

Statistical properties of the spectrum of the QCD Dirac operator at low energy

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Abstract

We analyze the statistical properties of the spectrum of the QCD Dirac operator at low energy in a finite box of volume L^4 by means of partially quenched Chiral Perturbation Theory (pqChPT), a low-energy effective field theory based on the symmetries of QCD. We derive the two-point spectral correlation function from the discontinuity of the chiral susceptibility. For eigenvalues much smaller than $E_c = F^2/\Sigma L^2$, where F is the pion decay constant and Σ is the absolute value of the quark condensate, our result for the two-point correlation function coincides with the result previously obtained from chiral Random Matrix Theory (chRMT). The departure from the chRMT result above that scale is described by the contribution of the nonzero momentum modes. In terms of the variance of the number of eigenvalues in an interval containing n eigenvalues on average, it results in a crossover from a $\log n$ -behavior to a $n^2 \log n$ -behavior.

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1 Introduction

Since the seminal work by Wigner, Dyson and Mehta [1, 2, 3] on the Random Matrix Theory description of level correlations in nuclei, the problem of level statistics has been analyzed in great detail for many different quantum systems (see [4] for a recent review). In QCD, a statistical analysis can be applied to the eigenvalues of the Dirac operator $i\mathcal{D}$ defined by

$$i\mathcal{D}\phi_k = i\lambda_k\phi_k. \quad (1.1)$$

The Euclidean QCD partition function for N_f quarks of mass m_f is given by

$$Z^{\text{QCD}} = \int [dA]_\nu \prod_{f=1}^{N_f} \det(i\mathcal{D} + m_f) e^{-S_{YM}[A]}, \quad (1.2)$$

and can thus be expressed as

$$Z^{\text{QCD}} = \langle \prod_{f=1}^{N_f} \prod_k (i\lambda_k + m_f) \rangle_{\text{YM}}. \quad (1.3)$$

Here, $\langle \cdots \rangle_{\text{YM}}$ denotes an average over the Yang-Mills partition function. This shows that, although the eigenvalues cannot be observed directly, their properties are of fundamental importance to the physics of QCD.

For small enough energies, below the so-called Thouless energy [5, 6, 7, 8], the eigenvalues are strongly correlated, and their correlations are given by chiral Random Matrix Theory (chRMT) [9, 10]. For energy differences much larger than the Thouless energy but much smaller than Λ_{QCD} , the eigenvalues of the QCD Dirac operator show much weaker correlations that are different from chRMT. In this domain, the eigenvalue correlations can be computed perturbatively by means of partially quenched Chiral Perturbation Theory (pqChPT). This is a low-energy effective theory based only on the symmetries of QCD formulated to probe the spectrum of the QCD Dirac operator. Finally, for energies beyond Λ_{QCD} the eigenvalues are uncorrelated. The Random Matrix behavior can also be understood from universality arguments [11, 12, 13, 14, 15, 16]. These different domains have been identified in lattice QCD simulations [17, 18, 19], and can also be understood in terms of random hopping models with the chiral symmetry of the QCD partition function [20, 21, 22, 23, 24].

In previous publications [7, 8, 25], we have studied the average spectral density $\rho(\lambda)$ of the QCD Dirac operator by means of pqChPT. We have shown that the chRMT and pqChPT results agree below an energy scale E_c given by $E_c = F^2/\Sigma L^2$, where F is the pion decay constant and Σ is the absolute value of the quark condensate. In the theory of disordered mesoscopic systems, this energy scale is known as the Thouless energy (see for example [26]). There are two other important energy scales in the spectrum of the QCD Dirac operator: the smallest nonzero eigenvalue λ_{min} and Λ_{QCD} . The smallest nonzero eigenvalue is directly related to the spectral density near zero virtuality and is therefore determined by the Banks-Casher formula [27]: $\lambda_{\text{min}} = 1/\rho(0) = \pi/\Sigma V$. With the terminology adopted from the study of disordered mesoscopic systems [26] we can thus distinguish four domains in the spectrum. The quantum domain and the ergodic domain are separated by λ_{min} , the ergodic domain and the diffusive domain are separated by the Thouless energy E_c , and the diffusive domain and the ballistic domain are separated by Λ_{QCD} .

In this article, we turn our attention to correlations of Dirac eigenvalues, i.e. the fluctuations about the average behavior of the spectrum. For this purpose we consider multi-level correlation

functions. We focus our study on the average connected two-point spectral correlation function, $\rho_c(\lambda_1, \lambda_2)$, defined by

$$\begin{aligned}
\rho_c(\lambda_1, \lambda_2) &= \langle \sum_{i,j} \delta(\lambda_1 - \lambda_i) \delta(\lambda_2 - \lambda_j) \rangle_{\text{QCD}} \\
&\quad - \langle \sum_k \delta(\lambda_1 - \lambda_k) \rangle_{\text{QCD}} \langle \sum_k \delta(\lambda_2 - \lambda_k) \rangle_{\text{QCD}} \\
&\equiv \delta(\lambda_1 - \lambda_2) \langle \sum_k \delta(\lambda_2 - \lambda_k) \rangle_{\text{QCD}} + R(\lambda_1, \lambda_2).
\end{aligned} \tag{1.4}$$

where $\langle \cdots \rangle_{\text{QCD}}$ means the average over the QCD partition function, and λ_k are the eigenvalues of the Dirac operator. In the last line of this equation we have decomposed the correlation function into a term containing the self-correlations and the two-point cluster function, $R(\lambda_1, \lambda_2)$. Both terms enter in the disconnected scalar susceptibility which is a more natural object in a field theory context. It is defined by

$$\begin{aligned}
\chi &= \frac{1}{V} \partial_{m_1} \partial_{m_2} \ln Z_{\text{QCD}} \\
&= \frac{1}{V} \langle \sum_k (i\lambda_k + m_1)^{-1} \sum_j (i\lambda_j + m_2)^{-1} \rangle_{\text{QCD}}
\end{aligned} \tag{1.5}$$

$$\begin{aligned}
&\quad - \frac{1}{V} \langle \sum_k (i\lambda_k + m_1)^{-1} \rangle_{\text{QCD}} \langle \sum_j (i\lambda_j + m_2)^{-1} \rangle_{\text{QCD}} \\
&= \frac{1}{V} \int_{-\infty}^{\infty} d\lambda_1 \int_{-\infty}^{\infty} d\lambda_2 \frac{\rho_c(\lambda_1, \lambda_2)}{(i\lambda_1 + m_1)(i\lambda_2 + m_2)}.
\end{aligned} \tag{1.6}$$

Because of the averaging over the QCD partition function, $\rho_c(\lambda_1, \lambda_2)$ depends on the quark masses. Therefore it is not possible to invert the relation (1.6) to derive the two-point correlation function. In order to invert this relation and to compute the correlation function one has to introduce special scalar sources that are unequivocally related to the eigenvalues of the QCD Dirac operator in the QCD partition function. This partition function contains extra degrees of freedom: one fermionic and one bosonic ghost-quark for each special scalar source related to an eigenvalue of the QCD Dirac operator. The pqQCD partition function with two special sources required for the calculation of the spectral two-point function is given by

$$\begin{aligned}
Z^{\text{pqQCD}} &= \int [dA]_{\nu} \frac{\det(i\not{D} + z_1 + j_1/2) \det(i\not{D} + z_2 + j_2/2)}{\det(i\not{D} + z_1 - j_1/2) \det(i\not{D} + z_2 - j_2/2)} \prod_{f=1}^{N_f} \det(i\not{D} + m_f) e^{-S_{\text{YM}}[A]} \\
&= \left\langle \prod_k \left(\prod_{f=1}^{N_f} (i\lambda_k + m_f) \right) \frac{i\lambda_k + z_1 + j_1/2}{i\lambda_k + z_1 - j_1/2} \frac{i\lambda_k + z_2 + j_2/2}{i\lambda_k + z_2 - j_2/2} \right\rangle_{\text{YM}}.
\end{aligned} \tag{1.7}$$

Notice that for $j_1 = j_2 = 0$ the pqQCD partition function reduces to the QCD partition function. Therefore, the derivatives of the pqQCD partition function with respect to the source terms at $j_1 = j_2 = 0$ are given by averages over the QCD partition function. This enlarged partition function was first introduced to study the quenched limit in QCD and is therefore known as partially quenched QCD (pqQCD) [28]. It has already been used to compute the spectral density of the QCD Dirac operator in [7, 8]. In that case, only one such special scalar source had to be introduced.

We will be interested in the disconnected scalar susceptibility defined by

$$\chi(z_1, z_2) = \frac{1}{V} \partial_{j_1} \partial_{j_2} \ln Z_{\text{pqQCD}} \Big|_{j_1=j_2=0} \quad (1.8)$$

$$\begin{aligned} &= \frac{1}{V} \left\langle \sum_k \frac{1}{i\lambda_k + z_1} \sum_j \frac{1}{i\lambda_j + z_2} \right\rangle_{\text{QCD}} \\ &\quad - \frac{1}{V} \left\langle \sum_k \frac{1}{i\lambda_k + z_1} \right\rangle_{\text{QCD}} \left\langle \sum_j \frac{1}{i\lambda_j + z_2} \right\rangle_{\text{QCD}}. \end{aligned} \quad (1.9)$$

It is straightforwardly related to the spectral two-point correlation function of the QCD Dirac operator (1.4) in the following way

$$\chi(z_1, z_2) = \frac{1}{V} \int_{-\infty}^{\infty} d\lambda_1 \int_{-\infty}^{\infty} d\lambda_2 \frac{\rho_c(\lambda_1, \lambda_2)}{(i\lambda_1 + z_1)(i\lambda_2 + z_2)}. \quad (1.10)$$

Because the average over the QCD partition function does not depend on z_1 and z_2 , this integral equation can be inverted. Using that $\chi(z_1, z_2)$ is odd in both z_1 and z_2 , we find that

$$\begin{aligned} \frac{1}{V} \rho_c(\lambda_1, \lambda_2) &= \frac{1}{4\pi^2} \text{Disc} \Big|_{z_1=i\lambda_1, z_2=i\lambda_2} \chi(z_1, z_2) \\ &= \frac{1}{4\pi^2} \lim_{\epsilon \rightarrow 0^+} \left(\chi(i\lambda_1 + \epsilon, i\lambda_2 + \epsilon) + \chi(-i\lambda_1 + \epsilon, i\lambda_2 + \epsilon) \right. \\ &\quad \left. + \chi(i\lambda_1 + \epsilon, -i\lambda_2 + \epsilon) + \chi(-i\lambda_1 + \epsilon, -i\lambda_2 + \epsilon) \right). \end{aligned} \quad (1.11)$$

At low energies, as already discussed in [7, 8], the properties of the spectrum of the QCD Dirac operator can be obtained from the low-energy limit of (1.7). As is the case for the usual chiral Lagrangian, this effective theory is completely determined by the symmetries of the pqQCD partition function. This topic will be discussed in the next section. An alternative way to derive the perturbative results for the scalar susceptibility is the use of the replica method [19, 29]. However, nonperturbative results cannot be obtained this way [30, 31, 32, 33, 34, 35, 36, 37].

Most of our results are based on partially quenched Chiral Perturbation Theory which will be introduced in section 2. The zero momentum sector of this theory will be analyzed in section 3. It will be shown that in this domain the two-point spectral correlation function for arbitrary topological charge is given by chiral Random Matrix Theory. This nonperturbative result generalizes a calculation in [38] to arbitrary topological charge without starting from chiral Random Matrix Theory, but rather from an effective chiral Lagrangian which is obtained from the symmetries of the theory. The contribution of the nonzero momentum modes to the scalar susceptibility and its discontinuity is calculated in section 4. In section 5 we evaluate the number variance for the different domains in the Dirac spectrum. Concluding remarks are made in section 6.

2 Partially Quenched Chiral Perturbation Theory

It is well-known that the low-energy limit of QCD is given by a theory of weakly interacting Goldstone bosons. The reason is two-fold: the spontaneous breaking of chiral symmetry and the existence of a mass gap for the non-Goldstone excitations. In QCD, chiral symmetry is maximally broken consistent with the Vafa-Witten theorem, implying that $SU_L(N_f) \times SU_R(N_f)$ is broken spontaneously to the diagonal subgroup $SU_V(N_f)$. The corresponding low-energy

effective theory has been investigated in great detail by means of Chiral Perturbation Theory. It describes successfully the strong interaction phenomenology at low energies [39, 40, 41].

In this section we construct the low-energy limit of the pqQCD. This low-energy effective theory, known as partially quenched Chiral Perturbation Theory (pqChPT), is again solely based on the symmetries of pqQCD and is given by a theory of Goldstone modes associated with the spontaneous symmetry breaking of chiral symmetry. Because the unitary symmetry is not a good symmetry for the bosonic ghost quarks, i.e., unitary transformations violate the complex conjugation of the fields necessary to obtain a convergent integral, we start from the complexified flavor symmetry group given by $Gl(N_f + 2|2) \times Gl(N_f + 2|2)$. The flavor symmetry group of the bosonic ghost quarks is then given by $Gl(2)/U(2) \times Gl(2)/U(2)$ which results in convergent bosonic integrals. The natural choice for the flavor symmetry group of the fermionic quarks is $U(N_f + 2) \times U(N_f + 2)$. As is the case for QCD we will assume that the axial symmetry of the pqQCD is maximally broken by the vacuum expectation value of the chiral condensate. We will also assume that supersymmetry is not spontaneously broken. Because of spontaneous breaking of chiral symmetry to the diagonal subgroup, the complete Goldstone manifold, which also includes fermionic Goldstone modes, is then given by the maximum super-Riemannian submanifold of $Gl(N_f + 2|2)$ [42]. It will be denoted by $\widehat{Gl}(N_f + 2|2)$. It consists of a fermion-fermion block given by $U(N_f + 2)$ and a boson-boson block given by $Gl(2)/U(2)$. The matrix elements of the boson-fermion and the fermion-boson blocks of this manifold are given by independent Grassmann variables. The unbroken chiral symmetry group of the pqQCD partition function is thus given by $\widehat{Gl}_L(N_f + 2|2) \times \widehat{Gl}_R(N_f + 2|2)$.

At low energies the relevant excitations are the Goldstone fields parameterized by

$$U = \exp(i\sqrt{2}\Pi^a T_a/F), \quad (2.12)$$

with T_a the generators of the Goldstone manifold $\widehat{Gl}(N_f + 2|2)$. Under a $\widehat{Gl}_L(N_f + 2|2) \times \widehat{Gl}_R(N_f + 2|2)$ transformation of the quark fields, the Goldstone fields transform as

$$U \rightarrow U_L U U_R^{-1}. \quad (2.13)$$

The low energy effective partition function is obtained from the requirement that its transformation properties are the same as for the pqQCD partition function. Under flavor transformations the pqQCD partition function transforms as

$$Z^{\text{pqQCD}}(U_L \mathcal{M} U_R^{-1}, \theta - i \log[\text{Sdet} U_L U_R^{-1}]) = Z^{\text{pqQCD}}(\mathcal{M}, \theta), \quad (2.14)$$

where θ is vacuum angle, and the quark mass matrix is denoted by \mathcal{M} . To lowest order in the momenta and quark masses, the effective partition function is thus given by

$$Z^{\text{eff}}(\mathcal{M}, \theta) = \int_{U \in \widehat{Gl}(N_f + 2|2)} dU e^{-\int d^4x \mathcal{L}(x)} \quad (2.15)$$

with the effective chiral Lagrangian given by [28, 7]

$$\mathcal{L} = \frac{F^2}{4} \text{Str}(\partial_\mu U \partial_\mu U^{-1}) - \frac{\Sigma}{2} \text{Str}(\mathcal{M}^\dagger U + \mathcal{M} U^{-1}) + \frac{\Sigma \bar{m}}{2} \left(\frac{\Phi}{F} - \theta \right)^2. \quad (2.16)$$

Here, the quark mass matrix is given by

$$\mathcal{M} = \text{diag}(z_1 + j_1/2, z_2 + j_2/2, \underbrace{m, \dots, m}_{N_f}, z_1 - j_1/2, z_2 - j_2/2). \quad (2.17)$$

The axial anomaly is included through a mass term for the super- η' field $\Phi = -iF\text{Str}\ln U$ while integrating over the full axial symmetry group $\widehat{Gl}(N_f + 2|2)$. This term serves as a constraint that projects out the flavor singlet channel. The first two terms in (2.16) also appear in ChPT to lowest order. However, in the case of pqChPT, there are both fermionic and bosonic Goldstone modes. Their masses are given by $\sqrt{2\Sigma m}/F$, $\sqrt{2\Sigma z_1}/F$, $\sqrt{2\Sigma z_2}/F$, $\sqrt{\Sigma(m+z_1)}/F$, $\sqrt{\Sigma(m+z_2)}/F$, and $\sqrt{\Sigma(z_1+z_2)}/F$, depending on the quark content.

The partially quenched effective partition function was first formulated for the supergroup $U(N_f + k|k)$ to study the quenched approximation in QCD [28]. This formulation is suitable for perturbative calculations. However, an effective partition function based on this group cannot be used for nonperturbative calculations of the group integral. For example, the supertrace leads to the appearance of both positive and negative masses. The correct integration manifold is the super-Riemannian manifold $\widehat{Gl}(N_f + 2|2)$ [42, 7, 8].

As already mentioned in the introduction, it is possible to distinguish different domains in the spectrum of the Dirac operator. First, the effective partition function is only valid for $z_1, z_2 \ll \Lambda_{\text{QCD}}$. An important scale is given by the Thouless energy defined as the quark mass scale for which the Compton wavelength of the lightest corresponding Goldstone mode is equal to the size of the box, i.e.

$$\frac{m_c \Sigma}{F^2} = \frac{1}{L^2}. \quad (2.18)$$

For $|z_i| \ll m_c$ the z_i -dependence of the condensate is determined by the fluctuations of the zero momentum modes. In this domain, the partition function factorizes into a zero momentum sector and a nonzero momentum sector [7, 43, 44]. In [7, 8] it was shown that in this domain the pqQCD partition function for the one-point function reduces to the chRMT partition function. In this article, we will show that the same is true for the spectral two-point correlation function. Inside this domain, we can distinguish a second scale: the smallest nonzero eigenvalue

$$\lambda_{\min} = \frac{\pi}{\Sigma V}. \quad (2.19)$$

For $|z_i| \gg \lambda_{\min}$ the group integrals can be evaluated perturbatively, whereas for smaller values of the z_i the group integrals have to be calculated exactly. In the domain $z_i \geq F^2/\Sigma L^2$, the nonzero momentum modes become important and have to be taken into account. These three domains for the spectral two-point function will be analyzed in detail in the remainder of this article.

3 Quantum and Ergodic domains

In these domains, corresponding to $|z_i| \ll m_c$, the zero momentum mode sector and the nonzero momentum sectors of the pqChPT partition function factorize [7]. The z_i -dependence of the partition function and therefore the spectral two-point function can be obtained from its zero momentum part only. It is given by the super-unitary matrix integral

$$Z_{\text{eff}}(\mathcal{M}, \theta) = \int_{U \in \widehat{Gl}(N_f + 2|2)} dU e^{\frac{\Sigma V}{2} \text{Str}(\mathcal{M}^\dagger U + \mathcal{M} U^{-1}) - \frac{\Sigma \bar{m} V}{2} (\frac{\Phi}{F} - \theta)^2}. \quad (3.20)$$

We decompose the partition function according to

$$Z_{\text{eff}}(\mathcal{M}, \theta) = \int d\nu e^{i\nu\theta} Z_\nu(\mathcal{M}), \quad (3.21)$$

with the partition function in the sector of topological charge ν given by

$$Z_\nu(\mathcal{M}) = C_1 e^{-\frac{\nu^2}{2m\Sigma V}} \int_{U \in \widehat{Gl}(N_f+2|2)} dU \text{Sdet}^\nu U e^{\frac{\Sigma V}{2} \text{Str}(\mathcal{M}U + \mathcal{M}^\dagger U^{-1})}. \quad (3.22)$$

Exact integrals over the Goldstone manifold $\widehat{Gl}(N_f+2|2)$ are mathematically quite complicated. In the following we will restrict ourselves to the simplest case, and calculate the scalar susceptibility in the quantum and ergodic domains only in the quenched limit. A calculation of the two-point correlation function based on the supersymmetric formulation of chiral Random Matrix Theory [45, 46] was first given in [38] for the sector of zero topological charge. In this article we start directly from the partially quenched chiral Lagrangian and extend the calculation to all values of the topological charge.

3.1 Scalar Susceptibility in the Quenched Limit

In the quenched limit and in a sector of topological charge ν , the zero-mode part of the effective partition function can be written as:

$$Z_{\text{eff}}^\nu(z_1, z_2, j_1, j_2) = \int_{\widehat{Gl}(2|2)} DU \text{Sdet} U^\nu \exp\left\{\frac{\Sigma V}{2} \text{Str} \mathcal{M}(U + U^{-1})\right\}, \quad (3.23)$$

where the mass matrix is given by

$$\Sigma V \mathcal{M} = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \quad (3.24)$$

with $M_i = \Sigma V \text{diag}(z_i + j_i/2, z_i - j_i/2)$. The Goldstone manifold $\widehat{Gl}(2|2)$ is the maximum Riemannian submanifold of $Gl(2|2)$ with fermion-fermion block given by $U(2)$ and boson-boson block by $Gl(2)/U(2)$ [42].

3.1.1 Parameterization of the Goldstone manifold

We parameterize the Goldstone manifold in terms of Goldstone modes related with the one-point functions corresponding to z_1 and z_2 , and Goldstone modes that describe two-point correlations. A convenient parametrization is given by [38],

$$U = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix} \begin{pmatrix} \sqrt{1-w\bar{w}} & w \\ -\bar{w} & \sqrt{1-\bar{w}w} \end{pmatrix} \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}, \quad (3.25)$$

where $w_{1,2} \in \widehat{Gl}(1|1)$ and $w, \bar{w} \in Gl(1|1)$. An explicit parametrization of these supermatrices will be given below. The advantage of the parameterization (3.25) becomes clear upon consideration of the supertrace that appears in the exponent in the effective partition function (3.23):

$$\frac{\Sigma V}{2} \text{Str} \mathcal{M}(U + U^{-1}) = \mathcal{S}_1 + \mathcal{S}_2, \quad (3.26)$$

with

$$\mathcal{S}_1 = \frac{1}{2} \text{Str} \left[M_1 \left(w_1 \sqrt{1-w\bar{w}} w_1 + w_1^{-1} \sqrt{1-\bar{w}w} w_1^{-1} \right) \right], \quad (3.27)$$

and

$$\mathcal{S}_2 = \frac{1}{2} \text{Str} \left[M_2 \left(w_2 \sqrt{1-\bar{w}w} w_2 + w_2^{-1} \sqrt{1-w\bar{w}} w_2^{-1} \right) \right]. \quad (3.28)$$

Furthermore, the source terms j_1 and j_2 only occur in combination with w_1 and w_2 , respectively. As we will see in the next subsection, the invariant measure of $\widehat{Gl}(2|2)$ in the parameterization (3.25) factorizes according to

$$\mu(w, \bar{w}, w_1, w_2) = \mu(w, \bar{w})\mu(w_1)\mu(w_2). \quad (3.29)$$

We thus find that the integrals over w_1 and w_2 in the computation of the scalar susceptibility factorize

$$\chi(z_1, z_2) = \frac{1}{V} \frac{\partial^2}{\partial j_1 \partial j_2} \log Z \Big|_{j_1=j_2=0} = \frac{1}{V} \int dw d\bar{w} \mu(w, \bar{w}) \mathcal{I}_1(w, \bar{w}) \mathcal{I}_2(w, \bar{w}), \quad (3.30)$$

where the integrals $\mathcal{I}_1(w, \bar{w})$ and $\mathcal{I}_2(w, \bar{w})$ are given by

$$\mathcal{I}_i = \int dw_i \mu(w_i) \partial_{j_i} \mathcal{S}_i e^{\mathcal{S}_i} \Big|_{j_i=0}, \quad i = 1, 2. \quad (3.31)$$

A further simplification arises by using a polar decomposition of the 2×2 supermatrices that appear in the parameterization (3.25),

$$w_i = v_i \Lambda_i v_i^{-1}, \quad i = 1, 2, \quad (3.32)$$

$$w = v S u^{-1}, \quad \bar{w} = u \bar{S} v^{-1}, \quad (3.33)$$

where $\Lambda_{1,2}$ and S are 2×2 diagonal supermatrices with commuting elements given by

$$\Lambda_i = \text{diag}(e^{i\psi_i/2}, e^{s_i/2}), \quad i = 1, 2, \quad (3.34)$$

$$S = \text{diag}(\sin \theta e^{i\rho}, i \sinh \phi e^{i\sigma}), \quad (3.35)$$

$$\bar{S} = \text{diag}(\sin \theta e^{-i\rho}, i \sinh \phi e^{-i\sigma}), \quad (3.36)$$

$$C = \sqrt{1 - S\bar{S}} = \text{diag}(\cos \theta, \cosh \phi), \quad (3.37)$$

and u, v, v_1 , and v_2 , all elements of $U(1|1)/U(1) \times U(1)$, can be conveniently parameterized according to

$$\begin{aligned} u &= \exp \begin{pmatrix} 0 & \zeta \\ \chi & 0 \end{pmatrix}, & v &= \exp \begin{pmatrix} 0 & \xi \\ \eta & 0 \end{pmatrix}, \\ v_i &= \exp \begin{pmatrix} 0 & \xi_i \\ \eta_i & 0 \end{pmatrix}, & i &= 1, 2. \end{aligned} \quad (3.38)$$

After these ‘‘Grassmann diagonalizations’’, the supertraces in (3.27, 3.28) are given by

$$\mathcal{S}_1 = \frac{1}{2} \text{Str} \left[v_1^{-1} v C v^{-1} v_1 \left(\Lambda_1 v_1^{-1} M_1 v_1 \Lambda_1 + \Lambda_1^{-1} v_1^{-1} M_1 v_1 \Lambda_1^{-1} \right) \right], \quad (3.39)$$

and

$$\mathcal{S}_2 = \frac{1}{2} \text{Str} \left[v_2^{-1} u C u^{-1} v_2 \left(\Lambda_2 v_2^{-1} M_2 v_2 \Lambda_2 + \Lambda_2^{-1} v_2^{-1} M_2 v_2 \Lambda_2^{-1} \right) \right]. \quad (3.40)$$

3.1.2 Measure

The parameterization of the Goldstone manifold is of the form

$$U = WTW = W^2W^{-1}TW \equiv W^2T'. \quad (3.41)$$

The Berezinian of the transformation from the variables T' to T is one. To calculate the invariant measure we thus consider

$$T'[U^{-1}dU]T'^{-1} = W^{-2}dW^2 + [dT']T'^{-1}. \quad (3.42)$$

The measure thus factorizes into a product of one factor depending only on W and another factor depending only on T' (and on T after the transformation from T' to T). The W -dependent part of the measure trivially factorizes into a w_1 -dependent piece and a w_2 -dependent piece. We thus find

$$\begin{aligned} d\mu(U) &= w_1^{-2}dw_1^2 w_2^{-2}dw_2^2 T^{-1}dT \\ &\equiv \mu(w_1)dw_1 \mu(w_2)dw_2 \mu(w, \bar{w})dw d\bar{w}. \end{aligned} \quad (3.43)$$

The first two integration measures simply follow from the invariant measure of $Gl(1|1)/U(1|1)$ whereas the integration measure of the T -integrations is given by the invariant measure of $Gl(1|1)$. Both measures will be calculated in the next part of this section.

The matrices w_i in coset $Gl(1|1)/U(1|1)$ have four independent parameters and can be parameterized as in (3.25, 3.38). We first calculate the measure $w_i^{-1}dw_i$. To obtain the measure $w_i^{-2}dw_i^2$ we only have to replace the diagonal elements by their square. The invariant measure is given by the Berezinian of the transformation from variables

$$\delta w'_i \equiv v_i^{-1}[w_i^{-1}dw_i]v_i \quad (3.44)$$

to variables $d\psi_i, ds_i, d\xi_i$ and $d\eta_i$. One easily derives that

$$\delta w'_i = \Lambda_i^{-1}\delta w''_i = \Lambda_i^{-1}[\delta v_i\Lambda_i + d\Lambda_i - \Lambda_i\delta v_i], \quad (3.45)$$

where $\delta v_i = v_i^{-1}dv_i$. The Berezinian from the transformation of the variables $\delta w''_i$ to the $d\Lambda_i$ and the offdiagonal elements of δv_i is given by

$$\begin{aligned} &\text{Sdet} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{s_i/2} - e^{i\psi_i/2} & 0 \\ 0 & 0 & 0 & e^{i\psi_i/2} - e^{s_i/2} \end{pmatrix} \\ &= \frac{1}{(e^{s_i/2} - e^{i\psi_i/2})(e^{i\psi_i/2} - e^{s_i/2})}. \end{aligned} \quad (3.46)$$

The Berezinian of the transformation from $\delta w''$ to $\delta w'$ is one (factors from the bosonic integrations cancel against factors from the fermionic integrations), and $(\delta v)_{12}(\delta v)_{21} = d\xi d\eta$. For the Berezinian, denoted by B , we thus find

$$\begin{aligned} w_i^{-1}dw_i \equiv B d\psi_i ds_i d\xi_i d\eta_i &= \frac{de^{s_i/2} de^{i\psi_i/2} d\xi_i d\eta_i}{(e^{s_i/2} - e^{i\psi_i/2})(e^{i\psi_i/2} - e^{s_i/2})} \\ &= \frac{id s_i d\psi_i d\xi_i d\eta_i}{4(e^{s_i/2} - e^{i\psi_i/2})(e^{-s_i/2} - e^{-i\psi_i/2})}. \end{aligned} \quad (3.47)$$

The integration measure for the w_i variables is then simply obtained by squaring the diagonal elements of Λ_i ,

$$\mu(w_i)dw_i = \frac{ids_i d\psi_i d\xi_i d\eta_i}{(e^{s_i} - e^{i\psi_i})(e^{-s_i} - e^{-i\psi_i})}. \quad (3.48)$$

Next we calculate the invariant measure of $Gl(1|1)$. This group has eight independent parameters and can be parameterized as in (3.25,3.38). With the observation that [30, 32] $T^{-1}dT = dw d\bar{w}$. we calculate the integration measure starting from

$$\begin{aligned} dw' &\equiv v^{-1}dwu = \delta v S + dS - S\delta u, \\ d\bar{w}' &\equiv u^{-1}d\bar{w}u = \delta u \bar{S} + d\bar{S} - \bar{S}\delta v, \end{aligned} \quad (3.49)$$

with $\delta u = u^{-1}du$ and $\delta v = v^{-1}dv$. The Berezinian of the transformation from the variables on the left hand side to the variables on the right hand side is given by

$$\frac{1}{(\sin^2 \theta + \sinh^2 \phi)^2} = \frac{1}{(\cos^2 \theta - \cosh^2 \phi)^2}. \quad (3.50)$$

The Jacobian for the transformation of the variables $\{S_{11}, S_{22}, S_{11}^*, S_{22}^*\}$ to the variables $\{\theta, \phi, \rho, \sigma\}$ is simply given by

$$i \sinh 2\phi \sin 2\theta. \quad (3.51)$$

For the measure of $Gl(1|1)$ we thus find

$$T^{-1}dT = \mu(w, \bar{w})dw d\bar{w} = \frac{i \sinh 2\phi \sin 2\theta}{(\cos^2 \theta - \cosh^2 \phi)^2} d\theta d\phi d\rho d\sigma d\zeta d\chi d\xi d\eta. \quad (3.52)$$

3.1.3 Integration over the Goldstone manifold

The first supertrace \mathcal{S}_1 in (3.39) appearing in the integral (3.30) can be written as

$$\begin{aligned} \frac{\mathcal{S}_1}{\Sigma V} &= (z_1 + \frac{1}{2}j_1) \cos \theta \cos \psi_1 - (z_1 - \frac{1}{2}j_1) \cosh \phi \cosh s_1 \\ &+ j_1 (\cos \theta \cos \psi_1 - \cosh \phi \cosh s_1) \xi_1 \eta_1 \\ &+ (\cos \theta - \cosh \phi) \left((z_1 + \frac{1}{2}j_1) \cos \psi_1 - (z_1 - \frac{1}{2}j_1) \cosh s_1 \right) (\xi - \xi_1)(\eta - \eta_1) \\ &+ j_1 \cos \psi_1 \cosh s_1 (\cos \theta - \cosh \phi) \left((\xi - \xi_1)\eta_1 + \xi_1(\eta - \eta_1) \right) \\ &+ j_1 (\cos \theta - \cosh \phi) (\cos \psi_1 - \cosh s_1) \xi \eta \xi_1 \eta_1. \end{aligned} \quad (3.53)$$

The Grassmann integrals in (3.30) are calculated by collecting the coefficient of $\xi \eta \xi_1 \eta_1$ in the expansion of $e^{\mathcal{S}_1}$. Only the terms linear in j_1 contribute to susceptibility. Using the nilpotency of the Grassmann variables, such as for example $(\xi - \xi_1)^2 = 0$, one easily shows that

$$\begin{aligned} \frac{1}{\Sigma V} \partial_{j_1} e^{\mathcal{S}_1} |_{j_1=0} &= \xi \eta \xi_1 \eta_1 e^{\Sigma V z_1 (\cos \theta \cos \psi - \cosh \phi \cosh s)} (\cos \theta - \cosh \phi) (\cos \psi_1 - \cosh s_1) \\ &\times (1 + \Sigma V z_1 \cos \theta \cosh \psi_1 - \cosh \phi \cosh s_1). \end{aligned} \quad (3.54)$$

An analogous result is obtained from the expansion of $e^{\mathcal{S}_2}$. The integration over the Grassmann variables in the disconnected scalar susceptibility (3.30) is now trivial. Taking into account the measures (3.52, 3.48) we arrive at

$$\chi(z_1, z_2) = 4\Sigma^2 V \int_0^1 dt \int_1^\infty dp \frac{tp}{(t^2 - p^2)^2} \mathcal{F}(\Sigma V z_1) \mathcal{F}(\Sigma V z_2), \quad (3.55)$$

where

$$\begin{aligned}\mathcal{F}(x) &= (t-p) \int_0^{2\pi} d\psi \int_0^\infty ds \frac{1}{(e^{i\psi} - e^s)(e^{-i\psi} - e^{-s})} e^{x(t \cos \psi - p \cosh s)} e^{\nu(i\psi - s)} \\ &\times (\cos \psi - \cosh s) \left(1 + x(t \cos \psi - p \cosh s)\right).\end{aligned}\quad (3.56)$$

To avoid Efetov-Wegner terms [47, 48] and problems related to the singularity in the integrand of $\mathcal{F}(x)$, we compute

$$\begin{aligned}\frac{1}{(t-p)} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial p} \right) \mathcal{F}(x) &= -\frac{x}{4} \int_0^{2\pi} d\psi \int_0^\infty ds e^{x(t \cos \psi - p \cosh s)} e^{\nu(i\psi - s)} \\ &\times (1 - e^{s+i\psi})(1 - e^{s-i\psi}) \left(2 + x(t \cos \psi - p \cosh s)\right) \\ &= \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial p} \right) \left((t+p) I_\nu(xt) K_\nu(xp) \right).\end{aligned}\quad (3.57)$$

In the first equality we have used the identity

$$\frac{\cos \psi - \cosh s}{e^{i\psi} - e^s} = \frac{1}{2} (1 - e^{-s-i\psi}), \quad (3.58)$$

and the second equality follows from relations for modified Bessel functions. We thus find that

$$\mathcal{F}(x) = (t^2 - p^2) I_\nu(xt) K_\nu(xp). \quad (3.59)$$

Inserting this expression into (3.55), we find that the disconnected scalar susceptibility is given by

$$\begin{aligned}\chi(z_1, z_2) &= 4\Sigma^4 V^3 z_1 z_2 \int_0^1 dt t I_\nu(\Sigma V z_1 t) I_\nu(\Sigma V z_2 t) \int_1^\infty dp p K_\nu(\Sigma V z_1 p) K_\nu(\Sigma V z_2 p) \\ &= \frac{4\Sigma^2 V z_1 z_2}{(z_1^2 - z_2^2)^2} \left(z_1 I_{\nu+1}(\Sigma V z_1) I_\nu(\Sigma V z_2) - z_2 I_{\nu+1}(\Sigma V z_2) I_\nu(\Sigma V z_1) \right) \\ &\times \left(z_1 K_{\nu+1}(\Sigma V z_1) K_\nu(\Sigma V z_2) - z_2 K_{\nu+1}(\Sigma V z_2) K_\nu(\Sigma V z_1) \right).\end{aligned}\quad (3.60)$$

In the limit $z_1 = z_2$, this expression coincides with the result obtained in [17].

3.2 Two-point correlation function

Finally, the two-point spectral correlation function is given by the discontinuity of the disconnected scalar susceptibility across the imaginary axis (1.11). This follows by using relations for Bessel functions such as

$$\begin{aligned}K_\nu(iz) &= \frac{\pi i}{2} e^{\pi i \nu / 2} (J_\nu(-z) + i N_\nu(-z)), \\ K_\nu(-iz) &= e^{-\pi i \nu} K_\nu(iz) - \pi i I_\nu(iz), \\ I_\nu(iz) &= e^{-\pi i \nu / 2} J_\nu(-z), \quad -\pi < \arg z \leq \frac{\pi}{2}, \\ -\frac{2}{\pi z} &= J_\nu(z) N_{\nu+1}(z) - J_{\nu+1}(z) N_\nu(z).\end{aligned}\quad (3.61)$$

In the quantum and ergodic domains, we then obtain the two-point correlation function

$$\begin{aligned}\frac{\rho_c(\lambda_1, \lambda_2)}{V^2 \Sigma^2} &= \delta(\lambda_1 - \lambda_2) \frac{\lambda_1}{2} [J_\nu^2(\Sigma V \lambda_1) - J_{\nu+1}(\Sigma V \lambda_1) J_{\nu-1}(\Sigma V \lambda_1)] \\ &- \frac{\lambda_1 \lambda_2}{(\lambda_1^2 - \lambda_2^2)^2} \left(\lambda_1 J_{\nu+1}(\Sigma V \lambda_1) J_\nu(\Sigma V \lambda_2) - \lambda_2 J_{\nu+1}(\Sigma V \lambda_2) J_\nu(\Sigma V \lambda_1) \right)^2.\end{aligned}\quad (3.62)$$

The first term represents the contribution due to the selfcorrelations of the eigenvalues. It comes from the non-trivial $\epsilon \rightarrow 0$ limit in the discontinuity of the susceptibility (1.11). This results coincides with the chRMT result [49]. In the ergodic domain where $\Sigma V \lambda_{1,2} \gg 1$ the two-point spectral correlation function reduces to

$$\rho_c(\lambda_1, \lambda_2) = \frac{1}{2\pi^2} \frac{1}{(\lambda_1 - \lambda_2)^2} + \dots, \quad (3.63)$$

and agrees with the asymptotic result for the two-point correlation function of the Wigner-Dyson ensembles.

4 Ergodic and Diffusive domains

In these domains, corresponding to $\lambda_{\min} \ll |z_i| \ll \Lambda_{\text{QCD}}$, the scalar susceptibility can be computed perturbatively within pqChPT with Lagrangian for $\theta = 0$ given by (2.16),

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4} \text{Str}(\partial_\mu U \partial_\mu U^{-1}) - \frac{\Sigma}{2} \text{Str}(\mathcal{M}(U + U^{-1})) + \frac{M_0^2}{2} \Phi^2 + \frac{\alpha}{2} \partial_\mu \Phi \partial_\mu \Phi. \quad (4.64)$$

The $(N_f + 4) \times (N_f + 4)$ matrix U is parameterized as

$$U = \exp(i\sqrt{2}\Pi^a T_a/F), \quad (4.65)$$

with T_a the generators of the Goldstone manifold $\widehat{Gl}(N_f + 2|2)$, and the singlet field is normalized as

$$\Phi = -iF \text{Str} \ln U. \quad (4.66)$$

Its mass is related to the topological susceptibility by [7]

$$M_0^2 = \frac{2\langle \nu^2 \rangle}{F^2 V} \equiv \frac{2\bar{m}\Sigma}{F^2}, \quad (4.67)$$

where \bar{m} has been introduced to simplify expressions below. For $\bar{m} = m/N_f$ this expression gives the topological susceptibility for N_f light quarks with mass m ,

$$\langle \nu^2 \rangle = \frac{mV\Sigma}{N_f}. \quad (4.68)$$

We can distinguish two types of Goldstone modes: those that correspond to the diagonal generators and those that correspond to the off-diagonal generators. To one-loop order, the off-diagonal Goldstone modes do not mix with the super-singlet field Φ . Therefore, their propagator is simply given by

$$G(p^2) = (p^2 + M^2)^{-1}, \quad (4.69)$$

where M is the mass of the corresponding Goldstone mode. The diagonal Goldstone modes, on the other hand, do mix with the super-singlet mode. It is still possible to diagonalize the quadratic form in the Goldstone fields (see [50, 28, 25]). This results in the following propagator for diagonal mesons in the sector of fermionic quarks ($1 < i, j < N_f + 2$),

$$G_{ij}(p^2) = \delta_{ij} \frac{1}{p^2 + M_{ii}^2} - \frac{(\alpha p^2 + M_0^2)(p^2 + M^2)}{(p^2 + M_{ii}^2)(p^2 + M_{jj}^2)((1 + N_f \alpha)p^2 + M^2 + N_f M_0^2)}, \quad (4.70)$$

where i denotes the flavor of the quarks of one of the diagonal mesons, and j the flavor of the quarks of the other one. The masses in the propagator are given by the Gell-Mann–Oakes–Renner relation, $M_{ii}^2 = 2\Sigma z_i/F^2$ for $i = 1, 2$, $M^2 \equiv M_{ii}^2 = 2\Sigma m/F^2$ for $i = 3, \dots, N_f + 2$.

We will consider two limits of this theory: $M \rightarrow \infty$ (i.e. $m \rightarrow \infty$), which is the quenched case, and $M_0 \rightarrow \infty$, which is QCD with N_f light flavors of quarks (and a completely decoupled singlet field). In the quenched case, the propagator matrix for the “diagonal mesons” is given by

$$G_{ij}(p^2) = \delta_{ij} \frac{1}{p^2 + M_{ii}^2} - \frac{(\alpha p^2 + M_0^2)}{(p^2 + M_{ii}^2)(p^2 + M_{jj}^2)}, \quad \text{for } 1 \leq i, j \leq 2, \quad (4.71)$$

whereas in the QCD limit we find

$$G_{ij}(p^2) = \delta_{ij} \frac{1}{p^2 + M_{ii}^2} - \frac{(p^2 + M^2)}{N_f(p^2 + M_{ii}^2)(p^2 + M_{jj}^2)}, \quad \text{for } 1 \leq i, j \leq N_f + 2. \quad (4.72)$$

Below we only need the propagators for $1 \leq i, j \leq 2$ so that both limits can be treated at the same time by introducing the propagator

$$G_{ij}(p^2) = \delta_{ij} \frac{1}{p^2 + M_{ii}^2} - \frac{(\eta p^2 + 2\Sigma m_\eta/F^2)}{(p^2 + M_{ii}^2)(p^2 + M_{jj}^2)}, \quad \text{for } 1 \leq i, j \leq 2. \quad (4.73)$$

In the QCD limit we have that $\eta = 1/N_f$ and $m_\eta = m/N_f$, whereas the quenched case is obtained by the substitution $\eta = \alpha$ and $m_\eta = \bar{m}$.

4.1 Disconnected scalar susceptibility

We compute the disconnected scalar susceptibility for spectral quarks with masses given by z_1 and z_2 . The mass of all N_f other quarks is taken to be equal to m . To one-loop order the contributions represented by the following diagrams have to be taken into account:

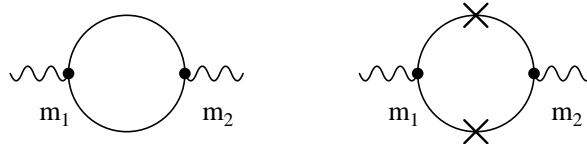


Figure 1: One-loop diagrams in pqChPT which contribute to $\chi(z_1, z_2)$. The lines represent either the “standard” propagator of an off-diagonal meson (4.69) or the propagator of a diagonal meson (4.73) (with a cross). The wiggly lines denote the two different scalar sources.

The propagators in the first diagram in Fig. 1 are given by $G(p^2) = (p^2 + M_{12}^2)^{-1}$ with $M_{12}^2 = \Sigma(z_1 + z_2)/F^2$. In the second diagram of Fig. 1, the lines with a cross denote the propagators that mix the diagonal mesons with the super- η' , $G_{12}(p)$ in (4.73). Such type of contributions were also essential in the calculation of the resolvent from the partially quenched chiral Lagrangian ([28, 7, 25]). The disconnected scalar susceptibility computed to one-loop order within pqChPT

is thus given by

$$\begin{aligned}\chi(z_1, z_2) &= \frac{\Sigma^2}{VF^4} \sum_p \left[G(p^2)^2 + 2G_{12}(p^2)^2 \right] \\ &= \frac{\Sigma^2}{VF^4} \sum_p \left[\frac{1}{(p^2 + M_{12}^2)^2} + 2 \frac{(\eta p^2 + 2\Sigma m_\eta/F^2)^2}{(p^2 + M_{11}^2)^2(p^2 + M_{22}^2)^2} \right].\end{aligned}\quad (4.74)$$

In the notation of [51] with

$$G_r(M^2) = \frac{\Gamma(r)}{V} \sum_p (p^2 + M^2)^{-r}, \quad (4.75)$$

the scalar susceptibility can be rewritten as

$$\begin{aligned}\chi(z_1, z_2) &= \frac{\Sigma^2}{F^4} \left[G_2(M_{12}^2) + \frac{2(\eta z_1 - m_\eta)^2}{(z_1 - z_2)^2} G_2(M_{11}^2) + \frac{2(\eta z_2 - m_\eta)^2}{(z_1 - z_2)^2} G_2(M_{22}^2) \right. \\ &\quad \left. + \frac{F^2}{\Sigma} \frac{2(\eta z_1 - m_\eta)(\eta z_2 - m_\eta)}{(z_1 - z_2)^3} (G_1(M_{11}^2) - G_1(M_{22}^2)) \right].\end{aligned}\quad (4.76)$$

The functions $G_1(M^2)$ and $G_2(M^2)$ for momenta in a finite box were analyzed in detail in [51]. They are obviously related to properties of the propagator of a free scalar particle at the origin (notice that $G_2(M^2) = -\partial_{M^2} G_1(M^2)$). For a box with volume L^4 and momentum cutoff Λ they are given by [40, 51],

$$\begin{aligned}G_1(M^2) &= \frac{1}{16\pi^2} M^2 \log \frac{M^2}{\Lambda^2} + g_1(M^2, L), \\ G_2(M^2) &= -\frac{1}{16\pi^2} (1 + \log \frac{M^2}{\Lambda^2}) + g_2(M^2, L),\end{aligned}\quad (4.77)$$

where

$$g_r(M^2, L) = \frac{1}{16\pi^2} \int_0^\infty d\lambda \lambda^{r-3} \sum_{\mathbf{n} \neq 0} e^{-\lambda M^2 - \mathbf{n}^2 L^2/4\lambda}, \quad (4.78)$$

and the sum is over a four dimensional lattice of integers. The functions $g_r(M^2)$ obviously vanish in the thermodynamic limit. For $1/\Lambda L^2 \ll M \ll 1/L$ they dominate the logarithmic terms in the propagators and can be expanded in powers of M resulting in

$$\begin{aligned}g_1(M^2, L) &= \frac{1}{M^2 L^4} + O(M^0/L^2), \\ g_2(M^2, L) &= \frac{1}{M^4 L^4} + O(\log(ML)).\end{aligned}\quad (4.79)$$

In the domain $1/\Lambda L^2 \ll M_{ii} \ll 1/L$ ($i = 1, 2$) the disconnected scalar susceptibility (4.74) is thus given by

$$\chi(z_1, z_2) = \frac{m_\eta^2}{2 z_1^2 z_2^2 V} + \frac{1}{(z_1 + z_2)^2 V} + \dots \quad (4.80)$$

Therefore in the quenched limit, using the relation (4.67), the result reads,

$$\chi(z_1, z_2) = \frac{\langle \nu^2 \rangle^2}{2 z_1^2 z_2^2 \Sigma^2 V^3} + \frac{1}{(z_1 + z_2)^2 V} + \dots, \quad (4.81)$$

and in the QCD limit, with $m_\eta = m/N_f$, one finds

$$\chi(z_1, z_2) = \frac{m^2}{2N_f^2 z_1^2 z_2^2 V} + \frac{1}{(z_1 + z_2)^2 V} + \dots \quad (4.82)$$

In the QCD limit, for spectral quarks with equal masses $z_1 = z_2 = m$, we indeed recover the ChPT result that can be easily derived from the results given in [52],

$$\chi = \frac{N_f^2 + 2}{4N_f^2} \frac{1}{m^2 V} + \dots \quad (4.83)$$

In the thermodynamic limit at fixed values of the quark masses, the finite volume corrections $g_{1,2}(M^2, L)$ in (4.77) can be ignored. This results in the susceptibility

$$\begin{aligned} \chi(z_1, z_2) = & -\frac{\Sigma^2}{16\pi^2 F^2} \left[\ln \frac{z_1 + z_2}{2\mu} \right. \\ & + \frac{2}{(z_1 - z_2)^3} \left\{ \left((\eta z_1 - m_\eta)[m_\eta(z_1 + z_2) + \eta z_1(z_1 - 3z_2)] \right) \ln \frac{z_1}{\mu} - z_1 \leftrightarrow z_2 \right. \\ & \left. \left. + (z_1 - z_2) \left(\eta^2(z_1^2 + z_2^2) - 2\eta m_\eta(z_1 + z_2) + 2m_\eta^2 \right) \right\} \right], \end{aligned} \quad (4.84)$$

where we have defined the scale $\mu = \Lambda^2 F^2 / 2\Sigma$ (compare to (4.77)). If the two spectral quarks have the same mass $z_1 = z_2 = z$, a somewhat simpler expression is obtained,

$$\chi(z) = \frac{\Sigma^2}{16\pi^2 F^4} \left[\frac{m_\eta^2}{3z^2} + \frac{4\eta m_\eta}{3z} - (1 + 2\eta^2) \ln \frac{z}{\mu} \right]. \quad (4.85)$$

In the limit $z \rightarrow 0$, the disconnected scalar susceptibility is singular (notice that this expression is only valid for $z \gg 1/V\Sigma$). This is due to the double pole occurring in the neutral meson propagator when $m_\eta \neq 0$. Such singularities are common in pqChPT. They also appear, for instance, in the small mass behavior of the scalar radius of the pion in the quenched approximation [53]. In the case $m_\eta = 0$, i.e. for $\langle \nu^2 \rangle = 0$ or in the chiral limit, the algebraic singularities in (4.84) cancel and only a logarithmic singularity remains.

The physically more interesting QCD limit of the disconnected scalar susceptibility, $\chi(z_1, z_2)$, is obtained from (4.84) by putting $m_\eta = m/N_f$ and $\eta = 1/N_f$,

$$\begin{aligned} \chi(z_1, z_2) = & -\frac{1}{16\pi^2} \left(\frac{\Sigma}{F^2} \right)^2 \left[\ln \frac{z_1 + z_2}{2\mu} + \frac{z_1^2 + z_2^2 + 2m^2 - 2(z_1 + z_2)m}{N_f^2(z_1 - z_2)^2} \right. \\ & \left. + \frac{2}{N_f^2(z_1 - z_2)^3} \left\{ \left(z_1^2(z_1 - 3z_2) + 4z_1 z_2 m - (z_1 + z_2)m^2 \right) \ln \frac{z_1}{\mu} - z_1 \leftrightarrow z_2 \right\} \right]. \end{aligned} \quad (4.86)$$

For $z_1 \rightarrow z_2 = z$ the apparent singularities cancel, and in this limit the expression (4.86) simplifies to

$$\chi(z) = \frac{\Sigma^2}{16\pi^2 N_f^2 F^4} \left(\frac{m^2}{3z^2} + \frac{4m}{3z} - (N_f^2 + 2) \ln \frac{z}{\mu} \right), \quad (4.87)$$

or, in the chiral limit,

$$\chi(z) = -\frac{\Sigma^2(N_f^2 + 2)}{16\pi^2 N_f^2 F^4} \ln \frac{z}{\mu}. \quad (4.88)$$

Finally, in the case $z_1 = z_2 = m$, we recover the result derived in [52] within ChPT.

4.2 Two-point correlation function

In this section, we calculate the two-point spectral correlation function from the discontinuities of the disconnected scalar susceptibility (see (1.11)) obtained perturbatively within pqChPT in the previous section. Therefore, our results for the two-point function are only valid within the domain of validity of perturbative pqChPT,

$$\begin{aligned} 1/\Sigma L^4 &\ll \lambda_i \ll \Lambda_{\text{QCD}} \quad i = 1, 2, \\ |\lambda_1 - \lambda_2| &\gg \lambda_{\min} \sim 1/\Sigma L^4. \end{aligned} \quad (4.89)$$

The second condition arises from the susceptibilities $\chi(\pm i\lambda_1 + \epsilon, \mp i\lambda_2 + \epsilon)$ that contribute to the two-point correlation function (see eq. (1.11)). In that case, we have zero-momentum modes with mass $\sim |\lambda_1 - \lambda_2|$ and a perturbative evaluation of the susceptibility is only possible for $\Sigma V|\lambda_1 - \lambda_2| \gg 1$. For eigenvalues inside the domain (4.89) satisfying in addition the condition $\lambda_i \ll F^2/\Sigma L^2$, that is the ergodic domain, only the zero momentum modes contribute to the scalar susceptibility which is then given by (4.80). For the connected two-point correlation function we then find

$$\rho_c(\lambda_1, \lambda_2) = -\frac{1}{2\pi^2} \left[\frac{1}{(\lambda_1 - \lambda_2)^2} + \frac{1}{(\lambda_1 + \lambda_2)^2} \right] + \dots \quad (4.90)$$

This result is also obtained from a perturbative expansion of the chRMT result, that is the expression (3.62) derived from the zero-momentum sector of the pqChPT partition function in Section 3.2. It does not depend on any of the low-energy coupling constants that appear in the effective Lagrangian (4.64) and is thus the same, independent of the number of flavors and the quark masses, provided that chiral symmetry is broken spontaneously.

The perturbative result for the two-point spectral correlation function is obtained from the discontinuity of (4.76). At finite volume, in the domain (4.89), it is given by

$$\begin{aligned} \rho_c(\lambda_1, \lambda_2) &= -\frac{1}{2\pi^2} \frac{1}{(\lambda_1 - \lambda_2)^2} - \frac{1}{2\pi^2} \frac{1}{(\lambda_1 + \lambda_2)^2} \\ &+ \frac{\Sigma^2}{2\pi^2 F^4} \sum_{\vec{p} \neq 0} \left\{ \frac{p^4 - \frac{\Sigma^2}{F^4}(\lambda_1 - \lambda_2)^2}{(p^4 + \frac{\Sigma^2}{F^4}(\lambda_1 - \lambda_2)^2)^2} + \frac{p^4 - \frac{\Sigma^2}{F^4}(\lambda_1 + \lambda_2)^2}{(p^4 + \frac{\Sigma^2}{F^4}(\lambda_1 + \lambda_2)^2)^2} \right\} \\ &+ \frac{\Sigma^2}{2\pi^2 F^4} \sum_{\vec{p} \neq 0} 4(\eta p^2 + 2\Sigma m_\eta/F^2)^2 \frac{p^4 - \frac{\Sigma^2}{F^4}\lambda_1^2}{(p^4 + \frac{\Sigma^2}{F^4}\lambda_1^2)^2} \frac{p^4 - \frac{\Sigma^2}{F^4}\lambda_2^2}{(p^4 + \frac{\Sigma^2}{F^4}\lambda_2^2)^2} \}. \end{aligned} \quad (4.91)$$

In the ergodic domain, only the contribution of the first two terms has to be taken into account. In the diffusive domain, the thermodynamic limit of (4.91) is given by

$$\begin{aligned} \rho_c(\lambda_1, \lambda_2) &= -\frac{\Sigma^2 V}{64\pi^4 F^4} \left[\ln \frac{(\lambda_1^2 - \lambda_2^2)^2}{16\mu^4} + 8 \frac{(\lambda_1^2 + \lambda_2^2)(\eta^2(\lambda_1^2 + \lambda_2^2) - 2m_\eta^2)}{(\lambda_1^2 - \lambda_2^2)^2} \right. \\ &- 16\pi \frac{\eta m_\eta |\lambda_1| |\lambda_2|}{(|\lambda_2| + |\lambda_1|)^3} + 4 \frac{1}{(\lambda_1^2 - \lambda_2^2)^3} \left\{ m_\eta^2 (\lambda_1^4 + 6\lambda_1^2 \lambda_2^2 + \lambda_2^4) \ln \frac{\lambda_1^2}{\lambda_2^2} \right. \\ &\left. \left. + \eta^2 \lambda_1^2 [\lambda_1^4 - 6\lambda_1^2 \lambda_2^2 - 3\lambda_2^4] \ln \frac{\lambda_1^2}{\mu^2} - \eta^2 \lambda_2^2 [\lambda_2^4 - 6\lambda_2^2 \lambda_1^2 - 3\lambda_1^4] \ln \frac{\lambda_2^2}{\mu^2} \right\} \right]. \end{aligned} \quad (4.92)$$

This result can also be obtained directly from the thermodynamic limit of the scalar susceptibility in the diffusive domain (4.84). The two-point correlation function is even in both λ_1 and λ_2

but is not translational invariant. These properties originate from the pairing of the Dirac eigenvalues. For massless quarks (or topological susceptibility equal to zero in the quenched case) the expression (4.92) simplifies to

$$\begin{aligned} \rho_c(\lambda_1, \lambda_2) = & -\frac{\Sigma^2 V}{64\pi^4 F^4} \left[\ln \frac{(\lambda_1^2 - \lambda_2^2)^2}{16\mu^4} + \frac{4\eta^2}{(\lambda_1^2 - \lambda_2^2)^2} \left\{ 2(\lambda_1^2 + \lambda_2^2)^2 \right. \right. \\ & \left. \left. + \frac{1}{\lambda_1^2 - \lambda_2^2} \left(\lambda_1^2 [\lambda_1^4 - 6\lambda_1^2 \lambda_2^2 - 3\lambda_2^4] \ln \frac{\lambda_1^2}{\mu^2} - \lambda_2^2 [\lambda_2^4 - 6\lambda_1^2 \lambda_2^2 - 3\lambda_1^4] \ln \frac{\lambda_2^2}{\mu^2} \right) \right\} \right], \end{aligned} \quad (4.93)$$

where $\eta = 1/N_f$ in the QCD case, and $\eta = \alpha$ in the quenched case. Notice that in the limit $\lambda_1 \rightarrow \lambda_2$ the terms proportional to η^2 are regular. For $|\lambda_1 - \lambda_2| \ll \lambda_{1,2}$ the infrared singular part of the correlation function (4.93) simplifies to

$$\rho_c(\lambda_1, \lambda_2) = -\frac{\Sigma^2}{32\pi^4 F^4} \log \frac{|\lambda_1 - \lambda_2|}{\mu}. \quad (4.94)$$

This result, derived for $m_\eta = 0$, does not depend on the parameter η . It is therefore valid for QCD with any number of flavors of massless quarks, even zero. It cannot be obtained from chRMT. Beyond the energy scale for which the kinetic term in the chiral Lagrangian cannot be neglected, also known as the Thouless energy, pqChPT differs from chRMT. This was observed earlier in the analysis of the spectral density in [5, 7] and will be discussed in greater detail in the next section.

5 Number Variance

In the study of disordered systems, a frequently used measure of the spectral correlations is the number variance of the eigenvalues. It is defined as the variance of the number of eigenvalues in an interval that contains n eigenvalues on average. If the actual number of eigenvalues in each such interval for the i -th member of the ensemble is given by n_i , the p -th moment of the number of eigenvalues is given by the ensemble average $\langle n_i^p \rangle$. With the number variance denoted by $\Sigma^2(n)$ we thus have

$$\begin{aligned} n &= \langle n_i \rangle, \\ \Sigma^2(n) &= \langle n_i^2 \rangle - \langle n_i \rangle^2. \end{aligned} \quad (5.95)$$

If we denote the eigenvalues for the i -th member of the ensemble by $\lambda_k^{(i)}$ the corresponding spectral density is given by

$$\rho_i(\lambda) = \sum_k \delta(\lambda - \lambda_k^{(i)}). \quad (5.96)$$

The number of eigenvalues inside the interval $[a, b]$ is equal to

$$n_i = \int_a^b d\lambda \rho_i(\lambda). \quad (5.97)$$

The average number of eigenvalues inside this interval is thus given by

$$n = \int_a^b d\lambda \langle \rho_i(\lambda) \rangle, \quad (5.98)$$

and the number variance can be written as

$$\Sigma^2(n) = \int_a^b \int_a^b d\lambda_1 d\lambda_2 \rho_c(\lambda_1, \lambda_2), \quad (5.99)$$

with the connected two-point correlation function given by

$$\rho_c(\lambda_1, \lambda_2) = \langle \rho_i(\lambda_1) \rho_i(\lambda_2) \rangle - \langle \rho_i(\lambda_1) \rangle \langle \rho_i(\lambda_2) \rangle. \quad (5.100)$$

If the average spectral density is not constant on the interval $[a, b]$ there will be a contribution to the number variance due to the variation of the average spectral density. We will always assume that this contribution has been eliminated by a procedure called unfolding [3].

The two-point correlation function can be decomposed as

$$\langle \rho_i(\lambda) \rho_i(\lambda') \rangle = \delta(\lambda - \lambda') \sum_k \langle \delta(\lambda - \lambda_k) \rangle + \sum_{k \neq l} \langle \delta(\lambda - \lambda_k) \delta(\lambda' - \lambda_l) \rangle. \quad (5.101)$$

The connected two-point correlation function can be written as

$$\rho_c(\lambda, \lambda') = \delta(\lambda - \lambda') \rho(\lambda) + R(\lambda, \lambda'), \quad (5.102)$$

where the two-point cluster function $R(\lambda, \lambda')$ contains only correlations of different eigenvalues and is regular for $\lambda \rightarrow \lambda'$.

For a finite total number of eigenvalues, we obtain by integration of the connected two-point correlation function the sum rule

$$\int d\lambda_1 \rho_c(\lambda_1, \lambda_2) = 0. \quad (5.103)$$

The contribution due to the self-correlations is cancelled by the contribution from the two-point cluster function. The coefficient of the linear term in the asymptotic expansion of the number variance is given by

$$\frac{d\Sigma^2(n)}{dn} = 2 \frac{\pi}{\Sigma V} \int_0^{\pi n / \Sigma V} d\lambda \rho_c\left(\frac{\pi n}{\Sigma V}, \lambda\right). \quad (5.104)$$

where we have used that $\rho_c(\lambda_1, \lambda_2) = \rho_c(\lambda_2, \lambda_1)$. This quantity is also known as the spectral compressibility. We observe that for large n the coefficient of the linear term vanishes provided that the correlation function approaches zero faster than $1/\lambda$, $1/\lambda'$, i.e. when the integration in the sum-rule (5.103) is convergent without imposing a cutoff on the total number of eigenvalues. If this is the case the correct asymptotic result for the number variance can only be obtained from the asymptotic result of the two-point cluster function if it is regularized such that the sum-rule (5.103) is satisfied.

In the previous sections, the two-point correlation function was derived under the assumption that $\lambda_1, \lambda_2 \ll \Lambda_{\text{QCD}}$. On this scale, the variations of the average spectral density [7] can be neglected. The average number of eigenvalues in the interval $[0, \Delta]$ is given by

$$n = \int_0^\Delta d\lambda \rho(\lambda). \quad (5.105)$$

For $m_\eta = 0$ the spectral density for $\lambda \ll \Lambda_{\text{QCD}}$ is given by the Banks-Casher relation

$$\rho(\lambda) = \frac{\Sigma V}{\pi} + \dots. \quad (5.106)$$

The average number of eigenvalues in the interval $[0, \Delta]$, of eigenvalues is thus given by

$$n = \frac{\Sigma V \Delta}{\pi} + \dots, \quad (5.107)$$

and the number variance can be written as

$$\begin{aligned} \Sigma^2(n) &= \int_0^{\pi n / \Sigma V} d\lambda_1 \int_0^{\pi n / \Sigma V} d\lambda_2 \rho_c(\lambda_1, \lambda_2) \\ &= n + \int_0^{\pi n / \Sigma V} d\lambda_1 \int_0^{\pi n / \Sigma V} d\lambda_2 R(\lambda_1, \lambda_2). \end{aligned} \quad (5.108)$$

Perturbative calculations of the spectral two-point function are only valid for $|\lambda_1 - \lambda_2| \gg \lambda_{\min}$ and do not include the term due to self-correlations. However, they are included in the nonperturbative result (3.60) valid in the ergodic domain and in the quantum domain. Therefore the number variance in the quantum domain and in the ergodic domain is obtained by simply inserting the non-perturbative result (3.62) into (5.108). We are not aware of any analytic expression for these integrals. However both in the case of small n and in the case of large n , one can derive concise analytic expressions for the number variance. With the small n -expansion of (3.62), we find that the number variance is given by

$$\Sigma^2(n) = n + O(n^2). \quad (5.109)$$

As discussed above the asymptotic large- n result of the number variance can only be obtained from the asymptotic result of the two-point correlation function after it has been regularized such that the sum-rule (5.103) is satisfied. In the ergodic domain this is achieved by

$$\rho_c(\lambda_1, \lambda_2) = \rho(\lambda)[\delta(\lambda - \lambda') + \delta(\lambda + \lambda')] - \frac{1}{2\pi^2} \left[\frac{1}{(\lambda_1 - \lambda_2)^2 + a^2} + \frac{1}{(\lambda_1 + \lambda_2)^2 + a^2} \right] + \dots \quad (5.110)$$

where a is determined by the sum rule (5.103),

$$a = \frac{1}{2\pi\rho(\lambda)}. \quad (5.111)$$

In the ergodic domain, we have that $\lambda \ll 1/L^2 \Lambda_{\text{QCD}}$ so that $\rho(\lambda)$ is well approximated by $\rho(0)$. Then $\rho(0)$ drops out of the expression for the number variance, and at leading order we find the familiar logarithmic dependence known from Random Matrix Theory,

$$\Sigma^2(n) = \frac{1}{2\pi^2} \log n + \dots \quad (5.112)$$

It coincides with the result derived from chRMT. Notice that in the bulk of the spectrum the coefficient of $\log n$ is a factor of 2 larger.

At finite volume, for eigenvalues much larger than the smallest eigenvalues but in the domain of chiral perturbation theory (i.e. $n_{\text{QCD}} \gg n \gg 1$), the number variance can be calculated from the expression for the two-point correlation function that includes the nonzero-momentum modes (4.91). We find

$$\begin{aligned} \Sigma^2(n) &= \frac{1}{2\pi^2} \log n + \frac{1}{4\pi^2} \sum_{\vec{k} \neq 0} \log \frac{k^4 + \left(\frac{n}{2\pi^2 n_c}\right)^2}{k^4} \\ &\quad + 8\pi^2 \left(\frac{n}{2\pi^2 n_c}\right)^2 \sum_{\vec{k} \neq 0} \left(\frac{\frac{\eta k^2}{4\pi^2} + \frac{2\Sigma m_\eta}{F^2}}{k^4 + \left(\frac{n}{2\pi^2 n_c}\right)^2} \right)^2, \end{aligned} \quad (5.113)$$

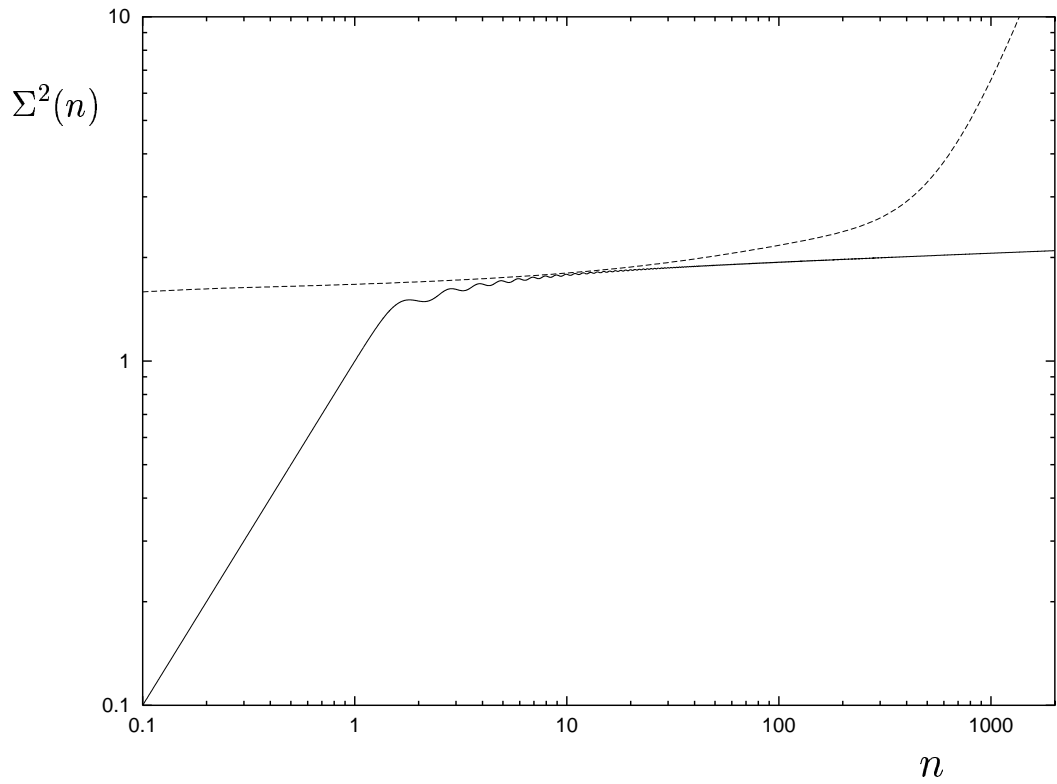


Figure 2: The number variance for levels under n_{QCD} computed within pqChPT for $N_f = 3$ massless quarks. The solid curve is the zero-mode sector of the partition function, or chRMT, and the dashed curve is the perturbative result. The volume of the box is $(20 \text{ fm})^4$, the Thouless scale is $n_c \approx 27.5$ and $n_{\text{QCD}} \approx 10^5$. Notice how late the asymptotic $\log n$ -behavior is reached, and how early the non-zero momentum corrections are visible. A strikingly similar figure has been obtained for a disordered metal in [54].

where $n_c = E_c/\lambda_{\min} = F^2 L^2/\pi$ is the dimensionless Thouless scale. We have used that $p_\mu = 2\pi k_\mu/L$ with k_μ a four-dimensional hypercubic lattice with unit cell of length one.

In the diffusive domain, the thermodynamic limit of (5.113) is given by

$$\Sigma^2(n) = -\frac{1 + 2\eta^2}{16\pi^4} \left(\frac{n}{n_c}\right)^2 \log \frac{n}{n_{\text{QCD}}}, \quad (5.114)$$

where $n_{\text{QCD}} \equiv \mu/\lambda_{\min} \sim \Lambda_{\text{QCD}}/\lambda_{\min}$. Therefore, in QCD with N_f flavors of massless quarks one finds that in the diffusive domain $n_c \ll n \ll n_{\text{QCD}}$, the number variance of the QCD Dirac spectrum given by

$$\Sigma^2(n) = -\frac{N_f^2 + 2}{16\pi^4 N_f^2} \left(\frac{n}{n_c}\right)^2 \log \frac{n}{n_{\text{QCD}}}. \quad (5.115)$$

The number variance computed within pqChPT for eigenvalues under Λ_{QCD} is shown in Fig. 2. The $n^2 \log n$ behavior saturates as n for $n \gg n_{\text{QCD}}$ where the interactions become weak because of asymptotic freedom.

Finally in the ballistic domain, for $n \gg n_{\text{QCD}}$ the eigenvalues of the Dirac operator in QCD are correlated as those for a free Dirac operator. The spectrum thus obeys Poisson statistics, and the number variance is given by

$$\Sigma^2(n) = n. \quad (5.116)$$

6 Conclusions

We have analyzed the fluctuations of the eigenvalues of the QCD Dirac operator by means of the partially quenched chiral susceptibility. From its discontinuity across the imaginary axis we get the two-point spectral correlation function. The variance, $\Sigma^2(n)$, of the number of eigenvalues in an interval containing n eigenvalues on average is obtained by integrating the spectral two-point correlation function. The generating function for the scalar susceptibility is given by the QCD partition function with two additional fermionic quarks and two additional bosonic ghost quarks with a mass equal to the spectral mass (also known as valence quark mass) that enters in the partially quenched chiral susceptibility. For spectral quark masses well below Λ_{QCD} the low-energy limit of this theory is completely determined by its global symmetries.

Based on this partition function we have distinguished three important scales in the Dirac spectrum, the smallest nonzero eigenvalue, $\lambda_{\min} = \pi/\Sigma V$, the spectral mass for which the Compton wavelength of the corresponding Goldstone boson is equal to the size of the box, $m_c = F^2/\Sigma L^2$, and Λ_{QCD} . The analogue of these scales are well-known in mesoscopic physics where they separate the quantum domain, the ergodic domain, the diffusive domain and the ballistic domain, respectively.

For spectral quark masses well below m_c , the kinetic term in the partially quenched chiral Lagrangian decouples from the zero momentum part, and only the latter part has to be taken into account. In this domain the partition function only depends on the combination $mV\Sigma$ or $zV\Sigma$. In the thermodynamic limit, a nontrivial finite result is obtained by keeping this combination constant. A perturbative expansion of the zero momentum part of the partition function fails for spectral quark masses that are not much larger than λ_{\min} . In this domain we have exactly calculated the super-integrals that appear in the partition function for the sector of topological charge ν . Our results for the scalar susceptibility and the corresponding spectral two-point correlation function in this domain are in complete agreement with chiral Random Matrix Theory. This result, together with earlier work on the microscopic spectral density, provides strong evidence that also all higher order correlation functions that can be derived from the low-energy limit of the QCD partition function coincide with the results from chiral Random Matrix Theory. The number variance in this domain is given by $\log n/2\pi^2$, showing that the fluctuations of the eigenvalues are strongly suppressed.

For spectral masses in the diffusive domain, $m_c \ll z \ll \Lambda_{\text{QCD}}$, a perturbative evaluation of the partition function is sufficient provided that the nonzero momentum modes are taken into account. In the ergodic domain the fluctuations of the QCD Dirac eigenvalues are completely determined by the number of massless flavors and the total topological charge. They only depend on the combination $\lambda V\Sigma$. At fixed λ in the thermodynamic limit, when the value of m_c approaches zero as $1/\sqrt{V}$, the nonzero momentum modes contribute to the chiral susceptibility. In this limit for $\lambda \ll \Lambda_{\text{QCD}}$ we have calculated the scalar susceptibility to one loop in chiral perturbation theory. We have found that in the case of massive quarks (including the quenched case) that the scalar susceptibility shows a quadratic infrared divergence. At finite volume this divergence is regulated by nonzero momentum modes, and thus only appears for valence quark masses beyond the Thouless energy. This divergence is the analogue of the quenched chiral logarithms that appear in the calculation of the chiral condensate. We expect that it will show up much more prominently in lattice QCD simulations than the quenched chiral logarithms.

The prediction for the behavior of the two-point correlation function in the diffusive regime is $\rho_c(\lambda_1 - \lambda_2) \sim 1/(\lambda_1 - \lambda_2)^{(d-4)/2}$ [55]. In agreement with this result we have found a logarithmic dependence of $\rho_c(\lambda_1 - \lambda_2)$ on $|\lambda_1 - \lambda_2|$. The corresponding number variance is given by $\Sigma^2(n) \sim n^2 \log n$. The proportionality constant is given by $(N_f^2 + 2)/(16\pi^2 N_f^2 F^2 L^2)$ and provides us with

an alternative method to determine the pion decay constant.

The behavior of the number variance is very different in the various domains we have defined. Deep in the quantum domain, for small n , the interval contains randomly either zero or one level and $\Sigma^2(n) = n$. When the first level is reached, the number variance almost stops increasing. It has a $\log n$ behavior up to the Thouless scale $n_c = 1/F^2 L^2$. Therefore the spectrum is quite stiff in this domain. Around the Thouless scale, the number variance increases much faster again. Well above n_c but below n_{QCD} , the level number corresponding to Λ_{QCD} , the number variance grows like $n^2 \log n$. Finally, for level number much larger than n_{QCD} , the Dirac spectrum is in essence a free particle spectrum because of asymptotic freedom. The levels are therefore randomly distributed and the number variance is again given by $\Sigma^2(n) = n$. This behavior is typical of a disordered metallic phase in condensed matter physics.

In conclusion, we have shown that the fluctuations of the QCD Dirac eigenvalues below Λ_{QCD} in the phase of broken chiral symmetry are completely determined by the symmetries of the QCD partition function.

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