology certainly can only make qualitative distinctions. Although topology has been studied extensively within the mathematical literature, much of it is too abstract to be of relevance to those attempting to formalise common-sense spatial reasoning. Although various qualitative spatial theories have been influenced by mathematical topology, there are number of reasons why such a wholesale importation seems undesirable in general [100], in particular, the absence of consideration of computational aspects, such as we consider below in Section 13.3. In fact mereotopology is the most studied aspect of QSR and for this reason we devote particular attention to it in this chapter.

[6] For the sake of uniformity, in a number of cases we have renamed predicate and other symbols in this chapter from the original formulation.

Although Whitehead tried to define topological notions within mereology [210], this is not possible, and requires some further primitive notions. Varzi [205, 204] presents a systematic account of the subtle relations between mereology and topology. He notes that whilst mereology is not sufficient by itself, there are theories in literature which have proposed integrating topology and mereology (henceforth, *mereotopology*). There are three main strategies of integrating the two:

- Generalise mereology by adding a topological primitive. Borgo et al. [21] add the topological primitive $\mathsf{SC}(x)$, i.e., x is a self connected (one-piece) spatial entity to the mereological part relation. Alternatively a single primitive can be used as in [205]: "x and y are connected parts of z". The main advantage of separate theories of mereology and topology is that it allows collocation without sharing parts[7] which is not possible in the second two approaches below.

- Topology is primal and mereology is a subtheory. For example in the topological theories based on $\mathsf{C}(x, y)$ (x is connected to y, discussed further below) one defines $\mathsf{P}(x, y)$ from $\mathsf{C}(x, y)$. This has the elegance of being a single unified theory, but collocation implies sharing of parts. These theories are normally boundaryless (i.e., without lower dimensional spatial entities) but this is not absolutely necessary [161, 4], as discussed further below.

- The final approach is that taken by [73], i.e., topology is introduced as a specialised domain specific subtheory of mereology. An additional primitive needs to be introduced. The idea is to use restricted quantification by introducing a sortal predicate, $\mathsf{Rg}(x)$, to denote a region. $\mathsf{C}(x, y)$ can then be defined thus: $\mathsf{C}(x, y) =_{df} \mathsf{O}(x, y) \wedge \mathsf{Rg}(x) \wedge \mathsf{Rg}(y)$.

In the remainder of this subsection, we concentrate on the first two approaches, which are largely based on approaches based on work to be found in the philosophical logic community in particular the work of Clarke [33, 34] which was in turn based on the theory of extensive connection outlined by Whitehead in Process and Reality [211]. Other work in this tradition is cited below and more extensively in [49], in each case building axiomatic theories of space which are predominantly topological in nature, and which take regions rather than points as primitive—indeed, this tradition has been termed as "pointless geometries" [96]. We concentrate here on overviewing the axiomatic approach to mereotopology; the reader is referred to [17] for a thorough treatment of the algebraic and axiomatic approaches to mereotopology and their relationship.

As has been pointed out [49], not all this work agrees in its basic terms; even where there is agreement on vocabulary, such as the use of a binary *connection* predicate, it is not always interpreted in the same way. A model-theoretic framework for investigating the logical space of mereotopological theories and comparing the main options in light of their intended models has been set out [49]. We now describe this framework further since it also provides an overview of the various approaches to mereotopology (for details see [49]).

All the theories are interpreted with respect to some topological space, T, on which a closure operator $\mathsf{c}(x)$ is axiomatised in a standard way:

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[7] For further discussion of this issue see [27, 58].

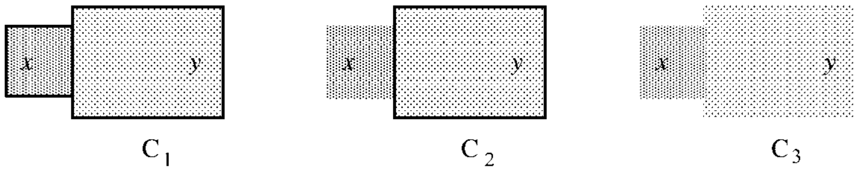

Figure 13.2: The three C relations (limit cases); a solid line indicates closure.

$(A0)\quad \emptyset = \mathsf{c}(\emptyset).$

$(A1)\quad x \subseteq \mathsf{c}(x).$

$(A2)\quad \mathsf{c}(\mathsf{c}(x)) \subseteq \mathsf{c}(x).$

$(A3)\quad \mathsf{c}(x) \cup \mathsf{c}(y) = \mathsf{c}(x \cup y).$

Three different notions of connection are then defined (which are illustrated in Fig. 13.2), the semantics which are given by:

$$\mathsf{C}_1(x, y) \Leftrightarrow x \cap y \neq \emptyset,$$
$$\mathsf{C}_2(x, y) \Leftrightarrow x \cap \mathsf{c}(y) \neq \emptyset \text{ or } \mathsf{c}(x) \cap y \neq \emptyset.$$
$$\mathsf{C}_3(x, y) \Leftrightarrow \mathsf{c}(x) \cap \mathsf{c}(y) \neq \emptyset,$$

However, since some mereotopologies (e.g., see the first of the three strategies outlined above) have multiple primitives, two further primitives are made available:

$$\mathsf{P}_n(x, y) =_{df} \forall z(\mathsf{C}_n(z, x) \to \mathsf{C}_n(z, y)) \quad (1 \leqslant n \leqslant 3)$$
$$\sigma_n x\phi =_{df} \iota z \forall y(\mathsf{C}_n(y, z) \leftrightarrow \exists x(\phi \wedge \mathsf{C}_n(y, x))) \quad (1 \leqslant n \leqslant 3)$$

Intuitively: x is part (P_n) of y iff whatever is connected (C_n) to x is also connected (C_n) to y, and the fusion (σ_n) of all ϕ-ers (where ϕ is some formula with x free) is that thing (if it exists at all) that connects$_n$ precisely to those things that ϕ (i.e., for which ϕ holds for that particular binding of x). Many theories define these notions in terms of the same connection relation that is assumed as a topological primitive, in which case the above reduce to ordinary definitions in the object language of the theory. However, this need not be the case, and in fact an important family of theories stem precisely from the intuition that parthood and connection cannot be defined in terms of each other. This effectively amounts to using two distinct primitives—two notions of connection (one of which is used in defining parthood), or a notion of connection and an independent notion of parthood. Accordingly, and more generally, the framework considers the entire space of mereotopological theories that result from the options determined by the above definitions when $1 \leqslant n \leqslant 3$. That is to say, in the object language all three connection predicates are available as primitives, and the framework models theories in which some such predicates are defined in term of others by adding suitable axioms in place of the corresponding definitions. The choice of which primitives are used will be indicated with a triple,[8] which is called a *type*, $\tau = \langle i, j, k \rangle$ (where $1 \leqslant i, j, k \leqslant 3$), the three components, respectively, indicating which C_i, P_j and σ_k relation is being

[8] In fact, in [49] a type is quadruple, but we ignore the final component here.

used in the corresponding τ-theory, thus:

$$\mathsf{C}_{\langle i,j,k\rangle}(x, y) =_{df} \mathsf{C}_i(x, y),$$

$$\mathsf{P}_{\langle i,j,k\rangle}(x, y) =_{df} \mathsf{P}_j(x, y),$$

$$\sigma_{\langle i,j,k\rangle} x\phi =_{df} \sigma_k x\phi.$$

There are a great many mereotopological relations which can be defined using these three primitives. We list some of the most common here:

$\mathsf{O}_\tau(x, y) =_{df} \exists z(\mathsf{P}_\tau(z, x) \wedge \mathsf{P}_\tau(z, y))$	x τ-overlaps y
$\mathsf{A}_\tau(x, y) =_{df} \mathsf{C}_\tau(x, y) \wedge \neg\mathsf{O}_\tau(x, y)$	x τ-abuts y
$\mathsf{E}_\tau(x, y) =_{df} \mathsf{P}_\tau(x, y) \wedge \mathsf{P}_\tau(y, x)$	x τ-equals y
$\mathsf{PP}_\tau(x, y) =_{df} \mathsf{P}_\tau(x, y) \wedge \neg\mathsf{P}_\tau(y, x)$	x is a proper τ-part of y
$\mathsf{TP}_\tau(x, y) =_{df} \mathsf{P}_\tau(x, y) \wedge \exists z(\mathsf{A}_\tau(z, x) \wedge \mathsf{A}_\tau(z, y))$	x is a tangential τ-part of y
$\mathsf{IP}_\tau(x, y) =_{df} \mathsf{P}_\tau(x, y) \wedge \neg\mathsf{TP}_\tau(x, y)$	x is an interior τ-part of y
$\mathsf{BP}_\tau(x, y) =_{df} \forall z(\mathsf{P}_\tau(z, x) \rightarrow \mathsf{TP}_\tau(z, y))$	x is a boundary τ-part of y
$\mathsf{PO}_\tau(x, y) =_{df} \mathsf{O}_\tau(x, y) \wedge \neg\mathsf{P}_\tau(x, y) \wedge \neg\mathsf{P}_\tau(y, x)$	x properly τ-overlaps y
$\mathsf{TO}_\tau(x, y) =_{df} \exists z(\mathsf{TP}_\tau(z, x) \wedge \mathsf{TP}_\tau(z, y))$	x tangentially τ-overlaps y
$\mathsf{IO}_\tau(x, y) =_{df} \exists z(\mathsf{IP}_\tau(z, x) \wedge \mathsf{IP}_\tau(z, y))$	x internally τ-overlaps y
$\mathsf{BO}_\tau(x, y) =_{df} \mathsf{O}_\tau(x, y) \wedge \neg\mathsf{IO}_\tau(x, y)$	x boundary τ-overlaps y
$\pi_\tau x\phi =_{df} \sigma_\tau z\forall x(\phi \rightarrow \mathsf{P}_\tau(z, x))$	τ-product of ϕ-ers
$x +_\tau y =_{df} \sigma_\tau z(\mathsf{P}_\tau(z, x) \vee \mathsf{P}_\tau(z, y))$	τ-sum of x and y
$x \times_\tau y =_{df} \sigma_\tau z(\mathsf{P}_\tau(z, x) \wedge \mathsf{P}_\tau(z, y))$	τ-product of x and y
$x -_\tau y =_{df} \sigma_\tau z(\mathsf{P}_\tau(z, x) \wedge \neg\mathsf{O}_\tau(z, y))$	τ-difference of x and y
$\mathsf{k}_\tau(x) =_{df} \sigma_\tau z\neg\mathsf{O}_\tau(z, x)$	τ-complement of x
$\mathsf{i}_\tau(x) =_{df} \sigma_\tau z\mathsf{IP}_\tau(z, x)$	τ-interior of x
$\mathsf{e}_\tau(x) =_{df} \mathsf{i}_\tau(\mathsf{k}_\tau(x))$	τ-exterior of x
$\mathsf{c}_\tau(x) =_{df} \mathsf{k}_\tau(\mathsf{e}_\tau(x))$	τ-closure of x
$\mathsf{b}_\tau(x) =_{df} \mathsf{c}_\tau(x) -_\tau \mathsf{i}_\tau(x)$	τ-boundary of x
$\mathsf{U}_\tau =_{df} \sigma_\tau z\mathsf{O}_\tau(z, z)$	τ-universe
$\mathsf{Bd}_\tau(x) =_{df} \exists y\mathsf{BP}_\tau(x, y)$	x is a τ-boundary
$\mathsf{Rg}_\tau(x) =_{df} \exists y\mathsf{IP}_\tau(y, x)$	x is a τ-region
$\mathsf{Op}_\tau(x) =_{df} \mathsf{E}_\tau(x, \mathsf{i}_\tau(x))$	x is τ-open
$\mathsf{Cl}_\tau(x) =_{df} \mathsf{E}_\tau(x, \mathsf{c}_\tau(x))$	x is τ-closed
$\mathsf{Re}_\tau(x) =_{df} \mathsf{E}_\tau(\mathsf{i}_\tau(x), \mathsf{i}_\tau(\mathsf{c}_\tau(x)))$	x is τ-regular
$\mathsf{Cn}_\tau(x) =_{df} \forall y\forall z(E_\tau(x, y +_\tau z) \rightarrow \mathsf{C}_\tau(y, z))$	x is τ-connected (i.e. in one piece)
$\mathsf{CP}_\tau(x, y) =_{df} \mathsf{P}_\tau(x, y) \wedge \mathsf{Cn}_\tau(x)$	x is a τ-connected part of y

Depending on the structure of τ, the notions thus defined may receive different interpretations, hence the gloss on the right should not be taken too strictly. One intended interpretation of the binary relations relative to the Euclidean plane R^2—an interpretation that justifies the gloss—is illustrated in Figs. 2 and 3 in [49]. However, the exact semantic consequence of these definitions may change radically from one framework to another, depending on the type τ and on the constraints in the model theory.

It is easy to see that the following formulas are true in every canonical model for all types τ (i.e., C_τ is reflexive and symmetric), and indeed these formulae are normally included as axioms in any mereotopology based on a binary connection relation:

$(C1_\tau)$ $\mathsf{C}_\tau(x, x)$.

$(C2_\tau)$ $\mathsf{C}_\tau(x, y) \rightarrow \mathsf{C}_\tau(y, x)$.

Similarly, the following are always logically true in view of the definition of P_τ (and are included as axioms if parthood is not defined in terms of connection, i.e., the first and second indices of the type are different):

$(P1_\tau)$ $\mathsf{P}_\tau(x, x)$.

$(P2_\tau)$ $(\mathsf{P}_\tau(x, y) \wedge \mathsf{P}_\tau(y, z)) \rightarrow \mathsf{P}_\tau(x, z)$.

Another important property that is often associated with parthood is antisymmetry. There are two formulations of this property, depending on whether we use τ-equality (E_τ) or plain equality ($=$). The first formulation:

$(P3_\tau)$ $(\mathsf{P}_\tau(x, y) \wedge \mathsf{P}_\tau(y, x)) \rightarrow \mathsf{E}_\tau(x, y)$

is obviously true by definition. However, the second formulation:

$(P3_{\tau=})$ $(\mathsf{P}_\tau(x, y) \wedge \mathsf{P}_\tau(y, x)) \rightarrow x = y$

is stronger and may fail in some models. Antisymmetry in the sense of $(P3_{\tau=})$ is logically equivalent to the requirement that parthood be extensional in the following sense:

$(P4_{\tau=})$ $\forall z(\mathsf{P}_\tau(z, x) \leftrightarrow \mathsf{P}_\tau(z, y)) \rightarrow x = y$,

which in turn is equivalent to the requirement that connection is likewise extensional:

$(C3_{\tau=})$ $\forall z(\mathsf{C}_\tau(z, x) \leftrightarrow \mathsf{C}_\tau(z, y)) \rightarrow x = y$.

These requirements are stronger than the corresponding versions for E_τ. These latter are logically true, but whether a model satisfies $(P4_{\tau=})$ and $(C3_{\tau=})$ depends crucially on the relevant closure operator c and on which sets are included in the universe U.

It can easily be shown that for any pair of types $\tau_1 = \langle i_1, j, k\rangle$ and $\tau_2 = \langle i_2, j, k\rangle$, the following holds whenever $i_1 \leqslant i_2$:

$(C4_{i_1 i_2})$ $\mathsf{C}_{\tau_1}(x, y) \rightarrow \mathsf{C}_{\tau_2}(x, y)$.

The three parthood predicates are not, in general, related in a similar fashion. In fact, no instance of the following *inclusion* schema is generally true when $\tau_1 \neq \tau_2$:

$(P5_{i_1 i_2})$ $\mathsf{P}_{\tau_1}(x, y) \rightarrow \mathsf{P}_{\tau_2}(x, y)$.

Some mereotopologies include boundaries (i.e., lower dimensional entities) in their domain of discourse; others do not; these cases are examined separately below.

Boundary-tolerant theories

It turns out that none of the cases where τ is uniform ($i = j = k$) are viable:

(a) The option $i = 1$ yields implausible topologies in which the boundary of a region is never connected to the region's interior (since the boundary and the interior never share any points).

(b) The option $i = 2$ yields implausible mereologies in which every boundary is part of its own complement (since anything connected to the former is connected to the latter).

(c) The option $i = 3$ yields implausible mereotopologies in which the interior of a region is always connected to its exterior (so that boundaries make no difference) and in which the closure of a region is always part of the regions interior.

There is also a sense in which these theories trivialise all mereotopological distinctions in the presence of boundaries. For (a)–(c) imply that if τ is uniform, any model that includes the boundaries of its elements satisfies the conditional: $\mathsf{C}_\tau(x, y) \rightarrow \mathsf{O}_\tau(x, y)$.

Hence, in every such model the τ-abut predicate A_τ defines the empty relation, and so do the predicates of tangential and boundary parthood (TP_τ, BP_τ) and tangential and boundary overlap (TO_τ, BO_τ). Thus if boundaries are admitted in the domain, uniformly typed theories appear to be inadequate. In fact, this applies not only to uniform types, but to all types where $i = j$. (See [18, 96] for related material.)

Moving on to non-uniform types, we may note that some theories have been explicitly proposed in the literature, specifically for the case $\tau = \langle 2, 1, 1\rangle$. An early example is to be found in [25], though the topological primitive there is Op_τ rather than C_τ. (One gets a definitionally equivalent characterisation of C_τ via the definitions above. A similar warning applies to some other theories discussed below.) Other examples are in [49]. Since parthood P_τ is not defined in terms of the connection primitive C_τ, these theories need at least two distinct primitives (corresponding to the parameters 1 and 2 in the type); but since fusion σ_τ is typically understood using the same primitive as parthood, a third primitive is not needed (whence the equality of the second and third coordinates in the type). These theories typically represent an attempt to reconstruct ordinary topological intuitions on top of a mereological basis. In fact, it is immediate from the definition that in this case C_τ corresponds to the notion of connection of ordinary point-set topology: two regions are connected if the closure of one intersects the other, or vice versa. Moreover, P_τ is typically assumed to satisfy the relevant extensionality and inclusion principles.

Thus, a minimal theory of this kind is typically axiomatised using $(C1_\tau)$, $(C2_\tau)$, $(P1_\tau)$, $(P2_\tau)$, $(P3_\tau)$, $(P5_{12})$. If a fusion principle is added, the result is a mereotopology subsuming what is known as classical extensional mereology [189, 27], in which P_τ defines a complete Boolean algebra with the null element deleted. Further adding:

$(A1')$ $\quad \mathsf{P}_\tau(x, \mathsf{c}_\tau(x))$.

$(A2')$ $\quad \mathsf{P}_\tau(\mathsf{c}_\tau(\mathsf{c}_\tau(x)), \mathsf{c}_\tau(x))$.

$(A3')$ $\quad \mathsf{E}_\tau(\mathsf{c}_\tau(x) +_\tau \mathsf{c}_\tau(y), \mathsf{c}_\tau(x +_\tau y))$

gives what may be called a full mereotopology, in which c_τ behaves like the standard Kuratowski closure operator. ($(A0)$ has no analogue due to the lack of a null element.)

All of these theories, of course, must account in some way for the intuitive difficulties that arise out of the notion of a boundary, and correspondingly of the distinction between open and closed entities. For instance, Smith [57] considers various ways of supplementing a full mereotopology with a rendering of the intuition that boundaries are ontologically dependent entities [190], i.e., can only exist as boundaries of some open entity (contrary to the ordinary set-theoretic conception). In the notation here the simplest formulation of this intuition is given by the axiom:

$$(B1) \qquad \mathsf{BP}_\tau(x, y) \to \exists z(\mathsf{Op}_\tau(z) \wedge \mathsf{BP}_\tau(x, \mathsf{c}_\tau(z))).$$

It is noteworthy that all theories of this sort have type $\langle 2, 1, 1\rangle$. It is conjectured [49] that this is indeed the only viable option.

Boundary-free theories

Though the idea of a uniform type appears to founder in the case of boundary-tolerant theories, it has been taken very seriously in the context of boundary-free theories, i.e., theories that leave out boundaries from the universe of discourse in the intended models. Theories of this sort are rooted in [210, 56] and have recently become popular under the impact of Clarke's formulation [33, 34] (see also [96]). Clarke's own is a $\langle 1, 1, 1\rangle$-theory, and some later authors followed this account (e.g., [4, 5, 161]). However, one also finds examples of theories of type $\langle 2, 2, 2\rangle$ (e.g., in [105, 156]) as well as of type $\langle 3, 3, 3\rangle$ (especially in the work of Cohn et al., [43, 48, 100, 163]) which has led to an extended body of results and applications in the area of spatial reasoning; see [81] for an independent example of a type $\langle 3, 3, 3\rangle$ theory. Indeed, all boundary-free theories in the literature appear to be uniformly typed: this is remarkable but not surprising, since the main difficulties in reducing mereology to topology lies precisely in the presence of boundaries. Now, by definition, a boundary-free τ-theory admits of no boundary elements. In axiomatic terms, this is typically accomplished by adding a further postulate to the effect that everything is a region (i.e., has interior parts):

$$(R) \qquad \forall x \mathsf{Rg}_\tau(x)$$

which implies the emptiness of the relations BP_τ and BO_τ, hence of Bd_τ. So $\mathsf{b}_\tau(x)$ is never defined in this case. It is worth noting that such theories typically afford some indirect way of modelling boundary talk, e.g., as talk about infinite series of extended regions (cf. [18, 34, 72]). In this sense, these theories do have room for boundary elements, albeit only as higher-order entities. Note also the discussion of points and regions above in Section 13.2.1.

Consider now the three main options mentioned in the previous section, where τ is a basic uniform type of the form $\langle i, i, i\rangle$. Unlike their boundary-tolerant counterparts, none of these options yields a collapse of the distinction between tangential and interior parthood (TP_τ, IP_τ) or between tangential and interior overlap (TO_τ, IO_τ). However, the three options diverge noticeably with regard to the distinction between open and closed regions (Op_τ, Cl_τ). The general picture is as follows.

(a) The case $i = 1$ allows for the open/closed distinction, yielding theories in which the relation of abutting (A_τ) is a prerogative of closed regions (open regions abut nothing). As a corollary, such theories determine non-standard mereologies that violate the supplementation principle given above in Section 13.2.3. This is a feature that some authors have found unpalatable: as Simons [189] put it, one can discriminate regions that differ by as little as a point, but one cannot discriminate the point. There are also some topological peculiarities that follow from the choice of C_1 as a connection relation. For instance, it follows immediately that no region is connected to its complement, hence that the universe is bound to be disconnected. This was noted in [4, 34], where the suggestion is made that self-connectedness should be redefined accordingly:

$$\mathsf{Cn}'_\tau(x) =_{df} \forall y \forall z (\mathsf{E}_\tau(x, y +_\tau z) \rightarrow \mathsf{C}_\tau(\mathsf{c}_\tau(y), \mathsf{c}_\tau(z))).$$

This, however, is just a way of saying that self-connectedness must be defined with reference to a different notion of connection (namely, the notion obtained by taking $i = 3$).

(b) The case $i = 2$ also allows for the open/closed distinction, but yields theories in which the relation of abutting may only hold between two regions one of which is open and the other closed in the relevant contact area. This results in a rather standard topological apparatus, modulo the absence of boundary elements. However, also in this case the mereology is bound to violate (WSP). (Again, just take y open and x equal to the closure of y.)

(c) The case $i = 3$ is the only one where the open/closed distinction dissolves: in this case every region turns out to be τ-equal to its interior as well as to its closure. This follows from (P3$_\tau$), i.e., equivalently, from (C3$_\tau$) or (P4$_\tau$). This means that τ-theories of this sort cannot be extensional—in fact, they yield highly non-standard mereologies. However, this is coherent with the fundamental idea of a boundary-free approach. For one of the main motivations for going boundary-free is precisely to avoid the many conundrums that seem to arise from the distinction between open and closed regions [100]. In addition, and for this very same reason, such theories can validate (SA3), thereby eschewing the problem mentioned in (a) and (b) above.

The best known case of (c), i.e., a mereotopology with type $\langle 3, 3, 3\rangle$ was first presented in [163], and elaborated subsequently in a series of papers including [43, 48, 100, 44], which has been called the *Region Connection Calculus* (*RCC*).[9]

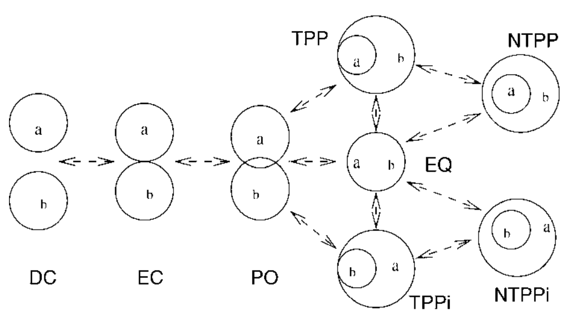

Figure 13.3: 2D illustrations of the relations of RCC-8 calculus and their continuous transitions (*conceptual neighbourhood*).

[9] Galton [92] coined this name.

In particular, a set of eight JEPD relations has been defined within the RCC mereotopology and this is now generally known as RCC-8, see Fig. 13.3.[10] The relation names used here differ from the relations defined above, but correspond thus (assuming the type is $\langle 3, 3, 3\rangle$ in each case): DC: ¬C, EC: A, PO: PO, TPP: TP ∧ ¬E, NTPP: IP, EQ: E; TPPi and NTPPi are simply the inverses of TPP and NTPP. The definitions of RCC-8 symbols, in particular $\mathsf{k}(x)$, differ from that given above—see [163], and in particular the discussion in [13, Section 3.3.3].

Examples of non-uniformly typed boundary-free theories are much rarer. However, one may imagine that such theories could also alleviate some of the unpalatable properties of the uniformly typed mereotopologies mentioned in (a) and (b) above. For example, a type of the form $\langle 1, 3, k\rangle$ would correspond to a mereotopology in which a type-1 notion of connection is combined with a type-3 parthood relation that satisfies the supplementation principle (WSP). Similarly with a type of the form $\langle 2, 3, k\rangle$. An example of a theory with a type 3 connection relation interpreted in boundary free models and a separate parthood relation is [128]—influenced by [177] this generalises the RCC system and the discrete mereotopology of Galton [91] to allow for discrete models of RCC (not possible in the standard theory cited above).

Topology via "n-intersections"

An alternative approach to representing and reasoning about topological relations has been promulgated via a series of papers including [65, 64, 69]. Three sets of points are associated with every region—its interior, boundary and complement. The relationship between any two regions can be characterised by a 3×3 matrix[11] called the *9-intersection* model, in which every entry in the matrix takes one of two values, denoting whether the intersection of the two point sets is empty or not; for example, the matrix in which every entry takes the non-empty value corresponds to the PO relation above.[12] Although it would seem that there are $2^9 = 512$ possible matrices, after taking into account the physical reality of 2D space and some specific assumptions about the nature of regions, it turns out that the there are exactly 8 remaining matrices, which correspond to the RCC-8 relations. Note, however, that the 9-intersection model only considers one-piece regions without holes in two-dimensional space, while RCC-8 allows much more general domains. Therefore, even though the two sets of relations appear similar, their computational properties differ considerably and reasoning in RCC-8 is much simpler than reasoning in the 9-intersection model [166]. One can also use the 9-intersection calculus to reason about regions which have holes by classifying the relationship not only between each pair of regions, but also the relationship between each hole of each region and the other region and each of its holes [68].

[10] A simpler, purely mereological calculus (usually called RCC-5), in which the distinctions between TPP and NTPP, TPPi and NTPPi, and DC and EC are collapsed has also been defined and investigated [127, 117].

[11] Actually, a simpler 2×2 matrix [65] known as the 4-intersection featuring just the interior and the boundary is sufficient to describe the eight RCC relations. However the 3×3 matrix allows more expressive sets of relations to be defined as noted below since it takes into account the relationship between the regions and its embedding space.

[12] The RCC-8 relations have different names in the 9-intersection model, in fact English words such as "overlap" instead of PO.

Different calculi with more JEPD relations can be derived by changing the underlying assumptions about what a region is and by allowing the matrix to represent the codimension of intersection. For example, one may derive a calculus for representing and reasoning about regions in Z^2 rather than R^2 [71]. Alternatively, one can extend the representation in each matrix cell by the specifying dimension of the intersection rather than simply whether it exists or not [36]. This allows one to enumerate all the relations between areas, lines and points and is known as the "dimension extended method" (DEM). A very large number of possible relationships may be defined in this way and a way termed as the "calculus based method" (CBM) to generate all these from a set of five polymorphic binary relations between a pair of spatial entities x and y: disjoint, touch, in, overlap, cross has been proposed [41]. A complex relation between x and y may then be formed by conjoining atomic propositions formed by using one of the five relations above, whose arguments may be either x or y or a boundary or endpoint operator applied to x or y. For the most expressive calculus (either the CBM or the combination of the 9-intersection and the DEM) there are 9 JEPD area/area relations, 31 line/area relations, 3 point/area relations, 33 line/line relations, 3 point/line relations and 2 point/point relations giving a total of 81 JEPD relations [41].

13.2.5 Between Mereotopology and Fully Metric Spatial Representation

Mereology and mereotopology can be seen as perhaps the most abstract and most qualitative spatial representations. However, there are many situations where mereotopological information alone is insufficient. The following subsections explore the different ways in which other qualitative information may be represented. After this, in Section 13.2.6 we look at how easily a spatial representation with a coordinate system and thus the full power of a geometry can be defined from qualitative primitives.

Direction and orientation

Direction relations describe the direction of one object to another, and can be defined in terms of three basic concepts: the primary object, the reference object and the frame of reference. Thus, unlike the mereotopological relations on spatial entities described in the preceding sections, a binary relation is not sufficient; i.e., if we want to specify the orientation of a *primary object* with respect to a *reference object*, then we need to have some kind of a *frame of reference*. This characterisation manifests itself in the display of qualitative direction calculi to be found in the literature: certain calculi have an explicit triadic relation while others presuppose an extrinsic frame of reference (such as the cardinal directions of $\mathrm{E}, \mathrm{N}, \mathrm{S}, \mathrm{W}$) [86, 112], or assume that objects have an intrinsic front (so that we can talk, for example, of being to the left of a person or vehicle); in this case we normally speak of *orientation* calculi, being the special case of a direction calculus when the primary object has an intrinsic front.

Of those with explicit triadic relations, a common scheme is to define (assuming attention is restricted to a 2D plane—as is usually the case in the literature) three relations between triples of points, denoting, clockwise, anti-clockwise or collinear ordering [184, 186, 176]. Schlieder developed a calculus [185] for reasoning about the relative orientation of pairs of line segments. Another triadic calculus is [116] which first defines binary relations on directed line segments using left/right relations

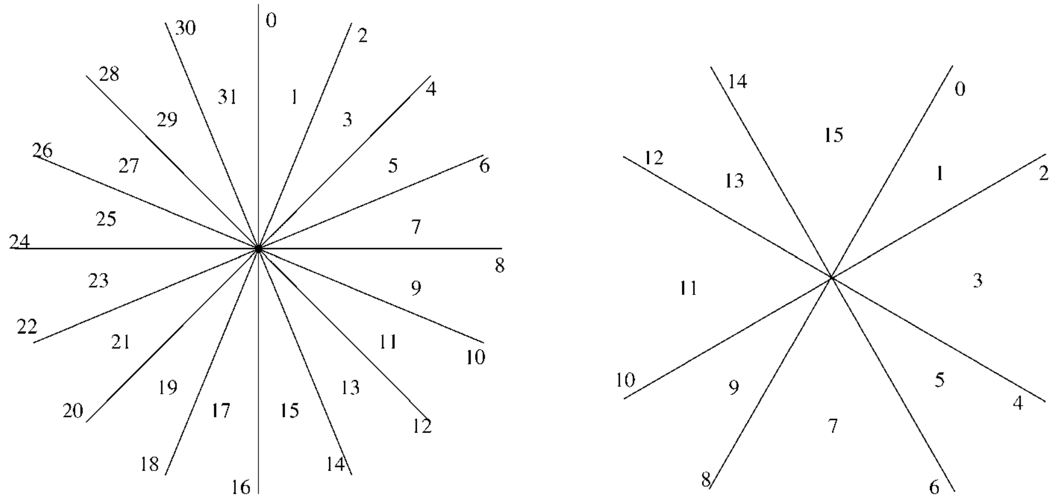

Figure 13.4: Different STAR calculi, the left one is defined using eight intersecting lines which result in 33 JEPD relations, the right one using four intersecting lines resulting in 17 JEPD relations. The STAR calculus allows any number and orientation of intersecting lines.

based on the intrinsic directedness of the line, and then defines ternary relations in terms of these, giving a 24 JEPD relation set, from which relations defining clockwise, anticlockwise and collinear can be recovered via disjunction.

For those calculi that use an extrinsic frame of reference, it is most common to use a given reference direction. This allows the orientation between two objects to be represented with respect to the reference direction using just binary relations. The first approaches described the directions of points in a 2D space. Frank [86] distinguished different ways of defining sectors for the different direction relations, cone-based and projection based (also called the cardinal direction algebra [130]), which both divide the plane into sectors relative to a point by using lines that intersect at the corresponding point. These calculi were later generalised for direction sectors generated by an arbitrary number of intersecting lines and form the STAR algebra [171] shown in Fig. 13.4. Interestingly, it turned out that once more than two intersecting lines are used for defining sectors, it is possible to generate a coordinate system and thus the distinction between qualitative and quantitative representation disappears. The solution to this dilemma is not to consider the lines as separate relations but to integrate them with sectors.

Most calculi for direction and orientation are based on points rather than regions, as calculi become rather coarse grained in the latter case. There are exceptions, for example, [101] or [135] in which directions *within* regions are considered (*London is in the south of England*). Directions for extended regions have mainly been developed for objects whose boundaries are parallel to the axes of the frame of reference, for example, the reference direction and the axis orthogonal to the reference direction, or by using a minimal bounding box which is parallel to the axes [8, 152]. A calculus which combines regions, mereotopology and a simple notion of unidimensional direction is the occlusion calculus of [164].

Distance and size

Spatial representations of distance can be divided into two main groups: those which measure on some "absolute" scale, and those which provide some kind of relative measurement. Of course, since traditional Qualitative Reasoning [209] is primarily concerned with dealing with linear quantity spaces, the qualitative algebras and the transitivity of such quantity spaces mentioned earlier can be used as a distance or size measuring representation, see Chapter 9.

Also of interest in this context are the order of magnitude calculi [140, 158] developed in the QR community. Most of these traditional QR formalisms are of the "absolute" kind of representations,[13] as in the delta calculus of [216]—which introduces a triadic relation: $x(>, d)y$ to note that x is larger/bigger than y by an amount d; terms such as $x(>, y)y$ mean that x is more than twice as big as y.

Of the "relative" representations specifically developed within the qualitative spatial reasoning community, perhaps the earliest is the triadic $\mathsf{CanConnect}(x, y, z)$ primitive [56]—which is true if body x can connect y and z by simple translation (i.e., without scaling, rotation or shape change). From this primitive it is easy to define notions such as equidistance, nearer than and farther than. This primitive allows a metric on the extent of regions to be defined: one region is larger than another if it can connect regions that the other cannot. Another method of determining the relative size of two objects relies on being able to translate regions (assumed to be shape and size invariant) and then exploit topological relationships—if a translation is possible so that one region becomes a proper part of another, then it must be smaller [143]; this idea is exploited in [51] to represent and reason about object location.

Of particular interest is the framework for representing distance [113] which has been extended to include orientation [40]. A distance system is composed of an ordered sequence of *distance relations* and a set of *structure relations* which give additional information about how the distance relations relate to each other. Each distance has an *acceptance area*; the distance between successive acceptance areas defines sequence of intervals: $\delta_1, \delta_2, \ldots$. The structure relations define relationships between these δ_i. Typical structure relations might specify a monotonicity property (the δ_i are increasing), or that each δ_i is greater than the sum of all the preceding δ_i. The structure relationships can also be used to specify order of magnitude relationships, e.g., that $\delta_i + \delta_j \sim \delta_i$ for $j < i$. The structure relationships are important in refining the *composition tables*.[14] In a *homogeneous* distance system all distance relations have the same structure relations; however this need not be the case in a *heterogeneous* distance system. The proposed system also allows for the fact that the context may affect the distance relationships: this is handled by having different frames of reference, each with its own distance system and with inferences in different frames of reference being composed using *articulation rules* (cf. [115]).

One obvious effect of moving from one scale, or context to another, is that qualitative distance terms such as "close" will vary greatly; more subtly, distances can behave in various "non-mathematical" ways in some contexts or spaces: e.g., distances may

[13] Actually it is straightforward to specify relative measurements given an "absolute" calculus: to say that $x > y$, one may simply write $x - y = +$.

[14] Section 13.3.2 introduces composition tables.

not be symmetrical.[15] Another "mathematical aberration" is that in some domains the shortest distance between two points may not be a straight line (e.g., because a lake or a building might be in the way), or the "Manhattan Distance" found in typical North American cities laid out in a grid system.

Shape

Shape is perhaps one of the most important characteristics of an object, and particularly difficult to describe qualitatively. In a purely mereotopological theory, very limited statements can be made about the shape of a region: e.g., whether it has holes, or interior voids, or whether it is one piece or not. It has been observed [92] that one can (weakly) constrain the shape of rigid objects by topological constraints using RCC-8 relations.

However, if an application demands finer grained distinctions, then some kind of semi-metric information has to be introduced.[16] For an explicit qualitative shape description one needs to go beyond mereotopology, introducing some kind of shape primitives whilst still retaining a qualitative representation. Of course, as [39] note: the mathematical community have developed many different geometries which are less expressive than Euclidean geometry, for example, projective and affine geometries, but have not necessarily investigated reasoning techniques for them (though see [7, 10, 35]).

A dichotomy can be drawn between representations which primarily describe the shape via the boundary of an object compared to those which represent its interior. Approaches to qualitative boundary description have been investigated using a variety of sets of primitives. The work of Meathrel and Galton [141] generalises much of this work. The basic idea is to consider the tangent at each point on the boundary of a 2D shape—it is either defined (D) or undefined (U)—in this latter case the boundary is at a cusp or kink point. If it is defined, then the rate of change of the tangent at that point can be considered (assuming a fixed (anticlockwise) traversal of the boundary), as can all the higher order derivatives (until it becomes undefined). Each derivative takes one of the qualitative values $+$, 0, $-$, and at the level of the first derivative denotes whether the shape is locally convex, straight or concave. Depending on how many higher order derivatives are considered, the description becomes progressively more and more detailed, and a greater variety of different shapes can be distinguished. The values $+$ and $-$ can only hold over a boundary segment, whereas 0 and U can hold at a single boundary point. Thus the description of a boundary starts at a particular point, and then proceeds, anticlockwise, to label maximal boundary segments having a particular qualitative value, and isolated points that may separate these. There are constraints on what sequences of descriptions are possible, and the rules for construction a *Token Ordering Graph* (which is an instance of the continuity networks/conceptual neighbourhoods discussed in Section 13.4 below) have been formulated. For example, a $+$ segment cannot directly transition to a $-$ segment without passing through a $U/0$ point or a 0 segment.

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[15] E.g., because distances are sometimes measured by time taken to travel, and an uphill journey may take longer than a return downhill journey [113].

[16] Of course, orientation and distance primitives as discussed above already add something to pure topology, but as already mentioned these are largely point based and thus not directly applicable to describing shape of a region.

Shape description by looking at global properties of the region rather than its boundary has been investigated too, for example, the work of [39] describes shape via properties such as compactness and elongation by using the minimum bounding rectangle of the shape and the order of magnitude calculus of [140]: elongation is computed via the ratio of the sides of the minimum bounding rectangle whilst compactness by comparing the area of the shape and its minimum bounding rectangle. The medial axis can also be used as a proxy for shape, and has been used extensively in the computer vision community, and within a KR setting in [179] for distinguishing lakes from rivers. The notion of a Voronoi hull has also been used (e.g., [63]).

Combinations of different aspects

Although we have attempted to present various aspects of spatial representation separately, in general they interact with each other. For example, knowing the relative size of two regions (smaller, larger, equal) can effect which mereotopological relationships are possible [95]. There is also a relationship between distance and the notion of orientation: e.g., distances cannot usually be summed unless they are in the same direction, and the distance between a point and a region may vary depending on the orientation. Thus it is perhaps not surprising that there have been a number of calculi which are based on a primitive which combines distance and orientation information. One straightforward idea [86] is to combine directions as represented by segments of the compass with a simple distance metric (*far, close*). A slightly more sophisticated idea is to introduce a primitive which defines the position of a third point with respect to a directed line segment between two other points [217] (generalised to the 3D case in [150]). Another approach that combines knowledge about distances and positions in a qualitative way—through a combination of the Delta-calculus [216] and orientation is presented in [215]. Liu [134] explicitly defines the semantics of qualitative distance and qualitative orientation angles and formulates a representation of qualitative trigonometry. A example of a combined distance and position calculus is [75]. A discussion of different ways to combine different aspects can be found in [174].

13.2.6 Mereogeometry

Just as mereotopology extends mereology with topological notions, so mereogeometry extends mereology with geometrical concepts. In principle one could add any of the notions of orientation or distance/size discussed above to mereology, but most of those are defined on points rather than regions which mereology presumes. In the style of [49] for mereotopology, Borgo and Masolo [22] compare and contrast a range of mereogeometries. The benchmark system is *Region Based Geometry* (RBG) [14, 16] which builds on the earlier work of Tarski [195]. This uses $\mathsf{P}(x, y)$ and $\mathsf{S}(x)$ (x is a sphere) as primitives, and captures full Euclidean geometry, in a region based setting. RBG is axiomatised in second order logic, and has been shown to be categorical [14]. Three other systems [21, 148, 56, 57] are shown to be equivalent, and all are termed *Full Mereogeometries*; these other systems have different sets of primitives, for example, the CanConnect primitive mentioned above in Section 13.2.5 or the primitive $\mathsf{CG}(x, y)$ (x is congruent to y). A fifth system [200, 6], which uses the primitive $\mathsf{Closer}(x, y, z)$ (x is closer to y than to z) reported there to be slightly weaker, is in fact also a full mereogeometry, a result which follows as an immediate consequence of the results in [54]. It is conjectured in [22] that the theory obtained by adding a convex hull primitive to mereotopology (as in extensions of RCC [43]) is strictly weaker. In fact, in [54] it is shown that this is indeed the case since such a language is invariant under affine transformations, and thus unable to express properties such as $\mathsf{S}(x)$ which is not invariant. This followed on from an earlier result, in which it was shown that in a constraint language [55] the primitives for adjacency, parthood and convexity are sufficient in combination to provide an affine geometry. A similar result is provided in [156] where it is demonstrated that the first order language with parthood and congruence of primitives also enables the distinction of any two regions not related by an affine transformation. Moreover, it is shown that a coordinate system can be defined in this language, thus raising the question of whether it deserves the label *qualitative*—and indeed this result and question also apply to any full mereogeometry. A similar observation has already been made above for the STAR calculus [171] described in Section 13.2.5, and also indeed for the affine mereogeometries based on convexity mentioned just above [54].

An application of a mereogeometry based on congruence and parthood to reasoning about the location of mobile rigid objects is [51]. A simple constraint language whose four primitives combine notions of congruence and mereology has been defined and investigated from a computational viewpoint [50]—the primitives are EQ, CGPP, CGPPi (congruent to a proper part, and the inverse relation), and CNO (where none of the other relations hold).

13.2.7 Spatial Vagueness

The problem of vagueness permeates almost every domain of knowledge representation. In the spatial domain, this is certainly true, for example, it is often hard to determine a region's boundaries (e.g., "southern England").

Vagueness of spatial concepts can be distinguished from that associated with spatially situated objects and the regions they occupy. An adequate treatment of vagueness in spatial information needs to account for vague regions as well as vague relationships [46]. Although there has been some philosophical debate concerning whether vague objects can exist [76], formal theories dealing with vagueness of extent are not well-established.

Existing techniques for representing and reasoning about vagueness such as supervaluation theory have been extended and applied in a spatial context [179] and [15], which also specifically addresses the issue of the preservation of object identity in the face of loss of 'small' parts.

There have also been extensions of existing spatial calculi specifically designed to address spatial indeterminacy. In particular there have been extensions of both the RCC calculus [45, 46] (called the "egg-yolk" calculus) and the 9-intersection [37]; the broad approach in each of these is essentially the same—to identify a core region which always belongs to the region in question (the *yolk* in the terminology of former), and an extended region which might or might not be part of it (together forming the *egg*). It turns out that if one generalises RCC-8 in this way [46] there are 252 JEPD relations between non-crisp regions which can be naturally clustered into 40 equivalence classes, and 46 JEPD relations, clustered into 13 equivalence classes in the case of the extension to the purely mereological RCC-5. The axiomatic presentation of the egg-yolk calculus in [46] extends the ontology of crisp regions with vague (non-crisp) ones and relies on an additional binary relation, 'x is crisper than region y'. An application of the egg-yolk calculus to reasoning about a non-spatial domain, class integration across databases, is [127].

It has been shown [38] that the extension of the 9-intersection model to model regions with broad boundaries can be used to reason not just about regions with indeterminate boundaries but also can be specialised to cover a number of other kinds of regions including convex hulls of regions, minimum bounding rectangles, buffer zones and rasters. (This last specialization generalises the application of the n-intersection model to rasters previously undertaken [71].)

Another notion of indefiniteness relates to locations. Bittner [19] deals with the notion of exact, part and rough location for spatial objects. The exact location is the region of space taken up by the object. The notion of part location (as introduced by [26]) relates parts of a spatial object to parts of spatial regions. The rough location of a spatial object is characterised by the part location of spatial objects with respect to a set of regions of space that form regional partitions. Consequently, the notion of rough location links parts of spatial objects to parts of partition regions.

Bittner [19] argues that the observations and measurements of location in physical reality yield knowledge about rough location: a vaguely defined object o is located within a regional partition consisting of the three concentric regions: 'core', 'wide boundary' and 'exterior'. In this context, the notion of rough location within a partition consisting of the three concentric regions coincides with the notion of vague regions introduced by [45].

It is worth noting the similarity of these ideas to rough sets [60], though the exact relationship has yet to be fully explored, though see, for example, [154, 20]. Other approaches to spatial uncertainty are to work with an indistinguishability relation which is not transitive and thus fails to generate equivalence classes [199, 118], and the development of nonmonotonic spatial logics [188, 3].

13.3 Spatial Reasoning

In the previous section we described some approaches to representing spatial information and gave different examples of spatial representations from the vast literature on this topic. For some purposes it is enough to have a representation for spatial knowledge, but what makes intelligent systems intelligent is their ability to reason about given knowledge. There are different reasoning tasks an intelligent system might have to perform. These include deriving new knowledge from the given information, checking consistency of given information, updating the given knowledge, or finding a minimal representation. Even though these reasoning problems are quite different, they can be transformed into each other, and algorithms developed for one reasoning problem can often easily be modified to solving other reasoning problems. Much of the research on spatial reasoning has therefore focused on one particular reasoning problem, the *consistency problem*, i.e., given some spatial information, is the given information consistent or inconsistent.

In principle, reasoning about spatial knowledge given in the form of a logical representation is not different from reasoning about other kinds of knowledge. However, much of the qualitative spatial knowledge we are dealing with is of a very particular form and can be represented as relations between spatial entities. We are usually considering binary and sometimes ternary relations which can be represented as constraints restricting the spatial properties of the entities we are describing. This constraint-based representation gives us the possibility to develop reasoning algorithms which are much more efficient than standard logical deduction, albeit less powerful.

A constraint-based representation of spatial knowledge takes the form of an existentially quantified first-order logical expression: $\exists x_1 \ldots \exists x_n \bigwedge_{i,j} \bigvee_{R \in \mathcal{A}} R(x_i, x_j)$, where $x_1, \ldots, x_n$ are variables over the domain of spatial entities, $\mathcal{A}$ is the set of available base relations, and $R(x_i, x_j)$ is a binary constraint which restricts the possible instantiations of x_i, x_j to the tuples of R. Solving this formula is basically a *constraint satisfaction problem* (*CSP*) as described in Chapter 4. One of the major differences of spatial relations and spatial constraints to those constraints described in Chapter 4 is that the domain of spatial entities is usually infinite, i.e., there is an infinite number of spatial entities that can be assigned to the variables $x_1, \ldots, x_n$ and which might have to be tested when deciding consistency of spatial information. While standard CSPs over finite domains are in general NP-complete, spatial CSPs over *infinite domains* are potentially undecidable.

Spatial reasoning with constraints and relations mainly relies on algebraic operators on the relations, the most important being the *composition* operator. Two relations R and S are composed according to the following definition: $R \circ S = \{(x, y) \mid \exists z\colon (x, z) \in R \text{ and } (z, y) \in S\}$. Composition has to be computed using the formal semantics of the relations. Due to the infinite domains, computing composition can be an undecidable problem. If the compositions of the base relations can be computed, they can be stored in a composition table and reasoning becomes a matter of table look-ups.

The main research topics in spatial reasoning in the past decade include the following:

- determining the complexity of reasoning over different spatial calculi,
- proving that a formalism is decidable and if so, possibly identifying tractable or even maximal tractable subsets of spatial calculi,
- finding representations of qualitative spatial knowledge which allow for more efficient reasoning,
- developing efficient algorithms for spatial reasoning as well as approximation methods and heuristics which lead to faster solutions in practice,
- developing methods for proving tractability,
- computing composition tables and verifying their correctness,
- determining whether a qualitative spatial description is *realisable*, i.e., whether a planar interpretation exists.

In this section we give an overview of some of the main achievements in this area. It is worth mentioning that some of these research questions originated in the area of temporal reasoning and most methods can be applied to both spatial and temporal reasoning (see Chapter 12).

13.3.1 Deduction

If the properties of the spatial relations and entities under consideration are represented axiomatically in a logical formalism, then of course a standard deduction mechanism for the formalism could be used for reasoning about the spatial knowledge so represented. As described in Section 13.2.4, the Region Connection Calculus was defined in first-order logic [163]. Even though reasoning in this first order representation of RCC (or indeed any first order mereotopology) is undecidable [104], first order theorem proving has been used to verify a number of theorems including those relating to the RCC-8 composition table [162] and its *conceptual neighbourhood* [110].

In order to create a decidable reasoning procedure, Bennett developed encodings of the RCC-8 relations first in propositional intuitionistic logic [11] and later an advanced encoding in propositional modal logic [12]. The encoding does not reflect the full expressive power of the first order RCC-8 theory, but does enable a decision procedure to be built. In the modal encoding, regions are represented as propositional atoms and a modal operator $\mathbf{I}$ is used to represent the interior of a region, i.e., if X represents a region, then $\mathbf{I}X$ represents the interior of X. The interior operator is an S4 modality and goes back to work by Tarski [194]. The usual propositional operators are used to represent intersection, union, or complement or regions. In addition, Bennett divides the propositional formulas into two types, *model constraints* which have to hold in all models of the encoding, and *entailment constraints* which are not allowed to hold in any model of the encoding. The model and entailment constraints are combined to a single formula using another modal operator $\Box$ which Bennett calls a strong S5 modality.

When encoding spatial relations in different logics, it is important to not only encode the properties of the relations, but also the properties of the spatial entities that are being used. Bennett's initial encodings were missing the regularity property of regions which was later added to the encoding [172]. The extended modal encoding was shown to be equivalent to the intended interpretation of the RCC-8 relations [149].

The intuitionistic and modal encodings were not only useful for providing a decidable decision procedure for reasoning about spatial information represented using RCC-8 relations, but also formed the basis for the subsequent computational analysis of RCC-8. Nebel [145] used the intuitionistic encoding for showing that the RCC-8 consistency problem is tractable if only base relations are used.[17] Renz and Nebel [172] used Bennett's modal encoding and transformed it into a classical propositional encoding. As well as performing actual spatial reasoning on an RCC-8 representation, the propositional encoding has also been used for analysing the computational properties of RCC-8. Since modern SAT solvers are extremely efficient, it might be possible that deductive reasoning can be used for obtaining efficient solutions to spatial reasoning problems. A similar analysis has been done by Pham et al. [153] who compared reasoning over the interval algebra using constraint-based reasoning methods with deductive reasoning using modern SAT solvers. First results indicate that deductive reasoning can be more efficient in some cases than constraint based reasoning.

[17] Due to the missing regularity conditions in the intuitionistic encoding, Nebel's result turned out to be incomplete.

There have been several extensions of the modal encoding of RCC-8 to deal with more expressive spatial and also with spatio-temporal information. BRCC-8 generalises the RCC-8 modal encoding to also cover Boolean combinations of spatial regions [212]. $\mathrm{S4}_u$ which is the propositional modal logic S4 extended with the universal modalities is the most general version and contains both BRCC and RCC-8 as fragments [213]. Several of these fragments have been combined with different temporal logics and compared with respect to their expressiveness and their complexity [89]. Modal logics are closely related to Description Logics, and in this context, we note that some research has been on spatial description logics [106].

Some work has also investigated constraint languages more expressive than mereotopology: it has been shown that the constraint language of $\mathsf{EC}(x, y)$, $\mathsf{PP}(x, y)$ and $\mathsf{conv}(x)$ is intractable (it is at least as hard as determining whether a set of algebraic constraints over the reals is consistent) [55].

13.3.2 Composition

Given a domain of spatial entities $\mathcal{D}$, spatial relations are subsets of the cross-product of $\mathcal{D}$ and may contain an infinite number of tuples, i.e., $R \subseteq \{(a, b) \mid a, b \in \mathcal{D}\}$, since $\mathcal{D}$ may itself be infinite. Having a set of jointly exhaustive and pairwise disjoint base relations $\mathcal{A}$ and considering the powerset $2^{\mathcal{A}}$ of the base relations as the set of possible relations, the algebraic operations union, intersection, and complement of relations are straightforward to compute. If the set of base relations is chosen in a way such that the converse relations of all base relations are also base relations, then the converse operator is also easy to compute. The most important algebraic operator which is the basis for reasoning over spatial relations is the *composition* operator which is defined as $R \circ S = \{(x, y) \mid \exists z\colon\ (x, z) \in R \text{ and } (z, y) \in S\}$ for two relations R and S. If composition is known for all pairs of base relations, then composition of all relations can be computed as the union of the pairwise compositions of all base relations contained in the relation, i.e., $R \circ S = \{R_i \circ S_j \mid R_i, S_j \in \mathcal{A}, R_i \subseteq R, S_j \subseteq S\}$. Therefore, if the composition and the converse of all base relations are known and if they are all contained in $2^{\mathcal{A}}$, i.e., if $2^{\mathcal{A}}$ is closed under composition and converse, then it is possible to reason about spatial relations without having to consider the tuples contained in the relations. The relations can then be treated as symbols that can be manipulated using the algebraic operators. In the following section we describe how this can be done using constraint-based reasoning methods.

The question remains how the composition of base relations can be computed if the domains are infinite. While it is possible to compute composition in situations where the domains can be ordered or are otherwise well-structured (for example, domains based on linear orders such as the Directed Interval Algebra [168] or the rectangle algebra [8]), in many cases it is not possible to compute composition effectively. This includes RCC-8 where it is possible to find example scenarios which show that the given composition table is not correct. One example given by Düntsch [61] considers three regions A, B, C in two-dimensional space where A is a doughnut and B its hole. It is not possible to find a region C which is externally connected to A and B and therefore the tuple (A, B) which is contained in the relation EC is not contained in $\mathsf{EC} \circ \mathsf{EC}$. So the composition of EC with EC does not contain EC even though this is specified in the RCC-8 composition table. In cases where it is not possible to compute composition or where a set of relations is not closed under composition, it is necessary to resort to a weaker form of composition in order to apply constraint-based reasoning mechanisms. Düntsch [61] proposed using *weak composition*. The weak composition of two relations $R, S \in 2^{\mathcal{A}}$ is the strongest relation of $2^{\mathcal{A}}$ which contains the actual composition, i.e., $R \circ_w S = \{B \mid B \in \mathcal{A}, B \cap (R \circ S) \neq \emptyset\}$. It is clear that any set $2^{\mathcal{A}}$ is always closed under weak-composition and therefore constraint-based reasoning methods can be applied to these relations. The RCC-8 composition table [162] is actually a weak composition table.

If only weak composition can be used, some of the inferences made by composing relations are not correct and might lead to wrong results. It has been shown that correctness of the inferences does not depend on whether composition or only weak composition is used, but on a different property, namely, whether a set of relations is closed under constraints [170]. A set of relations $2^{\mathcal{A}}$ is *closed under constraints* if for none of its base relations $R \in \mathcal{A}$ there exists two sets Θ_1, Θ_2 of constraints over $2^{\mathcal{A}}$ which both contain the constraint xRy such that the following property holds: Θ_1 and Θ_2 refine the constraint xRy to the constraints xR_1y and xR_2y, respectively, where $R_1, R_2 \subseteq R$ and $R_1 \cap R_2 = \emptyset$. That is none of the atomic relations can be refined to two non-overlapping sub-atomic relations by using arbitrary sets of constraints.

13.3.3 Constraint-based Spatial Reasoning

Using constraint-based methods for spatial reasoning gives the possibility to capture much of spatial reasoning within a unified framework. Even though qualitative spatial information is very diverse and covers different spatial aspects, it is usually expressed in terms of spatial relations between spatial entities which can be expressed using constraints. As mentioned in the introduction of this section, many different spatial reasoning tasks can be reduced to the *consistency problem*, on which we will focus on in this section.

Definition 1. *Let $\mathcal{A}$ be a finite set of JEPD binary relations and over a (possibly infinite) domain $\mathcal{D}$ and $\mathcal{S} \subseteq 2^{\mathcal{A}}$. The consistency problem* $\mathsf{CSPSAT}(\mathcal{S})$ *is defined as follows*:

Instance: *A finite set V of variables over the domain $\mathcal{D}$ and a finite set Θ of binary constraints xRy, where $R \in \mathcal{S}$ and $x, y \in V$.*

Question: *Is there an instantiation of all variables in Θ with values from $\mathcal{D}$ such that all constraints in Θ are satisfied*?

Constraint-based reasoning uses constraint propagation in order to eliminate values from the domains which are not consistent with the constraints (see Chapter 4). Since the domains used in spatial and temporal reasoning are usually infinite, restricting the domains is not feasible. Instead, it is possible to restrict the domains indirectly by restricting the relations that can hold between the spatial entities. This can only be done if there is a finite number of relations and an effective way of propagating relations, which is the case if we have a set of relations $\mathcal{S} \subseteq 2^{\mathcal{A}}$ which is closed under intersection, converse and weak composition. These operators are the only means we have for propagating constraints. While it is possible to use composition of higher arity, usually only binary composition is used for propagating constraints.

The best known constraint propagation algorithm for spatial CSPs is the *path-consistency algorithm* [136] (see also Chapter 4 of this Handbook). It is a local consistency algorithm which makes all triples of variables of Θ consistent by successively refining all constraints using the following operation until either a fixed point is reached or one constraint is refined to the empty relation: $\forall x, y, z. x\{R\}y := x\{R\}y \cap (x\{S\}z \circ z\{T\}y)$. If the empty relation occurs, then Θ is inconsistent, otherwise the resulting set is called *path-consistent*. If $2^{\mathcal{A}}$ is closed under composition, intersection and converse, then the path-consistency algorithm terminates in cubic time.

Path-consistency is equivalent to 3-consistency [88] which holds if for every consistent instantiation of two variables it is always possible to find an instantiation for any third variable such that the three variables together are consistent. 3-consistency can be generalised to *k-consistency* which holds if for any consistent instantiation of $k - 1$ variables there is always a consistent instantiation for any kth variable. In order to compute k-consistency, it is necessary to have $(k - 1)$-ary composition. In the following we restrict ourselves to 3-consistency and the associated path-consistency algorithm which uses binary composition.

In many cases, composition cannot be computed and only weak composition is available. In these cases, the path-consistency algorithm cannot be applied and a weaker algorithm, the *algebraic-closure algorithm* must be used [132]. Both algorithms are identical except that the path-consistency algorithm uses composition while the algebraic-closure algorithm uses weak composition. If the algebraic closure algorithm is applied to a set of constraints and a fixed point is reached, the resulting set is called *algebraically closed* or *a-closed*. It is clear that unless weak-composition is equivalent to composition, an a-closed set is usually not 3-consistent.

Local consistency algorithms such as path-consistency and algebraic-closure, and possible variants of these algorithms which make use of composition of higher arity, are the central methods that constraint-based reasoning offers to solving the consistency problem. It is highly desirable that for a given set of relations $2^{\mathcal{A}}$, the consistency problem for the base relations, i.e., CSPSAT($\mathcal{A}$), can be decided using a local consistency algorithm. It has been shown that algebraic-closure decides CSPSAT($\mathcal{A}$) if and only if $2^{\mathcal{A}}$ is closed under constraints [170]. While this is mainly useful for showing that algebraic closure does not decide CSPSAT($\mathcal{A}$), the other direction has to be manually proven for each set $\mathcal{A}$ and for each domain $\mathcal{D}$. If a decision procedure for CSPSAT($\mathcal{A}$) can be found, then the consistency problem for the full set of relations is also decidable and can be decided by backtracking over all sub-instances which contain only base relations.

The basic backtracking algorithm takes as input a set of constraints Θ over a set of relations $\mathcal{S} \subseteq 2^{\mathcal{A}}$, selects an unprocessed constraint $x\{R\}y$ of Θ, splits R into its base relations $B_1, \ldots, B_k$, replaces $x\{R\}y$ with $x\{B_i\}y$ and repeats this process recursively until all constraints are refined. If the resulting set of constraints is consistent, which can be shown using the local consistency algorithm, then Θ is consistent. Otherwise the algorithm backtracks and replaces the last constraint with the next possible base relation $x\{B_j\}y$. If all possible refinements of Θ are inconsistent, then Θ is inconsistent. The backtracking algorithm spans a search tree where each recursive call is a node and each leaf is a refinement of Θ which contains only base relations. If CSPSAT($\mathcal{A}$) can be decided in polynomial time, then CSPSAT($2^{\mathcal{A}}$) is in NP and the runtime of the backtracking algorithm is exponential in the worst case.

There are several ways of improving the performance of the backtracking algorithm. The easiest way is to apply the local consistency algorithm at every recursive step. This prunes the search tree by removing base relations that cannot lead to a solution. Nebel [147] has shown that the interleaved application of the path-consistency algorithm does not alter the outcome of the backtracking algorithm, but considerably speeds up its performance. The performance can also be improved by using heuristics for selecting the next unprocessed constraint and for selecting the next base relations. The first choice can reduce the size of the search tree while the second choice can help finding a consistent sub-instance earlier. While the basic backtracking algorithm refines a set Θ to sets containing only base relations, it is also possible to use any other set of relations $\mathcal{T}$ which contains all base relations and for which there is an algorithm which decides consistency for this set. If $\mathsf{CSPSAT}(\mathcal{T})$ can be decided in polynomial time, $\mathcal{T}$ is a *tractable subset* of $2^{\mathcal{A}}$. A tractable subset is a *maximal tractable subset*, if adding any other relation not contained in the tractable subset leads to an intractable subset. Tractable subsets can be used to improve backtracking by splitting each constraint $x\{R\}y \in \Theta$ into constraints $x\{T_1\}y, x\{T_2\}, \ldots, x\{T_s\}y$ such that $\bigcup_i T_i = R$ and all $T_i \in \mathcal{T}$, and by backtracking over these constraints. This considerably reduces the *branching factor* of the search tree. Instead of splitting each relation into all of its base relations, they can be split into sub-relations contained in $\mathcal{T}$ [126]. The average branching factor of the resulting search tree depends on how well $\mathcal{T}$ splits the relations of $2^{\mathcal{A}}$. The lower the average branching factor, the smaller the search tree.

It has been shown in detailed empirical analyses [173] that large tractable subsets combined with different heuristics can lead to very efficient solutions of the consistency problem. While it is not possible to determine in advance which choice of heuristics will be most successful for solving an instance of a spatial reasoning problem, it is clear that having large tractable subsets will always be an advantage. A lot of research effort has therefore been spent on identifying tractable subsets of spatial calculi.

The methods described above of using constraint propagation for determining local consistency and using backtracking for solving the general consistency problem can be applied to all kinds of spatial information if the spatial relations used are constructed from a set of base relations and the information is expressed in the form of constraints over these relations. This has the advantage that general methods and algorithms can be applied and that results for one set of spatial relations can be carried over to other sets. One problem with this approach is that spatial entities are treated as variables which have to be instantiated using values of an infinite domain. How to integrate this with settings where some spatial entities are known or can only be from a small domain is still unknown and is one of the main future challenges of constraint-based spatial reasoning.

13.3.4 Finding Efficient Reasoning Algorithms

As discussed in the previous section, large tractable subsets of spatial calculi are the most important part of efficient spatial reasoning. In order to find tractable subsets, or even maximal tractable subsets, several ingredients have to be provided:

1. One ingredient is a method for proving the complexity of a given subset, or slightly weaker, a sound method for proving that a given subset is tractable.

2. The second ingredient is a way of finding subsets that might be tractable subsets and for which the method described above can be used. A set of n base relations contains 2^n relations and $2^{(2^n)}$ different subsets. It is impossible to test all subsets for tractability, so the number of candidate sets should be made as small as possible.

3. In order to show that a tractable subset is a maximal tractable subset, it must be shown that any relation which is not contained in the tractable subset leads to an NP-hard subset when added to the tractable subset. For this it is necessary to have a method for proving NP-hardness of a given subset.

4. For a complete analysis of tractability, it must be shown that the identified tractable subsets are maximal tractable subsets and that no other subset which is not contained in one of the maximal tractable subsets is tractable.

In this section we are interested in finding tractable subsets of $2^{\mathcal{A}}$ for efficiently solving the consistency problem CSPSAT($\mathcal{A}$). We are therefore only interested in finding tractable subsets which contain all base relations as only these subsets can be used as split-sets in our backtracking algorithm. There has been a series of papers on finding tractable subsets of the Interval Algebra (e.g., [122]) and also of RCC-5 [117] which do not contain all base relations and which are mainly interesting for a theoretical understanding of what properties lead to intractability.

The number of subsets which have to be considered for analysing complexity can be greatly reduced by applying the closure property [172]: the closure of a set $\mathcal{S} \subseteq 2^{\mathcal{A}}$ under composition, intersection and converse has the same complexity as $\mathcal{S}$ itself. For finding tractable subsets this means that only subsets which are closed under the operators have to be considered, as all subsets of a tractable set are also tractable. This can only be applied if a set is closed under composition. Since in many cases only weak composition is known, it is not obvious that the closure under weak composition has the same complexity. It has only recently been shown [170] that whenever algebraic-closure decides consistency of CSPSAT($\mathcal{A}$), i.e., for atomic CSPs, then the closure under weak composition preserves complexity.

There have been several methods for finding tractable subsets of NP-hard sets of relations. The most obvious way is to find a polynomial one-to-one transformation of CSPSAT to another NP-hard problem for which tractable subclasses are known. The most popular problem is certainly the propositional satisfiability problem SAT for which two tractable subclasses are known, HORNSAT where each clause contains at most one positive literal, and 2SAT where each clause contains at most two literals. If CSPSAT($2^{\mathcal{A}}$) can be reduced to SAT and it is possible to find relations of $2^{\mathcal{A}}$ which lead to Horn clauses (HORNSAT) or Krom clauses (2SAT), respectively, then the set of all these relations is tractable. This method has first been applied by Nebel and Bürckert [146] for the Interval Algebra and later also by Nebel [145] and by Renz and Nebel [172] for RCC-8.

A different method has been proposed by Ligozat [129] who transformed the relations of the Interval Algebra to regions on a plane and to the lines that separate the regions. The dimension of a relation is the dimension to which a relation is transformed to, a two-dimensional region, a one-dimensional line, or a zero-dimensional point (the intersection of lines). Ligozat showed that the set of those relations that can be transformed to a convex set are tractable (the *convex relations*), and also those relations which do not yield a convex region but a region for which the convex closure adds only relations of lower dimension (the *preconvex relations*). This method has also been applied to other sets of relations, in particular those which are somehow derived from the interval algebra [131], but it seems that the preconvexity method cannot be generalised for every algebra.

These methods have in common that they can only be used for proving tractability of one or maybe two different particular subsets, but not for showing tractability for arbitrary subsets. Another method that has been proposed, the *refinement method* [167], is more general and can be applied to any subset. The refinement method takes as input a refinement strategy, which is a mapping of every relation of the to be tested subset $\mathcal{S}$ to a subset $\mathcal{T}$ for which it is known that algebraic closure decides consistency in polynomial time. The mapping must be a refinement, i.e., every relation $S \in \mathcal{S}$ must be mapped to a relation $T \in \mathcal{T}$ such that $T \subseteq S$. The refinement method then checks every a-closed triple of relations over $\mathcal{S}$ and tests whether making the refinements leads to an inconsistency. If none of the original refinements nor the new refinements obtained by applying the method result in an inconsistency, then algebraic closure also decides consistency for $\mathcal{S}$ and therefore $\mathcal{S}$ is a tractable subset. The refinement method relies upon finding a suitable refinement strategy. It has been shown that using the identity refinement strategy, i.e., removing all identity relations, was successful for all the tested subsets of RCC-8 and the interval algebra [167].

Even though the refinement method is very general, it does not help with finding candidate sets to which it can be applied. All candidates have in common that they must be closed under (weak) composition, converse and intersection and they must not contain any relation which is known to be an NP-hard relation. Therefore we also need methods for identifying NP-hard relations, i.e., relations that make the consistency problem NP-hard when combined with the base relations. In order to show NP-hardness of a set of relations $\mathcal{N} \subseteq 2^{\mathcal{A}}$, it is sufficient to find a known NP-hard problem which can be polynomially reduced to $\mathsf{CSPSAT}(\mathcal{N})$. This is a difficult problem and might require a different transformation from a different NP-hard problem for each different set $\mathcal{N}$. However, since CSPSAT has a common structure for all sets of relations, namely, a constraint graph where the labels on the edges are unions of base relations, it is possible to generate the transformations with computer assisted methods.

Renz and Nebel [172] proposed a scheme for transforming 3SAT variants to CSPSAT by translating variables, literals and clauses to a set of spatial constraints and to relations $R_t, R_f \in 2^{\mathcal{A}}$ which correspond to variables and literals being true (R_t) or false (R_f). For example, each variable p is transformed to the constraints $x_p^+\{R_t, R_f\}y_p^+$ and $x_p^-\{R_t, R_f\}y_p^-$ where the first constraint is refined to the relation R_t if p is true and the second one to R_f if p is true. In order to ensure this, additional *polarity constraints* between the remaining pairs of x_p^+, x_p^-, y_p^+ and y_p^- are needed. *Clause constraints* which ensure that the requirements imposed by the clauses hold for the spatial variables are also needed. The relations R_t and R_f as well as the relations contained in the polarity and clause constraints can be found by exhaustive search over all possible relations. If an assignment of relations of $2^{\mathcal{A}}$ to this constraint schema can be found and if it can be shown that the transformation preserves consistency, then the set $\mathcal{N}$ of all relations used in this schema is NP-hard.

Based on this NP-hard subset $\mathcal{N}$, it is possible to identify other NP-hard subsets using the closure property and a computer assisted enumeration of different subsets. Every subset of $2^{\mathcal{A}}$ whose closure contains $\mathcal{N}$ is also an NP-hard subset. Easier to compute and more useful is the property that for a known tractable subset $\mathcal{T}$ and every relation $R \in 2^{\mathcal{A}}$ which is not contained in $\mathcal{T}$, $\mathcal{T} \cup \{R\}$ is NP-hard if its closure contains a known NP-hard set. This property can be used to compute whether a tractable subset is a maximal tractable subset, namely, if every extension of the set is NP-hard.

By combining the presented methods, the closure property, the refinement method, the transformation schema and computer assisted enumerations, a complete analysis of tractability can be made. This has been demonstrated for RCC-8 [167] where three maximal tractable subsets were identified. These subsets combined with different backtracking heuristics lead to very efficient solutions of the RCC-8 consistency problem and most of the hardest randomly generated instances were solved very efficiently [173].

In a recent paper, Renz [169] extended the refinement method and presented a procedure which automatically identifies large tractable subsets given only the base relations $\mathcal{A}$ and the corresponding weak composition table. The sets generated by Renz's procedure are guaranteed to be tractable if algebraic-closure decides $\mathsf{CSPSAT}(\mathcal{A})$. The procedure automatically identified all maximal tractable subsets of RCC-8 in less than 5 minutes and for the Interval Algebra in less than one hour.

13.3.5 Planar Realizability

Given a metric spatial description it is a simple matter to display it. But given a purely qualitative spatial configuration then finding a metric interpretation which satisfies it is not, in general, trivial. A particular problem of interest here is whether mereotopological descriptions have planar realizations, where all the regions are simply connected; clearly this is not possible in general, since it is easy to specify a 5-clique using a set of externally connected regions, and a 5-clique graph is not realisable in the plane. This problem has been studied, initially in [103][18] which considers an RCC-8 like calculus and two simpler calculi and determines which of a number of different problem instances of relational consistency and planar realizability are tractable and which are not—the latter is the harder problem. Planar realizability is of particular interest for the 9-intersection calculus since it is defined for planar regions. Until recently it was unknown if the consistency problem for the 9-intersection calculus is decidable at all and it has only recently been shown that the problem is NP-complete [182].

13.4 Reasoning about Spatial Change

So far we have concentrated purely on static spatial calculi (although we briefly mentioned the combination of modal spatial and temporal logics above in Section 13.3.1). However it is important to develop calculi which combine space and time in an in-

[18] Claim 24 in this paper is subsequently admitted not to hold [28]; further work on this problem, generally known as the "map graph" recognition problem can be found in [29, 30, 197, 31].

tegrated fashion. We do not have the space here to deal with this topic in any detail. Galton's book [93] is an extended treatment of this topic.

As discussed in Chapter 9, an important aspect of qualitative reasoning is the standard assumption that change is continuous. A simple consequence is that while changing, a quantity must pass through all the intermediate values. For example, in the frequently used quantity space $\{-, 0, +\}$, a variable cannot transition from '$-$' to '$+$' without going through the intermediate value 0. In the relational spatial calculi we have concentrated on in this chapter, this requirement manifests itself in knowing which relations are neighbours in the sense that if the predicate holds at a particular time, then there is some continuous change possible such that the next predicate to hold will be a neighbour. *Continuity networks* defining such neighbours are often called *conceptual neighbourhoods* in the literature following the use of the term [87] to describe the structure of Allen's 13 JEPD relations [2] according to their conceptual closeness[19] (e.g., *meets* is a neighbour of both *overlaps* and *before*). Most of the qualitative spatial calculi reported in this paper have had conceptual neighbourhoods constructed for them,[20] for example, Fig. 13.3 illustrates the case for RCC-8. Continuity networks have been used as the basis of qualitative spatial simulations and reasoning about motion [52, 159, 67, 201, 202]. Continuity networks are presented essentially as axioms in most calculi; however there has been some work on inferring these from first principles [53, 110, 93].

There are two main approaches to spatio-temporal representation; in one, *snapshots* of the world at different instants of time are considered; alternatively, a true spatio-temporal ontology, typically a 4D region based representation is used, with time being one of the dimensions. Grenon and Smith discuss this *snap-scan* ontology [102] in more detail. Examples of 4D approaches to spatio-temporal representation include [144, 110, 111, 109].

13.5 Cognitive Validity

An issue that has not been much addressed yet in the QSR literature is the issue of cognitive validity. Claims are often made that qualitative reasoning is akin to human reasoning, but with little or no empirical justification. One exception to this is the study made of a calculus for representing topological relations between regions and lines [138]. Another study is [120] that has investigated the preferred Allen relation (interpreted as a 1D spatial relation) in the case that the composition table entry is a disjunction. Perhaps the fact that humans seem to have a preferred model explains why they are able to reason efficiently in the presence of the kind of ambiguity engendered by qualitative representations. In [119, 175] they extend their evaluation to topological relations.

[19] Note that one can lift this notion of closeness from individual relations to entire scenes via the set of relations between the common objects and thus gain some measure of their conceptual similarity as suggested by [23].

[20] A closely related notion is that of "closest topological distance" [67]—two predicates are neighbours if their respective n-intersection matrices differ by fewer entries than any other predicates; however the resulting neighbourhood graph is not identical to the true conceptual neighbourhood or continuity graph—some links are missing.

13.6 Final Remarks

In this paper we have presented some of the key ideas and results in the QSR literature, but space has certainly not allowed an exhaustive survey. A handbook on spatial logics [1] will cover some of the topics briefly described here in much more detail. As in so many other fields of knowledge representation it is unlikely that a single universal spatial representation language will emerge—rather, the best we can hope for is that the field will develop a library of representational and reasoning devices and some criteria for their most successful application. What we have outlined here are the major axes of the space of qualitative spatial representation and reasoning systems, and in particular the dimensions of variability, such as the choice of representational formalism (e.g., first order logic, modal logic, relation algebra), the ontology of spatial entities (e.g., points, lines, regions), the primitive relations and operators (such as the various JEPD sets of relations discussed above), and the different kinds of reasoning techniques (such as constraint based spatial reasoning).

As in the case of non-spatial qualitative reasoning, quantitative knowledge and reasoning must not be ignored—qualitative and quantitative reasoning are complementary techniques and research is needed to ensure they can be integrated—for example, by developing reliable ways of translating between the two kinds of formalisms[21]—this problem naturally presents itself when spatial information is acquired from sensors, in particular image/video data—i.e. how qualitative symbolic spatial representations are grounded in sensory and sensorimotor experience. Of particular interest is how to automatically learn appropriate spatial abstractions and representations, for example see [124, 90]. Equally, interfacing symbolic QSR to the techniques being developed by the diagrammatic reasoning community [97] is an interesting and important challenge.

In many situations, a hierarchical representation of space is desirable, for example, in robotics. Kuipers has promulgated the "Spatial Semantic Hierarchy" [123] as one such hierarchical model which consists of a number of distinct levels. Simply put, the "control level" is composed of sensor values, from which local 2D geometry and control laws can be determined. The next level is the "causal level"—a partially determined network in which actions determine transitions between states identified at the previous control level. The "topological level" describes space as being composed of paths, regions and places with relations between them such as we have described in this chapter. Being at a place corresponds to a distinct state of the causal layer. Finally the "metrical level" augments the topological level with metric information such as distance and orientation. There has also been work on hierarchical spatial reasoning in the context of a particular kind of spatial information, such as direction relations [151].

Another important part of future work in this area is to find general ways of combining different spatial calculi and analysing combined calculi. Most applications require more than just one spatial aspect. Even though many calculi are using constraint-based reasoning methods, combining constraints over different relations is a difficult problem as the relations have infinite domains. That means their interactions must be taken care of on a semantic level. This might require defining new relations which can negatively or positively affect properties of the combined calculi [95, 94, 74].

[21] Some existing research on this problem includes [82, 80, 192].

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