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A FIELD THEORY APPROACH TO THE t - J MODEL AND THE SPIN-CHARGE SEPARATION *

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Abstract

We analyze the t - J model using the CP^1 representation for the slave operators (holons and spinons) which is particularly suited to study the phenomenon of the spin-charge separation in strongly correlated electron systems. In particular, we show that for the one-dimensional t - J model below half-filling the low energy effective dynamics of the spin and charge degrees of freedom is represented in the continuum limit by a CP^1 model with a topological term, minimally coupled to a massless Dirac field with a four-fermion interaction. The bosonic term of this action describes the spin waves produced by the spinons, while the fermionic term represents the low energy charge excitations. This theory exhibits explicitly a local abelian gauge invariance.

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1. Introduction

The discovery of high temperature superconductivity [1] has determined an increasing interest in the study of strongly correlated electron systems, and in particular of the Hubbard model [2] and the t - J model [3]. The Hubbard model describes the dynamics of non-relativistic electrons moving on a lattice. Its Hamiltonian consists of a kinetic hopping term of strength t , and an on-site repulsion between up and down spins of strength U modeling the Coulomb interaction. At infinite U , one must pay infinite energy to put two electrons of opposite spins in the same point. Hence, double occupancy is strictly forbidden, and the electrons can move in the lattice only if some sites are vacant. Consequently, at half-filling, *i.e.* when the number of electrons equals the number of lattice points, the infinite U Hubbard model describes an insulator (of the Mott type) [4]. Notice that this insulating behavior occurs at half-filling, while ordinary insulators are always characterized by completely filled bands.

The Hubbard model at half-filling is known to describe a Mott insulator also when U is large but finite, *i.e.* in the strong-coupling limit [4]. In this case one can develop (degenerate) perturbation theory in the small parameter t^2/U and still use states with no double electron occupancy. To leading order the Hubbard interaction transforms into an antiferromagnetic Heisenberg interaction with a coupling constant $J \propto t^2/U$. Thus, to this order the Hubbard system can be effectively described by a Hamiltonian with a kinetic hopping of strength t and an antiferromagnetic Heisenberg spin-exchange interaction of strength J , both acting on states without double electron occupancy. This is the so-called t - J model. Of course, at half-filling the t - J model reduces to the Heisenberg model since the hopping term becomes ineffective, and describes a Mott insulator with an antiferromagnetic Néel order. If the band filling factor is lowered below $1/2$ by the introduction of holes (*i.e.* by doping), the system becomes a conductor, and eventually the Néel order gives way to superconductivity. Thus we can say that the t - J model below half-filling represents a conducting doped antiferromagnet which may become a superconductor [3].

In one space dimension, both the Hubbard model and the t - J model (with $J = \pm 2t$ or with $J/t \rightarrow 0$) can be exactly solved by the Bethe *Ansatz* [5,6,7], and the spectrum of their collective excitations can be completely determined. In particular for the t - J model below half-filling, two kinds of quasi-particle gapless excitations are identified above the anti-ferromagnetic ground state: one carries only charge and no spin and is called holon, the other carries only spin and no charge and is called spinon. In other words there is a separation between the spin and charge degrees of freedom. Even though this phenomenon can be rigorously established only using the exact solution provided by the Bethe *Ansatz*, it is believed that the spin-charge separation actually occurs also in non-exactly solvable models (like for example the one-dimensional t - J model with $J \neq \pm 2t$) and even in higher dimensions [8,9]. Therefore, in order to understand better the mechanism of the spin-charge separation, it would be desirable to develop techniques and formalisms that capture the essential dynamical features of strongly correlated electron systems without insisting on their exact solvability.

One way of implementing the spin-charge separation is to use the formalism of the slave operators [10] and write the electrons as products of one spinless antiholon and one neutral spinon of spin 1/2. This formalism is very useful since it allows the application to electron systems below half-filling of several standard techniques that would be otherwise inapplicable, such as path-integrals, mean field approximations, large N expansions and so on. The primary reason for this is that the anholonomic constraint which characterizes an electron system below half-filling, is transformed into a holonomic constraint on the slave operators (see for instance (2.1) and (2.9) below). Another fundamental property of this formalism is that from the outset it introduces the local freedom of choosing arbitrarily the phase of holons and spinons which therefore are naturally related to abelian gauge theories. On the basis of this very general observation, one expects that the effective action describing the low-energy dynamics of holons and spinons should be invariant under local gauge transformations. This is indeed what happens in many cases, like for example in the t - J model at half-filling, *i.e.* in the Heisenberg model [11,12].

In this paper we show that this happens also in the t - J model below half-filling, when both spin and charge degrees of freedom can be excited. In particular, by using the slave operators in the so-called CP^1 representation [13,14] we prove that the Hamiltonian of the one-dimensional t - J model in the continuum limit gives rise to an Euclidean field theory in two dimensions that is explicitly invariant under abelian gauge transformations. More precisely, this theory is a CP^1 model with topological term [15,16,17] for the spin degrees of freedom minimally coupled to a massless Dirac fermion describing the charge excitations.

Our results are similar to those of previous investigations on the dynamics of doped antiferromagnets in one or two dimensions [18,19,20,21]. However, a more detailed comparison with the existing literature (and specifically with [18]) shows several significant differences. Firstly, we use from the beginning the slave operator formalism in the CP^1 representation, and deduce everything from the Hamiltonian of the t - J model in such representation. Thus, the coupling between holons and spinons is not imposed by hand, but it is derived directly from the hopping term of the Hamiltonian of the t - J model. Secondly, we introduce only one type of holons (like it happens in the exact solution of the Bethe *Ansatz*), and do not have the need of distinguishing between two types of holons with opposite charges as in [18,19,20]. This is an explicit consequence of using the CP^1 representation, as we will see in detail. Finally, in our end result we find a four-fermion interaction for the holon field that previously was not emphasized. However, the general structure of our approach as well as some motivations are in agreement with what already exists in the literature.

This paper is organized as follows: In section 2 we introduce the CP^1 representation for the t - J model in any dimension and write the Hamiltonian explicitly in terms of holons and spinons. In Section 3 we study the low-energy effective action for the charge and spin degrees of freedom in one dimension, and show that in the continuum limit this action becomes that of a CP^1 model with a topological term minimally coupled to a massless Dirac fermion. The bosonic field of the CP^1 model describes the dynamics of the spin waves produced by the spinons, while the fermionic

field represents the low-energy holon excitations. Finally in Section 4 we rewrite the effective action by introducing an abelian gauge field to exhibit explicitly the local gauge invariance of the model, and then present our conclusions.

2. The \mathbb{CP}^1 Representation of the t - J Model

The cuprate oxides which some years ago allowed the discovery of high temperature superconductivity [1], share a common structural feature: the presence of layers made of copper and oxygen ions where all electric transport phenomena (and eventually superconductivity) seem to take place [8]. Each of these Cu-O layers can be represented as a lattice with the copper ions sitting on the sites and the oxygen ions on the bonds. In undoped materials every copper ion has one unpaired electron in its outer $3d$ shell, whilst all electrons in the outer $2p$ shell of the oxygen ions are paired. Therefore the Cu-O layer of an undoped oxide can be regarded as a lattice with exactly one electron per site, *i.e.* at half-filling. Clearly, in this case no electric current can flow and the system is an insulator. Spectroscopic experiments reveal that the spins of the unpaired electrons are antiferromagnetically ordered at low temperatures [22], and only spin-exchange interactions can occur. Thus, the physics of these undoped materials can be effectively described by an antiferromagnetic Heisenberg model [8].

Upon doping, some oxygen ions of the Cu-O layers lose one electron from their outer $2p$ shells so that new unpaired spins show up. However the hybridization between the $2p$ oxygen orbitals and the $3d$ copper orbitals strongly binds each one of these unpaired O electrons to one of the unpaired Cu electrons already present, in such a way that a spin singlet is formed [3]. Therefore the Cu-O layer of a doped oxide can be represented as a lattice where most of the sites are occupied by one electron and the remaining few sites are occupied by a (charged) spin singlet. The unpaired O electrons introduced by doping can actually move in the lattice and couple to different Cu electrons. Thus an electric current is generated and the material becomes a conductor and eventually a superconductor below a high critical temperature.

The electron dynamics in these doped cuprate oxides can be effectively described by the so-called t - J model which was originally proposed by Zhang and Rice [3]; together with the Hubbard model [2], it has become one of the most studied examples of strongly correlated electron systems. Let us denote by $\hat{c}_\alpha^\dagger(i)$ and $\hat{c}_\alpha(i)$ the fermionic operators which create and destroy an electron at site i with z -component of the spin $\alpha/2$ ($\alpha = \pm 1$). Since each site can accommodate at most one electron, we must require that

$$n(i) \equiv \sum_{\alpha=\pm 1} \hat{c}_\alpha^\dagger(i) \hat{c}_\alpha(i) \leq 1 , \quad (2.1)$$

i.e. we must exclude double occupancy of any lattice point. The spin-1/2 operators which generate local rotations on the electron spins, can be represented à la Schwinger

by

$$\hat{S}^a(i) = \frac{1}{2} \sum_{\alpha, \beta=\pm 1} \hat{c}_\alpha^\dagger(i) \sigma_{\alpha\beta}^a \hat{c}_\beta(i) \quad , \quad a = 1, 2, 3 \quad , \quad (2.2)$$

(σ^a being the Pauli matrices). It can be easily checked that the $\hat{S}^a(i)$'s satisfy the standard $SU(2)$ algebra. In terms of these operators, the Hamiltonian of the t - J model with chemical potential μ can be written as

$$H_{tJ} = \mathcal{P} \left[-t \sum_{\langle i,j \rangle} \sum_{\alpha=\pm 1} \left(\hat{c}_\alpha^\dagger(i) \hat{c}_\alpha(j) + \text{h. c.} \right) + J \sum_{\langle i,j \rangle} \left(\hat{\mathbf{S}}(i) \cdot \hat{\mathbf{S}}(j) - \frac{1}{4} n(i) n(j) \right) + \mu \sum_i n(i) \right] \mathcal{P} \quad , \quad (2.3)$$

where \mathcal{P} is the Gutzwiller projection operator [23] enforcing the constraint (2.1) and the symbol $\langle i, j \rangle$ denotes a pair of nearest-neighbor lattice sites. The parameters t and J are coupling constants which we take to be positive: t is the strength of the kinetic hopping term, and J is the strength of the spin-exchange antiferromagnetic interaction. The chemical potential μ in (2.3) must be fixed in such a way that

$$\langle n(i) \rangle = 1 - \delta \quad , \quad (2.4)$$

where the doping concentration δ is defined by

$$\delta = \frac{N - N_{\text{el}}}{N} \quad (2.5)$$

with N_{el} representing the number of unpaired electrons and N the number of lattice sites. At half-filling when $\delta = 0$, the t term of (2.3) becomes ineffective and only the J term survives, so that in the absence of doping the t - J model reduces to the (antiferromagnetic) Heisenberg model. The situation is clearly different below half-filling, where also the t term gives a non-trivial contribution to the electron dynamics. However, it should be emphasized that because of the Gutzwiller projection, the t and J terms of the Hamiltonian (2.3) always act disjointedly. In fact, given a couple of nearest-neighbor points, when both of them are occupied by one electron only the J term acts; on the other hand, when one of the two sites is empty the J term vanishes while the hopping term t is effective; finally when both sites are empty the t and J terms are both vanishing.

Several authors suggested that at small doping (*i.e.* $\delta \simeq 0.1 - 0.2$) a very interesting phenomenon should occur: the spin and charge degrees of freedom should separate, indicating that the whole system behaves as a Luttinger liquid [24]. In other words, the spectrum of the lowest lying excitations of the t - J model should consist of holons, which carry only charge and no spin, and spinons, which carry only spin and no charge [8,9]. Actually, a rigorous proof of the occurrence of this spin-charge

separation exists only for the one-dimensional t - J model since this can be exactly solved by Bethe Ansatz, at least when $J = \pm 2t$ (supersymmetric point) [6] or when $J/t \rightarrow 0$ [7]. For arbitrary values of t and J or for the two-dimensional model, there are no exact results available, but some approximate solutions [25] as well as recent numerical calculations [26] seem to support the conjecture that the spin-charge separation always occurs, and also that the peculiar Luttinger liquid properties of the one-dimensional model survive in higher dimensions.

One common way to formally implement the spin-charge separation is to use slave operators [10] and factorize the electron oscillators, which obviously carry both spin and charge, into products of holons and spinons. Specifically one writes

$$\hat{c}_\alpha(i) = \hat{s}_\alpha(i) \hat{h}^\dagger(i) \quad , \quad \hat{c}_\alpha^\dagger(i) = \hat{h}(i) \hat{s}_\alpha^\dagger(i) \quad , \quad (2.6)$$

where $\hat{h}^\dagger(i)$ and $\hat{h}(i)$ are respectively the creation and annihilation operators for charged spinless holons, and $\hat{s}_\alpha^\dagger(i)$ and $\hat{s}_\alpha(i)$ are the creation and annihilation operators for neutral spinons of spin $\alpha/2$.

The statistics of holons and spinons is not *a priori* determined. However, since the electron oscillators must be fermionic, there are in general only two possibilities: one is to take fermionic holons and bosonic spinons, the other is to take bosonic holons and fermionic spinons. In either case, the products of one holon and one spinon (like those in (2.6)) are always fermionic. Actually, in two dimensions where the statistics may be anything [27], it can be shown that holons and spinons may be anyons of arbitrary complementary statistics [28]. In this paper we choose to work with fermionic holons and bosonic spinons that obey the following (anti)commutation relations

$$\left\{ \hat{h}(i), \hat{h}^\dagger(j) \right\} = \delta(i,j) \quad , \quad \left[\hat{s}_\alpha(i), \hat{s}_\beta^\dagger(j) \right] = \delta_{\alpha\beta} \delta(i,j) \quad , \quad (2.7)$$

where $\delta(i,j)$ is the lattice δ -function. The reason for this choice is that, with fermionic holons,

$$\hat{c}_\alpha(i) \hat{c}_\beta(i) = \hat{c}_\alpha^\dagger(i) \hat{c}_\beta^\dagger(i) = 0 \quad (2.8)$$

for *all* α and β . Thus, double occupancy is always automatically forbidden at any lattice site as required by (2.1).

Another important point to emphasize is that holons and spinons are *not* completely independent but must be constrained if one wants to recover the correct electronic configurations. One possibility to achieve this goal is to impose that

$$\hat{h}^\dagger(i) \hat{h}(i) + \hat{s}_\alpha^\dagger(i) \hat{s}_\alpha(i) = 1 \quad . \quad (2.9)$$

(From now on, the summation symbol over repeated spin indices will be understood.) The holon and spinon operators subject to this relation form the so-called slave fermion representation¹. Notice that (2.9) implies that each lattice site is always

¹ When (2.9) is imposed on bosonic holons and fermionic spinons one realizes the so-called slave boson representation.

occupied either by a holon or by a spinon of spin up or down, and consequently at each point there are only three possible states, just as required by (2.1). Thus, in this slave operator formalism the anholonomic constraint (2.1) is transformed into the equality constraint (2.9). This is certainly a remarkable improvement.

In this representation the spin operators $\hat{S}^a(i)$ can be written only in terms of spinon oscillators as follows (see *e.g.* [28])

$$\hat{S}^a(i) = \frac{1}{2} \hat{s}_\alpha^\dagger(i) \sigma_{\alpha\beta}^a \hat{s}_\beta(i) \quad , \quad a = 1, 2, 3 \quad . \quad (2.10)$$

Upon using (2.6) and (2.10), and taking into account the constraint (2.9), the Hamiltonian of the t - J model in the slave fermion representation is

$$H_{tJ} = H_t + H_J + \mu \sum_i \left(1 - \hat{h}(i)^\dagger \hat{h}(i) \right) \quad , \quad (2.11)$$

where

$$H_t = -t \sum_{\langle i,j \rangle} \left(\hat{h}(i) \hat{s}_\alpha^\dagger(i) \hat{s}_\alpha(j) \hat{h}^\dagger(j) + \text{h. c.} \right) \quad , \quad (2.12)$$

and

$$H_J = \frac{J}{4} \sum_{\langle i,j \rangle} \left[\sum_{a=1}^3 \left(\hat{s}_\alpha^\dagger(i) \sigma_{\alpha\beta}^a \hat{s}_\beta(i) \hat{s}_{\alpha'}^\dagger(j) \sigma_{\alpha'\beta'}^a \hat{s}_{\beta'}(j) \right) \right. \\ \left. - \hat{s}_\alpha^\dagger(i) \hat{s}_\alpha(i) \hat{s}_{\alpha'}^\dagger(j) \hat{s}_{\alpha'}(j) \right] \quad . \quad (2.13)$$

The theory described by (2.11), (2.12) and (2.13) can be studied in the mean field approximation; however the reliability of this analysis is questionable and the results obtained with this method are purely qualitative since it is very hard to treat the *local* constraint (2.9) in a systematic and controllable way.

One possibility to overcome this problem is to partially liberate the slave holons and spinons, and use the so-called CP^1 representation [13,14] in which only the spinons are constrained while the holons are left free. More precisely, the CP^1 representation is characterized by the constraint

$$\hat{s}_\alpha^\dagger(i) \hat{s}_\alpha(i) = 1 \quad , \quad (2.14)$$

meaning that each lattice site is *always* occupied by a spinon; on the contrary, the holons may or may not be present. The average of the holon occupation number is directly related to the doping concentration δ ; in fact

$$\langle n(i) \rangle = \langle h(i) h^\dagger(i) s_\alpha^\dagger(i) s_\alpha(i) \rangle = 1 - \langle h^\dagger(i) h(i) \rangle \quad , \quad (2.15)$$

and by comparison with (2.4), we immediately deduce that

$$\langle h^\dagger(i) h(i) \rangle = \delta \quad . \quad (2.16)$$

We want to stress that there is a crucial difference between the slave operator representation considered earlier and the CP^1 representation introduced now. Indeed, as noted above, in the slave operator description the Hilbert space of holons and spinons at each lattice point is three-dimensional like the original electron Hilbert space; instead, in the CP^1 representation holons and spinons form a four-dimensional space, because each point must always have one spinon (up or down) but may or may not have a holon. To remove this discrepancy, one should correct the CP^1 representation with a suitable projection operator P in such a way that the Hilbert space of holons and spinons is reduced from four to three dimensions. Then, instead of (2.6) one should write [29]

$$\hat{c}_\alpha(i) = P \hat{s}_\alpha(i) \hat{h}^\dagger(i) P^\dagger \quad , \quad \hat{c}_\alpha^\dagger(i) = P \hat{h}(i) \hat{s}_\alpha^\dagger(i) P^\dagger \quad (2.17)$$

with the constraint (2.14). A systematic treatment of this operator P is cumbersome and difficult. However, it has been recently shown [29] that even without the projection one can obtain very good results both from a qualitative and a quantitative point of view. This is the same attitude that we take here, since we will not insert P in any formulas. It is precisely the freedom we gain in this way that allows us to make further progress.

In the CP^1 representation the spin operators $\hat{S}^a(i)$'s are realized with both holon and spinon oscillators as follows

$$\hat{S}^a(i) = \frac{1}{2} \left(1 - \hat{h}^\dagger(i) \hat{h}(i) \right) \hat{s}_\alpha^\dagger(i) \sigma_{\alpha\beta}^a \hat{s}_\beta(i) \quad , \quad a = 1, 2, 3 . \quad (2.18)$$

When no holon is present, $\hat{S}^a(i)$ receives a contribution only by the spinons, just like in (2.10); on the contrary, due to the prefactor, the spin vanishes in all sites occupied by holons ². Then, whenever $\hat{h}^\dagger(i) \hat{h}(i) = 1$, the spinon configuration can be arbitrarily chosen without changing the physical spin $\hat{S}^a(i)$ which remains zero; in other words, when the holons are present the CP^1 spinons are merely fictitious degrees of freedom which can be used at ease. This fact will play a crucial role in the following.

With this in mind, the Hamiltonian of the t - J model in the CP^1 representation can be written as in (2.11) with H_t given by (2.12) and H_J by

$$H_J = \frac{J}{4} \sum_{\langle i,j \rangle} \left\{ \left(1 - \hat{h}^\dagger(i) \hat{h}(i) \right) \left(1 - \hat{h}^\dagger(j) \hat{h}(j) \right) \cdot \left[\sum_{a=1}^3 \left(\hat{s}_\alpha^\dagger(i) \sigma_{\alpha\beta}^a \hat{s}_\beta(i) \hat{s}_{\alpha'}^\dagger(j) \sigma_{\alpha'\beta'}^a \hat{s}_{\beta'}(j) \right) - 1 \right] \right\} . \quad (2.19)$$

² Despite the appearance, the same is true for the slave fermion representation (2.10) if the constraint (2.9) is taken into account.

and with the spinons subject to the relation (2.14). To implement systematically this local constraint one may add to the Hamiltonian the following term

$$\sum_i \lambda(i) \left(\hat{s}_\alpha^\dagger(i) \hat{s}_\alpha(i) - 1 \right) , \quad (2.20)$$

where $\lambda(i)$ serves as a Lagrange multiplier over which we will integrate. However, before this integration is done the spinons are not constrained, and thus we can use the coherent state quantization method to analize the theory. The coherent states for holons and spinons, $|h\rangle$ and $|s\rangle$, are defined in such a way that

$$\begin{aligned} \hat{h}(i)|h\rangle &= h(i)|h\rangle \quad , \quad \langle h| \hat{h}^\dagger(i) = \langle h| h^*(i) \quad , \\ \hat{s}_\alpha(i)|s\rangle &= s_\alpha(i)|s\rangle \quad , \quad \langle s| \hat{s}_\alpha^\dagger(i) = \langle s| s_\alpha^*(i) \quad , \end{aligned} \quad (2.21)$$

where $h(i)$ and $s_\alpha(i)$ are respectively anticommuting and commuting complex fields.

Then, the partition function of the d -dimensional t - J model at temperature $1/\beta$ becomes an Euclidean path integral in $d+1$ dimensions

$$\mathcal{Z}_{tJ} = \int \mathcal{D}^2 h \mathcal{D}^2 s_1 \mathcal{D}^2 s_2 \mathcal{D}\lambda e^{-S_{tJ}} , \quad (2.22)$$

where

$$\begin{aligned} S_{tJ} = \int_0^\beta d\tau \left[\sum_i \left(h^*(i, \tau) \partial_\tau h(i, \tau) - s_\alpha^*(i, \tau) \partial_\tau s_\alpha(i, \tau) \right) + \mathcal{H}_{tJ} \right. \\ \left. + \sum_i \lambda(i, \tau) \left(s_\alpha^*(i, \tau) s_\alpha(i, \tau) - 1 \right) \right] . \end{aligned} \quad (2.23)$$

In this formula τ is an imaginary “time” parameter on which the fields h and s are made dependent, \mathcal{H}_{tJ} is the functional obtained from H_{tJ} in the CP^1 representation by replacing every occurrence of $\hat{h}(i)$, $\hat{h}^\dagger(i)$, $\hat{s}_\alpha(i)$ and $\hat{s}_\alpha^\dagger(i)$ with $h(i, \tau)$, $h^*(i, \tau)$, $s_\alpha(i, \tau)$ and $s_\alpha^*(i, \tau)$ respectively according to (2.21), and finally $\lambda(i, \tau)$ is the Lagrange multiplier which is integrated along the imaginary axis from $-\infty$ to $+\infty$ to enforce the local constraint

$$s_\alpha^*(i, \tau) s_\alpha(i, \tau) = 1 . \quad (2.24)$$

This is nothing but the expectation value in a coherent state of the original operator constraint (2.14), and also explains why (2.17) is called the CP^1 representation.

We remark that the τ -derivative terms in the first line of (2.23) represent the Berry phase for the holon and spinon fields which is usual in the coherent state quantization method³. Notice however that the spinon phase has a physical meaning only in those points that are not occupied by holons, *i.e.* where there is a physical spin to which the spinon is directly related. In the other points where holons are

³ The minus sign in the spinon term is the same that appears in [30].

present, the spinon phase has no direct significance since it corresponds to a fictitious spinon configuration. Finally, we recall that one has to impose periodic boundary conditions in τ on the bosonic fields, and antiperiodic boundary conditions on the fermionic ones in order to incorporate correctly the statistics effects.

At any site i , the spinon field $s_\alpha(i)$ can be used to define a three-dimensional unit vector $\mathbf{n}(i)$ with components

$$n^a(i) = s_\alpha^*(i) \sigma_{\alpha\beta}^a s_\beta(i) \quad , \quad a = 1, 2, 3 . \quad (2.25)$$

If no holon is present at i , then $\mathbf{n}(i)$ points along the spin direction (cf (2.18)). Thus, an antiferromagnetic arrangement of the spins corresponds to an antiferromagnetic order for the vector field $\mathbf{n}(i)$; we call this a Néel configuration. We now assume that the lattice where the model is defined is bipartite so that no frustration is present ⁴. Then, we can distinguish between even and odd sites, which we denote by a and b respectively. In a classical Néel configuration we have

$$\mathbf{n}(b) = -\mathbf{n}(a) . \quad (2.26)$$

This can be realized in terms of the spinon field by choosing for instance

$$s_\alpha(b) = \varepsilon_{\alpha\beta} s_\beta^*(a) e^{i\theta} , \quad (2.27)$$

where $\varepsilon_{\alpha\beta}$ is the completely antisymmetric tensor of $SU(2)$ and θ is an arbitrary phase. We notice that (2.27) corresponds to the usual choice of placing a certain spin representation on the even sites and its conjugate one on the odd sites [30].

Let us assume for a moment that (2.27) holds in the whole lattice, even in those points where holons are present and the spin is vanishing. If this is the case, then the holons are frozen and cannot move. This is most clearly seen by reinstating temporarily the holon operators so that the hopping Hamiltonian H_t becomes

$$H_t = -t \sum_{\langle a, b \rangle} \left(\hat{h}(a) s_\alpha^*(a) s_\alpha(b) \hat{h}^\dagger(b) + \text{h.c.} \right) . \quad (2.28)$$

If the spinons are arranged according to (2.27), then the hopping matrix elements are obviously vanishing. Therefore, one can say that the holons cannot move in a rigid Néel background and must distort the neighboring spinons in order to acquire kinetic energy. This is a rather well known fact [31,14]. We point out that in the gauge theory of [18,19,20] the hopping term (2.28) is substituted by hand with a next-to-nearest neighbor hopping Hamiltonian which forces the holons to jump only within one sublattice (even or odd) where the spinons are ferromagnetically ordered. However, in such a case one loses contact with the original t - J model in which a nearest-neighbor hopping term is the only one that is present to give kinetic energy to the holons.

⁴ This choice is customary in the study of antiferromagnetism and moreover the lattice representing the Cu-O layers of the cuprate oxides is in this class.

This puzzle can be overcome if one recalls that at the holon sites the spinon configuration is *a priori* not well defined. Therefore, assuming that (2.27) holds everywhere on the lattice is a too strong statement that is not really justified. In fact, an antiferromagnetic spinon arrangement is natural only for those links between two occupied sites, but not for those bonds connecting (at least) one holon. For example, let us consider the electron configuration represented in Fig. 1a. Using the holon-spinon language in the CP^1 representation we can describe Fig. 1a by saying that the even site a is occupied by a spinon of spin up and no holon, while the odd site b is occupied by a holon *and* a spinon. It seems that in principle there is no preferred choice for the orientation of the latter spinon since the spin is vanishing at b . However, if we consider the situation after the hole has hopped (Fig. 1b) we see that an up-spin appears in b and thus the spinon in b should be chosen upwards (*i.e.* in the same direction of the spinon in a). With this in mind, it is then clear that the spinons must be *chosen* in a ferromagnetic order on every link involving one holon and one spinon but must be in an antiferromagnetic configuration on those links without holons. However, for ease of notation and later convenience, it would be better not to distinguish between different kinds of links and, if possible, to deal only with one uniform order (either ferromagnetic or antiferromagnetic).

We can achieve this goal by flipping the physical spins on all odd sites. (This is customary in the study of antiferromagnetism.) After this is done, it is obvious that the spinon configuration becomes ferromagnetic in all occupied sites, and then it is no problem to choose the spinons on the empty sites as before in such a way that the spinon configuration be uniformly ferromagnetic. This procedure corresponds to making the following change of variable

$$s_\alpha(b) \longrightarrow \varepsilon_{\alpha\beta} s_\beta^*(b) \left(1 - \hat{h}^\dagger(b) \hat{h}(b) \right) + s_\alpha(b) \hat{h}^\dagger(b) \hat{h}(b) \quad (2.29)$$

in the Hamiltonian. From the explicit expressions, one can easily see that the t term of H_{tJ} does not change under (2.29) while the J term picks up an overall minus sign.

Also the spinon Berry phase is modified by (2.29). Indeed, we have

$$\begin{aligned} -s_\alpha^*(b, \tau) \partial_\tau s_\alpha(b, \tau) &\longrightarrow -s_\alpha(b, \tau) \partial_\tau s_\alpha^*(b, \tau) \left(1 - \hat{h}^\dagger(b) \hat{h}(b) \right) \\ &\quad - s_\alpha^*(b, \tau) \partial_\tau s_\alpha(b, \tau) \hat{h}^\dagger(b) \hat{h}(b) . \end{aligned} \quad (2.30)$$

From the constraint (2.24) it follows that $s_\alpha(b, \tau) \partial_\tau s_\alpha^*(b, \tau) = -s_\alpha^*(b, \tau) \partial_\tau s_\alpha(b, \tau)$, and thus the contribution of the odd sites to the spinon Berry phase can be written as

$$\sum_b \left(1 - 2h^*(b, \tau) h(b, \tau) \right) s_\alpha^*(b, \tau) \partial_\tau s_\alpha(b, \tau) . \quad (2.31)$$

Since the even sites are not affected by the spin flips, their contribution to the spinon phase remains as before.

Putting everything together, writing explicitly the sums over even and odd sites whenever is necessary, restoring the functional notation also for the holons and dropping the τ -dependence of the fields for ease of notation, we finally arrive at the

following action

$$S_{tJ} = \int_0^\beta d\tau \left\{ - \sum_a s_\alpha^*(a) \partial_\tau s_\alpha(a) + \sum_b \left(1 - 2h^*(b) h(b) \right) s_\alpha^*(b) \partial_\tau s_\alpha(b) \right. \\ \left. + \sum_i \left[h^*(i) \partial_\tau h(i) + \lambda(i) \left(s_\alpha^*(i) s_\alpha(i) - 1 \right) \right] + \mathcal{H}_{tJ} \right\}, \quad (2.32)$$

where

$$\mathcal{H}_{tJ} = \mathcal{H}_t + \mathcal{H}_J + \mu \sum_i \left(1 - h^*(i) h(i) \right), \quad (2.33)$$

with

$$\mathcal{H}_t = -t \sum_{\langle a,b \rangle} \left(h(a) s_\alpha^*(a) s_\alpha(b) h^*(b) + \text{c.c.} \right), \quad (2.34)$$

and

$$\mathcal{H}_J = -\frac{J}{2} \sum_{\langle a,b \rangle} \left(1 - h^*(a) h(a) \right) \left(1 - h^*(b) h(b) \right) s_\alpha^*(a) s_\alpha(b) s_\beta^*(b) s_\beta(a). \quad (2.35)$$

Notice the overall minus sign in \mathcal{H}_J that, as noted above, originates directly from the spin flip on the odd sites of the lattice realized by (2.29). Furthermore, in order to obtain the explicit expression (2.35) we used standard identities on the Pauli matrices. Finally, we remark again that in the action (2.32) the spinon fields are classically ordered in a ferromagnetic way everywhere.

This is the \mathbb{CP}^1 representation of the t - J model. When no holes are present (*i.e.* $h^*(i) h(i) = 0$ everywhere), (2.32) reduces to the well known \mathbb{CP}^1 representation of the Heisenberg model [32]. In the next section we will study the action (2.32) in the particular case of a one-dimensional lattice.

3. The Continuum Field Theory Description of the t - J Model in One Dimension

The action (2.32) is a field theory description for the t - J model that is valid in any dimension. However, to make further progress, from now on we will restrict our considerations only to a one-dimensional lattice (*i.e.* a chain) where some exact results can be obtained.

Let us begin our analysis by first setting $J = 0$ in (2.32)⁵. If we reinstate

⁵ We recall that when $J = 0$ the t - J model is equivalent to a Hubbard model with infinite on site repulsion.

temporarily the holon operators, the Hamiltonian becomes

$$H_{tJ}^{J=0} = -t \sum_{\langle a,b \rangle} \left(\hat{h}(a) s_\alpha^*(a) s_\alpha(b) \hat{h}^\dagger(b) + \text{h.c.} \right) - \mu \sum_i \hat{h}^\dagger(i) \hat{h}(i) + \mu N , \quad (3.1)$$

where N is the number of points in the chain. (Hereinafter we will drop the additive constant μN of (3.1).) We can interpret $H_{tJ}^{J=0}$ as the Hamiltonian for the hopping of holons in a background given by the spinons, which, as explained in the previous section, must be ferromagnetically ordered. If the spinons were constant classical fields, this would mean that

$$s_\alpha(a) = s_\alpha(b) \quad (3.2)$$

for any pair of nearest neighbor points a and b , and thus, because of (2.24) the Hamiltonian (3.1) would reduce simply to

$$H'_{tJ}^{J=0} = -t \sum_{\langle a,b \rangle} \left(\hat{h}(a) \hat{h}^\dagger(b) + \text{h.c.} \right) - \mu \sum_i \hat{h}^\dagger(i) \hat{h}(i) . \quad (3.3)$$

This is the Hamiltonian of a fermionic tight-binding model which can be easily diagonalized by Fourier transform. After this is done, one can see that it describes free fermions with dispersion relation

$$\epsilon(k) = 2t \cos(k\ell) - \mu , \quad (3.4)$$

where ℓ is the lattice spacing and the momentum k lies in the first Brillouin zone between 0 and 2π . If one requires that

$$\langle h^\dagger(i) h(i) \rangle = \delta \quad (3.5)$$

(cf (2.16)), the chemical potential at zero temperature must be fixed as

$$\mu = -2t \cos(\pi\delta) . \quad (3.6)$$

Correspondingly, in the first Brillouin zone there exist two Fermi points given by

$$k_F^\pm = \frac{1}{\ell} \pi (1 \pm \delta) \quad (3.7)$$

around which the dispersion relation $\epsilon(k)$ is linear with a Fermi velocity

$$v_F^\pm = -2t\ell \sin k_F^\pm = \pm 2t\ell \sin(\pi\delta) . \quad (3.8)$$

These are the typical values for spinless fermions with concentration δ , and moreover they are in full agreement with the exact Bethe Ansatz values of the one-dimensional infinite- U Hubbard model to which the t - J model reduces at $J=0$.

In the theory described by (3.3) it is rather straightforward to compute the correlation function $\langle h(a) h^\dagger(b) \rangle$ for any pair of nearest neighbor points a and b under

the assumption that it does not depend on the link (a, b) and is real. Indeed, by computing the derivative with respect to t of the free energy associated to $H'_{tJ}^{J=0}$ and taking into account (3.6), one finds that at zero temperature

$$\chi \equiv \langle h(a) h^\dagger(b) \rangle = \frac{\sin(\pi \delta)}{\pi} . \quad (3.9)$$

We remark that this same result could be obtained in the more general case in which the spinons are not constant by using the mean field approximation.

Since in physical processes one can excite only the particles whose momentum is close to the Fermi surface, we linearize our theory near the Fermi points (3.7) (for definiteness we choose k_F^+ which from now on we denote simply by k_F), and write the holon operator as

$$\hat{h}(i) = e^{+ik_F i} \hat{\psi}_+(i) + e^{-ik_F i} \hat{\psi}_-(i) , \quad (3.10)$$

where $\hat{\psi}_\pm(i)$ are smoothly varying on the lattice since the Fermi factors $e^{\pm ik_F i}$ have been pulled out. Then, upon inserting (3.10) into (3.3), and using (3.6) and (3.8), we get

$$H'_{tJ}^{J=0} = i \frac{v_F}{\ell} \sum_b \left[\hat{\psi}_-^\dagger(b) \hat{\psi}_-(b+\ell) - \hat{\psi}_+^\dagger(b) \hat{\psi}_+(b+\ell) - \hat{\psi}_-^\dagger(b) \hat{\psi}_-(b-\ell) + \hat{\psi}_+^\dagger(b) \hat{\psi}_+(b-\ell) \right] . \quad (3.11)$$

Notice that in deriving (3.11) we have dropped all terms with any rapidly oscillating factor according to the standard procedure [4], and also that the sum over all odd sites actually reproduces the sum over the whole lattice.

The Euclidean action that is equivalent to (3.11) in the functional formalism is, in obvious notations,

$$S'_{tJ}^{J=0} = \int_0^\beta d\tau \left\{ \sum_i \left[\psi_-^*(i, \tau) \partial_\tau \psi_-(i, \tau) + \psi_+^*(i, \tau) \partial_\tau \psi_+(i, \tau) \right] + i \frac{v_F}{\ell} \sum_b \left[\psi_-^*(b, \tau) \psi_-(b+\ell, \tau) - \psi_+^*(b, \tau) \psi_+(b+\ell, \tau) - \psi_-^*(b, \tau) \psi_-(b-\ell, \tau) + \psi_+^*(b, \tau) \psi_+(b-\ell, \tau) \right] \right\} . \quad (3.12)$$

It is now possible to take the continuum limit of (3.12). In order to do this correctly, we must remember that in one space dimension a fermionic field has engineering dimension 1/2. Therefore, we first replace ψ_\pm with $\sqrt{\ell} \psi_\pm$ so that

$$\frac{1}{\ell} \psi_+^*(b, \tau) \psi_+(b \pm \ell, \tau) \longrightarrow \psi_+^*(b, \tau) \psi_+(b, \tau) \pm \ell \psi_+^*(b, \tau) \partial_1 \psi_+(b, \tau) . \quad (3.13)$$

(A similar relation obviously holds for ψ_-). Then, for notational convenience we write ∂_τ as ∂_0 , and in the limit $\ell \rightarrow 0$ we replace $\left(2\ell \int_0^\beta d\tau \sum_b \right)$ with $\int d^2x$. As a result we

find

$$S'_{tJ}^{J=0} = \int d^2x \left\{ \bar{\psi}(x) \gamma^0 \partial_0 \psi(x) + v_F \bar{\psi}(x) \gamma^1 \partial_1 \psi(x) \right\} , \quad (3.14)$$

where we have introduced the notation

$$\psi(x) \equiv \begin{pmatrix} \psi_-(x) \\ \psi_+(x) \end{pmatrix} , \quad \bar{\psi}(x) \equiv \psi^\dagger(x) \gamma^0 = \begin{pmatrix} \psi_+^*(x) & \psi_-^*(x) \end{pmatrix} \quad (3.15)$$

with $\gamma^0 = \sigma^1$ and $\gamma^1 = \sigma^2$ being 2×2 Euclidean gamma matrices. Eq. (3.14) represents the action of a “relativistic” massless Dirac fermion in two dimensions with a characteristic velocity v_F . We have given the details of this standard derivation [4] so that it will be simpler to follow the later developments.

So far the spinons have been considered as a classical constant background in which the holons move like free fermions. If we want to take into account the presence of the spinons at the quantum level, we must study the Hamiltonian (3.1) with the spinons yielding a configuration that is not constant as in (3.2) but can fluctuate. Thus, the spinons should become true dynamical degrees of freedom which can take all possible values compatible with the constraint (2.24). However, in a sort of semiclassical approximation, we assume that the spinons despite their fluctuations, still define a ferromagnetic order in the lattice, which is broken only by (possibly antiferromagnetic) deviations that are small and of the order of the lattice spacing ℓ . Thus, following [11], for any site i we posit

$$s_\alpha(i) = z_\alpha(i) \sqrt{1 - \ell^2 \Delta_\beta^*(i) \Delta_\beta(i)} + \ell \Delta_\alpha(i) , \quad (3.16)$$

where z_α is a complex slowly varying field such that

$$z_\alpha^*(i) z_\alpha(i) = 1 , \quad (3.17)$$

and

$$z_\alpha(i \pm \ell) \simeq z_\alpha(i) \pm \ell \partial_1 z_\alpha(i) , \quad (3.18)$$

and Δ_α is a small staggered fluctuation such that

$$z_\alpha^*(i) \Delta_\alpha(i) + \Delta_\alpha^*(i) z_\alpha(i) = 0 , \quad (3.19)$$

and

$$\Delta_\alpha(i \pm \ell) \simeq -\Delta_\alpha(i) . \quad (3.20)$$

Notice that the square root factor in (3.16) together with (3.17) and (3.19), preserves the normalization of s_α to all orders in ℓ .

The meaning of these equations is quite straightforward. In fact, from (3.16) and (3.18) one can see that to order ℓ^0 that there is a ferromagnetic alignment among the spinons because $s_\alpha(a) \simeq s_\alpha(b)$. However to order ℓ^1 the spinon alignment is deformed both by the gradient of z_α and by the small fluctuation Δ_α which explicitly introduces an antiferromagnetic short range distortion due to the staggering sign

of (3.20). Of course we must remember that in those sites that are not occupied by holons, the ferromagnetic order of the z_α 's actually corresponds to a physical antiferromagnetic spin configuration, while the antiferromagnetic behavior of field Δ_α actually corresponds to a physical ferromagnetic deviation from a local Néel state. This is because we flipped the spinons on the odd sublattice (see (2.29)).

We now linearize the theory near the Fermi level of the holons and insert the decomposition (3.10) into the Hamiltonian (3.1) (from now on we will work only in the functional formalism, so that obvious change of notation should be made in these formulas). A simple calculation leads to the following result

$$\begin{aligned} \mathcal{H}_{tJ}^{J=0} = & t \sum_b \left\{ \left[e^{+ik_F \ell} \psi_+^*(b - \ell) \psi_+(b) s_\alpha^*(b) s_\alpha(b - \ell) \right. \right. \\ & + e^{-ik_F \ell} \psi_+^*(b + \ell) \psi_+(b) s_\alpha^*(b) s_\alpha(b + \ell) + \text{c.c.} \left. \left. \right] \right. \\ & + (\psi_+ \leftrightarrow \psi_-, \quad k_F \leftrightarrow -k_F) \left. \right\} \\ & - \frac{\mu}{2} \sum_b \left\{ \left[\psi_+^*(b - \ell) \psi_+(b - \ell) + \psi_+^*(b + \ell) \psi_+(b + \ell) \right. \right. \\ & \left. \left. + 2 \psi_+^*(b) \psi_+(b) \right] + (\psi_+ \leftrightarrow \psi_-) \right\} . \end{aligned} \quad (3.21)$$

Then, we decompose the spinon fields according to (3.16) and perform the continuum limit along the lines we discussed before, by keeping at most terms of order ℓ^2 . Notice that a bosonic field in one space dimension has zero engineering dimensions and thus no redefinition of the spinons is necessary. On the contrary, as noted above, the fermionic holons must be rescaled by a factor of $\sqrt{\ell}$ and therefore the square root factors of (3.16) can be simply put to one since they would yield contributions of higher order in ℓ . Taking this into account and using (3.18) and (3.20), it is possible to show that inside the curly brackets of (3.21) all the terms involving the fluctuation field Δ_α exactly cancel and so do also the terms of order ℓ . Then, the final result can be written in the continuum limit as

$$\int_0^\beta d\tau \mathcal{H}_{tJ}^{J=0} = v_F \int d^2x \left\{ \bar{\psi}(x) \gamma^1 [\partial_1 + i a_1(x)] \psi(x) \right\} , \quad (3.22)$$

where we have used the notation (3.15) and defined

$$a_1(x) = i z_\alpha^*(x) \partial_1 z_\alpha(x) . \quad (3.23)$$

Because of the constraint (3.17), $a_1(x)$ is real and looks like the space component of a gauge field to which the holons are minimally coupled. As we will see later this is indeed the correct interpretation.

Let us now consider the terms of the action $S_{tJ}^{J=0}$ which involve τ derivatives. From (2.32) we see that these can be distinguished into a spinon Berry phase

$$S_B = \int_0^\beta d\tau \left\{ \sum_b s_\alpha^*(b) \partial_\tau s_\alpha(b) - \sum_a s_\alpha^*(a) \partial_\tau s_\alpha(a) \right\} , \quad (3.24)$$

which does not involve holons, and into the remainder

$$\begin{aligned} S_h = \int_0^\beta d\tau \sum_b & \left\{ \frac{1}{2} \left[h^*(b-\ell) \partial_\tau h(b-\ell) + h^*(b+\ell) \partial_\tau h(b+\ell) \right] + h^*(b) \partial_\tau h(b) \right. \\ & \left. - 2h^*(b) h(b) s_\alpha^*(b) \partial_\tau s_\alpha(b) \right\} , \end{aligned} \quad (3.25)$$

which instead contains the holon field. The continuum limit of S_h can be easily performed as before and one gets

$$S_h = \int d^2x \left\{ \bar{\psi}(x) \gamma^0 [\partial_0 + i a_0(x)] \psi(x) \right\} , \quad (3.26)$$

where

$$a_0(x) = i z_\alpha^*(x) \partial_0 z_\alpha(x) . \quad (3.27)$$

As expected, (3.26) is the natural completion of (3.22). Thus we can say that in the continuum limit the action of the one-dimensional t - J model with $J = 0$ is given by [33]

$$S_{tJ}^{J=0} = \tilde{S}_B + S_\psi = \tilde{S}_B + \int d^2x \left\{ \bar{\psi}(\not{\partial} + i \not{a}) \psi \right\} , \quad (3.28)$$

where \tilde{S}_B is the continuum limit of the spinon Berry phase (3.24) to which we will return in a moment, and the “ / ” notation means

$$\not{\partial} = \gamma^0 \partial_0 + v_F \gamma^1 \partial_1 , \quad \not{a} = \gamma^0 a_0 + v_F \gamma^1 a_1 . \quad (3.29)$$

We remark that S_ψ is invariant under the following (standard) gauge transformation

$$\begin{aligned} \psi(x) &\longrightarrow \psi'(x) = e^{-i\Lambda(x)} \psi(x) \\ a_\mu(x) &\longrightarrow a'_\mu(x) = a_\mu(x) + \partial_\mu \Lambda(x) , \quad \mu = 0, 1 \end{aligned} \quad (3.30)$$

for an arbitrary function $\Lambda(x)$. Of course, in order to interpret (3.30) as a true local symmetry of the full action $S_{tJ}^{J=0}$ we need to examine in detail also the spinon Berry phase. Actually, the properties of \tilde{S}_B have already been investigated at length by many authors following [11]. Thus, we simply recall the main results referring to the original literature for their derivation [11,12,30,34]. If we substitute (3.16) into (3.24) and use (3.18) and (3.20), we can easily obtain

$$\tilde{S}_B = S_{top} + S' ,$$

where

$$S' = \int d^2x \left\{ \Delta_\alpha^*(x) \partial_0 z_\alpha(x) - \partial_0 z_\alpha^*(x) \Delta_\alpha(x) \right\} , \quad (3.31)$$

and S_{top} is the continuum limit of

$$\int_0^\beta d\tau \left\{ \sum_b z_\alpha^*(b) \partial_\tau z_\alpha(b) - \sum_a z_\alpha^*(a) \partial_\tau z_\alpha(a) \right\} . \quad (3.32)$$

At first sight it would seem that this expression vanishes; indeed since the z_α 's are ferromagnetically ordered, the contributions of the two sub-lattices seem to cancel each other. However, this cancellation occurs only for the bulk of the lattice leaving possible residual contributions at the boundaries. In fact, a careful analysis of (3.32) in the continuum limit (see for example [18,30] for details) leads to

$$S_{\text{top}} = \frac{i}{2} \int d^2x \left\{ \varepsilon^{\mu\nu} \partial_\mu a_\nu \right\} \quad (3.33)$$

with $a_\mu(x)$ defined by (3.27) and (3.23) for $\mu = 0, 1$. Notice that the integrand of (3.33) is a total derivative and thus S_{top} is a pure topological term which simply represents the flux associated with the vector field $a_\mu(x)$.

We summarize the results obtained so far by writing the full continuum action $S_{tJ}^{J=0}$, namely

$$S_{tJ}^{J=0} = S_{\text{top}} + \int d^2x \left\{ \bar{\psi} (\not{\partial} + i \not{a}) \psi + \Delta_\alpha^* \partial_0 z_\alpha - \partial_0 z_\alpha^* \Delta_\alpha \right\} . \quad (3.34)$$

To complete our analysis now we switch on the spin-exchange interaction (*i.e.* the J term (2.35)). We assume for simplicity that J be small as compared to t . (This is precisely the case of interest for the phenomenological applications of the t - J model.) As we can see from (2.35), \mathcal{H}_J is rather complicated: it contains quartic terms both in the holon and in the spinon fields. Thus, to render it tractable, we must make some approximations. To this aim, let us first observe that the presence of a non zero spin-exchange interaction certainly modifies the hopping of the holons with corrections of order J to the t term considered so far; but since $J \ll t$, this effect is negligible. Another consequence of a non zero J term is the appearance of a nearest-neighbor coupling between spinons which may induce spin exchanges in the chain. As is clear from (2.35), if in two neighboring sites a and b no holon is present, then the corresponding spinons interact and the spins in a and b can be exchanged. But, if a or b or both are occupied by a holon, then the J term vanishes. This means that the spinon dynamics is heavily influenced by the presence or the absence of the holons, and since these are mobile, two neighboring spinons may interact at some times but not at others. However, we must remember that $t \gg J$, which means that the holon motion is very quick as compared to the spinon dynamics. Therefore, to describe the latter it is reasonable to average over all possible holon configurations and replace \mathcal{H}_J with an effective spinon interaction, in which the coupling constant is reduced

with respect to J . In particular, this amounts to replace the holon factors of \mathcal{H}_J with the corresponding average values computed with the tight-binding Hamiltonian (3.3) that is independent of the spinons. More precisely inside \mathcal{H}_J we perform the following substitution

$$\left(1 - h^*(a) h(a)\right) \left(1 - h^*(b) h(b)\right) \longrightarrow \left\langle \left(1 - h^\dagger(a) h(a)\right) \left(1 - h^\dagger(b) h(b)\right) \right\rangle . \quad (3.35)$$

Using Wick's theorem and (3.5) and (3.9), the right hand side of (3.35) becomes

$$1 - 2\delta + \delta^2 - \frac{\sin^2(\pi\delta)}{\pi^2} .$$

Thus, within this approximation we can replace \mathcal{H}_J with

$$\tilde{\mathcal{H}}_J = -\frac{\tilde{J}}{2} \sum_{<a,b>} s_\alpha^*(a) s_\alpha(b) s_\beta^*(b) s_\beta(a) , \quad (3.36)$$

where the renormalized spin-exchange coupling constant \tilde{J} is defined by

$$\tilde{J} = J(1-\delta)^2 \left(1 - \frac{\sin^2(\pi\delta)}{\pi^2(1-\delta)^2}\right) . \quad (3.37)$$

Notice that \tilde{J} is smaller than J (as expected), and, like J , is positive. We remark that our approximation is different from the one that leads to the squeezed spin chain considered in [14,29], even though there are some similarities in the final formulas. In fact, in our case the effective Hamiltonian $\tilde{\mathcal{H}}_J$ is still defined on the original chain and not on a squeezed one like in [14,29]. This is because the spinons in the CP^1 representation are defined everywhere, even where holons are present. However, the differences between our approach and the squeezed spin chain approximation disappear in the continuum limit.

To complete our analysis, we now insert into (3.36) the decomposition (3.16) and keep all the terms up to order ℓ^2 . (The square root factors of (3.16) are now important since they yield contributions of order ℓ^2 .) Upon expanding the products in (3.36), we obviously produce terms quadratic in Δ , terms linear in Δ and terms without Δ . However, using (3.18), (3.19) and (3.20), after some elementary algebra we may show that all the terms linear in Δ exactly cancel each other, leaving us with the following expression

$$\begin{aligned} \tilde{\mathcal{H}}_J = & -\frac{\tilde{J}\ell}{2} \left\{ 2\ell \sum_b \left[\frac{1}{2} \partial_1^2 z_\alpha^*(b) z_\alpha(b) + \partial_1 z_\alpha^*(b) z_\alpha(b) z_\beta^*(b) \partial_1 z_\beta(b) + \frac{1}{2} z_\alpha^*(b) \partial_1^2 z_\alpha(b) \right. \right. \\ & \left. \left. + 4 \Delta_\alpha^*(b) A^{\alpha\beta}(z, z^*) \Delta_\beta(b) \right] \right\} - \tilde{J} N , \end{aligned} \quad (3.38)$$

where the matrix A is explicitly given by

$$A(z, z^*) = \begin{pmatrix} -z_2^* z_2 & z_2^* z_1 \\ z_1^* z_2 & -z_1^* z_1 \end{pmatrix} . \quad (3.39)$$

It is amusing to observe that

$$\frac{1}{2} \partial_1^2 z_\alpha^* z_\alpha + \partial_1 z_\alpha^* z_\alpha z_\beta^* \partial_1 z_\beta + \frac{1}{2} z_\alpha^* \partial_1^2 z_\alpha = - |(\partial_1 + i a_1) \mathbf{z}|^2 , \quad (3.40)$$

where in the right hand side we have introduced a standard vectorial notation for the two-component vector

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} . \quad (3.41)$$

Let us define for convenience the quantity

$$c = \tilde{J} \ell \quad (3.42)$$

which has the dimensions of a velocity. As we will see later, c can be interpreted as the characteristic velocity of the spin-waves produced by the fluctuating spinons.

Dropping the irrelevant additive constant $-\tilde{J} N$ from (3.38) and taking the continuum limit, we find that the action associated to $\tilde{\mathcal{H}}_J$ is

$$\tilde{S}_J = \frac{c}{2} \int d^2 x |(\partial_1 + i a_1) \mathbf{z}|^2 - 2c \int d^2 x \Delta_\alpha^* A^{\alpha\beta} \Delta_\beta . \quad (3.43)$$

If we collect everything together, we can write the effective partition function of the t - J model at small J as follows

$$\mathcal{Z}_{tJ} = \mathcal{N} \int \mathcal{D}^2 \psi \mathcal{D}^2 \mathbf{z} \mathcal{D}^2 \Delta_1 \mathcal{D}^2 \Delta_2 \mathcal{D} \lambda e^{-S_{tJ}} , \quad (3.44)$$

where \mathcal{N} is a normalization factor and

$$\begin{aligned} S_{tJ} = & S_{\text{top}} + \int d^2 x \left\{ \bar{\psi} (\not{\partial} + i \not{a}) \psi + \frac{c}{2} |(\partial_1 + i a_1) \mathbf{z}|^2 + \lambda (|\mathbf{z}|^2 - 1) \right\} \\ & - \int d^2 x \left\{ 2c \Delta_\alpha^* A^{\alpha\beta} \Delta_\beta - (\Delta_\alpha^* \partial_0 z_\alpha - \partial_0 z_\alpha^* \Delta_\alpha) \right\} . \end{aligned} \quad (3.45)$$

We remark that since we have taken the continuum limit, the integration over both the ‘‘background’’ \mathbf{z} and the fluctuations Δ in (3.44) does not over count the bosonic degrees of freedom, just like it happens in the background field formalism of quantum field theory. The ψ -term in S_{tJ} represents the low-energy effective action for the charge degrees of freedom. On the contrary, the spinon terms contain both long and short-distance effects, the former described by \mathbf{z} and the latter by Δ . To remove the short-distance effects and obtain the low-energy effective action also for the spin

degrees of freedom, we must integrate out the fluctuations Δ . This integration is feasible since (3.45) is simply a quadratic form in Δ ; in fact what we have to compute is

$$\tilde{I} = \int \mathcal{D}^2\Delta_1 \mathcal{D}^2\Delta_2 \exp \left[\int d^2x (2c \Delta_\alpha^* A^{\alpha\beta} \Delta_\beta - \Delta_\alpha^* \partial_0 z_\alpha + \partial_0 z_\alpha^* \Delta_\alpha) \right]. \quad (3.46)$$

It is worth pointing out that the determinant of the matrix A is zero and its trace is -1 , as we can immediately see from (3.39). This means that its eigenvalues are 0 and -1 . Due to the overall positive sign in the exponent (3.46), the presence of a negative eigenvalue is welcome, but the presence of a zero mode is disturbing since it may render \tilde{I} divergent. However, a careful analysis shows that this is not a real problem because the (divergent) integral over the zero mode turns out to be a constant that can be reabsorbed into the normalization \mathcal{N} of the partition function. To see this explicitly, let us denote by

$$\Lambda_\alpha^0 = z_\alpha \quad , \quad \Lambda_\alpha^- = -\varepsilon_{\alpha\beta} z_\beta^*$$

the eigenvectors of A corresponding respectively to eigenvalues 0 and -1 . Then, let us decompose Δ_α along these eigenvectors according to

$$\Delta_\alpha = \Lambda_0 \Lambda_\alpha^0 + \Lambda_- \Lambda_\alpha^-,$$

where

$$\Lambda_0 = z_\alpha^* \Delta_\alpha \quad , \quad \Lambda_- = \varepsilon_{\alpha\beta} z_\alpha \Delta_\beta. \quad (3.47)$$

With straightforward algebra, we can rewrite the exponent of (3.46) as

$$\begin{aligned} \int d^2x & \left(-2c |\Lambda_-|^2 - \Lambda_-^* \varepsilon_{\alpha\beta} z_\alpha \partial_0 z_\beta + \varepsilon_{\alpha\beta} z_\alpha^* \partial_0 z_\beta^* \Lambda_- \right. \\ & \left. - \Lambda_0^* z_\alpha^* \partial_0 z_\alpha + \partial_0 z_\alpha^* z_\alpha \Lambda_0 \right). \end{aligned} \quad (3.48)$$

Now, let us observe that the constraint (3.19) implies that

$$\Lambda_0 = -\Lambda_0^*,$$

and thus the second line of (3.48) identically vanishes. Therefore, the zero mode Λ_0 completely decouples from the dynamical fields and we find

$$\begin{aligned} \tilde{I} &= \left(\int \mathcal{D}^2\Lambda_0 \right) \int \mathcal{D}^2\Lambda_- \exp \left[\int d^2x \left(-2c |\Lambda_-|^2 - \Lambda_-^* \varepsilon_{\alpha\beta} z_\alpha \partial_0 z_\beta + \varepsilon_{\alpha\beta} z_\alpha^* \partial_0 z_\beta^* \Lambda_- \right) \right] \\ &= \text{const} \cdot \exp \left[-\frac{1}{2c} \int d^2x \left(\varepsilon_{\alpha\beta} z_\alpha^* \partial_0 z_\beta^* \right) \left(\varepsilon_{\alpha\beta} z_\alpha \partial_0 z_\beta \right) \right]. \end{aligned} \quad (3.49)$$

It is straightforward to show that

$$(\varepsilon_{\alpha\beta} z_\alpha^* \partial_0 z_\beta^*) (\varepsilon_{\alpha\beta} z_\alpha \partial_0 z_\beta) = |(\partial_0 + i a_0) \mathbf{z}|^2$$

which is the natural completion of the terms we found before.

Following the standard procedure, let us introduce the spin-wave stiffness

$$\rho = \frac{\tilde{J} \ell}{4} \quad (3.50)$$

which in one space dimension is simply related to the spin-wave velocity (cf (3.42)). Then, if we collect everything together we can write the effective partition function for the t - J model at small J in the following suggestive way

$$\mathcal{Z}_{tJ} = \mathcal{N}' \int \mathcal{D}^2 \psi \mathcal{D}^2 \mathbf{z} \mathcal{D}\lambda e^{-S_{tJ}^{\text{eff}}} , \quad (3.51)$$

where \mathcal{N}' is a new normalization factor and

$$\begin{aligned} S_{tJ}^{\text{eff}} = & S_{\text{top}} + \int d^2 x \left\{ \bar{\psi} (\not{\partial} + i \not{a}) \psi + \lambda (|\mathbf{z}|^2 - 1) \right. \\ & \left. + 2\rho \left[|(\partial_1 + i a_1) \mathbf{z}|^2 + \frac{1}{c^2} |(\partial_0 + i a_0) \mathbf{z}|^2 \right] \right\} . \end{aligned} \quad (3.52)$$

From this expression it is clear that c indeed represents the velocity of the spin-waves described by the spinon field \mathbf{z} as mentioned above. Of course, we are free to choose our units in such a way that $c = 1$ and the last part of the action becomes formally “Lorentz” invariant. Notice however that in doing this, the Fermi velocity of the holons which is used in the “/” notation becomes

$$v_F = \frac{2t}{\tilde{J}} \sin(\pi \delta) \quad (3.53)$$

that in general differs from one. Therefore, our statement about the “Lorentz” invariance of (3.52) must be suitably interpreted. Of course this is not unexpected, because in our problem there are two different characteristic fundamental velocities: one for the charge degrees of freedom and one for the spin degrees of freedom.

The action (3.52) is manifestly invariant under the local gauge transformation (3.30) supplemented by

$$z_\alpha(x) \longrightarrow z'_\alpha(x) = e^{-i\Lambda(x)} z_\alpha(x) . \quad (3.54)$$

Of course, (3.54) is compatible with (3.30), (3.27) and (3.23) as one can immediately verify. This local gauge invariance is not at all unexpected. Indeed, it has its roots in the fact that the original factorization of the electron operators in (2.6) leaves the

freedom of choosing arbitrarily the phase of holons and spinons in any point of the lattice. This freedom then translates into the gauge transformations (3.54) and (3.30) of the effective continuum field theory.

It is interesting to remark that (3.52) is nothing but the action of a CP^1 model [15,16,17] with a coupling constant proportional to $1/\rho$ and minimally coupled to a fermionic field. Therefore, we can formulate our results by saying that the effective dynamics of the one-dimensional t - J model at small J in the CP^1 representation is described by a CP^1 model for the spin degrees of freedom minimally coupled to a massless fermionic field representing the low-energy charge excitations.

4. Conclusions and Outlook

In the previous section we managed to represent the effective dynamics of the one-dimensional t - J model at small J by means of a continuum field theory with an explicit Abelian gauge invariance. However, as we can see from (3.27) and (3.23), the vector potential a_μ appearing in the action is not an independent field as is customary in gauge theories. On the contrary, it is a functional entirely determined by other fields, *viz.* by the slowly varying spinons. A similar situation occurred also in the original formulation of the CP^1 model and its large- N generalizations [16]; but there, with a suitable reinterpretation, it was possible to introduce the gauge potential as an independent degree of freedom, and then study its dynamics, for example using a large- N expansion. We recall that within the framework of quantum field theory the CP^1 model and its generalizations attracted a lot of attention as interesting examples of non trivial renormalizable gauge theories exhibiting confinement, dimensional transmutation and topological effects (for reviews see *e.g.* [17]). It would be very interesting to explore these issues and study the role of these models also in the context of the strongly correlated electron systems; to this aim, a good starting point is certainly represented by the action (3.52) that we derived directly from the t - J model.

To make the connection with [16,17] more explicit, let us fix $c = 1$ in (3.52) and perform a Wick rotation on x^0 to transform the two-dimensional Euclidean space considered so far into a two-dimensional Minkowski space-time; the resulting Lagrangian is then

$$L = \frac{1}{2g^2} (\partial^\mu - i a^\mu) \mathbf{z}^* \cdot (\partial_\mu + i a_\mu) \mathbf{z} + \lambda (\mathbf{z}^* \cdot \mathbf{z} - 1) + L_{\text{top}} \quad (4.1)$$

$$+ \bar{\psi} (i \not{\partial} - \not{a}) \psi .$$

Here we have introduced a dimensionless coupling constant g^2 for the CP^1 term; because of our choice of units, g^2 is actually fixed to be one (cf (3.50)), but it could be more convenient to think of it as a parameter. Furthermore, in (4.1) the space-time indices are contracted with the Minkowski metric $\eta_{11} = -\eta_{00} = 1$, and the gamma matrices used in the “/” symbol are $\tilde{\gamma}_0 = i \sigma^1$ and $\tilde{\gamma}_1 = v_F \sigma^2$ with v_F given

by (3.53). (For ease of notation we have incorporated the Fermi velocity into the definition of $\tilde{\gamma}_1$, and thus these gamma matrices satisfy a modified Clifford algebra.) Finally, L_{top} is the topological Lagrangian which is given by the integrand of (3.33) also in a Minkowski space-time.

If, for a moment, we consider a_μ as an independent field and compute its equation of motion from the first line of the Lagrangian (4.1) denoted by L_z , we get

$$\frac{\delta L_z}{\delta a_\mu} = 0 \implies a_\mu = i \mathbf{z}^* \cdot \partial_\mu \mathbf{z} , \quad (4.2)$$

i.e. precisely the (Wick rotated) definition given in (3.27) and (3.23). Notice that L_{top} , being a topological term, does not give any contribution to this field equation. The result in (4.2) is quite interesting because it shows that the wanted relation between the gauge field a_μ and the bosonic vector \mathbf{z} may be dynamically determined since it could be viewed as an equation of motion. However, (4.2) is not the true field equation of a_μ determined by the whole Lagrangian L , since also the fermionic terms in the second line of (4.1) give a contribution to it because the vector potential a_μ couples to ψ . This problem is easily cured if, instead of (4.1), we consider the following Lagrangian

$$L' = \frac{1}{2g^2} (\partial^\mu - i A^\mu) \mathbf{z}^* \cdot (\partial_\mu + i A_\mu) \mathbf{z} + \lambda (\mathbf{z}^* \cdot \mathbf{z} - 1) + L_{\text{top}} \\ + \bar{\psi} (i \not{\partial} - \not{A}) \psi + \frac{g^2}{2} \bar{\psi} \tilde{\gamma}^\mu \psi \bar{\psi} \tilde{\gamma}_\mu \psi , \quad (4.3)$$

where the vector field A_μ is completely independent of the other fields, and a new four-fermion interaction has been introduced. Notice that A_μ appears in L' without a kinetic term, and thus it can be effectively eliminated through its equation of motion, which is given by

$$\frac{\delta L'}{\delta A_\mu} = 0 \implies A_\mu = i \mathbf{z}^* \cdot \partial_\mu \mathbf{z} + g^2 \bar{\psi} \tilde{\gamma}_\mu \psi = a_\mu + g^2 \bar{\psi} \tilde{\gamma}_\mu \psi . \quad (4.4)$$

If we substitute this into L' , we exactly reproduce the original Lagrangian (4.1); in particular, the four-fermion interaction of (4.3) has been designed to cancel the fermionic terms arising when (4.4) is inserted into L' . Thus, we can say that (4.3) is equivalent to (4.1), but it is more useful than the latter since the gauge field A_μ in (4.3) is truly an independent degree of freedom, whilst a_μ in (4.1) is functionally determined by the bosonic vector \mathbf{z} . We point out that the four-fermion interaction of L' is renormalizable in two dimensions being marginal (indeed the coupling constant in front of it is dimensionless). It is one of the standard fermionic interactions that are usually considered in this context, the other being $(\bar{\psi} \psi)^2$ and $(\bar{\psi} \gamma_5 \psi)^2$ where $\gamma_5 = \sigma^3$.

The general structure of our final result is similar to that of previous investigations on the dynamics of holes in doped antiferromagnets; however a more detailed

comparison, in particular with [18,20], shows important and significant differences. First of all, contrarily to [18,20], we took as our starting point the t - J model in the CP^1 representation, and deduced everything from it. The coupling between holons and spinons is then determined by the structure of the Hamiltonian of t - J model and is not introduced by hand. In the low-energy limit, our model turns out to be described by the Lagrangian L' (or better by its Euclidean version) which, though similar, is different from that of [18,20] in many respects. Indeed, we have only one fermionic field describing the charge degrees of freedom instead of two, and moreover we have also a four-fermion interaction that is not present in [18]. The reason for having one holon field instead of two is because in the CP^1 representation it is possible to realize a nearest neighbor hopping of holons in the presence of a ferromagnetic arrangement of fictitious spinons, as we discussed at length in section 2. On the contrary, if one wants to describe the charge motion in doped antiferromagnets without using the CP^1 representation, it becomes necessary to introduce by hand a next-to-nearest neighbor holon hopping term as it was done in [18,20], and consequently to define two different types of holons each one moving only within one sub-lattice. However, a next-to-nearest neighbor hopping term is vanishing from the point of view of the t - J model and hence it would be unnatural for us.

We conclude this paper by pointing out that if we generalize the doublet (3.41) to an N -component vector

$$\mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix} \quad (4.5)$$

with $\mathbf{z}^* \cdot \mathbf{z} = 1$, the action (4.3) becomes that of a CP^{N-1} model coupled to one fermionic field. Since \mathbf{z} appears quadratically in this action, it can be integrated out to yield an effective action for the gauge field A_μ and the fermionic field ψ . This may be studied non-perturbatively with a large- N expansion [35]; in particular one may examine if the factorization of the electrons into holons and spinons is really consistent and survives loop corrections; furthermore, one may analyze the role of the topological term of the action in connection with Haldane's conjecture [11,34], and determine the structure of the renormalization group flow and the nature of the long-distance fixed points, and hopefully also shed some light on the dynamical mechanisms of the charge-spin separation from a field theory point of view. We leave these issues as well as a more complete analysis of the system described by (4.3) to future investigations.

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Figure Captions

Fig. 1a An electronic configuration in which the even site a is occupied by an electron of spin up and the nearest neighbor odd site b is empty.

Fig. 1b After hopping, the spin up electron has moved to b and the hole to a .