ARITHMETICAL PARAPHRASES*

BY

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I. Introduction

1. Let

$$\xi_i \equiv (x_{i1}, x_{i2}, \cdots, x_{ia_i}) \qquad (i = 1, 2, \cdots, r),$$

$$\eta_j \equiv (y_{j1}, y_{j2}, \cdots, y_{jb_j}) \qquad (j = 1, 2, \cdots, s),$$

denote $(r + s)$ one-rowed matrices of independent variables, no pair of matrices having a variable in common. Write

$$-\xi_i \equiv (-x_{i1}, -x_{i2}, \cdots, -x_{ia_i}) \equiv -(x_{i1}, x_{i2}, \cdots, x_{ia_i}),$$

and similarly for $-\eta_j$. If in η_j each $y = 0$, η_j is said to vanish. Let

$$f(\xi_1, \xi_2, \cdots, \xi_r | \eta_1, \eta_2, \cdots, \eta_s) \tag{1}$$

denote a function which exists and has a determinate value for all integral values $\gtreqless 0$ of the x, y in ξ, η; which remains unchanged in value when any one of the ξ is replaced by its negative, and which changes sign and vanishes with each of the η. Similarly

$$g(\xi_1, \xi_2, \cdots, \xi_r |), \qquad h(| \eta_1, \eta_2, \cdots, \eta_s) \tag{2}$$

exist and are determinate for all integral values of the x, y in ξ, η respectively; the value of g is unchanged when any one of the ξ is replaced by its negative; h changes sign and vanishes with each η.

It is emphasized, once for all, that beyond these restrictions f, g, h are wholly arbitrary.

As examples of the bar notation,

$$f(x, y|) = f(-x, y|) = f(x, -y|);$$
$$f(x|y) = f(-x|y) = -f(x|-y);$$
$$f(|x, y) = -f(|-x, y) = -f(|x, -y);$$
$$f((x, y)|z) = f((-x, -y)|z) = -f((x, y)|-z);$$
$$f((x, y, z), (u, w)|(t, v)) = f((-x, -y, -z), (u, w)|(t, v))$$
$$= f((x, y, z), (-u, -w)|(t, v)) = -f((x, y, z), (u, w)|(-t, -v)).$$

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2. The parity of the f in (1) is denoted by

$$p(a_1, a_2, \cdots, a_r | b_1, b_2, \cdots, b_s); \tag{3}$$

and the respective parities of g, h in (2) are

$$p(a_1, a_2, \cdots, a_r | 0), \qquad p(0 | b_1, b_2, \cdots, b_s), \tag{4}$$

the notation being obvious. The positive integers

$$\omega = \sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j, \qquad \delta = r + s, \tag{5}$$

are called the order and degree respectively of f. Similarly for g, h. When

$$a_i = 1 = b_j \quad (i = 1, 2, \cdots, r; j = 1, 2, \cdots, s),$$

the parities (3), (4) are written respectively:

$$p(1^r | 1^s), \; p(1^r | 0), \; p(0 | 1^s). \tag{6}$$

Likewise, if α_j of the a_i each $= a_j$, and β_i of the b_j each $= b_i$, the parities (3), (4) are written (the order of the a's or b's within $(|)$ is immaterial),

$$p(a_1^{\alpha_1}, a_2^{\alpha_2}, \cdots | b_1^{\beta_1}, b_2^{\beta_2}, \cdots); \; p(a_1^{\alpha_1}, a_2^{\alpha_2}, \cdots | 0); \; p(0 | b_1^{\beta_1}, b_2^{\beta_2}, \cdots). \tag{7}$$

From the definitions, an f whose parity is $p(1^r | 1^s)$ is a function of $(r + s)$ single independent variables, even separately in r of them, odd in each of the remaining s variables, and vanishing with each of the s. The corresponding statement for a function of parity $p(1^r | 0)$ follows on supposing $s = 0$; similarly for one of parity $p(0 | 1^s)$, on supposing $r = 0$. Henceforth we shall in general consider it unnecessary to give separate statements for f, g, h of (1), (2), regarding all as implicit in the statement for (1). The parity of a constant is considered $= p(0 | 0)$.

3. Without difficulty it may be shown* that an f of order ω and degree δ is

* For this result and that of § 4, cf. Bell, Bulletin of the American Mathematical Society, vol. 25 (1918–19), p. 313. The proofs follow readily from the fundamental identities (52), (53) of § 33, and (59), (60) of § 35. On account of its interest we add the following alternative proof. We are concerned in §§ 3, 4 with a generalization of the expression of a function as the sum of an odd and an even function. Thus

$$\begin{aligned} 2f(x) &= [f(x) + f(-x)] + [f(x) - f(-x)] \equiv \phi_0(x) + \phi_1(x), \\ 2f(-x) &= [f(x) + f(-x)] - [f(x) - f(-x)] \equiv \phi_0(x) - \phi_1(x). \end{aligned} \tag{A}$$

If now f is a general function of ω variables $x = x_1, \cdots, x_\omega$, then singling out x_1 we define $\phi_0(x)$, $\phi_1(x)$. In ϕ_0, ϕ_1 we single out x_2, and proceeding as in (A) obtain ϕ_{00}, ϕ_{01}, ϕ_{10}, ϕ_{11}, where 1, 0 indicates oddness or evenness respectively in the variables in order. Proceeding thus we have eventually 2^ω functions

$$\phi_{i_1 i_2 \cdots i_\omega} \qquad (i_j = 0, 1; j = 1, \cdots, \omega),$$

of parities $p(1^\alpha | 1^\beta)$, $(\alpha + \beta = \omega)$.

On the other hand we apply to $f(x_1, x_2, \cdots, x_\omega) = f_{000\cdots 0}$ the operations of the group G

linearly expressible in terms of $2^{\omega-\delta}$ suitably chosen functions, all of whose parities are of the form $p(1^\alpha|1^\beta)$, where $\alpha+\beta=\omega$.

4. Removing the restriction that the f in (1) shall vanish with each η, we get what we shall call a special f of parity (3). E.g., $wz/(x+y)$ is a special f of parity $p(0|1,1,2)$, $=p(0|1,2,1)$, etc. Clearly, parity has no relevance in regard to a perfectly arbitrary function of n variables; such a function is not necessarily even or odd in any one of its variables or in any matrix ξ, η of its variables. It is easy to show, however, that an arbitrary function of n variables is linearly expressible in terms of 2^n suitably chosen special f's, all of whose parities are of the form $p(1^\alpha|1^\beta)$ where $\alpha+\beta=n$. This result and that of § 3 are basic in the subsequent discussion.

5. In addition to the functions already defined, we shall consider others, ϕ, having the same parities as f, g, h but further restricted, e.g., as to alterance, invariance under the substitutions of a finite group on the x, y, etc., the essential feature being change or invariance of sign under permutation of the variables. For a reason appearing presently, all functions f, g, h, ϕ are called L-functions, where the L stands for Liouville. Functions ϕ, and functions F, G, H, Φ which satisfy the same conditions of parity as f, g, h, ϕ but which also implicitly satisfy further conditions, as e.g., continuity, differentiability, etc., with respect to some or all of the x, y variables, are called restricted L-functions. The explicit restrictions on a given ϕ, which so far as this paper is concerned* are only of the nature that ϕ is unaltered to within sign under permutations of the variables, will be exhibited by stating the equations which express them. Thus,

$$\phi((x,y)|z) = -\phi((y,x)|z),$$

expresses that $\phi((x,y)|z)$, of parity $p(2|1)$, in addition to satisfying the parity equations

$$\phi((x,y)|z) = \phi((-x,-y)|z) = -\phi((x,y)|-z),$$

of order 2^ω of changes of sign of the variables and obtain 2^ω functions

$$f_{i_1 i_2 \dots i_\omega} \qquad (i_j = 0, 1),$$

where $i_j=0,1$ according as in $f=f_{000\dots 0}$ the sign of x_i has not or has been changed. *Then*, by repeated application of (A) we obtain a linear transformation with coefficients ± 1 which expresses the set of 2^ω functions $2^\omega f_{i_1 i_2 \dots i_\omega}$ in terms of the 2^ω functions ϕ. This is true therefore of the one function $2^\omega f_{000\dots 0} = 2^\omega f$.

If in particular (§ 3) f has degree δ, then f is unaltered to within sign by a subgroup (necessarily invariant since G is abelian) of G of order 2^δ. An operation such as (A) becomes the identity when f itself has parity, and the number of functions $f_{i_1\dots i_\omega}$, $\phi_{i_1\dots i_\omega}$, reduces to $2^{\omega-\delta}$ and the linear transformation between them contains the integer factor $2^{\omega-\delta}$.

* Other restrictions of great use in applications are of the kinds (i) $\phi(x|)=1,0$ according as x is or is not the $(2r-1)$th power of an integer; (ii) $\phi(x|)=1,0$ according as x is or is not divisible by a given integer, and a similar restriction upon $\phi(|x)$; (iii) the obvious extensions of these to ϕ's of several variables. Examples of these will be given in papers to appear elsewhere.

which are implicit in the bar notation, is alternating in x, y.

A set of equations expressing restrictions may imply further restrictions. For example we find theorems for restricted L-functions, ϕ, of order 4, the restrictions first presenting themselves in the form:*

$$\phi(x, y, z, w) = \phi(y, x, z, -w) = -\phi(x, -y, w, z).$$

From these we infer, among others:

$$\phi(x, y, z, w) = \phi(-x, -y, -z, -w) = -\phi(y, -x, -w, z).$$

Hence $\phi(x, y, z, w)$ may be represented by $\phi((x, y, z, w)|)$; and we have the canonical set of restrictions:

$$\phi((x, y, z, w)|) = \phi((y, x, z, -w)|) = -\phi((x, -y, w, z)|);$$

a set being canonical when it includes the parity conditions and a minimum number of restrictions from which all may be inferred.†

It will be shown, when we consider restrictions in detail, that a canonical set for a restricted L-function ϕ, of order ω, may always be found by determining the group to which a certain algebraic form on ω letters associated with ϕ belongs. This, at first sight, is rather remarkable, as the L-functions (cf. § 1), are not necessarily algebraic.‡

6. With ξ, η as in § 1, consider the implicitly restricted L-functions:

$$F(\xi_1, \xi_2, \cdots, \xi_r | \eta_1, \eta_2, \cdots, \eta_s) \equiv \sum_{m=1}^{k} c_m \left[\prod_{i=1}^{r} \cos \left(\sum_{n=1}^{a_i} \alpha_{min}\, x_{in} \right) \cdot \prod_{j=1}^{s} \sin \left(\sum_{n=1}^{b_j} \beta_{mjn}\, y_{jn} \right) \right];$$

$$G(\xi_1, \xi_2, \cdots, \xi_r |) \equiv \sum_{m=1}^{k} c_m \left[\prod_{i=1}^{r} \cos \left(\sum_{n=1}^{a_i} \alpha_{min}\, x_{in} \right) \right];$$

$$H(|\eta_1, \eta_2, \cdots, \eta_s) \equiv \sum_{m=1}^{k} c_m \left[\prod_{j=1}^{s} \sin \left(\sum_{n=1}^{b_j} \beta_{mjn}\, y_{jn} \right) \right].$$

* An example occurs among the illustrations, § 15 (19a).

† The following alternative statement may be made. In the notation of § 3, footnote, the permutations of the variables under which f is unaltered to within sign generate with G an enlarged group Γ under which G is invariant to within sign. Thus a *canonical* set of restrictions may be described as one which gives the generators of G and a minimum number of generators of the factor group of G under Γ, i.e., the permutation group.

‡ Detailed consideration of this point having been omitted to save space, we shall give here sufficient indications of the course to be followed, from which the whole process can easily be reconstructed. The algebraic form mentioned is that which is deduced from the reduced invariant I_ϵ defined, Bulletin of the American Mathematical Society, vol. 26, p. 217, § 9, as follows: each k-au (ibid., p. 212, § 2) is to be replaced by the restricted L-functions F, G or H of this paper, § 6, on the same variables; the algebraic form is then the coefficient in this result of the general term in the x, y variables when the entire I_ϵ is expanded in powers of these variables.

Write

$$A_{mi} \equiv (\alpha_{mi1}, \alpha_{mi2}, \cdots, \alpha_{mia_i}) \qquad (i = 1, 2, \cdots, r);$$

$$B_{mj} \equiv (\beta_{mj1}, \beta_{mj2}, \cdots, \beta_{mjb_j}) \qquad (j = 1, 2, \cdots, s);$$

and let the c, α, β denote integers. Then, in its general form, the principle of paraphrase which we shall use is:

(i) If for all values of the x, y,

$$F(\xi_1, \xi_2, \cdots, \xi_r | \eta_1, \eta_2, \cdots, \eta_s) = 0, \tag{8}$$

then

$$\sum_{m=1}^{k} c_m f(A_{m1}, A_{m2}, \cdots, A_{mr} | B_{m1}, B_{m2}, \cdots, B_{ms}) = 0. \tag{8a}$$

(ii) If for all values of the x,

$$G(\xi_1, \xi_2, \cdots, \xi_r |) = 0, \tag{9}$$

then

$$\sum_{m=1}^{k} c_m g(A_{m1}, A_{m2}, \cdots, A_{mr} |) = 0. \tag{9a}$$

(iii) If for all values of the y,

$$H(| \eta_1, \eta_2, \cdots, \eta_s) = 0, \tag{10}$$

then

$$\sum_{m=1}^{k} c_m h(| B_{m1}, B_{m2}, \cdots, B_{ms}) = 0. \tag{10a}$$

In (8a), (9a), (10a), f, g, h are general L-functions as defined in § 1; and the principle asserts that the sine-cosine identities (8), (9), (10) may be paraphrased directly into (8a), (9a), (10a) respectively. By means of this simple principle, which we shall prove as required (cf. § 18 et seq.), the applications of the elliptic, hyperelliptic and theta functions to the theory of numbers are greatly extended. For, from the theories of these functions we write down identities (8), (9), (10) in which the A_{mi}, B_{mj} are matrices whose elements are linear functions of the divisors of integers belonging to certain linear or quadratic forms (more specifically defined in §§ 7, 8). The (8a), (9a), (10a) written down from the (8), (9), (10) then give, for special choices of the L-functions, as for example

$$f(x, y|) = y^{2n} + x^{2k} \cos \pi y, \qquad f(x, y|z) = (-1)^x y^{2k} \sin(z^3\pi/2),$$

$$f(|x) = x^{2n} \sin \frac{\pi}{x},$$

an inexhaustible source of arithmetical theorems. It will be noted that this principle effects the passage from circular to L-functions immediately without further analysis or transformations. Finally, it will be shown,* from a para-

* Cf. §§ 32–34.

phrase concerning L-functions of parity $p(a|0)$ that we can at once infer paraphrases in which the L-functions are of either of the parities $p(a_1, a_2|0)$, $p(0|a_1, a_2)$, where $a_1 + a_2 = a$. Similarly, from a paraphrase for L-functions of parity $p(0|b)$ follow immediately paraphrases for L-functions of parity $p(b_1|b_2)$, where $b_1 + b_2 = b$. Now obviously an L-function of parity $p(a_1, a_2, \cdots, a_r|b_1, b_2, \cdots, b_s)$ may be regarded as an L-function of any of the parities $p(a_i|0)$, $p(0|b_j)$, $(i = 1, 2, \cdots, r;\ j = 1, 2, \cdots, s)$. Applying the foregoing inferences successively to some or all of the a_i, b_j, we find that a paraphrase in which the L-functions are unrestricted of parity $p(a_1, a_2, \cdots, a_r|b_1, b_2, \cdots, b_s)$, degree δ, order ω, implies further paraphrases for unrestricted L-functions of order ω, and degree δ', where

$$\delta < \delta' \gtreqless \omega.$$

From the paraphrases for the functions of degree δ' may be readily built up paraphrases* for L-functions of order ω subject to restrictions as outlined in § 5.

7. Before illustrating the nature of the paraphrases we shall define the sense in which *separation* is used constantly throughout. Unless the contrary is explicitly stated, all integers now considered are positive and different from zero. Adopting Glaisher's convenient notation,† we use letters m to denote odd integers, letters n to denote arbitrary integers; and in reference to separations, m, n shall always, without further specification, have this significance. Letters d, δ denote positive integral divisors. Hence in $m = d\delta$ both d, δ are odd; in $n = d\delta$ either or both d, δ may be odd or even; and $n = 2^\alpha m$, in which $\alpha \geqq 0$, indicates the highest power of 2 that divides n. We shall be frequently concerned with three types of division, T_1, T_2, T_3:

$$(11) \qquad T_1 : m = d\delta; \qquad T_2 : n = 2^\alpha m,\ m = d\delta; \qquad T_3 : n = d\delta.$$

Let $n, c, c_1, c_2, \cdots, c_r, c_1', c_2', \cdots, c_s'$ denote fixed integers, $n, c > 0$, the rest $\geqq 0$; $n_1, n_2, \cdots, n_r, n_1', n_2', \cdots, n_s'$ variable integers. Then, a separation of cn is the totality, $[S]$, of all solutions, $(2^\alpha d,\ \delta,\ 2^\alpha d_1,\ \delta_1,\ \cdots,\ n_1',\ n_2',\ \cdots)$, of such a system as

$$\begin{aligned} &cn = c_1 n_1 + c_2 n_2 + \cdots + c_r n_r + c_1' n_1'^2 + c_2' n_2'^2 + \cdots + c_s' n_s'^2, \\ &n = 2^\alpha m,\ n_1 = 2^{\alpha_1} m_1,\ \cdots,\ n_r = 2^{\alpha_r} m_r, \\ (12) \quad &n_1' \gtreqless 0,\ n_2' \geqq 0,\ \cdots,\ n_s' \lesseqgtr 0, \\ &m = d\delta,\ m_1 = d_1 \delta_1,\ \cdots,\ m_r = d_r \delta_r, \\ &\alpha > 0,\ \alpha_1 \geqq 0,\ \cdots,\ \alpha_r = 0, \end{aligned}$$

whose essential characteristics are:

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* The process is illustrated in Bulletin of the American Mathematical Society, vol. 26 (1919–20), p. 218, § 10, and elsewhere in the same paper.

† Kronecker used a similar notation in his memoirs on class-number relations; cf. Journal für Mathematik, vol. 57 (1860), p. 248.

(i) T_j $(j = 1, 2, 3)$ is given for each of $n_1, n_2, \cdots, n_r$;

(ii) the range of permissible values for each of the $n'_1, n'_2, \cdots, n'_s$ is specified, when it is other than $+1$ to $+\infty$; viz., the range, which may be any of $\gtreqless 0$, > 0, $\geqq 0$ according to the case, of permissible values for each of the $n'_1, n'_2, \cdots, n'_s$ is specified in a given separation. Similarly for the α's, which may range > 0, $\geqq 0$. The actual set given in (12) is merely a specimen separation. Thus $n'_1 \geqq 0$, $n'_2 > 0$, $n'_3 \gtreqless 0$, $\alpha \geqq 0$, $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 \geqq 0$ characterizes one definite separation; $n'_1 > 0$, $n'_2 \gtreqless 0$, $n'_3 > 0$, $\alpha > 0$, $\alpha_1 \geqq 0$, $\alpha_2 \geqq 0$ characterizes another.

(iii) The coefficients c, c_i, c'_j are all positive.

When further conditions, e.g., $\delta_1 < \sqrt{m_1}$, are imposed, the separation is said to be restricted. The degree* of $[S]$ is the number of non-vanishing c_i, c'_j.

8. Let the degree of $[S]$ be ν; and denote by (S) a particular solution of (12):

$$(S) \equiv (\lambda_1, \lambda_2, \cdots, \lambda_\nu).$$

Form ω linear functions of the λ's:

$$\Lambda_i \equiv l_{i1}\lambda_1 + l_{i2}\lambda_2 + \cdots + l_{i\nu}\lambda_\nu \qquad (i = 1, 2, \cdots, \omega);$$

and denote by $F(z_1, z_2, \cdots, z_\omega)$ any L-function of order ω. Construct $F(\Lambda_1, \Lambda_2, \cdots, \Lambda_\omega)$ for each (S) in $[S]$. Since the $c_i, c'_j \geqq 0$, there will be only a finite number, k, of such F's; say

$$F(S_1),\ F(S_2),\ \cdots,\ F(S_k).$$

We shall be concerned with sums

$$\sum_{i=1}^{k} a_i F(S_i), \tag{13}$$

where the a_i denote constant integers, for L-functions of specified parities; and (13) is defined to be the integration of $a_i F(X_1, X_2, \cdots, X_\omega)$ over $[S]$, where

$$X_i \equiv l_{i1}x_1 + l_{i2}x_2 + \cdots + l_{i\nu}x_\nu \qquad (i = 1, 2, \cdots, \omega).$$

9. Separations are segregated into two main classes: linear, when $c'_1 = c'_2 = \cdots = c'_s = 0$; quadratic, when at least one $c'_j > 0$. Linear separations are further classified according to the types T_1, T_2, T_3; and quadratic, in addition to the specification of types for the η_i, according to the evenness or oddness

* The degree of $[S]$ expresses, as will be evident from the derivations of the paraphrases in Part II, section V, the greatest number of elliptic and theta series which are multiplied together in an identity furnishing L-function paraphrases whose integrations (§ 8) are over $[S]$. This has proved a useful clue in tracing certain of Liouville's more abstruse results to their elliptic-theta equivalents, cf. § 13.

of the n_i'. This classification is basic in connection with the subsequent classification and interlacing of the paraphrases, the latter depending naturally upon the former. The elliptic and theta series which we shall use are similarly classified before paraphrasing.

10. Paraphrases, which will be of the general form $\sum_{i=1}^{i=k} a_i F(S_i) = 0$, (cf. § 8), will be stated by giving the separations and corresponding integrations, which always, as in § 8, are with respect to the separations. For simplicity in writing, the L-functions under the $\sum$ will sometimes be indicated as follows:

$$f(x_1, x_2, \cdots, x_r, y_1, y_2, \cdots, y_s) \equiv F(x_1, x_2, \cdots, x_r | y_1, y_2, \cdots, y_s),$$

and the paraphrase written $\sum f() = 0$. Paraphrases in which the integrations are over several separations will be similarly written, the several separations being given separately by different systems of letters, thus:

$$n = m_1 + 2m_2; \qquad n = 2^{\alpha'} m' + m''; \qquad \cdots$$

$$m_1 = d_1 \delta_1, \qquad m_2 = d_2 \delta_2; \qquad m' = d' \delta', \qquad m'' = d'' \delta''; \qquad \cdots.$$

Always, unless it is explicitly given that they are restricted, the L-functions are general as defined in § 1.

11. To illustrate the concepts of this introduction we shall now give without proof* a few simple examples. These indicate the nature of the general formulas into which we later paraphrase certain parts of the theories of elliptic and theta functions. *References are at the end of this paper.*

As a first example we consider the following in detail. By a simple transformation it is easily shown to be identical with Liouville's 5, (f).

$$n = n' + n''; \qquad n = d\delta, \qquad n' = d'\delta', \qquad n'' = d''\delta'':$$

$$(14) \quad \begin{aligned} &\sum [f(d' - d'', \delta' + \delta''|) - f(d' + d'', \delta' - \delta''|)] \\ &= \sum [(d-1)\{f(0, d|) - f(d, 0|)\} + 2\sum_{r=1}^{d-1} \{f(\delta, r|) - f(r, \delta|)\}]. \end{aligned}$$

Here an L-function of parity $p(1^2|0)$ is integrated over a linear separation of degree 2, and of a type that may be conveniently designated by T_3^2. The precise nature of (14) will be evident from a numerical example. Let $n = 5$; then:

* The first example is proved in part II, § 23. The paraphrase of the $t(x, y)$ identity in § 14 is immediate from the series for the doubly periodic functions of the second kind given in Part II, § 16; that of (ii) is a translation of the trigonometric identity obtained on equating coefficients of q^m, the series for the functions being written down from those given by G. Humbert in Journal de Mathématiques pures et appliquées, (6) 3, vol. 72 (1907), p. 350, first formula in (5), and from Hermite, *Œuvres*, vol. 2, p. 244, formula 1.

$(n', n'') = (1,4), (2,3), (3,2), (4,1)$:

$$\begin{array}{r|l|l|l|l|}
(n',n'') = & (1,4) & (2,3) & (3,2) & (4,1) \\
(d',\delta') = & (1,1) & (1,2),(2,1) & (1,3),(3,1) & (1,4),(2,2),(4,1) \\
(d'',\delta'') = & (1,4),(2,2),(4,1) & (1,3),(3,1) & (1,2),(2,1) & (1,1)
\end{array},$$

whence, for the successive (n', n'') the values of $(d' \mp d'', \delta' \pm \delta'')$ are

$$\begin{array}{c|l|l|}
(n',n'') & (d'-d'',\delta'+\delta'') & (d'+d'',\delta'-\delta'') \\
(1,4) & (0,5),(-1,3),(-3,2) & (2,-3),(3,-1),(5,0) \\
(2,3) & (0,5),(-2,3),(1,4),(-1,2) & (2,-1),(4,1),(3,-2),(5,0) \\
(3,2) & (0,5),(-1,4),(2,3),(1,2) & (2,1),(3,2),(4,-1),(5,0) \\
(4,1) & (0,5),(1,3),(3,2) & (2,3),(3,1),(5,0)
\end{array}.$$

Since $f(x, y|) = f(-x, y|) = f(x, -y|)$, we have, on writing

$$f(x, y) \equiv f(x, y|):$$

$$[4f(0,5) + 2f(1,3) + 2f(3,2) + 2f(2,3) + 2f(1,4) + 2f(1,2)]$$
$$- [4f(5,0) + 2f(3,1) + 2f(3,2) + 2f(2,3) + 2f(4,1) + 2f(2,1)],$$

for the left of (14); and this reduces to:

$$4[f(0,5) - f(5,0)]$$
$$+ 2[f(1,2) - f(2,1) + f(1,3) - f(3,1) + f(1,4) - f(4,1)].$$

For $n = 5$, we have $(d, \delta) = (1,5), (5,1)$; and the right of (14) is:

$$(1-1)\{f(0,1) - (1,0)\} + (5-1)\{f(0,5) - f(5,0)\}$$
$$+ 2\sum_{r=1}^{4} \{f(1,r) - f(r,1)\},$$

which agrees with the value found for the left.

12. All of Liouville's formulas for functions whose order or degree exceeds unity have in common one feature which is truly remarkable. To see it for (14), an inspection of the numerical example will show that in each $f(d' \mp d'', \delta' \pm \delta''|)$, d', d'' are associated with *their own* conjugates δ', δ''. That is, if all the resolutions of n in the form $n = d'\delta' + d''\delta''$ are

$$n = d_1'\delta_1' + d_1''\delta_1'' = d_2'\delta_2' + d_2''\delta_2'' = \cdots = d_k'\delta_k' + d_k''\delta_k'',$$

the left of (14) is

$$\sum_{i=1}^{k}[f(d_i' - d_i'', \delta_i' + \delta_i''|) - f(d_i' + d_i'', \delta_i' - \delta_i''|)], \tag{14a}$$

and not (for instance) what the single $\sum$ notation might equally well be used to express:

$$\sum_{i=1}^{k}\sum_{j=1}^{k}[f(d_i' - d_i'', \delta_j' + \delta_j''|) - f(d_i' + d_i'', \delta_j' - \delta_j''|)]. \tag{14b}$$

Wherever in the sequel $d, \delta, d', \delta', \cdots$ are associated together in an L-function, the d, δ, the $d', \delta', \cdots$ are conjugates; and the $\sum$ has the meaning of (14a), never of (14b). When we come to examine the elliptic and theta series for paraphrases, we shall see that paraphrases involving sums of the kind (14b) may be written down with great ease, while those of the Liouville kind, in which the sums are of the form (14a) while also readily deducible from certain of the expansions, are much less common, and therefore of correspondingly greater interest. The applications of the (14a) kind seem also to be of more importance than those of the (14b). It is interesting to note that paraphrases for sums of L-functions of degrees or orders > 1, in which the divisors are associated with their own conjugates as arguments of the L-functions, are implicit in Jacobi's memoirs on rotation, also in many of Hermite's earlier (and some of his later) papers on elliptic functions,* but not in the *Fundamenta Nova*. Nor do they occur in Schwarz' '*Sammlung*,' although many of the lists in that work may be prepared easily in a form suitable for the deduction of such paraphrases. A few of Kronecker's uncollected notes on elliptic function series also contain developments leading to (14a) paraphrases.

13. Passing to a more significant illustration of (14), we choose for $f(x, y|)$ the (implicitly) restricted L-function $\cos 2xu \cos 2yv$ in which u, v are parameters. After some simple reductions, (14) becomes:

$$\begin{aligned} 2\sum \sin 2(d'u + \delta' v)\sin 2(d''u - \delta'' v) &\\ &= \sum d(\cos 2du - \cos 2dv) \\ &+ \sum(\cot v \cos 2du \sin 2\delta v - \cot u \sin 2du \cos 2\delta v), \end{aligned} \tag{15}$$

* References to which are given in Part II where the series are considered.

which is the result of equating coefficients of $q^{n/2}$ in:

$$(16)\quad \frac{\vartheta_1'\,\vartheta_1(u+v)}{\vartheta_1(u)\,\vartheta_1(v)}\cdot\frac{\vartheta_1'\,\vartheta_1(u-v)}{\vartheta_1(u)\,\vartheta_1(-v)} \equiv \left[\vartheta_2\,\vartheta_3\frac{\vartheta_0(u)}{\vartheta_1(u)}\right]^2 - \left[\vartheta_2\,\vartheta_3\frac{\vartheta_0(v)}{\vartheta_1(v)}\right]^2.$$

In paraphrasing these steps are reversed. We start with (16), deduce (15), change (15) by separating trigonometric products into sums to the form (9), and paraphrase the result by (9a) immediately into (14). We note that, (15) being a very special case of (14); and (16), when considered merely as an identity between series, being deducible from (15) by a simple reversal of the steps which lead from (16) to (15), in a sense (14) includes (16) as a special case. There is, however, nothing in (14) that gives any immediate information concerning the periodicity, pseudo or real, of the quotients in (16). From this point of view, (16) is more general than (14). Against this may be put the following remarks of Liouville, which accord with the first view: "En effet mes formules se rattachent aussi à la théorie des fonctions elliptiques, seulement elles contiennent plutôt cette théorie qu'elles n'en dependent. . . . On n'a pas plus peine à y arriver au moyen des fonctions elliptiques.* Il y a là un genre de traduction que l'habitude rend facile" (19; p. 44). Again, (speaking of his general formulas): "Elles donnent naissance à des équations entre des séries qui contiennent comme cas particulier celles de la théorie des fonctions elliptiques" (19; p. 41).

From the present standpoint, (14), (15) are abstractly identical; but (14), as shown by numerous applications made of it by Liouville and others, presents the arithmetical information implicit in (15) or (16) in the more suggestive and usable form.

14. The diversity of the paraphrases is evident from the two following, selected at random from those found systematically in the sequel. Each is but one of several interpretations of the corresponding theta formula from which it is deduced.

(i) Write $t(x,y) \equiv \vartheta_1'\,\vartheta_1(x+y)/\vartheta_0(x)\,\vartheta_0(y)$, and denote by $t_w(x,y)$ the w-derivative of $t(x,y)$. Then,

$$t_x(x,y) - t_y(x,y) = t(x,y)\left[\frac{\vartheta_0'(y)}{\vartheta_0(y)} - \frac{\vartheta_0'(x)}{\vartheta_0(x)}\right],$$

which paraphrases into the elegant result:

$$m = m_1 + 2n_2;\quad n_2 = 2^{a_2} m_2;\quad m = d\delta,\quad m_1 = d_1\,\delta_1,\quad m_2 = d_2\,\delta_2:$$

* The process of proof which Liouville suggests for the deduction from elliptic functions of his paraphrases concerning L-functions of order 1 cannot be extended to deduce paraphrases in which the order exceeds 1. Hence it will not be followed here. Again, regarding its proposed application to the functions of order 1, Liouville's method assumes that the functions are expansible in a Fourier series, an assumption which would not be justified for L-functions as defined in § 1. Liouville does not indicate from what elliptic function identities his theorems may be deduced.

$$(17)\quad 4\sum[\Phi(\delta_1, d_1 - 2^{\alpha_2+1} d_2) + \Phi(d_1 + 2^{\alpha_2+1} d_2, \delta_1)] = \sum(d - \delta)\Phi(d, \delta),$$

where Φ is any one of the restricted L-functions, ϕ, ψ, χ, defined by:

$$\phi((x, y)|) = -\phi((y, x)|); \qquad \psi(x, y|) = -\psi(y, x|);$$
$$\chi(|x, y) = -\chi(|y, x).$$

The respective parities of ϕ, ψ, χ are $p(2|0)$, $p(1^2|0)$, $p(0|1^2)$: and the functions are (explicitly) restricted because subject to one other condition, here change of sign with interchange of variables, in addition to those of parity. Illustrative of general processes considered in §§ 25, 32, 36, the paraphrase for ϕ implies both the ψ and the χ paraphrases, which are independent; and from ψ, χ together, it is easy to infer ϕ. Special cases of interest arise for the choices, obviously legitimate:

$$\phi((x, y)|) = f((x, y)|) - f((y, x)|);$$
$$\psi(x, y|) = f(x, y|) - f(y, x|);$$
$$\chi(|x, y) = f(|x, y) - f(|y, x).$$

In fact, the Φ-paraphrase first presents itself for this ϕ; and by the processes cited, the ϕ-paraphrase may be at once replaced by the Φ-form.

(ii) One paraphrase of the identity

$$\vartheta_2 \frac{\vartheta_1(x)\vartheta_2(x)}{\vartheta_0(x)} \cdot \vartheta_2\vartheta_3 \frac{\vartheta_1(x)}{\vartheta_0(x)} = \vartheta_2^2\vartheta_3 \frac{\vartheta_1^2(x)\vartheta_2(x)}{\vartheta_0^2(x)}$$

is for a restricted linear separation of degree 2, and a function of parity $p(1|0)$:

$$m = l_1 + 2m_2 = d\delta \equiv 1 \mod 4; \qquad m_1 \gtreqless \sqrt{m}; \qquad m_2 = d_2\,\delta_2;$$
$$l_1 = d_1\,\delta_1 \equiv -1 \mod 4; \qquad d_1 > \sqrt{l_1}:$$

$$(18)\quad \sum\left[f\left(\frac{d_1+\delta_1}{2} - d_2|\right) - f\left(\frac{d_1+\delta_1}{2} + d_2|\right)\right] = \sum\left[F(m - m_1^2)f(m_1|) - \left(\frac{d-\delta}{2}\right)f\left(\frac{d+\delta}{2}\Big|\right)\right],$$

where $F(n)$ is, with the usual conventions, the number of uneven classes, for the determinant $-n$, of binary quadratic forms. Such formulas in which the L-functions are of orders and degrees > 1, containing those in which the order or degree is 1 as special cases, may be derived with great ease on combining the series in Part II, § 15, with those given by Humbert (loc. cit.), and form the subject of a separate paper. For the L-functions suitably specialized these formulas give, among others, the class number formulas of Kronecker, Hermite, Liouville and others.

15. To have an illustration of the processes considered in §§ 35, 36, we transcribe the following.

$$m = m_1^2 + 8n_2; \qquad n_2 = d_2\,\delta_2:$$

$$\begin{aligned}(19)\quad 2\sum(-1)^{(m_1+1)/2}\phi(2d_2-m_1, 2\delta_2+m_1|) = \epsilon(m)(-1)^{(\sqrt{m}-1)/2}\Big[\phi(1, \sqrt{m}|) \\ + \sum_{r=1}^{(\sqrt{m}-1)/2}\{\phi(2r-1, \sqrt{m}|) - \phi(\sqrt{m}, 2r-1|)\}\Big],\end{aligned}$$

where $\epsilon(n) = 1$ or 0 according as n is or is not a square; and $\phi(x, y|)$ is subject to the restriction $\phi(x, y|) = -\phi(y, x|)$. The separation here is quadratic and unrestricted of degree 2. For the same separation, we find by a process of linear transformation of the variables in (19), the following transform of it:

$$\begin{aligned}&2\sum(-1)^{(m_1+1)/2}\phi((d_2+\delta_2, d_2-\delta_2-m_1, 2d_2-m_1, 2\delta_2+m_1)|) \\ &= \epsilon(m)(-1)^{(\sqrt{m}-1)/2}\Big[\phi\Big(\Big(\frac{1+\sqrt{m}}{2}, \frac{1-\sqrt{m}}{2}, 1, \sqrt{m}\Big)\Big|\Big) \\ (19a)\qquad &+ \sum_{r=1}^{(\sqrt{m}-1)/2}\Big\{\phi\Big(\Big(r+\frac{1+\sqrt{m}}{2}, r+\frac{1-\sqrt{m}}{2}, 2r+1, \sqrt{m}\Big)\Big|\Big) \\ &+ \phi\Big(\Big(r-\frac{1+\sqrt{m}}{2}, r-\frac{1-\sqrt{m}}{2}, 2r-1, \sqrt{m}\Big)\Big|\Big)\Big\}\Big],\end{aligned}$$

where ϕ is subject to the restrictions, forming a canonical set (§ 5):

$$\phi((x, y, z, w)|) = \phi((y, x, z, -w)|) = -\phi((x, -y, w, z)|).$$

The transformation for passing from (19) to (19*a*) is briefly indicated in § 35 (end). It is a good exercise in the bar notation to verify (19), (19*a*) for $m = 17, 25$.

For the same system of arguments, $d_2 + \delta_2$, etc., as in (19*a*), the linear transformation converting (19) into (19*a*) gives also 15 more paraphrases, seven of which are for restricted functions, and eight for unrestricted. This indicates the fertility of the method. These paraphrases, together with an infinity more, are all consequences of the obvious identity:

$$\vartheta_1(x-y)\cdot\frac{\vartheta_1'\,\vartheta_1(x+y)}{\vartheta_1(x)\vartheta_1(y)} + \vartheta_1(x+y)\cdot\frac{\vartheta_1'\,\vartheta_1(x-y)}{\vartheta_1(x)\vartheta_1(-y)} = 0.$$

From this identity, when $\vartheta_1'\,\vartheta_1(x \pm y)/\vartheta_1(x)\vartheta_1(\pm y)$ are replaced by their Fourier expansions given in Part II (or written out independently in the usual way), and § 36, the origin of the restriction imposed upon $\phi(x, y|)$ in (19), is sufficiently evident.

We may mention here some general results which form part of a later investigation. The example just given illustrates the concept of a class of paraphrases; two paraphrases being equivalent when either may be transformed into the other by a linear transformation of the variables, the coefficients of the transformation being rational. All paraphrases equivalent to one another constitute a class. In each class there is one and only one subclass, the reduced class, such that the order of the functions in any member of the class cannot be further reduced by linear transformations on the variables, and such that any member of the class may be transformed into any other by a transformation with coefficients ± 1 on the variables. The reduced class is said to be represented by any one of its members. In the above, (19) represents a reduced class; and (19*a*) is equivalent to (19). It is easily seen from §§ 31, 32 that (19*a*) includes (19) as a special case; but it is less obvious that (19) includes (19*a*).

16. Proofs for most of Liouville's general formulas will be found in the cited papers of Smith, Pepin, Mathews and Meissner. All of these use the method of Dirichlet in modified or extended form, to which Liouville himself repeatedly refers; but this method (cf. Bachmann, p. 366), offers no suggestion either as to proper assumptions to be made regarding the parity or restrictions of the functions, or to the constitution of the separations for a given function. It is, in fact, a process of à posteriori verification. By the method of paraphrase the questions concerning the nature of the functions and separations receive immediate answers on an examination of the class of series from which the paraphrases are derived. As it has been suggested by Bachmann (p. 433) that the source of Liouville's theorems was a consideration of the transformation of bilinear forms on four variables (as given, for example by Kronecker, *Werke*, vol. 1, p. 143), we shall state what seem the principal advantages of deriving them as paraphrases primarily of the elliptic-theta identities. Considering, for example, (16), it may be made, by simple algebraic or analytical transformations, to yield many more paraphrases in addition to (14), some of which are for quadratic separations, some for separations of degrees $3, 4$, and others for restricted or unrestricted functions of orders $1, 2, 3, 4, \cdots$ integrated over linear separations of degree 2. Even with the end-results before us, it is a matter of considerable difficulty to transform these into each other by Dirichlet's method as used (in amplified form) by Smith, Pepin, Meissner and others; and this method would seem to be the natural modification of Kronecker's transformation processes to be used for this purpose. But the most important advantage is that we have in the method of paraphrase what that of Dirichlet has not yet given, a direct and powerful means for the discovery of new paraphrases, which severally, as Bachmann says of this class of theorems (l. c., p. 366), " eine schier unerschöpfliche Fundgrube für zahlen-

theoretische Sätze darbieten." We shall not derive all of Liouville's general formulas en bloc by the method of paraphrase, although this may easily be done if desired, but shall derive them incidentally as they arise in applying the following developments to the elliptic and theta series.

17. An inspection of the numerical example in § 11, reveals the important fact, otherwise obvious from §§ 1, 6, that (14) is ultimately an identity between sets of absolute values of integers; two sets, $(|a_1|, |a_2|)$, $(|b_1|, |b_2|)$, being identical when and only when $|a_1| = |b_1|$ and $|a_2| = |b_2|$. The like, considerably generalized, will be evident for functions of parity

$$p(a_1, a_2, \cdots, a_r | b_1, b_2, \cdots, b_s).$$

Hence we next examine the properties of sets of matrices of absolute values. On them we shall base a proof, by new but simple considerations, of the legitimacy of the paraphrase process outlined in § 6, in sufficient detail to derive all the paraphrases first arising in the theory of the elliptic and theta functions. The process for functions of parity

$$p(a_1, a_2, \cdots, a_r | b_1, b_2, \cdots, b_s)$$

will appear as a corollary of that for functions of parity $p(a_1, a_2, \cdots, a_r | 0)$, and the latter as a corollary of the process for $p(a_1 | 0)$, which in turn follows from that for $p(1|0)$.

II. Sets of matrices and L-functions

18. The equality between matrices, $(a_1, a_2, \cdots, a_r) = (a_1', a_2', \cdots, a_s')$, implies $s = r$ and $a_i = a_i'$, $(i = 1, \cdots, r)$. If $a_i = 0$, $(i = 1, \cdots, r)$, the matrix is the zero matrix, $(0, 0, \cdots, 0)$, $\equiv (0)_r$. A set is a collection of things independently of their order. We shall write the matrix of absolute values

$$(|x_{i1}|, |x_{i2}|, \cdots, |x_{ir}|) \equiv (|x_i|)_r;$$

and the set of $(n - j)$ matrices

$$(|x_{j+1}|)_r, (|x_{j+2}|)_r, \cdots, (|x_n|)_r \qquad (n \geqq j + 1, j \geqq 0),$$

will be denoted by either of

$$\int_j^n (|x_i|)_r, \int_j^n (|x_{i1}|, |x_{i2}|, \cdots, |x_{ir}|);$$

and when all the $(n - j)$ matrices are zero, the set will be written, as convenient, in any of the forms

$$\int_j^n (0)_r, \int_j^n (0, 0, \cdots, 0), (n - j)\int (0)_r, (n - j)\int (0, 0, \cdots, 0),$$

the 0 in $(0, 0, \cdots, 0)$ being repeated r times. Two sets are equal:

$$\int_j^n (|x_i|)_r = \int_{j'}^{n'} (|x_i'|)_{r'} \tag{20}$$

when and only when the $(|x_i|)_r$ are a permutation of the $(|x_i'|)_{r'}$; and hence in particular only when $r' = r$ and $n' - j' = n - j$.

19. The sum (logical sum) of two sets is that set which consists of all the matrices in either set. Hence addition of sets is commutative and associative, and

$$\int_j^\lambda (|x_i|)_r + \int_\lambda^n (|x_i|)_r = \int_j^n (|x_i|)_r \qquad (0 \overline{\overline{<}} j < \lambda < n). \tag{21}$$

20. An obvious property of sets for which we shall have frequent use is that the same $|\alpha|$ may be inserted in homologous places of equal sets without destroying their equality, viz., (20) implies

$$\begin{aligned} &\int_j^n (|x_{i1}|, \cdots, |x_{is-1}|, |\alpha|, |x_{is}|, \cdots, |x_{ir}|) \\ &\qquad = \int_{j'}^{n'} (|x_{i1}'|, \cdots, |x_{is-1}'|, |\alpha|, |x_{is}'|, \cdots, |x_{ir}'|). \end{aligned} \tag{22}$$

Again, from the definitions, if $p, q, \cdots, t$ are any of the integers $1, 2, \cdots, r$, (20) implies

$$\int_j^n (|x_{ip}|, |x_{iq}|, \cdots, |x_{it}|) = \int_{j'}^{n'} (|x_{ip}'|, |x_{iq}'|, \cdots, |x_{it}'|). \tag{23}$$

21. Immediately from the definitions of §§ 1, 18:

LEMMA 1. *If* $\int_0^n (|x_i|)_r = \int_0^n (|x_i'|)_r$, *then*

$$\sum_{i=1}^n f(x_{i1}, x_{i2}, \cdots, x_{ir}|) = \sum_{i=1}^n f(x_{i1}', x_{i2}', \cdots, x_{ir}'|).$$

In the same way, or as an obvious corollary:

$$\begin{aligned} &\int_0^n (|x_i|)_r + \int_0^t (|y_i|)_r + \cdots + \int_0^p (|z_i|)_r \\ &\qquad = \int_0^n (|x_i'|)_r + \int_0^t (|y_i'|)_r + \cdots + \int_0^p (|z_i'|)_r \end{aligned}$$

implies

$$\begin{aligned} &\sum_{i=1}^n f(x_{i1}, x_{i2}, \cdots, x_{ir}|) + \sum_{i=1}^t f(y_{i1}, y_{i2}, \cdots, y_{ir}|) \\ &\qquad + \cdots + \sum_{i=1}^p f(z_{i1}, z_{i2}, \cdots, z_{ir}|) \end{aligned}$$

$$= \sum_{i=1}^{n} f(x'_{i1}, x'_{i2}, \cdots, x'_{ir}|) + \sum_{i=1}^{t} f(y'_{i1}, y'_{i2}, \cdots, y'_{ir}|) + \cdots + \sum_{i=1}^{p} f(z'_{i1}, z'_{i2}, \cdots, z'_{ir}|).$$

22. For the passage from circular to L-functions the following lemmas* are fundamental.

LEMMA 2. *If the a_i, b_j are integers $\gtrless 0$, and if there is an infinity of odd integers n, > 0, for which*

$$\sum_{i=1}^{r} a_i^n = \sum_{j=1}^{s} b_j^n,$$

then $s = r$, and the a_i are a permutation of the b_j.

LEMMA 3. *If a_i, $(i = 1, 2, \cdots, r)$ are integers $\gtreqless 0$, and b_j, $(j = 1, 2, \cdots, s)$ integers $\gtrless 0$; and if for all integral values > 0 of n,*

$$\sum_{i=1}^{r} a_i^{2n} = \sum_{j=1}^{s} b_j^{2n},$$

then, (i): *$r \geqq s$, and precisely $(r - s)$ of the $a_i = 0$.* (ii) *If, without loss of generality, the s non-zero a_i are $a_1, a_2, \cdots, a_s$, then by Lemma 2, the $a_1^2, a_2^2, \cdots, a_s^2$ are a permutation of the $b_1^2, b_2^2, \cdots, b_s^2$; and hence, by Lemma* 1:

(iii) $$\sum_{i=1}^{s} f(a_i|) = \sum_{j=1}^{s} f(b_j|); \qquad \sum_{i=1}^{r} f(a_i|) = (r - s) f(0|) + \sum_{j=1}^{s} f(b_j|).$$

The first part is an immediate consequence of Lemma 2.

LEMMA 4. *If a_{kj}, α_{lj}, m_k, μ_k, $(k = 1, \cdots, r; j = 1, \cdots, s; l = 1, \cdots, t)$, are integers $\gtrless 0$, and if for all integral values > 0 of n_j,*

$$\sum_{k=1}^{r} m_k a_{k1}^{2n_1} a_{k2}^{2n_2} \cdots a_{ks}^{2n_s} = \sum_{k=1}^{t} \mu_k \alpha_{k1}^{2n_1} \alpha_{k2}^{2n_2} \cdots \alpha_{ks}^{2n_s}, \tag{24}$$

then

$$\sum_{k=1}^{r} m_k f(a_{k1}, a_{k2}, \cdots, a_{ks}|) = \sum_{k=1}^{t} \mu_k f(\alpha_{k1}, \alpha_{k2}, \cdots, \alpha_{ks}|). \tag{25}$$

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* Lemma 2 was proposed as a problem by the writer in the American Mathematical Monthly; and a proof given ibid., vol. 24 (1917), p. 288, by Professor E. Swift. An independent proof of Lemma 3 is readily deduced from Newton's formulas in the theory of equations, on considering the a_i, b_j as the roots of two equations of degrees r, s respectively, and then showing these equations identical by the given conditions. Hence, (i), (ii) of Lemma 3 being sufficient for the proof of all following Lemmas, it follows that the paraphrase method depends only upon finite processes. The lemmas may be generalized by lightening the conditions; but as such generalizations have no application in the sequel they have been omitted. Professor C. F. Gummer has given (in a paper which has not yet appeared) some interesting developments of Lemma 1, based upon the extension of Descartes' rule of signs to transcendental equations. In particular he has shown that $\Sigma_{i=1}^{i=r} a_i^n = \Sigma_{j=1}^{j=s} b_j^n$ $(r \geqq s)$ for r distinct values of n are necessary and sufficient conditions for the identity of the a_i with the b_j, when n is odd.

Without loss of generality we may assume $m_k, \mu_k > 0$, the other case being immediately reducible by transposition to this. By repeating the terms a proper number of times the coefficients m_k, μ_k may be taken as unity. Now putting $n_i = n\nu_i$, $(i = 1, 2, \cdots, s)$, where ν_i is an arbitrary integer > 0, we infer from (24) by Lemma 3, a set of identities of the form

$$a_{i1}^{2\nu_1} a_{i2}^{2\nu_2} \cdots a_{is}^{2\nu_s} = \alpha_{j1}^{2\nu_1} \alpha_{j2}^{2\nu_2} \cdots \alpha_{js}^{2\nu_s} \quad (1 \gtreqless i \gtreqless r;\ 1 \gtreqless j \gtreqless t), \tag{25a}$$

valid for all integral values > 0 of the $\nu_1, \nu_2, \cdots, \nu_s$. Replacing in (25$a$) any one of the exponents by its double, say ν_r by $2\nu_r$, we get an identity, which, with (25a), gives, provided the a's and α's are not zero, for all $\nu_r > 0$, $a_{ir}^{2\nu_r} = \alpha_{jr}^{2\nu_r}$; and hence $a_{ir}^2 = \alpha_{jr}^2$. Hence (25a) gives $(|a_i|)_s = (|\alpha_j|)_s$, and the lemma follows at once by § 21. Obviously the condition $m_k, \mu_k \gtrless 0$ may be replaced by $m_k, \mu_k \gtreqless 0$; a remark of importance presently in passing to L-functions of parity $p(0|b_1, b_2, \cdots, b_s)$. We point out expressly that the replacing of the condition $a_{kj}, \alpha_{lj} \gtrless 0$ by $a_{kj}, \alpha_{lj} \gtreqless 0$, would invalidate the proof. There is a fundamental distinction between paraphrases involving zero matrices and those which do not. In passing from circular to L-functions, this amounts to distinguishing the paraphrase of homogeneous polynomials in sines and cosines from the paraphrase of the non-homogeneous. We take the former case first.

23. For the a, α, m, μ as in Lemma 4:

LEMMA 5. *If for all values of* $x_1, x_2, \cdots, x_s$,

$$\begin{aligned}\sum_{k=1}^{r} m_k \cos a_{k1} x_1 \cos a_{k2} x_2 \cdots \cos a_{ks} x_s \\ = \sum_{k=1}^{t} \mu_k \cos \alpha_{k1} x_1 \cos \alpha_{k2} x_2 \cdots \cos \alpha_{ks} x_s,\end{aligned} \tag{26}$$

then (25) *holds.*

For, equating coefficients of $x_1^{2n_1} x_2^{2n_2} \cdots x_s^{2n_s}$ in (26) we get (24).

LEMMA 6. *The notation being as in Lemma* 5, *and for all values of* $x_1, x_2, \cdots, x_s$,

$$\begin{aligned}\sum_{k=1}^{r} m_k \sin a_{k1} x_1 \sin a_{k2} x_2 \cdots \sin a_{ks} x_s \\ = \sum_{k=1}^{t} \mu_k \sin \alpha_{k1} x_1 \sin \alpha_{k2} x_2 \cdots \sin \alpha_{ks} x_s,\end{aligned} \tag{27}$$

then

$$\sum_{k=1}^{r} m_k\, g(|a_{k1}, a_{k2}, \cdots, a_{ks}) = \sum_{k=1}^{t} \mu_k\, g(|\alpha_{k1}, \alpha_{k2}, \cdots, \alpha_{ks}).$$

For, from the definitions in § 1 we may write

$$g(|z_1, z_2, \cdots, z_s) \equiv z_1 z_2 \cdots z_s f(z_1, z_2, \cdots, z_s|).$$

Operating on both sides of (27) with $\partial^s/\partial x_1\,\partial x_2\cdots\partial x_s$ reduces (27) to (26) with $m_k\,a_{k1}\cdots a_{ks}$, $\mu_k\,\alpha_{k1}\cdots\alpha_{ks}$ in place of m_k, μ_k respectively; and by Lemma 5 we deduce (25) with m_k, μ_k similarly changed:

$$\sum_{k=1}^{r} m_k\,a_{k1}\,a_{k2}\cdots a_{ks} f(a_{k1}, a_{k2}, \cdots, a_{ks}|)$$
$$= \sum_{k=1}^{t} \mu_k\,\alpha_{k1}\,\alpha_{k2}\cdots\alpha_{ks} f(\alpha_{k1}, \alpha_{k2}, \cdots, \alpha_{ks}|).$$

On replacing in this $z_1 z_2\cdots z_s f(z_1, z_2, \cdots, z_s|)$ by $g(|z_1, z_2, \cdots, z_s)$, the Lemma follows. By an obvious change in notation, Lemma 6 may be restated in the more convenient form:

The a_{ki} denoting integers $\geqq 0$, and the m_k integers $\gtreqless 0$, the identity in the x_i,

$$\sum_{k=1}^{r} m_k \sin a_{k1} x_1 \sin a_{k2} x_2 \cdots \sin a_{ks} x_s = 0$$

implies

$$\sum_{k=1}^{r} m_k\, g(|a_{k1}, a_{k2}, \cdots, a_{ks}) = 0.$$

Clearly, the preceding Lemmas may be similarly restated. In the same way, the proof of Lemma 7 can be based on Lemma 5 by operating on the identity in Lemma 7 by $\partial^s/\partial y_1\,\partial y_2\cdots\partial y_s$:

LEMMA 7. *The a_{ki}, b_{kj} denoting integers $\geqq 0$, and the m_k integers $\gtreqless 0$, the identity in the x_i, y_j,*

$$\sum_{k=1}^{n} m_k \left(\prod_{i=1}^{r} \cos a_{ki} x_i \cdot \prod_{j=1}^{s} \sin b_{kj} y_j \right) = 0,$$

implies

$$\sum_{k=1}^{n} m_k f(a_{k1}, a_{k2}, \cdots, a_{kr} | b_{k1}, b_{k2}, \cdots, b_{ks}) = 0.$$

A little reflection will show that the method of proof used in Lemmas 6, 7 is applicable when and only when the a_{ki}, b_{kj} are rational.* As there is no essential gain in generality by considering rational rather than integral variables in the paraphrases, we ignore the former.

24. The terms in any one of the trigonometric identities paraphrased in § 23 are all of the same parity. Thus, in Lemma 7, the parity of each sine-cosine term is $p(1^r|1^s)$. Passing to an important generalization we now consider the paraphrase of homogeneous sine-cosine identities whose terms are of several parities. In the following the notation is based upon that of § 6, for the proofs of the processes there stated are intimately connected with Lemma 8, next considered.

* Cf. § 35, footnote.

LEMMA 8. *Let the set of ω independent variables, $z_1, z_2, \cdots, z_\omega$, be separated in N ways into two sets, X_n, Y_n:*

$$X_n \equiv x_{1n}, x_{2n}, \cdots, x_{r_n n}; \qquad Y_n \equiv y_{1n}, y_{2n}, \cdots, y_{s_n n} \quad (n = 1, 2, \cdots, N),$$

so that $r_n + s_n = \omega$, and the x_{in}, y_{jn} are a permutation of the z_k. Let all the X_n, and consequently all the Y_n, be distinct among themselves, two sets being identical only when all the variables in either are also in the other. Write

$$\phi_m(n) \equiv \prod_{i=1}^{r_n} \cos \alpha_{min} x_{in} \cdot \prod_{j=1}^{s_n} \sin \beta_{mjn} y_{jn}; \tag{28}$$

$$\psi_m(n) \equiv f_n(\alpha_{m1n}, \alpha_{m2n}, \cdots, \alpha_{mr_n n} | \beta_{m1n}, \beta_{m2n}, \cdots, \beta_{ms_n n}); \tag{29}$$

$$\Phi(n) \equiv \sum_{m=1}^{t_n} c_{mn} \phi_m(n); \qquad \Psi(n) \equiv \sum_{m=1}^{t_n} c_{mn} \psi_m(n). \tag{30}$$

Then, the $\alpha_{min}, \beta_{mjn}$ denoting integers $\gtrless 0$, and the c_{mn} integers $\gtreqless 0$, the identity in the z_k,

$$\sum_{n=1}^{N} \Phi(n) = 0 \tag{30a}$$

implies

$$\sum_{n=1}^{N} \Psi(n) = 0, \tag{31}$$

and it will be shown that each term of this sum is zero, viz.,

$$\Psi(n) = 0 \qquad (n = 1, 2, \cdots, N). \tag{31a}$$

For, the X_n being distinct, after operation on (30a) with

$$\partial^{s_n}/\partial y_{1n}\, \partial y_{2n} \cdots \partial y_{s_n n},$$

every $\Phi'(k)$, $k \neq n$ will involve at least one sine factor in each of its terms, $c_{mk}\phi'_m(k)$; while each term, $c_{mn}\phi'_m(n)$, of $\Phi'(n)$ will be c_{mn} times a product of ω cosines. Hence in the differentiated (30a) only the terms arising from $\Phi(n)$ contribute to the coefficient of $z_1^{2p_1} z_2^{2p_2} \cdots z_\omega^{2p_\omega}$, $p_i > 0$, $(i = 1, 2, \cdots, \omega)$; and precisely as in the proofs of Lemmas 5, 6, 7, we conclude that $\Psi(n) = 0$, $(n = 1, 2, \cdots, N)$, and hence

$$\sum_{n=1}^{N} \Psi(n) = 0.$$

It is essential for our present purpose to note that (31a) is a system of identities for N general L-functions. That is, $f_1, f_2, \cdots, f_N$ in (31a) may denote the same or different L-functions, which, except that their parities are respectively identical with those of the $\phi_m(n)$, $(n = 1, 2, \cdots, N)$, are wholly arbitrary as defined in § 1.

25. Before proving the general result it will be well for clearness to give the proof in detail for a very simple case, unencumbered by the notation. The reasoning in the general case is of exactly the same kind. We shall now show that

$$(32)\qquad \sum c_i \cos(a_i x + b_i y) = 0$$

for all values of x, y implies

$$(32a)\qquad \sum c_i f((a_i, b_i)|) = 0,$$

the c_i denoting integers $\gtreqless 0$, and the a_i, b_i integers $\gtrless 0$. The significance of the parenthesis (a_i, b_i) will be evident on referring to § 1 and the examples of the bar notation there given.

(i) From (32):

$$\sum c_i [\cos a_i x \cos b_i y - \sin a_i x \sin b_i y] = 0;$$

whence, by Lemma 8:

$$(33)\qquad \sum c_i f_1(a_i, b_i|) = 0; \qquad \sum c_i f_2(|a_i, b_i) = 0,$$

in which f_1, f_2 are arbitrary, of the indicated parities $p(1^2|0)$, $p(0|1^2)$.

(ii) Now, it is shown* in the proof of the theorem stated in § 3 that the parities $p(1^\alpha|1^\beta)$ of the L-functions appropriate for the stated linear expression of the (general) $f(A_{m1}, A_{m2}, \cdots, A_{mr}|B_{m1}, B_{m2}, \cdots, B_{ms})$ in § 6, whose parity is $p(a_1, a_2, \cdots, a_r|b_1, b_2, \cdots, b_s)$, are precisely those of the several sine-cosine terms in the addition-theorem development and subsequent distribution of products in

$$(34)\qquad \prod_{i=1}^{r} \cos\left(\sum_{n=1}^{a_i} \alpha_{min} x_{in}\right) \cdot \prod_{j=1}^{s} \sin\left(\sum_{n=1}^{b_j} \beta_{mjn} y_{jn}\right).$$

It is shown, moreover, that the appropriate L-functions are of the form (29), corresponding to the individual terms of (34), the latter being of the form (28).

(iii) In the present case we have, therefore, that $f((a_i, b_i)|)$ is a linear function of suitably chosen $f_1(a_i, b_i|)$, $f_2(|a_i, b_i)$, say

$$(35)\qquad f((a_i, b_i)|) = k_1 f_1'(a_i, b_i|) + k_2 f_2'(|a_i, b_i).\dagger$$

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* Bulletin of the American Mathematical Society, vol. 25 (1918–19), p. 313.

† The actual forms of f_1', f_2' are given by:

$$2f_1'(a_i, b_i|) = f((a_i, b_i)|) + f((a_i, -b_i)|),$$

$$2f_2'(|a_i, b_i) = f((a_i, b_i)|) - f((a_i, -b_i)|); \qquad k_1 = k_2 = 1;$$

but these are not essential to the proof. The same applies to the general case: it is not necessary to have the linear expressions; it is sufficient to know that they exist.

Multiplying (35) throughout by c_i, and summing:

$$\sum c_i f((a_i, b_i)|) = k_1 \sum c_i f_1'(a_i, b_i|) + k_2 \sum f_2'(|a_i, b_i). \tag{35a}$$

But (33) holds for f_1, f_2 arbitrary of the indicated parities. Hence the right side of (35a) vanishes, and this establishes (32a).

26. Turning to § 6, we may write (8) in the form (30a) by the process outlined in § 25 (ii). In this case, $c_{mn} \equiv c_m$; and it is easy to see that $N = 2^{\omega-\delta}$, where ω, δ are as in § 2, (5). By Lemma 8 we get from (30a), corresponding to (31a):

$$\Psi(n) = 0 \qquad (n = 1, 2, 3, \cdots, 2^{\omega-\delta}). \tag{36}$$

Choosing for the f_n in (36) the L-functions f_n' appropriate for the linear expressions (§ 3) of $f(A_{m1}, A_{m2}, \cdots, A_{mr}|B_{m1}, B_{m2}, \cdots, B_{ms})$, and multiplying as in § 25 (iii) the successive identities of (36) by the appropriate constants, k_n, of the linear expression, we get on adding the results as in the special (35a), the identity (8a) of § 6.

27. The proofs for (9a), (10a) in the homogeneous case are precisely similar to that for (8a), and need not be written out. Again we emphasize that (8a), (9a), (10a) have so far been proved only for the case in which the α_{min}, β_{mjn} are non-zero integers. We next (cf. § 22) consider in less detail the paraphrase process for non-homogeneous sine-cosine polynomials. We shall give only so much of it as suffices for the paraphrases of identities first arising in the elliptic and theta functions; this includes all of the Liouville paraphrases and many more of kinds distinct from his. The most general non-homogeneous case may be similarly treated, but the notation becomes considerably more complicated, and it is best, by using the linear transformations outlined in § 35, to refer back to the homogeneous case. By the method of sets, much information not otherwise evident, is revealed concerning the ultimate nature of the paraphrases.

Lemma 9. *The identity in* x_1, x_2:

$$\begin{aligned}\sum_{i=1}^{r} (\cos a_{i1} x_1 \cos a_{i2} x_2 - \cos a_{i3} x_1 \cos a_{i4} x_2)& \\ = \sum_{i=1}^{s} (\cos a_{i5} x_2 - \cos a_{i6} x_1)&\end{aligned} \tag{37}$$

where the a's are integers, and $a_{i2}, a_{i3}, a_{i5}, a_{i6} \gtrless 0$, *implies*

$$\sum_{i=1}^{r} [f(a_{i1}, a_{i2}|) - f(a_{i3}, a_{i4}|)] = \sum_{i=1}^{s} [f(0, a_{i5}|) - f(a_{i6}, 0|)].$$

For, equating coefficients of x_1^{2n}, x_2^{2n}, $x_1^{2n_1}$, $x_2^{2n_1}$, $(n, n_1, n_2 > 0)$ in (37), we get:

$$\sum_{i=1}^{r} a_{i1}^{2n} + \sum_{i=1}^{s} a_{i6}^{2n} = \sum_{i=1}^{r} a_{i3}^{2n}, \tag{38}$$

$$\sum_{i=1}^{r} a_{i4}^{2n} + \sum_{i=1}^{s} a_{i5}^{2n} = \sum_{i=1}^{r} a_{i2}^{2n}, \tag{39}$$

$$\sum_{i=1}^{r} a_{i1}^{2n_1} a_{i2}^{2n_2} = \sum_{i=1}^{r} a_{i3}^{2n_1} a_{i4}^{2n_2} \tag{40}$$

for all integral $n, n_1, n_2 > 0$. Since* the a's, except perhaps some of the a_{i1}, a_{i4}, are not zero, we find from (38), (39) and Lemma 3 that precisely s each of the a_{i1}, a_{i4} are zero. Moreover as in Lemma 4 we find that in (40) *the pairs* $(|a_{i1}|, |a_{i2}|)$ *for which* $a_{i1} \neq 0$ *are merely a permutation of the pairs* $(|a_{i3}|, |a_{i4}|)$ *for which* $a_{i4} \neq 0$. Hence if $a_{i1} = 0$, then $a_{i4} = 0$. Suppose this true for $i = 1, \cdots, s$. Then

$$\sum_{i=1}^{r} [f(a_{i1}, a_{i2}|) - f(a_{i3}, a_{i4}|)] = \sum_{i=1}^{s} [f(0, a_{i2}|) - f(a_{i3}, 0|)].$$

But again from (38), (39) and Lemma 3 we see that the first s of the $|a_{i2}|$ are the $|a_{i5}|$, and the first s of the $|a_{i3}|$ are the $|a_{i6}|$, which proves the paraphrase. From this there is obviously the corollary:

LEMMA 10. *With the notation of Lemma 9, and b_i, c_i integers* $\geqq 0$,

$$\sum_{i=1}^{r} b_i (\cos a_{i1} x_1 \cos a_{i2} x_2 - \cos a_{i3} x_1 \cos a_{i4} x_2) = \sum_{i=1}^{s} c_i (\cos a_{i5} x_2 - \cos a_{i6} x_1) \tag{45}$$

for all values of x_1, x_2 implies

$$\sum_{i=1}^{r} b_i [f(a_{i1}, a_{i2}|) - f(a_{i3}, a_{i4}|)] = \sum_{i=1}^{s} c_i [f(0, a_{i5}|) - f(a_{i6}, 0|)]. \tag{46}$$

It is not difficult to prove this also in the case $b_i, c_i \gtreqless 0$; but this is not an immediate consequence of Lemma 9.

28. By a process of frequent use we get from Lemma 10 an important special case as a corollary. Obviously (46) is true for all integers for which (45) is true. But (45) is true for the integers $a_{i2} = a_{i4} = a_{i5} = 0$ (the other integers being the same), since this is the form which (45) takes when $x_2 = 0$. Hence

LEMMA 11. *The identity in* x_1:

$$\sum_{i=1}^{r} b_i (\cos a_{i1} x_1 - \cos a_{i2} x_1) = \sum_{i=1}^{s} c_i (1 - \cos a_{i3} x_1)$$

* An alternative proof by the method of sets is somewhat longer and has been omitted, but is not without interest. It may easily be reconstructed from (38)–(40), Lemma 3, and § 20 (21), (22). (To save renumbering formulas, (45) follows (40).)

implies

$$\sum_{i=1}^{r} b_i[f(a_{i1}|) - f(a_{i2}|)] = \sum_{i=1}^{s} c_i[f(0|) - f(a_{i3}|)].$$

This* may be proved independently by Lemma 9; or it follows almost at once from Lemma 3. It is of interest as covering the first paraphrase stated by Liouville, which follows from Jacobi's series for $\operatorname{sn}^2 u$ from the identity $\operatorname{sn} u \times \operatorname{sn} u = \operatorname{sn}^2 u$, on substituting for $\operatorname{sn} u$, $\operatorname{sn}^2 u$ their Fourier developments. The generalizations to functions of two variables in Liouville's first five memoirs follow from Lemmas 9, 10 applied to the appropriate series, which also were given by Jacobi, but not in the *Fundamenta Nova*. The formulas of Liouville's sixth memoir are paraphrases of

$$\operatorname{sn}^3 u = \operatorname{sn}^2 u \times \operatorname{sn} u = \operatorname{sn} u \times \operatorname{sn} u \times \operatorname{sn} u.$$

29. By differentiation as in §§ 23, 24 we may make the cases of non-homogeneous paraphrases for functions of parity $p(0|2)$, $p(0|a)$, $\cdots$ depend upon those for functions of parity $p(2|0)$, $p(a|0)$, $\cdots$. We shall consider it unnecessary to prove formally the legitimacy of paraphrasing non-homogeneous identities differing but slightly from those considered in §§ 27, 28; and for the present we may omit the paraphrase of identities involving tangents, cotangents, secants and cosecants, these depending upon sixteen simple identities which will be given with the elliptic series, and in no respect introducing considerations different in principle from the paraphrase of sine-cosine identities. We remark, however, that they are the source of all such paraphrases as those of Liouville which involve sums of *L*-functions whose arguments are in arithmetical or geometrical progression, such, for instance, as (14), (19), (19*a*).

III. Elementary transformations

30. Examining Liouville's theorems we note his frequent use of such transformations as $f_1(z|) = (-1)^{(z+1)/2} f_2(|z)$, where z is an odd integer, and f_1, f_2 arbitrary of the parities indicated. These are immediate translations of the effects of replacing the x-variables in the elliptic or theta identities from

* G. Humbert, (Paris Comptes Rendus, vol. 150; 21 Fév. 1910, p. 433) uses what is essentially a special case of Lemma 11, and refers for proof to a theorem of Borel: "*There exists an entire function of x, taking for integral values of the variable the same values as any given function.*" Liouville functions being not necessarily entire, Borel's theorem cannot be used to prove Lemma 11; and in any event it is preferable here to use some method, such as the above, which is applicable to functions of any number of variables. On the other hand, some writers (cf. Bachmann, l. c., p. 295), regard the paraphrase to functions of a single variable as self-evident. Our lemmas are, no doubt, obvious; but in view of the indicated difference of opinion as to what is or is not obvious in this regard, it seemed best to offer proofs for all cases.

which the paraphrases are derived by $x + \pi/2$, or in Weierstrass' notation, by $x + 1/2$. In the notation of §7, all such transformations follow from that next given, which may be verified by inspection. The functions in any pair are of the same parity, and the sign of transformation, $\sim$, indicates that in any paraphrase either function separated by the sign $\sim$ may be replaced throughout by the other, provided, of course, that the evenness or oddness of the integral arguments of the functions in the paraphrase is constant throughout. Thus, $\sum_i f(|m_i, 2n_i) = 0$ may be replaced by $\sum_i (-1)^{(m_i-1)/2} f(m_i|2n_i) = 0$;

$$\sum_i -(1)^{n_i} f(|m_i, 2n_i) = 0; \qquad \sum_i (-1)^{[n_i+(m_i-1/2)]} f(m_i|2n_i) = 0.$$

It is readily seen that if

$$\xi \equiv (m_1, m_2, \cdots, m_r, m_1', m_2', \cdots, m_s', 2n_1, 2n_2, \cdots, 2n_k, 2n_1', 2n_2', \cdots, 2n_t');$$

$$M = m_1 + m_2 + \cdots + m_r; \qquad N = n_1 + n_2 + \cdots + n_k:$$

then

$$f(|\xi) \sim (-1)^{(M-1)/2} f(\xi|), \quad f(\xi|) \sim (-1)^{(M+1)/2} f(|\xi), \quad \text{if } M \equiv 1 \bmod 2;$$

$$f(|\xi) \sim (-1)^{M/2} f(|\xi), \qquad f(\xi|) \sim (-1)^{M/2} f(\xi|), \qquad \text{if } M \equiv 0 \bmod 2;$$

$$f(\xi|) \sim (-1)^{N} f(\xi|), \qquad f(|\xi) \sim (-1)^{N} f(|\xi).$$

31. We may regard $f(\xi_1, \xi_2, \cdots, \xi_r|\eta_1, \eta_2, \cdots, \eta_s)$, the ξ, η being the matrices of § 1, as an L-function of ξ_1 alone, or of η_1 alone. Hence in the following sections we need consider only the behavior of functions of parity $p(a|0)$, $p(0|b)$, and need examine paraphrases for functions of those parities alone. By repeated application of the theorems below for functions of parity $p(a|0)$ and $p(0|b)$, the results for functions of parity

$$p(a_1, a_2, \cdots, a_r|b_1, b_2, \cdots, b_s)$$

may be written out if desired, cf. § 6.

32. Let $\xi_i \equiv (\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ia})$, $\eta_i \equiv (\beta_{i1}, \beta_{i2}, \cdots, \beta_{ib})$. Then the matrices $(\xi_i; \eta_i)$, $(\xi_i; -\eta_i)$, where

$$(\xi_i; \eta_i) \equiv (\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ia}, \beta_{i1}, \beta_{i2}, \cdots, \beta_{ib}),$$

$$(\xi_i; -\eta_i) \equiv (\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ia}, -\beta_{i1}, -\beta_{i2}, \cdots, -\beta_{ib}),$$

are termed the conjoints of ξ_i, η_i, of ξ_i, $-\eta_i$ respectively.

Consider now a paraphrase (over a given separation):

$$\sum_i a_i f((\xi_i; \eta_i)|) = 0. \tag{47}$$

Choose for $f((\xi_i; \eta_i)|)$ the implicitly restricted L-function:

$$\cos\left(\sum_{r=1}^{a} \alpha_{ir} x_r + \sum_{s=1}^{b} \beta_{is} y_s\right) \tag{48}$$

in which the x, y are parameters. Substituting (48) in (47), and applying Lemma 8 (§ 24), we infer by § 3, as in § 25, for f_1, f_2 arbitrary of the indicated parities $p(a, b|)$, $p(|b, a)$:

$$\sum a_i f_1(\xi_i, \eta_i|) = 0; \qquad \sum a_i f_2(|\xi_i, \eta_i) = 0, \tag{49}$$

as consequences of (47). Similarly, as consequences of

$$\sum_i a_i f(|(\xi_i; \eta_i)) = 0, \tag{50}$$

we find in precisely the same way:

$$\sum_i a_i f_1(\xi_i|\eta_i) = 0; \qquad \sum_i a_i f_2(\eta_i|\xi_i) = 0. \tag{51}$$

33. The results of § 32 are paraphrases, ultimately, of the addition theorems for the sine and cosine. So also are the following obvious identities, which are frequently useful, cf. § 3.

$$\begin{aligned} f((\xi_i; \eta_i)|) &= f_1(\xi_i, \eta_i|) - f_2(|\xi_i, \eta_i), \\ f(|(\xi_i; \eta_i)) &= f_3(\eta_i|\xi_i) + f_4(\xi_i|\eta_i); \end{aligned} \tag{52}$$

where

$$\begin{aligned} 2f_1(\xi_i, \eta_i|) &\equiv f((\xi_i; -\eta_i)|) + f((\xi_i; \eta_i)|), \\ 2f_2(|\xi_i, \eta_i) &\equiv f((\xi_i; -\eta_i)|) - f((\xi_i; \eta_i)|), \\ 2f_3(\eta_i|\xi_i) &\equiv f(|(\xi_i; \eta_i)) + f(|(\xi_i; -\eta_i)), \\ 2f_4(\xi_i|\eta_i) &\equiv f(|(\xi_i; \eta_i)) - f(|(\xi_i; -\eta_i)). \end{aligned} \tag{53}$$

That f_1, f_2, f_3, f_4 have the parities implied by their bar notations may be verified at once from the definitions of § 1.

34. Obviously $f(\xi_1, \xi_1, \cdots, \xi_1, \xi_2, \xi_3, \cdots, \xi_r|)$ is no more general than $f(\xi_1, \xi_2, \cdots, \xi_r|)$. Similarly any L-function may be formally reduced by omitting from its symbol redundant matrices. This obvious remark will appear in the sequel as the source of some of Liouville's most difficult paraphrases (from the standpoint of proof of Dirichlet's method). We next consider the complement of this process of reduction. It leads from paraphrases for functions of order ω to paraphrases in which the order of the function exceeds ω, again a process which seems to have been employed by Liouville to transform his simpler results.

35. To keep the writing simple, we may at this stage confine our attention to functions of order 2 integrated over separations of degree 3, deferring the general case, which is treated in the same way, and the theory of classes of paraphrases until we shall have in the next paper a considerable body of theorems for particular functions and separations by which to illustrate the processes involved. For simplicity, since we are to consider functions of

order 2, choose for the F of § 8 (13), $F \equiv f((z_1, z_2)|)$. The partition is to be of degree 3; hence in the notation of § 8, where now (cf. footnote) the l's are integral,

$$\Lambda_i = l_{i1}\lambda_1 + l_{i2}\lambda_2 + l_{i3}\lambda_3 \qquad (i = 1, 2);$$

and for the paraphrase (13), we have in the present case:

$$(54) \qquad \sum a_i f((l_{11}\lambda_1 + l_{12}\lambda_2 + l_{13}\lambda_3,\, l_{21}\lambda_1 + l_{22}\lambda_2 + l_{23}\lambda_3)|) = 0,$$

the $\sum$ extending over all $\lambda_1, \lambda_2, \lambda_3$ defined by the separation. Write

$$(55) \quad A \equiv \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_r x_r; \qquad B \equiv \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_r x_r,$$

the x's denoting parameters, and the α, β constant integers.* As in § 32, replace (54) by its special case:

$$(56) \quad \sum a_i \cos \{(l_{11}\lambda_1 + l_{12}\lambda_2 + l_{13}\lambda_3)\zeta_1 + (l_{21}\lambda_1 + l_{22}\lambda_2 + l_{23}\lambda_3)\zeta_2\} = 0,$$

an identity in ζ_1, ζ_2. Substitute for the parameters ζ_1, ζ_2 in (56), A, B, respectively; then there is the identity in the x's:

$$(57) \qquad \sum a_i \cos (L_1 x_1 + L_2 x_2 + \cdots + L_r x_r) = 0,$$

where

$$(58) \quad L_i \equiv (\alpha_i l_{11} + \beta_i l_{21})\lambda_1 + (\alpha_i l_{12} + \beta_i l_{22})\lambda_2 + (\alpha_i l_{13} + \beta_i l_{23})\lambda_3;$$

and (57) paraphrases into:

$$(59) \qquad \sum a_i f((L_1, L_2, \cdots, L_r)|) = 0.$$

By suitably choosing the constants α_i, β_i, the L_i may be taken equal to linear functions of the $\lambda_1, \lambda_2, \lambda_3$, to a certain extent predetermined; viz., if $L_i = l_i\lambda_1 + m_i\lambda_2 + n_i\lambda_3$, the values of any two of the l_i, m_i, n_i fix the value of the third. Applying § 32 to (59), we deduce from it paraphrases for functions of parity $p(r_1|r_2)$, where $r_1 + r_2 = r$.

If we had chosen $F \equiv f(z_1|z_2)$, we should have had in place of (56):

$$(56a) \quad \sum a_i \cos (l_{11}\lambda_1 + l_{12}\lambda_2 + l_{13}\lambda_3)\zeta_1 \cdot \sin (l_{21}\lambda_1 + l_{22}\lambda_2 + l_{23}\lambda_3)\zeta_2 = 0;$$

whence

$$\sum a_i [\sin \{(l_{21}\zeta_2 + l_{11}\zeta_1)\lambda_1 + (l_{22}\zeta_2 + l_{12}\zeta_1)\lambda_2 + (l_{23}\zeta_2 + l_{13}\zeta_1)\lambda_3\}$$
$$+ \sin \{(l_{21}\zeta_2 - l_{11}\zeta_1)\lambda_1 + (l_{22}\zeta_2 - l_{12}\zeta_1)\lambda_2 + (l_{23}\zeta_2 - l_{13}\zeta_1)\lambda_3\}] = 0,$$

* There is no difficulty in extending this to the case of α, β numerical constants from any field. A like remark applies to the lemmas of §§ 22–28. In particular, if the α, β in F, G, H of § 6 denote rational numbers, it is obviously possible on replacing the x, y variables in (8), (9), (10) by suitable integral multiples of themselves, to reduce (8), (9), (10) to forms in which the α, β are replaced by integers, and the paraphrases of these forms may be taken by definition as the equivalents of the paraphrases of the first forms in which the α, β were rational. The cases of transcendental α, β or α, β belonging to other fields are ignored because non-trivial identities (8), (9), (10) involving such numbers do not yet (apparently) exist. Cf. § 23.

and the work henceforth is of the same kind as above. It is an interesting exercise on this section and the next to show that (19) is transformed into (19*a*) by the substitution

$$x \sim \tfrac{1}{2}x + \tfrac{1}{2}y + z, \qquad y \sim \tfrac{1}{2}x - \tfrac{1}{2}y + w.$$

36. Without considering explicitly restricted L-functions in detail at this point, we may illustrate their origin by a simple example. Again the general case is of the same nature, and the work for it similar to that for the special example. The L-function

$$(60) \qquad f(x, y|) - f(y, x|), \qquad \equiv \phi(x, y|)$$

obviously satisfies $\phi(x \ y|) = -\phi(y, x|)$. Conversely, if it be required to determine the form of the most general L-function, $\psi(x, y)$, of parity $p(1^2|0)$, which changes sign with interchange of the variables, we have, expressing the parity conditions, $\psi(x, y) \equiv F(x, y|)$; and, by the given condition, $\psi(y, x) \equiv F(y, x|) = -\psi(x, y)$. Whence

$$2\psi(x, y) = F(x, y|) - F(y, x|);$$

F being unrestricted of parity $p(1^2|0)$. An arbitrary constant factor may clearly be absorbed in an L-function without changing its parity or diminishing its generality; hence, we may take $F(x, y|) = 2f(x, y|)$, f arbitrary of parity $p(1^2|0)$; and $\psi(x, y) \equiv \phi(x, y|)$.

The forms of restricted functions which it is profitable to investigate are suggested by the elliptic and theta identities. One of the chief uses of restricted L-functions is to sum up in compendious form paraphrases for unrestricted L-functions. Thus, the paraphrase* $\sum a_i[f(x_i, y_i|) - f(y_i, x_i|)]$, may be replaced by $\sum a_i \phi(x_i, y_i|) = 0$ where $\phi(x, y|) = -\phi(y, x|)$. Restricted paraphrases may be found directly from the elliptic or theta identities by permuting the variables, multiplying the results by ± 1 and adding and simplifying; or in many other ways that suggest themselves as we proceed. Illustrative of the first method, it may be verified without difficulty that the multitude of paraphrases to which Weierstrass' "equation of three terms" gives rise, are all equivalent to the following, or to special cases of it:

$$(61) \qquad \begin{gathered} 4n = m_1 + m_2 + m_3 + m_4; \qquad m_i = d_i \delta_i \qquad (i = 1, 2, 3, 4): \\ \sum \phi((d_1 - d_2, \delta_1 + \delta_2, d_3 - d_4, \delta_3 + \delta_4)|) = 0, \end{gathered}$$

where $\phi((x, y, z, w)|) = \phi((x, y, -z, -w)|)$, and $\phi((x, y, z, w)|)$ changes sign under each of the 12 odd substitutions on x, y, z, w.

37. Liouville (11; p. 301, (σ)) has given one example of a paraphrase in-

* § 15 (19) comes under this case.

volving a wholly arbitrary function of a single variable. By the present methods such paraphrases may be found for arbitrary functions of n variables.* For, let $f(x_1, x_2, \cdots, x_n)$ denote an arbitrary function, then:

(62) $2f(x_1, x_2, \cdots, x_n) \equiv f_1((x_1, x_2, \cdots, x_n)|) + f_2(|(x_1, x_2, \cdots, x_n))$,

where f_1, f_2 are given by:

$$\begin{aligned} &f_1((x_1, x_2, \cdots, x_n)|) \\ &\qquad = f(x_1, x_2, \cdots, x_n) + f(-x_1, -x_2, \cdots, -x_n), \\ &f_2(|(x_1, x_2, \cdots, x_n)) \\ &\qquad = f(x_1, x_2, \cdots, x_n) - f(-x_1, -x_2, \cdots, -x_n). \end{aligned} \tag{63}$$

Hence, if by any means we have deduced

$$\begin{aligned} &\sum c_i F((a_{i1}, a_{i2}, \cdots, a_{in})|) = 0, \\ &\sum c_i G(|(a_{i1}, a_{i2}, \cdots, a_{in})) = 0, \end{aligned} \tag{64}$$

in which F is arbitrary of parity $p(n|0)$, G arbitrary of parity $p(0|n)$, we may choose $F \equiv f_1$, $G \equiv f_2$, and by (62) infer

$$\sum c_i f(a_{i1}, a_{i2}, \cdots, a_{in}) = 0. \tag{65}$$

Pairs of paraphrases such as (64) are furnished by the elliptic and theta expansions; hence also paraphrases of the kind (65).

38. Returning for a moment to § 35, we shall illustrate the use of linear transformations in non-homogeneous paraphrases by giving an alternative proof of Lemma 9. The general case admits of similar treatment. Writing $x_1 = l_1 x + m_1 y$, $x_2 = l_2 x + m_2 y$ in (37), we infer, as in § 32 (49), l_1, m_1, l_2, m_2 denoting arbitrary integral constants:

$$\begin{aligned} \sum_{i=1}^{r} [&f(l_1 a_{i1} + l_2 a_{i2}, m_1 a_{i1} + m_2 a_{i2}|) + f(l_1 a_{i1} - l_2 a_{i2}, m_1 a_{i1} - m_2 a_{i2}|) \\ &- f(l_1 a_{i3} + l_2 a_{i4}, m_1 a_{i3} + m_2 a_{i4}|) - f(l_1 a_{i3} - l_2 a_{i4}, m_1 a_{i3} - m_2 a_{i4}|)] \\ &= 2 \sum_{i=1}^{s} [f(l_2 a_{i5}, m_2 a_{i5}|) - f(l_1 a_{i6}, m_1 a_{i6}|)]. \end{aligned}$$

Setting in this $l_1 = m_2 = 1$, $l_2 = m_1 = 0$, we find:

$$\sum_{i=1}^{r} [f(a_{i1}, a_{i2}|) - f(a_{i3}, a_{i4}|)] = \sum_{i=1}^{s} f(0, a_{i5}|) - f(a_{i6}, 0|)],$$

as stated in Lemma 9.

* Such paraphrases do not appear to be numerous for the elliptic functions. On the other hand they are of universal occurrence for the theta functions of more than one variable. An account of Kummer's surface from this point of view will be published elsewhere.

We have merely sketched a few of the principal transformation processes, which will be more fully developed when we have written out the elliptic and theta series in a form suitable for paraphrase, to which we pass next, translating as we go the results into paraphrases of the kind described in this paper.

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* Of this list 2.1, 3.2, 3.3, 3.4, 3.5, 3.6, 9, 10 were supplied by Professor A. J. Kempner from the galleygraphs of chapter eleven of Dickson's 2d vol. of the "*History of the Theory of Numbers*." With the exception of 2.1, all of these have been inaccessible to me. Quoting from Dickson, Kempner says in regard to 3.6, " N. V. Bougaief proved some of the theorems in Liouville's series of articles by showing that, if $F(x)$ is an even function, an identity $\Sigma_{m=0}^{\infty} A_m \cos mx = \Sigma_{n=0}^{\infty} B_n \cos nx$ implies $\int A_m F(m) = \int B_n F(n)$, and a similar theorem involving sines and an odd function $F_1(n)$." This would appear to be in accordance with Liouville's suggestions, cf. § 13, especially footnote.

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