# ON LEFT ABSOLUTELY FLAT BANDS

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ABSTRACT. A semigroup $S$ is called (left, right) absolutely flat if all of its (left, right) $S$-sets are flat. Let $S = \bigcup \{S_\gamma \colon \gamma \in \Gamma\}$ be the least semilattice decomposition of a band $S$. It is known that if $S$ is left absolutely flat then $S$ is right regular (that is, each $S_\gamma$ is right zero). In this paper it is shown that, in addition, whenever $\alpha, \beta \in \Gamma$, $\alpha < \beta$, and $F$ is a finite subset of $S_\beta \times S_\beta$, there exists $w \in S_\alpha$ such that $(wu, wv) \in \theta_R(F)$ for all $(u, v) \in F$ ($\theta_R(F)$ denotes the smallest right congruence on $S$ containing $F$). This condition in fact affords a characterization of left absolute flatness in certain classes of right regular bands (e.g. if $\Gamma$ is a chain, if all chains contained in $\Gamma$ have at most two elements, or if $S$ is right normal).

**1. Introduction.** Let $S$ be a band. Then $S$ is a semilattice of rectangular bands, and if $S$ is left absolutely flat then each rectangular component must be a right zero band ([3], or an easy adaptation of [1, Theorem 4.3]). Therefore every left absolutely flat band is right regular (the variety of right regular bands is defined by the equation $xyx = yx$). In the case where $S$ is right normal, $S$ is left absolutely flat if and only if the nonidentity structure maps are constant [2]. If $S$ is an arbitrary band a characterization of left absolute flatness which makes no reference to $S$-sets is not known, and the problem of finding one appears to be difficult. The main goal of this paper is to establish a condition which is necessary for a band to be left absolutely flat. This condition is also sufficient in special cases.

**2. A necessary condition for left absolute flatness.** For any semigroup $S$, $S$-Ens (respectively, Ens-$S$) will denote the class of all left (right) $S$-sets and $S^1$ will represent the monoid obtained by adjoining a new identity element 1 to $S$. Let $A \in$ Ens-$S$, $a, a' \in A$, $B \in S$-Ens, and $b, b' \in B$. Following Lemma 1.2 of [1] it is easy to show that $a \otimes b = a' \otimes b'$ in $A \otimes B$ if and only if there exist $a_1, \ldots, a_n \in A$, $b_2, \ldots, b_n \in B$, $s_1, \ldots, s_n, t_1, \ldots, t_n \in S^1$ such that

$$
\begin{aligned}
a &= a_1 s_1 \\
a_1 t_1 &= a_2 s_2 \qquad & s_1 b &= t_1 b_2 \\
a_2 t_2 &= a_3 s_3 \qquad & s_2 b_2 &= t_2 b_3 \\
&\vdots & &\vdots \\
a_n t_n &= a' \qquad & s_n b_n &= t_n b'
\end{aligned}
$$

(where it is assumed that $S^1$ acts unitally on $A$ and $B$). The system of equalities above is called a *scheme over $A$ and $B$ of length $n$ joining $(a, b)$ to $(a', b')$*. From this description we see that a left $S$-set $B$ is flat if and only if, for every right $S$-set $A$, and every $a, a' \in A$, $b, b' \in B$ such that there exists a scheme over $A$ and $B$ joining $(a, b)$ to $(a', b')$, there exists a scheme (of possibly different length) over $aS^1 \cup a'S^1$ and $B$ joining $(a, b)$ to $(a', b')$. (See Lemma 2.2 of [1].)

In the sequel $S = \bigcup\{S_\gamma \mid \gamma \in \Gamma\}$ will denote a right regular band with its associated greatest semilattice decomposition ($\Gamma$ is a semilattice, each $S_\gamma$ is a right zero semigroup, and $S_\gamma S_\delta \subseteq S_{\gamma\delta}$ for each $\gamma, \delta \in \Gamma$). If $F$ represents any subset of $S^2$ then $\theta_R(F)$ will indicate the smallest right congruence on $S$ containing $F$. The first condition ($I$) (for "insertion") in the following proposition is useful when considering schemes.

PROPOSITION. *Let $S = \bigcup\{S_\gamma \mid \gamma \in \Gamma\}$ be a right regular band. The following statements are equivalent.*

($I$) *For all $\alpha, \beta \in \Gamma$, $\alpha < \beta$, $u_1, v_1, \ldots, u_m$, $v_m \in S_\beta$ and for every $A \in \mathrm{Ens}\text{-}S$, $a_1, \ldots, a_{m+1} \in A$, if $a_i u_i = a_{i+1} v_i$ ($1 \le i \le m$) then $a_i w u_i = a_{i+1} w v_i$ ($1 \le i \le m$) for some $w \in S_\alpha$.*

($C$) *For all $\alpha, \beta \in \Gamma$, $\alpha < \beta$, $u_1, v_1, \ldots, u_m$, $v_m \in S_\beta$, there exists $w \in S_\alpha$ such that $(w u_i, w v_i) \in \theta_R((u_1, v_1), \ldots, (u_m, v_m))$ ($1 \le i \le m$).*

PROOF. ($I$) implies ($C$). Denote $\theta_R((u_1, v_1), \ldots, (u_m, v_m))$ by $\theta$, let $A = S/\theta$, suppose $\overline{u_1}$ represents the $\theta$-class containing $u_1$, note that $\overline{u_1} u_i = \overline{u_1} v_i$ ($1 \le i \le m$) in $A$ and apply ($I$). Then there exists $w \in S_\alpha$ such that $\overline{u_1} w u_i = \overline{u_1} w v_i$ ($1 \le i \le m$) and hence $(w u_i, w v_i) \in \theta$ ($1 \le i \le m$) since $u_1 w = w$.

($C$) implies ($I$). Suppose $A$ is a right $S$-set, $a_1, \ldots, a_{m+1}$ belong to $A$, and $a_i u_i = a_{i+1} v_i$ ($1 \le i \le m$). By right regularity it is clear that $a_i s = a_j s$ whenever $1 \le i, j \le m + 1$, $s \in S_\alpha \cup S_\beta$, and that $a_i u_i = a_{i+1} u_i = a_i v_i = a_{i+1} v_i$ ($1 \le i \le m$). Let $F = \{(u_1, v_1), (v_1, u_1), \ldots, (u_m, v_m), (v_m, u_m)\}$. By ($C$) there exists $w \in S_\alpha$ such that $(w u_i, w v_i) \in \theta_R(F)$ ($1 \le i \le m$). Now consider any $k \in \{1, \ldots, m\}$. We seek to prove $a_k w u_k = a_{k+1} w v_k$. If $w u_k = w v_k$ then $a_k w u_k = a_k u_k w u_k = a_{k+1} v_k w u_k = a_{k+1} w u_k = a_{k+1} w v_k$. Otherwise there exist $y_1, \ldots, y_n, z_1, \ldots, z_n \in S_\beta$ and $s_1, \ldots, s_n \in S^1$ such that $(y_i, z_i) \in F$ ($1 \le i \le n$) and

$$w u_k = y_1 s_1$$
$$z_1 s_1 = y_2 s_2$$
$$\vdots$$
$$z_n s_n = w v_k.$$

For each $i \in \{1, \ldots, n\}$ there exists $j_i \in \{1, \ldots, m\}$ such that $(y_i, z_i) \in \{(u_{j_i}, v_{j_i}), (v_{j_i}, u_{j_i})\}$. Then $a_k w u_k = a_k y_1 s_1 = a_{j_1} y_1 s_1 = a_{j_1} z_1 s_1 = a_{j_1} y_2 s_2 = a_{j_2} y_2 s_2 = a_{j_2} z_2 s_2 = \cdots = a_{j_n} z_n s_n = a_{j_n} w v_k = a_{k+1} w v_k$ as required. $\square$

Note that the second condition ($C$) (for "congruence") in the proposition above refers only to the semigroup $S$ itself and not to arbitrary $S$-sets. The following theorem, whose proof will appear in the next section, is the main result of this paper. (Recall that every left absolutely flat band is necessarily right regular.)

THEOREM. *Every left absolutely flat band satisfies condition ($C$).*

Let $S = \bigcup \{ S_\gamma \mid \gamma \in \Gamma \}$ be a right regular band and suppose that either $\Gamma$ is a chain or that $S$ is right normal. Then whenever $\alpha, \beta \in \Gamma$, $\alpha < \beta$, and $F \subseteq S_\beta^2$ it is true that $\theta_R(F) \cap S_\alpha^2 \subseteq \Delta$. Hence we have

COROLLARY 1 [2]. *If $S$ is a left absolutely flat normal band then the (nonidentity) structure maps of $S$ are constant.*

Recall that every band is partially ordered by the relation $e \leq f$ if and only if $ef = e = fe$. We shall say that a right regular band $S = \bigcup \{ S_\gamma \mid \alpha \in \Gamma \}$ has "the lower bound condition" if whenever $\alpha, \beta \in \Gamma$ and $\alpha < \beta$, every finite subset of $S_\beta$ has a lower bound in $S_\alpha$. This condition is equivalent to condition $(\mathcal{C})$ in case $\Gamma$ is a chain.

COROLLARY 2. *Let $S = \bigcup \{ S_\gamma \mid \gamma \in \Gamma \}$ be a right regular band such that $\Gamma$ is a chain. Then $S$ is left absolutely flat if and only if $S$ has the lower bound condition.*

PROOF. The necessity of the lower bound condition follows immediately from the theorem and the remark preceding Corollary 2.

Assume now that $S$ has the lower bound condition. Let

$$a = a_1 s_1$$
$$a_1 t_1 = a_2 s_2 \qquad s_1 b = t_1 b_2$$
$$\vdots \qquad \qquad \vdots$$
$$a_n t_n = a' \qquad s_n b_n = t_n b'$$

be a scheme of length $n$ joining $(a, b)$ to $(a', b')$ over $S$-sets $A$ and $B$ as described at the beginning of this section. We use induction on $n$ to show that a suitable replacement scheme over $aS^1 \cup a'S^1$ and $B$ can be found. When $n = 1$ the scheme

$$a = (as_1)s_1$$
$$(as_1)t_1 = (a's_1)t_1 \qquad s_1 b = t_1 b'$$
$$(a's_1)s_1 = a's_1 \qquad t_1 b' = s_1 b$$
$$a't_1 = a' \qquad s_1 b = t_1 b'$$

is appropriate (in fact these equalities hold for any right regular band). Assume now that $n > 1$ and that all schemes of length less than $n$ can be replaced. Suppose $s_1 \in S_\alpha$, $t_1 s_2 \cdots t_{n-1} s_n \in S_\beta$, and $t_n \in S_\gamma$. If $\alpha \geq \beta$ or $\gamma \geq \beta$ then $a_i t_i \in aS \cup a'S$ for some $i \in \{ 1, \ldots, n-1 \}$ and the scheme may be resolved into two shorter schemes. The proof can then be completed as shown in [2, proof of Theorem 2.4]. We therefore assume $\alpha, \gamma < \beta$ and without loss of generality suppose that $\alpha \geq \gamma$. The lower bound condition can be used to find an element $l \in S_\alpha$ such that $lt_1 = ls_2 = \cdots = lt_{n-1} = ls_n = l$. Then an appropriate replacement scheme is

$$a = as_1$$
$$al = al \qquad s_1 b = lb_n$$
$$at_n = a' \qquad lb_n = t_n b'$$

and the proof is complete. $\square$

In general the lower bound condition of this corollary is not necessary for a right regular band to be left absolutely flat, as the following example shows. Consider

the band $S = S_\alpha \cup S_\beta \cup S_\gamma$ where $S_\alpha = \{a,b\}$, $S_\beta = \{c\}$, and $S_\gamma = S_{\alpha\beta} = \{d,e\}$ with table

|   | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $b$ | $d$ | $d$ | $e$ |
| $b$ | $a$ | $b$ | $e$ | $d$ | $e$ |
| $c$ | $d$ | $e$ | $c$ | $d$ | $e$ |
| $d$ | $d$ | $e$ | $d$ | $d$ | $e$ |
| $e$ | $d$ | $e$ | $e$ | $d$ | $e$ |

Note that $\{a,b\}$ does not have a lower bound in $S_\gamma$ and that $S$ has condition $(C)$. The fact that this semigroup is left absolutely flat is a consequence of the following more general result. *If $S = \bigcup\{S_\gamma \mid \gamma \in \Gamma\}$ is a right regular band satisfying $(C)$ and if every chain in $\Gamma$ has no more than two elements then $S$ is left absolutely flat.* For simplicity our proof of this fact assumes $\Gamma$ has exactly two maximal elements $\alpha$ and $\beta$ (as is the case in the example above). Suppose

$$a = a_1 s_1$$
$$a_1 t_1 = a_2 s_2 \qquad\qquad s_1 b = t_1 b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_n t_n = a' \qquad\qquad s_n b_n = t_n b'$$

is a scheme of length $n$ over $S$-sets $A$ and $B$ joining $(a,b)$ to $(a',b')$. If $n = 1$ the replacement scheme of length 3 displayed in the proof of Corollary 2 suffices. Now assume $n > 1$. Without loss of generality we may assume $t_1, s_2, \ldots, t_{n-1}, s_n$ belong to $S_\alpha \cup S_\beta$ and in fact, $t_i, s_{i+1} \in S_\alpha$ or $t_i, s_{i+1} \in S_\beta$ for each $i \in \{1, \ldots, n-1\}$, because otherwise $a_i t_i \in aS \cup a'S$ for some $i$ and induction may be used (as in the proof of Corollary 2). Suppose $t_1 \in S_\alpha$. Then $s_1 \in S_\beta \cup S_\gamma$ (since, if not, $a_1 t_1 \in aS$). Let $i$ be the largest index such that $t_j \in S_\alpha$ for all $j$, $1 \le j \le i$. If $i = n$, $a_n s_n \in a'S$ would result and therefore we assume $i < n$. Because we have condition $(C)$ and, hence, $(I)$, there exists $w \in S_\gamma$ such that

$$a = a_1 s_1$$
$$a_1(wt_1) = a_2(ws_2) \qquad\qquad s_1 b = (wt_1)b_2$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$a_i(wt_i) = a_i(ws_{i+1}) \qquad\qquad (ws_i)b_i = (wt_i)b_{i+1}$$
$$a_i t_{i+1} = a_{i+1} s_{i+1} \qquad\qquad (ws_{i+1})b_{i+1} = t_{i+1}b_{i+2}$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$a_n t_n = a' \qquad\qquad\qquad s_n b_n = t_n b'.$$

Note that $s_1 b = t_1 s_1 b = w t_1 s_1 b = w t_1 b_2$ and the equality $(ws_{i+1})b_{i+1} = t_{i+1}b_{i+2}$ follows similarly. We now have a scheme of length $n$ over the $S$-sets $A$ and $B$ joining $(a,b)$ to $(a',b')$ in which $a_1(wt_1) \in aS$. Induction may be used to complete the proof. (If $t_1 \in S_\beta$ a similar proof serves.)

**3. Proof of the theorem.** Let $S = \bigcup\{S_\gamma \mid \alpha \in \Gamma\}$ be a left absolutely flat band. Then $S$ is right regular (i.e. each $S_\gamma$ is a right zero semigroup). Suppose $\alpha, \beta \in \Gamma$, $\alpha < \beta$, and suppose $F = \{(u_1, v_1), \ldots, (u_m, v_m)\}$ is a finite subset of $S_\beta^2$.

We will first show that condition $(\mathcal{C})$ holds under the assumption $\theta_R(F) \cap S_\alpha^2 \subseteq \Delta$; i.e. we will prove that, in this situation, there exists $w \in S_\alpha$ such that $wu_i = wv_i$ $(1 \le i \le m)$. Since all filters in $S$ are left absolutely flat (see [1]) we may suppose without loss of generality that $\alpha$ is the smallest element of $\Gamma$. Because $\theta_R(F) \cap S_\alpha^2 \subseteq \Delta$ there exists an embedding $S_\alpha \hookrightarrow S/\theta_R(F)$ of right $S$-sets. Select any $l \in S_\alpha$. Let $Y = S^1 y \, \dot\cup \, S^1 y_1 \, \dot\cup \cdots \dot\cup \, S^1 y_m \, \dot\cup \, S^1 y'$ be the free left $S$-set on free generators $\{y, y_1, \ldots, y_m, y'\}$, denote by $\psi$ the congruence on $Y$ generated by the set

$$G = \{(ly, u_1 y_1), (v_1 y_1, u_2 y_2), \ldots, (v_{m-1} y_{m-1}, u_m y_m), (v_m y_m, ly')\},$$

and let $B$ represent the left $S$-set $Y/\psi$. Note that $\bar{l} \otimes \bar{y} = \bar{l} \otimes \overline{y'}$ in $S/\theta_R(F) \otimes B$:

$$\bar{l} \otimes \bar{y} = \overline{u_1 l} \otimes \bar{y} = \overline{u_1} \otimes \overline{ly} = \overline{u_1} \otimes \overline{u_1 y_1} = \overline{u_1} \otimes \overline{y_1}$$
$$= \overline{v_1} \otimes \overline{y_1} = \overline{v_1} \otimes \overline{v_1 y_1} = \overline{v_1} \otimes \overline{u_2 y_2} = \overline{v_1 u_2} \otimes \overline{y_2} = \overline{u_2} \otimes \overline{y_2}$$
$$= \cdots = \overline{u_m} \otimes \overline{y_m} = \overline{v_m} \otimes \overline{y_m} = \overline{v_m} \otimes \overline{v_m y_m} = \overline{v_m} \otimes \overline{ly'} = \overline{v_m l} \otimes \overline{y'}$$
$$= \bar{l} \otimes \overline{y'}.$$

Since $B$ is flat it follows that $l \otimes \bar{y} = l \otimes \overline{y'}$ in $S_\alpha \otimes B$, hence also in $S^1 \otimes B \cong B$. Thus, $\overline{ly} = \overline{ly'}$ in $B$, or $(ly, ly') \in \psi$. This means there exist $s_1, s_2, \ldots, s_n \in S^1$, $x_1, x_2, \ldots, x_n \in Y$, and $z_1, z_2, \ldots, z_n \in Y$ such that

$$ly = s_1 x_1$$
$$s_1 z_1 = s_2 x_2$$

$(\Sigma)$
$$\vdots$$

$$s_{n-1} z_{n-1} = s_n x_n$$
$$s_n z_n = ly',$$

where $(x_i, z_i) \in G \cup G^{-1}$ $(1 \le i \le n)$.

Assume $n$ is the smallest positive integer for which such a scheme exists. We now prove that, in fact, $(x_i, z_i) \in G$ $(1 \le i \le n)$. Since $ly = s_1 x_1$, $x_1 = ly$ and hence, $(x_1, z_1) \in G$. Suppose (by way of contradiction) there exists an index $j$ such that $(x_j, z_j) \in G^{-1}$. Let $i$ be the smallest such index. Then since $s_{i-1} z_{i-1} = s_i x_i$, $z_{i-1} = x_i$ because $(x_{i-1}, z_{i-1}) \in G$. If $z_{i-1} = u_1 y_1$ then $s_i z_i = s_i ly = ly$ and, consequently, $ly$ and $ly'$ may be joined by a shorter scheme (contradiction). If there exists $j$ $(2 \le j \le m)$ such that $z_{i-1} = u_j y_j$ then $s_{i-1} z_{i-1} = s_i x_i$ becomes $s_{i-1} u_j y_j = s_i u_j y_j$ which shows $s_{i-1} u_j = s_i u_j$ and, hence, $s_{i-1} v_{j-1} = s_i v_{j-1}$. It now follows that $s_{i-1} x_{i-1}$ equals $s_i z_i$ and therefore, deletion of the equation $s_{i-1} z_{i-1} = s_i x_i$ yields a shorter scheme joining $ly$ and $ly'$ (contradiction). If $z_{i-1} = ly'$ then $s_i x_i = s_i ly' = ly'$ and again, $ly$ and $ly'$ are connected by a shorter scheme (contradiction). Therefore $(\Sigma)$ is in fact the scheme

$$ly = s_1 ly$$
$$s_1 u_1 y_1 = s_2 v_1 y_1$$
$$s_2 u_2 y_2 = s_3 v_2 y_2$$

$$\vdots$$

$$s_m u_m y_m = s_{m+1} v_m y_m$$
$$s_{m+1} ly' = ly'.$$

Consequently, in $S$

$$l = s_1 l$$
$$s_1 u_1 = s_2 v_1$$
$$s_2 u_2 = s_3 v_2$$
$$\vdots$$
$$s_m u_m = s_{m+1} v_m$$
$$s_{m+1} l = l.$$

Let $w = l s_1$. Then for each $i$ $(1 \leq i \leq m)$, $w u_i = l s_1 u_i = l s_1 u_1 u_i = l s_2 v_1 u_i = l s_2 u_2 u_i = \cdots = l s_i u_i u_i = l s_i u_i = l s_{i+1} v_i$ and hence, $w u_i = w v_i$ as required.

We return now to the general situation (in which the relation $\theta_R(F) \cap S_\alpha^2 \subseteq \Delta$ is not assumed). Consider the filter $S_{[\alpha)} = \bigcup \{ S_\gamma \mid \gamma \geq \alpha \}$, which is also left absolutely flat. The equivalence relation $\theta = (\theta_R(F) \cap S_\alpha^2) \cup \Delta$ is easily seen to be a (two-sided) congruence on $S_{[\alpha)}$. The previous argument may be applied to the left absolutely flat right regular band $S_{[\alpha)}/\theta$ to eventually obtain $w$ in $S_\alpha$ such that $(w u_i, w v_i) \in \theta_R(F)$ $(1 \leq i \leq m)$. $\square$

**4. Concluding remarks.** If $S$ is a right regular band satisfying condition $(\mathcal{C})$ then it is possible to prove that every scheme of length 2

$$a = a_1 s_1$$
$$a_1 t_1 = a_2 s_2 \qquad s_1 b = t_1 b_2$$
$$a_2 t_2 = a' \qquad s_2 b_2 = t_2 b'$$

joining $(a, b)$ to $(a', b')$ over $S$-sets $A$ and $B$ (see §2) may be replaced by a (symmetric) scheme of length 14 joining $(a, b)$ to $(a', b')$ over $aS^1 \cup a'S^1$ and $B$. In spite of this and the fact that condition $(\mathcal{C})$ implies left absolute flatness in special cases, the authors doubt that this implication holds for right regular bands in general. The investigation will be continued in a subsequent paper.

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