of this damage by stimulating others to improve portions of the proof in a more dramatic fashion and that the proof of the Classification will finally begin to receive the kind of second-generation attention which can lead to a better understanding of the simple groups.

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Mathematical Go: Chilling gets the last point, by Elwyn Berlekamp and David Wolfe. AK Peters, Wellesley, MA, 1994, xx + 234 pp., $34.95. ISBN 1-56881-032-6; also in paperback as *Mathematical Go endgames: Nightmares for the professional Go player*, Ishi Press International, San Jose, London, and Tokyo, $24.95. ISBN 0-923891-36-6

Last July, at the Combinatorial Games Workshop in Berkeley, David Blackwell suggested that most of mathematics may be chaotic, and that it is only the small part where we recognize patterns that we actually call mathematics. Combinatorial game theory tends to foster such a view. Are games with complete information and no chance moves of little interest because there are pure winning strategies? On the contrary, when we come to analyze even the simplest of such games, we often run into regions of complete chaos.

David Gale wrote in [4] that he had barely scratched the surface of combinatorial game theory, but

> the experience has left me with an overwhelming sense of awe at the unfathomable diversity of mathematics itself. The authors of [1] have created a mathematical fairyland ... [some] people say all of mathematics is really one ... I wonder if this belief in ultimate unity may not be just wishful thinking ... My own hunch is that mathematics (perhaps physics too) is not going to unify ... Combinatorial game theory is just one example of what unfettered mathematical imagination is capable of creating ... There will be many others ... Mathematics will continue to diversify in totally unpredictable ways ...

But now this fairyland turns out to be part of the real world. Go is the most significant of combinatorial games, whether you measure by popularity, playability, or resistance to computer attack. Yet Berlekamp, a mere 10-kyu (connotes "child") player, has been to Japan, set up endgame positions against

9-dan (connotes "adult") professional opponents and beaten them, not once but repeatedly, and then again from the same position with the colors reversed!

Although the mathematical methods exposed in this book cannot significantly improve the play at the non-expert level (the real battle in late-stage Go is only over a few points, usually only one – which explains the subtitle of the hardback edition of the book), the mathematical terminology substantially simplifies and also enriches the traditional terminology. For example, the notions of number, zero game, the game Star, and relatively colder or hotter game correspond respectively to the traditional notions of number of points a position is worth, *miai*, *dame*, and *gote* or *sente*. There is no traditional name corresponding to Up $=\uparrow=\{0\,|\,*\}$, which can be reduced to *miai* by a Black move and to *dame* by a White, although Up is a common occurrence as a "chilled" late-stage Go position.

Conway defined a **combinatorial game** as an ordered pair of sets of games

$$\{G^L|G^R\}$$

where G^L and G^R are **sets** of games, the set of Left options and the set of Right options. This definition is recursive and has the empty set as its basis, the simplest game being the **Endgame**, $\{\emptyset\,|\,\emptyset\}=0$. With the (recursive) definitions

$$G+H=\{G^L+H\,,\,G+H^L\,|\,G^R+H\,,\,G+H^R\}\quad\text{and}\quad -G=\{-G^R\,|\,-G^L\}$$

of **addition** and **negative**, games form a partially ordered commutative group. The partial order is given by defining wins for Left to be positive and wins for Right to be negative, where the winner is the last player to move: if you can't move, you lose. Games fall into four outcome classes, **positive** (wins for L = Belle Black in the book), **negative** (wins for R = Wright White), **zero** (wins for the second player), and **incomparable with zero** (wins for the first player). The prototypes for these four classes are "the games born on day one": $\{0\,|\,\emptyset\}=1$, $\{\emptyset\,|\,0\}=-1$, the already-born $\{\emptyset\,|\,\emptyset\}=0$, and the first nonnumber, **Star**, $\{0\,|\,0\}=*$. A game is a **number** if all its options are numbers and each Left option is less than each Right option. This leads not only to the real numbers but also to a far richer collection of **transfinite numbers** than that envisaged by Cantor and to a correspondingly rich collection of **infinitesimals**. The uninitiated reader will need to study [1] or [3] for a justification of these statements and of the use of familiar mathematical symbols in an entirely new context.

Numbers are **cold games**. It is the *fourth* class of games, the games which are a win for the *next* player, no matter whether Left or Right, that is the interesting one. Star is prototypical of **impartial games**, where the options are the same for each player and which are covered by the Sprague-Grundy theory [5, 8]. But impartial games are only **tepid**. The first **hot games**, those in which both players are *really* keen to make the next move, appear on day two; for example, $\pm 1=\{1\,|\,-1\}$. Most worthwhile positional games are of this type. Analysis of sums of hot games can be very complicated. To find the outcome requires a study of **thermography**: compare and contrast Chapter 9 of [3] with Chapter 6 of [1]. The essential idea of **incentive** [1, pp. 144, 246–249] can also be found in the earlier work of Milnor [7] and of Hanner [6].

But many hot games can be understood in terms of a process introduced by the authors which they call **chilling**.

To cool a game, we impose a tax payable at each move. The game G_t, i.e., G **cooled by** t, is defined for increasing values of t by

$$G_t = \{G_t^L - t \mid G_t^R + t\}$$

unless there is a smaller temperature t' for which $G_{t'}$ is infinitesimally close to a number x, in which case $G_t = x$ for all $t > t'$. An **infinitesimal game** is one which is less than any positive number (including infinitesimal ones!) and greater than any negative one. Remember that all the definitions are *recursive* and that symbols such as G_t^L represent *sets* of games.

The near-inverse of cooling is **heating**: the result of heating G through a temperature t is defined in [1] as

$$\int^t G = \left\{ \int^t G^L + t \,\middle|\, \int^t G^R - t \right\}$$

unless G is a *number*, in which case $\int^t G = G$. **Overheating** by X, where X may now be any positive *game*, was also defined in [1] by

$$\int_0^X G = \left\{ \int_0^X G^L + X \,\middle|\, \int_0^X G^R - X \right\}$$

even when G *is* a number. **Warming** is "overheating from s to t, where s and t are 1-ish" and "-ish" means "infinitesimally shifted" so that s and t are infinitesimally close to 1.

Chilling is cooling by 1, i.e., by one point. In Go the warming operator which inverts chilling is $\int_{1*}^{1}$ and is equivalent to a Norton multiply [1, p. 246] by $1*$ $(= 1 + * = \{1 \mid 1\})$. If we use a plain integral sign for this operator,

$$\int G = \begin{cases} G & \text{if } G \text{ is an even integer}, \\ G* \quad (= G + *) & \text{if } G \text{ is an odd integer}, \\ \{1 + \int G^L \mid -1 + \int G^R\} & \text{otherwise.} \end{cases}$$

A Go position is **even** (or **odd**) if the number of empty intersections plus the number of prisoners captured is even (or odd).

Mathematical Go does not consider the whole game of Go but just the fight over the last point. The situation is one in which no groups can be killed and only *territory* is being threatened. The type of territory considered is either a **corridor** or a **small room**. The analysis of small rooms is quite complicated. For corridors, most moves take away at least one point of territory, with larger threats eventually becoming available if the opponent does not reply. On each move either a point is taken or a point is saved: chilling focuses the attention on the minute value differences between different options. If the score is tied before this stylized endgame begins, then the person making the last move at least ties. It is the latent threats and their relative sizes that become important. Essentially, the **chilled game** gives each player the number of points that they would obtain if they replied to every move of the opponent, thereby saving their own points rather than taking away points from the opponent. The tax equals the gain of the move to within "-ish". What is left over are the infinitesimals. These are complicated and difficult to comprehend, indeed they are counterintuitive at first glance. In chess a threat is often more powerful than its execution, and in Go threats to save stones turn out to be more valuable than the actual saving.

For example, even invading corridors becomes complicated. It is almost always better to invade than to stop your opponent invading your territory. There is one case where a group of stones may be killed. This is when the invading group is connected to a live or immortal group by a false eye or socket. In this situation the last move by the invading group is answered by a blocking move of the opponent which threatens to remove all of the invading group. Thus the last threat of all the moves is against and not for the invader! This complicates the situation but only when the group is running very short of liberties.

Figure 1 is a simple (!?) example. White to move and win. The power of combinatorial game theory comes into play as the board separates into regions which do not influence one another. This is not quite true of the example, but, as the authors explain, it is true enough for practical purposes. Here are the values of the regions. In the unlabelled regions, Black has +6+2 at top left and bottom right and White has $-2-1$ at top right and bottom left. The *chilled* values of the labelled regions are $a = \downarrow - 2$, $b = \frac{1}{2}$, $c = * - 1$, $d = -\frac{1}{4} - 1$, $e = \downarrow - 2$, $f = -\frac{1}{4} - 1$, $g = \Uparrow * + 3$, $h = * - 1$, where $\downarrow =$ Down $= -\uparrow = \{* \mid 0\}$ and $\Uparrow * =$ Doubleupstar $= \uparrow + \uparrow + *$. As $* + * = 0$, the grand total is Star! So White's winning moves are at c, h, or g (best). All other moves lose.

For much more difficult examples, read the book and also see [2].

The appendices alone may be worth the price of the book to the Go enthusiast. The authors present a much needed comparative overview of the major rule systems, including their own newly introduced mathematical ruleset, called Universalist. In examining the different systems, the authors also touch upon etiquette and some of the philosophical concepts behind the game. The occasions on which different rulesets give different results are usually when there are *sekis* on the board. (A *seki* is a position in which neither player can play without killing his own group.) These situations are analyzed to find which rulesets give the same results. Several examples are given which show just how carefully the Japanese rules have to be applied. Indeed the written rules do not always suffice to determine the outcome of the game. Standardization of the rules is much needed. There is a good deal to be said for the adoption of the authors' Universalist rules.

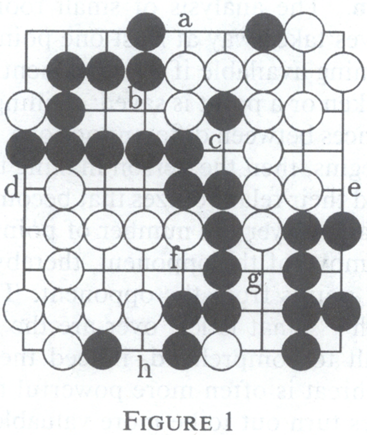

FIGURE 1

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Geometric analysis on symmetric spaces, by Sigurdur Helgason. Math. Surveys Monographs, vol. 39, Amer. Math. Soc., Providence, RI, 1994, xiv+611 pp., $71.00. ISBN 0-8218-1538-5

About the title

Among all combinations between the basic nouns of mathematics (algebra, analysis, geometry, topology, ...) and the corresponding adjectives, the grouping *geometric analysis* seems to be one of the most recently coined. Trying to define it might be difficult and unnecessarily reducing. But in the book under review (as for its predecessor *Groups and geometric analysis* by the same author [6]), "analysis" means study of differential operators and integral transforms, and "geometric" is to be understood as related to a group action. Thus the book discusses in depth such topics as Radon transforms, (generalized) Fourier transforms, invariant differential operators on homogeneous spaces of Lie groups, and links of these objects with representation theory. It cannot be considered