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Simulating Quantum Mechanics by Non-Contextual Hidden Variables

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No physical measurement can be performed with infinite precision. This leaves a loophole in the standard no-go arguments against non-contextual hidden variables. All such arguments rely on choosing special sets of quantum-mechanical observables with measurement outcomes that cannot be simulated non-contextually. As a consequence, these arguments do not exclude the hypothesis that the class of physical measurements in fact corresponds to a dense subset of all theoretically possible measurements with outcomes and quantum probabilities that *can* be recovered from a non-contextual hidden variable model. We show here by explicit construction that there are indeed such non-contextual hidden variable models, both for projection valued and positive operator valued measurements.

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I. INTRODUCTION

Bell's theorem [1] establishes that local hidden variable theories are committed to Bell inequalities violated by quantum correlations. Since violations of Bell inequalities can be verified without requiring that the observables whose correlations figure in the inequalities be measured with arbitrarily high precision, Bell's theorem yields a method of falsifying local hidden variable theories. Moreover, the evidence for violations of Bell inequalities very strongly suggests that such theories have indeed been falsified [2–5].

We are concerned here, however, not with locality but with non-contextuality. A non-contextual hidden variable theory ascribes a definite truth value to any projection or, in the case of generalised measurements, any positive operator, so that the truth values predict the outcome of any measurement involving the relevant operator and are independent of the other projections or positive operators involved in the measurement. Of course, non-contextual hidden variable theories that reproduce the quantum correlations between spatially separated systems must, by Bell's theorem, be non-local. However, our chief interest is in determining whether *non-relativistic* quantum theory can be simulated classically via non-contextual hidden variables. Since non-relativistic classical mechanics does not presuppose a light cone structure, non-locality is not a meaningful constraint on hidden variables in this context, and we shall henceforth ignore questions of non-locality altogether.

Unlike the arguments against local hidden variables, the known arguments against non-contextual hidden variables require observables to be measurable with perfect precision. These arguments derive from the work of Gleason [6], Bell [7], and Kochen and Specker [8]. (For more recent discussions see [9–13].)

Pitowsky [14,15] argued some time ago that these no-go arguments could be evaded by restricting attention to appropriately chosen subsets of the space of observables. While ingenious, Pitowsky's models, which are constructed via the axiom of choice and the continuum hypothesis, have the defect that they rely for their interpretation on a radically non-standard version of probability theory, according to which (for example) the conjunction of two probability one events can have probability zero.

More recently, Meyer [16] has emphasized that the fact that all physical measurements are of finite precision leaves a loophole in the arguments against non-contextual hidden variables. One could hypothesize that the class of possible physical measurements is only a dense subset of the full set of von Neumann or positive operator valued measurements. That is, in any given finite precision measurement there is a fact of the matter, unknown to us, as to which precise measurement is being carried out, and these realised measurements always belong to some particular dense subset, which again need not necessarily be known to us. Under this hypothesis, the arguments against non-contextual hidden variables, which rely on ascribing definite values to all projections in a real three-dimensional space (or to certain well chosen finite subsets of projections), have been shown not to go through [16], nor can any similar arguments be constructed for either projections or positive operator valued measurements in Hilbert spaces of three or higher dimensions [17]. These recent counterexamples [16,17] rely only on constructive set theory.

The aim of the present paper is more ambitious. We shall show that all the predictions of non-relativistic quantum mechanics that are verifiable to within any finite precision *can* be simulated classically by non-contextual hidden variable theories. That is, there are non-contextual hidden variable models whose predictions are practically indistinguishable from those of non-relativistic quantum mechanics for either projection valued or positive operator valued measurements. We give explicit examples, whose construction requires only constructive set theory and whose interpretation needs only elementary (standard) probability theory.

Before giving details, we should explain why we find the question interesting. We have no particular interest in advocating non-contextual hidden variable theories. However, we believe that it is important to distinguish strongly held theoretical beliefs from rigorously established facts in analysing the ways in which quantum theory is demonstrably non-classical. We share, too, with Meyer [16] another motivation: questions about the viability of hidden variable models for a particular quantum process translate into questions about the classical simulability of some particular aspect of quantum behaviour, and are interesting independently of the physical plausibility of the relevant models. In particular, from the point of view of quantum computation, the precision attainable in a measurement is a computational resource. Specifying infinite precision requires infinite resources and prevents any useful comparison with discrete classical computation. It is interesting to see that, once the assumption of infinite precision is relaxed, the outcomes and probabilities of quantum measurements can indeed be simulated classically and non-contextually.

II. OUTLINE OF RESULTS

We begin by reviewing the standard theoretical argument against non-contextual hidden variable models, which is based on infinite precision von Neumann, i.e. projection valued, measurements performed on a system represented by an n-dimensional Hilbert space H_n, where n is finite.

Determining a unique value for some measured observable O, with spectral projections $\{P_i\}$, is equivalent to distinguishing exactly one member of the set $\{P_i\}$ and assigning it value '1' (to signify that the corresponding eigenvalue of O is determined to occur on measurement), while assigning all the other projections in $\{P_i\}$ value '0'. Let $\mathcal{P}$ denote a set of projections, and $\overline{\mathcal{P}}$ denote the set of all observables whose spectral projections lie in $\mathcal{P}$. Then, whether there can be hidden variables that uniquely determine the measured values of all the observables in $\overline{\mathcal{P}}$ is equivalent to asking whether there exists a *truth function* $t : \mathcal{P} \to \{0, 1\}$ satisfying:

$$\sum_i t(P_i) = 1 \text{ whenever } \sum_i P_i = I \text{ and } \{P_i\} \subseteq \mathcal{P}. \tag{1}$$

If $n > 2$, and we take $\overline{\mathcal{P}}$ to be all observables of the system, it is an immediate consequence of Gleason's [6] theorem that $\mathcal{P}$ admits no truth function. A simpler proof of the relevant part of Gleason's result was given by Bell [7], who discussed its implications for hidden variable models.

Kochen and Specker [8] exhibited a finite set of spin-1 observables $\overline{\mathcal{P}}$ such that $\mathcal{P}$ admits no truth function, and arguments for the nonexistence of truth functions on finite sets of projections are often (summarily) called the 'Kochen-Specker Theorem'. Simpler examples of finite sets of observables admitting no truth function have since been given by Peres [10] and Zimba and Penrose [11], among others.

Clearly, then, even before statistical considerations enter, for a hidden variable theory to remain viable, its hidden variables cannot determine unique values for all observables, or even certain finite subsets of observables. There is a simple intuitive reason why contradictions can arise when one attempts to construct a truth function on some set of projections $\mathcal{P}$. Recall that two projections P, P' are said to be compatible if $[P, P'] = 0$. Call two resolutions of the identity operator $\sum_i P_i = I$ and $\sum_j P'_j = I$ *compatible* if $[P_i, P'_j] = 0$ for all i, j. If the various resolutions of the identity generated by $\mathcal{P}$ are all mutually compatible, then closing $\mathcal{P}$ under products of projections and complements yields a Boolean algebra under the operations $P \wedge Q = PQ$ and $P^{\perp} = I - P$, and Boolean algebras always possess truth functions. However, in dimension $n > 2$, one can choose $\mathcal{P}$ so that it generates resolutions of the identity with many projections in common between different *incompatible* resolutions (in particular, such $\mathcal{P}$'s fail to generate a Boolean algebra). This can drastically reduce the number of 'unknowns' relative to equations in (1) to the point where they have no solution.

Conversely, one *can* have a viable hidden variable theory (still assuming infinite measurement precision) only if one is prepared to assign a value to a projection that is not simply a function of the hidden variables, but also of the resolution of the identity that the projection is considered to be a member of. Physically, this would mean that the measured value of the projection would be allowed to depend upon the context in which it is measured, i.e., on the complete commuting set of observables that the projection is jointly measured with — hence the phrase 'contextual hidden variable theory'. Alternatively, one can adopt the approach Kochen and Specker themselves advocated, and view the nonexistence of truth functions as an argument for a quantum-logical conception of a system's properties.

However, when it comes to considering practical experiments, we are not actually forced either towards contextual hidden variables or quantum logic by the Kochen-Specker theorem. The Pitowsky models [14] mentioned above provide one possible alternative approach. Unfortunately, both the axiom of choice and the continuum hypothesis (or some weaker axiom in this direction) are needed to define Pitowsky's models. Moreover, the non-standard version of probability theory required for their interpretation has such bizarre properties that we doubt whether these models can reasonably be said to constitute a classical explanation or simulation of quantum theory. At the very least, these features decrease the value of the comparison with classical physics and classical computation.

However, the more recent constructive arguments by Meyer [16] and Kent [17] show that one can always find a subset $\mathcal{P}_d$ of projections on H_n, for any finite n, such that $\mathcal{P}_d$ admits a truth function t and generates a countable dense set of resolutions of the identity. By the latter, we mean that for any k-length resolution $\sum_{i=1}^{k} P_i = I$ (where $k \leq n$) and any $\epsilon > 0$, one can always find another k-length resolution $\sum_{i=1}^{k} P'_i = I$ such that $|P_i - P'_i| < \epsilon$ for all i and $\{P'_i\} \subseteq \mathcal{P}_d$. In particular, for any self-adjoint operator O and any $\epsilon > 0$, there is an $O' \in \overline{\mathcal{P}}_d$, with the same eigenspectrum as O, such that the probabilities for measurement outcomes of O' lie within ϵ of the corresponding probabilities for O.

Given this, the non-contextual hidden variable theorist is free to adopt the hypothesis that any *finite* precision measurement by which we attempt an approximate measurement of an observable O with spectral decomposition $\sum_{i=1}^{k} a_i P_i$ actually corresponds to a measurement of some other observable $O' = \sum_{i=1}^{k} a_i P'_i$ lying in the precision range and belonging to $\overline{\mathcal{P}}_d$. Moreover, one can allow this observable O' to be specified by the state and characteristics of the measuring device alone: O' need not depend either on the quantum state of the system or on the hidden variables of the particle being measured. The hypothesis, in other words, is that when we set up an experiment to carry out a finite precision measurement there is some fact of the matter as to which precise observable is actually being measured, and the measured observable in fact belongs to $\overline{\mathcal{P}}_d$. The precise observable being measured is presumably specified by hidden variables associated with the measuring apparatus. It is not known to us in any given experiment; we know only the precision range.

This, it should be stressed, does *not* amount to reintroducing contextualism into the hidden variable theory. While the distribution of hidden variables associated with the measured system needs to be related to the quantum state of that system in order to recover its quantum statistics, these variables are supposed to be independent of both the hidden variables associated with the measuring apparatus and *its* quantum state; in particular, they are not correlated with the unknown choice of O'. Note too that the fact that measurements are finite precision is not sufficient in itself to overthrow the standard analysis. If we assumed, as above, that some precise O' was always specified, and we allowed the possibility that O' could be chosen to be any observable sufficiently close to O, the Kochen-Specker theorem would again threaten. It is the restriction to observables in $\overline{\mathcal{P}}_d$, that has a set of projections *admitting* a truth function, that creates a loophole.

Thus, a finite precision attempt to measure O can have its outcome specified, in a noncontextual way, by the value picked out for O' by the truth function on $\mathcal{P}_d$. Invoking this toy noncontextual hidden variable model, the practical import of the theoretical argument against such models supplied by the Kochen-Specker theorem can be regarded as 'nullified' (to use Meyer's term).

Meyer and Kent leave open the question as to whether we can actually construct a non-contextual hidden variable model that is consistent with all the *statistical* predictions of quantum theory. Let us call a collection of truth functions on a set $\mathcal{P}$ *full* if for any two distinct $P, P' \in \mathcal{P}$, there exists a truth function t on $\mathcal{P}$ such that $t(P) \neq t(P')$. For example, if $\mathcal{P}$ generates a Boolean algebra, then it will possess a full set of truth valuations (one for each minimal nonzero projection of $\mathcal{P}$), but the converse fails. Unless $\mathcal{P}_d$ possesses a full set of truth valuations, the previous paragraph's toy model cannot satisfy the statistical predictions of quantum theory. For suppose that two distinct $P, P' \in \mathcal{P}_d$ are always mapped to the same truth value under all truth valuations. Then every set of hidden variables must dictate the same values for P and P', and therefore the theory will have to predict the same expectation values for P and P' in every quantum state of the system. But this is absurd: since P and P' are distinct, there is certainly some quantum state of the system (which we need only assume is prepared to within finite precision) in which the expectations of P and P' differ.

There certainly are sets of projections $\mathcal{P}$ that admit truth valuations, but not a full set (see the 'Kochen-Specker diagram' on p. 70 of [8], involving only 17 projections). Moreover, it is not obvious that any $\mathcal{P}_d$ with the above properties — for example those given by Meyer [16] and Kent [17] — must necessarily possess a full set of truth valuations. Hence the unfalsifiability of non-contextual hidden variables in the face of quantum statistics has yet to be established. The goal of the present paper is to establish this unfalsifiability.

In Section **III**, we shall prove the following result:

> **Theorem 1** *There exists a set of projections $\hat{\mathcal{P}}_d$ on H_n, closed under products of compatible projections and complements, that generates a countable dense set of resolutions of the identity with the property that no two compatible projections in $\hat{\mathcal{P}}_d$ are members of incompatible resolutions.*

Consider, first, the structure of $\hat{\mathcal{P}}_d$. Regard two resolutions of the identity in $\hat{\mathcal{P}}_d$ as equivalent if they are compatible. Then since compatible projections in $\hat{\mathcal{P}}_d$ can only figure in compatible resolutions, we obtain an equivalence relation. To see its transitivity, suppose $\{P_i\}$, $\{P'_j\}$ and $\{P'_j\}$, $\{P''_k\}$ are compatible pairs of resolutions in $\hat{\mathcal{P}}_d$, fix some arbitrary indices i, j, k, and consider the resolution $I = P_iP'_j + (I - P_i)P'_j + (I - P'_j)$. Since $[I - P'_j, P''_k] = 0$, we must have $[P_iP'_j, P''_k] = 0$, which in turn entails

$$P''_k P_i P'_j = P_i P'_j P''_k = P_i P''_k P'_j. \tag{2}$$

Summing this equation over j yields $[P''_k, P_i] = 0$, and hence the resolutions $\{P_i\}$, $\{P''_k\}$ must be compatible as well.

It follows that the relation of compatibility between resolutions of the identity generated by $\hat{\mathcal{P}}_d$ partitions that set into a collection of Boolean algebras that share only the projections 0 and I in common. In particular, the truth valuations on $\hat{\mathcal{P}}_d$ are full, since each of its Boolean subalgebras possesses a full set of truth valuations, and any collection of assignments of truth values to all the Boolean subalgebras of $\hat{\mathcal{P}}_d$ extends trivially to a truth valuation on the whole of $\hat{\mathcal{P}}_d$. Moreover, it should already be clear that the set of truth valuations on $\hat{\mathcal{P}}_d$ will be sufficiently rich to recover the statistics of any quantum state by averaging over the values of the hidden variables that determine the various truth valuations. For, given any state D, and nonzero projection $P \in \hat{\mathcal{P}}_d$, there will always be many truth valuations that map $t(P) = 1$, and we may assign the subset of hidden variables for which $t(P) = 1$ measure $\mathrm{Tr}(DP)$. This prescription also works for calculating joint probabilities of compatible projections $P, P' \in \hat{\mathcal{P}}_d$, since

$$\mathrm{Prob}_D(P=1,P'=1)=\mathrm{Prob}_D(PP'=1)=\mathrm{Tr}(DPP'),\ \text{and}\ PP'\in\hat{\mathcal{P}}_d. \tag{3}$$

To sum up, then, the existence of the set $\hat{\mathcal{P}}_d$ defeats the practical possibility of falsifying non-contextual hidden variables on either nonstatistical or statistical grounds.

The same goes for falsifying classical logic. Following Kochen and Specker, let us call a set of projections $\mathcal{P}$ a *partial Boolean algebra* if it is closed under products of compatible projections and complements, and a truth valuation $t:\mathcal{P}\to\{0,1\}$ will be called a *two valued homomorphism* if t also preserves compatible products and complements (i.e., $t(PP')=t(P)t(P')$ and $t(P^{\perp})=1-t(P)$). In their Theorem 0, Kochen and Specker establish that if $\mathcal{P}$ possess a full set of two valued homomorphisms, then $\mathcal{P}$ is imbeddable into a Boolean algebra. Since $\hat{\mathcal{P}}_d$ is closed under the relevant operations, it is a partial Boolean algebra, and it is clear that all its truth valuations will in fact be two valued homomorphisms. Thus $\hat{\mathcal{P}}_d$ can always be imbedded into a Boolean algebra, and this, from a practical point of view, nullifies any possible argument for a quantum logical conception of properties.

In Section **IV**, we rule out falsifications of non-contextual models based on generalized observables, represented by POV measures. Let $\mathcal{A}$ be a set of positive operators on H_n, and consider all the positive operator (PO) decompositions of the identity that $\mathcal{A}$ generates, i.e., decompositions $\sum_i A_i=I$ with $\{A_i\}\subseteq\mathcal{A}$. Since $\sum_i A_i=I$ does not entail $A_iA_j=0$ for $i\neq j$, there can be more members of $\{A_i\}$ than the dimension, n, of the space, and the POs in a resolution of the identity need not be mutually compatible. Still, we can ask the analogous question: does there exist a truth function $t:\mathcal{A}\to\{0,1\}$ satisfying

$$\sum_i t(A_i)=1 \text{ whenever } \sum_i A_i=I \text{ and } \{A_i\}\subseteq\mathcal{A}? \tag{4}$$

Moreover: does $\mathcal{A}$ possess enough truth valuations to recover the statistical predictions, prescribed by any state D, that pertain to the members of $\mathcal{A}$? Again, we show the answer is 'Yes' for some sets $\mathcal{A}$ containing countable dense sets of *finite* PO resolutions of the identity. By this we mean (just as in the projective resolution case) that for any k-length PO decomposition $\sum_i^k A_i=I$ and $\epsilon>0$, there is another k-length PO decomposition $\sum_i^k A_i'=I$ such that $|A_i-A_i'|<\epsilon$ for all i and $\{A_i'\}\subseteq\mathcal{A}$.

Note that from the point of view of practically performable measurements, there is no need to consider infinite PO resolutions of the identity $\sum_i^\infty A_i=I$ (which, of course, exist even in finite dimensions). The reason is that, for any state D of the system, a POV measurement amounts to ascertaining the values of the numbers $\mathrm{Tr}(DA_i)$. Since these numbers normalize to unity, all but a finite subset of them are measurable to within any finite precision. Thus any infinite POV measurement is always practically equivalent to a finite one. Similarly, any von Neumann measurement of an observable is always practically equivalent to the measurement of an observable with finite spectrum. For this reason, we have not sought to generalize any of our results to infinite-dimensional Hilbert spaces, though it might well be of theoretical interest to do so.

Specifically, in Section **IV** we shall establish:

> **Theorem 2** *There exists a set of POs $\hat{\mathcal{A}}_d$ on H_n that generates a countable dense set of finite PO resolutions of the identity with the property that no two resolutions share a common PO.*

Though this result is weaker than its analogue for projections in Theorem 1, it still insulates noncontextual models of POV measurements from falsification. Because the resolutions generated by $\hat{\mathcal{A}}_d$ fail to overlap, the truth values for POs within a resolution may be set quite independently of the values assigned to POs in other resolutions. Thus, it is clear that there will be sufficiently many such valuations to recover the statistics of any density operator D. And, as before, we can suppose that any purported POV measurement, of any length k, actually corresponds to a POV measurement corresponding to a k-length PO resolution (within the precision range of the measurement) that lies in $\hat{\mathcal{A}}_d$.

III. NON-CONTEXTUAL HIDDEN VARIABLES FOR PV MEASURES

Our goal in this section is to establish Theorem 1. Let H_n be an n-dimensional Hilbert space, and denote an (ordered) orthonormal basis of H_n by

$$\langle e_i\rangle=\{e_1,e_2,\ldots,e_n\}\,. \tag{5}$$

Let M be the metric space whose points are orthonormal bases, with the distance between two bases $\langle e_i\rangle$ and $\langle e_i'\rangle$ given by $|I-U|$, where U is the unitary operator mapping $e_i\mapsto e_i'$ for all i. Next, consider $U(n)$, the unitary group on H_n (endowed with the operator norm topology), and recall that $U(n)$ is compact. Fix a reference point $\langle e_i \rangle \in M$, and consider the mapping $\varphi : M \mapsto U(n)$ defined by $\varphi(\langle e'_i \rangle) = U$, where U is the unitary operator that maps $e_i \mapsto e'_i$ for all i. Evidently φ is a homeomorphism, thus M is a compact, complete, separable metric space.

Call two points $\langle e_i \rangle, \langle e'_i \rangle \in M$ *totally incompatible* whenever *every* projection onto a subspace generated by a nonempty proper subset of the vectors $\langle e_i \rangle$ is incompatible with *every* projection onto a subspace generated by a nonempty proper subset of the vectors $\langle e'_i \rangle$. We shall need to make use of the following:

Lemma 1: For any finite sequence

$$\langle e_i^{(1)} \rangle, \ldots, \langle e_i^{(m-1)} \rangle \in M, \tag{6}$$

the subset of points, $T^{(m)}$, that are totally incompatible with all members of the sequence, is dense in M.

Proof: Let the indices k and l range over the values of some fixed enumeration of the (proper) subsets of $\{1, \ldots, n\}$, let $P_k^{(j)}$ denote the projection onto the subspace generated by the kth subset of $\langle e_i^{(j)} \rangle$ (for $j = 1, ..., m-1$), and for an arbitrary unlabelled point $\langle f_i \rangle \in M$, let P_l be the projection onto the subspace generated by the lth subset of $\langle f_i \rangle$. Define:

$$I_{kl}^{(j)} \stackrel{\text{def}}{=} \left\{ \langle f_i \rangle \in M : [P_l, P_k^{(j)}] \neq 0 \right\}. \tag{7}$$

Clearly $T^{(m)}$ is just the finite intersection of all the sets of form $I_{kl}^{(j)}$ over all j, k, l. Now the intersection of any two open dense sets in M is again an open dense set. So if we can argue that each $I_{kl}^{(j)}$ the intersection of any two open dense sets in M is again an open dense set is both open and dense, then since M is homeomorphic to $U(n)$ it will follow that $T^{(m)}$ is dense in M.

So fix j, k, l once and for all. To see that $I_{kl}^{(j)}$ is open, pass to its complement $\overline{I}_{kl}^{(j)}$, and consider any Cauchy sequence $\{\langle f_i^{(p)} \rangle\}_{p=1}^{\infty} \subseteq \overline{I}_{kl}^{(j)}$ with limit $\langle f_i \rangle \in M$. We must show $\langle f_i \rangle \in \overline{I}_{kl}^{(j)}$, i.e., that $[P_l, P_k^{(j)}] = 0$. By hypothesis, $[P_l^{(p)}, P_k^{(j)}] = 0$ for all p, and $\langle f_i^{(p)} \rangle \to \langle f_i \rangle$. Let U_p be the unitary operator mapping $f_i \mapsto f_i^{(p)}$ for all i. Then $P_l^{(p)} = U_p P_l U_p^{-1}$ for all p, $U_p \to I$ (in operator norm), and we have:

$$\begin{aligned} 0 &= \lim_{p \to \infty} [P_l^{(p)}, P_k^{(j)}] \\ &= \lim_{p \to \infty} \left[U_p P_l U_p^{-1}, P_k^{(j)} \right] = [P_l, P_k^{(j)}]\,. \end{aligned} \tag{8}$$

To see that $I_{kl}^{(j)}$ is dense, fix an arbitrary point $\langle f_i \rangle \in M$, and arbitrary $\epsilon > 0$. We must show that one can always find a unitary U such that:

$$|I - U| < \epsilon \text{ and } [U P_l U^{-1}, P_k^{(j)}] \neq 0; \tag{9}$$

for then the point $\langle U f_i \rangle$ must lie inside $I_{kl}^{(j)}$ and within ϵ of $\langle f_i \rangle$. Since j, k, l are all fixed, we are free to set $P = P_l$ and $Q = P_k^{(j)}$ for simplicity, bearing in mind that $P, Q \neq 0$ or I. To establish (9), then, all we need to show is that assuming

$$|I - U| < \epsilon \quad \Rightarrow \quad [U P U^{-1}, Q] = 0 \tag{10}$$

leads to a contradiction.

First, we dispense with the case $P = Q$. Consider the one parameter group of unitaries $U_t \equiv e^{itH}$ where H is self-adjoint. Since $Q \neq 0, I$, we may suppose that $[H, Q] \neq 0$. By (10), $U_t Q U_{-t}$ and Q commute for all sufficiently small t, in which case we may write

$$Q = A_t + B_t, \quad U_t Q U_{-t} = A_t + C_t, \tag{11}$$

where A_t, B_t, and C_t are pairwise orthogonal projections. Then,

$$0 = \lim_{t \to 0} |U_t Q U_{-t} - Q| = \lim_{t \to 0} |B_t - C_t|. \tag{12}$$

By (12), we may choose a $\delta > 0$ so that $|B_t - C_t| < \frac{1}{2}$ for all $t < \delta$. If B_t were nonzero for some $t < \delta$, then we could choose a unit vector e in the range of B_t, and in that case we would have $\|(B_t - C_t)e\| = \|e\| = 1$. However, this contradicts the fact that $|B_t - C_t| < \frac{1}{2}$. Thus, in fact $B_t = 0$ for all $t < \delta$, and by symmetry $C_t = 0$ for all $t < \delta$. Hence, for all $t < \delta$, $U_t Q U_{-t} = A_t = Q$, i.e., $[U_t, Q] = 0$. However, since $\lim_{t\to 0} t^{-1}(U_t - I) = iH$, any operator that commutes with all U_t in a neighborhood of the identity must commute with H. Thus $[H, Q] = 0$, contrary to hypothesis.

Next, consider the case of general P and Q ($\neq 0, I$). Since $[P, Q] = 0$, we may write $Q = A + B$, $P = A + C$, where A, B and C are pairwise orthogonal projections. Without loss of generality, we may assume $A \neq 0$ (i.e., $PQ \neq 0$); for if not, then we may replace P by $I - P$ (in order to guarantee $PQ = A \neq 0$), and under that replacement (10) continues to hold. Similarly, we may assume that $A + B + C \neq I$; for if not, we could replace Q by $I - Q$. Since neither $A + B + C$ nor A equals 0 or I, there is a self-adjoint H' such that $[H', B] = [H', C] = 0$ but $[H', A] \neq 0$. Defining $U_t \equiv e^{itH'}$, we have that $[U_t P U_{-t}, Q] = [U_t A U_{-t}, A]$ for all t. Thus (10) implies:

$$|I - U_t| < \epsilon \quad \Rightarrow \quad [U_t A U_{-t}, A] = 0, \tag{13}$$

which, in turn, entails the contradiction $[H', A] = 0$ by the argument of the previous paragraph (with H' in place of H, and A in place of Q). *QED.*

Proposition 1: There is a countable dense subset of M whose members are pairwise totally incompatible.

Proof: Since M is separable, it possesses a countable dense set $T = \{\langle e_i^{(1)}\rangle, \ldots, \langle e_i^{(m-1)}\rangle, \ldots\}$. If, in moving along this sequence, some point $\langle e_i^{(m)}\rangle$ were found to be not totally incompatible with all previous members of the sequence, then, by Lemma 1, we could always discard $\langle e_i^{(m)}\rangle$ and replace it with a new point $\langle \hat{e}_i^{(m)}\rangle \in T^{(m)}$. The replacement point $\langle \hat{e}_i^{(m)}\rangle$ *will* be totally incompatible with all previous members, and (since $T^{(m)}$ is dense) it can always be chosen to lie within a distance 2^{-m} of $\langle e_i^{(m)}\rangle$. Moving down the sequence T, and replacing points in this way as many times as needed, we obtain a new countably infinite sequence $\hat{T}$ that is pairwise totally incompatible. And $\hat{T}$ is itself dense. For in every ball $B(p, \epsilon) = \{q : d(p, q) < \epsilon\}$ around any point $p \in M$ there must be infinitely many members of the dense set T. If one of those members did not need replacing, then clearly $B(p, \epsilon)$ contains an element of $\hat{T}$. But this must also be true if all of them needed replacing, because the replacement points $\langle \hat{e}_i^m\rangle$ lie closer and closer to $\langle e_i^m\rangle$ as $m \to \infty$. *QED.*

Now we can complete the proof of Theorem 1. Let $\{\langle \hat{e}_i^{(1)}\rangle, \langle \hat{e}_i^{(2)}\rangle, \ldots,\}$ be the dense subset of Proposition 1. Let the k in $\hat{P}_k^{(m)}$ (the projection onto the span of the kth subset of $\langle \hat{e}_i^{(m)}\rangle$) now range over an enumeration of *all* the subsets of $\{1, \ldots, n\}$ (including the empty set, corresponding to $k = 0$ and $\hat{P}_0^{(m)} = 0$, and the entire set, corresponding to $k = 2^n$ and $\hat{P}_{2^n}^{(m)} = I$). Define:

$$B_m \stackrel{\text{def}}{=} \left\{ \hat{P}_k^{(m)} : k = 1, \ldots, 2^n \right\}, \quad \hat{\mathcal{P}}_d \stackrel{\text{def}}{=} \bigcup_{m=1}^{\infty} B_m. \tag{14}$$

Clearly each B_m is a maximal Boolean algebra. Moreover, Proposition 1 assures us that any two (nontrivial) compatible projections $P, P' \in \hat{\mathcal{P}}_d$ must lie in the same B_m, as well as all the resolutions of the identity in which P, P' figure. Thus compatible projections in $\hat{\mathcal{P}}_d$ only appear in compatible resolutions. Trivially, $\hat{\mathcal{P}}_d$ is closed under compatible products and complements, since each B_m is so closed, and the projections contained in different B_m's (excepting 0 and I) are all incompatible. Finally, since $\{\langle \hat{e}_i^{(1)}\rangle, \langle \hat{e}_i^{(2)}\rangle, \ldots,\}$ is dense in M, it is immediate that $\hat{\mathcal{P}}_d$ generates a dense set of resolutions of the identity, and Theorem 1 is proved. Note, finally, that by specifying from the outset a particular countable dense subset of $U(n)$, all our arguments in the proof can be made constructively. (In particular, the argument for (9) could have been given directly, rather than via a reductio ad absurdem from (10).)

IV. NON-CONTEXTUAL HIDDEN VARIABLES FOR POV MEASURES

We turn, next, to establish Theorem 2. Let $\mathcal{O}$ be the set of all operators on H_n, endowed with the operator norm topology, and let $\mathcal{O}^+$ denote the PO's on H_n, a closed subset of $\mathcal{O}$. Fix, once and for all, some basis $B \subseteq H_n$, and consider the matrix representations of all operators in $\mathcal{O}$ relative to B. Define $\mathcal{Q}_C$ to be the set of all operators with complex rational matrix entries (relative to B), and $\mathcal{Q}_C^+$ to be the subset of positive operators therein. Clearly $\mathcal{Q}_C$ is dense in $\mathcal{O}$. It follows that $\mathcal{Q}_C^+$ is dense in $\mathcal{O}^+$. For consider any $A \geq 0$. Then there is an $X \in \mathcal{O}$ such that $A = X^*X$. Since there is a sequence $\{X_m\}_{m=1}^{\infty} \subseteq \mathcal{Q}_C$ converging to X, $\{X_m^* X_m\}_{m=1}^{\infty}$ is a sequence in $\mathcal{Q}_C^+$ converging to A. Next, define $\mathcal{A}$ to be the subset of positive operators in $\mathcal{Q}_C^+$ whose (complex rational) matrix entries are all *nonzero*. To see that $\mathcal{A}$ is also dense in $\mathcal{O}^+$, it suffices to observe that it is dense in $\mathcal{Q}_C^+$. So consider any $A \in \mathcal{Q}_C^+$, choose any $A' \in \mathcal{A}$, and let $\{t_m\}_{m=1}^{\infty}$ be a sequence of positive rationals tending to 0. Then clearly $A + t_m A' \to A$, each $A + t_m A' \in \mathcal{Q}_C^+$ (since the positive operators form a convex cone, and the rationals a field), and at most n^2 of the operators $\{A + t_m A'\}_{m=1}^{\infty}$ can have a zero matrix entry (since the operators A and A' are fixed).

We now need to establish:

Lemma 2: For any k, $\mathcal{A}$ generates a dense set of k-length PO resolutions of the identity.

Proof: Let $\sum_{i=1}^{k} A_i = I$ be any k-length PO resolution of the identity, and fix $\epsilon > 0$. Choose a rational $r \in (0, \epsilon)$ and set $\delta = r/(5+k) > 0$. Then, since $\mathcal{A}$ is dense in $\mathcal{O}^+$, we may choose k POs $\{A_i'\}_{i=1}^{k} \subseteq \mathcal{A}$ such that $|A_i - A_i'| < \delta$ for all i. With $H = \sum_{i=1}^{k} A_i'$, observe that

$$-\delta I < -|A_i' - A_i| I \leq A_i' - A_i \leq |A_i' - A_i| I < \delta I, \tag{15}$$

which, summed over $i = 1, \ldots, k$, yields

$$-\delta k I < H - I < \delta k I. \tag{16}$$

Next, introduce the new positive operators:

$$A_i'' = (1 + k\delta)^{-1} \left[A_i' + t_i \left((1 + k\delta) I - H \right) \right], \tag{17}$$

for all i. Here the t_i are positive rationals obeying $\sum_{i=1}^{k} t_i = 1$ and $t_i < 2/k$, and are chosen so that all the matrix entries of all the A_i'' are nonzero. Such a choice can always be made, since for each fixed i there are only finitely many choices one can make for t_i (in fact, at most n^2) such that a matrix entry of A_i'' can vanish. Each A_i'' is positive, since $A_i' \geq 0$, $(1 + k\delta) I - H > 0$ (by (16)), and the positive operators form a convex cone. Each A_i'' is also complex rational (hence lies in $\mathcal{Q}_C^+$), since each A_i' and, therefore, H is, as is I (obviously), and all of k, δ, t_i are rational. Thus, $\{A_i''\}_{i=1}^{k} \subseteq \mathcal{A}$. Summing (17) over $i = 1, \ldots, k$ reveals that $\sum_{i=1}^{k} A_i'' = I$. Finally, note that since

$$|H - I| = \inf\{\lambda > 0 : -\lambda I \leq H - I \leq \lambda I\}, \tag{18}$$

(16) entails $|H - I| \leq \delta k$. And, since $\sum_{i=1}^{k} A_i = I$, $|A_i| \leq 1$ for all i. With latter inequalities, the triangle inequality, $t_i < 2/k$, and the inequalities of the previous paragraph, we obtain:

$$\begin{aligned} |A_i - A_i''| \leq & \left| (1 + k\delta)^{-1} \left[A_i' + t_i \left((1 + k\delta) I - H \right) \right] - A_i \right| \\ \leq & (1 + k\delta)^{-1} |A_i' - (1 + k\delta) A_i| + t_i (1 + k\delta)^{-1} |(1 + k\delta) I - H| \\ \leq & (1 + k\delta)^{-1} |A_i' - A_i - k\delta A_i| + t_i (1 + k\delta)^{-1} |I - H + k\delta I| \\ \leq & (1 + k\delta)^{-1} (\delta + k\delta) + t_i (1 + k\delta)^{-1} (\delta k + k\delta) \\ \leq & \delta(5 + k)/(1 + k\delta) < \delta(5 + k) = r < \epsilon \,, \end{aligned} \tag{19}$$

for all i. *QED.*

Now since $\mathcal{A}$ consists only of complex rational POs, it is countable. The set of all finite subsets of a countable set is itself countable. Thus $\mathcal{A}$ can include at most countably many finite PO decompositions of the identity. Let the variable m range over an enumeration of these resolutions, denoting the mth resolution, which will have some length k_m, by $\{A_i^{(m)}\}_{i=1}^{k_m}$. For each m, define the unitary operator U_m to be given by the diagonal matrix $\mathrm{diag}(e^{i\theta_m}, 1, \ldots, 1)$ relative to the basis B, where

$$\sin\theta_m = (\pi/4)^m, \quad \cos\theta_m = (1 - (\pi/4)^{2m})^{1/2} \,, \tag{20}$$

and for definiteness we take the positive square root. For each m, define a new k_m-length PO resolution $\{\hat{A}_i^{(m)}\}_{i=1}^{k_m}$ by

$$\hat{A}_i^{(m)} = U_m A_i^{(m)} U_m^{-1}, \text{ for all } i = 1, \ldots, k_m. \tag{21}$$

Then we have:

Proposition 2: No two of the PO resolutions $\{\hat{A}_i^{(m)}\}_{i=1}^{k_m}$ (for different m) share a common PO.

Proof: Assuming that for some i, j and m, p we have $\hat{A}_i^{(m)} = \hat{A}_j^{(p)}$, we must show that $m = p$. By (21),

$$U_m A_i^{(m)} U_m^{-1} = U_p A_j^{(p)} U_p^{-1}. \tag{22}$$

Let (x_{ab}) and (y_{ab}) be the nonzero complex rational matrix coefficients (relative to B) of $A_i^{(m)}$ and $A_j^{(p)}$, respectively. Using the definition of U_m, (22) entails, in particular, that $x_{12}e^{i\theta_m} = y_{12}e^{i\theta_p}$. Since $x_{12} \neq 0$, we may write:

$$e^{i(\theta_m - \theta_p)} = c, \tag{23}$$

where c is a complex rational. Equating the real parts of (23), one obtains

$$\cos\theta_m \cos\theta_p = r - \sin\theta_m \sin\theta_p, \tag{24}$$

where r $(= \Re(c))$ is rational. Squaring both sides of (24), inserting the expressions (20), and rearranging, yields:

$$2r\left(\frac{\pi}{4}\right)^{m+p} - \left(\frac{\pi}{4}\right)^{2p} - \left(\frac{\pi}{4}\right)^{2m} = r^2 - 1. \tag{25}$$

The transcendentality of π requires that the coefficient of any given power of π in this equation must be zero. This cannot happen unless $m = p$, since if $m \neq p$, then π^{m+p}, π^{2p}, and π^{2m} are three distinct powers of π, and the latter two powers occur with nonzero coefficient. *QED.*

We finish with the argument for Theorem 2. All that remains to show is that the countable collection of finite PO resolutions contained in

$$\hat{\mathcal{A}}_d \stackrel{\mathrm{def}}{=} \bigcup_{m=1}^{\infty} \{\hat{A}_i^{(m)}\}_{i=1}^{k_m} \tag{26}$$

is dense. Fix k and let $\{A_i\}_{i=1}^k$ be an arbitrary k-length resolution. By Lemma 2, we can find k-length resolutions in $\mathcal{A}$ arbitrarily close to $\{A_i\}_{i=1}^k$. Since the unitary operators defined by (20) satisfy $U_m \to I$, by Proposition 2 we can find k-length resolutions in $\hat{\mathcal{A}}_d$ arbitrarily close to any k-length resolution in $\mathcal{A}$. This completes the proof.

V. DISCUSSION

To many physicists, the fact that quantum theory can most elegantly be expressed in a radically non-classical language makes explicit demonstrations of its non-classicality essentially redundant. This paper is addressed not to them, but to those who, like us, are interested in what we can establish for certain on the question. As we have already stressed, we hold no particular brief for non-contextual hidden variable theories. However, our conclusion, in the light of references [16,17] and the constructions above, is that there is no truly compelling argument establishing that non-relativistic quantum mechanics describes classically inexplicable physics. Only when quantum theory and relativity are combined can really compelling arguments be mounted against the possibility of classical simulation, and then — so far as is presently known — only against the particular class of simulations defined by local hidden variable theories.

It might be argued that the proofs and constructions above, though perhaps interesting, establish more than is strictly necessary. While we have defined a hidden variable simulation of quantum theory by constructing a dense subset of the set of projective decompositions with the property that no two compatible projections belong to incompatible resolutions, this last property is not necessary. To give a trivial example, one could also define a simulation on the basis of a dense set of projections exactly one of which belongs to two (or more) incompatible resolutions. Similar comments apply to the analysis for positive operator decompositions.

Our response would simply be that the constructions we give do the job. They are certainly not unique, and there may well be other constructions which differ in interesting ways, or even perhaps define more natural non-contextual hidden variable models.

A related argument that might also be made is that the constructions in references [16,17] already imply the simulability of quantum theory by non-contextual hidden variables, since they each describe a truth valuation t on a dense set in which both truth values occur densely (i.e., for any projection or positive operator mapped to 1 by t, there is another one arbitrarily close to it mapped to 0). One could imagine a model in which the particle, confronted by a measuring apparatus set with a particular precision range, first calculates the approximate probability that it should produce a '1' by evaluating the expectation in its quantum state of some projection randomly chosen from within the precision range, and then goes through some deterministic algorithm to *choose* a (generally different) projection from the range for which it will reveal its pre-determined measurement value. In this picture, it is left to the deterministic algorithm to simulate quantum statistics by choosing projections of value '0' or '1' in the right proportions.

Such models cannot be logically excluded, but they seem to us overly baroque: hidden variable theories in which the effects of the system and apparatus hidden variables can be separated seem to us cleaner and simpler than theories which rely on some conspiratorial interaction between those variables at the point of measurement.

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