# Higher Dimensional Szekeres' Space-time in Brans-Dicke Scalar Tensor Theory 

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#### Abstract

The generalized Szekeres family of solution for quasi-spherical space-time of higher dimensions are obtained in the scalar tensor theory of gravitation. Brans-Dicke field equations expressed in Dicke's revised units are exhaustively solved for all the subfamilies of the said family. A particular group of solutions may also be interpreted as due to the presence of the so-called C-field of Hoyle and Narlikar and for a chosen sign of the coupling parameter. The models show either expansion from a big bang type of singularity or a collapse with the turning point at a lower bound. There is one particular case which starts from the big bang, reaches a maximum and collapses with the in course of time to a crunch.


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## I. INTRODUCTION

The first step to solve the Einstein equations for the metric belonging to Szekeres family[1]

$$
\begin{equation*}
d s^{2}=d t^{2}-e^{2 \alpha} d r^{2}-e^{2 \beta}\left(d x^{2}+d y^{2}\right) \tag{1}
\end{equation*}
$$

was taken by Szekeres [1] with dust and $\Lambda=0$. Here $\alpha$ and $\beta$ are in general functions of $(t, r, x, y)$. Szekeres' result was generalized by Szafron et al [2] without any further assumption except for non-zero pressure. In further generalizations the perfect fluid has been replaced by a fluid with heat flow [3], viscosity [4] and also electromagnetic field [5]. Also Barrow et al [6] gave solutions for dust model with a cosmological constant and recently Chakraborty et al [7] gave solutions for perfect fluid model with a cosmological constant in $(n+2)$-D space-time. We are not aware of any generalization of the above class in more than 4 dimensions in Brans-Dicke scalar tensor theory of gravitation. In this paper, we work out solutions for dust in the presence of the cosmological constant $(\Lambda \neq 0)$ and the Brans-Dicke scalar field. We consider the scalar tensor theory in the Dicke's [8] revised version after unit transformation. In the revised version, $G$ does not vary while the masses of the elementary particles are varying. In this version, the trajectories of particles are not geodesics and the energy momentum tensor holds for combined matter and the scalar field. Suitable transformation of units used by Dicke are $\bar{g}_{\mu \nu}=\lambda g_{\mu \nu}, \bar{m}=\lambda^{-1 / 2} m, d \bar{s}=\lambda^{1 / 2} d s$ and the scalar field $\lambda=\phi / \phi_{0}$, where bars indicate the variables in the revised units, $\lambda$ is the scalar field in the new unit and $\phi_{0}$ is a constant. The field equations in the revised units are

$$
\begin{equation*}
R_{\mu \nu}-\frac{1}{2} g_{\mu \nu} R=8 \pi G\left(T_{\mu \nu}+S_{\mu \nu}\right)+\Lambda g_{\mu \nu} \tag{2}
\end{equation*}
$$

where $\quad S_{\mu \nu}=\frac{(2 \omega+3)}{16 \pi G}\left(C_{\mu} C_{\nu}-\frac{1}{2} g_{\mu \nu} C_{l} C^{l}\right) \quad$ and the scalar $\quad C=\ln \lambda=\ln \left(\phi / \phi_{0}\right)$.
In what follows we consider a more general $(n+2)$-dimensional Szekeres space-time, which can be expressed by the following metric form [7]

$$
\begin{equation*}
d s^{2}=d t^{2}-e^{2 \alpha} d r^{2}-e^{2 \beta} \sum_{i=1}^{n} d x_{i}^{2} \tag{3}
\end{equation*}
$$

[^0]where $\alpha$ and $\beta$ are now functions of all the $(n+2)$ space-time variables i.e.,
$$
\alpha=\alpha\left(t, r, x_{1}, \ldots, x_{n}\right), \quad \beta=\beta\left(t, r, x_{1}, \ldots, x_{n}\right)
$$

It is to be noted here that in the presence of the non-zero scalar field it is necessary to make at least one assumption in order to solve the field equations. We can either assume the form of scalar field or of the metric co-efficients. Here we have assumed $\beta_{x_{i}}^{\prime}=0$. Then there are two possibilities. In one of these cases the scalar field is in general a function of $r$ and $t$, whereas in the second case the scalar field may either be a function of $r$ or a function of $t$ alone. In fact the differential equation obtained in terms of the metric co-efficients for $C=C(t, r)$ is apparently not solvable. So in the next section we present exact solutions in the following different cases:
(i) $\beta^{\prime} \neq 0, \dot{\beta}_{x_{i}}=0, C=C(t, r)$,
(ii) $\beta^{\prime} \neq 0, \dot{\beta}_{x_{i}}=0, C=C(t)$,
(iii) $\beta^{\prime}=0, \dot{\beta}_{x_{i}}=0, C=C(t)$.

The case $C=C(r)$ is not consistent with the field equations. In all the above cases the asymptotic behaviour of the dust density $\rho$ and the shear scalar $\sigma$ are obtained in the limits. The cosmological models described in all the cases mentioned above are all inhomogeneous models.

The special cases of our solutions in the absence of the scalar field lead to those of Szekeres as shown in the subsequent sections. At this point one must note that Szekeres solutions are however given in co-ordinates different from those used in the present text.

In the last section an alternative interpretation is given for the above mentioned solutions in the special case of $C$ as a linear function of time. In this particular case the scalar field $C$ may be interpreted as the creation field first proposed by Hoyle and Narlikar [9] in order to obtain a steady state cosmology or later a quasi steady state cosmology.

## II. FIELD EQUATIONS IN THE REVISED VERSION OF THE BRANS-DICKE THEORY AND THEIR EXACT SOLUTIONS

The Einstein's field equations in the presence of the cosmological constant $\Lambda$ follow from (2) (we choose $8 \pi G=1$, see [7]):

$$
\begin{array}{r}
n \dot{\alpha} \dot{\beta}+\frac{1}{2} n(n-1) \dot{\beta}^{2}-e^{-2 \beta} \sum_{i=1}^{n}\left\{\alpha_{x_{i}}^{2}+\frac{1}{2}(n-1)(n-2) \beta_{x_{i}}^{2}+(n-2) \alpha_{x_{i}} \beta_{x_{i}}+\alpha_{x_{i} x_{i}}\right. \\
\left.+(n-1) \beta_{x_{i} x_{i}}\right\}+e^{-2 \alpha}\left\{n \alpha^{\prime} \beta^{\prime}-\frac{1}{2} n(n+1) \beta^{\prime 2}-n \beta^{\prime \prime}\right\}=\Lambda+\rho-\frac{1}{2} f\left(\dot{C}^{2}+e^{-2 \alpha} C^{\prime 2}\right) \\
\begin{array}{r}
\frac{1}{2} n(n+1) \dot{\beta}^{2}+n \ddot{\beta}-\frac{1}{2} n(n-1) e^{-2 \alpha} \beta^{\prime 2}-e^{-2 \beta} \sum_{i=1}^{n}\left\{\frac{1}{2}(n-1)(n-2) \beta_{x_{i}}^{2}+(n-1) \beta_{x_{i} x_{i}}\right\} \\
=\Lambda+\frac{1}{2} f\left(\dot{C}^{2}+e^{-2 \alpha} C^{\prime 2}\right)
\end{array}
\end{array}
$$

$$
\dot{\alpha}^{2}+\ddot{\alpha}+(n-1) \dot{\alpha} \dot{\beta}+\frac{1}{2} n(n-1) \dot{\beta}^{2}+(n-1) \ddot{\beta}+e^{-2 \alpha}\left\{(n-1) \alpha^{\prime} \beta^{\prime}-\frac{1}{2} n(n-1) \beta^{\prime 2}-\right.
$$

$$
\begin{gather*}
\left.(n-1) \beta^{\prime \prime}\right\}-e^{-2 \beta} \sum_{i \neq j=1}^{n}\left\{\alpha_{x_{j}}^{2}+\frac{1}{2}(n-2)(n-3) \beta_{x_{j}}^{2}+\alpha_{x_{j} x_{j}}+(n-2) \beta_{x_{j} x_{j}}+(n-3) \alpha_{x_{j}} \beta_{x_{j}}\right\} \\
-e^{-2 \beta}\left\{(n-1) \alpha_{x_{i}} \beta_{x_{i}}+\frac{1}{2}(n-1)(n-2) \beta_{x_{i}}^{2}\right\}=\Lambda+\frac{1}{2} f\left(\dot{C}^{2}+e^{-2 \alpha} C^{\prime 2}\right)  \tag{6}\\
\alpha_{x_{j}}\left(-\alpha_{x_{i}}+\beta_{x_{i}}\right)+\beta_{x_{j}}\left(\alpha_{x_{i}}+(n-2) \beta_{x_{i}}\right)-\alpha_{x_{i} x_{j}}-(n-2) \beta_{x_{i} x_{j}}=0, \quad(i \neq j)  \tag{7}\\
\dot{\alpha} \beta^{\prime}-\dot{\beta} \beta^{\prime}-\dot{\beta}^{\prime}=-\frac{1}{n} f \dot{C} C^{\prime}  \tag{8}\\
-\dot{\alpha} \alpha_{x_{i}}+\dot{\beta} \alpha_{x_{i}}-\dot{\alpha}_{x_{i}}-(n-1) \dot{\beta}_{x_{i}}=0  \tag{9}\\
\alpha_{x_{i}} \beta^{\prime}-\beta_{x_{i}}^{\prime}=0 \tag{10}
\end{gather*}
$$

where dot, dash and subscript stands for partial differentiation with respect to $t, r$ and the corresponding variables respectively (e.g., $\beta_{x_{i}}=\frac{\partial \beta}{\partial x_{i}}$ ) with $i, j=1,2, \ldots, n$ and $f=-\frac{1}{2}(2 \omega+3)$.

From equations (8) and (10) after differentiating with respect to $x_{i}$ and $t$ respectively, we have the integrability condition

$$
\begin{equation*}
\dot{\beta}_{x_{i}} \beta^{\prime 2}=-\frac{1}{n} f \dot{C} C^{\prime} \beta_{x_{i}}^{\prime}, \quad i=1,2, \ldots, n \tag{11}
\end{equation*}
$$

This equation cannot be solved without any specific assumption either on the metric co-efficients or on the scalar field $C$. Such an assumption is however necessary to obtain exact solutions for all the variables. One of such assumption is $\beta_{x_{i}}^{\prime}=0$, which leave us two possibilities: (i) $\beta^{\prime} \neq 0$ so that $\dot{\beta}_{x_{i}}=0$, (ii) $\beta^{\prime}=0$. The second choice in view of eq.(8) leads us to the condition $\dot{C} C^{\prime}=0$, which implies that $C$ can not be a function of both the co-ordinates $r$ and $t$. For general $C=C(t, r)$, first of all we consider $\beta^{\prime} \neq 0, \dot{\beta}_{x_{i}}=0$. So we obtain from the field equations (8) and (10), the metric co-efficients are in the following form:

$$
\begin{equation*}
e^{\beta}=R(t, r) e^{\nu\left(r, x_{1}, \ldots, x_{n}\right)} \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
e^{\alpha}=\frac{R^{\prime}+R \nu^{\prime}}{S(t, r)} \tag{13}
\end{equation*}
$$

where $R$ and $S$ are function of $t, r$ only. Now from the field equations (5) and (6) using equations (12) and (13), we have the differential equations for $R$ and $S$ :

$$
\begin{equation*}
2 R \ddot{R}+(n-1)\left(\dot{R}^{2}-b^{2} R^{2}\right)-\frac{1}{n}\left[2 \Lambda+f\left(\dot{C}^{2}+e^{-2 \alpha} C^{2}\right)\right] R^{2}=(n-1) K(r) \tag{14}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{f}{2 n} e^{-\alpha} \frac{\partial}{\partial r}\left[\dot{C}^{2}+e^{-2 \alpha} C^{2}\right]=\frac{\dot{S}}{R S^{2}} \frac{\partial}{\partial t}\left(R \dot{S} e^{2 \alpha}\right) \tag{15}
\end{equation*}
$$

where $K$ is a function of $r$ only. Since $\beta_{x_{i}}^{\prime}=0$, so $\alpha$ is a function of $t, r$ only and the form of $e^{\nu}$ suggests

$$
\begin{equation*}
e^{-\nu}=A(r)\left(\sum_{i=1}^{n}\left(x_{i}^{2}+x_{i}\right)+1\right) \tag{16}
\end{equation*}
$$

where $A(r)$ is arbitrary function of $r$ alone satisfying $(n-4) A^{2}(r)=K(r)$. It is to be noted that when $n \neq 4$ then $K(r) \neq 0$ and $n=4$ implies $K(r)=0$, which shows that $K(r)$ must vanish when we consider a six dimensional space-time. In other cases, however $K(r) \neq 0$. The field equation (7) is automatically satisfies by the above solutions. Also from the field equation (4), the expression for the energy density $\rho$ is

$$
\begin{equation*}
\rho\left(t, r, x_{1}, \ldots, x_{n}\right)=-\frac{n}{n-1}\left(\ddot{\alpha}+n \ddot{\beta}+\dot{\alpha}^{2}+n \dot{\beta}^{2}\right)+\frac{2 \Lambda}{n-1}+\frac{n f}{n-1}\left(\dot{C}^{2}+e^{-2 \alpha} C^{\prime 2}\right) \tag{17}
\end{equation*}
$$

and the expression for the shear scalar is

$$
\begin{equation*}
\sigma^{2}=\frac{n}{2(n+1)} \frac{\left(R \dot{R}^{\prime}-\dot{R} R^{\prime}\right)^{2}}{R^{2}\left(R^{\prime}+R \nu^{\prime}\right)^{2}} \tag{18}
\end{equation*}
$$

Now eliminating the terms containing the derivative of the scalar field $C$ between (14) and (15) we obtain the differential equation in $R$ and $S$ as

$$
\begin{equation*}
\frac{\partial}{\partial r}\left[\frac{1}{R^{2}}\left\{2 R \ddot{R}+(n-1)\left(\dot{R}^{2}-S^{2}-K\right)\right\}\right]=2 e^{\alpha} \frac{\dot{S}}{R S^{2}} \frac{\partial}{\partial t}\left(R \dot{S} e^{2 \alpha}\right) \tag{19}
\end{equation*}
$$

This differential equation in $R$ and $S$ is quite complicated and is apparently not solvable in closed form. For general $C=C(t, r)$, it is not possible to find out any exact solutions. So we may consider $C$ is a function of $t$ or $r$ alone. For $C=C(t)$, we arrive the following two cases from equation (11) in order to obtain explicit form of the metric co-efficients: (i) $\beta^{\prime} \neq 0, \dot{\beta}_{x_{i}}=0,(i i) \beta^{\prime}=0$.

Case I: $\quad \beta^{\prime} \neq 0, \dot{\beta}_{x_{i}}=0(i=1,2, \ldots, n), C=C(t):$
With the above condition being used we obtain from the field equations (8) and (10), the metric coefficients in the following form [7]:

$$
\begin{equation*}
e^{\beta}=R(t, r) e^{\nu\left(r, x_{1}, \ldots, x_{n}\right)} \tag{20}
\end{equation*}
$$

and

$$
\begin{equation*}
e^{\alpha}=R^{\prime}+R \nu^{\prime} \tag{21}
\end{equation*}
$$

Now using the relation (20) and (21) in equation (5), we obtain a differential equation for $R$ as follows

$$
\begin{equation*}
2 R \ddot{R}+(n-1) \dot{R}^{2}-\frac{1}{n}\left(2 \Lambda+f \dot{C}^{2}\right) R^{2}=(n-1) K \tag{22}
\end{equation*}
$$

and the solution for $\nu$ can be expressed as

$$
\begin{equation*}
e^{-\nu}=A \sum_{i=1}^{n} x_{i}^{2}+\sum_{i=1}^{n} B_{i} x_{i}+D \tag{23}
\end{equation*}
$$

There is, however, the following restriction

$$
\begin{equation*}
\sum_{i=1}^{n} B_{i}^{2}-4 A D=K-1 \tag{24}
\end{equation*}
$$

where $K, A, B_{i}$ 's and $D$ are all functions of the co-ordinate ' $r$ ' alone.
The first integral of the equation (22) may be given by

$$
\begin{equation*}
\dot{R}^{2}=K+\frac{F}{R^{n-1}}+\frac{2 \Lambda}{n(n+1)} R^{2}+\frac{1}{n} f R^{1-n} \int \dot{C}^{2} R^{n} \dot{R} d t \tag{25}
\end{equation*}
$$

where $F$ is an arbitrary functions of ' $r$ ' and $f$ is a constant which is already expressed earlier in terms of Brans-Dicke parameter $\omega$.

The following situations are considered separately:
(i) $C=C_{0} t, \quad K \neq 0$ and $n=3$, that is this satisfied $\lambda=e^{C_{0} t}$ where $C_{0}$ is a constant.

The integration of the equation (25) in the next step yields

$$
\begin{equation*}
2 z R^{2}+K=\left(4 z F-K^{2}\right)^{1 / 2} \operatorname{Sinh}\left\{2 z^{1 / 2}\left(t-t_{0}\right)\right\}, \quad z>0 \tag{26}
\end{equation*}
$$

and

$$
\begin{equation*}
2|z| R^{2}-K=\left(K^{2}+4|z| F\right)^{1 / 2} \operatorname{Sin}\left\{2|z|^{1 / 2}\left(t-t_{0}\right)\right\}, \quad z<0 \tag{27}
\end{equation*}
$$

In the above solution for $R$ is another arbitrary function of $r$ and $z$ is a constant given by $z=\frac{\left(2 \Lambda+f C_{0}^{2}\right)}{n(n+1)}$.
(ii) $C=C_{0} t, K=0, n$ is arbitrary:

The solution is given by

$$
\begin{equation*}
R^{\frac{n+1}{2}}=\left(\frac{F}{z}\right)^{1 / 2} \operatorname{Sinh}\left\{\frac{1}{2}(n+1) z^{1 / 2}\left(t-t_{0}\right)\right\}, \quad z>0 \tag{28}
\end{equation*}
$$

and

$$
\begin{equation*}
R^{\frac{n+1}{2}}=\left(\frac{F}{|z|}\right)^{1 / 2} \operatorname{Sin}\left\{\frac{1}{2}(n+1)|z|^{1 / 2}\left(t-t_{0}\right)\right\}, \quad z<0 \tag{29}
\end{equation*}
$$

Here of course there is another possibility. If $F<0$, we can obtain from (25) on integration a different solution expressed as

$$
\begin{equation*}
R^{\frac{n+1}{2}}=\left(\frac{|F|}{z}\right)^{1 / 2} \operatorname{Cosh}\left\{\frac{1}{2}(n+1) z^{1 / 2}\left(t-t_{0}\right)\right\}, \quad z>0 \tag{30}
\end{equation*}
$$

The solutions (28) and (30) have different properties and will be discussed subsequently.
(iii) $C=C_{0} t^{a}, K=0$, ' $a$ ' being a constant parameter. We put $R=G^{2 /(n+1)}$ in the equation (22) so that we obtain

$$
4 n S^{-1} \ddot{G}-(n+1)\left(2 \Lambda+f \dot{C}^{2}\right)=0
$$

which on assume $\Lambda=0$ may also be written as

$$
\begin{equation*}
\ddot{G}-y t^{2 a-2} G=0 \tag{31}
\end{equation*}
$$

where $y=\frac{(n+1)}{4 n} f C_{0}^{2} a^{2}$. In equation (31) after integration finally yields the following solution for the variable $R$,

$$
\begin{equation*}
G \equiv R^{\frac{n+1}{2}}=t^{1 / 2}\left[A_{1}(r) I_{\frac{1}{2 a}}\left[y^{1 / 2} t^{a} / a\right]+A_{2}(r) I_{-\frac{1}{2 a}}\left[y^{1 / 2} t^{a} / a\right]\right], \quad y>0 \tag{32}
\end{equation*}
$$

and

$$
\begin{equation*}
G \equiv R^{\frac{n+1}{2}}=t^{1 / 2}\left[A_{1}(r) J_{\frac{1}{2 a}}\left[|y|^{1 / 2} t^{a} / a\right]+A_{2}(r) J_{-\frac{1}{2 a}}\left[|y|^{1 / 2} t^{a} / a\right]\right], \quad y<0 \tag{33}
\end{equation*}
$$

It may be noted that when $a=1$, the solutions (32) reduces to either (28) or (30) but solution (33) reduces to (29) only. As usual $J_{m}(x)$ and $I_{m}(x)$ respectively stand for Bessel function and modified Bessel function of first kind of order $m$.

The expressions for the density $\rho$ and the shear scalar $\sigma$ are obtained from (17) and (18) and are given by

$$
\begin{equation*}
\rho\left(t, r, x_{1}, \ldots, x_{n}\right)=-\frac{n}{(n-1)}\left[\frac{\ddot{R}^{\prime}+\ddot{R} \nu^{\prime}}{R^{\prime}+R \nu^{\prime}}+\frac{n \ddot{R}}{R}\right]+\frac{\left(2 \Lambda+n f \dot{C}^{2}\right)}{(n-1)} \tag{34}
\end{equation*}
$$

and

$$
\begin{equation*}
\sigma^{2}=\frac{n}{2(n+1)} \frac{\left(R \dot{R}^{\prime}-\dot{R} R^{\prime}\right)^{2}}{R^{2}\left(R^{\prime}+R \nu^{\prime}\right)^{2}} \tag{35}
\end{equation*}
$$

The explicit forms of (34) and (35) are apparently very complicated. However one can discuss the behaviour of different models in the limits $t \rightarrow t_{0}$ and $t \rightarrow \infty$. In fact $t_{0}(r)$ depends the initial moment of evolution in each case. The evolution may or may not begin with a big bang singularity. Again since $t_{0}^{\prime} \neq 0$ the instant of singularity if it exists will be position dependent. These are apparent from the following facts.

In case (i) with $K \neq 0$ the dust density and the shear scalar both are finite at $t \rightarrow t_{0}$ but in the other extreme limit that is as $t \rightarrow \infty$ the dust density $\rho$ remains finite but the shear scalar $\sigma$ vanishes.

In case (ii), $\rho \rightarrow \infty$ and $\sigma \rightarrow \infty$ at the initial instant $t \rightarrow t_{0}$ but in course of time as $t \rightarrow \infty$ the dust density remains finite, whereas $\sigma \rightarrow 0$. This is true for the solution (28). On the other hand if we concentrate our attention on the solution (30). We find that the density $\rho$ remains finite throughout the evolution, whereas the shear starts from a finite magnitude and gradually disappear in course of evolution.

In case (iii), we note that there is singularity $(R=0)$ for the solution in equations (32) and (33) provided we choose arbitrary constants $A_{2}(r)=0$. Also asymptotically for large $t, R$ oscillates infinitely for the solution (33) while $R$ becomes infinite for the solution (32). In this case, for $0 \leq a \leq 1, \rho$ and $\sigma$ both explode initially whereas they attain finite magnitudes in course of evolution as $t \rightarrow \infty$.

Case II: $\quad \beta^{\prime}=0, C=C(t)$ :
In this case, for only the above choices, the exact solutions can not be found from the field equations (that shown by Szekeres [1]). So we need further assumption like $\dot{\beta}_{x_{i}}=0(i=1,2, \ldots, n)$, in order to obtain the exact solutions.

Now from the field equations we obtain the metric coefficients in the form

$$
\begin{equation*}
e^{\beta}=R(t) e^{\nu\left(x_{1}, x_{2}, \ldots, x_{n}\right)} \tag{36}
\end{equation*}
$$

and

$$
\begin{equation*}
e^{\alpha}=R(t) \eta\left(r, x_{1}, x_{2}, \ldots, x_{n}\right)+\mu(t, r) \tag{37}
\end{equation*}
$$

Then as before from the field equation (5), we have similar differential equations in $R$ as

$$
\begin{equation*}
2 R \ddot{R}+(n-1) \dot{R}^{2}-\frac{1}{n}\left(2 \Lambda+f \dot{C}^{2}\right) R^{2}=(n-1) K \tag{38}
\end{equation*}
$$

and the solution for $\nu$ as

$$
\begin{equation*}
e^{-\nu}=A \sum_{i=1}^{n} x_{i}^{2}+\sum_{i=1}^{n} B_{i} x_{i}+D \tag{39}
\end{equation*}
$$

along with the restriction

$$
\begin{equation*}
\sum_{i=1}^{n} B_{i}^{2}-4 A D=K \tag{40}
\end{equation*}
$$

Here $K, A, B_{i}$ 's and $D$ are all arbitrary constants.
Now from the field equation (7) we have the solution for $\eta$ as

$$
\begin{equation*}
e^{-\nu} \eta=u \sum_{i=1}^{n} x_{i}^{2}+\sum_{i=1}^{n} v_{i} x_{i}+w \tag{41}
\end{equation*}
$$

and the resulting differential equation in $\mu$ is

$$
\begin{equation*}
R \ddot{\mu}+(n-1) \dot{R} \dot{\mu}+\mu\left[\ddot{R}-\frac{1}{n}\left(2 \Lambda+f \dot{C}^{2}\right) R\right]=g(r) \tag{42}
\end{equation*}
$$

with

$$
\begin{equation*}
g(r)=(n-1)\left[2(u D+w A)-\sum_{i=1}^{n} v_{i} B_{i}\right] \tag{43}
\end{equation*}
$$

where $u, v_{i}$ 's and $w$ as arbitrary functions of the co-ordinate ' r ' alone.
For simplicity, let us choose $C=C_{0} t, n=3$. In this case the solutions of $R$ (as before) and $\mu$ are

$$
\begin{equation*}
2 z R^{2}+K=\left(4 z F-K^{2}\right)^{1 / 2} \operatorname{Sinh}\left\{2 z^{1 / 2}\left(t-t_{0}\right)\right\}, \quad z>0 \tag{44}
\end{equation*}
$$

$$
\begin{equation*}
2|z| R^{2}-K=\left(K^{2}-4 z F\right)^{1 / 2} \operatorname{Sin}\left\{2|z|^{1 / 2}\left(t-t_{0}\right)\right\}, \quad z<0 \tag{45}
\end{equation*}
$$

and

$$
\begin{align*}
2 z \mu R+g(r) & =\sqrt{4 z^{2} h(r)-g^{2}(r)} \operatorname{Sinh}\left\{\sqrt{2 z}\left(t-t_{1}(r)\right)\right\}, \quad z>0  \tag{46}\\
2|z| \mu R-g(r) & =\sqrt{g^{2}(r)-4 z^{2} h(r)} \operatorname{Sin}\left\{\sqrt{2|z|}\left(t-t_{1}(r)\right)\right\}, \quad z<0 \tag{47}
\end{align*}
$$

where $t_{0}, F$ are arbitrary constants, $h(r), t_{1}(r)$ are arbitrary functions of ' $r$ ' only and $z=\frac{1}{12}\left(2 \Lambda+f C_{0}^{2}\right)$. The solutions (44) and (45) is identical with (28) and (29) except for the fact that now $F$ and $t_{0}$ are no longer functions of $r$.

In this case the expressions for the density and the shear scalar are given by

$$
\begin{align*}
\rho\left(t, r, x_{1}, \ldots, x_{n}\right) & =-\frac{n}{(n-1)}\left[\frac{\ddot{\mu}+\ddot{R} \eta}{\mu+R \eta}+\frac{n \ddot{R}}{R}\right]+\frac{\left(2 \Lambda+n f \dot{C}^{2}\right)}{(n-1)}  \tag{48}\\
\sigma^{2} & =\frac{n}{2(n+1)} \frac{(R \dot{\mu}-\dot{R} \mu)^{2}}{R^{2}(\mu+R \eta)^{2}} \tag{49}
\end{align*}
$$

There are two distinct cases for either $K \neq 0$ or $K=0$. In the former case with $K<0$ both the dust density and the shear scalar remain finite when $t \rightarrow t_{0}$, but they are infinitely large at this instant in the second case $K=0$. On the other hand as $t \rightarrow \infty$ the density is finite even though the shear vanishes in course of evolution. When $K>0$ the situation differs. Here the density and shear explode at same initial instant other than $t=t_{0}$.

Next if we consider $C=C(r)$ then from equation (11), we may get also two possibilities: (i) $\beta^{\prime} \neq 0, \dot{\beta}_{x_{i}}=0$, (ii) $\beta^{\prime}=0$. In these cases the form of metric co-efficients are similar to case I and Case II, but the differential equations (22) and (38) are slightly different i.e.,

$$
2 R \ddot{R}+(n-1) \dot{R}^{2}-\frac{1}{n}\left(2 \Lambda+f e^{-2 \alpha} C^{2}\right) R^{2}=(n-1) K
$$

This equation can not be solved because $\alpha$ is a function of all space-time co-ordinates. So we are not going to further discussion on the choice $C=C(r)$.

## III. ALTERNATIVE INTERPRETATION OF THE SOLUTIONS IN TERMS OF C-FIELD COSMOLOGY

It is appropriate at this stage to interpret some of the previously described solutions as due to the presence of the creation field first introduced by Hoyle and Narlikar [9]. The field equations in this case are exactly identical with (2) except for the replacement of $S_{\mu \nu}$ by $T_{\mu \nu}^{(c)}$, where

$$
\begin{equation*}
T_{\mu \nu}^{(c)}=-f\left(C_{\mu} C_{\nu}-\frac{1}{2} g_{\mu \nu} C_{l} C^{l}\right) \tag{50}
\end{equation*}
$$

where $C_{\mu}=C,_{\mu}$ and $f(>0)$ is a coupling constant. The additional feature in $C$-field cosmology is that one must confirm that the $C$-field satisfies the source equation

$$
\begin{equation*}
f C_{; \mu}^{\mu}=j_{; \mu}^{\mu} \tag{51}
\end{equation*}
$$

with $j^{\mu}=\rho \frac{d x^{\mu}}{d s}$.
It is not difficult to check that the Bianchi identity and the source equation (51) lead to the relation $\dot{C}=1$, which determines the expression of the $C$-field scalar $C=t+\psi(r)$. So for obvious reasons all the previous solutions with the scalar field $C=\ln \lambda$ expressed as a linear function of time are also solutions of the $C$-field cosmology.

## IV. DISCUSSIONS

One must note that in all the models presents above some are singularity free and some have big bang type singularities at the starting point. Particularly for the function $K=0$ those which show the turning point $(\dot{R}=0)$ at some stage represent only the lower bound (since $\ddot{R}>0$ ) as are evident from from the equations (22) and (38). The singularity occurs in each case at an instant which is not fixed for different shells of different $r$. In fact, the singularity $R=0$ is position dependent in all the cases described above and the reason is that the system is inhomogeneous which is the limit of spherical dust reduces to Tolman-Bondi space-time. There is one particular case given by the solution (29) where there is an initial singularity at $t=t_{0}$. Subsequently $R(t, r)$ increases and reaches a maximum followed by a collapse to a crunch.

Another interesting point to observe is that for a few solutions such as (26), (28), (44) etc, we have $\ddot{R}$ initially less than zero but becomes positive in course of time indicating that the expansion starts from decelerating phase to an accelerating phase at late stage. It is to be noted that in the above set of solutions is a kind of singularity given by $e^{\alpha}=0$ or equivalently $\left(e^{\beta}\right)^{\prime}=0$, which are analogous to the shell crossing singularity in Tolman-Bondi [10] models. In this case also the density $\rho$ diverges. Finally it can be mentioned that one can obtain Szekeres 4 dimensional solutions from our solutions if we put $n=2, \Lambda=0$ and $C=0$. In fact the differential equations in $R$ (see equations (22) and (38)) after the above simplification becomes identical to the corresponding differential equations in Szekeres solutions [1]. Therefore our solutions are generalization of Szekeres solutions.

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