# CONTRIBUTIONS TO THE PROBLEM OF APPROXIMATION OF EQUIDISTANT DATA BY ANALYTIC FUNCTIONS*

## PART A.—ON THE PROBLEM OF SMOOTHING OR GRADUATION. A FIRST CLASS OF ANALYTIC APPROXIMATION FORMULAE

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**Introduction.** Let there be given a sequence of ordinates

$$\{y_n\} \qquad (n = 0, \pm 1, \pm 2, \cdots),$$

corresponding to all integral values of the variable $x=n$. If these ordinates are the values of a known analytic function $F(x)$, then the problem of interpolation between these ordinates has an obvious and precise meaning: we are required to compute intermediate values $F(x)$ to the same accuracy to which the ordinates are known. Undoubtedly, the most convenient tool for the solution of this problem is the polynomial central interpolation method. It uses the polynomial of degree $k-1$, interpolating $k$ successive ordinates, as an approximation to $F(x)$ only within a unit interval in $x$, centrally located with respect to its $k$ defining ordinates. Assuming $k$ fixed, successive approximating arcs for $F(x)$ are thus obtained which present discontinuities on passing from one arc to the next if $k$ is odd, or discontinuities in their first derivatives if $k$ is even (see section 2.121). Actually these discontinuities are irrelevant in our present case of an analytic function $F(x)$. Indeed, if the interpolated values obtained are sufficiently accurate, these discontinuities will be apparent only if we force the computation beyond the intrinsic accuracy of the $y_n$.

The situation is quite different if $y_n$ are empirical data. In this case we are to determine an approximation $F(x)$ which, for $x=n$, may disagree with $y_n$ by amounts depending on the accuracy of the data, provided we thereby improve the smoothness of the resulting approximation $F(x)$. In various applied fields such as Ballistics and Actuarial mathematics it is at times desirable to compute very smooth approximations $F(x)$ to an accuracy surpassing by far the accuracy to which the physical or statistical function involved may be determined. This physically unjustified accuracy becomes desirable whenever the approximation $F(x)$ enters into numerical processes of some complexity, such as the numerical solution of differential equations. Modern electronic computing machines, especially, require a good amount of forced mathematical accuracy in such auxiliary tables in order to avoid the excessive accumulation of rounding errors in the computation of the solution. These remarks justify the desirability of approximation methods to empirical data furnishing easily computed approximations $F(x)$ which are very smooth functions of $x$. Approximations meeting these requirements are of two kinds: 1. *Polynomial approximation*, where $F(x)$ is composed of a succession of polynomial arcs meeting with a certain number of continuous derivatives. 2. *Analytic approximations*, where $F(x)$ is an analytic and regular function of $x$ for all real values of $x$.

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Important work concerning *polynomial approximations* is to be found in the actuarial literature under the subject of *osculatory interpolation*. Of the extensive literature we mention especially the fundamental work of W. A. Jenkins and the valuable systematization of the subject by T. N. E. Greville.[1] Especially important are those formulae derived by these authors which do not strictly interpolate the given ordinates, but rather combine the operation of smoothing the data and the operation of interpolation in one formula. Mr. Jenkins discusses interpolation formulae written in the convenient Everett (or Steffensen) form. Mr. Greville's starting point is his elegant expression of each polynomial arc in terms of the end point values of those derivatives which are to be continuous on passing from one arc to the next. Each of these two modes of attack has its peculiar advantages and one or the other seem indispensable for an algebraic treatment of the subject. The present writer has found the Lagrange form (explicitly in terms of the ordinates $y_n$) of such formulae preferable for two reasons: 1. The Lagrange form seems better adapted to computation with modern desk computing machines and undoubtedly superior for computation with punch-card machines. 2. The Lagrange form suggests a treatment of the subject by means of elementary concepts of Fourier analysis which, firstly, affords a more exhaustive treatment of the problem of *polynomial approximations*, secondly, shows how to extend these methods so as to furnish *analytic approximations*.

The explicit Lagrange form of the $k$-point central interpolation method, as well as of all the interpolation formulae of osculatory interpolation, is extremely simple in its formal appearance. Indeed, to every such formula corresponds an *even* function $L(x)$, defined for all real values of $x$, in terms of which the corresponding formula may be written as follows

$$F(x) = \sum_{n=-\infty}^{\infty} y_n L(x - n). \tag{1}$$

The simplicity of this formula springs from the fact that it depends on the single function $L(x)$ which describes the formula completely. Incidentally $F(x) = L(x)$ if

$$y_0 = 1, \qquad y_n = 0 \quad (n \neq 0). \tag{2}$$

Thus every interpolation method of this kind exhibits its corresponding $L(x)$ if we apply the method to the ordinates (2) (for an example see section 2.121).

The polynomial interpolation formulae arise from (1) if $L(x)$ is a composite polynomial function of arcs defined by various polynomials in successive unit intervals, such that $L(x) = 0$ for sufficiently large values of $|x|$ (for an important example see chapter II, formula (11)). The number of continuous derivatives of $F(x)$ is, of course, equal to the number of continuous derivatives of $L(x)$ for all real $x$.

We obtain the formally simplest interpolation formula (1) if we choose

$$L(x) = \frac{\sin \pi x}{\pi x} \tag{3}$$

[1] W. A. Jenkins, *Osculatory interpolation: New derivation and formulae*, Record of the American Institute of Actuaries, **15**, 87 (1926).

Thomas N. E. Greville, *The general theory of osculatory interpolation*, Transactions of the Actuarial Society of America, **45**, 202–265 (1944).

W. A. Jenkins wrote four papers on this subject of which the above paper is the first. References to the other three papers are found in the excellent bibliography in Greville's paper.

in which case (1) becomes

$$F(x) = \sum_{n=-\infty}^{\infty} y_n \frac{\sin \pi(x-n)}{\pi(x-n)} \cdot \tag{4}$$

This expression which interpolates the ordinates $y_n$, is known to mathematicians under the name of the *cardinal series*.[2] For this reason we wish to call the general formula (1) a formula of the *cardinal type*, referring to $L(x)$ as the *basic function* of the formula.

The aim of the paper, of which the present Part A is the first, is twofold. Firstly, we propose to carry through to a certain stage of completion the important actuarial work concerning polynomial approximations. Incidentally, our work will answer Mr. Greville's conjecture (loc. cit. pp. 212–213) concerning the existence of an "ordinary" interpolation formula furnishing an approximation $F(x)$ composed of polynomial arcs of degree $m+2$, having $m$ continuous derivatives and such that if the data $y_n$ are the values of a polynomial of degree $m-1$ then $F(x)$ reduces identically to that polynomial. In Part B it will be shown how to obtain such formulae for every value of $m$. (The case of $m=2$ reduces to Jenkins' formula mentioned in section 2. 122.) Secondly, we shall derive formulae of the cardinal type (1) with basic functions $L(x)$ which are analytic and regular for all real or complex values of $x$. The classical basic function (3) is of course analytic; however, its excessively slow rate of damping, for increasing $x$, makes the classical cardinal series (4) inadequate for numerical purposes. Our analytic $L(x)$, derived in chapter IV, dampen out exponentially. In Part B we will derive similar $L(x)$ which will dampen out even faster: like $\exp(-C^2x^2)$.

The paper is divided into five chapters. In chapter I we discuss the general problem of smoothing by means of a linear compound formula. This discussion, by no means exhaustive, is to serve as a guide to what is likely to be useful among formulae of the cardinal type (1) which smooth and interpolate at the same time. It serves to restrict somewhat the arbitrariness of the problem. The rather obvious idea of using cosine polynomials (or series) in this connection affords the possibility of a brief exposition of this subject in the more scientific manner of E. de Forest, W. F. Sheppard, E. T. Whittaker, and others, and may be followed up elsewhere.

Chapters II and III form the common foundation of both parts A and B. In chapter II we describe the interpolatory properties of the formula (1) in terms of extremely simple properties of the Fourier-transform

$$g(u) = \int_{-\infty}^{\infty} L(x) \cos uxdx \tag{5}$$

of the basic function $L(x)$ (Theorem 4). Thus we are assured that our formula (1) will be exact for (i.e., reproduce) polynomials of degree $k-1$, provided $g(u)-1$ has a zero of order $k$ at $u=0$ and $g(u)$ has zeros of order $k$ at all points $u=2\pi n$ ($n=\pm 1, \pm 2, \cdots$). This elementary fact is reminiscent of N. Wiener's fundamental

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[2] See J. M. Whittaker, *Interpolatory function theory*, Cambridge Tracts in Mathematics, 1935, pp. 62–64, for a discussion of the relation between the cardinal series and Stirling's interpolation series. The cardinal series was probably first investigated in an important mémoire by Ch. J. de la Vallée Poussin, *Sur la convergence des formules d'interpolation entre ordonnées équidistantes*, Bull. Acad. Roy. Belgique, **1908**, 319–410.

description of the closure properties of the family of translation functions $\{L(x-\lambda)\}$ in terms of the zeros of $g(u)$. Chapter III contains a somewhat general discussion of polygonal lines, the individual arcs of which are polynomials of degree $k-1$, joined together with $k-2$ continuous derivatives. A general parametric representation of such curves is obtained (Theorem 5) which greatly facilitates their use for the purpose of approximation of data. For $k=4$ they represent approximately the curves drawn by means of a spline and for this reason we propose to call them *spline curves of order k.* These polynomial spline curves are finally smoothed out, by means of one-dimensional heat flow during the time interval $t$, into *analytic spline curves of order k.* An analytic spline curve of order $k$ is represented by a series of the cardinal type

$$F(x) = \sum_{n=-\infty}^{\infty} f_n M_k(x - n, t), \tag{6}$$

where the basic function $M_k(x, t)$ is defined as

$$M_k(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t(u/2)^2} \left(\frac{2 \sin u/2}{u}\right)^k \cos uxdx, \tag{7}$$

while the coefficients $f_n$ may be thought of as arbitrary parameters.

The family of functions (6) forms the basis of our work. Its principal advantages for purposes of numerical approximation spring from two sources: 1) *The basic function $M_k(x, t)$ dampens out like* $\exp(-x^2t^{-1})$ (see III, formula (39)). As seen from our Table I, for $k=4$ and $t=0.5$, we have $M_4(x, 1/2)=0$ to something like 10 decimal places for $|x| \geqq 5$. This causes the great flexibility of the graph of $F(x)$ on varying the parameters $f_n$ and the ease in computing $F(x)$. 2) *The family* (6) *contains* (or represents) *all polynomials of degree $k-1$.* The simplest analytic family of this type is obtained for $k=0$ and $t>0$ when (6) becomes

$$F(x) = \sum_{n=-\infty}^{\infty} f_n \frac{1}{\sqrt{\pi t}} e^{-(x-n)^2/t}. \tag{8}$$

This family obviously still enjoys the first property. However, (8) fails badly in its ability of representing even the simplest types of curves because of the low value of $k=0$. Indeed $F(x)\equiv 0$, for all $f_n=0$, is the only constant value (8) is capable of representing.

Chapter IV contains the chief results of the present Part A. We show how the family of curves (6) can be used to approximate given data. First we derive an analytic interpolation formula of the cardinal type (1) which leaves the given ordinates unchanged (Theorem 8). Secondly we extend the result to a family of formulae depending on a positive smoothing parameter $\epsilon$ such as to combine a certain variable amount of smoothing (depending on $\epsilon$) with the operation of interpolation (Theorem 9).

In collaboration with Lt. J. H. Levin, the author has had the opportunity of applying on a large scale this analytic approximation method at the Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland. The computations were performed on punch-card machines. The given equidistant data $y_n$ were the values of the drag coefficient of a projectile as a function of its velocity. Since very accurately computed values of the derivatives $F'(x)$, $F''(x)$ of the approximation $F(x)$ were also desired, it seems doubtful if any of the existing osculatory interpolation formulae would have furnished satisfactory results in view of the complicated trend of the data to be approximated.

In the last chapter we discuss procedures for the accurate computation of the functions and constants tabulated at the end of the paper. The most noteworthy problem encountered in this connection is the following: Let

$$F(z) = \sum_{-\infty}^{\infty} a_n z^n \tag{9}$$

be a Laurent series which converges in a ring $\alpha < |z| < \beta$. We assume furthermore that $f(z)$ does not vanish in this ring:

$$F(z) \neq 0, \qquad (\alpha < |z| < \beta). \tag{10}$$

Under these circumstances we have an expansion of the reciprocal

$$\frac{1}{F(z)} = \sum_{-\infty}^{\infty} \omega_n z^n. \tag{11}$$

If the coefficients $a_n$ of the expansion (9) are given numerically the problem consists in finding very accurate numerical values of the $\omega_n$.[3] A very efficient iteration method solving this problem has been developed by H. A. Rademacher and the author. It solves the similar problem of finding the expansion of the $n$th root of $f(z)$ and generally of any algebraic function of Laurent series. This subject will be discussed elsewhere in a joint publication with Professor Rademacher.

In a sequel to these papers we expect to discuss the fitting of curves of the form (6) to data, in the sense of least squares. This will be accomplished by constructing series of the cardinal type (1) which also enjoy the orthogonality property

$$\int_{-\infty}^{\infty} L(x)L(x-n)dx = \begin{cases} 1 & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n \neq 0. \end{cases}$$

This construction reduces to the problem of computing the Laurent expansion of the square root $\sqrt{f(z)}$ of an expansion (9).

The author wishes to express his appreciation for the encouraging interest shown in his work by Dr. A. N. Lowan of the Mathematical Tables Project. He has benefited much by the helpful advice of Dr. L. S. Dederick, Major A. A. Bennett, Lt. J. H. Levin and others. Especially valuable were the author's frequent discussions with Dr. C. B. Morrey. The tables were computed by Mrs. Mildred Young. The author takes this opportunity of expressing his thanks to the officials of the Ballistic Research Laboratory for their permission to publish these tables.

The reader who is mainly interested in the numerical applications, may pass directly from this point to the Appendix, where the use of the tables is fully explained and one example is worked out.

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[3] It should be remarked here that also A. C. Aitken's computation in 1925 of the coefficients of E. T. Whittaker's smoothing method amounted to the expansion in a Laurent series of a certain simple rational function. See E. T. Whittaker and G. Robinson, *Calculus of observations*, London and Glasgow, 1940, pp. 308–312.

## I. DEFINITIONS OF SMOOTHING AND SMOOTHING FORMULAE

**1.1. A definition of smoothing formulae.** Let $\{y_n\}$ $(n = \cdots -2, -1, 0, 1, 2, \cdots)$ be a given sequence or "table" which we wish to smooth. This smoothing operation is ordinarily performed by means of a formula of the following type

$$F_n = y_{n-p}L_p + \cdots + y_{n-1}L_1 + y_nL_0 + y_{n+1}L_{-1} + \cdots + y_{n+p}L_{-p}, \tag{1}$$

where the numerical coefficients $L_\nu$ are symmetric about the middle term $L_0$, i.e., $L_\nu = L_{-\nu}$. The linear transformation (1) if applied to the original sequence $\{y_n\}$ will transform it into the smoothed sequence $\{F_n\}$. By extending the definition of $L_\nu = 0$ for $|\nu| > p$ we may rewrite (1) as

$$F_n = \sum_{\nu=-\infty}^{\infty} y_\nu L_{n-\nu}. \tag{2}$$

If $y_\nu = \text{const.} = c$, we also wish that $F_n = c$; therefore

$$\sum_\nu L_\nu = 1 \tag{3}$$

is a natural requirement.

When does the formula (1) actually smooth? As an example let $p = 1$ and let the coefficients $L_\nu$ be $(-1, 3, -1)$. If we now apply the formula (1) to the periodic sequence

$$\{y_n\} = \{\cdots, 0, 1, 0, 1, 0, 1, \cdots\}$$

we obtain

$$\{F_n\} = \{\cdots, -2, 3, -2, 3, -2, 3, \cdots\}$$

which is a good deal rougher than the original sequence. Obviously this situation deserves some clarification.

There seems no doubt that the "smoothness" of a sequence $\{y_n\}$ depends in some way on its differences of higher order, especially on the sums of their square. We also notice that the formula (2) agrees with the rule of multiplication of Fourier series. This suggests the use of such series.

Let us assume for the moment that

$$\sum_{n=-\infty}^{\infty} |y_n| < \infty. \tag{4}$$

We now define a function $T(u)$ by

$$T(u) = \sum_{n=-\infty}^{\infty} y_n e^{inu} \tag{5}$$

and call it the *characteristic function* of our sequence $\{y_n\}$; it is a complex-valued continuous function of $u$ of period $2\pi$.

Now (5) implies

$$e^{-iu}T(u) = \sum y_n e^{i(n-1)u} = \sum y_{n+1}e^{inu}$$

and by subtracting (5) we get

$$(e^{-iu} - 1)T(u) = \sum_{n=-\infty}^{\infty} \Delta y_n e^{inu}. \tag{6}$$

This shows that we obtain the characteristic function of the sequence $\{\Delta y_n\}$ of first differences of $\{y_n\}$ by multiplying the characteristic function $T(u)$ of $\{y_n\}$ by the factor $e^{-iu}-1$. Generally

$$(e^{-iu} - 1)^m T(u) = \sum_{n=-\infty}^{\infty} \Delta^m y_n e^{inu} \qquad (m = 0, 1, 2, \cdots). \tag{7}$$

Since $|e^{-iu}-1| = 2\ |\sin(u/2)|$, the Parseval relation furnishes the equation

$$\sum_{n=-\infty}^{\infty} (\Delta^m y_n)^2 = \frac{1}{2\pi}\int_0^{2\pi} [2 \sin (u/2)]^{2m} |\, T(u)\,|^2 du \qquad (m \geqq 0). \tag{8}$$

These formulae furnish an expression for the sums of the squares of the differences of any order in terms of the characteristic function $T(u)$ of the sequence.

Let us now turn to the "smoothed" sequence $\{F_n\}$. Let

$$\phi(u) = \sum_{n=-\infty}^{\infty} L_n e^{inu} = L_0 + 2L_1 \cos u + 2L_2 \cos 2u + \cdots \tag{9}$$

be the characterstic function of the sequence $\{L_n\}$. We shall also refer to $\phi(u)$ as the *characteristic function of the smoothing formula* (2). Notice that $\phi(u)$ is always real and even. By multiplication of the two Fourier series (5) and (9) we obtain, in view of (2),

$$T(u)\phi(u) = \sum_{n=-\infty}^{\infty} F_n e^{inu}. \tag{10}$$

Hence the characteristic function of the "smoothed" sequence $\{F_n\}$ is obtained by multiplying the characteristic function $T(u)$ of $\{y_n\}$ by the characteristic function $\phi(u)$ of the smoothing formula (2). By now applying (8) to the sequence $\{F_n\}$ we obtain

$$\sum_{n=-\infty}^{\infty} (\Delta^m F_n)^2 = \frac{1}{2\pi}\int_0^{2\pi} (2 \sin (u/2))^{2m} |\, T(u)\,|^2 (\phi(u))^2 du, \qquad (m \geqq 0). \tag{11}$$

A comparison of the relations (8) and (11) will readily furnish an answer to the question: what is a smoothing formula? Indeed, we notice that the integrands in (8) and (11) differ only, for each fixed value of $m$, by the factor $\phi(u)^2$ in (11). This justifies the following definition.

DEFINITION 1. *Let $L_n$ be a symmetric sequence of coefficients, i.e., $L_{-n} = L_n$. The formation of the weighted means*

$$F_n = \sum_{\nu=-\infty}^{\infty} y_\nu L_{n-\nu} \qquad (n = 0, \pm 1, \pm 2, \cdots) \tag{12}$$

*is said to be a smoothing formula if*

$$\sum_n L_n = 1, \tag{13}$$

$$\sum_n |\, L_n\,| < \infty, \tag{14}$$

*while the characteristic function*

$$\phi(u) = \sum_{-\infty}^{\infty} L_n e^{inu} = L_0 + 2L_1 \cos u + 2L_2 \cos 2u + \cdots \tag{15}$$

*satisfies the condition*

$$-1 \leqq \phi(u) \leqq 1, \qquad (0 \leqq u \leqq 2\pi). \tag{16}$$

The necessity of the condition (16) is justified as follows: By a comparison of (8) and (11), in view of (16), we obtain the inequalities

$$\sum_{n=-\infty}^{\infty} (\Delta^m F_n)^2 \leqq \sum_{n=-\infty}^{\infty} (\Delta^m y_n)^2, \qquad (m = 0, 1, 2, \cdots).$$

Actually the equality sign in one of these relations will arise only under highly exceptional or else trivial conditions. This remark should make it clear why the smoothing quality of a formula violating (16) should be highly questionable.

So far we were concerned merely with the ability of a formula (2) to smooth the sequence. However, the discrepancies between the two sequences also deserve attention. By subtracting (10) from (5) we obtain

$$T(u)(1 - \phi(u)) = \sum (y_n - F_n)e^{inu}$$

and therefore

$$\sum_{n=-\infty}^{\infty} (y_n - F_n)^2 = \frac{1}{2\pi}\int_0^{2\pi} |T(u)|^2(1 - \phi(u))^2 du. \tag{17}$$

A comparison of the integrands of (17) and (11) reveals the obvious fact that strong smoothing may be achieved only if we allow relatively large discrepancies between $F_n$ and $y_n$. Indeed, the integral of (17) will be small only if $\phi(u)$ differs but little from 1, while strong smoothing requires as small a $\phi(u)$ as possible.

1.11. *Examples of smoothing formulae.* (a) Our trivial example $L_0=3$, $L_1=L_{-1}=-1$, $L_n=0$ $(n>1)$ has the characteristic function $\phi(u)=3-2\cos u$. We find $\phi(u)\geqq 1$, with $\phi(\pi)=5$, which rules it out as a smoothing formula.

(b) If $L_n \geqq 0$ for all $n$, and $\sum L_\nu=1$, then (12) is always a smoothing formula Indeed

$$|\phi(u)| = \left|\sum L_n e^{inu}\right| \leqq \sum |L_n| = 1.$$

Thus

$$F_n = (y_{n-1} + y_n + y_{n+1})/3 \tag{18}$$

is a smoothing formula with

$$\phi(u) = (1 + 2\cos u)/3.$$

Let

$$\phi_1(u) = |\phi(u)| = |1 + 2\cos u|/3 = \sum L_n^{(1)} \cos nu.$$

Since $(\phi(u))^2=(\phi_1(u))^2$ it is clear from (11) that the formula (18) and the formula of characteristic function $\phi_1(u)$ have identical smoothing powers. However, since $0<1-\phi_1(u)<1-\phi(u)$ for $2\pi/3<u<4\pi/3$, we see by (17) that the formula

$$F_n^{(1)} = \sum y_\nu L_{n-\nu}^{(1)}$$

will alter the sequence $\{y_n\}$ much less than (18) will.

(c) Generally, our formula (17) shows that it is desirable for an efficient smoothing formula to have its characteristic function satisfy the more restrictive condition

$$0 \leqq \phi(u) \leqq 1. \tag{19}$$

1.12. *A comparison of smoothing formulae.* Again our relations (11) justify the following definition.

DEFINITION 2. *Let $\phi_1(u)$ and $\phi_2(u)$ be the characteristic functions of two smoothing formulae. We say that the first is stronger than the second if*

$$|\phi_1(u)| \leqq |\phi_2(u)|. \tag{20}$$

*with the inequality sign holding for some value of $u$.*

Later in this paper we shall set up a basic sequence of smoothing formulae of progressively greater strength according to this definition. Here we remark only that two smoothing formulae cannot in general be compared on the basis of this definition. However, the following remark seems obvious. Let

$$F_n = \sum_\nu y_\nu L_{n-\nu} \tag{21}$$

be a smoothing formula of characteristic function $\phi(u)$. The iteration, or repetition, of (21) may be thought of as another smoothing formula and its characteristic function is found to be $(\phi(u))^2$. Since $|\phi(u)| \leqq 1$ obviously

$$(\phi(u))^2 \leqq |\phi(u)|.$$

This shows that the formula (21) and the sequence of its successive iterates form a sequence of smoothing formula of progressively increasing strength.

1.13. *Smoothing formulae which are exact for polynomial values of a given degree.* The following definition is in common use.

DEFINITION 3. *A smoothing formula* (2) *is said to be exact for the degree $m$ if it reproduces exactly the values $\{y_n\}$ of a polynomial of degree not exceeding $m$.*

If

$$F_n = \sum y_\nu L_{n-\nu} \tag{22}$$

is to be exact for the degree $m$, it is obviously sufficient that it be exact for the basic monomials $1, x, \cdots, x^m$. Thus the exactness for the degree $m$ is equivalent to the relations

$$n^s = \sum_{\nu=-\infty}^{\infty} \nu^s L_{n-\nu} \qquad (s = 0, 1, \cdots, m). \tag{23}$$

Let us now assume for simplicity that the sequence of coefficients $L_n$ tends to zero exponentially as $n \to \infty$, i.e., we assume the existence of two positive constants $A$ and $B$ such that

$$|L_n| \leqq Ae^{-B|n|}$$

for all values of $n$. This implies that the function

$$\phi(u) = \sum_n L_n e^{inu}$$

is regular in the strip

$$|Iu| < B$$

of the complex $u$-plane. Now

$$\phi(u) = \sum L_\nu e^{i\nu u} = \sum L_{\nu-n} e^{i\nu u} \cdot e^{-inu}$$

and

$$e^{inu}\phi(u) = \sum_\nu e^{i\nu u} L_{n-\nu}.$$

We now expand both sides in ascending powers of $u$ and compare like coefficients. Since

$$\phi(u) = 1 + \frac{u^2}{2!}\phi'' + \frac{u^4}{4!}\phi^{(4)} + \cdots,$$

we get the identities in $n$

$$n^s - \binom{s}{2}\phi'' n^{s-2} + \binom{s}{4}\phi^{(4)} n^{s-4} - \cdots = \sum_{\nu=-\infty}^{\infty} \nu^s L_{n-\nu} \qquad (s = 0, 1, 2, \cdots).$$

A comparison with (23) will show that a smoothing formula is always exact for a highest degree which is always odd. It also proves the following proposition which may evidently be established under conditions less stringent than the ones we used.

**Theorem 1.** *A smoothing formula* (22) *is exact for a degree* $2\nu+1$ *if and only if* $\phi(u)-1$ *has at* $u=0$ *a zero of order* $2\nu+2$, *i.e.*,

$$\phi''(0) = \phi^{(4)}(0) = \cdots = \phi^{(2\nu)}(0) = 0. \tag{24}$$

As an illustration we mention the formula

$$F_n = \frac{1}{32}(-y_{n-3} + 9y_{n-1} + 16y_n + 9y_{n+1} - y_{n+3}) = y_n - \frac{3}{16}\delta^4 y_n - \frac{1}{32}\delta^6 y_n \tag{25}$$

of characteristic function

$$\phi(u) = (8 + 9\cos u - \cos 3u)/16. \tag{26}$$

We find that $\phi''(0)=0$, hence (25) is exact for cubics. The symmetry property $\phi(u)+\phi(\pi-u)=1$ shows that

$$\phi(\pi) = \phi'(\pi) = \phi''(\pi) = \phi'''(\pi) = 0.$$

This results in rather strong smoothing power. The formula (25) is part of a sequence of formulae, the next one of this kind being

$$F_n = \frac{1}{512}(3y_{n-5} - 25y_{n-3} + 150y_{n-1} + 256y_n + 150y_{n+1} - 25y_{n+3} + 3y_{n+5}) \tag{27}$$

or

$$F_n = y_n + \frac{5}{32}\delta^6 y_n + \frac{15}{256}\delta^8 y_n + \frac{3}{512}\delta^{10} y_n.$$

Its characteristic function

$$\phi(u) = (128 + 150\cos u - 25\cos 3u + 3\cos 5u)/256$$

again enjoys the symmetry property $\phi(u)+\phi(\pi-u)=1$. Also $\phi(u)-1$ has a zero of order 6 at $u=0$, hence (25) is exact for quintics, while $\phi(u)$ has a zero of order 6 at $u=\pi$ resulting in strong smoothing power.

**1.2. Smoothing a finite table.** In 1.1 we have discussed the smoothing of an infinite table $\{y_n\}$ which is such that the series of the absolute values of its entries converges. By (8), (11) and the inequalities (16) we have found that the sum of the squares of the differences of order $m$ is diminished by smoothing. This is true for $m=0, 1, 2, \cdots$. Now we shall discuss briefly the practically most important case of a given finite table

$$\{y_n\} \qquad (n = 0, 1, \cdots, N). \tag{28}$$

To fix the ideas we assume the following simplest concrete situation: the third differences $\Delta^3 y_n$ are slowly varying and of slowly varying signs, while the $\Delta^4 y_n$ are of random signs. In this situation we naturally wish to minimize the 4th differences of the table. Now we form an average value of the $\Delta^3 y_n$ at each of the two ends of the table and we extend the column of $\Delta^3 y_n$ with the corresponding constant average value at each end.[4] Thus the $\Delta^3 y_n$ are defined for all $n$ having one constant value for $n > N-3$ and another constant value for $n<0$. Now we extend the definition of $y_n$ for all $n$ from the values of the third differences. Also, we compute the $\Delta^4 y_n$ for the extended infinite table. Clearly $\Delta^4 y_n = 0$ for $n < -1$ or $n > N-3$. Let

$$T_4(u) = \sum \Delta^4 y_n e^{inu}$$

be the characteristic function of the sequence of 4th differences, the series containing really a finite sum of terms only.

Let us now apply to the extended table $y_n$ a smoothing formula

$$F_n = \sum_{\nu} y_\nu L_{n-\nu} \tag{29}$$

of characteristic function $\phi(u)$, *which is exact for cubics*. The result is the new sequence $\{F_n\}$ $(-\infty < n < \infty)$. Evidently $y_n$ are the values of cubics for large $|n|$ and therefore $F_n = y_n$ for large $|n|$, hence also $\Delta^3 F_n = \Delta^3 y_n$ and $\Delta^4 F_n = 0$ for large $|n|$. Notice also that we may think of the sequence $\{\Delta^4 F_n\}$ as arising from $\{\Delta^4 y_n\}$ by the smoothing operation (29). Therefore

$$\sum_{n=-\infty}^{\infty} (\Delta^4 y_n)^2 = \frac{1}{2\pi}\int_0^{2\pi} |\, T_4(u)\,|^2 du$$

and

$$\sum_{n=-\infty}^{\infty} (\Delta^4 F_n)^2 = \frac{1}{2\pi}\int_0^{2\pi} |\, T_4(u)\,|^2 \phi(u)^2 du.$$

Generally

$$\sum_{n=-\infty}^{\infty} (\Delta^{4+m} y_n)^2 = \frac{1}{2\pi}\int_0^{2\pi} (2 \sin u/2)^{2m} |\, T_4(u)\,|^2 du, \qquad (m \geqq 0) \tag{30}$$

$$\sum_{n=-\infty}^{\infty} (\Delta^{4+m} F_n)^2 = \frac{1}{2\pi}\int_0^{2\pi} (2 \sin u/2)^{2m} |\, T_4(u)\,|^2 \phi(u)^2 du, \qquad (m \geqq 0). \tag{31}$$

---

[4] Compare G. J. Lidstone, *Note on the computation of terminal values in graduation by Jenkins' modified osculatory formula*, Transactions of the Faculty of Actuaries (Scotland), **12**, 277 (1930).

A comparison of (30) and (31) shows that the sums of the squares of the fourths and subsequent differences have been decreased by the smoothing operation. No such statement can or should be inferred concerning the finite sums of relevant differences of orders 0, 1, 2, and 3.

## II. INTERPOLATION FORMULAE

**2.1. Interpolation formulae of the cardinal type.** Let

$$F_n = \sum_{\nu} y_\nu L_{n-\nu} \tag{1}$$

be a smoothing formula. If we apply it to the "elementary" table

$$y_0 = 1, \qquad y_n = 0 \quad (n \neq 0), \tag{2}$$

then $F_n = L_n$. The even sequence $\{L_n\}$ may therefore be regarded as the smoothed version by (1) of the elementary table. Now suppose that we are given not only the even sequence of ordinates $L_n$ but an even function $L(x)$ defined for all real $x$ and such that $L(n) = L_n$. Then we may replace the integral variable $n$ in (1) by the continuous variable $x$ and we obtain the formula

$$F(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L(x - \nu). \tag{3}$$

We call $L(x)$ the basic function of the formula (3). The chief aim of this paper is to point out that the subject of interpolation is truly dominated by the formulae of the type (3), the kind of approximation desired depending only on the choice of the basic function $L(x)$. The particular basic function

$$L(x) = \frac{\sin \pi x}{\pi x} \tag{4}$$

gives rise to the series

$$F(x) = \sum_{\nu=-\infty}^{\infty} y_\nu \frac{\sin \pi(x - \nu)}{\pi(x - \nu)} \tag{5}$$

which is well known to mathematicians and referred to as the *cardinal series.* For this reason we wish to call (3) a series, or formula of *cardinal type.*

We notice here for further reference that the basic function (4) may also be written as a Fourier integral as follows.

$$\frac{\sin \pi x}{\pi x} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ixu} du. \tag{6}$$

2.11. *The two kinds of interpolation formulae, ordinary or smoothing.* For integral values of $x = n$ our formula (3) becomes

$$F(n) = \sum_{\nu=-\infty}^{\infty} y_\nu L(n - \nu). \tag{7}$$

Equation (3) is an interpolation formula in the usual sense if $F(n) = y_n$, for all $n$, and this is the case if and only if $L(x)$ satisfies the conditions

$$L(0) = 1, \qquad L(n) = 0 \quad (n \neq 0). \tag{8}$$

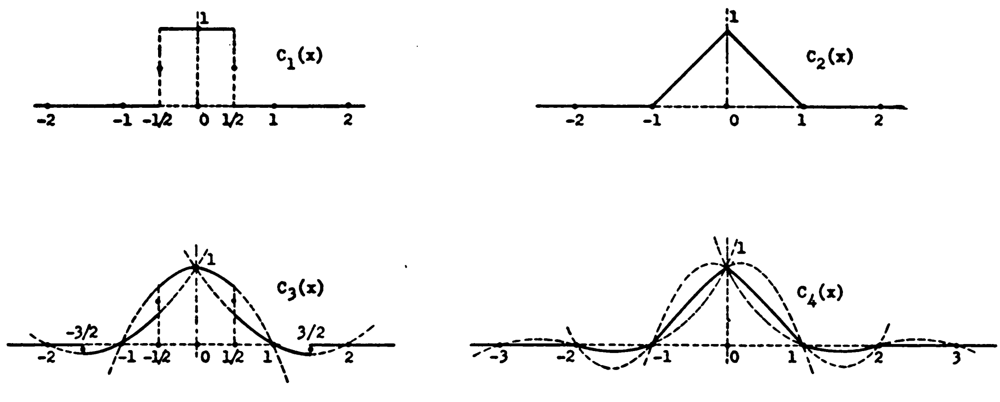
Otherwise, (7) is a smoothing formula. We shall follow the accepted actuarial practice of referring to (3) as an *ordinary interpolation formula* if (3) reproduces exactly the given ordinates $\{y_n\}$. Otherwise we call (3) a *smoothing interpolation formula.*

2.12. *Examples of interpolation formulae of the cardinal type.* Later in this paper we shall discuss various classes of such interpolation formulas all arising from a common general theory. For purposes of orientation and illustration we mention here a few concrete examples.

2.121. *The k-point central interpolation formula.* Let $k$ be a fixed integer $(=1, 2, 3, \cdots)$. By $k$-point central interpolation we mean the interpolation method whereby the polynomial of degree at most $k-1$, defined by $k$ consecutive ordinates $y_n$, is utilized within an interval of unit length centrally located with respect to the set of defining ordinates $y_n$. This set of $k$ defining ordinates $y_n$ is shifted up by one unit in the subscript for interpolation in the next unit interval. It seems obvious that this kind of interpolation is performed for any real value of $x$ by a formula of the cardinal type

$$F(x) = \sum_{n=-\infty}^{\infty} y_n C_k(x - n) \tag{9}$$

with a suitable function $C_k(x)$. To obtain this function, it is sufficient to interpolate the elementary table (2) by means of this method of $k$-point central interpolation. The graphs indicate the resulting $C_k(x)$ for $k=1, 2, 3$, and 4.[5] It is found that $C_k(x)$




[5] These graphs indicate geometrically the construction of the successive arcs of these curves. Thus $C_3(x)$ is defined in the interval $1/2 < x < 3/2$ by the parabola passing through the points (0, 1), (1, 0), (2, 0). Similarly $C_4(x)$ is defined in $-1 < x < 0$ by the cubic which takes the values 0, 0, 1, 0 at $x = -2, -1, 0, 1$ respectively. We mention incidentally the following general analytic expression of the basic function $C_k(x)$ of $k$-point central interpolation. In terms of the "central" factorial

$$x^{[k]-1} = \begin{cases} x(x^2 - 1^2)(x^2 - 2^2) \cdots (x^2 - (\nu - 1)^2) & \text{if } k = 2\nu, \\ \left(x^2 - \dfrac{1}{4}\right)\left(x^2 - \dfrac{9}{4}\right) \cdots \left(x^2 - \dfrac{(2\nu - 1)^2}{4}\right) & \text{if } k = 2\nu + 1 \end{cases}$$

we define the corresponding truncated function

$$x_+^{[k]-1} = \begin{cases} x^{[k]-1} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad (k = 1, 2, 3, \cdots).$$

is a polygonal line composed of arcs of degree $k-1$. Also $C_k(x)$ itself is continuous with a discontinuous first derivative if $k$ is even. For an odd $k$, $C_k(x)$ itself is discontinuous, the value assigned at a point of discontinuity being the arithmetic mean of the two local limits. Evidently for $k=2$ our formula (9) is identical with the method of linear interpolation and the graph of $F(x)$, as given by (9), is identical with the polygonal line of vertices $(n, y_n)$.

For further reference we mention the following formulae which are valid for all real values of $x$

$$
\begin{aligned}
C_1(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin u/2}{u/2}\, e^{iux}du,\\
C_2(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\sin u/2}{u/2}\right)^2 e^{iux}du,\\
C_3(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\sin u/2}{u/2}\right)^3 \left(1+\frac{1}{8}u^2\right) e^{iux}du,\\
C_4(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\sin u/2}{u/2}\right)^4 \left(1+\frac{1}{6}u^2\right) e^{iux}du.
\end{aligned} \tag{10}
$$

The $k$-point central interpolation method is the most important method for the interpolation and the construction of tables of analytic and regular functions. However, for the construction of tables of empirical functions, the low order of continuity of $C_k(x)$ is at times a serious limitation of this method. It seems indeed evident that the continuity properties of the linear compound

$$F(x) = \sum_n y_n L(x-n)$$

are directly dependent on the continuity properties of the basic function $L(x)$. We turn now to an interesting example of an "osculatory" interpolation formula having a basic $L(x)$ enjoying stronger continuity properties.

2.122. *An osculatory interpolation formula of W. A. Jenkins.* We define a basic function $L(x)$ for $x \leqq 0$ by

$$
L(x) = \begin{cases}
0 & \text{if} \quad x \leqq -3 \\
-\dfrac{1}{12}(x+3)^3(x+2) & \text{if} \quad -3 \leqq x \leqq -2 \\
\dfrac{1}{12}(x+1)(x+2)(x+3)(3x+7) & \text{if} \quad -2 \leqq x \leqq -1 \\
\dfrac{1}{6}(x+1)(6-6x-9x^2-x^3) & \text{if} \quad -1 \leqq x \leqq 0
\end{cases} \tag{11}
$$

---

The definition of this function is to be completed for $x=0$ by continuity, if $k$ is given, and by the arithmetic mean of the two limits, if $k$ is odd. Then

$$C_k(x) = \frac{1}{(k-1)!}\,\delta^k x_+^{[k]-1} \qquad (-\infty < x < \infty),$$

where $\delta^k$ is the symbol of the $k$th central difference of step unity (compare Theorem 3 of section 3.11). We will return to this subject in Part B where the extremely simple law of formation of the integrals (10) will also be given.

and extend its definition by the requirement $L(-x)=L(x)$ to all real values of $x$. Since the conditions (8) are visibly verified we see that

$$F(x) = \sum y_n L(x - n) \tag{11'}$$

is an ordinary interpolation formula. A closer inspection of the composite polynomial function (11) will show that $L(x)$, $L'(x)$, and $L''(x)$ are all continuous for all real values of $x$. Using a customary mathematical terminology we may say that $L(x)$ is of class $C''$. Moreover, in various ways it may be shown that the formula (11′) is exact if the $y_n$ are the ordinates of a polynomial of degree 3 or less, i.e., $F(x)$ becomes identical with that cubic polynomial.

It is of interest to compare Jenkins' formula (11′) with the 4-point central interpolation formula (9) (for $k=4$). Both are exact for cubics. $C_4(x)$ is of class $C$, while the present $L(x)$ of class $C''$. This was achieved by increasing the complexity of the basic function in two ways: 1) The interval where $L(x)$ is non-vanishing was increased from $|x| \leqq 2$ to $|x| \leqq 3$. 2) The degree of the polynomial arcs has increased from 3 to 4. Later Jenkins' formula (11′) will appear as a member of a sequence of interpolation formulae of similar characteiistics. Here we mention that the basic function (11) may be expressed in the form

$$L(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin u/2}{u/2}\right)^4\left(4 + \cos u - 4\frac{\sin u}{u}\right)e^{iux}du \tag{11''}$$

for all real values of $x$.

2.123. *A smoothing interpolation formula of W. A. Jenkins.* We define a basic function $L(x)$ by

$$L(x) = \begin{cases} 0 & \text{if } x \leqq -3, \\ -\dfrac{1}{36}(x+3)^3 & \text{if } -3 \leqq x \leqq -2, \\ \dfrac{1}{36}(69 + 117x + 63x^2 + 11x^3) & \text{if } -2 \leqq x \leqq -1, \\ \dfrac{1}{18}(15 - 27x^2 - 14x^3) & \text{if } -1 \leqq x \leqq 0, \end{cases} \tag{12}$$

$$L(-x) = L(x).$$

This particular $L(x)$, composed of cubic arcs, is of class $C''$. The formula[6]

[6] W. A. Jenkins writes his interpolation formula (11′) in the following Everett form

$$F(n+x) = y_n\xi + \delta^2 y_n\frac{\xi(\xi^2-1)}{6} - \delta^4 y_n\frac{\xi^3(\xi-1)}{12}$$
$$+ y_{n+1}x + \delta^2 y_{n+1}\frac{x(x^2-1)}{2} - \delta^4 y_{n+1}\frac{x^3(x-1)}{12}, \qquad (0 \leqq x \leqq 1,\ x+\xi = 1).$$

Likewise his formula (12′) takes the form

$$F(n+x) = y_n\xi + \delta^2 y_n\frac{\xi(\xi^2-1)}{6} - \delta^4 y_n\frac{\xi^3}{36}$$
$$+ y_{n+1}x + \delta^2 y_{n+1}\frac{x(x^2-1)}{6} - \delta^4 y_{n+1}\frac{x^3}{36}.$$

$$F(x) = \sum_{\nu} y_\nu L(x - \nu) \tag{12'}$$

corresponding to (12) is exact for polynomials of degree 3 or less. However, while (11′) was an ordinary interpolation formula, the formula (12′) is a smoothing interpolation formula. Since

$$L(0) = 15/18, \qquad L(1) = 2/18, \qquad L(2) = -1/36,$$

while $L(n) = 0$ for $n \geqq 3$, we see that for $x = n$ (12′) reduces to a smoothing formula of characteristic function

$$\phi(u) = \frac{1}{18}(15 + 4\cos u - \cos 2u).$$

We readily verify that $\phi''(0) = 0$ and $\phi(\pi) = 5/9 \leqq \phi(u) \leqq 1$. Hence (12′) reduces for integral $x = n$ to a smoothing formula, according to our Definition 1. On comparing Jenkins' two formulae (11′) and (12′) we notice that they are both exact for cubics, giving rise to curves of class $C''$. Since (12′) is only a *smoothing* interpolation formula while (11′) is an *ordinary* interpolation formula, it has been possible to lower the degree of $L(x)$ from 4 to 3. We finally mention that the function (12) may be expressed as

$$L(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin u/2}{u/2}\right)^4\left(\frac{4}{3} - \frac{1}{3}\cos u\right)e^{iux}du. \tag{12''}$$

**2.2. A general theory of interpolation formulae of the cardinal type.** In this section we shall discuss various characteristic properties of interpolation formulae of the cardinal type in terms of the Fourier transform of the basic function $L(x)$. This discussion will provide a sufficiently broad foundation for the subsequent development of specific formulae in the latter part of this paper.

2.21. *Characteristic properties of interpolation formulae.* Some of the following definitions have already occurred in the previous sections. For convenient reference we include them in our present enumeration of properties of an interpolation formula

$$F(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L(x - \nu). \tag{13}$$

a. We say that (13) is an *ordinary interpolation formula* if $F(x)$ interpolates exactly the given ordinates $y_n$, i.e., if

$$L(0) = 1, \qquad L(\nu) = 0 \quad (\nu \neq 0). \tag{14}$$

b. We say that (13) is a *smoothing interpolation formula* if for $x = n$ (13) turns into a smoothing formula

$$F(n) = \sum_{\nu} y_\nu L(n - \nu). \tag{15}$$

The term "smoothing formula" is meant, of course, in the sense of our Definition 1, section 1.1.

c. We say that (13) is *exact for the degree* $k-1$ if the relation

$$P(x) = \sum_{n=-\infty}^{\infty} P(n)L(x - n)$$

is an identity for any polynomial $P(x)$ of degree at most $k-1$. The last condition is in turn equivalent with the $k$ identities

$$x^\nu = \sum_{n=-\infty}^{\infty} n^\nu L(x-n) \qquad (\nu = 0, 1, \cdots, k-1) \tag{16}$$

out of which it can always be recovered by means of suitable linear combinations.

d. We say that (13) *preserves the degree* $k-1$,[7] if for any polynomial $P(x)$ of degree $\nu \leqq k-1$ we have an identity

$$\sum_{n=-\infty}^{\infty} P(n)L(x-n) = P(x) + (\text{a polynomial of degree} < \nu). \tag{17}$$

Notice that the leading term of $P(x)$ is not altered by (13). Again in terms of the monomials $x^\nu$ we may say: (13) preserves the degree $k-1$ whenever the $k$ functions

$$Q_\nu(x) = \sum_{n=-\infty}^{\infty} n^\nu L(x-n), \qquad (\nu = 0, 1, \cdots, k-1), \tag{18}$$

are polynomials of the form

$$Q_\nu(x) = x^\nu + a_{\nu 1}x^{\nu-1} + \cdots + a_{\nu\nu}, \qquad (\nu = 0, 1, \cdots, k-1). \tag{19}$$

e. We say that (13) is of *degree $m$ and of class $C^\mu$*, if the basic function $L(x)$ is a polygonal line of polynomial arcs of degree at most $m$ joining in such a way as to result in a function $L(x)$ having $\mu$ continuous derivatives. In the sequel, the junction points will always be either for integral values $x=n$ or else for $x=n+1/2$. Consequently no "condensation" of discontinuities will result by the formation of the linear compound (13). Hence the interpolation curve $F(x)$ will again be of degree $m$ and of class $C^\mu$. As examples we recall the formulae (11′) and (12′) of W. A. Jenkins, which are both of class $C''$ and of degree 4 and 3, respectively.

f. A formula (13) whose basic function $L(x)$ is composed of polynomial arcs will also be referred to as a *polynomial interpolation formula*. We shall say that it has the *span* $s$ if the even function $L(x)$ vanishes identically for $x>s/2$, but not for $x>s'$ with $0<s'<s/2$. Thus the $k$-point central interpolation formula (9) is of span $k$, while both formulae (11′), (12′) of Jenkins' have the span 6. For obvious practical reasons it is desirable to work with polynomial formulae having as small a span as possible.

g. We say that (13) *is an analytic interpolation formula* if the basic function $L(x)$ is analytic and regular for all real $x$. The original cardinal series (5) is an example of this type. Obviously no analytic formula can possibly have a finite span. The role of the span is taken over by the rate of damping of $L(x)$ as $x$ increases. For obvious practical reasons it is desirable to work with analytic $L(x)$ damping out as fast as possible.

2.22. *The characteristic function of the basic function* $L(x)$. It was shown in chapter

---

[7] This property of interpolation formulae seems to have been neglected so far. It represents an important weaker form of the condition of exactness for the degree $k-1$. Compare T. N. E. Greville, loc. cit. pp. 210–211, for our slight departure from the standard terminology. Jenkins speaks of a *modified* interpolation formula in case the formula is not *ordinary*. The term "modified" seems natural in view of Jenkins' construction of such formulae by *modifying* certain terms of Everett's formula (the author is indebted for this last remark to Mr. Chalmers L. Weaver). It seems, however, less desirable if their construction is, as here, otherwise performed.

I that various properties of a smoothing formula

$$F_n = \sum_\nu y_\nu L_{n-\nu}$$

are readily expressible in terms of its characteristic function

$$\phi(u) = \sum_n L_n e^{inu}.$$

Likewise, the properties of the interpolation formula

$$F(x) = \sum_\nu y_\nu L(x-\nu) \tag{20}$$

will largely depend on the behaviour of the function

$$g(u) = \int_{-\infty}^{\infty} L(x)e^{iux}dx = \int_{-\infty}^{\infty} L(x)\cos uxdx. \tag{21}$$

This even function $g(u)$ is the Fourier transform of $L(x)$. Following a terminology used in probability theory we shall refer to $g(u)$ as the *characteristic function* of $L(x)$.

Under certain general assumptions which will always be verified in our application, the relation (21) may be inverted[8] to

$$L(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)e^{iux}du. \tag{22}$$

However, it should be remarked that at times our integrals are not absolutely convergent and that they then converge only as a principal value in the sense of Cauchy: $\lim_{A\to\infty}\int_{-A}^{A}$. An example of this kind is our first formula (10)

$$C_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin u/2}{u/2} e^{iux}du.$$

Changing $u$ and $x$ to $2\pi u$ and $x/2\pi$ respectively we see

$$C_1\left(\frac{x}{2\pi}\right) = \int_{-\infty}^{\infty} \frac{\sin \pi u}{\pi u} e^{iux}du.$$

Inverting this relation we obtain

$$\frac{\sin \pi x}{\pi x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_1\left(\frac{u}{2\pi}\right)e^{iux}du = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iux}du \tag{23}$$

which is identical with (6) and shows that

$$g(u) = C_1\left(\frac{u}{2\pi}\right) = \begin{cases} 1 & \text{if} \quad |u| < \pi \\ \frac{1}{2} & \text{if} \quad |u| = \pi \\ 0 & \text{if} \quad |u| > \pi \end{cases}$$

is the characteristic function of the original cardinal series (5). It is precisely the discontinuity of its characteristic function which causes the extremely slow damping of the basic function (23). (See 2.21, g.)

[8] See e.g. S. Bochner, *Vorlesungen über Fouriersche Integrale*, Leipzig, 1932, Satz 11b on p. 42.

Similar reasons of slow damping will rule out the following rather obvious method of turning a given smoothing formula

$$F_n(x) = \sum_\nu y_\nu L_{n-\nu}$$

into a smoothing interpolation formula

$$F(x) = \sum_\nu y_\nu L(x-\nu).$$

From

$$\phi(u) = \sum_n L_n e^{inu}$$

we derive

$$L_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(u)e^{inu}du.$$

Now we simply define a basic function $L(x)$ by

$$L(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(u)e^{ixu}du.$$

The corresponding characteristic function $g(u)$ is found to be

$$g(u) = \begin{cases} \phi(u) & \text{if } |u| < \pi \\ 0 & \text{if } |u| > \pi. \end{cases}$$

Again the discontinuities of $g(u)$, or of one of its higher derivatives, will imply that the damping of $L(x)$ is too slow for numerical purposes. Indeed, by partial integrations, $L(x)$ is found to tend to zero as a certain negative power of $x$ only, as $x$ tends to infinity. (Concerning the order of magnitude of Fourier integrals for large values of $x$, see the theorem on page 11 of Bochner's book quoted in our footnote 8.)

2.23. *Fundamental criteria in terms of characteristic functions.* We shall now restrict ourselves to basic functions $L(x)$ which are everywhere continuous with the exception of possible "discontinuities of the first kind" (such as were exhibited by the basic functions $L(x)$ of section 2.121). Moreover, we shall assume that $L(x)$ dampens out exponentially. This means that we assume the existence of two positive constants $A$ and $B$ such that the inequality

$$|L(x)| < Ae^{-B|x|} \tag{24}$$

holds for all real values of $x$. This clearly rules out the basic function (23) of the cardinal series. The assumption (24) implies that the characteristic function

$$g(u) = \int_{-\infty}^{\infty} L(x)e^{iux}dx \tag{25}$$

is analytic and regular not only on the real $u$-axis but also in the infinite strip

$$|Iu| < B \tag{26}$$

of the complex $u$-plane. It also implies that the expression

$$\lim_{p\to\infty} \sum_{n=-p}^{p} g(u + 2\pi n)e^{2\pi i n x} \tag{27}$$

converges uniformly in a circular neighborhood of $u=0$ and for every real value of $x$.

The following theorem will demonstrate the usefulness of the characteristic function of an interpolation formula.

THEOREM 2. *Let the basic function* $L(x)$ *satisfy the condition* (24). *Let the corresponding interpolation formula be*

$$F(x) = \sum_{\nu} y_\nu L(x - \nu). \tag{28}$$

*For integral* $x=n$ (28) *reduces to the smoothing formula*

$$F(n) = \sum y_\nu L(n - \nu). \tag{29}$$

A. *The characteristic function* $\phi(u)$, *of the smoothing formula* (29) *is given by the relation*

$$\phi(u) = \sum_{\nu=-\infty}^{\infty} g(u + 2\pi\nu). \tag{30}$$

*In particular* (28) *is an ordinary interpolation formula* (see 2.21, a) *if and only if*

$$\sum_{\nu=-\infty}^{\infty} g(u + 2\pi\nu) \equiv 1. \tag{31}$$

B. *The formula* (28) *is exact for the degree* $k-1$ (see 2.21, c) *if the following two conditions hold simultaneously:*

$g(u) - 1$ *has a zero of order* $k$ *at* $u = 0$, (32)

$g(u)$ *has zeros of order* $k$ *for all non-vanishing integral multiples of* $2\pi$; $u=2\pi n$ $(n\neq 0)$. (33)

C. *The formula* (28) *preserves the degree* $k-1$ (see 2.21, d) *if the condition* (33) *holds, together with the additional condition*

$$g(0) = 1. \tag{34}$$

*Remark.* For some applications it is important to notice that an *ordinary* interpolation formula which *preserves* the degree $k-1$ is automatically *exact* for the degree $k-1$. This seems evident a priori. It is also evident in terms of our criteria, for (31) implies

$$g(u) - 1 = -\sum_{\nu\neq 0} g(u + 2\pi\nu)$$

and the right-hand side has a zero of order $k$ at $u=0$ by (33).

*Proof of* A. Our formula (25) implies

$$L(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)e^{ixu}du = \lim_{p\to\infty} \frac{1}{2\pi}\int_{-(2p+1)\pi}^{(2p+1)\pi} g(u)e^{ixu}du$$

$$= \lim_{p\to\infty} \frac{1}{2\pi} \sum_{\nu=-p}^{p} \int_{-\pi}^{\pi} g(u + 2\pi\nu)e^{iux}e^{2\pi i u x}du$$

$$= \lim_{p\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \sum_{\nu=-p}^{p} g(u + 2\pi\nu) e^{2\pi i x\nu} \right\} e^{iux} du,$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \sum_{\nu=-\infty}^{\infty} g(u + 2\pi\nu) e^{2\pi i x\nu} \right\} e^{iux} du.$$

In particular, if $x = n$ is an integer, we find

$$L(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \sum_{-\infty}^{\infty} g(u + 2\pi\nu) \right\} e^{inu} du. \tag{35}$$

Since the characteristic function $\phi(u)$ of (29) is by definition the function of Fourier coefficients $L(n)$ (see 1.1, (9)), the relation (30) is established.

*Proof of* B. We wish to apply Poisson's summation formula[9]

$$\sum_{n=-\infty}^{\infty} f(x - n) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x} \int_{-\infty}^{\infty} f(v) e^{-2\pi i v n} dv \tag{36}$$

to the function

$$f(x) = e^{-ixu} L(x). \tag{37}$$

By (37) and (25) we find

$$\int_{-\infty}^{\infty} f(v) e^{-2\pi i v n} dv = \int_{-\infty}^{\infty} L(v) e^{-i(u+2\pi n)v} dv = g(u + 2\pi n)$$

hence by (36)

$$e^{-ixu} \sum_{n=-\infty}^{\infty} e^{iun} L(x - n) = \sum_{n=-\infty}^{\infty} g(u + 2\pi n) e^{2\pi i n x}$$

and finally

$$\sum_{n=-\infty}^{\infty} e^{iun} L(x - n) = e^{iux} \sum_{n=-\infty}^{\infty} g(u + 2\pi n) e^{2\pi i n x}. \tag{38}$$

This identity actually holds for all real $x$ and all real or complex values of $u$ within the strip (26). It contains implicitly all the statements of Theorem 1. Thus for $x = 0$ it reduces to (30). To prove our statement B we assume $x$ fixed and regard both members of (38) as functions of $u$, which we expand in series of powers of $u$, then equating the respective coefficients on both sides. On the left-hand side we have the expansion

$$\sum_{\nu=0}^{\infty} \frac{i^\nu u^\nu}{\nu!} \sum_{n=-\infty}^{\infty} n^\nu L(x - n).$$

On the right-hand side, our assumption (33) implies that the terms $g(u + 2\pi n)e^{2\pi i n x}$ $(n \neq 0)$ do not contribute any terms in $u$ of order less than $k$. Thus our identity (38) becomes

$$\sum_{\nu=0}^{\infty} \frac{i^\nu u^\nu}{\nu!} \sum_{-\infty}^{\infty} n^\nu L(x - n) = e^{ixu} g(u) + u^k \text{ (regular function of } u). \tag{39}$$

On the other hand our assumption (32) amounts to

[9] See S. Bochner, loc. cit., theorem 10 on page 35.

$$g(u) = 1 + u^k \text{ (regular function).}$$

This and (39) imply

$$\sum_{\nu=0}^{\infty} \frac{i^\nu u^\nu}{\nu!} \sum_{n=-\infty}^{\infty} n^\nu L(x-n) = \sum_{\nu=0}^{\infty} \frac{i^\nu u^\nu}{\nu!} x^\nu + u^k \text{ (regular function).} \tag{40}$$

A comparison of the coefficients of the first $k$ terms on each side of (40) furnishes the identities (16). This concludes a proof of B.

*Proof of* C. Since $g(u)$ is regular at $u=0$, and even, it has in view of (34) an expansion of the form

$$g(u) = 1 - \frac{a_2}{2!} u^2 + \frac{a_4}{4!} u^4 - \frac{a_\mu}{6!} u_6 + \cdots .$$

We now define a sequence of polynomials by means of the generating function

$$e^{ixu} g(u) = \sum_{\nu=0}^{\infty} Q_\nu(x) \frac{(iu)^\nu}{\nu!} \tag{41}$$

or

$$e^{xu} g(u/i) = \sum_{\nu=0}^{\infty} Q_\nu(x) \frac{u^\nu}{\nu!} \tag{42}$$

where

$$g(u/i) = 1 + \frac{a_2}{2!} u^2 + \frac{a_4}{4!} u^4 + \cdots . \tag{43}$$

A comparison of terms on both sides of (42), using (43), shows that

$$Q_\nu(x) = x^\nu + \binom{\nu}{2} a_2 x^{\nu-2} + \binom{\nu}{4} a_4 x^{\nu-4} + \cdots . \tag{44}$$

On substituting the expansion (41) into the right-hand side of (39) and by comparison of the first $k$ terms on both sides we find that the identities (18) and (19) are established. This completes the proof of our theorem.

As a brief illustration of our criteria let us consider again Jenkins' smoothing interpolation formula (12′) of 2.123. Its basic function is

$$L(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2 \sin u/2}{u}\right)^4 \left(\frac{4}{3} - \frac{1}{3} \cos u\right) e^{iux} du. \tag{45}$$

A simple method of evaluating explicitly such integrals in order to find the polynomial expressions (12) will be discussed later. An inspection of the characteristic function

$$g(u) = \left(\frac{2 \sin u/2}{u}\right)^4 (4 - \cos u)/3$$

immediately reveals that our condition (33) is verified for $k=4$. Direct expansion shows that $g(u) = 1 - (7/240)u^4 + \cdots$ and (32) is also verified for $k=4$. The interpolation formula (12′) is therefore exact for cubics. Also the fact that $L(x)$ is of class $C''$ is revealed by an inspection of the integral (45). Indeed we notice that $g(u)$ vanishes for $u=\infty$ like $u^{-4}$. This implies that we may differentiate (45) twice under the integral sign and that the integral

$$L''(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)(iu)^2 e^{iux} du$$

is also continuous since it converges absolutely.

## III. THE THEORY OF SPLINE CURVES

The previous chapter provides a formal theory of interpolation formulae in terms of a basic function $L(x)$ which, as yet, is largely arbitrary. The present chapter will furnish the foundation for the derivation of special basic functions which are readily computed with great accuracy and lead to interpolation formulae enjoying the properties described in the previous chapter.

**3.1. Polynomial spline curves of order $k$.** A spline is a simple mechanical device for drawing smooth curves. It is a slender flexible bar made of wood or some other elastic material. The spline is place on the sheet of graph paper and held in place at various points by means of certain heavy objects (called "dogs" or "rats") such as to take the shape of the curve we wish to draw. Let us assume that the spline is so placed and supported as to take the shape of a curve which is nearly parallel to the $x$-axis. If we denote by $y=y(x)$ the equation of this curve then we may neglect its small slope $y'$, whereby its curvature becomes

$$1/R = y''/(1 + y'^2)^{3/2} \approx y''.$$

The elementary theory of the beam will then show that the curve $y=y(x)$ is a polygonal line composed of cubic arcs which join continuously, with a continuous first and second derivative.[10] These junction points are precisely the points where the heavy supporting objects are placed.

3.11. *Description of spline curves of order $k$.* Our last remark suggests the following definition.

DEFINITION 4. *A real function $F(x)$ defined for all real $x$ is called a spline curve of order $k$ and denoted by $\Pi_k(x)$ if it enjoys the following properties:*

1) *It is compressed of polynomial arcs of degree at most $k-1$.*

2) *It is of class $C^{k-2}$, i.e., $F(x)$ has $k-2$ continuous derivatives.*

3) *The only possible function points of the various polynomial arcs are the integer points $x=n$ if $k$ is even, or else the points $x=n+1/2$ if $k$ is odd.*

Thus a $\Pi_1(x)$ is a step function with possible discontinuities at the points $x=n+1/2$. A $\Pi_2(x)$ has an ordinary polygonal graph with vertices only at the integer points $x=n$. A $\Pi_4(x)$ corresponds to the elementary mathematical description of an ordinary (infinite) spline with the "dogs" placed at all or only some of the points with $x=n$.

It should be noticed that if a $\Pi_k(x)$ is of class $C^{k-1}$, then $\Pi^{(k-1)}(x)$ must necessarily be constant for all $x$. Thus such a $\Pi_k(x)$ reduces to a polynomial of degree $k-1$. It is just this relaxation of the requirement of the continuity of the $(k-1)$-order derivative of $\Pi_k(x)$ which turns the spline curve into a flexible and versatile instrument of approximation. Likewise, only the "dogs" (or "rats") enable the ordinary spline to trace curves differing from the graph of a cubic polynomial.

The special importance of spline curves will be due to the fact that by the addi-

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[10] The author is indebted for this suggestion to Professor L. H. Thomas of Ohio State University.

tion of several spline curves of successive orders we may get any desired polygonal line of given degree $m$ and class $C^\mu$.

3.12. *The evaluation of certain Fourier integrals.* Our further work is based on the consideration of the functions

$$M_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin u/2}{u}\right)^k e^{iux}du \qquad (k = 1, 2, \cdots; -\infty < x < \infty). \tag{1}$$

They have been evaluated explicitly for low values of $k$ by various authors.[11] The following general explicit representation is essentially due to Laplace (see J. V. Uspensky, *Introduction to mathematical probability*, 1937, Example 3, pp. 277–278).

THEOREM 3. *Let $k$ be a positive integer. Define the function $x_+^{k-1}$ by*

$$x_+^{k-1} = \begin{cases} x^{k-1} & \text{if } x \geqq 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{2}$$

*For $k=1$ and $x=0$ this definition is modified to $0_+^{k-1}=1/2$.*

*The following identity holds for all real values of $x$*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin u/2}{u}\right)^k e^{iux}du = \frac{1}{(k-1)!}\delta^k x_+^{k-1}, \tag{3}$$

*where $\delta^k$ stands for the usual symbol of the kth order central difference of step equal to unity.*

The identity (3) is correct for $k=1$. Indeed it is well known that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin u}{u}du = \tfrac{1}{2}.$$

On replacing $u$ by $ux$ we get that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin xu}{u}du = \begin{cases} \tfrac{1}{2} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\tfrac{1}{2} & \text{if } x < 0, \end{cases}$$

or

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin xu}{u}du = x_+^0 - \tfrac{1}{2}.$$

Therefore

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin u/2}{u}\cos uxdu = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin(x+\frac{1}{2})u}{u}du - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin(x-\frac{1}{2})u}{u}du$$

$$= \frac{1}{2\pi}\delta\int_{-\infty}^{\infty}\frac{\sin xu}{u}du = \delta(x_+^0 - \tfrac{1}{2}) = \delta x_+^0,$$

[11] See S. Bochner, *Fourier analysis*, Princeton University lectures, 1936–1937, where our $M_k(x)$ are worked out for $k=1, 2, 3$, and where the increasing smoothness qualities of these functions are clearly noted. Bochner also considers the integrals

$$M_k^*(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin u/2}{u}\right)^k\frac{\sin ux}{u}du.$$

Using Theorem 5 below, we readily obtain the identity $M_k^*(x) = -\frac{1}{2}+(1/k!)\delta^k x_+^k$.

and (3) is established for $k=1$.

Let us consider for the moment the sequence of functions

$$N_k(x) = \frac{1}{(k-1)!} \delta^k x_+^{k-1}. \tag{4}$$

We have already shown that

$$M_k(x) = N_k(x) \tag{5}$$

holds for $k=1$. Assume now that (5) holds. We wish to show that the similar identity for $k+1$, rather than $k$, arises from (5) by performing the operation

$$\int_{x-1/2}^{x+1/2}$$

on both sides of (5). This will be accomplished if we prove that

$$M_{k+1}(x) = \int_{x-1/2}^{x+1/2} M_k(x)dx \tag{6}$$

and

$$N_{k+1}(x) = \int_{x-1/2}^{x+1/2} N_k(x)dx. \tag{7}$$

In view of

$$\int_{x-1/2}^{x+1/2} e^{iux}dx = \frac{2\sin u/2}{u} e^{iux}$$

we obtain (6) by an integration under the integral sign as follows

$$\int_{x-1/2}^{x+1/2} M_k(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin u/2}{u}\right)^k \left(\int_{x-1/2}^{x+1/2} e^{iux}dx\right)du = M_{k+1}(x).$$

To prove (7) we notice that

$$\int_{x-1/2}^{x+1/2} N_k(x)dx = \delta\int_{-\infty}^{x} N_k(x)dx = \delta\int_{-\infty}^{x} \delta^k \frac{x_+^{k-1}}{(k-1)!}dx$$
$$= \delta\delta^k \int_{-\infty}^{x} \frac{x_+^{k-1}}{(k-1)!}dx = \delta^{k+1}\frac{x_+^k}{k!} = N_{k+1}(x).$$

This concludes the proof of Theorem 3.

3.13. *Explicit polynomial expressions for $M_k(x)$*. The formula

$$M_k(x) = \frac{1}{(k-1)!} \delta^k x_+^{k-1} \tag{8}$$

will readily show that *$y=M_k(x)$ represents a spline curve of order $k$*. Indeed, if $k$ is even, then

$$(x+n)_+^{k-1}$$

is a $\Pi_k$ and therefore also their linear combination (8). If $k$ is odd the same conclusion holds because

$$(x + n - \tfrac{1}{2})_+^{k-1}$$

is a $\Pi_k$.

It also follows from (8) that the explicit polynomial expressions for

$$(k-1)!M_k(x),$$

in successive unit intervals, are identical with the successive partial sums of the expansion

$$\delta^k x^{k-1} = \left(x+\frac{k}{2}\right)^{k-1} - \binom{k}{1}\left(x+\frac{k}{2}-1\right)^{k-1} + \cdots + (-1)^k\left(x-\frac{k}{2}\right)^{k-1},$$

an expression which incidentally vanishes identically, being the $k$th order difference of a polynomial of degree $k-1$. We thus get

$$(k-1)!M_k(x) = \begin{cases} 0 & \text{if} \quad x \leqq -\frac{k}{2} \\ \left(x+\frac{k}{2}\right)^{k-1} & \text{if} \quad -\frac{k}{2} \leqq x \leqq -\frac{k}{2}+1 \\ \left(x+\frac{k}{2}\right)^{k-1} - \binom{k}{1}\left(x+\frac{k}{2}-1\right)^{k-1} & \text{if} \quad -\frac{k}{2}+1 \leqq x \leqq -\frac{k}{2}+2 \\ \cdots\cdots\cdots & \\ \left(x+\frac{k}{2}\right)^{k-1} - \binom{k}{1}\left(x+\frac{k}{2}-1\right)^{k-1} + \cdots & \\ \quad + (-1)^{k-1}\binom{k}{1}\left(x-\frac{k}{2}+1\right)^{k-1} & \text{if} \quad \frac{k}{2}-1 \leqq x \leqq \frac{k}{2} \\ \delta^k x^{k-1} = 0 & \text{if} \quad \frac{k}{2} \leqq x. \end{cases} \tag{9}$$

For future reference we work out explicitly the cases $k=1, 2, 3,$ and $4$. The expansions

$$\begin{aligned} \delta x^0 &= 1 - 1 \\ \delta^2 x &= (x+1) - 2x + (x-1) \\ \delta^3 x^2 &= (x+\tfrac{3}{2})^2 - 3(x+\tfrac{1}{2})^2 + 3(x-\tfrac{1}{2})^2 - (x-\tfrac{3}{2})^2 \\ \delta^4 x^3 &= (x+2)^3 - 4(x+1)^3 + 6x^3 - 4(x-1)^3 + (x-2)^3 \end{aligned} \tag{10}$$

now furnish the following expressions

$$M_1(x) = \begin{cases} 0 & \text{if} \quad x < -\frac{1}{2} \\ 1 & \text{if} \quad -\frac{1}{2} < x < \frac{1}{2} \\ 0 & \text{if} \quad \frac{1}{2} < x, \end{cases} \tag{11}$$

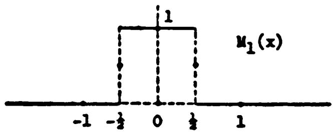
to which must be added $M_1(\pm 1/2) = 1/2$ as required by (1) for $k=1$.

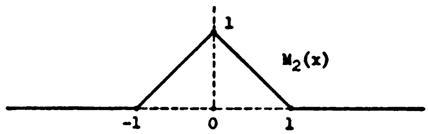
$$M_2(x) = \begin{cases} 0 & \text{if} \quad x \leqq -1 \\ x+1 & \text{if} \quad -1 \leqq x \leqq 0 \\ -x+1 & \text{if} \quad 0 \leqq x \leqq 1 \\ 0 & \text{if} \quad 1 \leqq x, \end{cases} \tag{12}$$

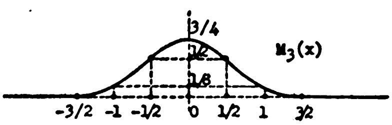
$$M_3(x) = \begin{cases} 0 & \text{if} \quad x \leqq -\tfrac{3}{2} \\ (1/2)(x + 3/2)^2 & \text{if} \quad -\tfrac{3}{2} \leqq x \leqq -\tfrac{1}{2} \\ (1/2)(x + 3/2)^2 - (3/2)(x + 1/2)^2 & \text{if} \quad -\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2} \\ (1/2)(-x + 3/2)^2 & \text{if} \quad \tfrac{1}{2} \leqq x \leqq \tfrac{3}{2} \\ 0 & \text{if} \quad \tfrac{3}{2} \leqq x, \end{cases} \tag{13}$$

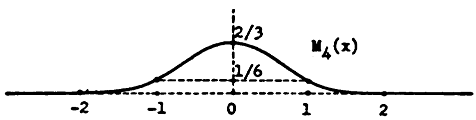
$$M_4(x) = \begin{cases} 0 & \text{if} \quad x \leqq -2 \\ (1/6)(x + 2)^3 & \text{if} \quad -2 \leqq x \leqq -1 \\ (1/6)(x + 2)^3 - (4/6)(x + 1)^3 & \text{if} \quad -1 \leqq x \leqq 0 \\ (1/6)(-x + 2)^3 - (4/6)(-x + 1)^3 & \text{if} \quad 0 \leqq x \leqq 1 \\ (1/6)(-x + 2)^3 & \text{if} \quad 1 \leqq x \leqq 2 \\ 0 & \text{if} \quad 2 \leqq x. \end{cases} \tag{14}$$

In deriving these expressions the expansions (10) were used up to the point from where the evenness of the functions $M_k(x)$ allowed us to complete their definition for all $x$ by symmetry.




3.14. *Interpolation formula with $M_k(x)$ as basic function.* The interpolation formula

$$F(x) = \sum_{\nu=-\infty}^{\infty} y_\nu M_k(x - \nu) \tag{15}$$

will play an important role in our subsequent work. We mention it at this place because it contributes to our investigation of $k$-order spline curves.

To the basic function

$$L(x) = M_k(x)$$

corresponds the characteristic function

$$g(u) = \left(\frac{2 \sin u/2}{u}\right)^k, \tag{16}$$

as seen by comparing III (1) with II (22). The characteristic function of the formula (15) for integral $x = n$ is

$$\phi_k(u) = M_k(0) + 2M_k(1) \cos u + 2M_k(2) \cos 2u + \cdots . \tag{17}$$

The $\{\phi_k(u)\}$ represent an interesting sequence of cosine polynomials which we will investigate more closely later in this paper. Here we mention without proof that

$$1 = \phi_1(u) = \phi_2(u) > \cdots > \phi_k(u) > \cdots > 0 \qquad (0 < u < 2\pi) \tag{18}$$

while, of course, $\phi_k(0) = 1$. Hence (15) is a smoothing interpolation formula of progressively increasing strength as $k$ increases. We assemble the various properties of (15) in the form of a theorem.

THEOREM 4.

$$F(x) = \sum_{\nu=-\infty}^{\infty} y_\nu M_k(x - \nu) \tag{19}$$

*is a polynomial smoothing interpolation formula of degree $k-1$, class $C^{k-2}$ and span $2s = k$ (see sections 2.21 and 3.13). It is exact for the degree 1 and preserves the degree $k-1$. The smoothing power of (19) increases progressively for increasing values of $k$.*

The exactness of (19) for the degree 1 and the preservation of the degree $k-1$ follow by Theorem 2 (B and C). Indeed, by (16), $g(u)-1$ has a double zero for $u=0$ while $g(u)$ has zeros of order $k$ for $u = 2\pi n$ $(n \neq 0)$. Since the preservation of the degree $k-1$ implies the identities II(18) and (19), the following corollary results.

COROLLARY. *Any given polynomial $P_{k-1}(x)$ of degree at most $k-1$ may be represented in the form*

$$P_{k-1}(x) = \sum_{n=-\infty}^{\infty} y_n M_k(x - n) \tag{20}$$

*where $\{y_n\}$ are the ordinates of some other suitably chosen polynomial of the same degree as $P_{k-1}$. This representation is unique.*

3.15. *The analytic representation of spline curves of order $k$.* We know that if $\{y_n\}$ is an arbitrary sequence of ordinates, then our interpolation formula

$$F(x) = \sum_n y_n M_k(x - n) \tag{21}$$

represents a spline curve of order $k$. This is true because all $M_k(x-n)$ are such curves. The following question arises: *Let $F(x)$ be a given $\Pi_k$; can we always represent it in the form* (21) *for an appropriate sequence* $\{y_n\}$?

This question is answered affirmatively by the following theorem.

THEOREM 5. *Any spline curve $\Pi_k(x)$ may be represented in one and only one way in the form*

$$\Pi_k(x) = \sum_{n=-\infty}^{\infty} y_n M_k(x - n) \tag{22}$$

*for appropriate values of the coefficients $y_n$. There are no convergence difficulties since $M_k(x)$ vanishes for $|x| > k/2$. Thus (22) represents a $\Pi_k$ for arbitrary $\{y_n\}$ and represents the most general one.*

In order to prove this theorem we return to the interpolation formula (21) and differentiate it repeatedly. By (8) we have

$$M_k'(x) = \delta^k \frac{d}{dx} \frac{1}{(k-1)!} x_+^{k-1} = \delta^k \frac{1}{(k-2)!} x_+^{k-2} = \delta M_{k-1}(x)$$

and repeating we get

$$M_k^{(\nu)}(x) = \delta^\nu M_{k-\nu}(x) \qquad (0 \leqq \nu \leqq k-1). \tag{23}$$

From (21) and (23) we obtain by partial summation

$$F'(x) = \sum_n y_n \delta M_{k-1}(x-n) = \sum_n \delta y_{n+1/2} M_{k-1}(x - n - \tfrac{1}{2})$$

or

$$F'(x + \tfrac{1}{2}) = \sum_n \delta y_{n+1/2} M_{k-1}(x-n). \tag{24}$$

If $k>2$, this formal rule of differentiation of a spline curve may now again be applied to (24) with the result

$$F''(x+1) = \sum \delta^2 y_{n+1} M_{k-2}(x-n)$$

or

$$F''(x) = \sum \delta^2 y_n M_{k-2}(x-n).$$

Generally for $0 \leqq \nu \leqq k-1$

$$F^{(\nu)}(x) = \begin{cases} \sum_n \delta^\nu y_n M_{k-\nu}(x-n) & \text{if } \nu \text{ is even,} \\ \sum_n \delta^\nu y_{n+1/2} M_{k-\nu}(x-n-\tfrac{1}{2}) & \text{if } \nu \text{ is odd.} \end{cases} \tag{25}$$

This result may be stated as follows: *The $\nu$th derivative of the spline curve* (21) *may be obtained directly by applying the same interpolation formula* (21) *with $k-\nu$, rather than $k$, to the sequence of the $\nu$th central differences $\delta^\nu y$ properly centered according to the parity of $\nu$. In particularly: $F^{(k-2)}(x)$ is obtained by interpolating linearly among the $\delta^{k-2}y$. $F^{(k-1)}(x)$ is a step function whose successive values agree with those of the corresponding $\delta^{k-1}y$.*

Now let $F(x)$ be a given $\Pi_k$. We are to show the existence of a sequence $\{y_n\}$ such that (21) holds identically. Suppose for the moment that such $y_n$ have been found which do make (21) hold. Then by (25) for $\nu = k-1$ we have

$$F^{(k-1)}(x) = \begin{cases} \delta^{k-1} y_n & \text{for } n - \tfrac{1}{2} < x < n + \tfrac{1}{2} \quad \text{if } k \text{ is odd.} \\ \delta^{k-1} y_{n+1/2} & \text{for } n < x < n+1 \quad \text{if } k \text{ is even.} \end{cases} \tag{26}$$

In either case the successive constant values of the step function $F^{(k-1)}(x)$ determine uniquely the values of the differences of order $k-1$ of the sequence of the as yet unknown coefficients $y_n$. These differences in turn determine the coefficients $y_n$ uniquely up to an additive sequence of vanishing differences of order $k-1$. Let $\bar{y}_n$ be one sequence such that

$$\delta^{k-1} \bar{y}_{n+1/2} = F^{(k-1)}(n + \tfrac{1}{2}) \quad \text{if} \quad k \text{ is even}$$

or else

$$\delta^{k-1} \bar{y}_n = F^{(k-1)}(n) \quad \text{if} \quad k \text{ is odd.}$$

Consider the $k$-order spline curve

$$\bar{F}(x) = \sum_n \bar{y}_n M_k(x - n)$$

and let

$$R(x) = F(x) - \bar{F}(x).$$

From the way $\bar{F}(x)$ was defined it is clear that $\bar{F}^{(k-1)}(x)$ and $F^{(k-1)}(x)$ agree in their successive unit intervals of constancy. Hence $R(x)$ is a $\Pi_k$ whose various polynomial arcs are of degree $k-2$ or lower. Therefore $R(x)$ is *identical* with a polynomial of degree $k-2$. As such it allows of a representation of the form (21) in view of our Corollary of section 3.14. Therefore also

$$F(x) = \bar{F}(x) + R(x)$$

may be represented by our formula (21).

The unicity of the representation (21) is readily established. Indeed two different such representations would imply a representation of zero

$$0 = \sum_n y_n M_k(x - n)$$

without all $y_n$ vanishing. However the $\delta^{k-1}y$ all vanish, and our conclusion would contradict the uniqueness of the representation (20) of polynomials.

A simple example might illustrate our proof of Theorem 5. Let us find the representation of the spline curve of order 4

$$F(x) = \frac{1}{3!} x_+^3.$$

By (26) we have

$$\delta^3 y_{n+1/2} = F'''(n + \tfrac{1}{2}) = (n + \tfrac{1}{2})_+^0 = \begin{cases} 1 & \text{if} \quad n \geq 0 \\ 0 & \text{if} \quad n < 0. \end{cases}$$

A sequence having these third differences is

$$y_n = \begin{cases} 0 & \text{if} \quad n < 0 \\ \binom{n+1}{3} = \dfrac{n(n^2 - 1)}{6} & \text{if} \quad n \geq 0. \end{cases}$$

Hence

$$x_+^3 = \sum_{n=0}^{\infty} n(n^2 - 1) M_4(x - n), \tag{27}$$

with the possibility still open that both member might differ by a third degree polynomial. However this possibility is excluded by the remark that both sides vanish identically for $x \leq 0$. Incidentally, (27) implies

$$-(-x)_+^3 = \sum_{n=-\infty}^{0} n(n^2 - 1) M_4(x - n)$$

and by addition and subtraction of the two relations we get the identities

$$x^3 = \sum_{n=-\infty}^{\infty} n(n^2 - 1)M_4(x - n), \qquad |x|^3 = \sum_{n=-\infty}^{\infty} |n(n^2 - 1)| M_4(x - n).$$

In later applications we shall frequently operate with polygonal lines $F(x)$ of degree $k-1$ not having continuity properties as strong as a $\Pi_k$. Thus Jenkins' $L(x)$ defined by II(11) is of degree 4 and class $C''$. Let $F(x)$ be a polygonal line of degree $k-1$, having vertices at integral point $x=n$, and being for all real $x$ a function of class $C^\mu$ $(-1 \leqq \mu \leqq k-2)$. We certainly obtain a curve of degree $k-1$ and class $C^\mu$ by addition of spline curves.

$$F(x) = \Pi_{\mu+2} + \Pi_{\mu+3} + \cdots + \Pi_k, \tag{28}$$

where $\Pi_\nu$ stands for $\Pi_\nu(x)$ or $\Pi_\nu(x+1/2)$, according to whether $\nu$ is even or odd.

THEOREM 6. *Any given polygonal line $F(x)$ of degree $k-1$ and class $C^\mu$ may be represented as a sum* (28) *of $k-\mu-1$ appropriate spline curves of orders $\mu+2, \mu+3, \cdots, k$.*

This theorem is a corollary to Theorem 5. Indeed, $F^{(\mu+1)}(x)$ may have certain discontinuities. We determine a $\Pi_{\mu+2}$ having the same discontinuities in its $(\mu+1)$st derivative. Then

$$F(x) - \Pi_{\mu+2}$$

is of degree $k-1$ and class $C^{\mu+1}$. Proceeding in this way the theorem is readily established.

Substituting for the $\Pi_\nu$ in (28) their expressions in terms of the $M_\nu$ we obtain an explicit (parametric) representation of such polygonal lines. Thus Jenkins' function II(11) may be represented as

$$L(x) = 4M_4(x) + \tfrac{1}{2}M_4(x + 1) + \tfrac{1}{2}M_4(x - 1) - 2M_5(x + \tfrac{1}{2}) - 2M_5(x - \tfrac{1}{2}).$$

At a glance we recognize a curve of degree 4, class $C''$, span $s=6$.

3.16. *A summation property of spline curves.* The degree of a polynomial is decreased or increased by one unit if we difference or else sum the polynomial. Not so with our spline curves $\Pi_k$. Indeed let

$$\Pi_k(x) = \sum_n y_n M_k(x - n) \tag{29}$$

be a given spline curve. Then

$$\Pi_k(x + 1) = \sum y_n M_k(x + 1 - n) = \sum y_{n+1} M_k(x - n)$$

and subtracting (29) we have

$$\Delta\Pi_k(x) = \Pi_k(x + 1) - \Pi_k(x) = \sum \Delta y_n M_k(x - n). \tag{30}$$

Hence $\Delta\Pi_k(x)$ will in general be also a $\Pi_k$. Now let the spline curve $\Pi_k^*(x)$ be given and we wish to find $\Pi_k(x)$ such that

$$\Delta\Pi_k(x) = \Pi_k^*(x). \tag{31}$$

By using Theorem 5 the solution is immediately found. Indeed let

$$\Pi_k^*(x) = \sum y_n^* M_k(x - n).$$

Now (29) will be a solution of (31) provided $\Delta y_n = y_n^*$, for all $n$. This relation defines the $y_n$ up to an additive constant which appears as an arbitrary additive constant in the solution $\Pi_k(x)$. It is thus seen that the operation of differencing or summing spline curves (29) reduces to performing the same operations on the sequence of the coefficients $\{y_n\}$.

3.17. *An interpretation of $M_k(x)$ in probability theory*. In concluding our discussion of polynomial spline curves we mention briefly the following interpretation of $M_k(x)$. As seen from its values as given by III(11) it is clear that $M_1(x)$ may be interpreted as the probability density function of the error committed on a random real variable $x$, if that variable is rounded off to its nearest integral value. Now III(1) shows that the characteristic function of $M_k(x)$ is the $k$th power of the characteristic function of $M_1(x)$. From a known proposition in probability theory we may conclude that $M_k(x)$ *is the density distribution function of the error committed on the sum*

$$x_1 + x_2 + \cdots + x_k$$

*of $k$ statistically independent real random variables $x_1, \cdots, x_k$, if each variable is replaced by its nearest integral value.*

This interpretation, otherwise entirely irrelevant for our purpose, does make a few of the properties of $M_k(x)$ intuitively obvious, such as

$$M_k(x) \begin{cases} > 0 & \text{for} \quad |x| > k/2 \\ = 0 & \text{for} \quad |x| < k/2, \end{cases}$$

$$\int_{-\infty}^{\infty} M_k(x)dx = 1.$$

In concluding we note the identity

$$\int_{x-1/2}^{x+1/2} dx_1 \int_{x_1-1/2}^{x_1+1/2} dx_2 \cdots \int_{x_{k-1}-1/2}^{x_{k-1}+1/2} f(x_k)dx_k = \int_{-\infty}^{\infty} M_k(u - x)f(u)du.$$

**3.2. Analytic spline curves of order $k$.** The polynomial spline curves $\Pi_k(x)$ described in section 3.1 will be shown to be sufficient for the derivation of polynomial approximations to equidistant data enjoying various desirable properties. These polynomial approximations will have any a priori assigned number of continuous derivatives. However, in order to obtain analytic approximations we shall now proceed to derive from our spline curves $\Pi_k(x)$ an analoguous family of analytic functions.

To achieve this end we shall smooth out our $\Pi_k(x)$ by means of one-dimensional heat flow. Consider an infinite homogeneous bar (the $x$-axis) in which the temperature at the point $x$ at the time $t$ is denoted by $F(x, t)$. We assume the flow of heat to be governed by the equation

$$\frac{\partial F}{\partial t} = \frac{1}{4} \frac{\partial^2 F}{\partial x^2}. \tag{32}$$

If $F(x) = F(x, 0)$ is given, i.e., the temperature distribution at the time $t = 0$ is known, then $F(x, t)$ is determined by the following integral[12]

$$F(x, t) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-(u-x)^2 t^{-1}} F(u, 0) du. \tag{33}$$

This result is easy to verify: by partial differentiations we find that $F(x, t)$ as defined by (33) indeed satisfies the differential equation (32) while familiar arguments originated by Weierstrass will show that (33) implies

$$\lim_{t \to +0} F(x, t) = F(x, 0),$$

provided $F(x, 0)$ is continuous and, e.g., bounded.

The solution of the problem of finding $F(x, t)$, if $F(x, 0)$ is given, is especially simple in the case when $F(x, 0)$ is defined by a Fourier integral

$$F(x, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(u) e^{iux} du. \tag{34}$$

Indeed, in this case we find

$$F(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t(u/2)^2} \psi(u) e^{iux} du. \tag{35}$$

Notice that the temperature $t$ enters only in the additional exponential factor. This can be proved in two ways, either by substituting (34) into (33) or else by verifying directly that (35) satisfies the differential equation (32). Obviously (35) reduces to (34) for $t=0$, as it should.

We may (and wish to) think of $F(x, t)$, for a fixed $t>0$, as a smoothed version of $F(x) = F(x; 0)$. In fact $F(x, t)$ is analytic and regular for all real or complex values of $x$ if $\psi(u)$ is, e.g., bounded.

If we now apply this heat-flow transformation to our basic $k$-order spline curve

$$M_k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2 \sin u/2}{u}\right)^k e^{iux} du \tag{36}$$

we obtain by (35) its smoothed version

$$M_k(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t(u/2)^2} \left(\frac{2 \sin u/2}{u}\right)^k e^{iux} du. \tag{37}$$

Obviously

$$M_k(x, 0) = M_k(x). \tag{38}$$

[12] See H. S. Carslaw, *Mathematical theory of the conduction of heat*, Dover Publications, New York, 1945, Chapter III, Section 16. Certain smoothing properties of heat flow were already noticed by Ch. Sturm in 1886. See in this connection G. Pólya, *Qualitatives über Wärmeausgleich*, Z. angew. Math. u. Mech. **13**, 125–128 (1933). It should be mentioned here that Weierstrass derived his famous approximation theorem by means of the integral (33). Finally see E. Czuber, *Wahrscheinlichkeitsrechnung*, vol. I, Leipzig-Berlin, 1924, pp. 417–418, for a brief sketch of a method of using (33) to derive analytic approximations to given data.

The graph of $y=M_k(x, t)$ $(t>0)$ is a bell-shaped curve which dampens out very fast. Later we shall learn how to compute its values very accurately. Here we mention that

$$0 < M_k(x, t) < \frac{1}{\sqrt{\pi t}} e^{-(x-k/2)^2 \cdot t^{-1}} \quad \text{if} \quad x \geqq k/2. \tag{39}$$

Also, the recurrence relation III(6) generalizes so that $M_{k+1}(x, t)$ is obtained by the averaging operation

$$M_{k+1}(x, t) = \int_{x-1/2}^{x+1/2} M_k(x, t)dx. \tag{40}$$

If now

$$\Pi_k(x) = \sum_{n=-\infty}^{\infty} y_n M_k(x - n) \tag{41}$$

is a spline curve of order $k$, then its heat-flow transform is

$$\Pi_k(x, t) = \sum_{n=-\infty}^{\infty} y_n M_k(x - n, t). \tag{42}$$

The graph of this function may be called an *analytic spline curve of order k*. We notice that the series (42) fails to converge only if $y_n$ increases very fast with $|n|$.

Summarizing we see that the curve (42) arises from the step function

$$\sum_n y_n M_1(x - n)$$

by $k-1$ successive applications of the averaging operation

$$\int_{x-1/2}^{x+1/2} (\quad)dx$$

followed by the heat-flow transformation during a time interval $t$. The order of application of these $k$ operations is of course irrelevant.

The remaining parts of the paper are devoted to the problem of utilizing the two families of curves (41) and (42) for the purpose of approximating given equidistant data.

## IV. A FIRST CLASS OF ANALYTIC INTERPOLATION FORMULAE

We shall now use the analytic spline curves III(42) to obtain

1. A smoothing interpolation formula which is exact for the degree 1 only.
2. An ordinary interpolation formula exact for the degree $k-1$.
3. A smoothing interpolation formula depending on a positive parameter $\epsilon$, of strictly increasing smoothing power as $\epsilon$ increases. For $\epsilon=0$ this formula reduces to 2, while for $\epsilon=\infty$ it is identical with 1.

**4.1. A smoothing interpolation formula exact for the degree 1.** A comparison of the formulae III(15) and III(42) immediately suggests the following analytic interpolation formula

$$F(x) = \sum_{n=-\infty}^{\infty} y_n M_k(x - n, t) \tag{1}$$

where $M_k(x, t)$ is defined by III(37). For the sequel we shall use the following notation

$$\psi_k(u, t) = e^{-t(u/2)^2}\left(\frac{2 \sin u/2}{u}\right)^k, \tag{2}$$

$$\psi_k(u) = \psi_k(u, 0) = \left(\frac{2 \sin u/2}{u}\right)^k, \tag{3}$$

in terms of which III(37) becomes

$$M_k(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_k(u, t)e^{iux}du. \tag{4}$$

The characteristic function of the smoothing formula (1) for integral $x$ is

$$\phi_k(u, t) = \sum_{n=-\infty}^{\infty} M_k(n, t) \cos nu. \tag{5}$$

The general relation II(30) furnishes the following equivalent expression

$$\phi_k(u, t) = \sum_{\nu=-\infty}^{\infty} \psi_k(u + 2\pi\nu, t). \tag{6}$$

The properties III(18) for the case $t=0$ generalize for $t>0$ as follows

$$1 > \phi_1(u, t) > \phi_2(u, t) > \cdots > \phi_k(u, t) > \cdots > 0, \qquad (0 < u < 2\pi, t > 0). \tag{7}$$

Moreover, for each fixed $u$, $0<u<2\pi$, $\phi_k(u, t)$ is strictly decreasing as $t$ increases. These last two properties we state without proving them here.

The arguments which lead to Theorem 4 now allow us to state the following theorem.

THEOREM 7. *For* $t>0$, $k=1, 2, \cdots$,

$$F(x) = \sum_{n=-\infty}^{\infty} y_n M_k(x - n, t) \tag{8}$$

*is an analytic smoothing interpolation formula which is exact for the degree* 1 *and preserves the degree* $k-1$. *The smoothing power of* (8) *increases whenever either* $k$ *or* $t$ *is increased.*

**4.2. An ordinary interpolation formula exact for the degree $k-1$.** We shall now use the important property (7) to the effect that the periodic function $\phi_k(u, t)$ is positive for real $u$. This allows us to define the basic function

$$L_k(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\psi_k(u, t)}{\phi_k(u, t)} e^{iux}du \tag{9}$$

whose characteristic function is

$$g(u) = \frac{\psi_k(u, t)}{\phi_k(u, t)}. \tag{10}$$

Our Theorem 2 of Chapter II will readily yield the following result.

THEOREM 8. *For* $t\geqq 0$, $k=1, 2, 3, \cdots$

$$F(x) = \sum_{n=-\infty}^{\infty} y_n L_k(x - n, t) \tag{11}$$

*is an ordinary interpolation formula which is exact for the degree $k-1$.*

Firstly we realize by (10) and (2) that the conditions II(33), (34) of Theorem 2 are verified. Therefore (11) preserves the degree $k-1$. Secondly we have by (6), (10), and II(30)

$$\phi(u) = \sum_{\nu} g(u + 2\pi\nu) = \sum_{\nu} \frac{\psi_k(u + 2\pi\nu, t)}{\phi_k(u + 2\pi\nu, t)} = \frac{1}{\phi_k(u, t)} \sum_{\nu} \psi_k(u + 2\pi\nu, t) = 1.$$

Thus II(31) holds and (11) is therefore an *ordinary* interpolation formula. This concludes the proof of our theorem. Indeed, by the remark following Theorem 2 our formula (11) must also be exact for the degree $k-1$.

We also mention without further details the following two limiting relations

$$\lim_{t\to\infty} \frac{\psi_k(u, t)}{\phi_k(u, t)} = \lim_{k\to\infty} \frac{\psi_k(u, t)}{\phi_k(u, t)} = \begin{cases} 1 & \text{if } |u| < \pi \\ 0 & \text{if } |u| > \pi. \end{cases} \tag{12}$$

They show, in view of the integral representation (9), that our present basic function $L_k(x, t)$ converges towards the basic function II(6) of the original cardinal series whenever either $k$ or else $t$ tends to infinity.

**4.3. A family of smoothing interpolation formulae depending on a smoothing parameter $\epsilon$.** In section 4.1 we have derived the smoothing interpolation formula (8) in the derivation of which no attempt was made to compromise between smoothness of results and goodness of fit. Such a compromise is afforded by the following basic function

$$L_k(x, t, \epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\epsilon + \phi_k(u, t)}{\epsilon + \phi_k(u, t)^2} \psi_k(u, t) e^{iux} du \qquad (0 \leqq \epsilon \leqq \infty) \tag{13}$$

which depends also on the smoothing parameter $\epsilon$. The corresponding interpolation formula

$$F(x) = \sum_{n=-\infty}^{\infty} y_n L_k(x - n, t, \epsilon) \tag{14}$$

includes our previous formula (8) and (11) as special cases. Indeed by (13) ,(4) and (9) we find

$$L_k(x, t, 0) = L_k(x, t), \tag{15}$$

$$L_k(x, t, \infty) = M_k(x, t). \tag{16}$$

Let us now investigate the characteristic function of the smoothing formula (14) for integral $x$. By (13) and II(30) this characteristic function is ($\phi_k(u, t)$ is periodic!):

$$\phi_k(u, t, \epsilon) = \frac{\epsilon + \phi_k(u, t)}{\epsilon + \phi_k(u, t)^2} \sum_{\nu} \psi_k(u + 2/\pi\nu, t).$$

This and (6) give

$$\phi_k(u, t, \epsilon) = (\epsilon\phi_k(u, t) + \phi_k(u, t)^2)/(\epsilon + \phi_k(u, t)^2). \tag{17}$$

On the other hand we have by (7) the inequalities

$$0 < \phi_k(u, t) < 1, \qquad (0 < u < 2\pi, t > 0). \tag{18}$$

Now (17), (18) imply

$$0 < \phi_k(u, t, \epsilon) < 1, \qquad (0 < u < 2\pi, t > 0, \epsilon > 0), \tag{19}$$

and therefore (14) is a smoothing interpolation formula in the sense of section 2.21 b. Moreover, we see from (17) that for fixed $t$ and $u$ $(0<u<2\pi)$ $\phi_k(u, t, \epsilon)$ decreases monotonically from

$$\phi_k(u, t, 0) = 1 \quad \text{to} \quad \phi_k(u, t, \infty) = \phi_k(u, t)$$

as $\epsilon$ varies from $\epsilon=0$ to $\epsilon=\infty$. Finally (14) is exact for the degree 1 and preserves the degree $k-1$ for the same reason as mentioned in the case of (8).

We summarize these properties in the following theorem.[13]

THEOREM 9. *For* $t\geqq 0$, $k=1, 2, \cdots$,

$$F(x) = \sum_{n=-\infty}^{\infty} y_n L_k(x - n, t, \epsilon), \tag{20}$$

*of basic function* (13), *is a smoothing interpolation formula which is exact for the degree* 1 *and preserves the degree* $k-1$. *For* $\epsilon=0$ (20) *is identical with the ordinary interpolation formula* (11). *For increasing values of* $\epsilon$ *it increases in smoothing power until for* $\epsilon=\infty$ (20) *is identical with our smoothing formula* (8).

4.31. *A property of the derivatives of the approximation* $F(x)$. Let the given data $\{y_n\}$ satisfy the additional condition

$$\sum_{n=-\infty}^{\infty} | y_n | < \infty. \tag{21}$$

Assume also $t>0$, $\epsilon>0$. We know that the sequence $\{F(n)\}$ obtained by (20) is smoother than $\{y_n\}$ in the sense of our discussion in I Section 1.1. However, it appears

[13] The method used in deriving Theorems 8 and 9 obviously generalizes as follows. Let

$$L(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)e^{iux}du$$

be a basic function. Let the periodic function

$$\phi(u) = \sum_{\nu} g(u + 2\pi\nu)$$

satisfy the inequality

$$0 < \phi(u) \leqq 1.$$

Then

$$L_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g(u)}{\phi(u)} e^{iux}du$$

is the basic function of an *ordinary* interpolation formula. Moreover

$$L_1(x, \epsilon) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\epsilon + \phi(u)}{\epsilon + \phi(u)^2} g(u)e^{iux}du$$

gives rise to a family of smoothing formulae of increasing strength, as $\epsilon$ increases.

to be of some interest to discuss here the smoothness of the function $F(x)$, rather than that of the sequence $\{F(n)\}$, as a function of the smoothing parameter $\epsilon$. In this connection we prove the following

THEOREM 10. *Denote by* $F(x, \epsilon)$ *the approximation* (20) *so as to indicate its dependence on* $\epsilon$. *The condition* (21) *insures the convergence of the integrals of the squares of the derivatives*

$$\int_{-\infty}^{\infty} (F^{(m)}(x, \epsilon))^2 dx, \qquad (m = 0, 1, 2, \cdots). \tag{22}$$

*Also each of these integrals is a monotonically decreasing function of* $\epsilon$ *in the range* $0 < \epsilon < \infty$.

Indeed, let

$$T(u) = \sum_{n=-\infty}^{\infty} y_n e^{inu} \tag{23}$$

be the characteristic function of the sequence $\{y_n\}$. For convenience we define

$$\Omega_k(u, t, \epsilon) = \frac{\epsilon + \phi_k(u, t)}{\epsilon + \phi_k(u, t)^2}. \tag{24}$$

Then (13) becomes

$$L_k(x, t, \epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega_k(u, t, \epsilon) \psi_k(u, t) e^{-iux} du. \tag{25}$$

By substitution of (25) into (20) we obtain

$$F(x, \epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega_k(u, t, \epsilon) \psi_k(u, t) T(u) e^{-iux} du. \tag{26}$$

We may evidently differentiate under the integral sign obtaining

$$F^{(m)}(x, \epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Omega_k(u, t, \epsilon) \psi_k(u, t) T(u) (-iu)^m e^{-iux} du.$$

This formula exhibits the Fourier transform of $F^{(m)}(x, \epsilon)$. We now use the analogue of the Parseval relation for Fourier integrals finally obtaining

$$\int_{-\infty}^{\infty} (F^{(m)}(x, \epsilon)^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\Omega_k(u, t, \epsilon) \psi_k(u, t))^2 \mid T(u) \mid^2 u^{2m} du. \tag{27}$$

This relation establishes our theorem. Indeed, by (24) the behaviour of the function $\Omega_k(u, t, \epsilon)$ of period of $2\pi$, is as follows: $\Omega_k(0, t, \epsilon) = 1$, while for each fixed $u$, $0 < u < 2\pi$, it decreases from

$$\Omega_k(u, t, 0) = 1/\phi_k(u, t) \quad \text{to} \quad \Omega_k(u, t, \infty) = 1,$$

as $\epsilon$ increases from $\epsilon = 0$ to $\epsilon = \infty$.

The discussion of I section 1.2 concerning the smoothing of a finite table also has an analogue concerning the derivatives of the approximations $F(x, \epsilon)$. We state the result without further details. We assume the concrete situation of I section 1.2 where a finite table was extended to an infinite table by constant third differences at each end. To this extended table we apply our formula (20) with such a value of $k$ which will insure that the formula (20) preserves cubics, i.e., $k \geqq 4$. Then we can prove that the integrals (22) converge for $m=4, 5, 6, \cdots$ and represent decreasing functions of $\epsilon$.

4.32. *Formula* (20) *as applied to subtabulation.* Our formula (20) is excellently suited for the systematic interpolation, or subtabulation of given ordinates $y_n$. It is less suited for interpolation. The reason is obvious: For subtabulation to tenths we need only a table of the basic function $L_k(x, t, \epsilon)$ for the step $h=0.1$ only, while interpolation would call for a much more elaborate table of this function.

The following transformation of the formula (20), in terms of the function $M_k(x, t)$, is of importance for numerical applications. First of all we expand the even periodic function (24) in a Fourier (cosine series)

$$\Omega_k(u, t, \epsilon) = \frac{\epsilon + \phi_k(u, t)}{\epsilon + \phi_k(u, t)^2} = \sum_{\nu=-\infty}^{\infty} \omega_\nu^{(k)}(t, \epsilon)e^{i\nu u} = \sum_{\nu=-\infty}^{\infty} \omega_\nu^{(k)}(t, \epsilon) \cos \nu u. \tag{28}$$

Substituting this expansion into (25) we get

$$L_k(x, t, \epsilon) = \sum_{\nu=-\infty}^{\infty} \omega_\nu^{(k)}(t, \epsilon) \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_k(u, t)e^{i(\nu-x)u}du,$$

which in view of (4) becomes

$$L_k(x, t, \epsilon) = \sum_{\nu=-\infty}^{\infty} \omega_\nu^{(k)}(t, \epsilon)M_k(x - \nu, t). \tag{29}$$

This formula expresses the basic function $L_k(x, t, \epsilon)$ in terms of the $M_k(x)$. If we substitute this expansion into our formula (20) we obtain

$$F(x) = \sum_n y_n L_k(x - n, t, \epsilon) = \sum_{n,\nu} y_n \omega_\nu^{(k)}(t, \epsilon)M_k(x - n - \nu, t)$$

and replacing $\nu$ by $\nu - n$

$$F(x) = \sum_{n,\nu} y_n \omega_{\nu-n}^{(k)}(t, \epsilon)M_k(x - \nu, t).$$

A first summation by $n$ introduces the sums

$$f_\nu = \sum_{n=-\infty}^{\infty} y_n \omega_{\nu-n}^{(k)}(t, \epsilon) \tag{30}$$

in terms of which our last expression becomes

$$F(x) = \sum_{\nu=-\infty}^{\infty} f_\nu M_k(x - \nu, t). \tag{31}$$

The pair of relations (30) and (31) is equivalent to (20) and represents its practical form. The reason for this is that the basic function $M_k(x, t)$ dampens out like

$\exp(-x^2)$ (see III (39)) while $L_k(x, t, \epsilon)$, $\epsilon < \infty$, dampens out like $\exp(-x)$ only. We notice incidentally that (31) is identical with our formula (8), to be applied to the new computed ordinates $\{f_n\}$ given by (30).

Frequently we require also tables of the derivatives $F'(x)$ and $F''(x)$, of the approximation $F(x)$. These are then computed by the formulae

$$F'(x) = \sum_\nu f_\nu M_k'(x - \nu, t), \tag{31'}$$

$$F''(x) = \sum_\nu f_\nu M_k''(x - \nu, t), \tag{31''}$$

from corresponding tables of $M_k'$ and $M_k''$.

4.33. *The least squares origin of formula* (20). We want to sketch briefly the genesis of our formula (20). Let the sequence $\{y_n\}$ be given and consider the spline curve $F(x)$ given by (31), where the coefficients $\{f_\nu\}$ are as yet unknown. If we try to determine these unknowns by the requirement that $F(x)$ should interpolate strictly the given ordinates $y_n$, i.e.,

$$F(n) = y_n \qquad (n = 0, \pm 1, \pm 2, \cdots), \tag{32}$$

we obtain an infinite system of linear equations in the unknown $f_\nu$, the solution of which was found to be given by

$$f_\nu = \sum_{n=-\infty}^{\infty} y_\nu \omega_{\nu-n}^{(k)}(t, 0).$$

This leads to our ordinary interpolation formula (11).

In view of Whittaker's well known smoothing method it seemed natural to proceed now as follows: *Let $\epsilon$ be a given positive number. To determine the unknown coefficients $f_\nu$ of* (31) *as solutions of the following minimal problem, we set*

$$\sum_n (F(n) - y_n)^2 + \epsilon \cdot \sum_n (f_n - y_\nu)^2 = \text{minimum}. \tag{33}$$

For $\epsilon = 0$ the solution is of course identical with the solution of the ordinary interpolation problem (32). For $\epsilon = \infty$ the solution is obviously $f_n = y_n$ in which case (31) reduces to (8). For $0 \leqq \epsilon \leqq \infty$ a system of "normal equations" arises whose solution is found to be given by (30). The explicit solution of these normal equations (matrix inversion) is performed by the numerical determination of the cosine coefficients $\omega_\nu$ of the expansion (28) for each given set of values of $k$, $t$ and $\epsilon$.

## V. THE COMPUTATION OF THE TABLES I, II, III

In this last chapter we shall discuss the methods used in the computation of our tables which allow us to use our formulae (30), (31) or

$$f_n = \sum_{\nu=-\infty}^{\infty} y_\nu \omega_{n-\nu}^{(k)}(t, \epsilon), \tag{1}$$

$$F(x) = \sum_{n=-\infty}^{\infty} f_n M_k(x - n, t), \tag{2}$$

for subtabulation to tenths.

**5.1. The computation of the function $M_k(x, t)$ and its derivatives.** This function is defined (see III(37)) by the integral

$$M_k(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t(u/2)^2}\left(\frac{2\sin u/2}{u}\right)^k e^{iux}du, \qquad (t > 0). \tag{3}$$

For the special case of $t=0$ and $k=1, 2, 3, \cdots$ we found previously the explicit polynomial expressions III(3). It now so happens that for our present case of $t>0$ (3) allows us to define our function also for $k=0$ as

$$M_0(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t(u/2)^2}e^{iux}du.$$

This last integral is known to be identical to

$$M_0(x, t) = \frac{1}{\sqrt{\pi t}}e^{-x^2/t}. \tag{4}$$

The recurrence relation III(40) shows that (3) is obtained from (4) by repeating $k$ times the averaging operation

$$\int_{x-1/2}^{x+1/2} \quad \text{or} \quad \delta\int_{-\infty}^{x}.$$

The result, however, is not changed if we perform all $k$ integrations first to be followed by the operation $\delta^k$ of $k$th central differencing. This proves the following result:

*If we define a sequence of functions $g_k(x, t)$ by*

$$g_0(x, t) = \frac{1}{\sqrt{\pi t}}e^{-x^2/t} \tag{5}$$

*and the recurrence relation*

$$g_k(x, t) = \int_{-\infty}^{x} g_{k-1}(x, t)dx \qquad (k = 1, 2, 3, \cdots), \tag{6}$$

*then*

$$M_k(x, t) = \delta^k g_k(x, t). \tag{7}$$

This relation reduces the problem to the problem of computing the repeated integral $g_k(x, t)$ of the error function (5). This we do as follows. It is easy to prove by induction or otherwise that (5) and (6) imply

$$g_k(x, t) = \frac{1}{(k-1)!}\frac{1}{\sqrt{\pi t}}\int_{-\infty}^{x}(x-u)^{k-1}e^{-u^2/t}du. \tag{8}$$

With $x-u=v$ this becomes

$$g_k(x, t) = \frac{1}{(k-1)!}\frac{1}{\sqrt{\pi t}}\int_{0}^{\infty} e^{-(x-v)^2/t}v^{k-1}dv. \tag{9}$$

By differentiating this with respect to $x$ we get

$$g_k'(x,t) = \frac{1}{(k-1)!}\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-(x-v)^2/t}v^{k-1}(-2xt^{-1}+2vt^{-1})dv$$

$$= -\frac{2x}{t}g_k(x,t) + \frac{2k}{t}g_{k+1}(x,t)$$

and therefore the recurrence relation

$$g_{k+1}(x,t) = \frac{t}{2k}g_k'(x,t) + \frac{x}{k}g_k(x,t), \qquad (k = 1, 2, \cdots) \tag{10}$$

which allows us to compute the successive values $g_k(x, t)$ by the operation of differentiation rather than integration. Indeed from (5) and (6) for $k=1$ we get

$$g_1(x,t) = \frac{1}{\sqrt{\pi t}}\int_{-\infty}^x e^{-x^2/t}dx. \tag{11}$$

Now (11) and (10) for $k=1$ will give

$$g_2(x,t) = \frac{t}{2}\frac{1}{\sqrt{\pi t}}e^{-x^2/t} + x\frac{1}{\sqrt{\pi t}}\int_{-\infty}^x e^{-x^2/t}dx$$

from which $g_3(x, t)$ is readily determined.

This progressive computation is greatly simplified if we realize that $g_k(x, t)$ will appear as an expression of the form

$$g_k(x,t) = P_k(x,t)g_0(x,t) + Q_k(x,t)g_1(x,t), \tag{12}$$

where $P_k$, $Q_k$ are polynomials in $x$ and $t$, while $g_0(x, t)$ is the error function (5) and $g_1(x, t)$ is the error integral (11). Substituting (12) into (10) we find

$$g_{k+1}(x,t) = \frac{t}{2k}(P_k' + Q_k)g_0(x,t) + \left(\frac{t}{2k}Q_k' + \frac{x}{k}Q_k\right)g_1(x,t).$$

On comparing with (12) for $k+1$, rather than $k$, we obtain the recurrence relations

$$P_{k+1} = \frac{t}{2k}(P_k' + Q_k)$$

$$Q_{k+1} = \frac{t}{2k}Q_k' + \frac{x}{k}Q_k \qquad (k = 1, 2, \cdots). \tag{13}$$

Since $P_1=0$, $Q_1=1$ we readily obtain the following explicit expressions

$$\begin{aligned}
P_2 &= t/2, & Q_2 &= x,\\
P_3 &= tx/4, & Q_3 &= t/4 + x^2/2,\\
P_4 &= t(t + x^2)/12, & Q_4 &= x(3t + 2x^2)/12,\\
P_5 &= tx(5t + 2x^2)/96, & Q_5 &= (3t^2 + 12tx^2 + 4x^4)/96,\\
P_6 &= t(4t^2 + 9tx^2 + 2x^4)/480, & Q_6 &= x(15t^2 + 20tx^2 + 4x^4)/480.
\end{aligned} \tag{14}$$

Excellent tables of the probability function (5) and its integral (11) are now available.

By means of these tables the formulae (12) and (14) allow us to compute readily the function $g_k(x, t)$.[14] It seems worth while to point out that the relation (7) goes over into III(8) if $t \to +0$. Indeed by an obvious change of variable we see that (11) becomes

$$g_1(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{t}} e^{-u^2} du$$

and therefore

$$\lim_{t \to +0} g_1(x, t) = \overset{0}{x_+}.$$

Now by induction we prove by (6), on letting $t \to +0$, that

$$\lim_{t \to +0} g_k(x, t) = \frac{1}{(k-1)!} \overset{k-1}{x_+}$$

which proves our last statement by continuity.

The computation of the derivatives of $M_k(x, t)$ is immediately settled by the relation

$$M_k^{(\nu)}(x, t) = \delta^k g_{k-\nu}(x, t), \tag{15}$$

which is implied by (6).

**5.2. The computation of the cosine coefficients $\omega_n^{(k)}(t, \epsilon)$.** By IV(5) and IV(28) we can see that the problem consists in computing the values of the coefficients of the cosine expansion of the function

$$\Omega_k(u, t, \epsilon) = \frac{\epsilon + \phi_k(u, t)}{\epsilon + \phi_k(u, t)^2} = \sum_{n=-\infty}^{\infty} \omega_n^{(k)}(t, \epsilon) \cos nu, \tag{16}$$

where the even periodic function $\phi_k(u, t)$ is defined by its cosine expansion

$$\phi_k(u, t) = \sum_{n=-\infty}^{\infty} M_k(n, t) \cos nu. \tag{17}$$

---

[14] I owe to D. H. Lehmer the reference to the functions $Hh_n(x)$ defined by

$$Hh_0(x) = \int_x^\infty e^{-x^2/2} dx, \qquad Hh_n(x) = \int_x^\infty Hh_{n-1}(x) dx.$$

Tables of these functions were published by J. R. Airey as Tables XV, Group IV, of the Mathematical Tables of the British Association for the Advancement of Science. The relation between our $g_k(x, t)$ and these new functions is

$$g_k(x, t) = \frac{1}{\sqrt{2\pi}} (t/2)^{(k-1)/2} Hh_{k-1}(-x\sqrt{2/t}).$$

This relation, for $k=4$, $t=\frac{1}{2}$, would readily allow us to compute our Table I by means of Airey's tables of $Hh_3$, $Hh_2$ and $Hh_1$. However, for other sets of values of $k$ and $t$, such as $k=8, t=\frac{1}{4}$, which are needed for other purposes, the range of $x$ in Airey's table becomes insufficient. In this case tables of $M_k(x, t)$ and its derivatives are computed by our formula (12) and the excellent *Tables of Probability Functions*, vol. I (1941), vol. II (1942), prepared by the Mathematical Tables Project under the direction of A. N. Lowan.

The coefficients $M_k(n, t)$ of this extremely fast convergent series are readily computed to 10 decimal places. The obvious procedure would be to compute from (17) a table of $\phi_k(u, t)$, then compute a similar table of $\Omega_k(u, t, \epsilon)$ which is then to be used in computing $\omega_n$ by some method of numerical harmonic analysis. It would be hard to achieve accuracy by this method and for this reason we proceeded differently. It should be born in mind that the cosine expansion of the denominator of (16) is readily obtained by the simple operation of multiplication of Fourier series. The only troublesome part is the computation of the expansion

$$\frac{1}{\epsilon + \phi_k(u, t)^2} = \sum_{-\infty}^{\infty} c_n \cos nu, \tag{18}$$

i.e., the reciprocation of a given cosine series. This was done as follows. The above-mentioned method of a 24-ordinate harmonic analysis scheme was used for obtaining values of the $c_n$'s accurate to 4–5 decimal places. These values were then improved to values accurate to 8 or 9 places by an iteration method developed by H. A. Rademacher and the author. This method is closely related to the method recommended by H. Hotelling for the reciprocation of ordinary matrices and will be described elsewhere.

In concluding this paper we want to point out two special cases of our ordinary interpolation formula (11), or (20) for $\epsilon = 0$, which are of mathematical interest. We mention first the case of $k = 0$, $t > 0$. This corresponds to interpolating our ordinates $y_n$ by means of a function $F(x)$ as described by the formula (8) of the Introduction. Although, as remarked there, the resulting interpolation formula is useless for practical purposes, it has the remarkable feature that the expansion coefficients $\omega_n^{(0)}(t, 0)$ of (16) may be obtained explicitly. Indeed the function $\phi_0(u, t)$ reduces to a Theta function which is a regular and uniform function of

$$z = e^{iu}$$

with singularities only at $z = 0$ and $z = \infty$. The simple zeros of this function are real, negative and form a geometric progression. As a result we are able to find explicitly the decomposition in partial fractions of the reciprocal

$$1/\phi_0(u, t).$$

The expansion of these partial fractions into geometric power series furnishes explicitly the Laurent expansion in powers of $z$ and therefore also the cosine expansion (16).

The second special case of interest is $k > 0$, $t = 0$. In this case our formula (11) reduces to an *ordinary polynomial interpolation formula of degree* $k-1$ *and class* $k-2$. This does not contradict Mr. Greville's statement (loc. cit. page 212) to the effect that such formulae do not exist. Indeed Mr. Greville considers only basic polynomial functions $L(x)$ of finite span $s$ only, while our basic $L_k(x, 0)$ are of infinite span. This case, which is of considerable interest, requires a more detailed investigation of the cosine polynomials $\phi_k(u, 0)$. We postpone this discussion to the second part B of this paper.

## APPENDIX

### Description of the tables and their use for the analytic approximation of equidistant data.

**Tables I and II.** In Table I we find the 8-place values of the even function

$$M(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-u^2/8}\left(\frac{2 \sin u/2}{u}\right)^4 \cos uxdu \tag{1}$$

and its derivatives $M'(x)$, $M''(x)$ for the step of $\Delta x = 0.1$. The graph of $M(x)$ is a bell-shaped curve and $M(x)$ vanishes to 8 places for $x \leqq -4.3$ and $x \geqq 4.3$. We now define a function of period $2\pi$ by the cosine series

$$\phi(u) = M(0) + 2M(1) \cos u + 2M(2) \cos 2u + \cdots \tag{2}$$

and expand in cosine series the following functions

$$\frac{\epsilon + \phi(u)}{\epsilon + \phi(u)^2} = \omega_0(\epsilon) + 2\omega_1(\epsilon) \cos u + 2\omega_2(\epsilon) \cos 2u + \cdots , \tag{3}$$

where $\epsilon$ is a non-negative parameter. Our Table II gives the 8-plane values of these coefficients for $\epsilon = 0, 0.1, 0.2, \cdots, 1.0$.

These tables may be used as follows to obtain an analytic approximation $F(x)$ to our ordinates $y_n$. We discuss first the case when $F(x)$ is to interpolate the ordinates, in the usual sense, i.e.,

$$F(n) = y_n. \tag{4}$$

For this end we compute first from the sequence $\{y_n\}$ a new sequence of coefficients $\{f_n\}$ by means of the formula

$$f_r = \cdots + y_{n-2}\omega_2(0) + y_{n-1}\omega_1(0) + y_n\omega_0(0) + y_{n+1}\omega_1(0) + y_{n+2}\omega_2(0) + \cdots \tag{5}$$

or

$$f_n = \sum_\nu y_\nu \omega_{n-\nu}(0), \tag{5'}$$

where $\omega_m = \omega_{-m}$. The analytic approximation of the ordinates $y_n$ is then given by

$$F(x) = \sum_n f_n M(x - n). \tag{6}$$

The values of $F(x)$, $x$ to even tenths, are now readily computed. Thus

$$\begin{aligned} F(2.3) = {} & f_{-1}M(3.3) + f_0M(2.3) + f_1M(1.3) + f_2M(0.3) \\ & + f_3M(-1 + .3) + f_4M(-2 + .3) + f_5M(-3 + .3) + f_6M(-4 + .3). \end{aligned}$$

The tabular values of $M(x)$ are so arranged that all 8 values needed in this computation are found in the fourth column headed $x + .3$. Generally, if the values of $f_n$ are written in a vertical column, we compute the values of $F(m + \nu \cdot 10^{-1})$ $(\nu = 0, 1, \cdots, 9)$ by matching the column of values of $f_n$ with the $\nu$th column of the table of $M(x)$ in such a way that $f_m$ corresponds to the row for $x = 0$. The products

$$f_n M(m - n + \nu \cdot 10^{-1})$$

are then accumulated in the products counter of a desk computing machine. Also the values $f_n$ are best computed by (5′) in a similar way if the column of values of $\omega_n(0)$ is extended upwards by symmetry for negative values of $n$.

From the tables of $M'(x)$ and $M''(x)$ we may likewise compute tables of the derivatives of $F(x)$ by

$$F^{(\nu)}(x) = \sum_n f_n M^{(\nu)}(x - n). \tag{7}$$

A check of the computation of the coefficients $f_n$ is afforded by (4). Indeed the values $F(n)$ computed by (6) should agree with the $y_n$ to about eight significant figures.

The formula (6) is exact for cubics, i.e., if the $y_n$ are the ordinates of a polynomial of degree at most 3, then $F(x)$ is identical with that polynomial.

If the conditions (4) of strict interpolation are not required, then we have the possibility of obtaining an approximation $F(x)$ which is such that the sequence $\{F(n)\}$ is smoother than the given $\{y_n\}$. The approximation $F(x)$ is then given by the pair of formulae

$$f_n = \sum_\nu y_\nu \omega_{n-\nu}(\epsilon), \tag{8}$$

$$F(x) = \sum_n f_n M(x - n), \tag{9}$$

which are applied as above. The choice of the value of the smoothing parameter $\epsilon$ depends on the amount of smoothing desired. The strongest smoothing afforded by our table is obtained for $\epsilon = +\infty$. Then (3) shows that $\omega_0(\infty) = 1$, $\omega_1(\infty) = \omega_2(\infty) = \cdots = 0$. Thus (8) becomes $f_n = y_n$ and (9) reduces to

$$F(x) = \sum_n y_n M(x - n). \tag{10}$$

This formula is especially simple to apply. It should be remarked however that, if $\epsilon > 0$, our formula (9) is exact only for linear functions and the same is true of (10).

**Table** III. We may eliminate the coefficients $f_n$ between (8) and (9). In terms of the new even function

$$L(x, \epsilon) = \sum_{n=-\infty}^{\infty} \omega_n(\epsilon) M(x - n), \tag{11}$$

our formulae (8), (9), then reduce to

$$F(x) = \sum_n y_n L(x - n, \epsilon). \tag{12}$$

Table III gives the values of $L(x, \epsilon)$ and $L''(x, \epsilon)$ for $\epsilon = 0, 0.1, \cdots, 1.0$ for the step $\Delta x = 0.5$. These may be used for subtabulation to halves in preference to (5), (6) or (8), (9). For subtabulation to fifths or tens, the use of formulae (8), (9) is preferable because of the slower damping of the function $L(x, \epsilon)$. Even so, formula (12) and Table III allow us to estimate quickly how well $F(x)$ approximates the $y_n$. By (12) we have

$$F''(x) = \sum_n y_n L''(x - n, \epsilon). \tag{13}$$

The table of $L''(x, \epsilon)$ then allows us to compute quickly a table of $F''(x)$ for the step $\Delta x = 0.5$ or else only isolated values if such are needed.

**Example of subtabulation to tenths.** We consider the following fairly smooth sequence of 64 ordinates $y_n$:

| $n$ | $y_n$ | $n$ | $y_n$ | $n$ | $y_n$ | $n$ | $y_n$ | $n$ | $y_n$ | $n$ | $y_n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 24614 | 12 | 25370 | 23 | 29290 | 34 | 73820 | 45 | 82450 | 56 | 79962 |
| 2 | 24644 | 13 | 25504 | 24 | 30160 | 35 | 77830 | 46 | 82290 | 57 | 79698 |
| 3 | 24680 | 14 | 25660 | 25 | 31320 | 36 | 80240 | 47 | 82110 | 58 | 79431 |
| 4 | 24723 | 15 | 25850 | 26 | 32840 | 37 | 81660 | 48 | 81911 | 59 | 79161 |
| 5 | 24772 | 16 | 26080 | 27 | 34790 | 38 | 82330 | 49 | 81699 | 60 | 78889 |
| 6 | 24828 | 17 | 26350 | 28 | 37260 | 39 | 82680 | 50 | 81472 | 61 | 78614 |
| 7 | 24892 | 18 | 26660 | 29 | 40440 | 40 | 82840 | 51 | 81234 | 62 | 78338 |
| 8 | 24966 | 19 | 27040 | 30 | 44750 | 41 | 82830 | 52 | 80987 | 63 | 78060 |
| 9 | 25048 | 20 | 27490 | 31 | 51120 | 42 | 82780 | 53 | 80736 | 64 | 77780 |
| 10 | 25143 | 21 | 28010 | 32 | 59390 | 43 | 82700 | 54 | 80481 | | |
| 11 | 25250 | 22 | 28600 | 33 | 67550 | 44 | 82590 | 55 | 80223 | | |

The differences of the section of this table with which we will be concerned are as follows:

| $n$ | $y_n$ | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $\Delta^5$ |
|---|---|---|---|---|---|---|
| 27 | 34790 | | | | | |
| 28 | 37260 | 2470 | | | | |
| 29 | 40440 | 3180 | 710 | | | |
| 30 | 44750 | 4310 | 1130 | 420 | | |
| 31 | 51120 | 6370 | 2060 | 930 | 510 | |
| 32 | 59390 | 8270 | 1900 | −160 | −1090 | −1600 |
| 33 | 67550 | 8160 | −110 | −2010 | −1850 | −760 |
| 34 | 73820 | 6270 | −1890 | −1780 | 230 | 2080 |
| 35 | 77830 | 4010 | −2260 | −370 | 1410 | 1180 |
| 36 | 80240 | 2410 | −1600 | 660 | 1030 | −380 |
| 37 | 81660 | 1420 | −990 | 610 | −50 | −1080 |
| 38 | 82330 | 670 | −750 | 240 | −370 | −320 |

We illustrate the case of strict interpolation, i.e., we use our Tables II for $\epsilon=0$. From our formula (5) and the values of $\omega_n$ as given in the column of Table II, with the heading $\epsilon=0$, we obtain the following coefficients.

| $n$ | $f_n$ |
|---|---|
| 27 | 34662.222 |
| 28 | 37031.355 |
| 29 | 40215.195 |
| 30 | 44060.182 |
| 31 | 50349.304 |
| 32 | 59490.524 |
| 33 | 68212.510 |
| 34 | 74566.216 |
| 35 | 78283.074 |
| 36 | 80460.234 |
| 37 | 81953.811 |
| 38 | 82356.888 |

From these values and our Table I of $M(x)$ and $M''(x)$, we obtain by the formulae (6) and (7) the following tables of $F(x)$ and $F''(x)$ with their differences.

Table of the function $F(x)$ and of its second derivative $F''(x)$.

| $x$ | $F(x)$ | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $F''(x)$ | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 31.0 | 51120.00 | | | | | 2117.97 | | | | |
| 31.1 | 51884.17 | 76417 | | | | 1966.48 | −15149 | | | |
| 31.2 | 52667.97 | 78380 | 1963 | | | 1787.44 | −17904 | −2755 | | |
| 31.3 | 53469.63 | 80166 | 1786 | −177 | | 1583.71 | −20373 | −2469 | 286 | |
| 31.4 | 54287.11 | 81748 | 1582 | −204 | −27 | 1359.15 | −22456 | −2083 | 386 | 100 |
| 31.5 | 55118.17 | 83106 | 1358 | −224 | −20 | 1118.30 | −24085 | −1629 | 454 | 68 |
| 31.6 | 55960.40 | 84223 | 1117 | −241 | −17 | 866.08 | −25222 | −1137 | 492 | 38 |
| 31.7 | 56811.29 | 85089 | 866 | −251 | −10 | 607.04 | −25868 | − 646 | 491 | − 1 |
| 31.8 | 57668.25 | 85696 | 607 | −259 | − 8 | 346.89 | −26051 | − 183 | 463 | −28 |
| 31.9 | 58528.68 | 86043 | 347 | −260 | − 1 | 88.63 | −25826 | 225 | 408 | −55 |
| 32.0 | 59390.00 | 86132 | 89 | −258 | 2 | − 163.98 | −25261 | 565 | 340 | −68 |
| 32.1 | 60249.69 | 85969 | − 163 | −252 | 6 | − 408.22 | −24424 | 837 | 272 | −68 |
| 32.2 | 61105.30 | 85561 | − 408 | −245 | 7 | − 642.14 | −23392 | 1032 | 195 | −77 |
| 32.3 | 61954.51 | 84921 | − 640 | −232 | 13 | − 864.26 | −22212 | 1180 | 148 | −47 |
| 32.4 | 62795.08 | 84057 | − 864 | −224 | 8 | −1073.51 | −20925 | 1287 | 107 | −41 |
| 32.5 | 63624.93 | 82985 | −1072 | −208 | 16 | −1269.11 | −19560 | 1365 | 78 | −29 |
| 32.6 | 64442.10 | 81717 | −1268 | −196 | 12 | −1450.39 | −18128 | 1432 | 67 | −11 |
| 32.7 | 65244.77 | 80267 | −1450 | −182 | 14 | −1616.76 | −16637 | 1491 | 59 | − 8 |
| 32.8 | 66031.30 | 78653 | −1614 | −164 | 18 | −1767.70 | −15094 | 1543 | 52 | − 7 |
| 32.9 | 66800.16 | 76886 | −1767 | −153 | 11 | −1902.77 | −13507 | 1587 | 44 | − 8 |
| 33.0 | 67550.00 | 74984 | −1902 | −135 | 18 | −2021.68 | −11891 | 1616 | 29 | −15 |
| 33.1 | 68279.64 | 72964 | −2020 | −118 | 17 | −2124.30 | −10262 | 1629 | 13 | −16 |
| 33.2 | 68988.05 | 70841 | −2123 | −103 | 15 | −2210.71 | − 8641 | 1621 | − 8 | −21 |
| 33.3 | 69674.37 | 68632 | −2209 | − 86 | 17 | −2281.13 | − 7042 | 1599 | −22 | −14 |
| 33.4 | 70337.91 | 65354 | −2278 | − 69 | 17 | −2335.91 | − 5478 | 1564 | −35 | −13 |
| 33.5 | 70978.07 | 64016 | −2338 | − 60 | 9 | −2375.46 | − 3955 | 1523 | −41 | − 6 |
| 33.6 | 71594.50 | 61643 | −2373 | − 35 | 25 | −2400.17 | − 2471 | 1484 | −39 | 2 |
| 33.7 | 72186.94 | 59244 | −2399 | − 26 | 9 | −2410.41 | − 1024 | 1447 | −37 | 2 |
| 33.8 | 72755.29 | 56835 | −2409 | − 10 | 16 | −2406.55 | 386 | 1410 | −37 | 0 |
| 33.9 | 73299.58 | 54429 | −2406 | 3 | 13 | −2389.01 | 1754 | 1368 | −42 | − 5 |
| 34.0 | 73820.00 | 52042 | −2387 | 19 | 16 | −2358.32 | 3069 | 1315 | −53 | −11 |

An inspection of these tables shows that they are very smooth and that they define $F(x)$ and $F''(x)$ to 7 significant figures by 4-point central interpolation. We have chosen on purpose an example for which it would be hard to obtain similar results by standard methods, if we are to maintain the forced accuracy requirement, and the same high degree of consistency between the function $F(x)$ and its second derivative $F''(x)$. For purposes of comparison we show also the interpolated values $F_c(x)$ for the range $x = 31.6 - 32.5$ obtained by the 10-point central interpolation method.
On comparing with our table of $F(x)$ we notice that

$$F_c(x) < F(x)$$

throughout this range, with the exception of the point $x = 32.0$ where, of course, both values agree. The curve $F_c(x)$ has a corner at $x = 32$. This is the typical discontinuity in the first derivative due to central interpolation methods (see the first paragraph of our Introduction).

| $x$ | $F_c(x)$ | $\Delta$ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ |
|---|---|---|---|---|---|
| 31.6 | 55959.90 | | | | |
| 31.7 | 56810.60 | 85070 | | | |
| 31.8 | 57667.55 | 85695 | 625 | | |
| 31.9 | 58528.22 | 86067 | 372 | −253 | |
| 32.0 | 59390.00 | 86178 | 111 | −261 | − 8 |
| 32.1 | 60248.72 | 85872 | − 306 | −417 | −156 |
| 32.2 | 61103.61 | 85489 | − 383 | − 77 | 340 |
| 32.3 | 61952.37 | 84876 | − 613 | −230 | −153 |
| 32.4 | 62792.77 | 84040 | − 836 | −223 | 7 |
| 32.5 | 63622.68 | 82991 | −1049 | −213 | 10 |

Notice that we needed 12 coefficients $f_n$ for the subtabulation of three panels. Each additional coefficient $f_n$ ($n=39, 40, \cdots$) allows the subtabulation of an additional panel.

It should be remarked that 53 ordinates $y_n$ enter into the computation of each coefficient $f_n$. This is due to the slow rate of damping of the $\omega_n(\epsilon)$ for $\epsilon=0$. Thus for $\epsilon=.1$ (very moderate smoothing) only 35 ordinates $y_n$ are needed, for $\epsilon=1.0$ only 23, for $\epsilon=\infty$ only 1. Concerning the important matter of dealing with the ends of a table see section 1.2 and the last paragraph of section 4.31.

TABLE I: $M_k(x, t)$, $M_k'(x, t)$, $M_k''(x, t)$ for $k=4$, $t=0.5$, $\Delta x=0.1$.
$M_4(x, 1/2)$

| $x$ | $x+0$ | $x+.1$ | $x+.2$ | $x+.3$ | $x+.4$ |
|---|---|---|---|---|---|
| 4 | .00000004 | .00000002 | .00000001 | | |
| 3 | .00011325 | .00005910 | .00002991 | .00001467 | .00000697 |
| 2 | .01616917 | .01105340 | .00737858 | .00480621 | .00305258 |
| 1 | .22597004 | .18940616 | .15590118 | .12596479 | .09986387 |
| 0 | .51549499 | .51132566 | .49901141 | .47911917 | .45254731 |
| −1 | .22597004 | .26483185 | .30499058 | .34523755 | .38420963 |
| −2 | .01616917 | .02311310 | .03230776 | .04418973 | .05917998 |
| −3 | .00011325 | .00021062 | .00038032 | .00066726 | .00113822 |
| −4 | .00000004 | .00000010 | .00000026 | .00000062 | .00000143 |

| $x$ | $x+.5$ | $x+.6$ | $x+.7$ | $x+.8$ | $x+.9$ |
|---|---|---|---|---|---|
| 3 | .00000321 | .00000143 | .00000062 | .00000026 | .00000010 |
| 2 | .00188907 | .00113822 | .00066726 | .00038032 | .00021062 |
| 1 | .07764689 | .05917998 | .04418973 | .03230776 | .02311310 |
| 0 | .42046084 | .38420963 | .34523755 | .30499058 | .26483185 |
| −1 | .42046084 | .45254731 | .47911917 | .49901141 | .51132566 |
| −2 | .07764689 | .09986387 | .12596479 | .15590118 | .18940616 |
| −3 | .00188907 | .00305258 | .00480621 | .00737858 | .01105340 |
| −4 | .00000321 | .00000697 | .00001467 | .00002991 | .00005910 |
| −5 | | | | .00000001 | .00000002 |

$M_4'(x, 1/2)$

| $x$ | $x+0$ | $x+.1$ | $x+.2$ | $x+.3$ | $x+.4$ |
|---|---|---|---|---|---|
| 4 | −.00000039 | −.00000015 | −.00000006 | −.00000002 | −.00000001 |
| 3 | −.00071955 | −.00039340 | −.00020833 | −.00010680 | −.00005298 |
| 2 | −.05961795 | −.04334506 | −.03071644 | −.02120376 | −.01424920 |
| 1 | −.37860391 | −.35140346 | −.31784825 | −.28043885 | −.24150489 |
| 0 | .00000000 | −.08306134 | −.16227165 | −.23406492 | −.29541674 |
| −1 | .37860391 | .39695855 | .40419276 | .39846265 | .37855467 |
| −2 | .05961795 | .07996844 | .10465732 | .13368990 | .16673619 |
| −3 | .00071955 | .00127546 | .00219236 | .00365652 | .00592117 |
| −4 | .00000039 | .00000096 | .00000229 | .00000529 | .00001179 |

| $x$ | $x+.5$ | $x+.6$ | $x+.7$ | $x+.8$ | $x+.9$ |
|---|---|---|---|---|---|
| 3 | −.00002542 | −.00001179 | −.00000529 | −.00000229 | −.00000096 |
| 2 | −.00931577 | −.00592117 | −.00365652 | −.00219236 | −.00217546 |
| 1 | −.20306520 | −.16673619 | −.13368990 | −.10465732 | −.07996844 |
| 0 | −.34404758 | −.37855467 | −.39846265 | −.40419276 | −.39695855 |
| −1 | .34404758 | .29541674 | .23406492 | .16227165 | .08306134 |
| −2 | .20306520 | .24150489 | .28043885 | .31784825 | .35140346 |
| −3 | .00931577 | .01424920 | .02120376 | .03071644 | .04334506 |
| −4 | .00002542 | .00005298 | .00010680 | .00020833 | .00039340 |
| −5 | | .00000001 | .00000002 | .00000006 | .00000015 |

$M_4''(x, 1/2)$

| $x$ | $x+.0$ | $x+.1$ | $x+.2$ | $x+.3$ | $x+.4$ |
|---|---|---|---|---|---|
| 4 | .00000357 | .00000145 | .00000056 | .00000021 | .00000008 |
| 3 | .00423106 | .00243772 | .00135797 | .00073109 | .00038024 |
| 2 | .18251117 | .14368197 | .10978191 | .08140988 | .05858190 |
| 1 | .23181861 | .30800376 | .35890239 | .38537940 | .38991971 |
| 0 | −.83712882 | −.81763132 | −.76058819 | −.67020231 | −.55301267 |
| −1 | .23181861 | .13144694 | .01013023 | −.12678241 | −.27209706 |
| −2 | .18251117 | .22494722 | .26885281 | .31126005 | .34845442 |
| −3 | .00423106 | .00710375 | .01154277 | .01816077 | .02768079 |
| −4 | .00000357 | .00000851 | .00001955 | .00004332 | .00009259 |
| −5 | | | | | .00000001 |

| $x$ | $x+.5$ | $x+.6$ | $x+.7$ | $x+.8$ | $x+.9$ |
|---|---|---|---|---|---|
| 4 | .00000003 | .00000001 | | | |
| 3 | .00019097 | .00009259 | .00004332 | .00001955 | .00000851 |
| 2 | .04089359 | .02768079 | .01816077 | .01154277 | .00710375 |
| 1 | .37617315 | .34845442 | .31126005 | .26885281 | .22494722 |
| 0 | −.41725773 | −.27209706 | −.12678241 | .01013023 | .13144694 |
| −1 | −.41725773 | −.55301267 | −.67020231 | −.76058819 | −.81763132 |
| −2 | .37617315 | .38991971 | .38537940 | .35890239 | .30800376 |
| −3 | .04089359 | .05858190 | .08140988 | .10978191 | .14368197 |
| −4 | .00019097 | .00038024 | .00073109 | .00135797 | .00243772 |
| −5 | .00000003 | .00000008 | .00000021 | .00000056 | .00000145 |

TABLE II: $\omega_n^{(k)}(t, \epsilon)$ for $k=4$, $t=0.5$, $\epsilon=0$ (0.1) 1.0.

| $n$ | $\epsilon=.0$ | $\epsilon=.1$ | $\epsilon=.2$ | $\epsilon=.3$ | $\epsilon=.4$ |
|---|---|---|---|---|---|
| 0 | 3.50637741 | 1.61378653 | 1.39953009 | 1.30308904 | 1.24631521 |
| 1 | −1.84900618 | −.26890929 | −.14132793 | −.09293505 | −.06784110 |
| 2 | .87238793 | −.08981772 | −.09332063 | −.08242675 | −.07223261 |
| 3 | −.40443570 | .07027891 | .04023694 | .02480160 | .01624479 |
| 4 | .18693997 | −.02078617 | −.00397484 | .00050538 | .00184671 |
| 5 | −.08636451 | .00133949 | −.00223908 | −.00188468 | −.00132999 |
| 6 | .03989615 | .00169234 | .00099749 | .00037463 | .00010684 |
| 7 | −.01842978 | −.00088114 | −.00010447 | .00005298 | .00006541 |
| 8 | .00851350 | .00019734 | −.00005372 | −.00003828 | −.00001761 |
| 9 | −.00393275 | .00001073 | .00002469 | .00000455 | −.00000116 |
| 10 | .00181670 | −.00002625 | −.00000273 | .00000174 | .00000130 |
| 11 | −.00083921 | .00001022 | −.00000129 | −.00000070 | −.00000014 |
| 12 | .00038767 | −.00000156 | .00000061 | .00000003 | −.00000006 |
| 13 | −.00017908 | −.00000044 | −.00000007 | .00000004 | .00000002 |
| 14 | .00008272 | .00000036 | −.00000003 | −.00000001 | |
| 15 | −.00003821 | −.00000011 | .00000002 | | |
| 16 | .00001765 | .00000001 | | | |
| 17 | −.00000815 | .00000001 | | | |
| 18 | .00000377 | | | | |
| 19 | −.00000174 | | | | |
| 20 | .00000080 | | | | |
| 21 | −.00000037 | | | | |
| 22 | .00000017 | | | | |
| 23 | −.00000008 | | | | |
| 24 | .00000004 | | | | |
| 25 | −.00000002 | | | | |
| 26 | .00000001 | | | | |

| $n$ | $\epsilon=.5$ | $\epsilon=.6$ | $\epsilon=.7$ | $\epsilon=.8$ | $\epsilon=.9$ | $\epsilon=1.0$ |
|---|---|---|---|---|---|---|
| 0 | 1.20834767 | 1.18095463 | 1.16016154 | 1.14379093 | 1.13054096 | 1.11958158 |
| 1 | −.05268720 | −.04264921 | −.03556878 | −.03034057 | −.02634263 | −.02319971 |
| 2 | −.06387537 | −.05710538 | −.05156939 | −.04698035 | −.04312407 | −.03984269 |
| 3 | .01107108 | .00773445 | .00547641 | .00389078 | .00274451 | .00189634 |
| 4 | .00219079 | .00218246 | .00204868 | .00187682 | .00170183 | .00153738 |
| 5 | −.00091344 | −.00062674 | −.00043120 | −.00029659 | −.00020265 | −.00013620 |
| 6 | −.00000351 | −.00004704 | −.00006163 | −.00006358 | −.00006018 | −.00005476 |
| 7 | .00005180 | .00003700 | .00002545 | .00001716 | .00001138 | .00000739 |
| 8 | −.00000656 | −.00000140 | .00000085 | .00000171 | .00000193 | .00000187 |
| 9 | −.00000206 | −.00000175 | −.00000127 | −.00000087 | −.00000057 | −.00000036 |
| 10 | .00000064 | .00000025 | .00000006 | −.00000002 | −.00000005 | −.00000006 |
| 11 | .00000003 | .00000006 | .00000006 | .00000004 | .00000002 | .00000001 |
| 12 | −.00000004 | −.00000002 | −.00000001 | | | |

TABLE III: $L_k(x, t, \epsilon)$, $L_k''(x, t, \epsilon)$ for $k=4$, $t=0.5$, $\epsilon=0$ (0.1) 1.0, $\Delta x=0.5$.
$L_4(x, 1/2, \epsilon)$

| $x$ | $\epsilon=.0$ | $\epsilon=.1$ | $\epsilon=.2$ | $\epsilon=.3$ | $\epsilon=.4$ |
|---|---|---|---|---|---|
| 0.0 | 1.00000000 | .70747935 | .65457028 | .62707488 | .60947694 |
| 0.5 | .62191163 | .53757743 | .51070485 | .49509729 | .48452022 |
| 1.0 | .00000000 | .20252568 | .22066478 | .22681461 | .22949350 |
| 1.5 | −.17291085 | −.02061576 | .01285810 | .02919893 | .03901344 |
| 2.0 | .00000000 | −.06545791 | −.04840114 | −.03681939 | −.02872086 |
| 2.5 | .07415615 | −.02765903 | −.03096280 | −.02894823 | −.02631330 |
| 3.0 | .00000000 | .00709183 | −.00340667 | −.00711221 | −.00850828 |
| 3.5 | −.03382251 | .01344008 | .00756627 | .00392340 | .00177090 |
| 4.0 | .00000000 | .00401304 | .00502862 | .00410188 | .00314843 |
| 4.5 | .01558996 | −.00276042 | .00041208 | .00121901 | .00135036 |
| 5.0 | .00000000 | −.00251219 | −.00118870 | −.00038015 | .00001138 |
| 5.5 | −.00719897 | −.00027479 | −.00076318 | −.00054505 | −.00033539 |
| 6.0 | .00000000 | .00065102 | −.00007596 | −.00021043 | −.00019927 |
| 6.5 | .00332530 | .00042139 | .00019012 | .00003152 | −.00002867 |
| 7.0 | .00000000 | −.00000773 | .00012316 | .00007297 | .00003257 |
| 7.5 | −.00153608 | −.00015286 | .00000861 | .00003207 | .00002580 |
| 8.0 | .00000000 | −.00006787 | −.00002989 | −.00000086 | .00000704 |
| 8.5 | .00070958 | .00002025 | −.00001865 | −.00000923 | −.00000251 |
| 9.0 | .00000000 | .00003031 | −.00000162 | −.00000502 | −.00000321 |
| 9.5 | −.00032779 | .00000793 | .00000477 | −.00000028 | −.00000120 |
| 10.0 | .00000000 | −.00000573 | .00000301 | .00000115 | .00000010 |
| 10.5 | .00015142 | −.00000566 | .00000017 | .00000072 | .00000036 |
| 11.0 | .00000000 | −.00000083 | −.00000075 | .00000011 | .00000018 |
| 11.5 | −.00006995 | .00000159 | −.00000045 | −.00000014 | .00000001 |
| 12.0 | .00000000 | .00000099 | −.00000003 | −.00000011 | −.00000004 |
| 12.5 | .00003231 | −.00000007 | .00000012 | −.00000002 | −.00000002 |
| 13.0 | .00000000 | −.00000034 | .00000007 | .00000001 | −.00000001 |
| 13.5 | −.00001492 | −.00000014 | .00000000 | .00000001 | |
| 14.0 | .00000000 | .00000004 | −.00000002 | | |
| 14.5 | .00000690 | .00000007 | −.00000001 | | |
| 15.0 | .00000000 | .00000002 | | | |
| 15.5 | −.00000318 | −.00000001 | | | |
| 16.0 | .00000000 | −.00000001 | | | |
| 16.5 | .00000147 | | | | |
| 17.0 | .00000000 | | | | |
| 17.5 | −.00000068 | | | | |
| 18.0 | .00000000 | | | | |
| 18.5 | .00000031 | | | | |
| 19.0 | .00000000 | | | | |
| 19.5 | −.00000014 | | | | |
| 20.0 | .00000000 | | | | |
| 20.5 | .00000007 | | | | |
| 21.0 | .00000000 | | | | |
| 21.5 | −.00000003 | | | | |
| 22.0 | .00000000 | | | | |
| 22.5 | .00000001 | | | | |
| 23.0 | .00000000 | | | | |
| 23.5 | −.00000001 | | | | |

$L_4(x, 1/2, \epsilon)$

| $x$ | $\epsilon=.5$ | $\epsilon=.6$ | $\epsilon=.7$ | $\epsilon=.8$ | $\epsilon=.9$ | $\epsilon=1.0$ |
|---|---|---|---|---|---|---|
| 0.0 | .59702260 | .58765637 | .58031608 | .57438799 | .56948898 | .56536580 |
| 0.5 | .47675954 | .47077398 | .46599416 | .46207717 | .45880202 | .45601892 |
| 1.0 | .23077657 | .23140004 | .23168090 | .23177314 | .23175789 | .23168050 |
| 1.5 | .04557847 | .05027848 | .05380648 | .05654957 | .05874137 | .06053136 |
| 2.0 | −.02276409 | −.01820178 | −.01459584 | −.01167397 | −.00925829 | −.00722771 |
| 2.5 | −.02384227 | −.02167108 | −.01979233 | −.01816756 | −.01675611 | −.01552233 |
| 3.0 | −.00896161 | −.00898986 | −.00881784 | −.00855221 | −.00824659 | −.00792883 |
| 3.5 | .00044979 | −.00038991 | −.00093726 | −.00129962 | −.00154092 | −.00170083 |
| 4.0 | .00238588 | .00180229 | .00135734 | .00101562 | .00075048 | .00054256 |
| 4.5 | .00127570 | .00114308 | .00100305 | .00087278 | .00075725 | .00065681 |
| 5.0 | .00019600 | .00027865 | .00030985 | .00031471 | .00030617 | .00029109 |
| 5.5 | −.00019071 | −.00009654 | −.00003598 | .00000283 | .00002756 | .00004311 |
| 6.0 | −.00015995 | −.00012137 | −.00008970 | −.00006510 | −.00004639 | −.00003223 |
| 6.5 | −.00004696 | −.00004885 | −.00004474 | −.00003886 | −.00003288 | −.00002744 |
| 7.0 | .00000987 | −.00000179 | −.00000738 | −.00000973 | −.00001039 | −.00001018 |
| 7.5 | .00001687 | .00001000 | .00000536 | .00000238 | .00000050 | −.00000064 |
| 8.0 | .00000771 | .00000641 | .00000486 | .00000350 | .00000244 | .00000165 |
| 8.5 | .00000044 | .00000148 | .00000168 | .00000156 | .00000134 | .00000110 |
| 9.0 | −.00000156 | −.00000057 | −.00000004 | .00000021 | .00000031 | .00000034 |
| 9.5 | −.00000101 | −.00000067 | −.00000039 | −.00000020 | −.00000009 | −.00000002 |
| 10.0 | −.00000023 | −.00000027 | −.00000022 | −.00000017 | −.00000011 | −.00000008 |
| 10.5 | .00000011 | −.00000001 | −.00000005 | −.00000006 | −.00000005 | −.00000004 |
| 11.0 | .00000012 | .00000006 | .00000002 | .00000000 | −.00000001 | −.00000001 |
| 11.5 | .00000004 | .00000004 | .00000002 | .00000001 | .00000001 | |
| 12.0 | .00000000 | .00000001 | .00000001 | .00000001 | | |
| 12.5 | −.00000001 | | | | | |
| 13.0 | −.00000001 | | | | | |

$L_4''(x, 1/2, \epsilon)$

| $x$ | $\epsilon=.0$ | $\epsilon=.1$ | $\epsilon=.2$ | $\epsilon=.3$ | $\epsilon=.4$ |
|---|---|---|---|---|---|
| 0.0 | −3.47753764 | −1.50781449 | −1.27083552 | −1.16381926 | −1.10100908 |
| 0.5 | −1.03983382 | −.69689346 | −.61542693 | −.57326418 | −.54670549 |
| 1.0 | 2.15613767 | .54167607 | .40225158 | .34798919 | .31925128 |
| 1.5 | 1.50655095 | .77131824 | .63355043 | .56889199 | .53067521 |
| 2.0 | −.58680458 | .31875073 | .30878327 | .29072626 | .27601758 |
| 2.5 | −.68097024 | −.03482567 | .02460402 | .04246822 | .04943501 |
| 3.0 | .24590776 | −.12647279 | −.07651577 | −.05154399 | −.03726679 |
| 3.5 | .31311773 | −.06455381 | −.05654901 | −.04581150 | −.03775297 |
| 4.0 | −.11165611 | .01678358 | −.00530735 | −.01047426 | −.01153447 |
| 4.5 | −.14452585 | .03142765 | .01425664 | .00665986 | .00297134 |
| 5.0 | .05142591 | .00673809 | .00811324 | .00596871 | .00423710 |
| 5.5 | .06675341 | −.00655085 | .00060896 | .00183704 | .00187944 |
| 6.0 | −.02374375 | −.00477107 | −.00194355 | −.00054832 | .00001970 |
| 6.5 | −.03083551 | −.00059653 | −.00138896 | −.00087620 | −.00050023 |
| 7.0 | .01096729 | .00133431 | −.00011460 | −.00030729 | −.00026914 |
| 7.5 | .01424417 | .00097571 | .00035809 | .00005960 | −.00003417 |
| 8.0 | −.00506618 | −.00005726 | .00019854 | .00010609 | .00004375 |
| 8.5 | −.00657998 | −.00035828 | .00001209 | .00004959 | .00003703 |
| 9.0 | .00234028 | −.00012112 | −.00004884 | −.00000116 | .00000954 |
| 9.5 | .00303957 | .00004878 | −.00003391 | −.00001505 | −.00000404 |
| 10.0 | −.00108107 | .00005884 | −.00000242 | −.00000732 | −.00000433 |
| 10.5 | −.00140411 | .00001802 | .00000898 | −.00000026 | −.00000164 |
| 11.0 | .00049939 | −.00001229 | .00000485 | .00000168 | .00000013 |
| 11.5 | .00064862 | −.00001316 | .00000023 | .00000113 | .00000053 |
| 12.0 | −.00023069 | −.00000110 | −.00000123 | .00000016 | .00000025 |
| 12.5 | −.00029962 | .00000374 | −.00000083 | −.00000023 | .00000001 |
| 13.0 | .00010657 | .00000182 | −.00000005 | −.00000015 | −.00000005 |
| 13.5 | .00013841 | −.00000018 | .00000022 | −.00000003 | −.00000004 |
| 14.0 | −.00004923 | −.00000067 | .00000012 | .00000002 | −.00000001 |
| 14.5 | −.00006394 | −.00000033 | .00000000 | .00000002 | .00000001 |
| 15.0 | .00002274 | .00000009 | −.00000003 | .00000001 | |
| 15.5 | .00002954 | .00000016 | −.00000002 | | |
| 16.0 | −.00001050 | .00000003 | | | |
| 16.5 | −.00001364 | −.00000003 | | | |
| 17.0 | .00000485 | −.00000002 | | | |
| 17.5 | .00000630 | .00000000 | | | |
| 18.0 | −.00000224 | .00000001 | | | |
| 18.5 | −.00000291 | | | | |
| 19.0 | .00000104 | | | | |
| 19.5 | .00000134 | | | | |
| 20.0 | −.00000048 | | | | |
| 20.5 | −.00000062 | | | | |
| 21.0 | .00000022 | | | | |
| 21.5 | .00000029 | | | | |
| 22.0 | −.00000010 | | | | |
| 22.5 | −.00000013 | | | | |
| 23.0 | .00000005 | | | | |
| 23.5 | .00000006 | | | | |
| 24.0 | −.00000002 | | | | |
| 24.5 | −.00000003 | | | | |
| 25.0 | .00000001 | | | | |
| 25.5 | .00000001 | | | | |
| 26.0 | −.00000001 | | | | |

$L_4'(x, 1/2, \epsilon)$

| $x$ | $\epsilon=.5$ | $\epsilon=.6$ | $\epsilon=.7$ | $\epsilon=.8$ | $\epsilon=.9$ | $\epsilon=1.0$ |
|---|---|---|---|---|---|---|
| 0.0 | −1.05919265 | −1.02916420 | −1.00647330 | −.98868331 | −.97433986 | −.96251767 |
| 0.5 | −.52821280 | −.51450880 | −.50390754 | −.49544281 | −.48851722 | −.48273978 |
| 1.0 | .30155959 | .28962736 | .28106624 | .27464186 | .26965348 | .26567454 |
| 1.5 | .50527188 | .48711044 | .47346009 | .46281678 | .45428116 | .44728133 |
| 2.0 | .26453429 | .25546310 | .24815810 | .24216442 | .23716440 | .23293281 |
| 2.5 | .05240385 | .05363758 | .05404202 | .05402794 | .05379802 | .05345832 |
| 3.0 | −.02823788 | −.02210889 | −.01772635 | −.01446547 | −.01196172 | −.00998973 |
| 3.5 | −.03182937 | −.02737565 | −.02393662 | −.02121579 | −.01901742 | −.01720888 |
| 4.0 | −.01135624 | −.01078633 | −.01011492 | −.00944820 | −.00882438 | −.00825481 |
| 4.5 | .00101034 | −.00009918 | −.00075259 | −.00114593 | −.00138383 | −.00152531 |
| 5.0 | .00303129 | .00219629 | .00160744 | .00118299 | .00087064 | .00063651 |
| 5.5 | .00166647 | .00142136 | .00120000 | .00101295 | .00085821 | .00073079 |
| 6.0 | .00024844 | .00033324 | .00035457 | .00034759 | .00032845 | .00030479 |
| 6.5 | −.00027461 | −.00014145 | −.00006202 | −.00001410 | .00001495 | .00003245 |
| 7.0 | −.00020347 | −.00014764 | −.00010563 | −.00007505 | −.00005291 | −.00003686 |
| 7.5 | −.00005788 | −.00005841 | −.00005187 | −.00004389 | −.00003637 | −.00002987 |
| 8.0 | .00001265 | −.00000184 | −.00000816 | −.00001051 | −.00001095 | −.00001051 |
| 8.5 | .00002301 | .00001329 | .00000717 | .00000344 | .00000121 | −.00000014 |
| 9.0 | .00000979 | .00000777 | .00000567 | .00000399 | .00000274 | .00000184 |
| 9.5 | .00000025 | .00000163 | .00000187 | .00000172 | .00000145 | .00000117 |
| 10.0 | −.00000199 | −.00000071 | −.00000008 | .00000021 | .00000032 | .00000034 |
| 10.5 | −.00000133 | −.00000085 | −.00000049 | −.00000026 | −.00000012 | −.00000004 |
| 11.0 | −.00000029 | −.00000033 | −.00000026 | −.00000019 | −.00000013 | −.00000008 |
| 11.5 | .00000016 | .00000000 | −.00000005 | −.00000006 | −.00000005 | −.00000004 |
| 12.0 | .00000015 | .00000007 | .00000002 | .00000000 | −.00000001 | −.00000001 |
| 12.5 | .00000005 | .00000004 | .00000003 | .00000001 | .00000001 | |
| 13.0 | .00000000 | .00000001 | .00000001 | .00000001 | | |
| 13.5 | −.00000002 | | | | | |
| 14.0 | −.00000001 | | | | | |