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SUPERGAUGE TRANSFORMATIONS IN FOUR DIMENSIONS

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A B S T R A C T

Supergauge transformations are defined in four space - time dimensions. Their commutators are shown to generate γ_5 transformations and conformal transformations. Various kinds of multiplets are described and examples of their combinations to new representations are given. The relevance of supergauge transformations for Lagrangian field theory is explained. Finally, the abstract group theoretic structure is discussed.

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1. INTRODUCTION

Supergauge transformations have been studied until now in dual models, especially in their formulation as two-dimensional field theories [1,2,3,4]. They transform scalar (in general tensor) fields into spinors and boson fields into fermion fields. This is possible because the parameters of the supergauge transformation are themselves totally anticommuting spinors. The commutator of two infinitesimal supergauge transformations is a conformal transformation in two dimensions. Invariance under supergauge transformations is closely connected to the absence of ghost states in the two dimensional field theory.

It is natural to ask whether one can define supergauge transformations in four dimensional space-time. In this paper we show that this is indeed possible, although the generalization from two to four dimensions is not completely straight forward and presents some interesting new features. In four dimensions the commutator of two infinitesimal supergauge transformations turns out to be a combination of a conformal transformation and a γ_5 transformation. Supergauge transformations can be represented on multiplets of fields, a given multiplet containing some tensors and some spinors. Examples of such representations are given in section 3.

A representation is characterized not only by a given multiplet of fields but also by a weight, much as an ordinary tensor representation corresponds to a tensor density having a given index structure as well as a certain weight. Representations for supergauges can be combined into other representations, as exemplified in section 3. Using the multiplet of a representation one can construct an invariant Lagrangian. More precisely, the Lagrangian transforms by a total derivative as one of the members of a supergauge density; the four-dimensional action integral is invariant. Two examples of this are shown in section 4. The examples given in this paper are the simplest cases of a "tensor" calculus which generalizes to supergauge transformations the ordinary tensor calculus of coordinate transformations.

From any particular representation one can abstract the "group" structure containing the supergauge transformations, the conformal transformations and the γ_5 transformations. Actually the corresponding algebraic structure is not a group or a Lie algebra in the conventional sense, since the parameters of a supergauge transformation are not c-numbers but rather completely anticommuting quantities belonging to a Grassmann algebra. Nevertheless, one can find the parameter composition law and verify that it satisfies the Jacobi identity. This is done in section 5.

The Lagrangian example given in this paper is a free field theory. Nevertheless it is possible, by combining representations, to construct interacting field theories invariant under supergauge transformations and consequently under conformal and γ_5 transformations. Supergauge transformations should prove a useful tool for the study of theories with massless particles or in the approximation in which the mass can be neglected. They may also provide a natural way for the formulation of higher internal symmetries linking mesons and baryons. We hope to come back to these questions in a later publication.

In two dimensions it is possible to define |5| generalized supergauge transformations which have as commutator general coordinate transformations, rather than conformal transformations. It is likely that such generalized supergauge transformations exist also in four-dimensional space time. Their commutators would generate an algebra containing that of general (Einstein) coordinate transformations. We are planning to come back at some later time to this very interesting question.

2. CONDITIONS ON THE PARAMETERS

It is convenient to study separately in this section the properties of the infinitesimal parameters of a supergauge transformation. As we shall see, they will be Majorana spinors $\alpha(x)$ subject to the condition

$$\left(\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu - \frac{1}{2}\eta_{\mu\nu}\gamma^\lambda \partial_\lambda\right)\alpha = 0 \tag{1}$$

This equation resembles the condition on the infinitesimal parameters ξ_μ of a conformal transformation

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{1}{2}\eta_{\mu\nu}\partial_\lambda \xi^\lambda = 0 \tag{2}$$

and indeed, if α_1 and α_2 satisfy (1), then

$$\xi_\mu = 2i\bar{\alpha}_1 \gamma_\mu \alpha_2 \tag{3}$$

satisfies (2). It is well known that (2) can be solved explicitly; the same is true for (1).

First multiply (1) with γ^μ obtaining

$$\partial_\mu \alpha = \frac{1}{4}\gamma_\mu \gamma^\lambda \partial_\lambda \alpha \quad . \tag{4}$$

From this, multiplying by ∂^μ , it follows

$$\Box \alpha = 0 .$$

Then, multiplying (4) with $\gamma^\nu \partial_\nu$, we find that

$$\partial_\mu \gamma^\nu \partial_\nu \alpha = 0 .$$

Therefore, always from (4),

$$\partial_\mu \partial_\nu \alpha = 0 ,$$

α is at most linear in x . Imposing (4) once more, it follows that

$$\alpha = \alpha^{(0)} + \gamma_\mu x^\mu \alpha^{(1)} \qquad (5)$$

where α_o and α_1 are x independent spinors.
The forms (1) and (4) of the condition on α are equivalent. It is easy to see that a third equivalent form, sometimes more convenient, is

$$\gamma^\mu \gamma^\nu \partial_\mu \alpha + \tfrac{1}{2} \gamma^\nu \gamma^\mu \partial_\mu \alpha = 0 . \qquad (6)$$

Observe that, if one assumes that α_1 and α_2 in (3) both have the form (5), ξ^μ takes the form, well known for a con-

formal transformation,

$$\xi_\mu = c_\mu + \omega_{\mu\nu} x^\nu + \varepsilon x_\mu + a_\mu x^2 - 2 x_\mu \alpha\cdot x$$

where

$$c_\mu = 2i\,\bar{\alpha}_1^{(0)} \gamma_\mu \alpha_2^{(0)}$$

$$\omega_{\mu\nu} = -\omega_{\nu\mu} = i\bar{\alpha}_1^{(0)}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\alpha_2^{(1)} - i\bar{\alpha}_2^{(0)}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\alpha_1^{(1)}$$

$$\varepsilon = 2i\left(\bar{\alpha}_1^{(0)}\alpha_2^{(1)} - \bar{\alpha}_2^{(0)}\alpha_1^{(1)}\right)$$

$$a_\mu = 2i\,\bar{\alpha}_1^{(1)} \gamma_\mu \alpha_2^{(1)}$$

For later use we calculate the expression

$$\eta = i\partial_\mu \bar{\alpha}_1 \gamma_5 \gamma^\mu \alpha_2 - i\partial_\mu \bar{\alpha}_2 \gamma_5 \gamma^\mu \alpha_1 \quad . \tag{7}$$

Using the explicit form for α_1 and α_2 one finds

$$\eta = 4i\left(\bar{\alpha}_1^{(1)}\gamma_5\alpha_2^{(0)} - \bar{\alpha}_2^{(1)}\gamma_5\alpha_1^{(0)}\right).$$

Therefore η is x-independent.

3. SUPERGAUGE TRANSFORMATIONS

Consider a multiplet consisting of a Majorana spinor ψ and four scalar fields A,B,F and G. Let us define an infinitesimal supergauge transformation by

$$\delta A = i\bar{\alpha}\psi$$
$$\delta B = i\bar{\alpha}\gamma_5\psi$$
$$\delta\psi = \partial_\mu(A-\gamma_5 B)\gamma^\mu\alpha + n(A-\gamma_5 B)\gamma^\mu\partial_\mu\alpha + F\alpha + G\gamma_5\alpha \qquad (8)$$
$$\delta F = i\bar{\alpha}\gamma^\mu\partial_\mu\psi + i(n-\tfrac{1}{2})\partial_\mu\bar{\alpha}\gamma^\mu\psi$$
$$\delta G = i\bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\psi + i(n-\tfrac{1}{2})\partial_\mu\bar{\alpha}\gamma_5\gamma^\mu\psi$$

where the parameter $\alpha(x)$ is an infinitesimal spinor which anti-commutes with itself and with ψ and commutes with the other fields. Furthermore α satisfies the differential equation

discussed in the previous section. The number n is arbitrary (it need not be an integer) and gives the weight of the multiplet. We say of a multiplet transforming as in (8) that it belongs to a scalar representation of weight n.

The supergauge transformations generate a closed algebraic structure, similar to a Lie algebra. To see this, let us evaluate the commutator of two infinitesimal supergauge transformations δ_1 and δ_2, of parameters α_1 and α_2. Starting with the scalar field A we have

$$\delta_2 \delta_1 A = i\bar{\alpha}_1(\delta_2\psi) = i\bar{\alpha}_1\Big(\partial_\mu(A-\gamma_5 B)\gamma^\mu\alpha_2 + n(A-\gamma_5 B)\gamma^\mu\partial_\mu\alpha_2 + F\alpha_2 + G\gamma_5\alpha_2\Big)$$

If we use the relations

$$\bar{\alpha}_1\alpha_2 = \bar{\alpha}_2\alpha_1 \qquad \bar{\alpha}_1\gamma_5\alpha_2 = \bar{\alpha}_2\gamma_5\alpha_1$$

we see that the fields F and G drop out in the commutator $(\delta_2\,\delta_1 - \delta_1\,\delta_2)A$. On the other hand, using

$$\bar{\alpha}_1 \gamma_\mu \alpha_2 = - \bar{\alpha}_2 \gamma_\mu \alpha_1$$

$$\bar{\alpha}_1 \gamma^\mu \partial_\mu \alpha_2 - \bar{\alpha}_2 \gamma^\mu \partial_\mu \alpha_1 = \partial_\mu (\bar{\alpha}_1 \gamma^\mu \alpha_2)$$

we obtain

$$(\delta_2 \delta_1 - \delta_1 \delta_2) A = \xi^\mu \partial_\mu A + \frac{n}{2} \partial_\mu \xi^\mu A + n \eta B$$

where, as in the previous section,

$$\xi_\mu = 2i \bar{\alpha}_1 \gamma_\mu \alpha_2$$

$$\eta = i \partial_\mu \bar{\alpha}_1 \gamma_5 \gamma^\mu \alpha_2 - \partial_\mu \bar{\alpha}_2 \gamma_5 \gamma^\mu \alpha_1 .$$

For the other fields the result is

$$[\delta_2, \delta_1] B = \xi^\mu \partial_\mu B + \frac{n}{2} \partial_\mu \xi^\mu B - n \eta A$$

$$[\delta_2, \delta_1] \psi = \xi^\mu \partial_\mu \psi + \left(\frac{n}{2} + \frac{1}{8}\right) \partial_\mu \xi^\mu \psi +$$

$$+ \frac{1}{4} (\partial_\mu \xi_\nu - \partial_\nu \xi_\mu) \Sigma^{\mu\nu} \psi - \left(\frac{3}{4} - n\right) \eta \gamma_5 \psi$$

$$[\delta_2, \delta_1] F = \xi^\mu \partial_\mu F + \left(\frac{n}{2} + \frac{1}{4}\right) \partial_\mu \xi^\mu F + \left(\frac{3}{2} - n\right) \eta G$$

$$[\delta_2, \delta_1] G = \xi^\mu \partial_\mu G + \left(\frac{n}{2} + \frac{1}{4}\right) \partial_\mu \xi^\mu G - \left(\frac{3}{2} - n\right) \eta F,$$

where

$$\Sigma^{\mu\nu} = \frac{1}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu).$$

Remember that η is independent of x.

The commutator of two supergauge transformations is a conformal transformation combined with a γ_5 transformation on ψ and a mixing of A with B and of F with G . It is also easy to verify that the commutator of a conformal transformation with a supergauge transformation or of a γ_5 transformation with a supergauge transformation are again supergauge transformations. In this sense the algebra closes.

To explain further how the above result for the commutator emerges, we indicate here as an example the evaluation of some terms of the commutator on ψ . One has

$$\delta_2\delta_1\psi = i\partial_\mu(\bar\alpha_2\psi)\gamma^\mu\alpha_1 - i\partial_\mu(\bar\alpha_2\gamma_5\psi)\gamma_5\gamma^\mu\alpha_1 +$$

$$+ in(\bar\alpha_2\psi)\gamma^\mu\partial_\mu\alpha_1 - in(\bar\alpha_2\gamma_5\psi)\gamma_5\gamma^\mu\partial_\mu\alpha_1 +$$

$$+ i(\bar\alpha_2\gamma^\mu\partial_\mu\psi)\alpha_1 + i(n-\tfrac{1}{2})(\partial_\mu\bar\alpha_2\gamma^\mu\psi)\alpha_1 +$$

$$+ i(\bar\alpha_2\gamma_5\gamma^\mu\partial_\mu\psi)\gamma_5\alpha_1 + i(n-\tfrac{1}{2})(\partial_\mu\bar\alpha_1\gamma_5\gamma^\mu\psi)\gamma_5\alpha_1$$

Let us consider separately the terms with derivatives of ψ. They are

$$i(\bar{\alpha}_2 \partial_\mu \psi)\gamma^\mu \alpha_1 - i\bar{\alpha}_2 \gamma_5 \partial_\mu \psi \gamma_5 \gamma^\mu \alpha_1 +$$
$$+ i(\bar{\alpha}_2 \gamma^\mu \partial_\mu \psi)\alpha_1 + i(\bar{\alpha}_2 \gamma_5 \gamma^\mu \partial_\mu \psi)\gamma_5 \alpha_1$$

Using the rearrangement formula of the appendix, the first of these terms can be transformed as

$$i(\bar{\alpha}_2 \partial_\mu \psi)\gamma^\mu \alpha_1 = -\frac{i}{4}(\bar{\alpha}_2 \gamma_A \alpha_1)\gamma^\mu \gamma^A \partial_\mu \psi$$

where we imply summation over all sixteen matrices γ_A. Similarly, for the other terms,

$$-i(\bar{\alpha}_2 \gamma_5 \partial_\mu \psi)\gamma_5 \gamma^\mu \alpha_1 = \frac{i}{4}(\bar{\alpha}_2 \gamma_A \alpha_1)\gamma_5 \gamma^\mu \gamma^A \gamma_5 \partial_\mu \psi$$

$$i(\bar{\alpha}_2 \gamma^\mu \partial_\mu \psi)\alpha_1 = -\frac{i}{4}(\bar{\alpha}_2 \gamma_A \alpha_1)\gamma^A \gamma^\mu \partial_\mu \psi$$

$$i(\bar{\alpha}_2 \gamma_5 \gamma^\mu \partial_\mu \psi)\gamma_5 \alpha_1 = -\frac{i}{4}(\bar{\alpha}_2 \gamma_A \alpha_1)\gamma_5 \gamma^A \gamma_5 \gamma^\mu \partial_\mu \psi$$

For the commutator $(\delta_2 \delta_1 - \delta_1 \delta_2)\psi$, in all these terms $\bar{\alpha}_2 \gamma_A \alpha_1$ is replaced by $\bar{\alpha}_2(\gamma_A - \tilde{\gamma}_A)\alpha_1$. Therefore only $\gamma_A = \gamma_\nu$ and $\gamma_A = \gamma_\nu \gamma_\sigma$ $(\nu < \sigma)$ survive. If we take the

latter, the four terms cancel each other. If we take the former we obtain finally the contribution to the commutator

$$-i(\bar{\alpha}_2\gamma_\nu\alpha_1)(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)\partial_\mu\psi = 2i\bar{\alpha}_1\gamma^\mu\alpha_2\partial_\mu\psi.$$

The other terms in the commutator on ψ are evaluated in a similar way. In the rest of this paper we shall not give all the details of the calculations, which are often long and rather tedious. They are based on the well known properties of the γ matrices and on the rearrangement formula collected in the appendix.

Two supergauge representations can be combined to a third. Let the fields A_1, B_1, ψ_1, F_1, G_1 belong to a scalar representation of weight n_1, as described above, and let the fields A_2, B_2, ψ_2, F_2, G_2 belong to a scalar representation of weight n_2. The new multiplet A, B, ψ, F, G defined by

$$\begin{aligned}
A &= A_1A_2 - B_1B_2 \\
B &= A_1B_2 + B_1A_2 \\
\psi &= (A_1 - \gamma_5 B_1)\psi_2 + (A_2 - \gamma_5 B_2)\psi_1 \\
F &= F_1A_2 + F_2A_1 + G_1B_2 + G_2B_1 - i\bar{\psi}_1\psi_2 \qquad (9) \\
G &= G_1A_2 + G_2A_1 - F_1B_2 - F_2B_1 + i\bar{\psi}_1\gamma_5\psi_2
\end{aligned}$$

belongs to a scalar representation of weight $n_1 + n_2$, which means that it transforms like (8) with $n = n_1 + n_2$. This can be verified directly. For instance

$$\delta A = A_1 i\bar{\alpha}\psi_2 + A_2 i\bar{\alpha}\psi_1 - B_1 i\bar{\alpha}\gamma_5\psi_2 - B_2 i\bar{\alpha}\gamma_5\psi_1 =$$

$$= i\bar{\alpha}(A_1\psi_2 + A_2\psi_1 - B_1\gamma_5\psi_2 - B_2\gamma_5\psi_1) = i\bar{\alpha}\psi$$

Similarly for the other fields. For the spinor field one must use the rearrangement formula described in the appendix. The combination of two scalar representations into a third scalar representation just described gives only the simplest example of a generalized tensor calculus for supergauge transformations, the theory of which we have not yet fully developed. It is clearly the main tool for the construction of invariant interactions.

Before closing this section we give another example of representation. Let us consider a multiplet consisting of four scalar fields D,C,M,N, of the vector field v_μ and of the two spinor fields χ and λ . We call it a vector multiplet, because of the presence in it of the vector field v_μ . We shall say that this vector multiplet transforms according to a vector representation of weight n if, under a supergauge transformation,

$$
\begin{aligned}
\delta D &= i\bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\lambda + \tfrac{i}{2}(n-1)\partial_\mu\bar{\alpha}\gamma_5\gamma^\mu\lambda + \\
&\quad + i(n-1)\partial^\mu\bar{\alpha}\gamma_5\partial_\mu\chi \\
\delta C &= i\bar{\alpha}\gamma_5\chi \\
\delta M &= i\bar{\alpha}\lambda + i\bar{\alpha}\gamma^\mu\partial_\mu\chi + \tfrac{i}{4}(n-3)\partial_\mu\bar{\alpha}\gamma^\mu\chi \\
\delta N &= i\bar{\alpha}\gamma_5\lambda + i\bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\chi + \tfrac{i}{4}(n-3)\partial_\mu\bar{\alpha}\gamma_5\gamma^\mu\chi \\
\delta\theta_\mu &= i\bar{\alpha}\gamma_\mu\lambda + i\bar{\alpha}\partial_\mu\chi + in\partial_\mu\bar{\alpha}\chi
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\delta\chi &= \gamma^\mu\theta_\mu\alpha - \partial_\mu C\gamma_5\gamma^\mu\alpha - \tfrac{1}{2}(n-1)C\gamma_5\gamma^\mu\partial_\mu\alpha + \\
&\quad + (M+\gamma_5 N)\alpha \\
\delta\lambda &= -\tfrac{1}{2}(\partial_\mu\theta_\nu - \partial_\nu\theta_\mu)\gamma^\mu\gamma^\nu\alpha + D\gamma_5\alpha + (n-1)\theta^\mu\partial_\mu\alpha + \\
&\quad + \tfrac{1}{2}(n-1)(M-\gamma_5 N)\gamma^\mu\partial_\mu\alpha - (n-1)\partial^\mu C\gamma_5\partial_\mu\alpha
\end{aligned}
$$

Just as for the scalar representation, also for the vector representation one can calculate the commutator of two infinitesimal supergauges. One finds

$$[\delta_2,\delta_1]D = \xi^\mu\partial_\mu D + \tfrac{1}{4}(n+1)\partial_\mu\xi^\mu D + \tfrac{1}{2}(n-1)\Box\xi^\mu\partial_\mu C$$

$$[\delta_2,\delta_1]C = \xi^\mu\partial_\mu C + \tfrac{1}{4}(n-1)\partial_\mu\xi^\mu C$$

$$[\delta_2,\delta_1]M = \xi^\mu\partial_\mu M + \tfrac{n}{4}\partial_\mu\xi^\mu M + \tfrac{3}{2}\eta N$$

$$[\delta_2,\delta_1]N = \xi^\mu\partial_\mu N + \tfrac{n}{4}\partial_\mu\xi^\mu N - \tfrac{3}{2}\eta M$$

$$[\delta_2,\delta_1]v_\nu = \xi^\mu\partial_\mu v_\nu + \partial_\nu\xi^\mu\omega_\mu + \tfrac{1}{4}(n-1)\partial_\mu\xi^\mu v_\nu$$

$$[\delta_2,\delta_1]\chi = \xi^\mu\partial_\mu\chi + \tfrac{1}{4}(n-\tfrac{1}{2})\partial_\mu\xi^\mu\chi +$$
$$+\tfrac{1}{4}(\partial_\mu\xi_\nu - \partial_\nu\xi_\mu)\Sigma^{\mu\nu}\chi - \tfrac{3}{4}\eta\gamma_5\chi \qquad (11)$$

$$[\delta_2,\delta_1]\lambda = \xi^\mu\partial_\mu\lambda + \tfrac{1}{4}(n+\tfrac{1}{2})\partial_\mu\xi^\mu\lambda +$$
$$+\tfrac{1}{4}(\partial_\mu\xi_\nu - \partial_\nu\xi_\mu)\Sigma^{\mu\nu}\lambda + \tfrac{3}{4}\eta\gamma_5\lambda$$
$$+\tfrac{1}{4}(n-1)\Box\xi^\mu\gamma_\mu\chi$$

Observe that, except for $n = 1$, the transformation law for D and for λ is not simply that of a conformal transformation, but has additional terms containing $\Box\varphi^{\mu}$. In the next section we shall see that, for the case of D, this corresponds to the very well known fact that the free Lagrangian for a scalar field does not transform simply as a density under conformal transformations. The additional term corresponds exactly to the additional term in δD above. These additional terms do not alter the algebraic structure described in section 5.

4. TWO LAGRANGIAN EXAMPLES

Consider a scalar multiplet of weight $\frac{1}{2}$. It consists of fields A,B,ψ,F,G transforming as in (8) with $n = \frac{1}{2}$. We claim that the Lagrangian

$$L = -\frac{1}{2}\partial_\mu A\,\partial^\mu A - \frac{1}{2}\partial_\mu B\,\partial^\mu B - \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{1}{2}(F^2+G^2) \qquad (12)$$

gives rise to an invariant action integral. The Lagrangian is not itself invariant. Rather, it is one of the members of a multiplet and it changes by a total derivative under a super-gauge transformation. Observe that the above Lagrangian is essentially the sum of the free Lagrangians for the fields A,B and ψ . The variational equations for the fields F and G are simply $F = o$ and $G = o$. Nevertheless, the fields F and G are essential for the transformation properties and especially for the closing of the commutator algebra described in section 3.

The multiplet to which the above Lagrangian (12) belongs is simply a vector multiplet of weight 3. Indeed one can verify that, when the fields A,B,ψ,F and G transform according to (8) with $n = \frac{1}{2}$ (scalar representation of weight $\frac{1}{2}$), then the fields D,C etc. defined by

$$
\begin{aligned}
\mathcal{D} &= 2L \\
C &= \tfrac{1}{2}(A^2 + B^2) \\
M &= AG + BF \\
N &= BG - AF \\
v_\mu &= B\partial_\mu A - A\partial_\mu B - \tfrac{i}{2}\bar{\psi}\gamma_5\gamma_\mu\psi \\
\chi &= B - \gamma_5 A \\
\lambda &= \{G + \gamma_5 F - \gamma^\mu\partial_\mu(B - \gamma_5 A)\}\psi
\end{aligned}
\tag{13}
$$

transform according to a vector representation of weight 3 (given by (10) with n = 3). The first of these fields is twice the Lagrangian. It follows, in particular, that

$$
\delta L = \frac{i}{2}\partial_\mu\left(\bar{\alpha}\gamma_5\gamma^\mu\lambda + 2\partial^\mu\bar{\alpha}\gamma_5\chi\right)
$$

and the four-dimensional action integral is invariant. Here we have used the fact that second derivatives of α vanish. The above formulas (13) give an example of how, starting from a representation, one can construct a new one. In (13) derivatives of the fields occurr, while in our earlier example (9) (in which the two representations could be taken to be identical) the fields occurred without derivatives.

Having treated the Lagrangian for a scalar multiplet, one may ask whether one can construct a Lagrangian for a vector multiplet. The answer is affirmative. Let the fields D, C, M, N, v_μ, χ and λ belong to a vector multiplet of weight $n = 1$ (representation given by (10) with $n = 1$). Then the fields A, B etc. defined by

$$
\begin{aligned}
A &= \frac{i}{2}\bar{\lambda}\lambda \\
B &= \frac{i}{2}\bar{\lambda}\gamma_5\lambda \\
\psi &= D\gamma_5\lambda + \frac{1}{2}v_{\mu\nu}\gamma^\mu\gamma^\nu\lambda \\
F &= \frac{1}{2}v_{\mu\nu}v^{\mu\nu} + i\bar{\lambda}\gamma^\mu\partial_\mu\lambda - D^2 \\
G &= -\frac{1}{4}\varepsilon^{\mu\nu\alpha\beta}v_{\mu\nu}v_{\alpha\beta} - i\bar{\lambda}\gamma_5\gamma^\mu\partial_\mu\lambda
\end{aligned}
\qquad (14)
$$

transform according to a scalar representation of weight $\frac{3}{2}$ (given by (8) for $n = \frac{3}{2}$) . Here

$$v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu \ , \quad \varepsilon_{0123} = 1$$

Now

$$\mathcal{L} = -\frac{1}{2}F = -\frac{1}{4} v_{\mu\nu} v^{\mu\nu} - \frac{i}{2}\bar{\lambda}\gamma^\mu \partial_\mu \lambda + \frac{1}{2} D^2$$

is essentially the sum of the free Lagrangians for the vector field v_μ and for the spinor field λ , the extra term $\frac{1}{2} D^2$ giving as variational equation simply $D = 0$. According to (8) with $n = \frac{3}{2}$,

$$\delta \mathcal{L} = -\frac{i}{2} \partial_\mu (\bar{\alpha}\gamma^\mu \lambda)$$

and the action integral is invariant. Observe that in (14) the other fields of the vector multiplet, namely C,N,M and χ , do not occur. This can happen because, for $n = 1$, none of them enters in the transformation laws of λ and D , and χ enters in that of v_μ but only in a gauge transformation. In fact, when $n = 1$, one can restrict the vector multiplet to the fields $v_{\mu\nu}$, λ and D , drop the additional fields, and write the transformation law for the restricted vector multiplet simply as

$$\delta D = i\,\bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\lambda$$

$$\delta v_{\mu\nu} = i\partial_\mu(\bar{\alpha}\gamma_\nu\lambda) - i\partial_\nu(\bar{\alpha}\gamma_\mu\lambda)$$

$$\delta\lambda = -\tfrac{1}{2}v_{\mu\nu}\gamma^\mu\gamma^\nu\alpha + D\gamma_5\alpha\ .$$

Now, of course, one must impose the restriction

$$\partial_\sigma v_{\mu\nu} + \partial_\mu v_{\nu\sigma} + \partial_\nu v_{\sigma\mu} = 0,$$

which, however, is respected by the supergauge transformation. For $n \neq 1$, on the other hand, the other fields of the vector multiplet are necessary to give the correct definition of the vector representation and, in particular, for the closing of the commutator algebra.

The two Lagrangian examples given in this section show that Lagrangians giving rise to action integrals invariant under supergauge transformations can belong to different representations. Here we found that the Lagrangian for a scalar multiplet is a member of a vector multiplet while the Lagrangian

for a vector multiplet is a member of a scalar multiplet. In both cases the weight is such that the Lagrangian transforms by the addition of a total derivative. Presumably the same applies in the case of invariant interactions, and they also can belong to different types of multiplets.

5. ALGEBRAIC STRUCTURE

In this section we abstract from the transformation law (8) for a scalar multiplet the composition law of the parameters. This is the analogue of finding the structure constants of a Lie algebra, except that our algebraic structure is a kind of generalized Lie algebra, because of the anti-commutation property of the parameters α .

Under a combined supergauge, conformal and γ_5 transformation of parameters α, ξ_μ, η, the particular members A,B and ψ of the multiplet, for instance, transform as follows

$$\delta A = i\bar{\alpha}\psi + \xi^\mu \partial_\mu A + \frac{n}{2}\partial_\mu \xi^\mu A + n\eta B$$

$$\delta B = i\bar{\alpha}\gamma_5\psi + \xi^\mu \partial_\mu B + \frac{n}{2}\partial_\mu \xi^\mu B - n\eta A$$

$$\delta\psi = \partial_\mu (A - \gamma_5 B)\gamma^\mu \alpha + n(A - \gamma_5 B)\gamma^\mu \partial_\mu \alpha +$$

$$+ F\alpha + G\gamma_5\alpha + \xi^\mu \partial_\mu \psi + \left(\frac{n}{2} + \frac{1}{8}\right)\partial_\mu \xi^\mu \psi +$$

$$+ \frac{1}{4}(\partial_\mu \xi_\nu - \partial_\nu \xi_\mu)\Sigma^{\mu\nu}\psi - \left(\frac{3}{4} - n\right)\eta\gamma_5\psi$$

Here the commuting parameters ξ_μ and η are additional parameters, independent of α. While η is x-independent, α and ξ_μ satisfy respectively (1) and (2). We can now evaluate the commutator of two such transformations, the first δ_1 of parameters α_1, $\xi_1{}^\mu$, η_1, and the second δ_2 of parameters α_2, $\xi_2{}^\mu$, η_2, for instance on the field A. One sees easily that the commutator $[\delta_2, \delta_1]$ is again a transformation of the same kind, with parameters α, ξ, η given by

$$\alpha = \xi_1^\mu \partial_\mu \alpha_2 - \frac{1}{8} \partial_\mu \xi_1^\mu \alpha_2 + \frac{1}{4} (\partial_\mu \xi_{1\nu} - \partial_\nu \xi_{1\mu}) \Sigma^{\mu\nu} \alpha_2$$

$$+ \frac{3}{4} \eta_1 \gamma_5 \alpha_2 - (1 \text{ and } 2 \text{ exchanged})$$

$$\xi^\nu = \xi_1^\mu \partial_\mu \xi_2^\nu - \xi_2^\mu \partial_\mu \xi_1^\nu + 2i \bar{\alpha}_1 \gamma^\nu \alpha_2 \qquad (15)$$

$$\eta = i \partial_\mu \bar{\alpha}_1 \gamma_5 \gamma^\mu \alpha_2 - i \partial_\mu \bar{\alpha}_2 \gamma_5 \gamma^\mu \alpha_1$$

This composition law for the parameters defines the algebraic structure and is independent of the particular representation from which we have abstracted it. We could have used, for in-

stance, the vector representation (12) described in section 3.

Clearly, if one exchanges α_1, ξ_1^μ, η_1 with $\alpha_2, \xi_2^\mu, \eta_2$, α, ξ^μ, η change sign. A further condition which must be satisfied is the Jacobi identity for the composition of three transformations. For instance, if we define

$$\eta_{3,12} = i\partial_\mu \bar{\alpha}_3 \gamma_5 \gamma^\mu \alpha - i\partial_\mu \bar{\alpha} \gamma_5 \gamma^\mu \alpha_3$$

with α given by (15), it must be identically

$$\eta_{3,12} + \eta_{1,23} + \eta_{2,31} = 0$$

The same relation must be valid for $\xi^\mu_{3,12}$ and $\alpha_{3,12}$, defined in an analogous manner. The Jacobi identity can naturally be expected to be satisfied, since we have derived the composition law from a particular representation, but we have verified it directly, again using the formulas collected in the appendix.

The fact that the conformal algebra in four dimensions can be extended to the algebraic structure described above, including γ_5 and supergauge transformations, and that the entire algebraic structure is generated by the supergauge transformations, does not appear to have been realized before. It is interesting in itself, irrespective of the field theoretic applications we have in mind.

6. APPENDIX

We use a Majorana representation. The four γ matrices have real elements and satisfy

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}$$

with $\eta_{oo} = -1$. The sixteen basic matrices γ_A (A= 1,2,...16) are 1, $\gamma_5 = \gamma_o \gamma_1 \gamma_2 \gamma_3$, γ_μ, $\gamma_5 \gamma_\mu$, $\gamma_\mu \gamma_\nu$ $(\mu < \nu)$. They are all real and $\gamma_5{}^2 = -1$, $\gamma_5{}^T = -\gamma_5$, where T denotes the ordinary transposed of a matrix.

For any four by four matrix Γ define the adjoint

$$\tilde{\Gamma} = -\gamma_o \Gamma^T \gamma_o$$

Then, for any two hermitian anticommuting spinors α_1 and α_2,

$$\bar{\alpha}_1 \Gamma \alpha_2 = \bar{\alpha}_2 \tilde{\Gamma} \alpha_1 \quad , \quad \bar{\alpha} = \alpha \gamma^o$$

Now $\tilde{\gamma}_A = \gamma_A$ for 1, γ_5, $\gamma_5 \gamma_\mu$, while $\tilde{\gamma}_A = -\gamma_A$ for γ_μ, $\gamma_\mu \gamma_\nu$ $(\mu < \nu)$.

The sixteen matrices γ_A have squares equal to ± 1. For any γ_A define γ^A so that $\gamma_A \gamma^A = +1$ (no summation).

The following rearrangement formula is then valid

$$(\bar{\alpha}_1 \psi)\alpha_2 = -\frac{1}{4}\sum_A (\bar{\alpha}_1 \gamma_A \alpha_2)\gamma^A \psi ,$$

where α_1, α_2 and ψ are any three spinors. The minus sign comes from the anticommutation property of the spinors.

The formulas collected in this appendix are sufficient to derive all the results given in the text of this paper.

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