

Production of Vortices by Scattering Particles

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Abstract:

We give an action that can be used to describe the production of global vortices in 2+1 dimensions by scattering Nambu-Goldstone bosons. At strong self-coupling the action reduces to scalar QED with particular values of the coupling constants and the production cross-section is explicitly found. We also consider the production of gauged vortices by scattering particles that have an Aharonov-Bohm interaction with the vortex.

Solitons occur in a wide range of condensed matter systems and also in many particle physics models. Usually a soliton is treated as an object separate from the particle-like excitations that the theory contains and the connection between particles and solitons remains an unsolved problem. In particular, it is not known how to calculate the cross-section for producing solitons in particle collisions. In this paper, we shall attempt to establish what is at least a partial connection by first considering the production of global vortices in the scattering of Nambu-Goldstone bosons and then considering the production of gauged vortices in the scattering of particles having an Aharonov-Bohm interaction with the vortices. Our analysis only applies to 2+1 dimensions.

The simplest model that gives rise to global vortices is described by the action:

$$S_{gl} = \int d^3x \left[\frac{1}{2} |\partial_\mu \phi|^2 - \frac{\lambda}{4} (|\phi|^2 - \eta^2)^2 \right] \quad (1)$$

where ϕ is a complex scalar field and the Greek indices run from 0 to 2. If we write

$$\phi = \rho e^{i\alpha}, \quad (2)$$

the field α is massless and is the Nambu-Goldstone boson of the theory. The field ρ acquires a vacuum expectation value and the excitations of ρ about this value are the massive degrees of freedom with mass $m_\rho = \sqrt{2\lambda}\eta$. (Note that, since we are working in 3 dimensions, λ has dimensions of mass and η of $\sqrt{\text{mass}}$.) The equation of motion for α is

$$\partial^\mu [\rho^2 \partial_\mu \alpha] = 0 \quad (3)$$

and can be solved by setting

$$\rho^2 \partial_\mu \alpha = \frac{1}{2} \eta \epsilon_{\mu\nu\lambda} H^{\nu\lambda} \quad (4)$$

where

$$H^{\nu\lambda} = \partial^\nu B^\lambda - \partial^\lambda B^\nu \quad (5)$$

for arbitrary gauge potentials B^μ .

This connection between a massless scalar field and the dual tensor field is well-known and has been used extensively to discuss the behaviour of global vortices^{1,2,3,4}. We can write the action S_{gl} in terms of the gauge field B_μ as⁴:

$$S_{gl} = \int d^3x \left[\frac{1}{2}(\partial_\mu\rho)^2 - \frac{\lambda}{4}(\rho^2 - \eta^2)^2 - \frac{1}{4}\frac{\eta^2}{\rho^2}H_{\mu\nu}H^{\mu\nu} \right] + g \int dx_v^\mu B_\mu \quad (6)$$

where, x_v^μ is the position of the vortex and the interaction between B_μ and the vortex is given by the coupling constant^{2,3}

$$g = 2\pi\eta . \quad (7)$$

In writing (6), no approximation has been made regarding the structure of the vortex; the only step in going from (1) to (6) has been the transformation of variables from α to B_μ . In doing so, the vortex degree of freedom had to be introduced so as to enforce the condition that α must wind around the vortex. In (6) we have assumed the presence of only one vortex; the presence of several vortices can be accommodated by including a sum over vortices in the interaction term. But we want to go further: we want to include the possibility that the number of vortices in the system can change. Hence, we must replace the vortex degree of freedom with a vortex field V that can be second quantized and whose quanta would represent vortices. To do this, note that the interaction of the vortex with B_μ is precisely the interaction of a current, $g\dot{x}_v^\mu$, with the gauge field B_μ . As things stand, the vortex is only identified with the point around which α winds and hence the vortex is massless. Therefore the vortex field is to be added such that it gives the correct current-gauge field interaction, is massless and respects the gauge invariance:

$$B_\mu \rightarrow B_\mu + \partial_\mu\Lambda . \quad (8)$$

(This gauge symmetry has nothing to do with the global U(1) symmetry of S_{gl} but is purely due to an introduction of redundant variables in the transformation (4).) This

leads to the action:

$$S_{gl+V} = \int d^3x \left[\frac{1}{2}(\partial_\mu\rho)^2 - \frac{\lambda}{4}(\rho^2 - \eta^2)^2 + |(\partial_\mu + igB_\mu)V|^2 - \frac{1}{4}\frac{\eta^2}{\rho^2}H_{\mu\nu}H^{\mu\nu} \right]. \quad (9)$$

In S_{gl+V} , a vortex only labels a point around which the original field α circulates and arbitrary excitations of the structure of the classical vortex are included since we have retained the field ρ . The most unusual feature of S_{gl+V} is the $\eta^2\rho^{-2}$ in front of the field strength term and, the physical interpretation of this term is that $\eta^2\rho^{-2}$ plays the role of a dynamical dielectric constant and $\eta^{-2}\rho^2$ the role of a dynamical magnetic permeability.

The action S_{gl+V} is our central result. In S_{gl+V} , the vortices appear explicitly as ordinary quanta would in a conventional quantum field theory. By second quantizing the field V it should be possible to describe the creation and annihilation of vortices. However, the peculiar interaction of ρ with B_μ does not immediately lend itself to a simple treatment of processes such as vortex production and therefore it is necessary to consider various approximation schemes.

The simplest approximation to consider is the case when the vortex is completely classical and the structure of the vortex can be considered to be rigid. Then we may take ρ to be always given by the classical solution ρ_v and integrate out the ρ degree of freedom. If we do this in (6), the vortex gets a mass, and the result is the Kalb-Ramond action⁵ (suitably reduced from 4 to 3 dimensions):

$$S_{KR} = -\mu \int d\tau + g \int dx_v^\mu B_\mu - \frac{1}{4} \int d^3x H_{\mu\nu}H^{\mu\nu} \quad (10)$$

where τ is the proper time of the vortex. The energy per unit length of the vortex is⁴

$$\mu = \int d^2x \left[\frac{1}{2}(\partial_i\rho_v)^2 - \frac{1}{2}\rho_v^2 \left(1 - \frac{\rho_v^2}{\eta^2} \right) (\partial_i\alpha_v)^2 + \frac{\lambda}{4}(\rho_v^2 - \eta^2)^2 \right] \quad (11)$$

The fields labelled by a subscript v refer to the field configurations for a static vortex and the index $i = 1, 2$. By numerically solving for the classical vortex profile, we have found μ :

$$\mu = 0.18\pi\eta^2 \sim \eta^2 \quad (12)$$

and it is easy to see, by rescaling arguments, that μ is independent of λ .

Now we introduce the vortex field in the Kalb-Ramond action. This simply leads to massive scalar QED without any vortex self-interactions:

$$S_V = \int d^3x \left[|(\partial_\mu + igB_\mu)V|^2 - \mu^2|V|^2 - \frac{1}{4}H_{\mu\nu}H^{\mu\nu} \right]. \quad (13)$$

With this action, it is possible to consider the production of vortices - quanta of V - in processes involving Nambu-Goldstone bosons - that is, quanta of B_μ (denoted by γ). The calculations that need to be done are standard scalar QED calculations. At high energies, perturbative calculations can be done and the lowest order cross-section for $\gamma\gamma \rightarrow VV$ is:

$$\frac{d\sigma}{d\theta} = \left(\frac{g^2}{E}\right)^2 \frac{1}{4\pi E} \left[1 - \frac{2(1-s^2)\sin^2\theta}{(1-s^2)\sin^2\theta + s^2} \right]^2 \quad (14)$$

where, $E \geq 2\mu$ is the energy in the center of mass system, $s = 2\mu/E \leq 1$, θ is the angle between the incoming and outgoing beams and the γ polarizations have been taken to be aligned.

The production of vortices requires an energy $E > 2\mu$ while, in deriving the Kalb-Ramond action, we have used $E < m_\rho$ since we have assumed that the vortex structure is rigid and, hence, quanta of the field ρ in S_{gl} remain unexcited. Therefore our result for the production cross-section is only valid in the regime where $m_\rho > 2\mu$, or, in terms of the parameters, where $\sqrt{\lambda} \gtrsim \eta$. This is in the strong coupling regime of the theory described by S_{gl} . The only way out of the strong coupling regime would seem to be to stick with the action S_{gl+V} or derive some approximation scheme that interpolates between the two extreme cases represented by S_{gl+V} and S_V .

The reader might wonder if the result (14) is believable since generally not much can be said of theories at strong coupling. However, in our case, the only step in going from (9) to (13) is an integration over the massive degree of freedom ρ and, in particular, no perturbative expansion in λ has been utilised. Also note that a simple rescaling argument is sufficient to show that μ is independent of λ and so the mass of the vortex does not depend on the value of the coupling constant. Once we have obtained (13), the result (14) follows within the energy domain $m_\rho > E > 2\mu$. It is also helpful to repeat this argument in terms of quantum fluctuations about the vortex. The fluctuations will consist of zero modes corresponding to translations of the vortex and of massive modes corresponding to fluctuations of the vortex profile. The latter fluctuations can be viewed as massive ρ quanta in the background of the vortex. In the strong coupling regime, the mass of ρ becomes large and hence the massive fluctuations cannot be excited at the energy scales we are interested in. Therefore the zero modes are the the only remaining fluctuations and these are already included in (10) as the dynamical degrees of freedom of the vortex.

An observation relevant to this discussion is that some features of 4He can be described using the Kalb-Ramond action even though 4He is in the strong coupling regime with (see Ref. [6]) $\lambda_4 \sim 10^9$. (The subscript 4 on λ indicates that the coupling is for the theory in 4 dimensions.) Hence, it may be possible to test the validity of this approach by doing experiments in the laboratory.

Observe that there is no *classical* production of vortices in the model (13) and also that, since we are working in three dimensions, the theory is confining. The confinement is most simply seen by noting that the electric force between two charges in 3 dimensions falls off as $1/r$ and the potential diverges logarithmically. A consequence of the weak confining force is that, after the vortices are produced, they will not move away indefinitely from each other but will eventually turn around and recollapse. However, we may define that

vortices are produced whenever the separation of the vortices exceeds the width of the vortices (roughly equal to m_ρ^{-1}). Then the logarithmic inter-vortex potential has an effect on the dynamics of the vortices that have already been produced but does not affect the production process itself.

Outside the strong coupling regime, a method that suggests itself is to treat the vortex as consisting of an inner core that remains classical and an outer shell which is quantum. Suppose that the classical core has a radius R . Inside the classical core we can take $\rho = \rho_v$ as was done in the Kalb-Ramond case while outside the core we write

$$\rho = \eta + \sigma . \quad (15)$$

Then, the full action S_{gl+V} may be approximated by,

$$S_{V+\sigma} = \int d^3x \left[|(\partial_\mu + igB_\mu)V|^2 - \mu_R^2|V|^2 - \frac{1}{4} \left(1 + \frac{\sigma}{\eta}\right)^{-2} H_{\mu\nu}H^{\mu\nu} + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{\lambda}{4}(2\eta + \sigma)^2\sigma^2 \right] \quad (16)$$

where, μ_R is the mass of the classical vortex within the core of radius R and we have ignored terms of order E^2R^2 and higher where E is the energy scale of interest. The σ field represents the dressing on the bare classical vortex and we can expand the coefficient of the field strength term in increasing powers of σ/η . (We expect that this expansion will make sense since $|\sigma|/\eta < 1$.) An extreme case of (16) is when $R = 0$ and the whole vortex is considered as being dressing. In this case, $\mu_R = 0$ and, at the center of the vortex, we necessarily have $\sigma = -\eta$. Therefore, if we expand out the coefficient of the field strength term, we would be forced to consider the entire expansion with its infinite number of terms. The general policy is that we would like R to be small compared to E^{-1} but not too small, as that would entail retaining large orders in σ/η . Another extreme case is the large coupling case already considered where we can simultaneously take E^2R^2 to be small and also truncate the σ expansion to the lowest order.

A physical picture of the production process is that the $\gamma\gamma$ collision produces the innermost cores (of size R such that $ER < 1$) of the vortex-anti-vortex pair. The relative momenta with which the vortices in the pair are moving away from each other at the moment of production is $\sqrt{E^2 - 4\mu_R^2}$. As the pair is trying to separate, the B_μ and σ fields provide an attractive potential that tries to prevent the separation. This attraction is due to the need to create the outer shell of the vortices and the accompanying gauge field configuration as the vortices move apart. If the initial momenta of the innermost cores is large enough to overcome the attractive potential and the energy of the inner cores is not transferred to excitations of the outer shell of the vortices or quanta that can be radiated away, the pair will separate out to distances larger than the vortex core width and we can say that a vortex pair has been produced. We know that the total energy required to separate the pair beyond the core width is $2(\mu - \mu_R)$ and so we at least need $E > 2\mu$ to create the vortices. However the rate at which the energy of the vortices is lost to excitations of the outer shells or to quanta that are radiated away will depend on the coupling λ and needs to be calculated from action such as $S_{V+\sigma}$. The only case when quanta of σ will not be excited is when the residual energy after producing the inner cores is not large enough to produce any σ quanta: $E/2 - \mu_R < m$. At threshold, $E/2 = \mu$ and then, a translation of the inequality in terms of the coupling constants leads to: $\sqrt{\lambda}/\eta > 1 - \mu_R/\mu$. If we take $R = \eta^{-2}$, and evaluate μ_R/μ numerically, we find that the inequality is satisfied for $\sqrt{\lambda}/\eta > 0.7$.

At first sight, the result (14) seems puzzling as one expects that the vortex production cross-section should be exponentially suppressed since the vortex is an extended object. However, there is no exponential suppression in (14) since the result is only valid in the thin vortex limit where the vortex can be treated as if it had no structure. To find the production cross-section in the case when the vortex does have structure, one would have

to find methods to analyze the action S_{gl+V} for arbitrary values of λ . What has emerged from the analysis here is that the production cross-section must go over to (14) in the limit that λ becomes large.

Next we shall study the possibility of producing gauged vortices. Consider the Abelian-Higgs model:

$$S_{AH} = \int d^3x \left[|(\partial_\mu + ieA_\mu)\phi|^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \lambda(|\phi|^2 - \eta^2)^2 \right]. \quad (17)$$

It is well-known that this model admits vortex solutions⁷. But now, when ϕ gets an expectation value, all the fields become massive. In the limit that the masses are very large compared to the energies we are interested in, the action for the vortices is simply that of a point particle:

$$S_p = -\mu \int d\tau \quad (18)$$

where, as before, $\mu \sim \eta^2$ is the vortex mass. (μ also depends weakly on the ratio e^2/λ which we assume to be of order one.) Therefore if we replace the single vortex by a field of vortices and ignore vortex-vortex interactions, the result is a free field theory:

$$S_{free} = \int d^3x [(\partial_\mu V)^2 - \mu^2|V|^2], \quad (19)$$

and vortices cannot be produced in this model because there are no particles to produce them with. Therefore to get something interesting, it is necessary to introduce other fields in the Abelian-Higgs model or to consider excitations of the scalar and vector fields already present in S_{AH} . Here we will only consider the first of these two options.

It should be mentioned that one can always introduce gravity in the model (19) and hope to get vortex production in a non-trivial gravitational background. In four spacetime dimensions this has already been considered in the context of nucleation of topological defects in de Sitter space⁸.

Let us consider the case when there is an extra scalar field present in S_{AH} :

$$S_{AH+} = S_{AH} + \int d^3x \left[\frac{1}{2} |(\partial_\mu + ie' A_\mu)\psi|^2 - U(|\phi|, |\psi|) \right]. \quad (20)$$

It is important to note that the gauge coupling e' is chosen to be different from e . Also, the modulus of the scalar field $|\phi|$ occurring in the potential may be replaced by the vacuum expectation value η in the zero width vortex limit.

Now let us integrate out the massive degrees of freedom:

$$S_{AH+} \rightarrow -\mu \int d\tau + \int d^3x \left[\frac{1}{2} \xi^2 \left\{ \partial_\mu \left(\alpha - \frac{e'}{e} \theta \right) \right\}^2 + \frac{1}{2} (\partial_\mu \xi)^2 - U(\xi) \right], \quad (21)$$

where, we have used (18), together with

$$A_\mu = -\frac{1}{e} \partial_\mu \theta, \quad (22)$$

where, θ is the polar angle about the vortex, and

$$\psi = \xi e^{i\alpha}. \quad (23)$$

We repeat the same steps that led to the Kalb-Ramond action for the global vortex and define:

$$\xi^2 \partial_\mu \left(\alpha - \frac{e'}{e} \theta \right) = \frac{1}{2} \eta' \epsilon_{\mu\nu\lambda} H^{\nu\lambda} \quad (24)$$

where $(\eta')^2$ is an arbitrary mass scale. This transformation of variables leads to a Kalb-Ramond type of action:

$$S_{KR+} = -\mu \int d\tau + q \int dx^\mu B_\mu + \int d^3x \left[-\frac{1}{4} \frac{\eta'^2}{\xi^2} H_{\mu\nu} H^{\mu\nu} + \frac{1}{2} (\partial_\mu \xi)^2 - U(\xi) \right], \quad (25)$$

where, the charge q will now be determined.

If we wish to find a non-trivial field configuration ψ in the background of the vortex, single-valuedness of ψ requires that

$$\oint_{C_\infty} dx^\mu \partial_\mu \alpha = 2\pi n \quad (26)$$

for some integer n . Therefore, if we let

$$\beta = \alpha - \frac{e'}{e}\theta \quad (27)$$

then, a constraint on the field configuration ψ is

$$\oint_{C_\infty} dx^\mu \partial_\mu \beta = 2\pi \left(n - \frac{e'}{e} \right) \quad (28)$$

where, C_∞ denotes a closed contour encircling the vortex. We will only consider the case when e'/e is not an integer and hence, the quanta of ψ have an Aharonov-Bohm interaction with the vortex.

The constraint (28) can now be expressed in terms of the gauge fields introduced in (24) by the manipulations in Refs. 2,3. This results in the coupling of the gauge field B_μ with the vortex as in (25) and where the charge q is given by

$$q = 2\pi \left[n - \frac{e'}{e} \right] \eta' . \quad (29)$$

Furthermore, we expect the most weakly coupled field configurations to be the most important and so n should be chosen to minimize q .

Next we wish to introduce the vortex field V so that we can describe vortex production. This is done using the minimal prescription and ignoring vortex-vortex interactions:

$$S_{V+} = \int d^3x \left[|(\partial_\mu + iqB_\mu)V|^2 - \mu^2|V|^2 - \frac{1}{4}\frac{\eta'^2}{\xi^2}H_{\mu\nu}H^{\mu\nu} + \frac{1}{2}(\partial_\mu\xi)^2 - U(\xi) \right] . \quad (30)$$

Here $\eta'^2\xi^{-2}$ plays the role of a dynamical dielectric constant and $\eta'^{-2}\xi^2$ the role of magnetic permeability. In contrast to the global case (see below eq. (9)), $\xi = 0$ in vacuum, and so the dielectric constant is infinite while the magnetic permeability is zero. Consequently the electric and magnetic fields must vanish in the vacuum.

The situation becomes more interesting if we consider the case when the potential $U(\xi)$ can be ignored, and a background of the field ψ is present:

$$\psi = Ae^{i\alpha(t, \vec{x})} \quad (31)$$

where, A is a constant. (This could happen if we were to consider the scattering of waves of α , or, if there were a non-trivial potential $U(\xi)$ and the global U(1) symmetry associated with phase rotations of ψ was broken, giving a vacuum expectation value to ψ .) Then we have $\xi = A$ and the relevant action (30) is scalar QED once again. Hence the cross-section for the production of vortices is identical to that in the global vortex case except for the value of the charge. The cross-section will be given by (14) with $g = -2\pi Ae'/e$, where we have taken e' to be much smaller than e and set $n = 0$.

The model (30) can be useful for calculating the production rates of vortices only in the strong coupling regime (large e and λ). If these couplings are not large, however, we would have to retain the possibility that quanta of the massive fields ϕ and A_μ could be produced in the scattering of ψ particles. For this, we would have to deal with an action similar to S_{gl+V} in (9). This action can be written down by adding S_{AH} to S_{V+} , including the possibility of ρ, ξ interactions in the potential and setting $\mu = 0$:

$$S_{AH+V} = S_{AH} + \int d^3x \left[|(\partial_\mu + iqB_\mu)V|^2 - \frac{1}{4} \frac{\eta'^2}{\xi^2} H_{\mu\nu}H^{\mu\nu} + \frac{1}{2}(\partial_\mu\xi)^2 - U(\rho, \xi) \right]. \quad (32)$$

How can one extend this analysis to 3+1 dimensions? Clearly the 2+1 dimensional results can describe vortices that are infinite along the added third spatial dimension. In a realistic set-up, however, the scattering is done with finite beam sizes and the vortices cannot be infinite. The vortices cannot end and hence would have to form a closed loop. To deal with this situation, one would have to construct a formalism capable of describing the creation and annihilation of one-dimensional extended systems. In the context of vortices

in condensed matter systems or fluids, the system is finite and it may be possible to create vortices that span the whole system. Such vortices are effectively infinite.

The ideas described above rest on the duality transformation relating a scalar to a gauge field. While the transformation may be reliable at a classical level, it is not certain that the transformation will give equivalent theories even at the quantum level. On the other hand, however, these ideas may be testable in the laboratory if (1), or some refinement of it, provides an adequate description of fluids or condensed matter systems such as 4He . This remains to be seen. Another possibility that comes to mind is the production of electroweak vortices in the scattering of leptons⁹. All the ingredients for the production of gauged vortices seem to be present in the electroweak model: vortex solutions and particles interacting with the vortex via an Aharanov-Bohm effect are also present. There are some complications too: the vortices are not topological and the particles are fermions. Another requirement that is not met, is the strong coupling limit and methods will have to be found to be able to treat actions such as S_{gl+V} and S_{AH+V} . Efforts in this direction are in progress.

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