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The period and Q of the Chandler wobble

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Sir, you have but two topicks . . .
I am sick of both.

Boswell, *Life of Samuel Johnson*

Summary. We have extended our calculation of the theoretical period of the Chandler wobble to account for the non-hydrostatic portion of the Earth's equatorial bulge and the effect of the fluid core upon the lengthening of the period due to the pole tide. We find the theoretical period of a realistic perfectly elastic Earth with an equilibrium pole tide to be 426.7 sidereal days, which is 8.5 day shorter than the observed period of 435.2 day. Using Rayleigh's principle for a rotating Earth, we exploit this discrepancy together with the observed Chandler Q to place constraints on the frequency dependence of mantle anelasticity. If Q_μ in the mantle varies with frequency σ as σ^α between 30 s and 14 months and if Q_μ in the lower mantle is of order 225 at 30 s, we find that $0.04 \leqq \alpha \leqq 0.11$; if instead Q_μ in the lower mantle is of order 350 near 200 s, we find that $0.11 \leqq \alpha \leqq 0.19$. In all cases these limits arise from exceeding the 68 per cent confidence limits of ± 2.6 day in the observed period. Since slight departures from an equilibrium pole tide affect the Q much more strongly than the period we believe these limits to be robust.

1 Introduction

For over a decade, the inversion of the observed periods and Q's of the Earth's free oscillations has been a significant and fruitful geophysical industry. The data necessary to this endeavour have been acquired in every instance from the intersection of very large seismic events with the availability either of extensive world-wide seismographic networks or of sophisticated long-period instruments. Ironically, the one normal mode which is continually excited and which has been observed for about a century, the Chandler wobble, has not yet yielded much additional insight into the nature of the Earth's interior. There is no *a priori* reason why this should be so; the period and Q of the Chandler wobble, like those of any other free oscillation, depend only upon the Earth's physical properties (its spin rate, density distribution, rheology, geometry, etc.) and not upon any aspect of the normal mode's energy source (which in the case of the Chandler wobble is unknown). In fact, the purpose of this paper is to enlist the Chandler wobble into the normal mode enterprise.

More than 1000 of the Earth's free oscillations have now been observed and positively identified (Gilbert & Dziewonski 1975), and it might seem inappropriate to devote an entire paper to but one more mode, however exotic. For some purposes, however, the Chandler wobble is almost as important as all the rest of the Earth's normal modes put together. Fig. 1 shows why. Conventional seismology only enables us to sample the Earth's transient linear rheology over a limited period range from 54 min, the period of the mode ${}_0S_2$, to a fraction of a second, the practical limit for studying short period teleseismic body waves. The Chandler wobble samples the Earth at a frequency which, on a logarithmic scale, is as far below this seismic band as the entire band is wide. As a result the data provided by the Chandler wobble are uniquely valuable for investigating the frequency dependence of the Earth's rheology.

It is fitting that this topic be discussed in an issue honouring Sir Harold Jeffreys, since it was he who first pointed out the Chandler wobble's importance in this regard. In a series of papers beginning in 1958, Jeffreys advanced the idea that the transient rheology of the mantle could be modelled by the 'modified Lomnitz law' of creep. In essence, this law states that the Q of the mantle varies with angular frequency σ as σ^{α}, where α is a constant. His first paper on this subject (Jeffreys 1958a) was published, appropriately enough, in the inaugural issue of the *Geophysical Journal*. In that and a companion paper (Jeffreys 1958b), he employed the then accepted value of the Chandler wobble Q together with the elementary observation that teleseismic S wave pulses are not noticeably dispersed to determine that '$\alpha = 0.17$ nearly'. In subsequent papers (Jeffreys & Crampin 1960, 1970; Jeffreys 1972, 1978), an error in the original paper was corrected, and a revised estimate of the Chandler wobble Q (Jeffreys 1968) as well as information about the damping of surface waves and other free oscillations was taken into account. His most recent conclusion (upon which we comment briefly in Appendix A) is that $\alpha = 0.2 \pm 0.05$ (Jeffreys 1978).

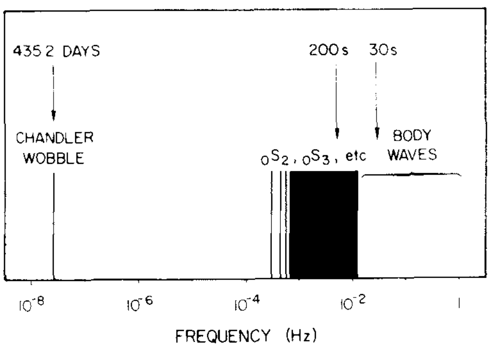

Figure 1. Schematic line spectrum of the observed free oscillations of the Earth showing the Chandler wobble and the elastic normal mode and teleseismic body wave regimes on a logarithmic frequency scale. The luni-solar body tides would appear about in the middle of the spectral gap but, as discussed in the text, they are not likely to shed any additional light on the Earth's anelasticity.

Jeffreys' ideas have been pursued and elaborated by Anderson & Minster (1979), who conclude 'the data can accommodate α from about 0.2 to 0.4'. Their preferred value is $\alpha = 1/3$, a value first proposed by Andrade (1910) for the transient creep of metals. Such intermediate values of α are, as Anderson & Minster point out, also consistent with laboratory measurements of transient creep and internal friction of rocks at high temperature. A value of $\alpha = 0.27$ has, e.g. just been measured for a mantle lherzolite subject to zero confining pressure by Berckhemer, Auer & Drisler (1979).

In this paper, we shall subject the hypothesis $Q \sim \sigma^\alpha$ to a much more rigorous test than has yet been carried out, and we shall show that it is definitely consistent with both the observed period and Q of the Chandler wobble. We assume implicitly that a single absorption band extends from seismic frequencies, where it is responsible for the observed damping of seismic waves and the Earth's free oscillations, down to 14 months, and that assumption too is shown to be consistent with the Chandler wobble data. The large gap in Fig. 1 between ${}_0S_2$ and the Chandler wobble is glaring, and raises the question whether there is any geophysical measurement which could provide mantle Q information between 54 min and 14 months. The most obvious possibility is the semi-diurnal and diurnal Earth tides, but in fact their utility is extremely limited. The quantity which must be measured to determine Q at tidal frequencies is the phase shift of the observed tide behind equilibrium. For Q in the range 200–300 the expected phase shifts on gravimeters, strainmeters and tiltmeters are, according to Zschau (1979), of order $0^\circ.01$ or less, and these will almost certainly by masked by poorly known ocean loading effects. We regard the outlook for obtaining direct Q information in the gap between ${}_0S_2$ and the Chandler wobble as dim.

The variation of mantle Q with frequency we find here is less than either Jeffreys or Anderson & Minster have indicated. Our preferred value of α is between 0.09 and 0.15, depending upon whether the average shear Q in the lower mantle is nearer 225, as indicated by the $\bar{Q}_{ScS}$ observations of Sipkin & Jordan (1980) or 350, as indicated by the normal mode observations of Sailor & Dziewonski (1978). Some uncertainty must be attached to these values because we have modelled the pole tide in the oceans as a strictly equilibrium tide. However, even if there are large departures from equilibrium, values as high as $\alpha = 1/3$ can definitely be eliminated. If the pole tide is equilibrium and if the lower mantle seismic shear Q is nearer to 225 than to 350, the value $\alpha = 0$, i.e. a frequency-independent Q, will be shown to be consistent with the 90 per cent confidence intervals of the data.

If the shear Q of the mantle is assumed to be constant with depth, we find the ratio of the Q of the Chandler wobble to that of the mantle at 14 months period to be

$$Q_{\text{wobble}}/Q_{\text{mantle}} = 1.83$$

if the strain due to pole tide loading is ignored and

$$Q_{\text{wobble}}/Q_{\text{mantle}} = 1.62$$

if it is taken into account. This conclusion may be surprising to many readers, as it was at first sight to us, since it is in conflict with a widely cited qualitative argument which maintains that the ratio should be of order 10, because the bulk of the wobble's energy is kinetic energy of rigid body rotation. A simple version of this argument is given by Stacey (1970, 1977) and, quite recently, Merriam & Lambeck (1979) have attempted a refinement. The method we use here to calculate the Q of the Chandler wobble given a Q distribution in the mantle has a rigorous theoretical basis, namely Rayleigh's principle or normal-mode perturbation theory. Both Jeffreys (1978) and Anderson & Minster (1979) have in the course of their arguments introduced *ad hoc* 'corrections' to the observed Q of the Chandler wobble in an attempt to account for its presumed unusual partition of energy. Those 'corrections' are based on the amount by which mantle elasticity acts to lengthen the period, and we show here that they are unnecessary. Since, in most cases, the discrepancies between our results and those of previous workers can be traced to an incorrect accounting of the Chandler wobble energy budget by the latter, we present in Appendix A an alternative derivation of our results based upon an energy argument.

Although the emphasis in this paper is on using the Chandler wobble to investigate the transient rheology of the mantle, we have attempted to provide a reasonably complete discussion of the various other factors which either definitely do or conceivably could have an influence on its period and Q. Much of the discussion is pedagogical in nature.

2 Summary of recent observations

Unlike the other free oscillations of the Earth, the Chandler wobble has never been usefully observed on conventional seismic instruments such as gravimeters or strainmeters. It is manifested instead as a periodic motion of the Earth's geographic axis in inertial space which produces a variation in the latitude of astronomical observatories. The International Latitude Service (ILS), which has monitored polar motion continuously since 1900, is one of the oldest international cooperative scientific projects on record. This service still operates, although it has been largely superseded by the International Polar Motion Service (IPMS), organized in 1962, and by the Bureau International de l'Heure (BIH), which added polar motion determinations to its timekeeping service in 1955. Since about 1970, Doppler observations of artificial satellites have provided data superior to that obtained by the methods of classical astrometry, and satellite determinations of polar motion are currently being provided on a routine basis by the US Defense Mapping Agency (DMA). Since 1972, the DMA results have been incorporated into the polar motion determinations published by the BIH (Guinot 1978).

Let us denote the observed period and Q of the Chandler wobble by T_0 and Q_0. Accuracy requires as long a record of polar motion as possible, and for that reason most determinations have been based almost entirely on the ILS series going back to the turn of the century. Quite possibly, this series has been subjected to more analyses by a wider variety of methods than any other geophysical time series. In Table 1, we give a summary of four recent determinations of T_0 and Q_0.

Jeffreys (1968) employed a standard version of the method of maximum likelihood 'with a small correction for observational error'. Currie (1974, 1975) used the maximum entropy method to calculate the Chandler wobble amplitude spectrum, and then measured the central frequency and width of the resonance peak. Wilson & Haubrich (1976) also used the method of maximum likelihood, but based their estimates only upon the signal

Table 1. Summary of recent determinations of T_0 and Q_0.

Investigator	Method	T_0 (sidereal days)	Q_0 (best estimate)	Q_0 (68 per cent confidence interval)
Jeffreys (1968)	maximum likelihood	434.3 ± 2.2	61	37–193
Currie (1974, 1975)	maximum entropy	434.1 ± 1.0	72 ± 20	
Wilson & Haubrich (1976)	narrow band maximum likelihood, Monte Carlo	435.2 ± 2.6	100	50–400
Ooe (1978)	complex ARMA	436.0 ± 2.0	96	50–300

contained in a narrow band near the Chandler frequency in order to reduce the effects of noise. They also made use of Monte Carlo studies, both to correct for a bias in the estimate of Q_0 inherent to the narrow band method and to determine error bounds on Q_0 and T_0. The most recent analysis is that of Ooe (1978), who employed a complex autoregressive-moving average model of the polar motion data.

All four analyses yield substantially identical results for the Chandler period T_0. The parameter Q_0 is much more difficult to estimate accurately, because the Chandler wobble is being continually re-excited by some unknown, irregular process. Meaningful error bounds on Q_0 are asymmetrical, since it is really the quantity Q_0^{-1} which is most likely to be nearly normally distributed. The error bounds given by Currie for both T_0 and Q_0 are smaller than those obtained in the other three analyses and are symmetric; they have, however, been derived by comparing results obtained with prediction filters of different length, an *ad hoc* procedure which has no statistical basis known to us.

It is reassuring that the last two analyses, by Wilson & Haubrich and by Ooe, although performed independently and by quite different methods, have obtained very similar results. Both agree that the confidence limits for Q_0 are about twice as large as those given earlier by Jeffreys. For specific comparisons in this paper, we shall adopt the values given by Wilson & Haubrich. The two data we shall be trying to fit here, with their 68 per cent confidence limits, are thus $T_0 = 435.2 \pm 2.6$ day and $Q_0 = 50\text{–}400$, with a preferred value of 100. The corresponding 90 per cent confidence limits, from Wilson & Haubrich, are ± 5.2 day for T_0 and 33–1500 for Q_0. Throughout this paper, the word *day* will always mean sidereal days of 86164.10 s each.

3 Wobble kinematics, geodesy and Earth models

This section begins our theoretical assault on the problem of computing T_0 and Q_0 for some geophysically interesting Earth model. We first discuss the elementary dynamics and kinematics of wobble, and qualitatively discuss why the Earth is not 'elementary'. We then summarize the important constraints geodesy places upon our calculations. Finally we discuss the properties of the perfectly elastic Earth model we use as a 'ground state' in this study and how, as is well known, the hydrostatic properties of such a model are in conflict with geodetic observations. The accurate calculation of the dissipationless Chandler period in the face of that conflict is the subject of the next two sections.

3.1 THE ELEMENTARY PHYSICS OF WOBBLE

The Chandler wobble is the only non-trivial free oscillation the Earth would possess if it were perfectly rigid. Its angular frequency $\sigma_0 = 2\pi/T_0$ would in that case be given by Euler's classic formula (see, e.g. Landau & Lifshitz 1969),

$$\sigma_0 = \left[\frac{(C-A)(C-B)}{AB}\right]^{1/2} \Omega. \tag{3.1}$$

The quantity Ω is the Earth's angular speed of rotation, while A, B and C are, respectively, its least, intermediate and greatest principal moments of inertia. For the Earth, the equatorial difference $B-A$ is much smaller than $C - ½(A+B)$, and equation (3.1) can be adequately approximate (we shall be more precise below) by

$$\sigma_0 = \frac{C - ½(A+B)}{½(A+B)} \Omega. \tag{3.2}$$

Equation (3.2) gives the Chandler wobble eigenfrequency of a nearly axisymmetric rigid body. If the body is exactly axisymmetric, so that $A = B$, both equations (3.1) and (3.2) reduce to the even more widely referenced formula (see, e.g. Goldstein 1950 or Munk & MacDonald 1960)

$$\sigma_0 = \frac{C - A}{A}\Omega. \tag{3.3}$$

The nature of the Chandler wobble motion is easily described in the rigid body case. Let the instantaneous angular velocity of rotation of the Earth be $\boldsymbol{\omega}$ and its instantaneous angular momentum be $\mathbf{H}$. In the absence of external torques, the latter must be a constant vector in inertial space. Suppose that we are observing the Earth from a platform fixed in inertial space. The Chandler wobble will appear to us as a counter-clockwise precession of the axis C of greatest inertia of the Earth about the fixed axis of angular momentum $\mathbf{H}$ (see Fig. 2). The rotation axis $\boldsymbol{\omega}$ also precesses about $\mathbf{H}$ in such a way that the three axes C, $\mathbf{H}$ and $\boldsymbol{\omega}$ are always coplanar, but the angular offset between $\boldsymbol{\omega}$ and $\mathbf{H}$ is much smaller than the angle between C and $\mathbf{H}$. (The ratio of the two angles is exactly σ_0/Ω for a rigid body.) A typical value for the angle between C and $\mathbf{H}$ is 0.14 arcsec, which amounts to about 4 m of linear displacement at the surface of the Earth. The corresponding linear distance between $\boldsymbol{\omega}$ and $\mathbf{H}$ at the surface of the Earth is (with sufficient accuracy for our purposes here) about 400 times smaller, i.e. about 1 cm. Thus to a very good approximation the $\boldsymbol{\omega}$ and $\mathbf{H}$ axes may be considered parallel during wobble. The very slight misalignment of $\boldsymbol{\omega}$ and $\mathbf{H}$ has been labelled 'sway' by Munk & MacDonald (1960).

An earthbound observer is essentially affixed to the C axis, and will observe a motion of the $\boldsymbol{\omega}$ and $\mathbf{H}$ axes relative to the solid body of the Earth. In the absence of dissipation and for an axisymmetric Earth, the intersection of $\mathbf{H}$ (or $\boldsymbol{\omega}$) and the Earth's surface will be a circular path. (Attenuation turns this path into an inward spiral; non-axisymmetry makes it elliptical.) Polar motion services such as the ILS, IPMS and BIH provide the coordinates of the rotation axis $\boldsymbol{\omega}$ relative to the so-called Conventional International Origin (CIO), which is simply a selected axis near the C axis (the C axis of the Earth actually varies with respect to the CIO, since the Earth deforms).

The numerator $C - \frac{1}{2}(A + B)$ in equation (3.2) is a measure of the size of the Earth's equatorial bulge, which for a perfectly rigid Earth produces the sole 'restoring force' for the

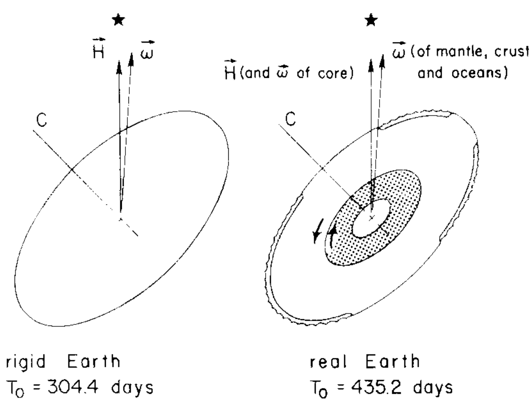

Figure 2. Schematic diagram illustrating the nature of the Chandler motion for a rigid Earth and a realistic Earth, as viewed by an observer in inertial space.

Chandler wobble. Uniform rotation of a rigid body about the C axis is a dynamically stable state. If a body in that state is somehow induced to rotate instantaneously about any other nearby axis, a condition depicted in Fig. 2, there is a restoring torque due to the equatorial bulge which seeks to return the body to its stable state. Since the body is rotating it responds to this torque (as would any gyroscope) by moving at right angles to it; the resulting precessing motion is the Chandler wobble. Just as the numerator in equation (3.2) corresponds to the Chandler wobble 'restoring force', the denominator $\frac{1}{2}(A+B)$ plays the role of the Chandler wobble 'inertia'. The Chandler wobble of a rigid Earth can be equivalently described as an instantaneous incremental rotation about an equatorial axis which itself rotates about the Earth with period T_0. Thus it is reasonable that the average equatorial moment of inertia $\frac{1}{2}(A+B)$ appears as the appropriate measure of the Chandler wobble 'inertia'.

The quantity

$$\eta = \frac{C - \frac{1}{2}(A+B)}{\frac{1}{2}(A+B)} \tag{3.4}$$

is the rigid-body Chandler wobble eigenfrequency measured in cpd. Fortunately η is related to the rate of luni-solar precession, and as a result it can be measured. For the Earth $\eta = 1/304.4$ (see below for more details). The Chandler period of a rigid Earth should thus be $T_0 = 304.4$ day, much less than the observed period $T_0 = 435.2$ day. The basic reason for this discrepancy has been known, at least qualitatively, for almost a century (Newcomb 1892); it is of course that the Earth is not rigid. A correct theory for the period T_0 must take into account not only the Earth's equatorial bulge as in equation (3.2) but also the elasticity of its solid mantle and crust, the presence of its fluid outer core and oceans and, finally, the slight physical dispersion associated with mantle anelasticity. We shall examine systematically the influence of each of these additional factors. The effect of each on the period T_0 can be interpreted, as we shall see, in terms of a change in either the 'restoring force' or the 'inertia' of the Chandler wobble. This interpretation is one of the major themes of this paper.

Although the various deviations of the Earth from perfect rigidity combine to lengthen the period of the Chandler wobble from 304.4 to 435.2 day, the motion itself is changed very little. The biggest change is that the core, to a very good approximation, does not partake in the wobble. Decoupling of the core by slip at the core–mantle boundary is virtually complete, so that while the mantle, crust and oceans wobble with instantaneous angular velocity $\boldsymbol{\omega}$, the core's angular velocity remains aligned nearly along $\mathbf{H}$ (see Fig. 2). This was first deduced, and its effect on the Chandler wobble period T_0 determined, in an elegant paper by Hough (1895); we shall make use of his analysis below. The motion of the elastic mantle and crust and probably that of the oceans as well differs little from a rigid body wobble. Roughly, the deformation each suffers is about a factor of σ_0/Ω (i.e. about 400 times) smaller than the magnitude of the rigid body rotation. Mantle displacements relative to the 4 m wobble are thus of the order of 1 cm, and the associated strains are of order 10^{-9}, about an order of magnitude less than typical luni-solar tidal strains. The polar motion services such as the ILS, IPMS, etc., all ignore this very slight deformation in combining data from various observatories; this can be justified, at least at the present time, by the same argument which allows us to ignore 'sway'. Rather surprisingly, even though the deformation of the wobbling portion of the Earth is quite small, it has a very important effect on the period T_0. It is in fact responsible for all of the lengthening from 304.4 to 435.2 day and more, since the effect of the core is not to lengthen the period but to shorten it by about 1 month, as we shall see.

3.2 PERTINENT GEODETIC RESULTS

In the ensuing discussion, we shall need the values of several terrestrial parameters. In general these fall into two categories: those that can be inferred from geodetic observations, and those which must be inferred from some seismologically derived Earth model. The geodetic parameters we shall refer to have been collected in Table 2; the first two of these, including η, have been mentioned above. In the fourth parameter $a^5\Omega^2/[\tfrac{3}{2}G(A+B)]$, a denotes the mean radius of the Earth, i.e. the radius of the sphere having the same volume, and G is Newton's constant. The parameter which is actually determined from the observed rate of luni-solar precession is not η, but rather the so-called dynamical ellipticity

$$H = [C - \tfrac{1}{2}(A+B)]/C. \tag{3.5}$$

These two quantities are related by $\eta = H/(1-H)$. The tabulated value of η corresponds to $H = 0.0032739935 = 1/305.43738$, inferred by Kinoshita *et al.* (1979) using the rigid Earth precessional theory developed recently by Kinoshita (1977).

It is important for our purposes that the parameter η be accurately measured since, in a sense, almost three-quarters of the observed period T_0 depends only on this parameter, while the remaining quarter depends on other properties of the Earth. The current uncertainty in H, according to Kinoshita *et al.* (1979), is about $\pm 7 \times 10^{-8}$, largely due to the uncertainty in the observed precessional rate, now that the uncertainty in the mass of the Moon has been reduced by radio tracking data from *Mariner* spacecraft. This uncertainty in H translates into an uncertainty of only ± 0.006 in the reciprocal η^{-1}, and is negligible compared with the ± 2.6 day error in T_0.

The small value of the ratio $[B-A]/[C-\tfrac{1}{2}(A+B)]$ was invoked above to reduce equation (3.1) to (3.2). The error incurred in that reduction is of the order of that small quantity squared, which with the value in Table 2 amounts to an error of only 0.14 day in the rigid Earth period T_0. Since this too is negligible with respect to the observational uncertainty, we shall for simplicity develop the subsequent theory in this paper for an axially symmetric Earth model, with $A = B$. Whenever a numerical value of the equatorial moment A is required, we shall always substitute the measured average moment $\tfrac{1}{2}(A+B)$. This recipe converts the formula (3.3) for an axially symmetric rigid Earth into equation (3.2), and it can be easily justified in the non-rigid case as well.

3.3 A MODEL OF THE EARTH

A model of the radial distribution of the Earth's density ρ_0 and elastic properties κ and μ is essential for the accurate calculation of the theoretical Chandler period. A variety of

Table 2. Observed geodetic constants of the Earth.

Quantity	Source	Adopted value
$[C-\tfrac{1}{2}(A+B)]/[\tfrac{1}{2}(A+B)]$	luni-solar precession; Kinoshita *et al.* (1979)	$0.00328475 = 1/304.437$
$[B-A]/[C-\tfrac{1}{2}(A+B)]$	MacCullagh's formula; GEM 8 model; Wagner *et al.* (1977)	6.692×10^{-3}
Ω	rounded value; Moritz (1979)	$7.292115 \times 10^{-5}\,s^{-1}$
$a^5\Omega^2/[\tfrac{3}{2}G(A+B)]$	simple combination of H, Ω and GEM 8 model parameters	$0.00348118 = 1/287.259$
ϵ_a	observed surface ellipticity; Wagner *et al.* (1977)	$1/298.255$

modern models, each constructed to be consistent with a large body of seismological data, is now available (see, e.g. Gilbert & Dziewonski 1975; Dziewonski, Hales & Lapwood 1975; Anderson & Hart 1976). Since we shall separately account for the existence and geographical distribution of the oceans, we desire a model which does not have an oceanic layer. We have performed calculations for both of models 1066A and 1066B of Gilbert & Dziewonski (1975). Until further notice, however, we shall refer exclusively to model 1066A. In Section 5 we will make use of our results for 1066B as a rough gauge of the sensitivity of T_0 to the Earth model. We shall find that sensitivity to be slight, which reflects the fact that any modern Earth model which fits the long-period low-degree free oscillation data will have the same planetary elastic-gravitational response, and thus will wobble in the same way, as any other. The Chandler wobble will not help us differentiate among competing models of the Earth's elastic structure; rather it can be used, as we shall see, to constrain the Earth's anelastic and associated dispersive properties.

The squared Brunt–Vaisala frequency, N^2, in the fluid outer core of model 1066A fluctuates over a range of about $\pm 9 \times 10^{-8}\,\mathrm{s}^{-2}$ with radius. For reasons discussed by Smith (1974, 1977) this is computationally inconvenient (and physically unlikely) and, so, we have slightly altered the original model by smoothing the density distribution in the outer core to make $N^2 = 0$ throughout. The density contrast at the inner core–outer core boundary was simultaneously changed to keep the mass and moment of the entire core (and thus of the Earth as a whole) unchanged. The net effect is to change the mass of the inner core by about 0.5 per cent. The squared Brunt–Vaisala frequency in the fluid core has recently been constrained quite closely by Masters (1979), and in particular he has shown that the uniform value $N^2 = 0$ is consistent with existing seismological data. We shall refer to his study again in Section 5 in discussing the sensitivity of our results to choice of Earth model. Hereafter, references to model 1066A will always be references to our $N^2 = 0$ variant of the original.

The hydrostatic ellipticity $\epsilon(r)$ of model 1066A has been calculated by numerical integration of Clairaut's equation (Jeffreys 1970) using the value of Ω given in Table 2; Radau's approximation was not used (although we found agreement between the two approaches to be quite good). The resulting $\epsilon(r)$ was used, in turn, to compute a number of other quantities which depend only upon the hydrostatic ellipsoidal density distribution. Those results, not all of which are actually needed here, are given in Table 3. The results tabulated are insensitive to our slight alteration of 1066A (except that for the unaltered model 1066A

Table 3. Inferred properties of hydrostatic ellipsoidal model 1066A.

Quantity		Value (cgs)
$\bar{\rho}_0$	mean density	5.5170
I/Ma^2		0.33089
C	polar moment of inertia	8.0438×10^{44}
A	equatorial moment of inertia	8.0177×10^{44}
C_M	mantle only	7.1242×10^{44}
A_M		7.1005×10^{44}
C_C	core (inner plus outer) only	9.1332×10^{43}
A_C		9.1100×10^{43}
C_I	inner core only	6.174×10^{41}
A_I		6.159×10^{41}
ϵ_a	surface ellipticity	1/299.8
ϵ_b	core–mantle boundary	1/393.0
ϵ_c	inner core–outer core	1/414.9
k	Love number	0.30088

the ellipticity of the inner core–outer core boundary is $\epsilon_c = 1/416.5$). The quantity $M = (4/3)\pi\bar{\rho}_0 a^3$ is the model's mass, and $I = (C + 2A)/3$ is its mean moment of inertia; both are in agreement with observed values. The polar and equatorial moments C and A were computed using the first-order formulae (Sasao, Okubo & Saito 1980)

$$C = \frac{8\pi}{3}\left[\int_0^a \rho_0 r^4\, dr + \frac{2}{15}\int_0^a \rho_0(r\partial_r\epsilon + 5\epsilon)r^4\, dr\right] \tag{3.6a}$$

and

$$A = \frac{8\pi}{3}\left[\int_0^a \rho_0 r^4\, dr - \frac{1}{15}\int_0^a \rho_0(r\partial_r\epsilon + 5\epsilon)r^4\, dr\right] \tag{3.6b}$$

with similar formulae (only the limits of integration are changed) for C_M, A_M, etc. The final quantity tabulated in Table 3, k, is not related to the Earth's hydrostatic ellipticity, but is the spherical Earth's static Love number taken from Dahlen (1976).

The hydrostatic value of $\eta = (C - A)/A$ for model 1066A, according to Table 3, is

$$\eta_{1066\text{A}} = \frac{1}{307.8}, \tag{3.7}$$

while the observed value for the Earth from Table 2 is

$$\eta_{\text{Earth}} = \frac{1}{304.4}. \tag{3.8}$$

The 1 per cent discrepancy between $\eta_{1066\text{A}}$ and η_{Earth} occurs because the Earth is not in perfect hydrostatic equilibrium. If it were, as Jeffreys (1970) has pointed out, 'the solid surface would be a level surface, the oceans would cover it, and we should have no interest in the matter'. The Earth, then, insists upon η_{Earth} while the internal hydrostatic theory of Clairaut insists on $\eta_{1066\text{A}}$; the difference must be somehow supported by global deviatoric stresses. The 0.5 per cent discrepancy between the observed surface ellipticity $\epsilon_a = 1/298.255$ and the corresponding hydrostatic value $\epsilon_a = 1/299.8$ is a more familiar indication of the Earth's departure from hydrostatic equilibrium, since it was the first major discovery of satellite geodesy (Henriksen 1960; Jeffreys 1963). It does not seem to be as well known that the dynamical ellipticity H, and therefore $\eta = H/(1 - H)$, depart from their respective hydrostatic values by about twice as much. This important discrepancy was not perceived by Smith (1977), and his results for the Chandler period T_0 as well as those of Dahlen (1980a) must be modified accordingly. That modification is the topic of the next section.

4 The oceanless, dissipation-free Chandler period

In order to make geophysical hay of the Earth's observed T_0 we must be able to predict theoretically the T_0 associated with a particular model of the Earth's properties with sufficient accuracy to discriminate between competing, geophysically interesting, possibilities. (An accuracy of order 1 month, for example, is sufficient to reveal that the Earth has a fluid outer core; unfortunately, models lacking such a feature are not geophysically interesting.) In the present instance we wish to know theoretical T_0 with significantly better accuracy than observed T_0. As we shall see, current uncertainty about the density and elastic structure of the Earth will place a limit of about ½ day on the accuracy of our theoretical calculations.

In this section we concentrate upon the computation of T_0 for an oceanless, perfectly elastic, but otherwise complete, Earth (oceans are the topic of the next section). There are two ways to undertake such a calculation, neither of which is entirely satisfactory by itself. We discuss both here and show how they may be combined to arrive at a satisfactory result.

The first of these techniques is based upon the Liouville equation, a global statement of the conservation of angular momentum. This equation forms the basis for an approximate (but as we shall see remarkably accurate) calculation which shows clearly the role of such properties of the Earth as core fluidity, mantle elasticity, and dynamical and geometrical ellipticity. Since this approach depends upon an *ad hoc* allowance for each of the principal physical processes contributing to wobble, it is difficult to generalize beyond the level we develop below.

The second technique is an application of elastic-gravitational normal mode theory. This is a rigorous calculation which is, in principle, capable of arbitrarily high accuracy provided that the Earth satisfies certain assumptions. The most important of these, here, is that the uniformly rotating Earth is in a state of hydrostatic equilibrium. The departure of the real Earth from hydrostatic equilibrium, as noted earlier, introduces an unacceptable inconsistency into the normal mode results. Since almost three-quarters of T_0 depends on only the parameter η with the remaining quarter depending upon other properties of the Earth, no matter how accurately normal mode theory may take account of some of those other properties, its results are not useful if it is applied indiscriminately to an Earth model (such as hydrostatic ellipsoidal model 1066A) having an incorrect value of η.

This discussion will lead naturally to two different reference frames from which to view the wobbling Earth, and it is worth while devoting some attention to these before introducing the complications of dynamics. For brevity, we shall often use the word mantle to denote both mantle and crust. The first frame, denoted F_{I}, is the invariably rotating frame. Let $\hat{\mathbf{z}}$ be a fixed vector in inertial space and suppose that, in its equilibrium state, the Earth has angular velocity

$$\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}},$$

where Ω is a constant. We suppose F_{I} to rotate with the constant angular velocity $\boldsymbol{\Omega}$ and to be aligned so that the fixed unit vector $\hat{\mathbf{z}}$ is one of the coordinate vectors of F_{I}. The Earth, in its equilibrium state, is stationary when viewed from F_{I} (it is not stationary in inertial space; it is spinning). No matter what befalls the Earth, F_{I} continues to spin with the constant angular velocity $\boldsymbol{\Omega}$, and the Earth as viewed from F_{I} is not generally stationary; earthbound observatories appear to move about with respect to their equilibrium positions etc. The frame F_{I} has two principal virtues: it is conceptually the simplest of all rotating frames, and if the Earth is not subject to external torques its net angular momentum $\mathbf{H}$ is forever constant in F_{I} and aligned along $\hat{\mathbf{z}}$.

The other frame, denoted F_{M}, is the mantle Tisserand frame, and is uniquely defined as that frame in which the mantle's relative angular momentum (defined below) vanishes at all times and which coincides with F_{I} when the Earth is in equilibrium. Let $\boldsymbol{\omega}(t)$ be the instantaneous angular velocity of F_{M}; in general, if the Earth is in a state of motion,

$$\boldsymbol{\omega}(t) \neq \Omega\hat{\mathbf{z}}.$$

The frame F_{M} is precisely defined below but, to visualize it, it is sufficient to regard it as the frame tied to the 'mean mantle' (i.e. the mantle apart from non-rigid rotational distortion). We have chosen a mantle frame rather than a whole Earth Tisserand frame (Munk & MacDonald 1960) in anticipation of the lack of participation of the core in the wobble.

4.1 EULER, HOUGH, LOVE AND LARMOR

Suppose that the Earth is undergoing some slight disturbance about its equilibrium state of steady diurnal rotation. Pick a reference frame which also varies only slightly (and in an as yet undescribed manner) from a state of steady rotation. Since the frame varies only slightly with respect to the Earth, the instantaneous position of any particle in the Earth can be described by

$$\mathbf{r}(\mathbf{x}, t) = \mathbf{x} + \mathbf{s}'(\mathbf{x}, t), \tag{4.1}$$

where $\mathbf{x}$ is the initial or reference position of the particle currently at r and $\mathbf{s}'$ is guaranteed to be small. (The choice of a frame which deviates only slightly from the Earth itself is necessary if we are to have both $\mathbf{s}'$ small and $\mathbf{x}$ constant.) The instantaneous inertia tensor $\mathbf{C}$ is given by

$$\mathbf{C} = \int_{\mathrm{E}} \rho_0 [(\mathbf{x} + \mathbf{s}')\cdot(\mathbf{x} + \mathbf{s}')\mathbf{I} - (\mathbf{x} + \mathbf{s}')(\mathbf{x} + \mathbf{s}')]\, dv \tag{4.2}$$

and the instantaneous relative angular momentum by

$$\mathbf{h}(t) = \int_{\mathrm{E}} \rho_0 (\mathbf{x} + \mathbf{s}') \times \partial_t s'\, dv \tag{4.3}$$

where E is the Earth. If $\hat{\mathbf{e}}_i$ form a Cartesian basis for our reference frame, we may represent its instantaneous angular velocity as

$$\boldsymbol{\omega}(t) = \Omega \hat{\mathbf{e}}_3 + \Omega \mathbf{m} \tag{4.4}$$

where $|\mathbf{m}|$ is small compared with unity. Then the total angular momentum of the Earth is

$$\mathbf{H}(t) = \mathbf{C}(t)\cdot\boldsymbol{\omega}(t) + \mathbf{h}(t). \tag{4.5}$$

The vector $\mathbf{H}$ is independent of how we choose the frame, although $\mathbf{C}$, $\boldsymbol{\omega}$ and $\mathbf{h}$ are not. For definiteness, we select the frame so that the contribution to $\mathbf{h}$ from the mantle vanishes, i.e. so that

$$\mathbf{h}_{\mathrm{M}} = \int_{\mathrm{M}} \rho_0 (\mathbf{x} + \mathbf{s}') \times \partial_t \mathbf{s}'\, dv = 0$$

where M denotes the mantle. In that case our frame is precisely the frame F_{M} described above.

It is useful to decompose $\mathbf{C}$ into

$$\mathbf{C} = A(\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + B\hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2) + C\hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3 + \mathbf{c}(t) \tag{4.6}$$

where $\mathbf{c}$ is the assemblage of all terms in equation (4.2) containing $\mathbf{s}'$. The tensor $\mathbf{c}$ is the contribution to the instantaneous inertia tensor $\mathbf{C}$ arising from any movement of mass relative to F_{M}. In general, it arises from both non-rigid rotational distortion as well as rigid rotation of the aspherical Earth with respect to F_{M}. In the present instance, it will be largely due to elastic deformation of the mantle and, to a lesser extent, deformation of the core.

A linearized Liouville equation is obtained by applying the conservation principle

$$\partial_t \mathbf{H} + \boldsymbol{\omega} \times \mathbf{H} = 0 \tag{4.7}$$

to equation (4.5), inserting equations (4.2)–(4.4) and (4.6), and linearizing in $\mathbf{s}'$. The result is

$$i\sigma_0 m_1 + \left(\frac{C-A}{A}\right)\Omega m_2 = -\frac{1}{A}\left[i\sigma_0 c_{13} - \Omega c_{23} + i(\sigma_0/\Omega)h_1 - h_2\right]$$

$$i\sigma_0 m_2 - \left(\frac{C-A}{A}\right)\Omega m_1 = -\frac{1}{A}\left[i\sigma_0 c_{23} + \Omega c_{13} + i(\sigma_0/\Omega)h_2 + h_1\right] \tag{4.8}$$

$$i\sigma_0 m_3 = -\frac{1}{C}(i\sigma_0 c_{33} + h_3),$$

where c_{ij} are the Cartesian components of $\mathbf{c}$, etc., and we have assumed time-dependence of the form $\exp(i\sigma_0 t)$. Equations (4.8) are a complete accounting of the angular momentum balance of the Earth. Together with rules for computing h_i and c_{ij}, equation (4.8) is an exact equation of motion for any normal mode of the Earth which imparts non-vanishing net rigid rotation to the mantle. Equations (4.8) are symmetric in m_1 and m_2 but not in m_3. The absence of complete symmetry reflects the alignment of m_3 along the axis of steady rotation. (If $\Omega \to 0$, (4.8) becomes wholly symmetric.)

The ease with which we reached (4.8) suggests that the real effort must lie in computing $\mathbf{h}$ and $\mathbf{c}$, and that is so. For any motion small enough that the Earth's response is linear, $\mathbf{s}'$ must be linear in $\mathbf{m}$. Since $\mathbf{h}$ and $\mathbf{c}$ in turn are linear in $\mathbf{s}'$, there must be a linear relation connecting each of $\mathbf{h}$ and $\mathbf{c}$ to $\mathbf{m}$, so long as the motion is infinitesimally small. In general, then, there is a second-order tensor transforming $\mathbf{m}$ into $\mathbf{h}$ and a second-order tensor transforming $\mathbf{m}$ into $\mathbf{c}\cdot\hat{\mathbf{e}}_3$ (which is all we need to know for 4.8). These two possibly complex and frequency-dependent tensors, together with the values of A and C, completely determine the Chandler period T_0. They thus convey everything we need to know about the density, geometry and rheology of the Earth. This information is encapsulated in the nine coefficients D_{ij} of

$$c_{i3} = D_{ij} m_j \tag{4.9a}$$

and the nine coefficients E_{ij} of

$$h_i = \sigma_0 E_{ij} m_j, \tag{4.9b}$$

where we have extracted a factor of σ_0 from E_{ij} for later convenience.

For the moment we shall consider only the case of an Earth without oceans, and in that case the above connection can be greatly simplified. Specifically, we may then take the Earth to be both dynamically as well as geometrically axisymmetric. Dynamical axisymmetry not only means that $A = B$, but also imposes powerful symmetry relations on D_{ij} and E_{ij}. Those relations are given by Dahlen (1976) for D_{ij}, and here take the form

$$D_{ij} = D(\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2}) + D'\delta_{i3}\delta_{j3}, \tag{4.10a}$$

where D and D' are scalars. The corresponding symmetries of E_{ij} are somewhat more complicated, because an m_1 spin of the mantle can induce both an h_1 and h_2 component in the core's relative angular momentum. The most general axisymmetric form of E_{ij} is, as a result,

$$E_{ij} = E(\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2}) + iE'\epsilon_{ij3} + \tilde{E}\delta_{i3}\delta_{j3}, \tag{4.10b}$$

where E, E', and $\tilde{E}$ are scalars. We have extracted a factor of i from E' so that, in the treatment which follows, both E and E' will be real.

Furthermore, in treating the Chandler wobble of an axisymmetric Earth, we may consider only motion for which $m_3 = 0$; this may be done without loss of generality since axisymmetry prohibits spin–wobble coupling. For our purposes, then, the general form (4.9) reduces to

$$\begin{bmatrix} c_{13} \\ c_{23} \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \tag{4.11a}$$

and

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \sigma_0 \begin{bmatrix} E & iE' \\ -iE' & E \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}. \tag{4.11b}$$

Only the three scalars D, E and E' appear in (4.11); the other two, D' and $\tilde{E}$, are absent since $m_3 = 0$.

Using (4.11) it is straightforward to reduce (4.8) to a characteristic or secular equation for σ_0. That equation is

$$\Delta(\sigma_0) = \det \begin{vmatrix} i\sigma_0[A + D + E' + (\sigma_0/\Omega)E] & \Omega[C - A - D] - \sigma_0[E + (\sigma_0/\Omega)E'] \\ -\Omega[C - A - D] + \sigma_0[E + (\sigma_0/\Omega)E'] & i\sigma_0[A + D + E' + (\sigma_0/\Omega)E] \end{vmatrix} = 0. \tag{4.12}$$

Given C, A, Ω and D, E and E', (4.12) determines σ_0. In deducing (4.12), we exploited axisymmetry and supposed the wobble to be linearly small. Within those limits, however, (4.12) is both exact and general. It has no *a priori* notions of the rheology or internal dynamics of the Earth. We shall exploit (4.12) to estimate the Chandler period of an Earth model which is everywhere elastic, and which has a fluid outer core. We will approach this result by examining a sequence of calculations which account for increasingly complex behaviour in the Earth.

The simplest possible case is a rigid Earth. Such a body cannot deform, nor can different portions of it possess relative angular momentum; thus D, E and E' vanish and (4.12) reduces to

$$\sigma_0 = \left(\frac{C - A}{A}\right)\Omega, \tag{4.13}$$

the classical result for an axisymmetric rigid body. For the Earth this gives, as noted earlier,

$T_0 = 304.4$ day,

about 70 per cent of the observed period.

A more sophisticated calculation is due to Love (1909) and Larmor (1909). Both computed the contribution to **c** from elastic yielding by the wobbling Earth. This calculation was made possible by the approximation that the Earth responds to the centripetal potential associated with wobble exactly as a non-rotating Earth model would to a static potential of the same amplitude and type. This approximation neglects the effects of rotation upon the Earth's elasto-dynamics. Further, in any reasonable application of Love-Larmor theory, the Earth's response is computed using a spherically symmetric Earth model, and thus ellipticity as it modifies the Earth's elastic response is also ignored.

The reasoning behind the Love–Larmor calculation is insightful. In the absence of wobble, F_M and F_I coincide and the Earth is in strain-free equilibrium with both its own gravitational field and the centripetal forces due to steady rotation. At any instant during wobble, the mantle is tilted away from the $\hat{z}$ axis of F_I but the Earth's instantaneous angular velocity, as discussed in Section 3, has hardly changed at all. In F_M, which has moved with the Earth, the mean mantle appears stationary but the direction of the rotation axis has changed. In F_M, then, the Earth sees a slightly different centripetal potential from the one which defined its equilibrium state; the difference is a potential associated with a spherical harmonic of the type $Y_2^{\pm 1}$, to which the Earth responds elastically. In F_M, this elastic bulge moves around with the period instantaneous rotation axis. In F_I, the rotation axis and the elastic bulge are nearly stationary. Thus the elastic portion of the bulge does not move with the wobbling Earth and this decreases the effective 'restoring force' $C–A$.

In present terms, the content of spherical Love-Larmor theory is that

$$D = ka^5\Omega^2/3G \tag{4.14}$$

where k is the conventional Love number of degree 2; E and E' are not accounted for in this calculation. With (4.14), the secular equation (4.12) gives the famous result (see, e.g. Jeffreys 1970)

$$\sigma_0 = \frac{C - A - D}{A + D}\,\Omega = \frac{C - A - ka^5\Omega^2/3G}{A + ka^5\Omega^2/3G}, \tag{4.15}$$

which in turn yields

$$T_0 = 447.4 \text{ day}$$

or 103 per cent of the observed period (a misleading concordance as we see below). The denominator of (4.15), $A + D$, is often approximated (see, e.g. Munk & MacDonald 1960) as simply A since the difference is slight (0.3 per cent). The large effect of D, a 47 per cent increase in period, is due to its role in reducing the effective spring constant from $C–A$ to $C – A – D$.

The Love-Larmor calculation is often regarded as being appropriate to an elastic, everywhere solid Earth, a view founded in the knowledge that its principal shortcoming is that it neglects the dynamic response of the rotating fluid in the core. This view is correct, but it is worth noting that the static, elastic response of a non-rotating spherically symmetric fluid core is accounted for in modern calculations of the Love number k such as that in Table 3. If the Earth were everywhere solid, the Love–Larmor result (4.15) would give a very accurate accounting of the period T_0, but the numerical value of the Love number k would be different from that appropriate to the Earth as we know it. The relatively good agreement between the period T_0 produced by (4.15) and the observed T_0 is coincidental. Neither the dynamics of the fluid core nor that of the oceans has yet been taken into account; it turns out they both affect the period by about 1 month but in opposite directions.

The importance of the fluid core is immediately evident if we consider a planet with a rigid mantle, an inviscid core, and a spherical core–mantle boundary. In this limiting case the core and mantle are perfectly decoupled and the mantle wobbles alone. Then

$$\sigma_0 = \frac{C_M - A_M}{A_M}\,\Omega.$$

Another consequence of the spherical core–mantle boundary is that $C_M - A_M = C - A$ so

$$\sigma_0 = \frac{C - A}{A_M}\,\Omega,$$

and the effect of the fluidity of the core is to reduce the effective 'inertia' from A to A_M (and thereby shorten the period by the ratio A_M/A). To see what happens when the core–mantle boundary is ellipsoidal we must turn to fluid mechanics.

The influence of a fluid core, in the absence of elasticity, was rigorously examined by Hough (1895) in an elegant study of the free wobble and nutation of an Earth model comprising a rigid mantle and an ellipsoidal core of homogeneous, incompressible fluid. In F_M, the rigid mantle is stationary and does not contribute to **c**. Also, in this frame the boundary of the core is stationary. Since the fluid in the latter is homogeneous, the core cannot contribute to **c**. Therefore, **c** vanishes and the effects due to the presence of a fluid core must appear as contributions to **h**.

Hough solved explicitly for the motion of the core resulting from rigid rotation of the mantle and it is easy to infer the coefficients E and E' of (4.11b) from his results. Appendix B quotes Hough's exact results, and uses them to discuss briefly another rotational mode of the Earth, the nearly diurnal free wobble. Here it is sufficient, and much more convenient, to make use of the fact that $\sigma_0 \ll \Omega$ and to use terms through first order in ellipticity. In that case we show in Appendix B that

$$E = (\sigma_0/\Omega)A_C \tag{4.16a}$$

and

$$E' = -(1 - \epsilon_b)A_C, \tag{4.16b}$$

where A_C and ϵ_b are the equatorial moment of inertia and surface ellipticity of the core (see Table 3). Note that E is of order ellipticity (since σ_0/Ω is), but E' is not.

Using Hough's values (4.16) for E and E' and setting $D = 0$, (4.12) yields

$$\sigma_0 = \frac{C - A}{A_M + \epsilon_b A_C}\,\Omega \tag{4.17}$$

for the Chandler eigenfrequency of an Earth with a fluid core and a rigid mantle. Using $(C-A)/A$ from geodesy (Table 2) and A_M/A, A_C/A and ϵ_b for model 1066A (Table 3), gives

$$T_0 = 269.7 \text{ day},$$

a shortening of 34.7 day from the value for a perfectly rigid Earth.

As a first approximation, Hough's result is still that the core does not participate in the Chandler wobble, and the effective 'inertia' (the denominator of 4.17) is reduced from A to A_M, as was the case for the spherical core. The 'restoring force', however, is still $C - A$ even though

$$C - A \neq C_M - A_M$$

and there is a small additional term, $\epsilon_b A_C$ in the denominator. Since the term $\epsilon_b A_C$ is so small, we may without significant error use the hydrostatic value of ϵ_b in evaluating it.

We now combine both Hough and Love–Larmor to estimate the Chandler period for an elastic Earth with a fluid core. We regard the offspring, Hough–Love–Larmor (HLL) theory, in the following light. Hough's calculation is exact (save for the damage-of-convenience we have inflicted by linearizing in ellipticity). Let us now suppose Hough's rigid, incompressible, wobbling planet is allowed to deform elastically. Then it will yield in response to the centripetal potential due to wobble in the fashion supposed by Love–Larmor theory. The resultant distortion will be linearly superimposed upon the motions described by Hough. This is not an exact solution: Love–Larmor theory is approximate and Hough's results are only exact for a rigid mantle and a homogeneous incompressible fluid core. The phenomena we are combining, however, are quite distinct and we expect that HLL theory will be more accurate than either of its constituents.

One might suppose (as the authors in fact did for a frustrating two-week period during the course of this research) that since the core is not wobbling with the mantle the centripetal potential should only be allowed to act on the mantle and, thus, that k should be some sort of 'mantle only' Love number. That is, however, not so, as a glance at Fig. 2 clearly shows. From the point of view of an observer in the frame F_M, the angular velocities $\boldsymbol{\omega}$ of the core and the mantle are very nearly the same, to within a part in 400 or so. As a result, the centripetal potential due to wobble is sensibly the same everywhere in the Earth whether the core wobbles or not. Therefore the standard Love number is the correct Love number to use in HLL theory. This same reasoning explains the presence of $C-A$ rather than $C_M - A_M$ in Hough's result.

Solving (4.12) with D, E and E' present and retaining terms through order ellipticity yields the HLL hybrid formula

$$\sigma_0 = \frac{C-A-ka^5\Omega^2/3G}{A_M+\epsilon_b A_C+ka^5\Omega^2/3G}\,\Omega. \tag{4.18}$$

Using the observed values of $(C-A)/A$ together with values of the other parameters from model 1066A gives

$T_0 = 396.4$ day

for the Earth. This, as we shall see below, turns out to be quite accurate.

Sasao *et al.* (1980) have developed a theoretical description of the free and forced nutations of an Earth with a stratified fluid core and dissipative core–mantle coupling. Their results amount to a rather detailed version of HLL theory with the significant additions of dissipative core–mantle coupling and provision for deformation of the core–mantle boundary, as well as a more general approach to motion in the fluid core than was taken by Hough. However, the wobble-specific version we have developed here is both adequate for our purposes and simpler.

4.2 NORMAL MODE THEORY

The HLL calculation has two great virtues: it is relatively simple, and the answer unveils itself in a form that clearly depicts the manner in which the Earth's major features (its dynamical ellipticity, elastic rheology and fluid core) affect the Chandler period. It also has a number of drawbacks, the greatest of which is that it is difficult to see how to extend the calculation. HLL, as we have used it so far, ignores the influence of rotation and ellipticity of figure upon the Earth's elastic response and also the effects of elasticity and structure upon the behaviour of the core during wobble.

In this section we exploit an alternative method, elastic-gravitational normal mode theory in the frame F_I, which in principle can account for the complete behaviour of an Earth model which is initially in hydrostatic equilibrium. This technique has its own shortcomings: the Earth is not sufficiently close to hydrostatic equilibrium for our purposes and normal mode theory is not susceptible to simple modification to account for that discrepancy. What we shall do is to use the normal mode calculation to arrive at a wobble-effective Love number, k_{W}, which accounts for the various phenomena HLL neglects, and then combine k_{W} with the HLL result (4.18) to estimate T_0.

Normal mode theory for a non-rotating, spherically symmetric, self-gravitating, hydrostatically prestressed Earth with an isotropic, perfectly elastic constitutive relation was extended in an obvious way by Smith (1974) to account for rotation and slight ellipticity of figure. An intrinsic feature of this approach is that the Earth's ellipticity of figure must be the unique ellipticity given by Clairaut's equation for a planet in hydrostatic equilibrium.

Smith (1977) exploited this approach to study theoretically the Earth's free wobble and nutation, and we only summarize the calculation here. If $\mathbf{s}(\mathbf{x}, t)$ is the Lagrangian particle displacement of the particle with initial position $\mathbf{x}$ in F_{I}, and if the Earth is undergoing a free motion with angular frequency σ_0, then $(\mathbf{s}, \sigma_0)$ must satisfy

$$-\rho_0\sigma_0^2\mathbf{s} + 2i\sigma_0\rho_0\boldsymbol{\Omega}\times\mathbf{s} = \boldsymbol{H}(\mathbf{s}) \tag{4.19}$$

where

$$\boldsymbol{H}(\mathbf{s}) = -\rho_1\nabla(\phi_0+\psi) - \rho_0\nabla\phi_1 + \nabla\cdot\mathbf{T} - \nabla\left[\rho_0\mathbf{s}\cdot\nabla(\phi_0+\psi)\right] \tag{4.20a}$$

$$\rho_1 = -\nabla\cdot(\rho_0\mathbf{s}) \tag{4.20b}$$

$$\phi_1 = -G\int_{\mathrm{E}}\rho_0\mathbf{s}\cdot\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dv' \tag{4.20c}$$

$$\mathbf{T} = (\kappa - \tfrac{2}{3}\mu)(\nabla\cdot\mathbf{s})\mathbf{I} + 2\mu\boldsymbol{\epsilon} \tag{4.20d}$$

$$\boldsymbol{\epsilon} = \tfrac{1}{2}\left[(\nabla\mathbf{s}) + (\nabla\mathbf{s})^{\mathrm{T}}\right]. \tag{4.20e}$$

Here ρ_0 is the Earth's (ellipsoidal) equilibrium density field, κ and μ are the incompressibility and shear modulus, ϕ_0 is the equilibrium gravitational potential, ψ is the centripetal potential due to steady diurnal rotation, ρ_1 is the incremental Eulerian density due to deformation, ϕ_1 is the incremental gravitational potential, $\mathbf{T}$ is the Cauchy (conventional) stress tensor, and $\boldsymbol{\epsilon}$ is the infinitesimal strain tensor.

If rotation ceases, ellipticity also vanishes in order to maintain hydrostatic equilibrium. The Earth, and therefore equations (4.19) and (4.20), become perfectly spherically symmetric. In this limit, normal mode solutions can be found using spherical harmonics, which serve as a set of perfectly decoupling basis functions. Since the ellipticity vanishes so does the Chandler wobble 'restoring force' $C-A$, and so therefore does the Chandler eigenfrequency; its eigenfunction $\mathbf{s}$ becomes simply a pure rigid body equatorial rotation even though the Earth itself is not rigid. If rotation is present, both the Earth and equations (4.19) and (4.20) lose spherical symmetry and the general solution requires an infinite sum of spherical harmonic terms.

Smith's (1977) study used a three-term series of the form

$$\mathbf{s} = \boldsymbol{\tau}_1^1 + \boldsymbol{\sigma}_2^1 + \boldsymbol{\tau}_3^1 \tag{4.21}$$

where $\boldsymbol{\tau}_1^1$ is a toroidal vector field of degree and order 1, $\boldsymbol{\sigma}_2^1$ is a spheroidal vector field of degree 2 and order 1, and $\boldsymbol{\tau}_3^1$ is a toroidal vector field of degree 3 and order 1. Smith (1974) argued on analytic grounds, and Smith (1977) demonstrated by numerical example, that the representation (4.21) is sufficiently general to yield either the Hough or Love–Larmor results as special cases. The $\boldsymbol{\tau}_1^1$ term accounts for the wobble itself and $\boldsymbol{\sigma}_2^1$ represents the associated deformation of core and mantle. Smith (1974) has shown that the exact eigenfunctions s must be of the form

$$\mathbf{s} = \boldsymbol{\tau}_1^1 + \boldsymbol{\sigma}_2^1 + \boldsymbol{\tau}_3^1 + \boldsymbol{\sigma}_4^1 + \ldots,$$

and since it does not greatly increase the cost, $\boldsymbol{\tau}_3^1$ has been added to the calculations discussed by Smith (1977) and those reported here for good measure.

The normal-mode calculation is effected by inserting the representation (4.21) into (4.19) and (4.20). This leads to a tenth-order system of ordinary differential equations over radius governing the amplitudes of various components of $\boldsymbol{\tau}_1^1, \boldsymbol{\sigma}_2^1$ and $\boldsymbol{\tau}_3^1$. For simplicity (a precious commodity indeed in this rather expensive calculation), the differential equations are linearized in ellipticity, but not in the centripetal potential ψ, which is retained exactly. The result is a tenth-order differential eigenvalue problem, the eigenvalues of which are σ_0 and the eigenfunctions of which are s in the frame F_{I}.

The normal-mode calculation has several virtues. First (and least important here) it can be extended in principle to arbitrary accuracy by adding more terms to (4.21). Second (and most important here) the calculation as we have executed it accounts for the effects of rotation and ellipticity on the Earth's elastic response and the effects of structure in the core. It is not possible to assess the accuracy with which the last effect is computed since we lack *a priori* knowledge of its importance but we can safely assume that the normal mode calculation is more accurate than HLL theory.

Subjecting model 1066A to the normal mode calculation yields.

$$T_0 = 403.4 \text{ day.}$$

We would adopt this value as the best available theoretical estimate of the Chandler period of an oceanless but otherwise complete Earth except for the discrepancy discussed earlier between η_{Earth} and $\eta_{1066\text{A}}$. Our first thought was that we could safely account for this discrepancy by simply multiplying the hydrostatic ellipticity $\epsilon(r)$ everywhere by the ratio 307.8/304.4, to make $\eta_{1066\text{A}}$ agree with η_{Earth}. We discovered, however, by exploring quasi-rigid test cases similar to those discussed by Smith (1977), that hydrostatic normal mode theory will not abide such tinkering with its fundamental assumptions. Specifically, if model 1066A is made quasi-rigid by everywhere setting $\lambda = \kappa - 2/3\mu = \mu = 10^{15}$ dyn cm^{-2}, its wobble period as calculated by normal mode theory shortens to 308.4 day. (The slight difference between this and $\eta_{1066\text{A}} = 1/307.8$ can be attributed to the imperfect rigidity of this model; a rough estimate based on Love (1911) is that $k \approx 1.6 \times 10^{-3}$ for such a model, and if this value together with $\eta_{1066\text{A}}$ is used in (4.15), we find that $T_0 \approx 308.4$ day, in good agreement.) Upon altering this quasi-rigid model by increasing ϵ throughout by 307.8/304.4, we computed a normal mode period of $T_0 = 300.6$ day, a change almost exactly twice as large as we expected. We attribute this to a breakdown of the normal mode calculation in the face of an inconsistent model. This collapse at the 1 per cent level is surprising but, on reflection, we do not believe that we have a just cause for complaint. To be internally consistent, the ellipticity ϵ cannot be modified without introducing initial deviatoric stresses into the operator $\boldsymbol{H}$ in (4.20). Evidently, the calculation fails because these terms are absent, and since we do not know what the deviatoric stresses in the Earth are, we cannot correctly account for them with a strictly normal-mode approach.

4.3 A HYBRID CALCULATION

We resolve this impasse by turning to HLL theory. The HLL secular equation (4.12) is exact to the extent that we are able to exactly determine the tensor components D_{ij} and E_{ij}. We propose to use the normal mode calculation to do this. Suppose we choose a reference frame $F_{M'}$ in which the relative angular momentum is exactly equal in amplitude to that predicted by Hough's calculation for an Earth model with an incompressible homogeneous core. Such a choice is admissible because the results of the last section, except for the specific values of D, E and E', are frame independent. In choosing $F_{M'}$, we are simply choosing the frame that keeps E' equal to its HLL value. This assures that the difference between HLL and normal-mode theory can be expressed solely as a change in D. The new frame may no longer exactly follow the mantle (presumably it will very nearly do so), but for the purpose of calculating T_0 we do not care. Now the period equation (4.18) together with η_{1066A} can be used to infer an oceanless wobble-effective Love number k_w. We take this to be simply the value of k which in (4.18) leads to a Chandler period of 403.4 day. That value is

$$k_w = 0.30158,$$

or 0.2 per cent greater than the ordinary static Love number k for 1066A (see Table 3). (The quantity k_w is not a new tidal Love number and is not a better measure of the Earth's response to tidal forces than k. It has no meaning outside of this calculation.) Now use k_w in (4.18) together with the observed value η_{Earth} to compute T_0. The result

$$\sigma_0 = \frac{C - A - k_w a^5 \Omega^2/3G}{A_M + \epsilon_b A_C + k_w a^5 \Omega^2/3G}\,\Omega \tag{4.21}$$

yields a period

$$T_0 = 396.9 \text{ day}.$$

We believe this to be the best currently available estimate of the Chandler period of the Earth not accounting for the effects of oceans of anelasticity.

This value, as the reader will doubtless note, is only 0.5 day greater than the simple HLL estimate, and one might conclude that we have gone to rather a lot of trouble (the normal mode calculation is both difficult and costly) for little benefit. That is not so. Until we had done the normal mode calculation, we could not know at the fraction-of-a-day level precisely how good HLL was. What we have gained beyond a more accurate result is the assurance of its accuracy. (Also, we will exploit the normal mode eigenfunction s in Section 8.)

5 The pole tide

The principal kinematic feature of the Chandler wobble is the large rigid rotation of the mantle. The importance of the oceans in wobble is largely determined by the extent to which they do, or do not, partake of the mantle's motion.

Suppose that the oceans wholly abstained from wobble and for simplicity suppose for a moment that the Earth were spherical. An observer at a mid-ocean tide pole would perceive horizontal flow as the tide pole (which is affixed to the mantle) moved back and forth during the wobble, but he would not observe a change in sea level and thus would conclude that there is no tide associated with the Chandler wobble. (The ellipticity of figure of mantle and oceans would actually produce a change in sea level, in this case, about a factor of ϵ smaller than the local horizontal motions but this effect is slight.) In this case, the effect on T_0 would, much like the effect of the core, principally consist of a reduction in the effective 'inertia' A with no sensible alteration of the 'restoring force' $C-A$. The moment of inertia of the oceans is roughly that of a shell of mass M and radius a, i.e. $A_{\text{oceans}} \approx 2/3Ma^2$. Using $M = 1.4 \times 10^{24}$ g (Sverdrup, Johnson & Fleming 1942), we find that $A_{\text{oceans}}/A \approx 5 \times 10^{-4}$. Lack of participation of the oceans in the wobble would thus *decrease* T_0 by about 0.2 day.

Suppose on the other hand that there is no slip at the ocean bottom, so that the oceans are essentially affixed to the mantle and wobble with it. In that case, there will be a visible pole tide since, as a parcel of ocean moves with respect to the rotation axis, it will experience a varying centripetal force. This is exactly the effect which gives rise to the slight deformation of the mantle and core during wobble, and the pole tide is in fact nothing more than the corresponding deformation of the oceans. The pole tide amplitude should thus be of the same order of magnitude as the deformation of the rest of the Earth, i.e. about σ_0/Ω times the wobble amplitude or about 1 cm. The effect of this small tide will be to *increase* the period T_0, since the 'restoring force' $C - A$ will be diminished not only by the deformation of the mantle and core but also by that of the oceans.

5.1 IS THE POLE TIDE AN EQUILIBRIUM TIDE?

If the question 'Do the oceans wobble with the mantle?' is rephrased in the language of tidal theory, it becomes: 'Is the pole tide an equilibrium tide?' Because this question has received some recent attention (Dickman 1979; Naito 1979), and because it will prove in the end to be our greatest obstacle to placing firm constraints on the transient rheology of the mantle, we shall now consider this in some detail. If the Earth were everywhere solid, we could answer the equilibrium question for the corresponding solid Earth 'pole tide' without hesitation. In that case, the gravest natural resonance of the system (apart from the Chandler wobble itself) would be the mode ${}_0S_2$, with a period of about 1 hr. Since the period of Chandler forcing is much greater than that (see Fig. 3), the associated response could surely be treated as quasi-static, or equilibrium. This simple argument is of course the basis of the Love–Larmor theory leading to equation (4.15). If the Earth were everywhere solid, that equation in terms of the *static* Love number k would yield a very accurate estimate of the Chandler period.

The above argument breaks down in the presence of the core and oceans, since they endow the Earth with a wide variety of additional modes of arbitrarily low frequency. In a homogeneous, incompressible core, only one such mode turns out to be important (see the discussion by Smith 1977), and it is precisely this mode which is taken into account when Hough's theory is combined with that of Love and Larmor. The modes of the ocean which are important in this context are of two types: barotropic Rossby modes of low but non-zero frequency and zero-frequency geostrophic currents. To decide whether the pole tide is equilibrium, we must examine the excitation of each of these classes. Baroclinic Rossby waves associated with the ocean's stratification cannot be excited directly by a tidal potential and presumably play a minor role in the pole tide problem (the qualifier 'presumably' is required here since baroclinic modes can be excited by interaction of barotropic motions with sea-floor topography).

Historically, the question of the long-period tidal response of the oceans first arose in connection with Kelvin and Darwin's well-known determination of the Earth's mean rigidity from the height of the lunar fortnightly tide. To test the equilibrium hypothesis (which formed the basis of the rigidity determination), Darwin (1886) and Hough (1897) solved Laplace's tidal equations in the long-period limit on a water-covered globe. They considered an external potential of the form Y_2^0, as appropriate for a long-period luni-solar tide, rather than the form $Y_2^{\pm 1}$ appropriate to wobble and nutation. The solution they found consisted primarily of a zonal current system with an associated surface displacement which was

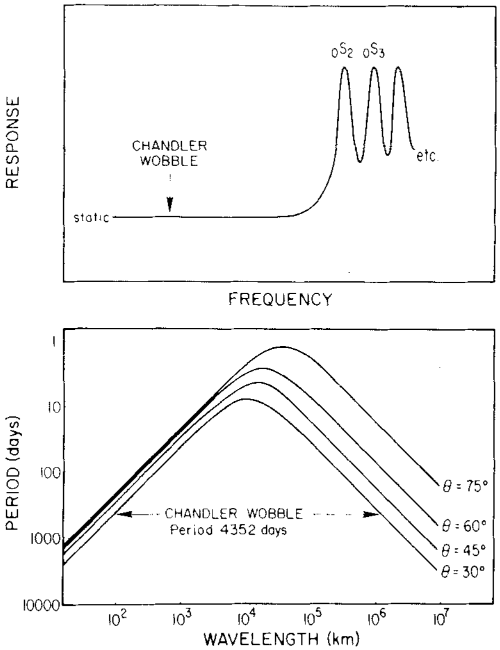

Figure 3. Top: if the Earth were everywhere solid the Chandler forcing would be well down on the flank of the resonant elastic response and the solid-Earth pole tide would be quasi-static or equilibrium. Bottom: dispersion of barotropic Rossby waves on a β-plane ocean of depth $h = 4$ km situated at colatitude ϑ. At a period of 14 month, there are waves with wavelengths $2\pi/k$ longer than 10^6 km which are non-dispersive ($\sigma \approx \beta ghk/f^2$) and have westward group velocities, and waves with wavelengths shorter than 100 km which are dispersive ($\sigma \approx \beta/k$) and have eastward group velocities.

markedly less than equilibrium, particularly away from the equator (an illustration of their solution may be seen in Wunsch 1967). In effect, this departure from equilibrium in the water-covered case can be attributed to the excitation of a nearly-resonant geostrophic mode (although, since it is forced, the current system which comprises the solution is not strictly geostrophic). Confronted with these results, Darwin himself came to doubt whether 'it will ever be possible to evaluate the effective rigidity of the earth's mass by means of tidal observations'. His pessimism was, however, countered by Rayleigh (1903), who pointed out that the Earth is not wholly water-covered and that continental barriers would impede the Darwin–Hough zonal current system. Arguing that geostrophic modes would not be efficiently excited in the presence of continental barriers, he concluded that the fortnightly tide should not differ materially from equilibrium. He cautioned, however, that this conclusion might be altered if the oceans possessed other normal modes with small but non-zero frequencies. It is now known that such modes, namely Rossby modes, do exist, and we shall describe below how their travelling wave counterparts interact with coastlines to play an important role in the establishment of equilibrium. In any case the non-equilibrium character of the Darwin–Hough solution cannot be used as an argument against a non-equilibrium pole tide, since the corresponding long-period response of a water-covered globe to $Y_2^{\pm 1}$ forcing has recently been shown by J. Wahr (private communication 1980) to be equilibrium. Although this distinction between Y_2^0 and $Y_2^{\pm 1}$ forcing is significant, it does not by itself constitute a proof that the pole tide in the real oceans will be equilibrium as well, since the interaction of Rossby waves with coastlines where boundary conditions must be satisfied still requires consideration.

Rossby modes for closed oceanic basins with flat bottoms and simple shapes were first calculated by Longuet-Higgins (1964, 1965) and by Longuet-Higgins & Pond (1970). Very roughly, for a basin of lateral dimension L centred on colatitude ϑ, the fastest such mode has a characteristic frequency of order $\sigma \approx \beta L/2\pi$, where $\beta = 2\Omega \sin \vartheta/a$ is the horizontal gradient of the Coriolis parameter $f = 2\Omega \sin \vartheta$. Higher modes have shorter wavelengths and longer periods, with an accumulation point at zero frequency and zero wavelength. In more realistic ocean basins, bottom topography affects the Rossby modes profoundly; the influence of variable water depth h has been studied by Rhines (1969a, b), Rhines & Bretherton (1973) and Ripa (1978), among others. In general, the role played in the flat-bottomed case by β is taken over in the presence of topography by $\beta^* = h|\nabla_a(f/h)|$, where $\nabla_a = a^{-1}[\hat{\boldsymbol{\vartheta}}\partial_\vartheta + \hat{\boldsymbol{\varphi}}(\sin\vartheta)^{-1}\partial_\phi]$ is the horizontal gradient operator on $r = a$. The relative importance of topography and the planetary β-effect is therefore given by the ratio of $h|\nabla_a h|$ to β/f. Except within a few degrees of the equator, β/f is of order a^{-1}, and topography should dominate, particularly on continental slopes and on the flanks of mid-ocean ridges. Such features will tend to give rise to trapped modes, with length-scales L of the order of the width of the topography and with associated frequencies of order $\sigma \approx \beta^* L/2\pi$. In general, since β^* may be large, some of these frequencies may be considerably higher than in the corresponding flat-bottomed case. The most concerted attempt to date to calculate any Rossby modes for a realistic configuration of the oceans has been that of Platzman (1975, 1978) for the Atlantic and Indian Oceans. Because of the strong dependence of these modes on resolution of the topography, he found a rather poor agreement between the Rossby modes predicted by his earlier finite-difference and his later finite-element models, much poorer than for the higher frequency gravity modes. The fastest Rossby mode in the north Atlantic Ocean was in both cases found to be associated with the Grand Banks topography, but the periods were quite different (55 hr versus 36 hr). Quantitative understanding of the oceans' Rossby mode spectrum near 14 months is clearly still a long way off. An atmospherically forced 4–6 day basin-wide Rossby mode of the Pacific ocean has been detected recently in island station sea level records by Luther (1980).

Regardless of the richness or nature of the low-frequency resonant structure of the oceans, the response at long periods will be equilibrium if there is sufficient friction. Proudman (1960) has used a simple semi-empirical law for turbulent dissipation in the benthic boundary layer to try to estimate roughly what the relevant frictional time-scale τ_f might be. Locally, the law predicts that τ_f should be proportional to h/V, where V is the speed of the modulating short-period (semi-diurnal and diurnal) tidal currents. Assuming typical values, Proudman finds that in shallow seas $\tau_f \approx 5$ day whereas in mid-ocean $\tau_f \approx 5$ yr. The effective average value, if shallow seas occupy one-tenth of the world's oceans, would thus be of order $\langle\tau_f\rangle \approx 50$ day. Tides with periods much longer than this should, according to Proudman, follow an equilibrium law. This would suggest, if it is correct, that departures from equilibrium at the Chandler period $T_0 \approx 14$ month should not be much larger than $\langle\tau_f\rangle/T_0$, or about 10 per cent. This estimate is obviously quite crude, since it ignores completely any of the dynamical details of the oceans' low-frequency response.

A related problem, in which the role of dynamics is now understood reasonably well, is the response of the oceans to slowly varying winds. In several respects, however, this problem differs rather fundamentally from the tidal problem. In the first place, the quantity which acts as the driving force in the wind problem is the curl of the surface wind stress whereas the tidal potential, since it is a potential, is curl-free. In addition, the wind stress is able to excite baroclinic as well as barotropic modes directly and these typically play an important role in the response. In spite of these differences, it is possible to draw some qualitative inferences about the tidal problem from the wind stress studies. The problem which has been most frequently studied is the initial value (spin-up) problem of the response to a wind which suddenly starts and continues to blow. The nature of the response of an idealized mid-latitude ocean has been summarized nicely by Anderson & Gill (1975), and Lighthill (1969) has considered some of the special features which arise near the equator. A series of numerical models with realistic wind forcing and North Atlantic topography have been described recently by Anderson *et al.* (1979). Although the transient response of an enclosed basin can be found in terms of its Rossby modes, the alternative formulation in terms of propagating Rossby waves provides more insight in this case. At long periods, there are two distinct types of Rossby waves: very long-wavelength waves which are non-dispersive with group velocities to the west, and very short dispersive waves with group velocities to the east. The wavelength of the former at $T_0 \approx 14$ month is much larger than the radius of the Earth, whereas the latter are typically shorter than 100 km (see Fig. 3). The steady-state solution which plays the role of the equilibrium tide in the wind stress problem is known as Sverdrup balance. The fundamental result of all the wind adjustment studies is that, after an initial acceleration, a Sverdrup balance is established upon the arrival of the long non-dispersive Rossby waves which are generated at the eastern coasts of basins. Presumably, if a tidal potential were to be applied to an ocean basin suddenly, an equilibrium tide would be established by the same mechanism. The important time-scale in that case would not be the average frictional time-scale $\langle \tau_f \rangle$, but rather a dynamical time-scale τ_d set by the group velocity $\beta gh/f^2$ of the long Rossby waves. At a distance L from the eastern coast of a basin, the time scale τ_d is of order $\tau_d \approx Lf^2/\beta gh$. Since $\beta gh/f^2$ is typically of order 4000 km day^{-1} in mid-latitudes, τ_d should be less than 4 days over most of the oceans. The simple formula $\tau_d \approx Lf^2/\beta gh$ is not applicable near the equator, due to the existence of trapped equatorial waves (Lighthill 1969; Philander 1978), but the equatorial response is even more rapid, as the formal limit $\tau_d \to 0$ suggests.

The above theoretical arguments, based on the analogy with the spin-up problem, suggest that over most of the oceans the pole tide should not depart from equilibrium by more than τ_d/T_0, or about 1 per cent. Local departures from equilibrium may arise in small nearly enclosed basins or other regions of peculiar topography where the dynamics could differ radically from the open-ocean dynamics described above. Departures from equilibrium in the open ocean would, in this picture, be associated primarily with the short-wavelength Rossby waves which are generated along western boundaries, where they give rise to a western boundary current. Some friction is still required to damp out these departures, but it can be substantially less than that estimated by Proudman since their short wavelengths will make them very prone to dissipation. Topography should act to scatter these waves and break them up even more, thereby assisting the equilibrium adjustment process; this has in fact been noted in the numerical models of Anderson *et al.* (1979).

Theoretical argument is seldom a good substitute for direct observation, and there have been several attempts to test the equilibrium hypothesis by analysing sea level records for the pole tide. This is clearly a marginal undertaking in view of the relative paucity of long tidal records, the expected small amplitude of the signal, and the rather high level of background meteorological noise at the Chandler period. Recent attempts have nevertheless been made by Haubrich & Munk (1959), Miller & Wunsch (1973), Currie (1975), Hosoyama, Naito & Sato (1976) and Naito (1977), among others. One consistent finding has been that the pole tide in both the North and Baltic Seas is significantly enhanced above its equilibrium level. This enhancement has been modelled by Wunsch (1974) as a response of the local basin topography to forcing by a non-equilibrium open-ocean pole tide at its edge. Since at the present time no other model has been proposed, this could perhaps be taken as evidence that the open-ocean pole tide in the Atlantic does depart substantially from equilibrium.

On the other hand, it has not been shown that this assumption is required to be consistent with the observations, and on theoretical grounds a non-equilibrium open-ocean pole tide at the edge of the North Sea would appear to be particularly unlikely as it is on the eastern boundary of the Atlantic. Alternative explanations of the North and Baltic Sea anomalies should, in our opinion, be sought.

The real question of interest for determining the Chandler period is whether the *globally averaged* pole tide is in equilibrium, and on this question the data founder. Miller & Wunsch conclude reluctantly that 'ports with lengthy records are too few, and too far apart, to allow one to say anything about the global structure of the pole tide ... There is no evidence in the data to either confirm or deny the equilibrium hypothesis'. Hosoyama *et al.* disagree with this conclusion, and claim to see a consistent departure from the equilibrium amplitude at both low and high latitudes, with an approach to equilibrium values in the northern hemisphere near 45° N. It is, however, at 45° N and 45° S that the equilibrium tide, being roughly proportional to $Y_2^{\pm 1}$, is maximum, decreasing to zero at both poles and the equator. Hosoyama *et al.*, unlike Miller & Wunsch, have not made any allowance for the bias in their observed amplitudes due to noise. We suspect, as a result, that rather than a latitude-dependent departure from equilibrium, they may have actually observed nothing more than the expected latitude-dependence of the signal-to-noise ratio.

Studies of the long-period tides, particularly the lunar fortnightly tide *Mf* and monthly tide *Mm*, have an obvious bearing on the equilibrium hypothesis. Since both of these tides are long compared with 4 days we would expect them both to be close to equilibrium in the open ocean if the above theoretical argument is valid. *Mf* and *Mm* are somewhat easier to study observationally than the pole tide, since the amplitudes are 2–4 times larger and the restrictions imposed by finite record lengths are diminished. The first modern observational study of *Mf* and *Mm* was undertaken by Wunsch (1967), using island stations in the Pacific Ocean. Despite high noise levels in the data, he reported substantial deviations from equilibrium, of order 50 per cent in amplitude and ± 60° in phase, with fluctuations over distances of order 3000 km. Since this fluctuation scale-length is roughly equal to the wave-length of short fortnightly and monthly Rossby waves, he concluded that the tides were exciting the Rossby modes of the Pacific Ocean.

More recently, longer tidal records from some of Wunsch's stations as well as records from a number of other island stations in both the Indian and Pacific oceans have been re-analysed by Luther (1980). Upon comparing these observations with the self-consistent equilibrium tidal calculations of Agnew & Farrell (1978), he finds a much better agreement with the equilibrium theory than did Wunsch. In particular there is no evidence in his analysis for the short-scale (3000 km) fluctuations found by Wunsch; rather the observed departures are basin-wide in scale and of order 20 per cent in amplitude and ± 20° in phase. This disagreement with the earlier study by Wunsch can probably be attributed to the higher noise levels expected with shorter records. The 20 per cent departure at a period of 14–28 days observed by Luther is roughly consistent with the departure expected if the time-scale for establishing equilibrium is $\tau_d \approx 4$ day as estimated above. The argument that the departure at $T_0 \approx 14$ month should be no more than 1 per cent is thus supported by his observations. More such analyses in other oceans could help to confirm this.

One additional recent bit of evidence that the *Mf* and *Mm* tides are close to equilibrium comes from studies of the fortnightly and monthly changes in the length of day; these are due to changes in the Earth's moment of inertia C produced by tidal deformation of the solid Earth and oceans. The combined results of Agnew & Farrell (1978) and Wahr, Sasao & Smith (1980) demonstrate convincingly that the observed changes Δ(lod) are consistent with the equilibrium hypothesis. This is an important conclusion for the present application, since it more than any other available observation pertains truly to the globally averaged *Mf* and *Mm* tides.

On the basis of all the above evidence, we think we are justified in adopting an equilibrium model of the pole tide. All of our quantitative estimates of the Chandler wobble period, and any implications we draw from them regarding mantle rheology will rest upon this assumption. In Section 10 we shall make a few remarks about the possible consequences of slight departures from equilibrium. As we shall see, our conclusions are only mildly weakened even if we admit that possibility.

5.2 THE EFFECT OF AN EQUILIBRIUM POLE TIDE ON THE PERIOD

The equilibrium hypothesis is attractive because determining the equilibrium pole tide is a relatively straightforward matter, requiring only a purely static calculation of the deformation of equipotential surfaces, together with the constraint that the total mass of the oceans is conserved. A self-consistent calculation, which takes oceanic self-attraction as well as tidal loading into account, has been performed by Dahlen (1976). By employing MacCullagh's theorem, the tensor D_{ij} in equation (4.9a) can be found. Because of the irregular geophysical distribution of the oceans, D_{ij} can no longer be written in the form (4.10a) appropriate to the axisymmetric case. We can, however, still write D_{ij} in the form

$$D_{ij} = (a^5\Omega^2/3G)d_{ij}, \tag{5.1}$$

where the coefficients d_{ij} are dimensionless. One effect of the oceans is that wobble and changes in the lod are coupled by the small but non-vanishing values of D_{13} and D_{23}; this effect, however, is minute (Dahlen 1976) and we shall ignore it here. If spin–wobble coupling is neglected the only coefficients required are d_{11}, d_{22} and $d_{12} = d_{21}$. The values of these for model 1066A are

$$\begin{aligned} d_{11} &= 0.35092, \\ d_{22} &= 0.34051, \\ d_{12} = d_{21} &= -0.00109. \end{aligned} \tag{5.2}$$

If the pole tide is equilibrium, the oceans remain affixed exactly to the mantle, so their relative angular momentum in the frame F_M is zero. The tensor E_{ij} is thus still due only to the core, and it retains the value (4.10b). If spin–wobble coupling is neglected the secular equation (4.12) generalizes in the absence of (4.10a) to

$$\Delta(\sigma_0) = \det\begin{vmatrix} i\sigma_0[A+D_{11}+E'+(\sigma_0/\Omega)E]-\Omega D_{21} & \Omega[C-A-D_{22}]-\sigma_0[E+(\sigma_0/\Omega)E']+i\sigma_0 D_{12} \\ -\Omega[C-A-D_{11}]+\sigma_0[E+(\sigma_0/\Omega)E']+i\sigma_0 D_{21} & i\sigma_0[A+D_{22}+E'+(\sigma_0/\Omega)E]+\Omega D_{21} \end{vmatrix} = 0. \tag{5.3}$$

Substituting (4.10b), (4.16), (5.1) and (5.2) into (5.3) yields the Chandler eigenfrequency of a Hough–Love–Larmor Earth with equilibrium oceans.

Inasmuch as both d_{12} and $d_{11} - d_{22}$ are small, we use the approximate solution

$$\sigma_0 = \frac{C - A - \tfrac{1}{2}(d_{11}+d_{22})a^5\Omega^2/3G}{A_\mathrm{M} + \epsilon_\mathrm{b} A_\mathrm{C} + \tfrac{1}{2}(d_{11}+d_{22})a^5\Omega^2/3G}\,\Omega. \tag{5.4}$$

This differs from the HLL result (4.15) in the absence of oceans only by the replacement of the Love number $k = 0.30088$ by the factor $\frac{1}{2}(d_{11} + d_{22}) = 0.34572$. The difference

$$\Delta k = \tfrac{1}{2}(d_{11} + d_{22}) - k = 0.04484$$

reflects the increased fluid–elastic bulge and the consequent decrease in 'restoring force' due to the equilibrium pole tide. It should be noted that equation (5.3) is not consistent with the procedure used to determine the ocean's influence on the period in both Dahlen (1976) and Dahlen (1980a), since in both of those earlier treatments the core was not properly accounted for. The upshot is that the increase in T_0 due to the oceans is essentially a factor of A/A_m larger than previously estimated. This error was also pointed out to us by T. Sasao (private communication 1980) independently of our own discovery. The error involved in approximating (5.3) by (5.4) is of order d_{12}^2 or $(d_{11} - d_{22})^2$; the latter, which is larger, is about 10^{-4}, so the corresponding error in T_0 due to the approximation is less than 0.1 day.

Our best estimate of the period can be obtained by adding the equilibrium oceanic correction Δk to the dynamical wobble Love number k_w. We are thus led to define a new wobble-effective Love number

$$k_e = k_w + \Delta k, \tag{5.6}$$

which takes the oceans into account. The corresponding improved eigenfrequency, which we shall now denote by σ_e, is then

$$\sigma_e = \frac{C - A - k_e a^5 \Omega^2/3G}{A_M + \epsilon_b A_C + k_e a^5 \Omega^2/3G}\,\Omega. \tag{5.7}$$

The value of k_e for model 1066A is

$$k_e = 0.34642$$

and the corresponding Chandler period is

$$T_e = 426.7 \text{ day}.$$

This is the Chandler period of an Earth which is everywhere perfectly elastic, and which has an equilibrium pole tide. The corresponding Chandler Q for such an Earth is of course $Q_e^{-1} = 0$ or $Q_e = \infty$. The discrepancy between these two theoretical values T_e and Q_e and the corresponding observed values T_0 and Q_0 is substantive. If $T_0 = 435.2$ day, as estimated by Wilson & Haubrich, the difference $T_0 - T_e$ is 8.5 day. Both T_e and Q_e thus differ from the observed values by more than three standard deviations. It is this discrepancy we shall now exploit to place onstraints on mantle anelasticity.

First, however, let us consider the accuracy of the elastic period T_e. According to (5.7), the factors which are relevant are $(C - A)/A$, $a^5\Omega^2/3GA$, Ω, A_M/A, k_e and $(1 + \epsilon_b A_C/A_M + k_e a^5 \Omega^2/3GA_M)$. As discussed in Section 3, the first three of these have all been determined geodetically to an accuracy which is more than adequate for our purposes. The final factor $(1 + \epsilon_b A_C/A_M + k_e a^5 \Omega^2/3GA_M) = 1.0017$ for model 1066A, and is responsible for only a 0.7 day increase in the period. It too is thus certainly known well enough that any possible error in its value can be ignored. The only really important parameters in estimating the possible error in T_e are thus the ratio $R = A_M/A$ and k_e. From equation (5.7), we find that the factional change $\delta T_e/T_e$ due to changes $\delta R/R$ and $\delta k_e/k_e$ is given by

$$\delta T_e/T_e = \delta R/R + 0.468\,(\delta k_e/k_e).$$

To estimate the possible uncertainties in $R = A_m/A$ and k_e, all we can do easily is to compare values for different seismologically derived Earth models. For model 1066B of Gilbert & Dziewonski (1975), we find that both R and k_e differ from the corresponding values for model 1066A used here by about 0.03 per cent. To be conservative, let us say that the actual uncertainties are about three times this, i.e. that both $\delta R/R$ and $\delta k_e/k_e$ are of order 10^{-3}. The overall uncertainty δT_e is then, according to (5.8), about ± 0.5 day. This is five times smaller than the current observational uncertainty of ± 2.6 day, and it will henceforth be ignored.

One of the concerns of Smith (1977) was the effect of the squared Brunt–Vaisala frequency N^2 of the core on the Chandler period. He found that if the core were highly stably stratified ($N^2 = 3.4 \times 10^{-4}\,\mathrm{s}^{-2}$), the period would be increased by 1.6 day over that corresponding to a neutral ($N^2 = 0$) core. Since then, Masters (1979) has undertaken an extensive study of the stratification in the core, using a variant of free oscillation inversion theory. He has determined that N^2 is close to zero, and has an uncertainty of about $\pm 5 \times 10^{-8}\,\mathrm{s}^{-2}$. The hypothetical model considered by Smith was thus, in this sense, at least 10 times more stable than seismic data will allow. The uncertainty in T_e due to our lack of knowledge of core stratification is, thanks to Masters, small: less than ± 0.2 day.

Finally, we should question whether or not we can trust the normal-mode theory, which was linearized in ellipticity, at the fraction-of-a-day level. We think that we can. We actually used the normal-mode calculation to determine the oceanless wobble-effective Love number k_w correct through effects of order ellipticity. This quantity in turn affects only about one-quarter of the total Chandler period. Thus errors from this source should not exceed about ± 0.3 day.

6 Damping of the Chandler wobble

An important feature of the Chandler wobble is, of course, that it has a finite Q and thus must be subject to one or more irreversible processes. In order to exploit the geophysical implications of Q_0 we must seek to identify the processes which contribute to it. All of the plausible candidates for sinks of the wobble's energy known to us fall into one of three categories:

(i) dissipation in the oceans;
(ii) bodily imperfections of elasticity; and
(iii) core–mantle coupling processes (such as viscosity, electromagnetism and topography) which were ignored in the calculations of Section 4.

This section briefly introduces (i) and (ii), since they are discussed in detail later, and discusses (iii) at greater length. Fig. 4 indicates these possibilities schematically.

With regard to (i), if the response of the oceans to wobble is exactly a state of instantaneous static equilibrium, there will be no dissipation. However, any departure from equilibrium will generally be accompanied by dissipation, if for no other reason than that there will then be slip at the ocean bottom. (It is important to draw a distinction between active and passive motions in the oceans. The former are the various currents, etc. which are driven primarily by the Earth's meteorological heat engine. These have no effect on the period and Q of the Chandler wobble; they could, however, provide a means of excitation (Wilson & Haubrich 1976; O'Connor 1980). The latter are the motions, not necessarily equilibrium, which arise directly in response to the instantaneous wobble of the mantle. These are a part of the Chandler wobble eigenfunction **s**, and they can affect its period and Q.) With regard to (ii), since the Earth is deformable, the Chandler wobble induces elastic

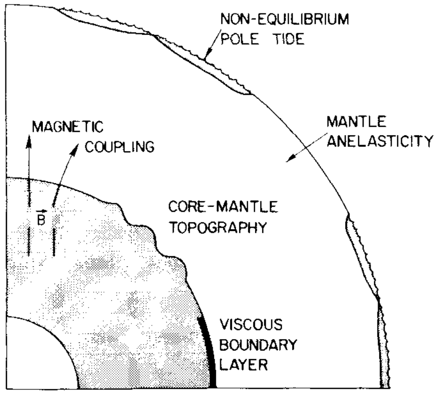

Figure 4. Schematic cross-section of the Earth illustrating the possible locations and mechanisms for dissipation of the Chandler wobble.

strain most of which, as we discuss in Section 8, is shear strain in the mantle. Most of the rest of this paper is devoted to exploring the consequences of shear loss in the mantle.

The remaining processes are core–mantle coupling mechanisms. The Chandler wobble is characterized by large relative rigid rotations of the core and mantle and thus by large transverse slip at the core–mantle boundary. In the perfectly elastic limit, only two relatively weak mechanisms tend to couple the rigid rotations of core and mantle. One of these is inertial or pressure coupling arising from the ellipticity of the core–mantle boundary, and the other is the gravitational torque generated by, and acting upon, the elliptically stratified density fields of core and mantle. Neither process is dissipative and both are accounted for in the normal mode calculations of Section 4. There are additional processes acting in the real Earth for which we have not accounted which may be important. Those of which we are aware (see Rochester 1970, for a review) are electromagnetic coupling associated with the field of the geodynamo, viscous coupling due to viscosity in the core, and topographic coupling arising from possible non-elliptical topography of the core–mantle interface.

Of the three, topographic coupling is probably the most enigmatic. We do not currently know much about the magnitude, scale-length or geography of core–mantle topography nor do we clearly understand the hydrodynamics associated with this mechanism. A critical question, e.g. is the extent to which the Taylor–Proudman theorem may dominate circulation in the core during wobble. This theorem, which strictly applies to steady (non-oscillatory) disturbances in a homogeneous rotating fluid, suggests that small bumps in the surface of the container (the core–mantle boundary) may entrain quasi-rigid columns of fluid extending deep into the core. Such a circumstance would clearly play havoc with the dynamical notions of the Chandler wobble we have outlined, but the extent to which this concept is applicable to non-steady motions of stratified compressible fluids is not clear. The best we can do here is take refuge in the claim that in the course of this study we encounter no phenomenon which requires topographic coupling for its explanation.

Viscous coupling occurs because a viscous fluid cannot slip freely past an interface but rather is dragged along by motion of the latter. For periods long compared with 1 day the thickness of the viscous Ekman boundary layer in the core is of order

$$h = (\nu/\Omega)^{1/2}, \tag{6.1}$$

where ν is the kinematic viscosity. For a rough estimate of the magnitude of the influence of core viscosity on the Chandler wobble, we greatly simplify the details of the boundary layer solution and suppose that the boundary layer behaves as a rigid shell of thickness h (in reality the transverse displacements decay exponentially with distance from the interface and h is the exponential scale length). If υ is the amplitude of rigid rotation of the mantle during wobble, and υ_B is the consequent rigid rotation of the boundary layer, then since this is a linear process we must have

$$\upsilon_B = \lambda \vartheta \tag{6.2}$$

where λ is a complex constant. In the absence of internal resonances in the core, we expect that

$$|\lambda| \leqq 1. \tag{6.3}$$

If λ vanishes, the boundary layer (and therefore the entire core) is stationary and the mantle wobbles freely by itself. If $\lambda \neq 0$, the mantle must drag the boundary layer about, thus affecting the period and, if λ is not purely real, the Q of the wobble. We can estimate this effect by computing the torque the mantle must exert to drive the boundary layer. Since $2\pi/\sigma_0$ is long compared with a day, the angular momentum conservation law for the shell can be written as

$$-\sigma_0 A_B \Omega \upsilon_B = \mathbf{\Gamma}_B, \tag{6.4}$$

where

$$A_B = (8\pi/3)\rho_0 b^4 h \tag{6.5}$$

is the moment of inertia of the boundary layer (assumed spherical, of radius b and density ρ_0) and $\mathbf{\Gamma}_B$ is the torque exerted *on* the boundary layer *by* the mantle. The torque experienced by the mantle, $\mathbf{\Gamma}_M$, must simply be $\mathbf{\Gamma}_M = -\mathbf{\Gamma}_B$, i.e.

$$\mathbf{\Gamma}_M = \sigma_0 A_B \Omega \lambda \upsilon. \tag{6.6}$$

Equation (6.6) describes the torque the mantle experiences as it drags the boundary layer along. That torque, naturally, depends upon the complex parameter λ and upon the size of the boundary layer through A_B. By adding the homogeneous torque term (6.6) to Liouville's equation for a rigid wobbling mantle with a spherical core–mantle boundary, it is straightforward to compute the complex perturbation in the mantle's wobble eigenfrequency. If $\lambda = \lambda_R + i\lambda_I$, that calculation yields

$$\delta T_0/T_0 = -\delta\sigma_0/\sigma_0 = \left[\frac{(8\pi/3)\rho_0 b^4 h}{A_M}\right]\lambda_R \tag{6.7}$$

and

$$Q_0^{-1} = \left[\frac{(16\pi/3)\rho_0 b^4 h}{A_M}\right]\lambda_I. \tag{6.8}$$

The real part of λ, equation (6.7), describes the in-phase relation of υ_B to υ and contributes only to a real change in frequency σ_0, while the imaginary part of λ, equation (6.8), describes the out-of-phase relation of υ_B to υ and contributes only to damping. If we accept that $|\lambda| \leqq 1$, then the maximum possible effect of a boundary layer of thickness h must be bounded by

$$\delta T_0/T_0 \leqq \frac{(8\pi/3)\rho_0 b^4 h}{A_M} \tag{6.9}$$

and

$$Q_0 \geqq \frac{A_M}{(16\pi/3)\rho_0 b^4 h}. \tag{6.10}$$

Note that (6.9) and (6.10) depend explicitly only upon h and not upon the mechanism which produces the boundary layer or determines h. Physically, the main assumptions we have made are that the boundary layer passively follows the mantle and that all of the dissipation occurs through mechanical work on the boundary layer. Order-of-magnitude bounds on Q_0 similar to (6.10) have been given before, using essentially the same argument, notably by Munk & MacDonald (1960) and by Jeffreys (1970). Our estimate of Q_0 is, if anything, a worst case since we have bounded λ_I by one (which would require a $\pi/2$ phase lag of the boundary layer). The bound (6.9) on $\delta T_0/T_0$ could alternatively be derived by noting that the effect of an in-phase boundary layer is simply to increase the Chandler 'inertia' A_M by the amount A_B.

Toomre (1974) has inferred an upper bound of $10^5\ \mathrm{cm^2\,s^{-1}}$ on the viscosity of the core from an estimate by Fedorov (1963) of the possible phase lag of the 18.6 yr nutation. A preliminary analysis (R. Gross, private communication 1980) of the recently recalculated and homogenized ILS latitude observations (Yumi & Yokoyama 1980) has provided a tighter bound than Fedorov's on the 18.6 yr phase lag and thus a more stringent bound on ν, *viz.*

$$\nu \leqq 4 \times 10^4\ \mathrm{cm^2\,s^{-1}}. \tag{6.11}$$

This bound, which we believe is reliable and which applies at a period of 24 hr, implies a viscous boundary layer thickness of

$$h \leqq 250\ \mathrm{m}. \tag{6.12}$$

Applying (6.9) and (6.10) to the Chandler wobble, with h from (6.12), shows that the effects of viscous coupling must be bounded by

$$\delta T_0 \leqq 0.02\ \mathrm{day} \tag{6.13}$$

and

$$Q \geqq 12\,000, \tag{6.14}$$

which we regard as negligible.

A better estimate of the kinematic viscosity of the core is thought to be that due to Gans (1972) based on the Andrade melting hypothesis; it suggests that $\nu = 10^{-2}\ \mathrm{cm^2\,s^{-1}}$ or even slightly smaller, which would greatly reduce the possibility of viscous coupling. The much larger bound on ν given by the nutation analysis of Gross is, however, important, because it is valid even if flow in the boundary layer is turbulent rather than laminar. In that case the pertinent parameter, rather than the molecular viscosity ν, would be some eddy viscosity ν_{eddy} which could be much larger than ν due to the more efficient momentum transport by macroscopic fluid parcels. At a period of 1 day, however, we know that the bound (6.11) must pertain to both ν_{eddy} and ν, whichever may be appropriate. The mechanics of turbulent coupling is not well enough understood to know how strongly ν_{eddy} may depend on frequency, but if that dependence is relatively weak, the bounds (6.13) and (6.14) show that the effect of turbulent coupling on the Chandler wobble should be small.

A correct laminar viscous boundary layer calculation accounting for the details of the distribution of fluid flow within the boundary layer has been performed recently by

T. Sasao (private communication, 1979). Using his results to estimate δT_0 and Q_0 for $\nu = 4 \times 10^4$ gives

$$\delta T_0 = 0.01 \text{ day},$$

$$Q_0 = 35\,000.$$

The agreement with our crude estimates is so good as to be clearly somewhat fortuitous, but not displeasing.

Now consider electromagnetic coupling arising from the interaction of the dynamo field and the conducting mantle. As with the oceans, it is important to distinguish between active and passive mechanisms. In the first case secular variations in the main geomagnetic field induce currents in the mantle which in turn are subject to Lorentz forces from the dynamo field. (This same mechanism causes conducting rings to fly into the air in undergraduate physics labs when those rings are subjected to a rapidly changing magnetic field.) This process is a conceivable energy source for the wobble (see, e.g. Rochester 1970) but does not damp it. Passive coupling, on the other hand, occurs when the mantle moves relative to the dynamo field. In the case of the Chandler wobble the mantle's motion causes currents to be induced in it which, as above, lead to mechanical forces by interaction with the dynamo field. The essential distinction is that in the latter case the retarding force experienced by the mantle is linear in its motion while forces induced by fluctuations in the dynamo field do not arise from motion of the mantle. The passive coupling process is similar to the damping of the oscillatory motion of a conducting pendulum in the presence of a magnetic field and, as with the pendulum, it can affect both the period and Q of the motion.

Current flowing in the mantle also leads to equal but opposite Lorentz forces acting on the conducting fluid of the core. Electromagnetic skin effects cause these stresses to be concentrated near the core's outer surface and this leads to a boundary layer coupled electromagnetically to the mantle. This boundary layer has the important property that its thickness scales like the electromagnetic skin depth in the core, or $h \sim \sigma_0^{-1/2}$, i.e. as period grows longer, h grows larger.

A correct calculation of core–mantle electromagnetic interaction is not only difficult (Loper 1975) but, more to the point, it depends upon unknown properties of the magnetic field at the core–mantle boundary. We propose, instead, to place a rather loose bound on this effect. Suppose that the non-inertial coupling at all frequencies between core and mantle is due to electromagnetic interaction. Gross' bound (6.11) on viscosity can in that case be regarded instead as a bound (6.12) on the thickness of the electromagnetic boundary layer at a period of 1 day. Then since h scales as $\sigma_0^{-1/2}$, we expect, at 14 months,

$$h \leqq 5 \text{ km}, \tag{6.13}$$

approximately. Using the latter value of h in (6.10) and (6.11) gives

$$\delta T_0 \leqq 0.4 \text{ day}$$

and

$$Q_0 \geqq 600.$$

By this estimate, the maximum effect on the period of the wobble is about 15 per cent of the current observational uncertainty and the maximum effect on Q_0 is about 15 per cent of the observed Q_0. We believe these estimates to be conservative, and we shall hereafter ignore the possibility of electromagnetic coupling between core and mantle. As we shall see below, we are supported in this decision by the observation that all of the Chandler wobble's energy can without difficulty be dissipated in mantle shear losses.

7 Models of mantle anelasticity and dispersion

There is good reason to believe that the mechanism for the dissipation of elastic energy within the Earth's mantle is linear. The best direct evidence in this regard comes from the recent study of Earth strain tides carried out by Agnew (1979). By looking for higher harmonics of the tides, he was able to demonstrate that 'for peak strains of 10^{-8} the response of the Earth is linear to 1 part in 1000'. The strains associated with the solid Earth pole tide, like those associated with the seismic normal modes after even the largest earthquakes, are at least an order of magnitude smaller than the luni-solar tidal strains. The assumption that mantle anelasticity is linear should therefore be justified in treating the attenuation of all the normal modes of the Earth, including the Chandler wobble. Roughly the same amount of extrapolation in frequency is required to make this argument in the two situations of interest, *viz.* down from the tides to the Chandler wobble or up from the tides to the seismic free oscillations.

The phenomenology of linear anelasticity has been reviewed many times, notably by Zener (1948), Gross (1953) and Nowick & Berry (1972). In an isotropic solid, dissipation can occur independently in shear or in compression, but by adopting a general notation we can discuss both simultaneously. In what follows, we shall use M to denote either the real shear modulus μ or the real bulk modulus κ, and we shall denote the corresponding quality factors Q_μ and Q_κ by Q. We shall restrict attention here to spherically symmetric models of the Earth's Q structure. The resulting errors of order ellipticity in our subsequent theoretical calculations of $T_0 - T_e$ and Q_0 are clearly negligible compared with the observational uncertainty of T_0 and Q_0. As well as depending upon radius r in the mantle, Q can depend upon frequency σ. If dissipation at a given radius is limited to a certain absorption band, as we shall assume, then outside that band Q^{-1} must look like one flank of a single Lorentzian response, that is it must be proportional to σ below the absorption band and to σ^{-1} above it. Within the band, however, Q may be prescribed arbitrarily provided its dependence on frequency is nowhere more extreme than σ or σ^{-1}. Once $Q(\sigma)$ has been specified, the corresponding dispersion or frequency dependence of the associated radial elastic structure M is determined by the restriction that the response be causal. Very approximately, we must have

$$d(\ln M)/d(\ln \sigma) \approx 2/\pi Q, \tag{7.1}$$

which shows clearly that, since Q is positive, M must decrease with decreasing frequency. The approximation (7.1) is fairly good as long as Q is large and reasonably independent of σ; if those conditions are not met, more elaborate methods must be used to determine an accurate description of the dispersion.

In this paper we shall explore the hypothesis that there is a single absorption band in which Q varies only weakly with frequency all the way from the top of the seismic frequency band, i.e. periods of the order of a fraction of a second, down to the Chandler period of 14 month. For simplicity, we shall confine attention to the one-parameter family of models for which Q varies as σ^α within this absorption band. More precisely, if σ_1 and σ_2 are two frequencies located well within the absorption band, we shall assume that

$$Q(\sigma_2)/Q(\sigma_1) = (\sigma_2/\sigma_1)^\alpha, \tag{7.2}$$

where $-1 < \alpha < 1$. The special case $\alpha = 0$ corresponds to frequency-independent Q, a model which is commonly invoked in seismic attenuation studies. In that case, the associated absorption band dispersion is described by the familiar result popularized by Kanamori &

Anderson (1977), *viz.*

$$M(\sigma_2)/M(\sigma_1) = 1 + (2/\pi Q) \ln (\sigma_2/\sigma_1). \tag{7.3}$$

Equation (7.3), which follows immediately from (7.1), is actually an approximation valid in the limit $Q \gg 1$. The exact form of the constant Q dispersion law is now known (Kjartansson 1979), but for the values of Q we shall consider, the approximation (7.3) will be adequate.

In the general case of a non-zero α, the dispersion can be described by

$$M(\sigma_2)/M(\sigma_1) = 1 - \cot (\alpha\pi/2) Q^{-1}(\sigma_1)[(\sigma_1/\sigma_2)^{\alpha} - 1], \tag{7.4}$$

which in the limit $\alpha \to 0$ reduces to (7.3). Equation (7.4) is a valid approximation so long as both σ_1 and σ_2 are located within the absorption band, and provided that $Q(\sigma_1) \gg 1$ and $Q(\sigma_2) \gg 1$. If α is small, so that cot $(\alpha\pi/2)$ can be replaced by $(2/\pi\alpha)$, then equation (7.4) follows from (7.1). The more general result for arbitrary $|\alpha| < 1$ has been obtained by first determining the strain retardation spectrum (Nowick & Berry 1972) which gives rise to the law (7.2). That spectrum is in fact given by Anderson & Minster (1979), but the dispersion law they give does not agree with (7.4), and is either in error or a misprint.

Equations (7.2)–(7.4) describe the frequency dependence of Q and the associated modulus M at a single point within the mantle. In general, then, the exponent α could be an arbitrary function of radius. In principle, we could explore this possibility quantitatively by combining attenuation data from a wide variety of the Earth's free oscillations, but in this paper we shall consider only a single value of α for the mantle as a whole. For a given α, the attenuation structure throughout the entire absorption band is then determined completely by the radial distribution of Q at some given reference frequency within the seismic band. A large number of Q models derived from free oscillation, surface wave and body wave data are now available. We shall in this paper confine attention to two extremely simple recent models: model QMU of Sailor & Dziewonski (1978) and model B of Sipkin & Jordan (1980).

Model QMU was obtained by parameter-space inversion of 38 selected normal mode attenuation measurements. It is a two-shell model with the boundary at 670 km depth within the mantle. The value of Q_μ in the upper mantle is 111, and that in the lower mantle is 350. There is no bulk dissipation, i.e. $Q_\kappa = \infty$ throughout. This simple two-shell model satisfied their data set as well as did models with more detail. The radial modes, particularly ${}_0S_0$, were not well satisfied by model QMU, and that discrepancy was used by Sailor & Dziewonski to argue for bulk dissipation. The Q of ${}_0S_0$ has, however, since been re-determined by Knopoff *et al.* (1979) and Riedesel *et al.* (1980) to be about 50 per cent higher than the value adopted by Sailor & Dziewonski, and that weakens their argument for significant bulk dissipation accordingly. Model QMU, being derived from normal mode observations, is indicative of the Q distribution within the seismic normal mode band; we shall assume it is an appropriate description at a period of 200 s, the approximate midpoint of that band.

The average value of Q_μ at 200 s in the upper mantle is probably fairly well determined. In particular, the value $Q_\mu = 111$ of model QMU is consistent with a wide body of recently acquired surface wave and free oscillation data which sample primarily the upper mantle (see Anderson & Hart 1978, for a recent review). The Chandler wobble is, however, as we shall see, much more sensitive to the Q of the lower mantle than that of the upper mantle. The lower mantle value $Q_\mu = 350$ of model QMU was probably primarily determined by the measured Q's of a rather small number of low-frequency modes such as ${}_0S_2$. Measurements of the Q of ${}_0S_2$ range from 370 to 815 (see the summary by Stein & Geller 1978), and the value $Q_\mu = 350$ for the lower mantle may not be as well determined as is Q_μ of the upper mantle. It is not for example consistent with the recent careful measurements of $\bar{Q}_{ScS}$ made by Sipkin & Jordan (1980). The predicted value of $\bar{Q}_{ScS}$ for model QMU is 213, whereas Sipkin & Jordan's best estimate for the average Earth is $\bar{Q}_{ScS} = 170 \pm 20$. This discrepancy led Sipkin & Jordan to introduce their model B, which is also a two-shell model with the boundary at 670 km depth. It has $Q_\mu = 108$ in the upper mantle, $Q_\mu = 225$ in the lower mantle, and predicts $\bar{Q}_{ScS} = 170$ exactly. Since *ScS* waves do not involve compression, Sipkin & Jordan do not stipulate Q_κ, but we shall again assume that $Q_\kappa = \infty$. The *ScS* waves they studied were recorded on High Gain Long Period seismometers, operating in the frequency interval 6–60 mHz. The approximate midpoint of this band is 30 s, and we take that value as the reference period for model B.

As Sipkin & Jordan have pointed out, the discrepancy between the normal mode model QMU and the observation that $\bar{Q}_{ScS} = 170$ could be explained in terms of a frequency-dependence of Q_μ in the lower mantle between 30 s and 200 s. If that is so, it cannot be consistent with equation (7.2) unless the exponent α is negative; in particular it implies $\alpha \approx -0.2$. A positive value of α is, on the other hand, definitely implied by the Chandler wobble observations, $T_0 - T_e$ and Q_0, as we shall see. There is of course no reason why a single empirical law like (7.2) should be expected to hold with a constant value of α across a span of eight decades from a fraction of a second to 14 month, and indeed the above inconsistency may imply that it does not. The conclusions we shall draw here regarding α should be indicative of the gross frequency dependence of Q_μ in the lower mantle in the band between a few tens or hundreds of seconds and 14 month. It should be possible in the very near future to make use of improved estimates of the Q's of such modes as ${}_0S_2$ to determine the frequency dependence of Q_μ in the seismic band, so that we may see with more certainty whether it is or is not consistent with the conclusions we draw here from the Chandler wobble. Our only reason for making use of the two Q models QMU and B in this study is to determine the sensitivity of our conclusions to the value of Q_μ in the lower mantle, since that value may not be well determined. The fact that model QMU is thought to be appropriate at 200 s and model B at 30 s is relatively inconsequential in this application since, from the vantage point of 14 month, those two periods are fairly close.

The elastic moduli κ and μ of model 1066A have been obtained by the inversion of free oscillation eigenfrequencies, without allowing for dispersion (Gilbert & Dziewonski 1975). They therefore also correspond to some average picture of the elastic structure within the normal mode band, and we shall take them too to be the description appropriate at 200 s, regardless of whether model QMU or model B is used as the attenuation model. Density ρ_0 does not vary since it reflects simply the atomic composition of the Earth.

In summary, the complete specification of the anelastic, dispersive Earth at the Chandler period is uniquely determined by:

(i) specification of a perfectly elastic Earth model ρ_0, κ, μ and the period at which it is appropriate (in our case model 1066A at 200 s);

(ii) specification of a Q model Q_κ, Q_μ and the period at which it is appropriate (in our case model QMU at 200 s or model B at 30 s); and

(iii) specification of the frequency dependence of Q_κ and Q_μ over a domain containing, but not limited to, both seismic and Chandler periods, i.e. in our case specification of the exponent α in equation (7.2).

Each of the two Q models QMU and B leads, via equations (7.2)–(7.4), to a one-dimensional family of frequency-dependent anelastic Earth models parametrized by α. Since in this paper we always assume that $Q_\kappa = \infty$, only the rigidity μ exhibits dispersion. In the case of model B, the Q_μ structure appropriate at 30 s must be adjusted to 200 s before being employed in equation (7.4) to determine the shear modulus discrepancy $\delta\mu$.

8 The period and Q for a dissipative Earth

The number of theoretical Chandler periods and corrections to them which we have employed has gotten rather large. In defence, we would like to take refuge in Jeffrey's famous assertion that responsibility for this complexity lies with the Earth, and not with the authors. Fig. 5 summarizes the path from rigid to dissipative Earth. It schematically portrays the rigid-Earth Chandler period, equation (4.13), and the sequence of modifications (4.13) → (4.15) → (4.22) → (5.7) we have pursued so far to the elastic period T_e. This section deals with the uppermost, smallest and, above all, last modification shown, that due to anelastic dispersion.

The manner in which the period T_e will be affected by dispersion is, from a physical point of view, clear. We see from equation (7.1) that the rigidity μ will decrease with decreasing frequency if the Earth is dissipative. On an oceanless Earth, a decrease in μ will decrease the Chandler wobble 'restoring force' since the instantaneous elastic part of the bulge will be larger. On an Earth with oceans there is an additional effect, *viz.* the amplitude of the pole tide will also be decreased, and this by itself will act to increase the wobble 'restoring force' since the pole tide's contribution to the instantaneous bulge will be less. The relative importance of these two effects is roughly in the ratio of k_e to Δk, i.e. the direct effect of the larger elastic bulge is about seven times greater than the effect of the decreased pole tide, and so the net effect of a positive α will be an increase in the elastic period T_e. A further effect of the oceans is to give rise to additional mantle damping due to the strains associated with the pole tide loading. The relative importance of this effect compared with the mantle damping due to the solid Earth pole tide itself is, as we shall see, also of order $\Delta k/k_e$. To use $T_0 - T_e$ and Q_0 to constrain α, we must be able to compute the theoretical change in T_0 due to a change in μ as well as the theoretical Q_0 associated with a given distribution of Q_μ. In Section 8.1 we shall show how to do this on an oceanless Earth, and in Section 8.2 we shall consider the effect of the oceans.

8.1 NORMAL MODE PERTURBATION THEORY FOR AN OCEANLESS EARTH

Estimation of the effect of small perturbations in the Earth's rheology upon T_0 and, especially, Q_0 of the Chandler wobble has been a perilous and, in retrospect, unrewarding undertaking for a variety of investigators. These estimates have usually been based upon

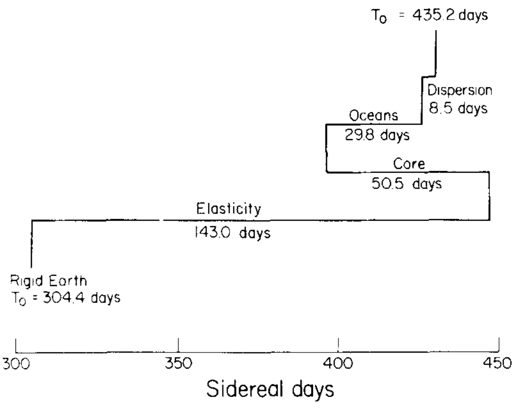

Figure 5. Schematic summary of the extent to which the rigid Earth Chandler period T_0 = 304.4 day is modified by the effects of elasticity, the fluid core, equilibrium oceans and mantle dispersion to its observed value T_0 = 435.2 day.

attempts to unravel the difficult question of energy partition during wobble in an elastic, gravitating Earth. In this section we give a unified and rigorous treatment of the problem, consisting of an application of normal-mode perturbation theory or, equivalently, Rayleigh's principle.

We begin with a stationary condition on the normal mode eigensolution of a rotating Earth (Dahlen & Smith 1975). Define the inner product

$$(\mathbf{s}', \mathbf{s}) = \int_E \rho_0 \mathbf{s}' \cdot \mathbf{s}^* \, dv \tag{8.1}$$

where the asterisk denotes complex conjugation and E denotes the Earth. If $(\sigma_0, \mathbf{s})$ is a normal mode eigensolution then σ_0 and $\mathbf{s}$ must satisfy

$$\sigma_0^2 T(\mathbf{s}, \mathbf{s}) - 2\sigma_0 W(\mathbf{s}, \mathbf{s}) - E(\mathbf{s}, \mathbf{s}) - \Gamma(\mathbf{s}, \mathbf{s}) = 0, \tag{8.2}$$

where

$$T(\mathbf{s}, \mathbf{s}) = (\mathbf{s}, \mathbf{s}), \tag{8.3}$$

$$W(\mathbf{s}, \mathbf{s}) = (\mathbf{s}, i\boldsymbol{\Omega} \times \mathbf{s}), \tag{8.4}$$

$$E(\mathbf{s}, \mathbf{s}) = \int_E \mathbf{T} : \boldsymbol{\epsilon}^* \, dv, \tag{8.5}$$

and

$$\Gamma(\mathbf{s}, \mathbf{s}) = \int_E [\rho_0 \mathbf{s} \cdot \nabla \phi_1^* + p_0(\nabla \mathbf{s} : \nabla \mathbf{s}^* - |\nabla \cdot \mathbf{s}|^2) + \rho_0 \mathbf{s} \cdot \nabla \nabla (\phi_0 + \psi) \cdot \mathbf{s}^*] \, dv \tag{8.6}$$

where ψ is the centripetal potential and p_0 is the hydrostatic prestress field satisfying

$$\nabla p_0 + \rho_0 \nabla(\phi_0 + \psi) = 0. \tag{8.6}$$

The inner product T and the other three bilinear forms W, E and Γ are bilinear functionals of $\mathbf{s}$ if we stipulate that $\mathbf{T}$ and ϕ_1 are defined by equation (4.20).

The important property of equation (8.2) is that it is stationary under small perturbations in the eigenfunction $\mathbf{s}$. If $(\sigma_0, \mathbf{s})$ is a normal mode solution, then (8.2) holds with an error of order $|\boldsymbol{\delta}\mathbf{s}|^2$ when we replace $\mathbf{s}$ by $\mathbf{s} + \boldsymbol{\delta}\mathbf{s}$ where $\boldsymbol{\delta}\mathbf{s}$ is small but otherwise arbitrary; this is Rayleigh's principle for a rotating Earth. Suppose $\oplus$ is an Earth model, and that T, W, E and Γ are its associated bilinear forms and $(\sigma_0, \mathbf{s})$ is one of its normal modes. Form a new Earth model $\oplus + \delta\oplus$ by perturbing $\oplus$ and let $\oplus + \delta\oplus$ have eigensolution $(\sigma_0 + \delta\sigma_0, \mathbf{s} + \boldsymbol{\delta}\mathbf{s})$ with functionals $T + \delta T$, etc. Rayleigh's principle means that for a small perturbation $\delta\oplus$ equation (8.2) remains true with only a second-order error if we replace σ_0 by $\sigma_0 + \delta\sigma_0$, each functional with its perturbed form, but do not replace $\mathbf{s}$ by $\mathbf{s} + \boldsymbol{\delta}\mathbf{s}$. This leads to

$$\frac{2\delta\sigma_0}{\sigma_0} = \frac{\delta E(\mathbf{s}, \mathbf{s}) + \delta\Gamma(\mathbf{s}, \mathbf{s}) - \sigma_0^2 \delta T(\mathbf{s}, \mathbf{s}) + 2\sigma_0 \delta W(\mathbf{s}, \mathbf{s})}{\sigma_0^2 T(\mathbf{s}, \mathbf{s}) - \sigma_0 W(\mathbf{s}, \mathbf{s})} \tag{8.8}$$

which must be correct through terms of first order in the perturbation.

We are interested in the effects of complex perturbations $\delta\kappa$ and $\delta\mu$ in κ and μ. In this instance, δT, δW and $\delta\Gamma$ vanish and it is easy to show that

$$\delta E = \int_E [\delta\kappa |\nabla \cdot \mathbf{s}|^2 + 2\delta\mu \boldsymbol{\delta} : \boldsymbol{\delta}^*] \, dv, \tag{8.9}$$

where

$$\boldsymbol{\delta} = \boldsymbol{\epsilon} - \frac{1}{3}(\nabla \cdot \mathbf{s})\mathbf{I}$$

is the deviatoric strain. If all effects of order ellipticity are ignored in evaluating the volume integral in (8.9) it can be simplified considerably. This procedure gives rise to an error of about 0.3 per cent in $\delta\sigma_0/\sigma_0$ and Q_0^{-1}, which is acceptable since $T_0 - T_e$ is only 8.5 day. Let us rewrite (8.9) in the form

$$\delta E = (\sigma_0^2 T - \sigma_0 W) \int_0^a [(\delta\kappa/\kappa)\kappa K + (\delta\mu/\mu)\mu M] r^2\, dr, \tag{8.10}$$

where $(\sigma_0^2 T - \sigma_0 W)$ is a scalar depending upon $(\sigma_0, \mathbf{s})$ and the Earth model, and K and M are scalar functions of radius determined from s and computed in a straightforward way from (8.9).

Combining (8.10) with (8.8) gives

$$2\delta\sigma/\sigma_0 = \int_0^a [(\delta\kappa/\kappa)\kappa K + (\delta\mu/\mu)\mu M] r^2\, dr, \tag{8.11}$$

which is what we need. If $\delta\kappa, \delta\mu$ are real, (8.11) enables us to compute their effect upon the normal mode eigenfrequency σ_0. If Q_κ, Q_μ are the Q's of compression and shear at the frequency σ_0, then (8.11) leads to Q_0, the Q of the normal mode

$$Q_0^{-1} = \int_0^a [Q_\kappa^{-1}\kappa K + Q_\mu^{-1}\mu M] r^2\, dr. \tag{8.12}$$

Table 4 lists values of κK and μM (in km^{-3}) versus radius for the Chandler eigenfunction of 1066A. The hybrid oceanless period of 396.9 day has been used to evaluate $\sigma_0^2 T - \sigma_0 W$. The contributions to the strain from $\boldsymbol{\tau}_1^1$ and $\boldsymbol{\tau}_3^1$ as well as that of the dominant term $\boldsymbol{\sigma}_2^1$

Table 4. Model 1066A: Chandler wobble Fréchet kernels for κ and μ.

Radius	μM(km^{-3})	κK(km^{-3})	L_M	L_K
3484.3	0.36663×10^9	0.10285×10^8	0.552154×10^{-5}	0.168988×10^{-1}
3691.4	0.26478×10^9	0.32558×10^7	0.824828×10^{-1}	0.185632×10^{-1}
3899.2	0.20774×10^9	0.47182×10^6	0.152385×10^0	0.190382×10^{-1}
4107.0	0.16942×10^9	0.49832×10^5	0.214764×10^0	0.190769×10^{-1}
4349.4	0.13527×10^9	0.11553×10^7	0.280374×10^0	0.193013×10^{-1}
4557.2	0.11132×10^9	0.27481×10^7	0.330981×10^0	0.200990×10^{-1}
4730.3	0.93965×10^8	0.42688×10^7	0.369203×10^0	0.214098×10^{-1}
4938.1	0.75445×10^8	0.61331×10^7	0.410171×10^0	0.239436×10^{-1}
5145.9	0.59438×10^8	0.79180×10^7	0.445656×10^0	0.276699×10^{-1}
5353.7	0.45371×10^8	0.93310×10^7	0.475475×10^0	0.326351×10^{-1}
5561.5	0.35434×10^8	0.10616×10^8	0.500332×10^0	0.388334×10^{-1}
5700.0	0.26278×10^8	0.98723×10^7	0.513975×10^0	0.434070×10^{-1}
5700.0	0.26278×10^8	0.98721×10^7	0.513975×10^0	0.434070×10^{-1}
5950.0	0.15123×10^8	0.76197×10^7	0.530727×10^0	0.507335×10^{-1}
6180.6	0.94746×10^7	0.55573×10^7	0.540773×10^0	0.561755×10^{-1}
6360.0	0.85823×10^7	0.57084×10^7	0.546960×10^0	0.600996×10^{-1}
6360.0	0.21104×10^7	0.11498×10^7	0.546960×10^0	0.600996×10^{-1}
6371.0	0.21741×10^7	0.10924×10^7	0.547055×10^0	0.601496×10^{-1}

have been accounted for in calculating κK and μM. Both kernels κK and μM are non-vanishing in the inner-core and κK is non-vanishing in the outer core, but the contributions from these two regions are slight and we do not show them. The shear kernel μM is everywhere larger than κK and, as we shall see, the Chandler wobble is about nine times more sensitive to fluctuations in rigidity than to fluctuations in incompressibility.

The last two columns of Table 4 are numerical values of the integrals

$$L_K(r) = \int_0^r \kappa K r^2\, dr \tag{8.13}$$

and

$$L_M(r) = \int_0^r \mu M r^2\, dr. \tag{8.14}$$

These quantities ease the evaluation of (8.11) and (8.12) when $\delta\kappa/\kappa$, $\delta\mu/\mu$, Q_κ or Q_μ are piecewise constant functions of radius. For example, suppose we wish to compute Q_0 from (8.12), given that $Q_\kappa = \infty$ and

$$Q_\mu^{-1} = \begin{cases} 350 & 3484 \leqq r \leqq 5700 \\ 111 & 5700 \leqq r \leqq 6371 \end{cases}.$$

In that case we find

$$Q_0^{-1} = \frac{1}{350}\,[L_M(5700) - L_M(3484)] + \frac{1}{111}\,[L_M(6371) - L_M(5700)]$$

or

$$Q_0 = 566.$$

Since the parameters chosen are those of model QMU, the value $Q_0 = 566$ is the predicted Chandler Q for that model if Q_μ is frequency-independent and pole-tide loading is ignored. The corresponding value for model B is

$$Q_0 = 386.$$

The relative importance of shear as compared with bulk dissipation in damping the Chandler wobble is illustrated by the ratio

$$\frac{L_M(6371)}{L_K(6371)} = 9.1.$$

Also note that

$$\frac{L_M(6371) - L_M(5700)}{L_M(5700) - L_M(3484)} = \frac{1}{15.5},$$

showing the strong dominance of the lower mantle.

Fig. 6 shows the results of several different calculations of $\mu M(r)$ for the mantle of 1066A. The solid curve labelled CW is the result for the Chandler wobble eigenfunction using the theory outlined above and the rotating normal-mode eigenfunction. It is the most accurate answer for an oceanless Earth and is the quantity tabulated in Table 5. The long-dashed curve labelled HLL is an estimate of $\mu M(r)$ based on HLL theory. The Chandler

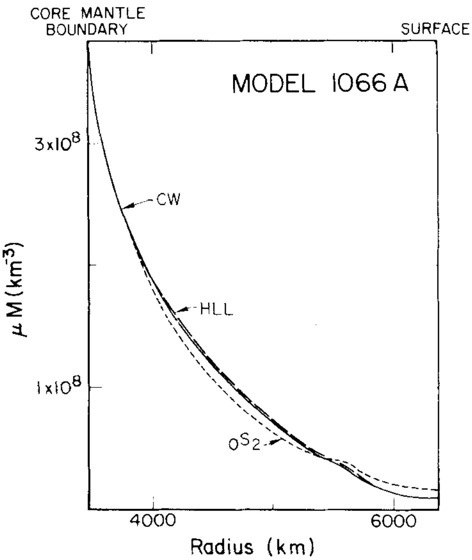

Figure 6. Plot of the Chandler wobble's shear modulus Fréchet kernel μM in the mantle for model 1066A calculated using both normal mode and HLL theory and compared with the corresponding kernel for the ${}_0S_2$ spheroidal mode. Note the steep rise with depth in the lower mantle emphasizing the relative importance of that region in determining the period and Q of both normal modes. If pole-tide straining is ignored, the Q's of the Chandler wobble and of ${}_0S_2$ can be compared directly with no additional 'corrections'.

wobble eigenfunction was taken to be exactly a rigid rotation plus the $\boldsymbol{\sigma}_2^1$ static tidal deformation caused by the wobble's varying centripetal potential, and that eigenfunction was used in the perturbation theoretic expressions given above. Note that since only the elastic properties of the model are being perturbed, the numerator of (8.7) will involve only $\boldsymbol{\sigma}_2^1$ and not rigid rotational motion. The denominator $\sigma_0^2 T - \sigma_0 W$ generally involves both but it is easy to show that in the case of the Chandler wobble where $\sigma_0 \ll \Omega$ this quantity is almost entirely due to that portion of W associated with rigid rotation. In constructing the HLL curve shown, we have used the approximate equation (A.21) in Appendix A for $\sigma_0^2 T - \sigma_0 W$. This calculation did not require the full panoply of rotating normal-mode theory, but is in extremely good agreement with it and, as such, provides a way to reconstruct or expand the results presented here without recourse to the numerical complexities of rotating normal-mode calculations.

The short-dashed curve labelled ${}_0S_2$ is the functional $\mu M(r)$ for the fundamental elastic mode of model 1066A. This result is based upon standard spherically symmetric normal-mode theory. (Note that the quantity being perturbed is now the angular eigenfrequency

Table 5. Value of $\chi = [\delta(\Delta\sigma_0)/\Delta\sigma_0]/[(\delta\mu/\mu)_{\text{lower mantle}}]$ for models QMU, B, and for a constant Q_μ model.

$\dfrac{(\delta\mu/\mu)_{\text{upper mantle}}}{(\delta\mu/\mu)_{\text{lower mantle}}}$	Corresponding Q_μ model	χ
111/350 = 0.317	model QMU of Sailor & Dziewonski	0.50
108/225 = 0.480	model B of Sipkin & Jordan	0.47
1	Q_μ independent of depth	0.45

of ${}_0S_2$, not that of the Chandler wobble.) The remarkable similarity between the latter result and that for CW means that these two normal modes sample the mantle and reflect its anelasticity at their separate periods in very similar ways. If it were not for the oceans, their respective Q's would be essentially measures of the same depth-average of the shear Q of the mantle at two very different frequencies. Since the effect of pole-tide loading is to decrease the theoretical Q_0 for a given distribution Q_μ by a little over 10 per cent and since there is no such effect on ${}_0S_2$, the similarity between the two modes is actually somewhat less than Fig. 6 indicates.

For the two Q models we consider here $Q_\kappa = \infty$ and so only the rigidity μ exhibits dispersion. In addition $Q_\mu = \infty$ in the inner core so the integration in (8.11) and (8.12) only needs to be carried out over the mantle and crust $b \leqq r \leqq a$. Let σ_{m} be the Q model's defining frequency and let $Q_\mu(r, \sigma_{\mathrm{m}})$ be the model's shear Q at that frequency. Define q by

$$q = \int_b^a Q_\mu^{-1}(r, \sigma_{\mathrm{m}})\mu(r)M(r)r^2\,dr. \tag{8.15}$$

Then the Chandler Q in the absence of oceans is obtained from

$$Q_0^{-1} = (\sigma_{\mathrm{m}}/\sigma_0)^\alpha q, \tag{8.16}$$

and the fractional change in rigidity at radius r is, from equation (7.4), given by

$$\delta\mu(r)/\mu(r) = -\cot(\alpha\pi/2)\,[(\sigma_{\mathrm{m}}/\sigma_0)^\alpha - 1]\,Q_\mu^{-1}(r, \sigma_{\mathrm{m}}). \tag{8.17}$$

Putting (8.17) into equation (8.11) yields the fractional change in Chandler eigenfrequency

$$\delta\sigma_0/\sigma_0 = -\tfrac{1}{2}\cot(\alpha\pi/2)\,[(\sigma_{\mathrm{m}}/\sigma_0)^\alpha - 1]q \tag{8.18}$$

apart from the perturbation in the ocean correction.

8.2 THE ADDITIONAL EFFECT OF THE OCEANS

Since the additional effect of the oceans is slight, we shall resort to a simple (but surprisingly good) approximation to calculate it. Let us denote the oceanic contribution to the response tensor D_{ij} by ΔD_{ij}. On a dissipationless Earth, as we have seen in Section 5, ΔD_{ij} is given by

$$\Delta D_{ij} = (d_{ij} - k\delta_{ij})(a^5\Omega^2/3G). \tag{8.19}$$

From (5.2), the components of $\Delta d_{ij} = d_{ij} - k\delta_{ij}$ for model 1066A are

$$\begin{aligned} \Delta d_{11} &= 0.05004, \\ \Delta d_{22} &= 0.03963, \\ \Delta d_{12} &= \Delta d_{21} = -0.00109. \end{aligned} \tag{8.20}$$

The amount $\Delta\sigma_0$ by which the oceans reduce the frequency σ_0 is given very nearly, according to (4.21) and (5.7), by

$$\Delta\sigma_0 = -\tfrac{1}{2}(\Delta d_{11} + \Delta d_{22})(a^5\Omega^2/3GA_{\mathrm{M}})\Omega. \tag{8.21}$$

The effect of dispersion will be to decrease the absolute value $|\Delta\sigma_0|$, since the amplitude of the equilibrium pole tide will be smaller on a less rigid Earth (on a perfectly fluid Earth, there would be no ocean bottom and thus no distinct pole tide at all). The lengthening of the period by the oceans will thus be less than it would be on a dissipationless and therefore dispersion-free Earth. An exact calculation of the dispersive perturbation to $\Delta\sigma_0$ would require us to calculate the perturbation to the tensor Δd_{ij}. This would be a rather large task, and instead we shall proceed as follows.

If the Earth were completely covered by oceans, the equilibrium pole tide would be a pure $Y_2^{\pm 1}$ harmonic, and the tensor Δd_{ij} would be given by (Dahlen 1976) $\Delta d_{ij} = \Delta d_{ij}\delta_{ij}$, where

$$\Delta d_0 = \frac{3}{5}\,(\rho_{\mathrm{w}}/\bar{\rho}_0)\left[\frac{(1+k')(1+k-h)}{1-\tfrac{3}{5}(\rho_{\mathrm{w}}/\bar{\rho}_0)(1+k'-h')}\right]. \tag{8.22}$$

The factors k, h, k' and h' are the Love numbers and load Love numbers of degree two, $\rho_{\mathrm{w}} = 1.025\ \mathrm{g\ cm^{-3}}$ is the density of sea water and $\bar{\rho}_0 = 5.517\ \mathrm{g\ cm^{-3}}$ is the mean density of the Earth. Tidal loading and the gravitational self-attraction of the oceans have both been taken into account in the formula (8.22), i.e. it is exact for a water-covered Earth. If the Earth were water-covered, the frequency shift $\Delta\sigma_0$ would, by (8.21), be given by

$$\Delta\sigma_0 = -\,\Delta d_0(a^5\Omega^2/3GA_{\mathrm{M}})\Omega. \tag{8.23}$$

To account approximately for the continents, we shall simply multiply this by the fraction of the Earth's surface area covered by oceans; according to Sverdrup *et al.* (1942), that fraction is 0.708. The approximation we shall use is thus

$$\Delta\sigma_0 = -\,0.708\,\Delta d_0(a^5\Omega^2/3GA_{\mathrm{M}})\Omega. \tag{8.24}$$

A similar approximation was first used by Larmor (1915). For model 1066A, the quantity $0.708\Delta d_0 = 0.0465$, which agrees rather well with the exact value $\tfrac{1}{2}(d_{11}+d_{22}) = 0.04484$, so the approximation is fairly good. Its virtue of course is that it allows us to consider the perturbations to only the degree 2 Love numbers k, h, k' and h', without taking the others into account.

For convenience, let us introduce the notation $\lambda = 1+k-h$ and $\lambda' = 1+k'-h'$. From (8.22) and (8.24), we find that the perturbation $\delta(\Delta\sigma_0)$ in $\Delta\sigma_0$ due to perturbations $\delta\lambda$ and $\delta\lambda'$ is given by

$$\delta(\Delta\sigma_0)/\Delta\sigma_0 = 2(\delta\lambda/\lambda) + 0.234\,(\delta\lambda'/\lambda'). \tag{8.25}$$

In deriving (8.25), we have made use of the reciprocity relation $k' = k - h$ established independently by Molodensky (1977) and Saito (1978). We have calculated $\delta\lambda$ and $\delta\lambda'$ due to a perturbation $\delta\mu$ in the rigidity of the mantle of model 1066A by straightforward numerical differentiation. Three separate cases have been considered; in every case $\delta\mu/\mu$ was taken to be constant in the lower mantle below 670 km depth and in the upper mantle above 670 km depth; the ratio of the two constant values has, however, been varied. The three cases considered and the final results are shown in Table 5 in the easily used form

$$\delta(\Delta\sigma_0)/\Delta\sigma_0 = \chi(\delta\mu/\mu)_{\text{lower mantle}}. \tag{8.26}$$

The factor χ changes very little, from $\chi = 0.50$ for model QMU to $\chi = 0.45$ if Q_μ is independent of depth in the mantle. Depending on the exponent α, the rigidity of the lower mantle may be as much as 10 per cent less at the Chandler period than in the seismic band.

The corresponding decrease in the period T_0 due to the effect of dispersion on the amplitude of the pole tide could thus be as much as a day and a half.

The amount $\delta(Q_0^{-1})$ by which Q_0^{-1} is increased by the additional mantle damping due to the pole tide loading strain can be readily determined by considering an imaginary perturbation $\delta\mu/\mu = iQ_\mu^{-1}$ in (8.26). This leads to the result

$$\delta(Q_0^{-1}) = 2\chi|\Delta\sigma_0/\sigma_0|(Q_\mu^{-1})_{\text{lower mantle}}, \tag{8.27}$$

where, as in (8.26), the factor χ is given in Table 5.

9 Implications for mantle anelasticity

The final expression for the theoretical Chandler eigenfrequency, when both anelastic effects are included, is

$$\sigma_0 = \sigma_e + \delta\sigma_0 + \delta(\Delta\sigma_0). \tag{9.1}$$

The perturbation $\delta\sigma_0$ is calculated by multiplying $\delta\sigma_0/\sigma_0$ from equation (8.18) times the elastic oceanless eigenfrequency $2\pi/(396.8$ day) calculated in Section 4 while $\delta(\Delta\sigma_0)$ is calculated by multiplying equations (8.21) and (8.25). The two perturbations $\delta\sigma_0$ and $\delta(\Delta\sigma_0)$ are of opposite sign, the former negative because a decrease in rigidity decreases the size of the elastic bulge following the wobble, and the latter positive because the pole tide, which also follows the wobble, is reduced. The final expression for the theoretical Q_0^{-1} is, likewise, given by the sum of equations (8.16) and (8.27). Both contributions to Q_0^{-1} are positive, since both processes are energy sinks.

Fig. 7 shows the locus of theoretical (T_0, Q_0) for model 1066A for each of the Q models QMU and B, as a function of the exponent α. The best estimate of Wilson & Haubrich for

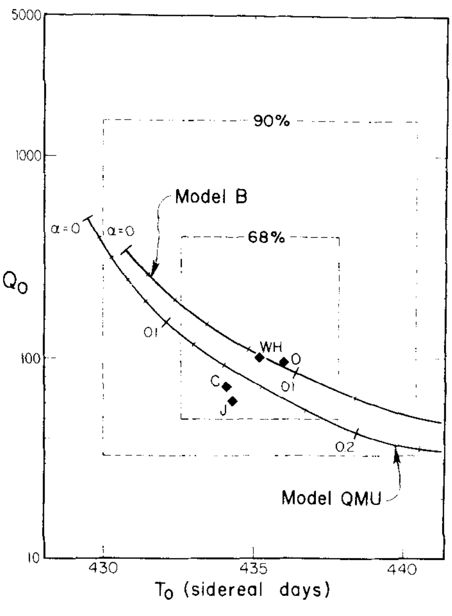

Figure 7. Plot of T_0 versus Q_0 for the Chandler wobble showing the preferred estimates of Wilson & Haubrich (WH) with their 68 and 90 per cent confidence limits as well as the preferred estimates of Currie (C), Jeffreys (J), and Ooe (O). The predicted values of T_0 and Q_0 for dissipation models QMU and B are shown as solid curves parametrized by α with tick marks at intervals of 0.02.

observed (T_0, Q_0) is shown (as well as values found by other investigators) together with the 68 per cent and 90 per cent confidence boxes. Fig. 7, the principal result of this paper, has a number of interesting features.

In particular, it can be seen that frequency independent Q_μ models (the points for which $\alpha = 0$) are not terribly bad from a Chandler wobble point of view. Model B, in particular, lies inside the 90 per cent box; QMU is farther off, being just outside the 90 per cent box, because of its greater lower-mantle Q_μ. The effect of extra pole-tide straining is to reduce the $\alpha = 0$ values of Q_0 from 566 to 504 for model QMU and from 386 to 343 for model B. The latter falls as shown below the upper 68 per cent confidence limit of Wilson & Haubrich.

For model B, the 68 per cent box implies a range

$$0.04 \leqq \alpha \leqq 0.11, \tag{9.2}$$

with $\alpha = 0.09$ preferred, while for QMU

$$0.11 \leqq \alpha \leqq 0.19, \tag{9.3}$$

with $\alpha = 0.15$ preferred. Somewhat surprisingly, in both cases the bounds are established by the observed period and *not* the observed Q.

As mentioned in Section 8, the effects of anelasticity upon (T_0, Q_0) are dominated by the contribution from the lower mantle. In the absence of pathological behaviour in the upper mantle and at the risk of some oversimplification, we might consider the bounds on α to be those appropriate to the lower mantle (below 670 km depth). The sensitivity of the bounds on α to the value Q_μ in the lower mantle can be easily gauged from a comparison of models QMU and B; as shown in Fig. 7 that sensitivity is significant but not extreme. In general the lower Q_μ in the lower mantle is, the less frequency dependence is required to be consistent with the data.

The constraints placed by (9.2) and (9.3) on the frequency dependence of Q_μ in the lower mantle are lower than those inferred in previous investigations, in particular by Jeffreys (1978) and Anderson & Minster (1979). In addition, and more importantly, they have been derived in a more rigorous manner. The two hypotheses that the pole tide is equilibrium and that the Chandler wobble is entirely damped by mantle anelasticity have been shown to be definitely consistent with the observed Chandler period and Q. Only a slight decrease in Q_μ with decreasing frequency is required to obtain excellent agreement with both T_0 and Q_0. The implied frequency dependence of Q_μ within the seismic band is shown in Fig. 8 for model QMU with its best-fitting value of α, *viz.* $\alpha = 0.15$, and for model B with its best-fitting value $\alpha = 0.09$. In the first case, Q_μ decreases from 350 and 111 at 200 s to 57 and 18 at 435.2 day. In the second, it decreases from 225 and 108 at 30 s to 64 and 31 at 435.2 day. The variation produced by an exponent as high as $\alpha = 0.15$ is sufficiently strong that it could perhaps be detected by a few high-quality attenuation observations widely spaced within the seismic band itself. The variation produced by $\alpha = 0.09$ is weaker and would be more difficult to detect. It is interesting to note that although model QMU with $\alpha = 0.15$ does not agree with the long-period (30 s) $\bar{Q}_{ScS}$ observations of Sipkin & Jordan (1980), it does agree rather well with the short-period observations collected earlier by Sipkin & Jordan (1979). They found that $\bar{Q}_{ScS}$ at high frequencies (0.3 to 3 s period) was of order 750, and interpreted this as evidence that the top of the absorption band was near 1 s. An alternative interpretation may be that a period of 1 s is still within the absorption band and that α is fairly large, near $\alpha = 0.15$.

The associated dispersion $\delta\mu/\mu$ within the seismic band is shown in Fig. 9 for models QMU and B with $\alpha = 0.15$ and $\alpha = 0.09$, respectively, as well as that which would be present in both cases if Q_μ were frequency-independent ($\alpha = 0$). It can be seen that the differences

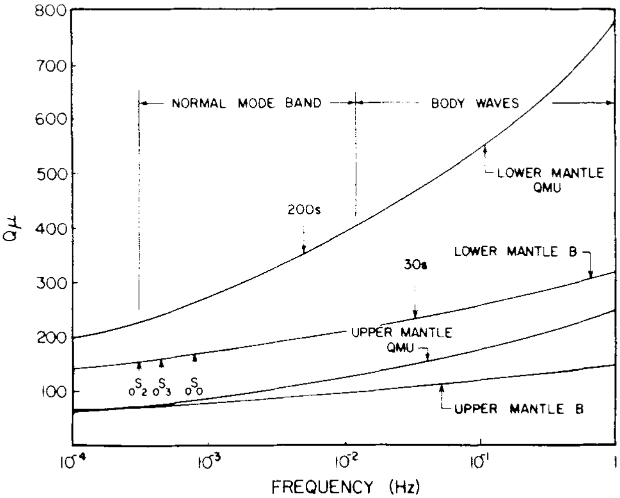

Figure 8. The frequency dependence of Q_μ in the seismic frequency band for model QMU with $\alpha = 0.15$ and for model B with $\alpha = 0.09$.

between constant Q_μ dispersion and that associated with the weakly frequency-dependent Q_μ found here are not too great. In particular either $\alpha = 0$ or α in the range $0.09 \leqq \alpha \leqq 0.15$ can successfully explain the observed travel-time baseline discrepancy of normal-mode derived Earth models such as 1066A or 1066B (Akopyan, Zharkov & Lyubimov 1975, 1976) at its current level of observational uncertainty. For model QMU with $\alpha = 0.15$ the

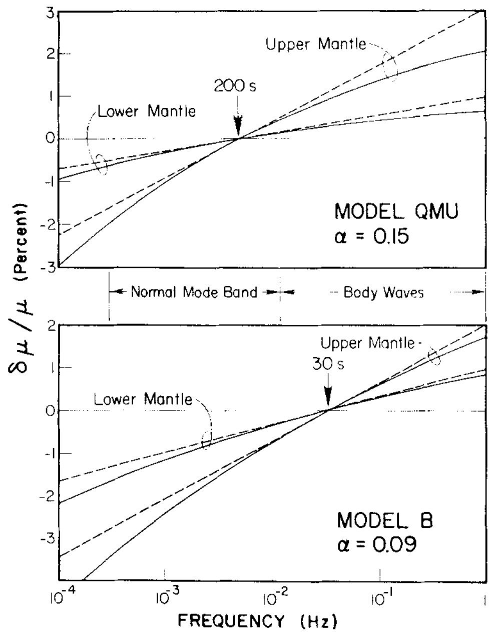

Figure 9. The solid curves show the seismic-band dispersion $\delta\mu/\mu$ in per cent for model QMU with $\alpha = 0.15$ and for model B with $\alpha = 0.09$ and the dashed curves show the corresponding constant-Q_μ or $\alpha = 0$ dispersion. The difference between solid and dashed curves is sufficiently slight that it will be difficult to detect using conventional seismological methods.

discrepancy $\delta\mu/\mu$ at the Chandler period $T_0 = 435.2$ day is 6.2 per cent in the lower mantle and 19.4 per cent in the upper mantle; the corresponding values for model B with $\alpha = 0.09$ are 7.9 and 16.5 per cent, respectively.

10 A caveat: non-equilibrium pole tides

As we have emphasized, the results found above are based upon the assumption that the global open pole tide is equilibrium. Let us now try to see how critical this assumption is. We can get a rough idea by considering the simplest possible hypothetical departure from equilibrium. Suppose the equilibrium tide is $\zeta = \zeta_0 \exp(i\sigma_0 t)$, and that the departure consists of a uniform amplification or reduction combined with a uniform phase lag over the Earth's oceans, i.e. suppose the tide is $\zeta = \gamma\zeta_0 \exp[i(\sigma_0 t - \epsilon)]$ where the amplification γ and the phase lag ϵ are constants. The effect of such a uniform departure from equilibrium on the period T_0 and the quality factor Q_0 is easily determined, since the tensor Δd_{ij} is clearly just changed to $\gamma\Delta d_{ij} \exp(-i\epsilon)$. Inserting this into the secular equation (5.3) yields a complex equation whose real and imaginary parts can be solved for σ_0 and Q_0 separately.

If the phase lag is small, the solution is very simple and in fact intuitively obvious. As before, let us use $\Delta\sigma_0$ to denote the amount by which the equilibrium tide $\zeta = \zeta_0 \exp(i\sigma_0 t)$ reduces the frequency σ_0. The corresponding reduction due to $\zeta = \gamma\zeta_0 \exp[i(\sigma_0 t - \epsilon)]$ is then simply $\gamma\Delta\sigma_0$, if $\epsilon \ll 1$. Under the same circumstances, the quality factor Q_0 (due to the oceans alone) is

$$Q_0^{-1} = \gamma|\Delta\sigma_0/\sigma_0|\epsilon. \tag{10.1}$$

Correct to first order, a slight uniform phase lag ϵ thus has no effect on T_0 while the quantity Q_0^{-1} is proportional to ϵ. Very roughly, if the tides are amplified or reduced by a factor γ not too different from 1, then the amount ΔT_0 by which the oceans act to increase the period will be multiplied by the same factor since $\Delta T_0/T_0 \sim -\Delta\sigma_0/\sigma_0$. For a strictly equilibrium tide, we have seen that $\Delta T_0 = 29.8$ day. Thus if the global departure from equilibrium is as little as 1 per cent, as the arguments in Section 5 suggest, the resulting effect on the period would only be about 0.3 day, which is negligible. The currently ambiguous state of direct observations of the pole tide however leaves some room for doubt here, especially since no explanation of the anomalous local pole tide observations in the North and Baltic Seas has been advanced which does not invoke departures from equilibrium in the open ocean at the edge of those seas. If the global departure from equilibrium is as high as 30 per cent, which cannot really be ruled out on the basis of pole tide observations alone, the period could be affected by as much as the total difference between T_0 and T_e, *viz.* 8.5 day. Although we are inclined to believe the actual uncertainty is really much less than this, it is fair to say that we cannot be absolutely certain.

Small departures from equilibrium can affect the damping factor Q_0 much more than the period T_0. If $\gamma = 1$, the phase lag required to give $Q_0 = 100$, according to (10.1), is $3^\circ.7$ or 4.5 day. If the globally averaged phase lag is of this order, then the oceans alone could completely account for the damping. Since this is roughly equal to the time-scale on which we expect equilibrium of the oceans to be established, this is clearly a possibility which must be taken seriously. A global phase lag of 4.5 day amounts to a departure from equilibrium of only 1 per cent, and is currently far below the level of direct detection.

In summary, if the global departures from equilibrium are as small as we have argued, then their influence on T_0 should be negligible, but their contribution to the damping might be profound. The hypothetical case of a uniform amplification together with a uniform phase lag is clearly too simplistic, but it does suggest the order of magnitude of these effects.

Such models have in fact been considered before by both Dickman (1979) and Naito (1979). Since Dickman seems willing to contemplate much larger departures from equilibrium than we are, his conclusions about the extent to which they might influence T_0 are somewhat more sensational than ours. In addition to considering the hypothetical model treated here, he has also estimated the effect of the anomalous tides observed in the North and Baltic Seas and, not surprisingly, finds their effect on T_0 to be thoroughly negligible even when the small accompanying tides which must be present in the rest of oceans to conserve mass are taken into account. Both he and Wunsch (1974) find, however, that dissipation in the North and Baltic Seas may alone account for a significant fraction of the observed Chandler wobble damping. By means of rough extrapolation to the rest of the world's shallow seas, Wunsch concludes that 'the ocean is unlikely to be a primary sink of energy, but it still cannot be ruled out'.

The conclusions we have drawn about the frequency dependence of Q_μ may be in error if the oceans, through departures from equilibrium, contribute measurably to the damping. We can, however, still place some useful bounds on the possible frequency dependence even if we admit this possibility. Let us now use Q_M to denote the contribution to Q_0 in the absence of any pole tide whatsoever; Q_M^{-1} is thus just the right hand side of equation (8.15), i.e.

$$Q_M^{-1} = \int_b^a [Q_\kappa^{-1}(\kappa K) + Q_\mu^{-1}(\mu M)] r^2 \, dr. \tag{10.2}$$

The observed quality factor Q_0 must be less than Q_M, since the purely passive pole tide plus the mantle strains it creates can only act as a sink and not a source of energy, regardless of the details or extent of any departures from equilibrium. This allows us to put an upper bound on the exponent α. If $Q_M = 566$ at 200 s as is the case for model QMU, then with 68 per cent confidence we can say that $\alpha < 0.20$ and with 90 per cent confidence $\alpha < 0.23$. If instead $Q_M = 386$ at 30 s as is the case for model B, the bounds are even stricter. In that case $\alpha < 0.15$ with 68 per cent confidence and $\alpha < 0.18$ with 90 per cent confidence. These bounds are based on the lower 68 and 90 per cent confidence limits placed on Q_0 by Wilson & Haubrich, *viz.* $Q_0 > 50$ and $Q_0 > 33$ respectively. Values as high as $\alpha = 1/3$, as advocated by Anderson & Minster (1979), seem by the above argument to be out of the question. We believe this conclusion is based on sufficiently few possibly unwarranted assumptions that it is essentially incontrovertible.

11 Conclusions

Models of $Q_\mu(r, \sigma)$ and $Q_\kappa(r, \sigma)$ will no doubt be refined in the future as more and more accurate attenuation data is collected. In this paper we have shown how the measured period T_0 and the measured quality factor Q_0 of the Chandler wobble can be used in this refinement process. There are two results in this paper we think will stand the test of time and prove to be of value in future Earth modelling studies. The first is the theoretical value of the Chandler period T_e for a dissipationless Earth with an equilibrium pole tide. For model 1066A of Gilbert & Dziewonski (1975), we have found that $T_e = 426.7$ sidereal days. The uncertainty in this value due to current uncertainty about the internal structure of the Earth is thought to be no more than 0.1 per cent, i.e. about 0.5 day. The discrepancy between T_e and the observed period $T_0 = 435.2 \pm 2.6$ day determined by Wilson & Haubrich (1976) must be due either to dispersion associated with mantle anelasticity or to departures of the pole tide from equilibrium if, as we have argued, other effects are small.

The second result which should prove useful is the algorithm we have presented for calculating Q_0 and $\delta\sigma_0/\sigma_0$ due to a given model of mantle anelasticity Q_κ, Q_μ and its associated dispersion $\delta\kappa/\kappa$, $\delta\mu/\mu$. Our algorithm is based on a straightforward application of normal mode perturbation theory for a rotating, oceanless Earth model. The kernels κK and μM which relate Q_0^{-1} to Q_κ^{-1} and Q_μ^{-1} and $2(\delta\sigma_0/\sigma_0)$ to $\delta\kappa/\kappa$ and $\delta\mu/\mu$ have been tabulated (for model 1066A) for future reference. Several previous attempts to relate Q_0^{-1} to Q_μ^{-1} using energy arguments have been shown to be in error.

The discrepancy $T_0 - T_e$ and Q_0 can be used to place constraints on $Q_\kappa(r, \sigma)$ and $Q_\mu(r, \sigma)$ in a number of different ways, depending on how conservative one wishes to be about the pole tide. Theoretical considerations together with *Mf* and *Mm* tidal studies suggest but do not prove that global departures of the pole tide from equilibrium should be quite small. If one is willing to assume there is no departure from equilibrium whatsoever, Q_κ and Q_μ can be constrained most tightly; both $T_0 - T_e$ and Q_0 must in that case be attributed solely to mantle anelasticity and dispersion. Given the present state of our oceanographic knowledge, a more conservative position might be thought appropriate. In that case, one may wish to place more reliance on a good fit to $T_0 - T_e$ rather than Q_0, since a 1 per cent departure from equilibrium should affect the former by only about 0.3 day, but could potentially account for all of the dissipation. On the other hand, since the oceans can act only as a sink, any model of Q_κ, Q_μ which predicts too low a value of Q_0 must definitely be rejected. This is in fact the most conservative position of all, since it is independent of how large any departures from equilibrium might be.

In this paper we have assumed that $Q_\kappa = \infty$, and have investigated the frequency dependence of Q_μ under the assumption that the mantle has a single absorption band in which $Q_\mu \sim \sigma^\alpha$, where α is a constant independent of depth. Future studies will undoubtedly wish to investigate the effect of relaxing these assumptions. Subject to these assumptions, however, we have found that the overall frequency dependence of Q_μ between a few tens or hundreds of seconds and 14 months can be constrained significantly. We can say with virtual certainty that α must be less than 1/3, the value associated with the so-called Andrade creep law preferred by Anderson & Minster (1979). If the pole tide is exactly equilibrium, our preferred value for α lies in the range 0.09–0.15, depending on whether the average Q_μ in the lower mantle is taken as 225 at 30 s (Sipkin & Jordan 1980) or as 350 at 200 s (Sailor & Dziewonski 1978). If the model of Sipkin & Jordan at 30 s is used, then frequency independence ($\alpha = 0$) down to 14 months is consistent with the data at the 90 per cent confidence level (the predicted Q_0 is actually consistent at the 68 per cent level but the predicted T_0 is not).

In essence, all we have done here is to substantiate and refine a conclusion first reached by Sir Harold Jeffreys over two decades ago. Our preferred value of α is slightly lower than his, i.e. we find that even less frequency dependence of Q_μ is required to explain the data than he thought. An attenuation of the form $Q_\mu \sim \sigma^\alpha$ implies a transient creep response, subsequent to the instantaneous elastic response, of the form $\epsilon(t) \sim t^\alpha$. If the law $Q_\mu \sim \sigma^\alpha$ persists down to geological periods, so will the transient creep law $\epsilon(t) \sim t^\alpha$. As Jeffreys (1972) has pointed out, thermal convection is precluded for any material which behaves for all t like $\epsilon(t) \sim t^\alpha$, if $\alpha < 1$. He has used this as his principal argument against plate tectonics and continental drift. In spite of the admiration we feel for his prescience regarding the implications of the damping of the Chandler wobble for mantle anelasticity, this is where we part company. In our opinion, the extrapolation involved in going from the Chandler wobble to either post-glacial rebound or plate tectonics cannot be justified.

From a strictly phenomenological point of view, this extrapolation may not at first sight seem all that improbable. The time scale for post-glacial rebound, if placed on the logarithmic scale of Fig. 1, is after all only about as far below the Chandler wobble as the Chandler wobble is below the seismic band. It must, however, be borne in mind that substantial extrapolation in stress is also required. The strains involved in the Chandler wobble are of order 10^{-9}, which means the stresses are only about 10^{-3} bars. The stresses associated with post-glacial rebound and mantle convection are several orders of magnitude larger, in the range 10–1000 bars. The basic microscopic mechanism which governs the attenuation in the mantle of the Chandler wobble is likely as a result to be altogether different from that which governs high-temperature steady-state creep. For example the former may be governed by bowing in the glide plane of dislocations which are pinned as suggested by Anderson & Minster (1979) while the latter, which must take over above some critical stress, is thought to be due to dislocation climb (Weertman 1970; Stocker & Ashby 1973).

The strongest argument against the validity of the extrapolation is, in our opinion, the wide body of evidence accumulated in the last 15 years in support of plate tectonics. Jeffreys (1972) has dismissed this evidence with the remark 'I think most of it can be explained more convincingly otherwise'. We do not agree. In fact the knowledge gained from studies of post-glacial rebound that the mantle can creep with an effective viscosity of order $\eta = 10^{22}$ poise can be used to constrain further the frequency dependence of its anelasticity. In particular this tells us that at a frequency near $\mu/\eta = 10^{-9}$ Hz both the rigidity μ and Q_μ must exhibit a drop to very low values as described by Goetze (1977).

There is one means whereby Jeffreys' arguments based on the Chandler wobble could conceivably be rationalized with plate tectonics. He himself has, in a sense, pointed it out in the statement 'A consequence of this law of creep is that *where it holds* (our italics) convection cannot take place' (Jeffreys 1972). The Chandler wobble is, as we have shown here, much more sensitive to attenuation in the lower rather than the upper mantle. Thus although we have assumed that α is depth-independent, the bounds we have determined actually pertain primarily to only the lower mantle. Advocates of upper mantle as opposed to whole mantle convection may wish to use this to bolster their case, but their adversaries are not likely to find the argument very persuasive in view of the strong probability that the microscopic mechanisms governing steady-state creep and the damping of the Chandler wobble are not the same.

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Appendix A: energetics and dissipation of the Chandler wobble

The contribution to Q_0 from mantle anelasticity is given in terms of Q_κ and Q_μ in the mantle by equation (8.12), which was obtained using normal mode perturbation theory. In this section we shall re-derive equation (8.12) from first principles and comment on several past attempts to do so.

We shall need a few more properties of the normal modes of rotating systems from Dahlen & Smith (1975). The first is that any two eigensolutions (σ_0, s) and (σ_0', s$'$) must satisfy the global relation

$$\sigma_0^2 T(\mathrm{s}', \mathrm{s}) - 2\sigma_0 W(\mathrm{s}', \mathrm{s}) - E(\mathrm{s}', \mathrm{s}) - \Gamma(\mathrm{s}', \mathrm{s}) = 0, \tag{A1}$$

where

$$T(\mathrm{s}', \mathrm{s}) = (\mathrm{s}', \mathrm{s}), \tag{A2a}$$

$$W(\mathrm{s}', \mathrm{s}) = (\mathrm{s}', i\boldsymbol{\Omega} \times \mathrm{s}), \tag{A2b}$$

$$E(\mathrm{s}', \mathrm{s}) = \int_{\mathrm{E}} [\kappa(\nabla\cdot\mathrm{s}')(\nabla\cdot\mathrm{s}^*) + 2\mu\boldsymbol{\delta}':\boldsymbol{\delta}^*]\,dv, \tag{A2c}$$

$$\Gamma(\mathrm{s}', \mathrm{s}) = \int_{\mathrm{E}} [\rho_0\mathrm{s}'\cdot\nabla\phi_1^* + p_0(\nabla\mathrm{s}':\nabla\mathrm{s}^* - (\nabla\cdot\mathrm{s}')(\nabla\cdot\mathrm{s}^*)) + \rho_0\mathrm{s}'\cdot\nabla\nabla(\phi_0+\psi)\cdot\mathrm{s}^*]\,dv. \tag{A2d}$$

If (σ_0, s) and (σ_0', s$'$) are the same, equation (A1) reduces to (8.2), which was exploited in doing perturbation theory. If on the other hand σ_0 and σ_0' are distinct (A1) implies that the associated eigenfunctions s and s$'$ are orthogonal in the sense

$$T(\mathrm{s}', \mathrm{s}) - 2(\sigma_0 + \sigma_0')^{-1} W(\mathrm{s}', \mathrm{s}) = 0. \tag{A3}$$

Finally, we note that every normal mode including the Chandler wobble has two associated eigensolutions, since if (σ_0, s) is a solution so is ($-\sigma_0$, s*). In the special case that s$'$ = s*, the orthogonality relation (A3) reduces to

$$W(\mathrm{s}^*, \mathrm{s}) = 0. \tag{A4}$$

In the remainder of this section, we shall use (σ_0, s) and ($-\sigma_0$, s*), with normalization unspecified, to refer exclusively to the Chandler wobble.

Suppose that at $t = 0$ the Earth is in the initial configuration

$$\left.\begin{aligned} &\mathrm{s}(\mathbf{x}, 0) = \mathrm{Re}\,[\mathrm{s}(\mathbf{x})] \\ &\partial_t \mathrm{s}(\mathbf{x}, 0) = \mathbf{0} \end{aligned}\right\} \tag{A5}$$

where s(**x**) is the Chandler wobble eigenfunction. Upon release it will oscillate in a single mode, the Chandler wobble; in fact the motion for $t \geqq 0$ in the absence of dissipation will be

$$\mathrm{s}(\mathbf{x}, t) = \mathrm{Re}\,[\mathrm{s}(\mathbf{x})\exp(i\sigma_0 t)]. \tag{A6}$$

We wish to calculate the energy associated with this motion. If there is no dissipation s(**x**, t) must satisfy

$$\rho_0\partial_t^2\mathrm{s} + 2\rho_0\boldsymbol{\Omega}\times\partial_t\mathrm{s} = \boldsymbol{H}(\mathrm{s}), \tag{A7}$$

where $\boldsymbol{H}$(s) is defined by equations (4.20). Upon taking the dot product of (A7) with ∂_ts and integrating over the entire Earth E, we obtain after some manipulation the result

$$d\mathscr{E}/dt = 0 \tag{A8}$$

where

$$\mathscr{E} = \tfrac{1}{2} \int_E [\rho_0 \partial_t \mathbf{s} \cdot \partial_t \mathbf{s} + \rho_0 \mathbf{s} \cdot \nabla \phi_1 + p_0(\nabla \mathbf{s} : \nabla \mathbf{s} - (\nabla \cdot \mathbf{s})^2) + \rho_0 \mathbf{s} \cdot \nabla\nabla(\phi_0 + \psi) \cdot \mathbf{s}$$
$$+ \kappa(\nabla \cdot \mathbf{s})^2 + 2\mu\, \boldsymbol{\delta} : \boldsymbol{\delta}\,]\, dv. \tag{A9}$$

It is clear that $\mathscr{E}$ must be the energy of oscillation we seek. The quantity $\tfrac{1}{2}\rho_0 \partial_t \mathbf{s} \cdot \partial_t \mathbf{s}$ is the relative kinetic energy density and the remaining terms in (A9), all of which are bilinear in $\mathbf{s}(\mathbf{x}, t)$, comprise the elastic-gravitational potential energy density. Their sum, when integrated over the entire Earth, is according to (A8), conserved in the absence of dissipation. The absence of a Coriolis term in (A9) is expected since the Coriolis force always acts at right angles to the motion and thus can do no work.

If a small amount of dissipation is now introduced the motion $\mathbf{s}(\mathbf{x}, t)$ and the associated energy of oscillation will decay exponentially, the former at a rate $\sigma_0/2Q_0$ and the latter at twice that rate. In the limit $t \to \infty$, the total energy dissipated must be precisely $\mathscr{E}$, the initial energy of oscillation. Upon substituting equation (A6) into (A9), we can rewrite $\mathscr{E}$ in the form

$$\mathscr{E} = \tfrac{1}{4}[\sigma_0^2 T(\mathbf{s}, \mathbf{s}) + E(\mathbf{s}, \mathbf{s}) + \Gamma(\mathbf{s}, \mathbf{s})]. \tag{A10}$$

This is independent of time as equation (A8) asserts; the terms which would otherwise oscillate as $\exp(\pm 2i\sigma_0 t)$ vanish identically by virtue of (A1) and (A4). To reiterate, equation (A10) gives the total energy dissipated in the Earth during the interval $0 \leqq t < \infty$ if the motion $\mathbf{s}(\mathbf{x}, t)$ is a slowly decaying version of (A6). Note that (A10) contains a relative kinetic energy term $\sigma_0^2 T$, an elastic energy term E and a gravitational energy term Γ; there is no Coriolis term (which comprises most of the kinetic energy of rotation) for the reason stated above. We may, however, easily write $\mathscr{E}$ in two forms which do contain W by exploiting the identity (7.2). In particular we can eliminate T and write

$$\mathscr{E} = \tfrac{1}{2}[\sigma_0 W(\mathbf{s}, \mathbf{s}) + E(\mathbf{s}, \mathbf{s}) + \Gamma(\mathbf{s}, \mathbf{s})] \tag{A11}$$

or we can eliminate E and Γ and write

$$\mathscr{E} = \tfrac{1}{2}[\sigma_0^2 T(\mathbf{s}, \mathbf{s}) - \sigma_0 W(\mathbf{s}, \mathbf{s})]. \tag{A12}$$

The latter form for the total energy dissipated by a mode has been derived previously from a somewhat different point of view by Dahlen (1978, 1980b). He considered the more realistic (and more complicated) problem of excitation by an earthquake; see in particular equation (45′) of Dahlen (1980b), wherein the normalization $T(\mathbf{s}, \mathbf{s}) - \sigma_0^{-1} W(\mathbf{s}, \mathbf{s}) = 1$ is employed.

By definition $2\pi Q_0^{-1}$ is the fractional energy dissipated per cycle. The rate at which energy is dissipated per cycle during the first few cycles is (Nowick & Berry 1972; O'Connell & Budiansky 1978)

$$\oint \dot{\mathscr{E}}\, dt = \pi \int_E [\kappa Q_\kappa^{-1} (\nabla \cdot \mathbf{s})(\nabla \cdot \mathbf{s}^*) + 2\mu Q_\mu^{-1} (\boldsymbol{\delta} : \boldsymbol{\delta})^*]\, dv. \tag{A13}$$

From the definition

$$Q_0^{-1} = (1/2\pi\mathscr{E}) \oint \dot{\mathscr{E}}\, dt, \tag{A14}$$

we find upon using (A12) the result

$$Q_0^{-1} = \frac{\displaystyle\int_E [\kappa Q_\kappa^{-1}(\nabla\cdot\mathbf{s})(\nabla\cdot\mathbf{s}^*) + 2\mu Q_\mu^{-1}(\boldsymbol{\delta}:\boldsymbol{\delta}^*)]\,dv}{\sigma_0^2 T(\mathbf{s},\mathbf{s}) - \sigma_0 W(\mathbf{s},\mathbf{s})}. \tag{A15}$$

This is precisely equation (8.12) derived earlier using normal mode perturbation theory and employed in this study. Normal mode perturbation theory is thus consistent with the energetics of the Chandler wobble, as of course it must be.

Suppose now that we decompose $E(\mathbf{s},\mathbf{s})$ into its compressional and shear parts, *viz.*

$$E = E_c + E_s,$$

where

$$E_c = \int_E [\kappa(\nabla\cdot\mathbf{s})(\nabla\cdot\mathbf{s}^*)]\,dv,$$

$$E_s = \int_E [2\mu\,\boldsymbol{\delta}:\boldsymbol{\delta}^*]\,dv.$$

To compare (A15) with previous approaches to this problem we shall restrict attention to the simple special case that $Q_\kappa = \infty$ everywhere and Q_μ = constant throughout the solid regions of the Earth. Equation (A15) reduces then to

$$Q_0/Q_\mu = 2\,\mathscr{E}/E_s \tag{A16}$$

where

$$2\mathscr{E} = \sigma_0^2 T - \sigma_0 W = \tfrac{1}{2}[\sigma_0^2 T + E_c + E_s + \Gamma] = \sigma_0 W + E_c + E_s + \Gamma. \tag{A17}$$

Equations of this same general form giving Q_0/Q_μ as a ratio of various energy expressions have been either derived or obtained on an *ad hoc* basis in a number of past investigations. If the Earth were axisymmetric and perfectly rigid, the eigenfrequency σ_0 would be given by (3.3) and the associated eigenfunction $\mathbf{s}$ in the frame F_I would be (Smith 1977)

$$\mathbf{s} = [\vartheta_0(\hat{\mathbf{x}} - i\hat{\mathbf{y}})] \times \mathbf{x}, \tag{A18}$$

where ϑ_0 is an arbitrary amplitude. The corresponding wobble eigenfunction $\mathbf{m}$ in the frame F_M is

$$\mathbf{m} = im_0(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \tag{A19a}$$

where

$$m_0 = \left(\frac{\sigma_0 + \Omega}{\Omega}\right)\vartheta_0 = (C/A)\vartheta_0. \tag{A19b}$$

It is easily shown that the quantity $\mathscr{E}$ is given (exactly) in this perfectly rigid case by

$$\mathscr{E} = \tfrac{1}{2}(A/C)(C-A)\Omega^2 m_0^2. \tag{A20}$$

As shown by Jeffreys (1970; p. 324) and Munk & MacDonald (1960; p. 167), this is precisely the incremental rotational kinetic energy of rigid body wobble.

Of the various previous attempts to estimate the ratio Q_0/Q_μ, the one with which we can find the least fault is one of the earliest and most widely known, namely that of Stacey (1970). He wrote Q_0/Q_μ in the form (A16) and used the rigid body value (A18) for $\mathscr{E}$, except that he replaced A/C by A_M/C, presumably to allow for the fact that the core does not participate in the wobble. Since the contribution to **s** from the Earth's deformation is so small compared with (A18), this is not too bad an approximation. A slightly better approximation, which we have used in conjunction with the HLL theory to construct the kernel shown in Fig. 7, is to note that since $|\sigma_0^2 T| \ll |\sigma_0 W|$,

$$\mathscr{E} \approx -\tfrac{1}{2}\sigma_0 W \approx \tfrac{1}{2}\sigma_0 A_\mathrm{m}\Omega m_0^2. \tag{A21}$$

Comparing this with the expression used by Stacey, we see that he has overestimated $\mathscr{E}$ only slightly, by about a factor of 4/3. For E_s, Stacey uses a very simple order-of-magnitude estimate, *viz.*

$$E_\mathrm{s} \approx \bar{\mu}(ka^5\Omega^2/3GA)^2 V\vartheta_0^2, \tag{A22}$$

where $\bar{\mu}$ is the mean rigidity of the mantle and V is its volume. This leads him in Stacey (1970) to the estimate $Q_0/Q_\mu \approx 7.5$, while in Stacey (1977; p. 65), using different numbers but essentially the same argument, he finds instead that $Q_0/Q_\mu \approx 10$. The actual value from Table 4 is

$$Q_0/Q_\mu = 1.83. \tag{A23}$$

HLL theory with the approximate value (A21) for $\mathscr{E}$ gives very nearly the same result, namely $Q_0/Q_\mu = 1.84$. The effect of pole-tide loading in the constant-Q_μ case is, from equation (8.27) and Table 5, to decrease (A23) to

$$Q_0/Q_\mu = 1.62.$$

In retrospect, the only thing wrong with Stacey's argument is that the approximation (A22) is too crude. He essentially approximates the Chandler strain energy as a constant throughout the entire mantle whereas, as Fig. 6 shows, it varies by more than an order of magnitude from top to bottom. On the other hand, his calculation is only intended as an order-of-magnitude estimate, and it is within an order of magnitude of the actual answer. That, as Jeffreys wrote in 1924, is 'all that the method claims'. As it happens, the direction in which it errs makes the mantle appear to be an unpromising contender as the sink of Chandler energy, whereas in fact very little frequency dependence of Q_μ is required for it to be the sole mechanism.

Anderson & Minster (1979) tried to determine the exponent α by comparing the Q of the Chandler wobble to that of the mode ${}_0S_2$. Before making the comparison, however, they 'correct' the observed Q of the Chandler wobble by dividing it by four. This, they argue, is necessary because in the absence of oceans the Earth's deformation acts to increase the period from about 300 to about 400 day, 'which means that only 24 per cent of the available energy is actually stored as deformational energy'. In fact, this correction is uncalled for since, as Fig. 8 shows, the kernels μM for the Chandler wobble and ${}_0S_2$ are very similar without any alteration. The correct way to compare the Chandler wobble with ${}_0S_2$, if one is willing to ignore pole-tide loading, is directly.

A similar *ad hoc* correction to Q_0 seems to have been employed by Jeffreys (1978) in his most recent discussion of this problem. Taking 306 day as the rigid Chandler period and

433 day as the observed, he states that 'Seventy-one per cent of the energy of the free nutation is due to rotation as a rigid body. Only the remaining 29 per cent is elastic and subject to damping. Thus the factor 0.29 must be taken out, to give $1/Q$ for motion unaffected by rotation'. He then goes on to compare the resulting corrected value of Q_0^{-1} with a theoretical value obtained by weighted integration of a Q_μ^{-1} distribution due to Berzon, Passechnik & Polikarpov (1974). That distribution is based on the observed attenuation of short-period teleseismic P waves, and is assumed by Jeffreys to be valid at a period of 1 s. Jeffreys' description is, characteristically, somewhat terse and it is not clear to us how the weighting function M (his version of our kernel μM) was obtained, but we think the only conceptual error in his procedure is the 'correction' factor 0.29. This, together with an arithmetical error in his equation (7) and the seemingly rather high Q_μ values of Berzon *et al.* (1974) all combine to yield the relatively high value of α (namely 0.20 ± 0.05) he obtains.

The most ambitious discussion to date of the energy budget of the Chandler wobble is that of Merriam & Lambeck (1979). They attempted to improve Stacey's order-of-magnitude estimate in the constant-Q_μ case by numerically integrating the elastic shear energy distribution and by taking gravitational energy into account. Unfortunately, they too do so in a somewhat *ad hoc* fashion, without using the equipartition relation $\sigma_0^2 T - 2\sigma_0 W - E_c - E_s - \Gamma = 0$. Their expression E which corresponds to the numerator $2\mathscr{E}$ in equation (A16) consists of a sum of rotational, elastic and gravitational energies, all of which are positive. Our numerator can be written in a form rather similar to theirs, namely as $2\mathscr{E} = \sigma_0 W + E_c + E_s + \Gamma$, but in that expression the factor $\sigma_0 W$ is negative. Their conclusion that $Q_0/Q_\mu = 8.75$ is, as a result of their overestimation of $2\mathscr{E}$, too large by almost a factor of 5.

Appendix B: exact Hough

Hough (1895) undertook to find the small oscillations of a rigid mantle of arbitrary triaxiality enclosing in its centre a triaxial ellipsoidal cavity filled with homogeneous incompressible fluid. A remarkable feature of Hough's original solution, which is given in his Appendix, is that it not only yields the free mantle wobbles of interest but also points to the existence of internal modes in the fluid core which were later studied extensively by Greenspan (1964) and co-workers. In the body of his paper Hough discusses a much simpler, but still exact, approach which is restricted to the case of normal modes which impart rigid rotation to the mantle.

We follow Hough (1895) in adopting a mantle Tisserand frame, F_M. As in Section 4 let

$$\boldsymbol{\omega}(t) = \Omega \hat{\mathbf{e}}_3 + \Omega \mathbf{m}$$

be the frame's, and therefore the mantle's, instantaneous angular velocity. The motion of the fluid core comprises an instantaneous rigid rotation and an irrotational flow driven by the ellipticity of the core–mantle boundary in conjunction with differential rigid rotation of the core and mantle. Let the instantaneous angular velocity of the core be $\Omega(\mathbf{m} + \boldsymbol{\eta})$, so $\boldsymbol{\eta}$ is the difference in angular velocities of core and mantle.

Hough showed that the total angular momentum of such a planet is given by

$$\begin{aligned} H_1 &= A\Omega m_1 + (A_C - q)\Omega \eta_1, \\ H_2 &= A\Omega m_2 + (A_C - q)\Omega \eta_2, \qquad \text{(B1)} \\ H_3 &= C\Omega, \end{aligned}$$

where we have specialized to the axisymmetric case; the terms $q\Omega\eta_1$ and $q\Omega\eta_2$ arise from the irrotational flow in the core. From Hough (1895), and after some algebra, we have that

$$q = A_C\epsilon_b^2\left[\frac{(1-\epsilon_b/6)^2}{(1-\epsilon_b/3+5\epsilon_b^2/18)^2}\right]. \tag{B2}$$

Clearly the relative angular momentum in F_M is simply

$$h_i = (A_C - q)\Omega\eta_i. \tag{B3}$$

In order to use (B2) and (B3) to infer E and E' of equation (4.11b), we must connect $\boldsymbol{\eta}$ to $\mathbf{m}$. That connection, also inferred from Hough compounded with some manipulation, is

$$\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} = \frac{1}{\Omega^2P^2-\sigma^2}\begin{bmatrix}\sigma^2 & -i\sigma\\ i\sigma & \sigma^2\end{bmatrix}\begin{bmatrix}m_1\\ m_2\end{bmatrix}, \tag{B4}$$

where

$$P^2 = \frac{(1+\epsilon_b/3)^2}{(1-\epsilon_b/3+5\epsilon_b^2/18)^2}. \tag{B5}$$

Combining (B3)–(B5) gives

$$E = -A_C(1-\Lambda)\sigma\Omega/(\sigma^2 - \Xi^2\Omega^2) \tag{B6}$$

and

$$E' = A_C(1-\Lambda)\Xi\Omega^2/(\sigma^2-\Xi^2\Omega^2), \tag{B7}$$

where

$$\Lambda = \frac{\epsilon_b^2(1-\epsilon_b/6)^2}{(1-\epsilon_b/3+5\epsilon_b^2/18)^2} \tag{B8}$$

and

$$\Xi = \frac{(1+\epsilon_b/3)^2}{(1-\epsilon_b/3+5\epsilon_b^2/18)}. \tag{B9}$$

These results are exact in the core–mantle boundary ellipticity ϵ_b.

In the limit of small ϵ_b

$$\Lambda = \epsilon_b^2 + O(\epsilon_b^3) \tag{B10}$$

and

$$\Xi = 1 + \epsilon_b + O(\epsilon_b^2) \tag{B11}$$

and we have, through terms of first order in ϵ_b

$$E = \frac{-A_C\sigma\Omega}{(\sigma^2-\Omega^2)[1+2\epsilon_b\Omega^2/(\sigma^2-\Omega^2)]} \tag{B12}$$

and

$$E' = \frac{A_C\Omega^2}{(\sigma^2 - \Omega^2)[1 + \epsilon_b + 2\epsilon_b\Omega^2/(\sigma^2 - \Omega^2)]} \tag{B13}$$

Equations (B12) and (B13) are appropriate for small ϵ_b but arbitrary σ (in particular, they remain valid if $\sigma \approx \Omega$). For the Chandler wobble, $\sigma \ll \Omega$ and (B12) and (B13) simplify further to yield (4.16a) and (4.16b).

By employing equations (B12) and (B13) in the secular equation (4.12) we can look for wobble modes of the Earth other than the Chandler wobble. In the rigid body case ($D = 0$) treated by Hough it is known that there is only one other solution, the nearly diurnal free wobble (NDFW) or free core nutation. By employing $D = ka^5\Omega^2/3G$ together with (B10) and (B11) in (4.12) we can investigate the influence of mantle elasticity on this mode. The eigenfrequency of the NDFW is near Ω and that is where we shall look. In particular, E and E' are singular at

$$\sigma = \pm \Xi\Omega,$$

so we look for a solution of the form

$$\sigma = \Xi\Omega + \zeta\Omega \tag{B14}$$

where we suppose, and can *a posteriori* verify, that $\zeta = O(\epsilon_b)$. We obtain, after some algebra,

$$\zeta = \frac{A_C}{A - A_C}\epsilon_b \tag{B15}$$

and thus

$$\sigma = \Omega + \epsilon_b\left(1 + \frac{A_C}{A_M}\right)\Omega. \tag{B16}$$

The result (B16) for the NDFW eigenfrequency σ has several interesting features. If $\epsilon_b \to 0$, $\sigma \to \Omega$ and this mode becomes the tilt-over mode of the core (see Dahlen & Smith 1975; Smith 1977) which is now perfectly decoupled from the mantle. More generally the extent to which σ depends on ϵ_b and on the relative moments of inertia of core and mantle is explicit; for the Earth the quantity $A_C/A_M = 0.13$.

The result of most interest to us is that the elastic deformation term D is absent; in fact (B16) is precisely Hough's rigid mantle formula. This implies that the eigenfrequency is, to first order in ellipticity, completely unaffected by the elasticity of the mantle. That, in turn, implies that this normal mode has virtually no strain energy and that, even if we could some day measure its period, it would not be likely to contribute appreciably to our knowledge either of mantle anelasticity or dispersion.

T. Sasao (private communication, 1980) has pointed out to us, however, that this conclusion, based on our version of HLL theory, does not survive a more accurate test unscathed. In particular, a more general form of HLL theory (Sasao *et al.* 1980) shows that elastic deformation of the core–mantle boundary (which we have neglected) does modify the NDFW eigenfrequency, by an amount of order 25 per cent of the effect of core–mantle boundary ellipticity. It is not inconceivable, then, that sufficiently accurate measurements of its eigenfrequency in the future could provide information about the Earth's anelastic rheology at diurnal periods. It should be remembered, however, that to date the NDFW has not been incontrovertibly detected.