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# A NOTE ON THE PRICING OF AMERICAN OPTIONS

В данной статье мы возвращаемся к задаче оптимальной остановки в применении к определению цены американских опционов с бесконечным временны́м горизонтом в биномиальной модели с дискретным временем. Вычислена функция цены для случая непрерывного пространства состояний и произвольного начального состояния $x > 0$. Полученная функция цены сравнивается с недавно опубликованным решением для некоторого дискретного пространства состояний.

*Ключевые слова и фразы*: американские опционы, оптимальная остановка.

**1. Introduction.** In this note we revisit the pricing of perpetual American options in discrete time. The market model is a special case of the classical binomial model, in which the return process takes only two values $\lambda$ and $\lambda^{-1}$. According to the no-arbitrage approach to option pricing, the fair or rational price of an American option should be given by the value of a certain optimal stopping problem. We calculate the value function on the continuous state space $E = (0, \infty)$ by deriving a series of conditions the value function must satisfy and which, taken together, fix it uniquely. It turns out that the value function is piecewise linear. Its restriction to the discrete state space $E_1 = \{\lambda^k \colon k \in \mathbf{Z}\}$ is compared with the solution obtained in [4] and [5].

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**2. Problem formulation.** We consider the market model laid down in [5], i.e., a *bank account* $B = (B_n)_{n\geqslant 0}$ and a *stock* $S = (S_n)_{n\geqslant 0}$ evolving according to the dynamic equations

$$B_n = (1+r)\,B_{n-1}, \quad B_0 > 0, \qquad S_n = (1+\rho_n)\,S_{n-1}, \quad S_0 > 0.$$

$r > 0$ is the nonrandom interest rate. The shocks $\rho_n$ are independent and identically distributed (i.i.d.) random variables, defined on some basic probability space $(\Omega, \mathscr{F}, \mathbf{P})$, taking the two values $\lambda - 1$ and $\lambda^{-1} - 1$ with probabilities $p$ and $q = 1 - p$, respectively, with $0 < p < 1$ and $\lambda > 1$. Note that the latter condition ensures that $S_n$ is positive for all $n$. Consider an American call or put option

$$f_n = (\alpha\beta)^n g(S_n), \qquad n \geqslant 0,$$

where either $g(x) = (x-1)^+$ (call) or $g(x) = (1-x)^+$ (put). The exercise price has been normalized to 1 without restriction of generality. Here $\alpha = (1+r)^{-1}$ is the discount factor, and $0 < \beta \leqslant 1$. Let $\overline{\mathfrak{M}}$ denote the class of all $(\mathscr{F}_n^S)$-stopping times $\tau$, $\mathfrak{M}$ the subclass of all a.s. finite $\tau$ and $\mathfrak{M}_N$ the further subclass of all $\tau \leqslant N$. The no-arbitrage approach to option pricing then leads to the result that the *fair* or *rational* price of $f = (f_n)$ should be

$$V_N = \sup_{\tau \in \mathfrak{M}_N} \mathbf{E}^*(\alpha\beta)^\tau g(S_\tau) \tag{2.1}$$

for finite expiration time and

$$V = \sup_\tau \mathbf{E}^*(\alpha\beta)^\tau g(S_\tau) \tag{2.2}$$

for the perpetual options, where in the latter problem $\tau$ ranges over $\mathfrak{M}$ or $\overline{\mathfrak{M}}$. In (2.1) and (2.2), $\mathbf{E}^*$ denotes expectation with respect to an *equivalent martingale measure* $\mathbf{P}^*$. That is, $\mathbf{P}^*$ is any probability measure on $(\Omega, \mathscr{F})$ such that the $\rho_n$ are i.i.d. and $S^* = (S_n^*)$ with $S_n^* = \alpha^n S_n$ is a martingale under $\mathbf{P}^*$. In other words, $p^* = \mathbf{P}^*\{\rho_n = \lambda - 1\}$ and $q^* = 1 - p^* = \mathbf{P}^*\{\rho_n = \lambda^{-1} - 1\}$ should satisfy $0 < p^* < 1$ and $q^*(\lambda^{-1} - 1) + p^*(\lambda - 1) = r$, i.e.,

$$p^* = \frac{\lambda - \alpha}{\alpha(\lambda^2 - 1)}, \qquad q^* = \frac{\lambda(\alpha\lambda - 1)}{\alpha(\lambda^2 - 1)}. \tag{2.3}$$

Note that the condition $\alpha\lambda > 1$ must be satisfied in order to ensure that $q^* > 0$ ($p^* > 0$ is automatically satisfied since $\lambda > 1 \geqslant \alpha$). A nice exposition of the arguments supporting (2.1) and (2.2) can be found in [4].

It turns out convenient to parametrize the dynamic equation (2.1) by the starting value $S_0 = x$ and thereby embed the problem in the setting of homogeneous Markov chains. We then arrive at a family $\{\mathbf{P}_x^*\colon x > 0\}$ of probability measures such that, for all $x > 0$,

$$\mathbf{P}_x^*\{\rho_n = \lambda - 1\} = p^*, \quad \mathbf{P}_x^*\{\rho_n = \lambda^{-1} - 1\} = q^* \quad \text{and} \quad \mathbf{P}_x^*\{S_0 = x\} = 1.$$

The corresponding *value functions* for (2.1) and (2.2) (with $\mathbf{E}^*$ replaced by $\mathbf{E}_x^*$) will be denoted by $V_N(x)$ and $V(x)$, respectively.

As a consequence, $S_n$ allows the convenient representation

$$S_n = x\lambda^{\varepsilon_1 + \cdots + \varepsilon_N} \tag{2.4}$$

(under $\mathbf{P}_x^*$), where the $\varepsilon_n$ are i.i.d. random $\pm 1$-variables defined by $\varepsilon_n = 1_{\{\rho_n = \lambda - 1\}} - 1_{\{\rho_n = \lambda^{-1} - 1\}}$. In particular, for starting value $x$, the «active» state space for the Markov chain $S$ is $E_x = \{x\lambda^k\colon k \in \mathbf{Z}\}$.

Since, from now on, we shall only work with the equivalent martingale measures $\mathbf{P}_x^*$, we shall drop the index $^*$. In the rest of this section, we shall collect some basic results from the theory of optimal stopping of homogeneous Markov chains. They will not be presented in full generality, but rather in a form adapted to our setting. In particular, nonnegativity of the process to be stopped obtains, so that we need not worry about lower bounds. References for the results listed below are [2] and [3]. Introduce the *transition operator*

$$Tf(x) = \mathbf{E}_x f(S_1) = pf(\lambda x) + qf(\lambda^{-1}x).$$

Then the following is true.

(i) $V_N \nearrow V$ as $N \nearrow \infty$.

(ii) $V$ is the minimal solution of the *stationary Bellman equation* $V(x) = \max\{g(x), \alpha\beta TV(x)\}$.

(iii) The $V_N$ and hence $V$ are convex functions (if $g$ is convex).

The sets $C = \{x\colon V(x) > g(x)\}$ and $D = \{x\colon V(x) = g(x)\}$ are called the *continuation region* and the *stopping region*, respectively (for the infinite horizon problem).

(iv) Assume that

$$\mathbf{E}_x \sup_n (\alpha\beta)^n g(S_n) < \infty \tag{2.5}$$

holds. Then the stopping time $\tau^* = \inf\{n \geqslant 0\colon V(S_n) = g(S_n)\}$ (with $\inf\varnothing = \infty$) is optimal for (2.2) (with $\mathbf{E}_x$ instead of $\mathbf{E}^*$) in the class $\overline{\mathfrak{M}}$: $\mathbf{E}_x(\alpha\beta)^{\tau^*} g(S_{\tau^*}) = V(x)$, where, on $\{\tau^* = \infty\}$, we define $(\alpha\beta)^{\tau^*} g(S_{\tau^*}) = \overline{\lim}_{n\to\infty}(\alpha\beta)^n g(S_n)$ (actually, in our setting, the $\overline{\lim}$ will always be a proper lim). If $\tau^*$ is a.s. finite, then it is optimal in class $\mathfrak{M}$.

**3. American calls.** In the following we shall derive a list of properties the value function must possess and which will finally determine it uniquely.

(i) $V(x) \leqslant x$ for all $x > 0$.

This is an immediate consequence of (i) in Section 2 and the fact that $V_N(x) \leqslant x$. The latter property follows from the estimate

$$\mathbf{E}_x(\alpha\beta)^\tau (S_\tau - 1)^+ \leqslant \mathbf{E}_x\, \alpha^\tau (S_\tau - 1)^+ \leqslant \mathbf{E}_x\, \alpha^\tau S_\tau = \mathbf{E}_x S_\tau^* = x,$$

which is valid for all bounded stopping times $\tau$.

(ii) $V(x) > 0$ for all $x > 0$.

P r o o f. Choose $N$ in such a way that $\lambda^N x > 1$. Then, making use of (2.4),

$$\begin{aligned}\mathbf{E}_x(\alpha\beta)^N(S_N - 1)^+ &= (\alpha\beta)^N \mathbf{E}(x\lambda^{\varepsilon_1+\cdots+\varepsilon_N} - 1)^+ \\ &> (\alpha\beta)^N(\lambda^N x - 1)\,\mathbf{P}\{\varepsilon_1 + \cdots + \varepsilon_N = N\} = (\alpha\beta)^N(\lambda^N x - 1)\,p^N > 0.\end{aligned}$$

*Consequence*: $V(x) > g(x) = (x-1)^+$ for all $x < x^*$, for some $x^* > 1$.

From now on, we work on the state space $E_x = \{\lambda^k x\colon k \in \mathbf{Z}\}$.

(iii) There exists an $y^* \in E_x \cup \{\infty\}, y^* > 1$, such that $V(y) > g(y) = (y-1)^+$ for all $y \in E_x$ such that $y < y^*$.

Consider now the stationary Bellman equation (cf. (ii) in Section 2):

$$V(y) = \max\{g(y),\ \alpha\beta TV(y)\}, \qquad y \in E_x.$$

In our scenario, $TV(y)$ is given by

$$TV(y) = \frac{1}{\alpha(\lambda^2 - 1)}\left[(\lambda - \alpha)\, V(\lambda y) + \lambda(\alpha\lambda - 1)\, V\Big(\frac{y}{\lambda}\Big)\right].$$

If, for some $y = \lambda^n x \in E_x$, we have $V(y) > g(y)$, i.e., the maximum is adopted at $\alpha\beta TV(y)$, then

$$\begin{aligned} V(\lambda^n x) &= \alpha\beta TV(\lambda^n x) \\ &= \frac{\beta}{\lambda^2 - 1}\,[(\lambda - \alpha)\, V(\lambda^{n+1}x) + \lambda(\alpha\lambda - 1)\, V(\lambda^{n-1}x)] \\ &\iff v_n = \frac{\beta}{\lambda^2 - 1}\,[(\lambda - \alpha)\, v_{n+1} + \lambda(\alpha\lambda - 1)\, v_{n-1}] \\ &\iff v_{n+1} = a_1 v_n + a_2 v_{n-1},\end{aligned}$$

where we have put

$$v_n = v_n(x) = V(\lambda^n x), \qquad a_1 = \frac{\lambda^2 - 1}{\beta(\lambda - \alpha)}, \qquad a_2 = -\frac{\lambda(\alpha\lambda - 1)}{\lambda - \alpha}.$$

The general solution of this difference equation is given by

$$v_n = C_+\mu_+^n + C_-\mu_-^n, \tag{3.1}$$

where $\mu_\pm = a_1/2 \pm \sqrt{a_1^2/4 + a_2} = \frac{1}{2}\,[a_1 \pm \sqrt{a_1^2 + 4a_2}\,]$ are the roots of the characteristic polynomial $a(z) = z^2 - a_1 z - a_2$. By elementary calculations (cf. [1] for details) it can be checked that the discriminant $D = a_1^2 + 4a_2 = \beta^{-2}(\lambda-\alpha)^{-2}[(\lambda^2-1)^2 - 4\beta^2\lambda(\lambda-\alpha)(\alpha\lambda-1)]$ is positive, $0 < \mu_- < \mu_+$, $\mu_- = (\alpha\lambda - 1)/(\lambda - \alpha)$ and $\mu_+ = \lambda$ for $\beta = 1$, as well as, for $\beta < 1$, $\mu_+ > \lambda$ and $\mu_- < 1$.

Therefore the following is true.

(iv) On any interval in $E_x$ on which $V(y) > g(y)$, $V(y)$ is of the form (3.1) for $y = \lambda^n x$. On the initial interval $(0, y^*]$ (cf. (iii)), we have the boundary condition

(B1) $\lim_{n\to-\infty} v_n = 0$,

as well as a matching condition (B2) at the right end-point which will emerge below (cf. (v)). (B1) is a direct consequence of (i).

Boundary condition (B1) implies that $C_- = 0$. Hence $V$ is of the form

$$V(y) = C\mu^n \qquad \text{for} \quad y = \lambda^n x < y^* \tag{3.2}$$

for some $y^* \in E_x \cup \{\infty\}$ (with $C = C_+, \mu = \mu_+$).

(v) Assume $\beta < 1$. Then there is an $y^* \in E_x$ (finite), $y^* > 1$, such that

$$V(y) = \begin{cases} C\mu^n & \text{for} \quad y = \lambda^n x < y^*, \\ y - 1 & \text{for} \quad y \geqslant y^*. \end{cases}$$

P r o o f. Suppose that $V(y) > (y-1)^+$ for all $y \in E_x$. Then $V$ is of the form (3.2) for all $y$, with $C > 0$. But then

$$\lim_{y\to\infty} \frac{V(y)}{y} = \frac{C}{x} \lim_{n\to\infty} \left(\frac{\mu}{\lambda}\right)^n = \infty,$$

contradicting (i) which implies that $\lim_{y\to\infty}[V(y)/y] \leqslant 1$. Let now $n^*$ be any index such that $y^* = \lambda^{n^*} x > 1$ and $V(y^*) = g(y^*) = y^* - 1$. Suppose that

$$V(\lambda y^*) > g(\lambda y^*).$$

Since $V$ must be convex (since the $V_N$ are), all points $(y, V(y))$, $y > y^*$, must lie above the line through $(y^*,\, g(y^*))$ and $(\lambda y^*,\, V(\lambda y^*))$, whose slope is greater than 1. Hence

$$V(y) > g(y) = y - 1 \qquad \text{for all} \quad y > y^*. \tag{3.3}$$

As a consequence, $V(y) = \alpha\beta T V(y)$, $y > y^*$, so that $V$ is of the form

$$V(\lambda^n y^*) = v_n = C_+\mu_+^n + C_-\mu_-^n$$

on $(y^*, \infty)$ (cf. (iv)). Since (3.3) holds for arbitrarily large $y$, $C_+ > 0$. Since $\mu_+ > \lambda$, $\mu_- < 1$,

$$\lim_{n\to\infty} \frac{V(\lambda^n y^*)}{\lambda^n y^*} = \frac{C_+}{y^*} \lim_{n\to\infty} \left(\frac{\mu_+}{\lambda}\right)^n = \infty,$$

contradicting (i): $\lim_{n\to\infty} V(x)/x \leqslant 1$. Hence, once $V(y^*) = y^* - 1$ for some $y^* > 1$, then $V(y) = y - 1$ for *all* $y > y^*$, which completes the proof.

For $\beta = 1$, the argument used in the first part of the proof of (v) does not work (since $\mu = \lambda$). In fact, $V(y) > g(y)$ holds for *all* $y \in E_x$. To see this, suppose there is an $y^*$ such that $V(y^*) = g(y^*) = y^* - 1$ and $V(y^*/\lambda) > g(y^*/\lambda)$. Then, since $g(y^*/\lambda) \geqslant y^*/\lambda - 1$, a simple calculation shows that $\alpha T V(y^*) > y^* - \alpha$ (cf. [1] for details). On the other hand, $V(y^*) = g(y^*) = y^* - 1$ implies

$$\alpha T V(y^*) \leqslant y^* - 1,$$

contradicting the strict inequality above. Hence, since $\mu = \lambda$ and $\mu_- \leqslant 1$, the value function must be of the form $V(y) = C\lambda^n$ for all $y = \lambda^n x$, and

$$1 \leqslant \lim_{y\to\infty} \frac{V(y)}{g(y)} = \lim_{n\to\infty} \frac{C\lambda^n}{\lambda^n x - 1} = \frac{C}{x}.$$

On the other hand, by virtue of (i), the limit must be $\leqslant 1$, so that $C = x$, $V(y) = y$.

The property (v) describes the general form of the value function. It reflects the intuitive notion that one should exercise the call immediately once the stock price is high enough while for low prices it is advantageous to wait. We are left with the task to determine $C$ and $y^*$. As above, let $y^* = \lambda^{n^*} x$ denote the *smallest* $y \in E_x$ for which $V(y) = g(y) = y - 1$. Around $y^*$, two *matching conditions* must be satisfied.

1) At $y = y^*/\lambda$:

$$\begin{aligned} \alpha\beta T V\left(\frac{y^*}{\lambda}\right) &= \frac{\beta}{\lambda^2 - 1}\left[(\lambda - \alpha)(\lambda^{n^*} x - 1) + \lambda(\alpha\lambda - 1)\, C\mu^{n^*-2}\right] \\ &= V\left(\frac{y^*}{\lambda}\right) = C\mu^{n^*-1} > \lambda^{n^*-1} x - 1. \end{aligned} \tag{3.4}$$

Equality of the 2-nd and the 4-th terms implies

$$\beta\left[(\lambda-\alpha)(\lambda^{n^*}x-1)+\lambda(\alpha\lambda-1)\,C\mu^{n^*-2}\right]=(\lambda^2-1)\,C\mu^{n^*-1}$$

or, equivalently, $C\mu^{n^*-1}[\mu(\lambda^2-1)-\beta\lambda(\alpha\lambda-1)]=\mu\beta(\lambda-\alpha)(\lambda^{n^*}x-1)$. But $\mu(\lambda^2-1)-\beta\lambda(\alpha\lambda-1)=\beta\mu^2(\lambda-\alpha)$ (since $\mu$ is a solution of the equation $a(z)=0$). Hence

$$C\mu^{n^*-1}=\frac{1}{\mu}(\lambda^{n^*}x-1) \tag{3.5}$$

or $C\mu^{n^*}=\lambda^{n^*}x-1$. This is the announced boundary condition (B2). Once $n^*$ is given, this determines $C$:

$$C=\frac{\lambda^{n^*}x-1}{\mu^{n^*}}. \tag{3.6}$$

Note that $C$ depends on $x$. The inequality in (3.4) may then be written in the form

$$\lambda^{n^*}x-1>\mu(\lambda^{n^*-1}x-1)\qquad\text{or, equivalently,}\quad(\mu-\lambda)\,\lambda^{n^*-1}x<\mu-1.$$

Hence $n^*$ must satisfy the condition

$$\lambda^{n^*}x<\lambda\frac{\mu-1}{\mu-\lambda}. \tag{3.7}$$

2) At $y=y^*$:

$$\alpha\beta TV(y^*)=\frac{\beta}{\lambda^2-1}\left[(\lambda-\alpha)(\lambda^{n^*+1}x-1)+\lambda(\alpha\lambda-1)\,C\mu^{n^*-1}\right]\leqslant V(y^*)=\lambda^{n^*}x-1.$$

Inserting the right-hand side of (3.5) for $C\mu^{n^*-1}$, we obtain the following chain of equivalences for the inequality:

$$\begin{aligned}&\beta\left[(\lambda-\alpha)(\lambda^{n^*+1}x-1)+\lambda(\alpha\lambda-1)\frac{1}{\mu}(\lambda^{n^*}x-1)\right]\leqslant(\lambda^2-1)(\lambda^{n^*}x-1)\\&\iff\lambda^{n^*}x\left[\mu(\lambda^2-1)-\beta\lambda(\alpha\lambda-1)-\beta\lambda\mu(\lambda-\alpha)\right]\\&\qquad\geqslant\mu(\lambda^2-1)-\beta\lambda(\alpha\lambda-1)-\beta\mu(\lambda-\alpha)\\&\iff\lambda^{n^*}x\left[\beta\mu^2(\lambda-\alpha)-\beta\lambda\mu(\lambda-\alpha)\right]\geqslant\beta\mu^2(\lambda-\alpha)-\beta\mu(\lambda-\alpha)\\&\iff\lambda^{n^*}x(\mu-\lambda)\geqslant\mu-1.\end{aligned}$$

Hence $n^*$ must satisfy the condition

$$\lambda^{n^*}x\geqslant\frac{\mu-1}{\mu-\lambda}. \tag{3.8}$$

Since the interval $[(\mu-1)/(\mu-\lambda),\ \lambda(\mu-1)/(\mu-\lambda))$ contains *exactly one* $y=\lambda^n x$, $n^*$ is uniquely determined by (3.7) and (3.8). Putting $\gamma=(\mu-1)/(\mu-\lambda)$, this may be written in the form

$$\gamma\leqslant\lambda^{n^*}x<\lambda\gamma$$

or, with $A=\log_\lambda(\gamma/x)$, $A\leqslant n^*<A+1$, i.e.,

$$n^*=[A], \tag{3.9}$$

with $[A]$ denoting the smallest integer $\geqslant A$. Note that $n^*=n^*(x)$.

Let us go back to the continuous state space $E=(0,\infty)$ and derive a closed form expression for the value function. To this end, recall (v) in this section, which states that, for $y=\lambda^n x$,

$$V(y)=\begin{cases}C(x)\mu^n & \text{for}\ \ n<n^*(x),\\ y-1 & \text{for}\ \ n\geqslant n^*(x),\end{cases}$$

where we have used the trivial equivalence $y=\lambda^n x<y^*=\lambda^{n^*}x\iff n<n^*(x)$. In particular, for $n=0$,

$$V(y)=V(x)=\begin{cases}C(x) & \text{for}\ \ n^*(x)>0,\\ x-1 & \text{for}\ \ n^*(x)\leqslant 0.\end{cases}$$

Consider now any state $x\in[\lambda^k\gamma,\lambda^{k+1}\gamma)$. Then

1) $n^*(x) = -k$ (by (3.9)),
2) $C(x) = \mu^k(\lambda^{-k}x - 1)$ (by (3.6)),
3) $n^*(x) > 0 \Longleftrightarrow k < 0$.
Hence
$$V(x) = \begin{cases} \mu^k(\lambda^{-k}x - 1) & \text{for } \; k < 0, \\ x - 1 & \text{for } \; k \geqslant 0. \end{cases}$$
Thus we have proved the first part of the following theorem.

**Theorem 1.** (i) *For American calls with $\beta < 1$, the value function is of the form*
$$V(x) = \begin{cases} \mu^k(\lambda^{-k}x - 1) & \textit{for } \; \lambda^k\gamma \leqslant x < \lambda^{k+1}\gamma \textit{ with } k < 0, \\ x - 1 & \textit{for } \; x \geqslant \gamma. \end{cases}$$
*In particular, the stopping region is given by $[\gamma, \infty)$. If $\beta = 1$, $V(x) = x$.*

(ii) *The stopping time $\tau^* = \inf\{n \geqslant 0\colon S_n \geqslant \gamma\}$ satisfies*
$$\mathbf{P}_x\{\tau^* < \infty\} = \begin{cases} 1 & \textit{if } \; x \geqslant \gamma \textit{ or } \lambda \leqslant \lambda_\alpha, \\ \left(\dfrac{\lambda - \alpha}{\lambda(\alpha\lambda - 1)}\right)^{[A]} & \textit{if } \; x < \gamma \textit{ and } \lambda > \lambda_\alpha, \end{cases} \tag{3.10}$$
*where $\lambda_\alpha = \alpha^{-1}(1 + \sqrt{1 - \alpha^2})$. In the first case, for $\beta < 1$, $\tau^*$ is an optimal stopping time in $\mathfrak{M}$ (and hence also in $\overline{\mathfrak{M}}$), in the second case it is optimal in $\overline{\mathfrak{M}}$. For $\beta = 1$, no optimal stopping time exists.*

Proof. It remains to prove (ii). For $x \geqslant \gamma$, $\tau^*$ is trivially finite ($\equiv 0$). Consider therefore the case $x < \gamma$. The condition $\lambda \leqslant \lambda_\alpha$ is equivalent to $p \geqslant q$. Put $Z_n = \varepsilon_1 + \cdots + \varepsilon_n$. Then $\overline{\lim}_{n\to\infty} Z_n = \infty$ $\mathbf{P}_x$-a.s. if $p \geqslant q$. Therefore, using the representation (2.4), $\overline{\lim}_{n\to\infty} S_n = \infty$ $\mathbf{P}_x$-a.s., implying that $\tau^* < \infty$ $\mathbf{P}_x$-a.s. If $p < q$, note that, under $\mathbf{P}_x$,
$$\tau^* = \inf\{n \geqslant 0\colon x\lambda^{Z_n} \geqslant \gamma\} = \inf\{n \geqslant 0\colon Z_n \geqslant A\} = \inf\{n \geqslant 0\colon Z_n = [A]\}. \tag{3.11}$$
As is well known, $\mathbf{P}_x\{\tau^* < \infty\} = (p/q)^{[A]}$. Inserting $p$ and $q$ from (2.3) completes the proof of (3.10). As to the optimality of $\tau^*$, we would like to apply property (iv) of Section 2. To be able to do so, we have to check condition (2.5). To this end, note first that
$$\mathbf{E}_x \sup_n (\alpha\beta)^n g(S_n) \leqslant \mathbf{E}_x \sup_n (\alpha\beta)^n S_n.$$
Next, $(M_n^{1+\delta})_{n\geqslant 0}$ with $M_n = (\alpha\beta)^n S_n$ is a positive supermartingale for $\delta > 0$ small enough. This can be seen as follows. Since $p\lambda + q\lambda^{-1} = \alpha^{-1}$, the function $f(\delta) = (\alpha\beta)^{1+\delta}(p\lambda^{1+\delta} + q\lambda^{-(1+\delta)})$ satisfies $f(\delta) = \beta + O(\delta)$ as $\delta \searrow 0$. In particular, for $\delta > 0$ small enough, $f(\delta) < 1$ (since $\beta < 1$). But then
$$\mathbf{E}_x[M_n^{1+\delta} \mid \mathscr{F}_{n-1}^S] = M_{n-1}^{1+\delta}(\alpha\beta)^{1+\delta}\mathbf{E}_x[\lambda^{(1+\delta)\varepsilon_n}] = M_{n-1}^{1+\delta} f(\delta) < M_{n-1}^{1+\delta}.$$
By the maximal inequality for positive supermartingales (cf. [2]),
$$\mathbf{P}_x\Big\{\sup_n M_n \geqslant m\Big\} = \mathbf{P}_x\Big\{\sup_n M_n^{1+\delta} \geqslant m^{1+\delta}\Big\} \leqslant \min\left(\frac{x^{1+\delta}}{m^{1+\delta}}, 1\right)$$
for all $m \in \mathbf{N}$. Hence
$$\sum_{m=1}^{\infty} \mathbf{P}_x\Big\{\sup_n M_n \geqslant m\Big\} < \infty,$$
which means that $\sup_n M_n$ and, *a fortiori*, $\sup_n (\alpha\beta)^n g(S_n)$ is integrable. In case $\beta = 1$, note that for every $\tau \in \overline{\mathfrak{M}}$, $\alpha^\tau g(S_\tau) < \alpha^\tau S_\tau$ on $\{\tau < \infty\}$. Moreover, it follows from the law of large numbers that
$$\lim_{n\to\infty} \alpha^n g(S_n) = \lim_{n\to\infty} \alpha^n S_n = 0 \qquad \mathbf{P}_x\text{-a.s.}$$
(note that $p < q$ for $\alpha = 1$). Since $S_n^* = \alpha^n S_n$ is a positive martingale, $\mathbf{E}_x S_\tau^* \leqslant x$. Hence
$$\begin{aligned} \mathbf{E}_x\, \alpha^\tau g(S_\tau) &< \mathbf{E}_x S_\tau^* \leqslant x = V(x) && \text{if } \; \mathbf{P}_x\{\tau < \infty\} > 0, \\ \mathbf{E}_x\, \alpha^\tau g(S_\tau) &= 0 < V(x) = x && \text{if } \; \mathbf{P}_x\{\tau = \infty\} = 1. \end{aligned}$$
Hence there exists no optimal stopping time. Theorem 1 is proved.

Let us compare the value function in Theorem 1 with the value function $V^*$ obtained in [5] for state space $E_1 = \{\lambda^k \colon k \in \mathbf{Z}\}$. For $x = \lambda^k$, Theorem 1 yields

$$V(x) = \begin{cases} \mu^{-n^*}(\lambda^{n^*} - 1)\mu^k & \text{for } k < n^*, \\ \lambda^k - 1 & \text{for } k \geqslant n^*, \end{cases}$$

with $n^* = n^*(1) = \lceil \log_\lambda \gamma \rceil$ (= smallest integer $\geqslant \log_\lambda \gamma$). Note that $n^* \geqslant 1$ (since $\gamma > 1$). For the Shiryaev et al. solution, one has to calculate

$$\gamma_1 = \log_\lambda \mu, \qquad \widetilde{x} = \frac{\gamma_1}{\gamma_1 - 1} = \frac{\ln \mu}{\ln \mu - \ln \lambda},$$

$$\widetilde{k} = \text{integer part of } \log_\lambda \widetilde{x},$$

$$B_1 = \mu^{-\widetilde{k}}(\lambda^{\widetilde{k}} - 1), \quad B_2 = \mu^{-(\widetilde{k}+1)}(\lambda^{\widetilde{k}+1} - 1), \qquad C^* = \min(B_1, B_2).$$

Then

$$k^* = \begin{cases} \widetilde{k} & \text{if } B_1 < B_2, \\ \widetilde{k} + 1 & \text{if } B_1 > B_2, \end{cases}$$

and, for $x = \lambda^k$,

$$V^*(x) = \begin{cases} C^* \mu^k & \text{for } k < k^*, \\ \lambda^k - 1 & \text{for } k \geqslant k^*. \end{cases}$$

In order to compare $V$ and $V^*$, the following observations are useful.

a) $\gamma < \widetilde{x} < \lambda\gamma$ (and hence $\widetilde{k} \geqslant 0$).

This follows easily from the strict monotonicity of the functions $(\ln \mu)/(\mu - 1)$ (decreasing) and $(\mu \ln \mu)/(\mu - 1)$ (increasing) for $\mu > 1$.

b) $B_1 <$ (=, >, respectively) $B_2 \iff \widetilde{k} <$ (=, >, respectively) $\log_\lambda \gamma$.

In particular, $B_1 = B_2$ holds if and only if $\log_\lambda \gamma$ is an integer. From a), it follows that $\log_\lambda \gamma < \log_\lambda \widetilde{x} < \log_\lambda \gamma + 1$ and hence

$$\widetilde{k} = n^* - 1 \quad \text{or} \quad \widetilde{k} = n^*. \tag{3.12}$$

From b), if $\log_\lambda \gamma$ is not an integer,

$$B_1 < B_2 \iff \widetilde{k} \leqslant n^* - 1, \quad B_1 > B_2 \iff \widetilde{k} \geqslant n^*. \tag{3.13}$$

Combining (3.12), (3.13) and the definition of $k^*$, we find that

$$k^* = \begin{cases} n^* - 1 & \text{if } B_1 < B_2, \\ n^* + 1 & \text{if } B_1 > B_2, \end{cases}$$

and therefore, for $x = \lambda^k$,

$$V^*(x) = \begin{cases} \mu^{-(n^*-1)}(\lambda^{n^*-1} - 1)\,\mu^k & \text{for } k < n^* - 1, \\ \lambda^k - 1 & \text{for } k \geqslant n^* - 1, \end{cases}$$

in case $B_1 < B_2$ and

$$V^*(x) = \begin{cases} \mu^{-(n^*+1)}(\lambda^{n^*+1} - 1)\,\mu^k & \text{for } k < n^* + 1, \\ \lambda^k - 1 & \text{for } k \geqslant n^* + 1, \end{cases}$$

in case $B_1 > B_2$. As a consequence, unless $\log_\lambda \gamma$ is an integer, $V$ and $V^*$ will not coincide and therefore, if one believes in the correctness of Theorem 1, $V^*$ cannot be the true value function.

In order to see what may go wrong with $V^*$, consider parameter constellations, where $n^* = 1$. Then, if $B_1 < B_2$, it turns out that $k^* = 0$ and hence

$$V^*(x) = 0 \qquad \text{for } x = \lambda^k, \quad k \leqslant 0,$$

whereas the true value function must be positive. If $B_1 > B_2$, we have $k^* = 2$ and

$$V^*(\lambda) = \frac{\lambda^2 - 1}{\mu^2}\,\mu = \frac{\lambda^2 - 1}{\mu} < \lambda - 1 = g(\lambda)$$

if $\lambda + 1 < \mu$, so that $V^*(\lambda)$ does not majorize $g(\lambda)$ and therefore does not qualify for the true value function. As an example of a parameter constellation of the above type, consider the case $\alpha = 1$, $\beta = \frac{4}{5}$ (there is nothing peculiar about this choice, many others will also do). Then, for $\lambda = 2$ and $\lambda = 15$, we find that $n^* = 1$ (actually, $n^* = 1$ for all $\lambda$). For $\lambda = 2$, calculations yields $\mu = 3.10610$, $B_1 = 0.321946 > B_2 = 0.310948$ and hence $k^* = 2$ and $V^*(2) = 0.96584 < 1 = g(2)$. For $\lambda = 15$, the corresponding values are $B_1 = 0$, $B_2 = 0.72842$, $k^* = 0$ and hence $V^*(x) = 0$ for $x = \lambda^k$, $k \leqslant 0$.

**4. American puts.** The approach is basically the same as for American calls. Therefore we shall only indicate the modifications to be made in the analysis performed in Section 3.

(i) $V(x) \leqslant 1$.

(ii) Remains the same (choose $N$ such that $\lambda^N x < 1$).

(iii) There exists an $y^* \in E_x \cup \{0\}$, $y^* < 1$, such that $V(y) > g(y) = (1-y)^+$ for all $y \in E_x$ such that $y > y^*$.

(iv) Remains valid without a change.

The «good» root now is $\mu = \mu_-$ (since $\mu_+ \geqslant \lambda$ is not compatible with (i) for large $y$).

(v) Assume $\alpha < 1$ or $\beta < 1$. Then there is an $y^* \in E_x$, $y^* < 1$, such that

$$V(y) = \begin{cases} 1-y & \text{for } \ y \leqslant y^*, \\ C\mu^n & \text{for } \ y = \lambda^n x > y^*. \end{cases}$$

The proof runs along the same lines as that of (v) in Section 3 (cf. [1] for details).

Let $y^* = \lambda^{n^*} x < 1$ denote the *largest* $y \in E_x$ such that $V(y) = g(y) = 1-y$. The two matching conditions to determine $n^*$ and $C$ are now evaluated at $y = \lambda y^*$ and $y = y^*$. Proceeding as in Section 3 (with some obvious modifications) we are lead to the following conditions fixing $C$ and $n^*$:

$$C = \frac{1-\lambda^{n^*}x}{\mu^{n^*}}, \qquad \gamma < \lambda^{n^*} x \leqslant \lambda\gamma,$$

where now $\gamma$ is defined as $\gamma = (1-\mu)/(\lambda-\mu)$. (Note that $\gamma\lambda < 1$.) This means that, with $A$ defined as in Section 3, $A < n^* \leqslant A+1$, i.e., $n^* =$ *largest* integer $\leqslant A+1$. In other words, $n^* = [A]$ (as defined above) unless $A$ is an integer, in which case $n^* = [A]+1$. In particular, for $\lambda^{k-1}\gamma < x \leqslant \lambda^k\gamma$, $n^* = -k+1$. Hence $n^* < 0 \iff k > 1$. Note that $\lambda\gamma < 1$.

We then have the following result.

**Theorem 2.** (i) *For American puts with $\alpha < 1$ or $\beta < 1$, the value function is of the form*

$$V(x) = \begin{cases} \mu^{k-1}(1-\lambda^{-k+1}x) & \textit{for } \ \lambda^{k-1}\gamma < x \leqslant \lambda^k\gamma \textit{ with } k > 1, \\ 1-x & \textit{for } \ x \leqslant \lambda\gamma. \end{cases} \tag{4.1}$$

*In particular, the stopping region is given by $(0, \lambda\gamma]$. If $\alpha = \beta = 1$, $V(x) \equiv 1$.*

(ii) *The stopping time $\tau^* = \inf\{n \geqslant 0\colon S_n \leqslant \lambda\gamma\}$ satisfies*

$$\mathbf{P}_x\{\tau^* < \infty\} = \begin{cases} 1 & \textit{if } \ x \leqslant \lambda\gamma \textit{ or } \lambda \geqslant \lambda_\alpha, \\ \left(\dfrac{\lambda-\alpha}{\lambda(\alpha\lambda-1)}\right)^{\{A\}+1} & \textit{if } \ x > \lambda\gamma \textit{ and } \lambda < \lambda_\alpha, \end{cases} \tag{4.2}$$

*where $\lambda_\alpha = \alpha^{-1}(1+\sqrt{1-\alpha^2})$ and $\{A\}$ is the largest integer smaller than $A$. In the first case, for $\alpha < 1$ or $\beta < 1$, $\tau^*$ is an optimal stopping time in $\mathfrak{M}$ (and hence also in $\overline{\mathfrak{M}}$), in the second case it is optimal in $\overline{\mathfrak{M}}$. For $\alpha = \beta = 1$, $\tau^* = \infty$ is optimal in $\overline{\mathfrak{M}}$ and no finite optimal stopping time exists.*

P r o o f. Of (i), only the last statement remains to be proved. If $\alpha = 1$, $p = 1/(\lambda+1)$, $q = \lambda/(\lambda+1)$. Hence $\mathbf{E}_x\varepsilon_i = p - q < 0$ and, by the law of large numbers, $Z_n \to -\infty$ a.s. as $n \to \infty$. Consequently, $S_n \to 0$ $\mathbf{P}_x$-a.s. for all $x > 0$, so that (for $\beta = 1$)

$$\lim_{n\to\infty} \mathbf{E}_x g(S_n) = \lim_{n\to\infty} \mathbf{E}_x(1-S_n)^+ = 1$$

by bounded convergence. This at the same time shows that $\tau^* = \infty$ is optimal. As to (ii), note that

$$\tau^* = \inf\{n \geqslant 0\colon x\lambda^{Z_n} \leqslant \lambda\gamma\} = \inf\{n \geqslant 0\colon Z_n \leqslant A+1\} = \inf\{n \geqslant 0\colon Z_n = \{A\}+1\}.$$

(4.2) then is a standard result for random walks. Optimality of $\tau^*$ is an immediate consequence of (iv) in Section 2, since condition (2.5) is trivially satisfied for American puts.

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 *KALLSEN J.**

# $\sigma$-LOCALIZATION AND $\sigma$-MARTINGALES

В статье вводится понятие $\sigma$-*локализации*, обобщающее понятие локализации в общей теории случайных процессов. $\sigma$-локализационный класс, связанный с множеством мартингалов, есть класс $\sigma$-мартингалов, который играет важную роль в финансовой математике. Подробно рассматриваются эти процессы и соответствующие $\sigma$-*мартингальные меры*. Обобщая понятие стохастического интеграла по компенсированным случайным мерам, мы выводим каноническое представление для $\sigma$-мартингалов.

*Ключевые слова и фразы*: $\sigma$-локализация, $\sigma$-мартингал, стохастический интеграл, каноническое представление, $\sigma$-мартингальная мера.

**1. Introduction.** $\sigma$-Martingales have been introduced by Chou [3] and were investigated further by Emery [6]. They play a key role in the general statement of the fundamental theorems of asset pricing in [5], [12], and [2]. $\sigma$-Martingales can be interpreted quite naturally as semimartingales with vanishing drift. Similar to local martingales, the set of $\sigma$-martingales may be obtained from the class of martingales by a localization procedure, but here localization has to be understood in a broader sense than usually (cf. [4, I.1d]). This concept of $\sigma$-*localization* is introduced in Section 2. The subsequent section treats the set of $\sigma$-martingales and their properties. By extending the stochastic integral relative to compensated random measures, the canonical local martingale representation $X = X_0 + X^c + x * (\mu - \nu)$ is generalized to $\sigma$-martingales in Section 4. Finally, $\sigma$-martingale measures are characterized in terms of semimartingale characteristics.

Throughout the paper, we use the notation of [4] and [9], [10]. In particular, we work with a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbf{R}_+}, P)$. The transposed of a vector $x$ or matrix is denoted by $x^\top$ and its components by superscripts. Increasing processes are identified with their corresponding Lebesgue–Stieltjes measure.

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