# Time-Space Noncommutative Abelian Solitons 

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#### Abstract

We demonstrate the construction of solitons for a time-space Moyal-deformed integrable $\mathrm{U}(n)$ sigma model (the Ward model) in $2+1$ dimensions. These solitons cannot travel parallel to the noncommutative spatial direction. For the U(1) case, the rank-one single-soliton configuration is constructed explicitly and is singular in the commutative limit. The projection to $1+1$ dimensions reduces it to a noncommutative instanton-like configuration. The latter is governed by a new integrable equation, which describes a Moyal-deformed sigma model with a particular Euclidean metric and a magnetic field.


## 1 Introduction and results

According to Ward's conjecture [1] all integrable equations can be obtained by dimensional and algebraic reduction of the self-dual Yang-Mills (SDYM) equations in four dimensions. This supposition extends to the Moyal-Weyl deformation of integrable systems [2]. Indeed, maximal integrability seems to be guaranteed only for those noncommutative equations which descend from SDYM theories on noncommutative $\mathbb{R}^{4,0}$ or $\mathbb{R}^{2,2}[3]$.

When the mother theory is non-abelian, the solitonic solutions of the reduced model connect smoothly to their undeformed cousins in the commutative limit. If one starts with the deformed U(1) SDYM equations, however, this will no longer be the case, because the latter trivializes in the commutative limit. Instead, noncommutative solitons stemming from an abelian mother theory must have a singular commutative limit, analogous to the celebrated noncommutative abelian instantons [4.

A known example of this phenomenon occurs for the $2+1$ dimensional Yang-Mills-Higgs system whose solitons are provided by a modified sigma model (the Ward model 5]). For the space-space noncommutative Ward model, multi-solitons have been constructed and studied [6] by passing to the operator formalism using the Moyal-Weyl map. The subclass of static abelian solitons solves the noncommutative standard sigma-model equation in $2+0$ dimensions [7]. For the timespace noncommutative case, much less is known, partially due to the lack of an apparent operator formalism. For instance, the reduction of the $\mathrm{U}(2)$ Ward model to various noncommutative sineGordon systems in $1+1$ dimensions has been investigated [3]. For other examples and further references, see e.g. [8]. However, the $\mathrm{U}(1)$ case remains illusive so far.

Therefore, the question arises as to whether there exist time-space noncommutative integrable equations which stem from the abelian SDYM theory on the Moyal-deformed $\mathbb{R}^{2,2}$. To proceed, one may step down to the $2+1$ dimensional time-space noncommutative $\mathrm{U}(1)$ Ward model and look there for specific classical solutions which have a chance to descend to noncommutative abelian solitons in $1+1$ dimensions. Equivalently, one may search for an integrable dimensional reduction of the deformed abelian Ward model's equation of motion, in order to obtain an integrable system in two dimensions featuring time-space noncommutative solitons with a singular commutative limit. In this letter, we will follow this strategy to construct the desired $1+1$ dimensional integrable system.

We summarize the organization and results of the letter as follows. In the next section, we give a brief review of the Ward model. Next, we demonstrate how to construct multi-center solutions of the time-space noncommutative $\mathrm{U}(n)$ Ward model by employing a linear system and the socalled dressing method [9]. Important to our construction is the observation that, despite the fact that time is noncommuting, a specific co-moving coordinate system can be introduced such that the coordinate commutators form the standard Heisenberg algebra, and hence a Fock-space construction is possible. The rank-one single-soliton configuration is written down explicitly, and its properties are discussed. In particular, we find that the construction degenerates when the soliton moves in the deformed spatial direction. We remark that the Ward soliton satisfies a BPStype equation and enjoys the standard shape invariance which implies a time-like translational symmetry. Therefore, it depends only on two of the three co-moving coordinates. This fact allows us to perform a dimensional reduction in the symmetry direction to obtain a $1+1$ dimensional noncommutative system. On the Ward soliton, this reduction takes a twisted form and produces an instanton-like configuration in the Moyal-deformed two-dimensional Minkowski space. The corresponding novel integrable equation of motion employs a Euclidean metric although we start out in Minkowski space. This flip of signature is a generic feature of the integrable reduction in the one-soliton sector.

## 2 The Ward model

The $\mathrm{U}(n)$ Yang-Mills-Higgs equations in $2+1$ dimensions are implied by the Bogomolnyi equations

$$
\begin{equation*}
\frac{1}{2} \varepsilon^{a b c}\left(\partial_{[b} A_{c]}+A_{[b} A_{c]}\right)=\partial^{a} \varphi+\left[A^{a}, \varphi\right] \quad \text { with } \quad a, b, c \in\{0,1,2\} \tag{1}
\end{equation*}
$$

for the gauge potential $A=A_{a} \mathrm{~d} x^{a}$ and the Higgs field $\varphi$, all Lie-algebra valued. Contractions employ the Minkowski metric $\left(\eta_{a b}\right)=\operatorname{diag}(-1,+1,+1)$. With the ansatz and light-cone gaugefixing

$$
\begin{array}{ll}
A_{t}-A_{y}=0, & A_{t}+A_{y}=\Phi^{\dagger}\left(\partial_{t}+\partial_{y}\right) \Phi  \tag{2}\\
A_{x}+\varphi=0, & A_{x}-\varphi=\Phi^{\dagger} \partial_{x} \Phi,
\end{array}
$$

introducing a $\mathrm{U}(n)$-valued prepotential $\Phi$, the Bogomolnyi equations (11) reduce to the Yang-type equation

$$
\begin{equation*}
-\partial_{t}\left(\Phi^{\dagger} \partial_{t} \Phi\right)+\partial_{x}\left(\Phi^{\dagger} \partial_{x} \Phi\right)+\partial_{y}\left(\Phi^{\dagger} \partial_{y} \Phi\right)+\partial_{y}\left(\Phi^{\dagger} \partial_{t} \Phi\right)-\partial_{t}\left(\Phi^{\dagger} \partial_{y} \Phi\right)=0 \tag{3}
\end{equation*}
$$

making explicit the breaking of $S O(2,1)$ Lorentz invariance by the ansatz. This can be more succinctly written as

$$
\begin{equation*}
\left(\eta^{a b}+k_{c} \varepsilon^{c a b}\right) \partial_{a}\left(\Phi^{\dagger} \partial_{b} \Phi\right)=0 \quad \text { for } \quad\left(k_{c}\right)=(0,1,0) \tag{4}
\end{equation*}
$$

which is the equation of motion for a (WZW-modified) nonlinear sigma model known as Ward's model [5]. The modification may be interpreted as a background magnetic field $b^{a b}=k_{c} \varepsilon^{c a b}$ in the $k$ direction. This model is integrable at the price of foregoing Lorentz invariance.

It is convenient to introduce light-cone coordinates

$$
\begin{equation*}
u:=\frac{1}{2}(t+y) \quad, \quad v:=\frac{1}{2}(t-y) \quad, \quad \partial_{u}=\partial_{t}+\partial_{y} \quad, \quad \partial_{v}=\partial_{t}-\partial_{y} \tag{5}
\end{equation*}
$$

in terms of which the Ward equation (3) simplifies to

$$
\begin{equation*}
\partial_{x}\left(\Phi^{\dagger} \partial_{x} \Phi\right)-\partial_{v}\left(\Phi^{\dagger} \partial_{u} \Phi\right)=0 \tag{6}
\end{equation*}
$$

## 3 Linear system and single-pole ansatz

It is easy to see that (6) is the compatibility condition of two linear equations,

$$
\begin{equation*}
\left(\zeta \partial_{x}-\partial_{u}\right) \psi=\left(\Phi^{\dagger} \partial_{u} \Phi\right) \psi \quad \text { and } \quad\left(\zeta \partial_{v}-\partial_{x}\right) \psi=\left(\Phi^{\dagger} \partial_{x} \Phi\right) \psi \tag{7}
\end{equation*}
$$

which can be obtained from the Lax pair for the SDYM equations in $\mathbb{R}^{2,2}$ 10 by gauge-fixing and dimensional reduction. Here, $\psi$ is $\mathrm{U}(n)$ valued and depends on $(x, u, v, \zeta)$. The spectral parameter $\zeta$ lies in the extended complex plane $\mathbb{C} P^{1}$. The auxiliary function $\psi$ is subject to the reality condition [5]

$$
\begin{equation*}
\psi(x, u, v, \zeta)[\psi(x, u, v, \bar{\zeta})]^{\dagger}=\mathbb{1} . \tag{8}
\end{equation*}
$$

We also impose on $\psi$ the standard asymptotic conditions

$$
\begin{equation*}
\psi(x, u, v, \zeta \rightarrow \infty)=\mathbb{1} \quad \text { and } \quad \psi(x, u, v, \zeta \rightarrow 0)=\Phi^{\dagger}(x, u, v) \tag{9}
\end{equation*}
$$

Once we have a solution $\psi$ to (7) the prepotential $\Phi$ may be reconstructed from its asymptotic value via (9), but the gauge potentials $A_{u}=\Phi^{\dagger} \partial_{u} \Phi$ and $A_{x}-\varphi=\Phi^{\dagger} \partial_{x} \Phi$ can also be found from

$$
\begin{align*}
-\psi(x, u, v, \zeta)\left(\zeta \partial_{x}-\partial_{u}\right)[\psi(x, u, v, \bar{\zeta})]^{\dagger} & =\left(\Phi^{\dagger} \partial_{u} \Phi\right)(x, u, v)  \tag{10}\\
-\psi(x, u, v, \zeta)\left(\zeta \partial_{v}-\partial_{x}\right)[\psi(x, u, v, \bar{\zeta})]^{\dagger} & =\left(\Phi^{\dagger} \partial_{x} \Phi\right)(x, u, v) \tag{11}
\end{align*}
$$

A nontrivial solution $\psi$ holomorphic in $\zeta$ cannot be analytic on $\mathbb{C} P^{1}$, hence must have poles, say at $\zeta=\mu_{k} \in \mathbb{C}$ with $k=1, \ldots, m$. Contrasting the ensueing $\zeta$ dependence of the left-hand sides of (8), (10) and (11) with the $\zeta$ independence of their right-hand sides, consistency requires that the residues at $\zeta=\mu_{k}$ and $\zeta=\bar{\mu}_{k}$ must vanish. In fact this condition also suffices to guarantee the existence of a solution. It is known that such solutions describe multi-soliton configurations, with the velocities of the individual solitons for $\mu=\mu_{k}$ being given by

$$
\begin{equation*}
\left(v_{x}, v_{y}\right)=-\left(\frac{\mu+\bar{\mu}}{\mu \bar{\mu}+1}, \frac{\mu \bar{\mu}-1}{\mu \bar{\mu}+1}\right)=-\left(\mu-\bar{\mu}-\frac{1}{\mu}+\frac{1}{\bar{\mu}}\right)^{-1}\left(\frac{\mu}{\bar{\mu}}-\frac{\bar{\mu}}{\mu}, \mu-\bar{\mu}+\frac{1}{\mu}-\frac{1}{\bar{\mu}}\right) \tag{12}
\end{equation*}
$$

This relation carries over to the noncommutative realm 6].
In this letter we only consider the simplest case of a single soliton, i.e. $m=1$, and comment on its generalization at the end. Accordingly, we make the single-pole ansatz (dropping the index $k$ )

$$
\begin{equation*}
\psi(x, u, v, \zeta)=\mathbb{1}+\frac{\mu-\bar{\mu}}{\zeta-\mu} P(x, u, v) \tag{13}
\end{equation*}
$$

The group-valued function $P$ is determined from the zero-residue conditions

$$
\begin{align*}
(\mathbb{1}-P) P^{\dagger} & =0,  \tag{14}\\
(\mathbb{1}-P)\left(\bar{\mu} \partial_{x}-\partial_{u}\right) P^{\dagger} & =0,  \tag{15}\\
(\mathbb{1}-P)\left(\bar{\mu} \partial_{v}-\partial_{x}\right) P^{\dagger} & =0 \tag{16}
\end{align*}
$$

and their hermitian conjugates. It follows from (14) that $P$ is a hermitian projector,

$$
\begin{equation*}
P^{2}=P=P^{\dagger} \tag{17}
\end{equation*}
$$

so that the two differential operators in (15) and (16) are seen to map the image of $P$ into itself.

## 4 Co-moving coordinates

The remaining conditions (15) and (16) restrict the coordinate dependence of $P$. They can be simplified with an appropriate linear coordinate transformation $(u, v, x) \mapsto(w, \bar{w}, s)$. In general, the coordinate transformation takes the form

$$
\begin{equation*}
w:=\nu x+z \quad, \quad \bar{w}:=\nu x+\bar{z}, \quad s:=c x+\gamma z+\bar{\gamma} \bar{z} \tag{18}
\end{equation*}
$$

where the complex combinations

$$
\begin{equation*}
z:=\alpha u+\beta v \quad \text { and } \quad \bar{z}:=\bar{\alpha} u+\bar{\beta} v \tag{19}
\end{equation*}
$$

are introduced for later convenience (see section 6), and the coefficients $\nu, \alpha, \beta, \gamma \in \mathbb{C}$ and $c \in \mathbb{R}$ are to be chosen. In terms of the new coordinates, the differential operators in (15) and (16) become

$$
\begin{align*}
\partial_{x}-\frac{1}{\bar{\mu}} \partial_{u} & =\left(\nu-\frac{\alpha}{\bar{\mu}}\right) \partial_{w}+\left(\bar{\nu}-\frac{\bar{\alpha}}{\bar{\mu}}\right) \partial_{\bar{w}}+\left(c-\frac{\gamma \alpha}{\bar{\mu}}-\frac{\bar{\gamma} \bar{\alpha}}{\bar{\mu}}\right) \partial_{s}  \tag{20}\\
\partial_{x}-\bar{\mu} \partial_{v} & =(\nu-\beta \bar{\mu}) \partial_{w}+(\bar{\nu}-\bar{\beta} \bar{\mu}) \partial_{\bar{w}}+(c-\gamma \beta \bar{\mu}-\bar{\gamma} \bar{\beta} \bar{\mu}) \partial_{s}
\end{align*}
$$

In general, $P$ depends on $w, \bar{w}$ and $s$. As we will see, the system (15), (16) becomes exactly solvable if there is a translational symmetry. In our case with noncommutativity, and with the goal to look for a Fock-space construction of solutions, we take this symmetry to be generated by $\partial_{s}$ so that our solutions depend on the pair $(w, \bar{w})$ alone. In this case, nontrivial solutions can be obtained if
the two differential operators in (20) with $\partial_{s}=0$ become collinear. Without loss of generality, this happens when the coefficients of $\partial_{w}$ vanish, i.e. for

$$
\begin{equation*}
\alpha=\nu \bar{\mu} \quad \text { and } \quad \beta=\frac{\nu}{\bar{\mu}} \quad \text { with } \quad \nu \text { left free } . \tag{21}
\end{equation*}
$$

For convenience, we choose the time-like vector $\partial_{s}$ to be normalized to one and orthogonal to the $w \bar{w}$ plane, which fixes the coefficients $c$ and $\gamma$. Our coordinate transformation then reads

$$
\left.\left.\begin{array}{rl}
w & =\nu\left[\bar{\mu} u+\frac{1}{\bar{\mu}} v+x\right] \\
\bar{w} & =\nu\left[x+\frac{1}{2}\left(\bar{\mu}-\frac{1}{\bar{\mu}}\right) y+\frac{1}{2}\left(\bar{\mu}+\frac{1}{\bar{\mu}}\right) t\right]  \tag{22}\\
s & =\frac{-2 \mathrm{i}}{\mu-\bar{\mu}}\left[\mu \bar{\mu} u+\frac{1}{\mu} v+x\right]
\end{array}\right] \bar{\nu}\left[x+\frac{1}{2}\left(\mu-\frac{1}{\mu}\right) y+\frac{1}{2}\left(\mu+\frac{1}{\mu}\right) t\right], ~ . ~(\mu+\bar{\mu}) x\right]=\frac{-\mathrm{i}}{\mu-\bar{\mu}}[(\mu+\bar{\mu}) x+(\mu \bar{\mu}-1) y+(\mu \bar{\mu}+1) t] .
$$

For the vector fields we arrive at

$$
\begin{align*}
& \partial_{w}=\frac{1}{\nu} \frac{\mu \bar{\mu}}{(\mu-\bar{\mu})^{2}}\left[\frac{1}{\mu} \partial_{u}+\mu \partial_{v}-2 \partial_{x}\right] \quad, \quad \partial_{u}=\nu \bar{\mu} \partial_{w}+\bar{\nu} \mu \partial_{\bar{w}}-\frac{2 \mathrm{i} \mu \bar{\mu}}{\mu-\bar{\mu}} \partial_{s}, \\
& \partial_{\bar{w}}=\frac{1}{\bar{\nu}} \frac{\mu \bar{\mu}}{(\mu-\bar{\mu})^{2}}\left[\frac{1}{\bar{\mu}} \partial_{u}+\bar{\mu} \partial_{v}-2 \partial_{x}\right], \quad \partial_{v}=\frac{\nu}{\bar{\mu}} \partial_{w}+\frac{\bar{\nu}}{\mu} \partial_{\bar{w}}-\frac{2 \mathrm{i}}{\mu-\bar{\mu}} \partial_{s},  \tag{23}\\
& \partial_{s}=\quad \frac{-\mathrm{i}}{\mu-\bar{\mu}}\left[\partial_{u}+\mu \bar{\mu} \partial_{v}-(\mu+\bar{\mu}) \partial_{x}\right] \quad, \quad \partial_{x}=\nu \partial_{w}+\bar{\nu} \partial_{\bar{w}}-\frac{\mathrm{i}(\mu+\bar{\mu})}{\mu-\bar{\mu}} \partial_{s},
\end{align*}
$$

revealing degeneracy for $\mu \in \mathbb{R}$. The Euclidean metric induced on the $w \bar{w}$ plane is proportional to $(\operatorname{Im} \mu)^{-2}$. We remark that, up to normalization, the expression for $\partial_{s}$ is independent of the value taken for $c$ or $\gamma$ in (20), so that the choice of $s$ does not matter for the rest of the letter.

Looking at (12), we notice that

$$
\begin{equation*}
\operatorname{Im} \mu=0 \quad \Leftrightarrow \quad v_{x}^{2}+v_{y}^{2}=1 \quad \text { and } \quad \mu= \pm \mathrm{i} \quad \Leftrightarrow \quad v_{x}=v_{y}=0 \tag{24}
\end{equation*}
$$

so the transformation (22) degenerates for light-like solitons. Let us take $\operatorname{Im} \mu<0$ from now on. Note that for static solitons $(\mu=-\mathrm{i})$ the above relations simplify to

$$
\begin{equation*}
w=\nu(x+\mathrm{i} y) \quad, \quad \bar{w}=\bar{\nu}(x-\mathrm{i} y), \quad s=t \tag{25}
\end{equation*}
$$

In terms of the new coordinates, the differential equations (15) and (16), with (17), combine to

$$
\begin{equation*}
(\mathbb{1}-P) \partial_{\bar{w}} P=0 \quad \text { and } \quad(\mathbb{1}-P) \partial_{s} P=0 \tag{26}
\end{equation*}
$$

As indicated above, we are interested in $s$ independent solutions. Hence, one is looking for projectors $P(w, \bar{w})$ satisfying a single "BPS" equation

$$
\begin{equation*}
\partial_{\bar{w}} P=P \gamma \quad \text { for some } \quad \gamma . \tag{27}
\end{equation*}
$$

Physically, although the solitons of the Ward model are neither Galilei nor Lorentz invariant, they depend - like all solitons - only on their moving-frame coordinates. The combinations $(w, \bar{w})$ are precisely those co-moving coordinates. Consequently, the vector field $\partial_{s}$ generates a symmetry of the one-soliton configuration, i.e. $\partial_{s} P=0$. An obvious solution to (27) is to build $P$ from a holomorphic function $T(w)$ which spans the image of $P$. This kind of solution extends to the noncommutative case when the gauge group is nonabelian. In the abelian situation, however, only genuinely noncommutative solutions exist, which we will discuss now.

## 5 Time-space noncommutativity and abelian solitons

Up to this point we have basically followed 5 which discusses the commutative case. The space-space noncommutative deformation was analyzed in [6], with

$$
\begin{equation*}
[x, y]=\mathrm{i} \theta \quad \Longrightarrow \quad[w, \bar{w}]=2 \theta>0 \quad \text { for } \quad \nu \bar{\nu}=-4 \mathrm{i}\left(\mu-\bar{\mu}-\frac{1}{\mu}+\frac{1}{\bar{\mu}}\right)^{-1} . \tag{28}
\end{equation*}
$$

Due to the anisotropy of the Ward model, there are many inequivalent choices of time-space deformations. The generic cases are ${ }^{1}$

$$
\begin{array}{llll}
{[t, x]=\mathrm{i} \theta} & \Longrightarrow & {[w, \bar{w}]=2 \theta>0} & \text { for } \quad \nu \bar{\nu}=4 \mathrm{i}\left(\mu-\bar{\mu}+\frac{1}{\mu}-\frac{1}{\bar{\mu}}\right)^{-1} \\
{[t, y]=\mathrm{i} \theta} & \Longrightarrow \quad[w, \bar{w}]=2 \theta>0 & \text { for } \quad \nu \bar{\nu}=-4 \mathrm{i}\left(\frac{\mu}{\bar{\mu}}-\frac{\bar{\mu}}{\mu}\right)^{-1} . \tag{30}
\end{array}
$$

Interestingly, the first case becomes singular for $|\mu|=1$, i.e. $v_{y}=0$, while for the second case this happens when $\operatorname{Re} \mu=0$, i.e. $v_{x}=0$. More generally, we find that

$$
\begin{equation*}
[t, x \cos \vartheta+y \sin \vartheta]=\mathrm{i} \theta \quad \Longrightarrow \quad[w, \bar{w}] \propto \nu \bar{\nu}\left(v_{y} \cos \vartheta-v_{x} \sin \vartheta\right) \tag{31}
\end{equation*}
$$

Apparently, solitonic motion in the direction of the deformed spatial coordinate is prohibited, ruling out static solutions in particular.

In order to describe abelian solitons, we specialize the target to $\mathrm{U}(n=1)$. It is convenient to employ the operator formalism, which realizes the noncommutativity via operator-valuedness in an auxiliary Fock space $\mathcal{H}$. Then, the image of the projector $P$ in the single-pole ansatz (13) is spanned by a collection (row vector) of kets,

$$
|T\rangle=\left(\begin{array}{llll}
\left|T^{1}\right\rangle & \left|T^{2}\right\rangle & \ldots & \left|T^{r}\right\rangle \tag{32}
\end{array}\right) \quad \text { so that } \quad P=|T\rangle\langle T|
$$

with $r=\operatorname{rank}(P)$ and normalization $\langle T \mid T\rangle=\mathbb{1}_{r \times r}$. Next, the BPS equation (27) reduces to

$$
\begin{equation*}
\partial_{\bar{w}}|T\rangle=|T\rangle \Gamma \quad \text { for some } r \times r \text { matrix } \Gamma . \tag{33}
\end{equation*}
$$

If we take

$$
\begin{equation*}
\Gamma=\operatorname{diag}\left(\alpha^{1}, \alpha^{2}, \ldots, \alpha^{r}\right) \quad \text { with } \quad \alpha^{i} \in \mathbb{C} \tag{34}
\end{equation*}
$$

then we arrive at the conditions

$$
\begin{equation*}
\partial_{\bar{w}}\left|T^{i}\right\rangle=\alpha^{i}\left|T^{i}\right\rangle \quad \text { for } \quad i=1, \ldots, r \tag{35}
\end{equation*}
$$

This identifies the kets as coherent states

$$
\begin{equation*}
\left|T^{i}\right\rangle=\mathrm{e}^{\alpha^{i} \bar{w}-\bar{\alpha}^{i} w}|0\rangle=:\left|\alpha^{i}\right\rangle \tag{36}
\end{equation*}
$$

based on the Fock vacuum of the algebra (29) or (30):

$$
\begin{equation*}
w|0\rangle=0 \tag{37}
\end{equation*}
$$

The rank-one solution reads (dropping the superscript)

$$
\begin{equation*}
\Phi=\mathbb{1}_{\mathcal{H}}-\left(1-\frac{\mu}{\bar{\mu}}\right) P=\mathbb{1}_{\mathcal{H}}-\left(1-\frac{\mu}{\bar{\mu}}\right)|\alpha\rangle\langle\alpha|=\mathrm{e}^{\alpha \bar{w}-\bar{\alpha} w}\left[\mathbb{1}_{\mathcal{H}}-\left(1-\frac{\mu}{\bar{\mu}}\right)|0\rangle\langle 0|\right] \mathrm{e}^{\bar{\alpha} w-\alpha \bar{w}} \tag{38}
\end{equation*}
$$

[^0]At this point, it is instructive to recall that the initial problem with time-space noncommutativity was the lack of a Fock-space formalism. We circumvented the problem by passing to an appropriately defined co-moving coordinate system. Although the commutation relations take a more complicated form as a whole, they become simpler in a two dimensional subspace and in fact take on the standard Heisenberg algebra form enabling a Fock-space construction.

To reveal the coordinate dependence of our soliton (38) and interpret the moduli $\alpha$, we pass to the star-product formulation using the Moyal-Weyl map, ${ }^{2}$

$$
\begin{equation*}
\Phi_{\star}=1-2\left(1-\frac{\mu}{\bar{\mu}}\right) \mathrm{e}^{-|w-2 \theta \alpha|^{2} / \theta}=1-2\left(1-\frac{\mu}{\bar{\mu}}\right) \exp \left\{-\frac{1}{\theta}\left|\nu x+\frac{\nu}{2}\left(\bar{\mu}-\frac{1}{\bar{\mu}}\right) y+\frac{\nu}{2}\left(\bar{\mu}+\frac{1}{\bar{\mu}}\right) t-2 \theta \alpha\right|^{2}\right\} . \tag{39}
\end{equation*}
$$

This is a Gaussian profile which peaks at

$$
\begin{align*}
& x(t)=\left(\mu-\bar{\mu}-\frac{1}{\mu}+\frac{1}{\bar{\mu}}\right)^{-1}\left[2 \theta\left(\frac{\mu \alpha}{\nu}-\frac{\bar{\mu} \bar{\alpha}}{\bar{\nu}}-\frac{\alpha}{\mu \nu}+\frac{\bar{\alpha}}{\bar{\mu} \bar{\nu}}\right)-\left(\frac{\mu}{\bar{\mu}}-\frac{\bar{\mu}}{\mu}\right) t\right], \\
& y(t)=\left(\mu-\bar{\mu}-\frac{1}{\mu}+\frac{1}{\bar{\mu}}\right)^{-1}\left[4 \theta\left(\frac{\bar{\alpha}}{\bar{\nu}}-\frac{\alpha}{\nu}\right)-\left(\mu-\bar{\mu}+\frac{1}{\mu}-\frac{1}{\bar{\mu}}\right) t\right] . \tag{40}
\end{align*}
$$

Even without computing its energy density, it is clear that this lump moves with constant velocity across the $x y$ plane. The velocity is determined by $\mu$ as given in (12), the zero-time position by $\alpha$. When the velocity points in the deformed direction, the Euclidean $w \bar{w}$ plane collapses and we lose the operator formalism.

## 6 Reduction to two dimensions

Although the Ward model lives in $2+1$ dimensions, our noncommutative abelian soliton (38) is effectively a two-dimensional configuration, since it does not depend on the $s$ coordinate. For a commutative soliton this is by no means surprising but a direct consequence of its shape invariance. In the co-moving frame, where the soliton is at rest, it is obvious that this symmetry is just time translation. Despite the absence of rest frames in the time-space Moyal-deformed Ward model, it remains true that the one-soliton sector enjoys the translational invariance generated by $\partial_{s}$. This yields a reduced coordinate dependence for the one-soliton configuration: Putting $\partial_{s}=0$ in (23), we learn that

$$
\begin{equation*}
\partial_{u}+\mu \bar{\mu} \partial_{v}-(\mu+\bar{\mu}) \partial_{x}=0 \quad \Longleftrightarrow \quad \partial_{x}=\nu \partial_{w}+\bar{\nu} \partial_{\bar{w}} \tag{41}
\end{equation*}
$$

which is actually obeyed on our solution (39).
What happens if we perform a reduction to $1+1$ dimensions, retaining the time-space Moyal deformation? Let us take the example of $t y$ noncommutativity and attempt to reduce along the $x$ coordinate. Simply demanding $\partial_{x}=0$ in (15) and (16) yields

$$
\begin{equation*}
(\mathbb{1}-P) \partial_{u} P=0 \quad \text { and } \quad(\mathbb{1}-P) \partial_{v} P=0 \tag{42}
\end{equation*}
$$

which overconstrains the system (unless $\mu=0$ or $\mu=\infty$ ). To remain compatible with the linear system, we must reproduce the $x=0$ slice of the soliton configuration. Therefore, we put $x=0$ in the coordinate transformation (22), and the reduced coordinates are

$$
\begin{align*}
& z:=\nu\left[\bar{\mu} u+\frac{1}{\bar{\mu}} v\right]=\nu\left[\frac{1}{2}\left(\bar{\mu}-\frac{1}{\bar{\mu}}\right) y+\frac{1}{2}\left(\bar{\mu}+\frac{1}{\bar{\mu}}\right) t\right], \\
& \bar{z}:=\bar{\nu}\left[\mu u+\frac{1}{\mu} v\right]=\bar{\nu}\left[\frac{1}{2}\left(\mu-\frac{1}{\mu}\right) y+\frac{1}{2}\left(\mu+\frac{1}{\mu}\right) t\right], \tag{43}
\end{align*}
$$

[^1]but this does not mean that $\partial_{x}$ vanishes! Rather, (41) yields the relation
\[

$$
\begin{equation*}
\partial_{x}=\nu \partial_{z}+\bar{\nu} \partial_{\bar{z}} \tag{44}
\end{equation*}
$$

\]

which fixes the $x$ dependence in terms of the reduced coordinates.
Time-space noncommutativity for $\operatorname{Im} \mu^{2} \neq 0$ now takes the form

$$
\begin{equation*}
[t, y]=\mathrm{i} \theta \quad \Longrightarrow \quad[z, \bar{z}]=2 \theta>0 \tag{45}
\end{equation*}
$$

like in the Euclidean Moyal plane. The reduction along $x$ is most easily implemented by (37) and the relation between $w$ and $z$ :

$$
\begin{equation*}
w=z+\nu x \quad \Longrightarrow \quad z|0\rangle=-\nu x|0\rangle \tag{46}
\end{equation*}
$$

and so

$$
\begin{equation*}
|0\rangle=\mathrm{e}^{x(\bar{\nu} z-\nu \bar{z}) / 2 \theta}|\Omega\rangle \quad \text { with } \quad z|\Omega\rangle=0 . \tag{47}
\end{equation*}
$$

Therefore, expressing the " $w$ vacuum" $|0\rangle$ through the " $z$ vacuum" $|\Omega\rangle$ produces a specific phase factor linear in $x$ as well as in the reduced coordinates. Further $x$ dependence, tied to the moduli $\alpha$, is seen to cancel in (38). Hence, all $x$ dependence in our soliton factorizes, and the reduction takes the twisted form

$$
\begin{equation*}
\Phi(x, u, v)=\mathrm{e}^{x(\bar{\nu} z-\nu \bar{z}) / 2 \theta} g(u, v) \mathrm{e}^{-x(\bar{\nu} z-\nu \bar{z}) / 2 \theta}=\mathrm{e}^{\mathrm{i} x f(u, v)} g(u, v) \mathrm{e}^{-\mathrm{i} x f(u, v)} \tag{48}
\end{equation*}
$$

with $f(u, v):=(\bar{\nu} z-\nu \bar{z}) / 2 \mathrm{i} \theta$. One quickly verifies that

$$
\begin{equation*}
\partial_{x} \Phi=\frac{1}{2 \theta}[\bar{\nu} z-\nu \bar{z}, \Phi]=\left(\nu \partial_{z}+\bar{\nu} \partial_{\bar{z}}\right) \Phi \tag{49}
\end{equation*}
$$

as it should be. Our soliton configuration (38) cleanly descends to the two-dimensional unitary configuration

$$
\begin{equation*}
g(u, v)=\mathrm{e}^{\alpha \bar{z}-\bar{\alpha} z}\left[\mathbb{1}_{\mathcal{H}}-\left(1-\frac{\mu}{\mu}\right)|\Omega\rangle\langle\Omega|\right] \mathrm{e}^{-\alpha \bar{z}+\bar{\alpha} z} . \tag{50}
\end{equation*}
$$

From its star-product form, obtained by simply putting $x=0$ in (39), we obtain again a Gaussian profile, but now peaked at

$$
\begin{align*}
y & =-2 \theta\left(\frac{\mu}{\bar{\mu}}-\frac{\bar{\mu}}{\mu}\right)^{-1}\left(\frac{\mu \alpha}{\nu}-\frac{\bar{\mu} \bar{\alpha}}{\bar{\nu}}+\frac{\alpha}{\mu \nu}-\frac{\bar{\alpha}}{\bar{\mu} \bar{\nu}}\right) \\
t & =+2 \theta\left(\frac{\mu}{\bar{\mu}}-\frac{\bar{\mu}}{\mu}\right)^{-1}\left(\frac{\mu \alpha}{\nu}-\frac{\overline{\bar{\omega}}}{\bar{\nu}}-\frac{\alpha}{\mu \nu}+\frac{\bar{\alpha}}{\bar{\mu} \bar{\nu}}\right) . \tag{51}
\end{align*}
$$

Thus, what we have constructed is not a soliton, but rather a noncommutative instanton. It is indeed the $x=0$ slice of the $2+1$ dimensional soliton constructed above, and thus has lost one of its world-volume dimensions. The commutative limit is singular as had to be expected.

Other choices of the deformed spatial direction confirm the generic picture: Putting to zero the commutative coordinate yields a time-space noncommutative integrable model in $1+1$ dimensions and slices our one-soliton solution orthogonal to it, producing an instanton configuration. To obtain from (38) a noncommutative soliton in $1+1$ dimensions one would have to slice it parallel to the velocity, i.e. to take the velocity in the deformed direction. ${ }^{3}$ However, this is precisely the singular situation encountered earlier, where only the star-product formulation is available. No soliton solutions are known for this case.

[^2]
## 7 A new integrable equation

Which is the field equation that governs the reduced configurations? Inserting our reduction ansatz (48) into the Ward equation (6) yields

$$
\begin{equation*}
\partial_{v}\left(g^{\dagger} \partial_{u} g\right)+\left[f, g^{\dagger}[f, g]\right]=0 \tag{52}
\end{equation*}
$$

Rewriting the commutator term with the help of

$$
\begin{equation*}
[f, .]=\frac{1}{2 \mathrm{i} \theta}[\bar{\nu} z-\nu \bar{z}, .]=\frac{1}{\mathrm{i}}\left(\nu \partial_{z}+\bar{\nu} \partial_{\bar{z}}\right) \tag{53}
\end{equation*}
$$

and also converting the first term via

$$
\begin{equation*}
\partial_{u}=\nu \bar{\mu} \partial_{z}+\bar{\nu} \mu \partial_{\bar{z}} \quad \text { and } \quad \partial_{v}=\frac{\nu}{\bar{\mu}} \partial_{z}+\frac{\bar{\nu}}{\mu} \partial_{\bar{z}} \tag{54}
\end{equation*}
$$

we arrive (after dividing by $\nu \bar{\nu}$ ) at

$$
\begin{equation*}
\left(\frac{\mu}{\bar{\mu}}-1\right) \partial_{z}\left(g^{\dagger} \partial_{\bar{z}} g\right)+\left(\frac{\bar{\mu}}{\mu}-1\right) \partial_{\bar{z}}\left(g^{\dagger} \partial_{z} g\right)=0 . \tag{55}
\end{equation*}
$$

Another way to obtain this equation is to express the Ward equation (6) in the co-moving coordinates,

$$
\begin{equation*}
\left(\frac{\mu}{\bar{\mu}}-1\right) \partial_{w}\left(\Phi^{\dagger} \partial_{\bar{w}} \Phi\right)+\left(\frac{\bar{\mu}}{\mu}-1\right) \partial_{\bar{w}}\left(\Phi^{\dagger} \partial_{w} \Phi\right)=0 \tag{56}
\end{equation*}
$$

and to simply pass to the reduced coordinates $(z, \bar{z})$.
The new equation can be simplified to

$$
\begin{equation*}
\mu \partial_{z}\left(g^{\dagger} \partial_{\bar{z}} g\right)-\bar{\mu} \partial_{\bar{z}}\left(g^{\dagger} \partial_{z} g\right)=0 \tag{57}
\end{equation*}
$$

or, parameterizing $g=\mathrm{e}^{\mathrm{i} \phi}$ by a noncommutative real scalar field $\phi(u, v)$, be written as

$$
\begin{equation*}
\mu \partial_{z}\left(\mathrm{e}^{-\mathrm{i} \phi} \partial_{\bar{z}} \mathrm{e}^{\mathrm{i} \phi}\right)-\bar{\mu} \partial_{\bar{z}}\left(\mathrm{e}^{-\mathrm{i} \phi} \partial_{z} \mathrm{e}^{\mathrm{i} \phi}\right)=0 \tag{58}
\end{equation*}
$$

It is instructive to interpret this as a sigma model with metric $h$ (symmetric) and magnetic field $b$ (antisymmetric) by comparing with

$$
\begin{equation*}
h^{i j} \partial_{(i}\left(g^{\dagger} \partial_{j)} g\right)+b^{i j} \partial_{[i}\left(g^{\dagger} \partial_{j]} g\right)=\left(h^{i j}+b^{i j}\right) \partial_{i}\left(g^{\dagger} \partial_{j} g\right)=0 \quad \text { for } \quad i, j=z, \bar{z} . \tag{59}
\end{equation*}
$$

Explicitly adding the contributions from the first and second term in (52), we have

$$
\begin{align*}
& \left(\begin{array}{l}
h^{z z}
\end{array} h^{z \bar{z}}, \begin{array}{cc}
\nu^{2} & \frac{\nu \overline{2}}{2}\left(\frac{\mu}{\bar{\mu}}+\frac{\bar{\mu}}{\mu}\right) \\
h^{\bar{z} z} & h^{\bar{z} \bar{z}}
\end{array}\right)=\left(\begin{array}{cc}
\nu^{2} & \nu \bar{\nu} \\
\frac{\nu \bar{\nu}}{2}\left(\frac{\mu}{\bar{\mu}}+\frac{\mu}{\mu}\right) & \bar{\nu}^{2}
\end{array}\right)=\nu \bar{\nu} \frac{(\mu-\bar{\mu})^{2}}{2 \mu \bar{\mu}}\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right),  \tag{60}\\
& \left(\begin{array}{ll}
b^{z z} & b^{z \bar{z}} \\
b^{\bar{z} z} & b^{\bar{z} \bar{z}}
\end{array}\right)=\frac{\nu \bar{\nu}}{2}\left(\frac{\mu}{\bar{\mu}}-\frac{\bar{\mu}}{\mu}\right)\left(\begin{array}{rr}
0 & 1 \\
-1 & 0
\end{array}\right)+\left(\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right)=\nu \bar{\nu} \frac{\mu^{2}-\bar{\mu}^{2}}{2 \mu \bar{\mu}}\left(\begin{array}{rr}
0 & 1 \\
-1 & 0
\end{array}\right),
\end{align*}
$$

showing that the commutator term is of rank one only but serves to cancel the diagonal metric contribution. At the same time, it modifies the original Minkowski metric to a Euclidean one, as is most transparent in the $(t, y)$ basis:

$$
\left(\begin{array}{cc}
h^{t t} & h^{t y}  \tag{61}\\
h^{y t} & h^{y y}
\end{array}\right)=\frac{-\mu \bar{\mu}}{(\mu+\bar{\mu})^{2}}\left(\begin{array}{cc}
\left|\frac{1}{\mu}-\mu\right|^{2} & \left|\frac{1}{\mu}\right|^{2}-|\mu|^{2} \\
\left|\frac{1}{\mu}\right|^{2}-|\mu|^{2} & \left|\frac{1}{\mu}+\mu\right|^{2}
\end{array}\right) \quad \text { and } \quad\left(\begin{array}{c}
b^{t t} \\
b^{t y} \\
b^{y t} \\
b^{y y}
\end{array}\right)=\left(\begin{array}{rr}
0 & 1 \\
-1 & 0
\end{array}\right)
$$

while the first term in (52) provides only $h=\sigma_{3}$ and $b=\mathrm{i} \sigma_{2}$. In these coordinates, our integrable equation (57) then reads

$$
\begin{align*}
& \quad\left|\frac{1}{\mu}-\mu\right|^{2} \partial_{t}\left(g^{\dagger} \partial_{t} g\right)+\left(\left|\frac{1}{\mu}\right|^{2}-|\mu|^{2}-\left|1+\frac{\mu}{\bar{\mu}}\right|^{2}\right) \partial_{t}\left(g^{\dagger} \partial_{y} g\right)+ \\
& +\left(\left|\frac{1}{\mu}\right|^{2}-|\mu|^{2}+\left|1+\frac{\mu}{\bar{\mu}}\right|^{2}\right) \partial_{y}\left(g^{\dagger} \partial_{t} g\right)+\left|\frac{1}{\mu}+\mu\right|^{2} \partial_{y}\left(g^{\dagger} \partial_{y} g\right)=0 . \tag{62}
\end{align*}
$$

It inherits a hidden global $\mathrm{U}(1)$ invariance from the global phase symmetry $z \mapsto \mathrm{e}^{\mathrm{i} \gamma} z$ of (57). To summarize this paragraph, the integrable reduction (48) effectively not only transforms the metric but also flips its signature! It is easily checked that this phenomenon does not depend on the spatial direction chosen for the integrable reduction.

Our novel equation (57) may also be obtained from an action principle. For the reduction (48), the Nair-Schiff sigma-model-type action [12, 13] for SDYM in $2+2$ dimensions descends to

$$
\begin{align*}
S[g] & =-\int \mathrm{d} t \mathrm{~d} y h^{z \bar{z}}\left(g^{\dagger} \partial_{z} g\right)\left(g^{\dagger} \partial_{\bar{z}} g\right)-\int \mathrm{d} t \mathrm{~d} y \mathrm{~d} \lambda b^{z \bar{z}}\left[\tilde{g}^{\dagger} \partial_{z} \tilde{g}, \tilde{g}^{\dagger} \partial_{\bar{z}} \tilde{g}\right] \tilde{g}^{\dagger} \partial_{\lambda} \tilde{g}  \tag{63}\\
& =\int \mathrm{d} t \mathrm{~d} y\left(h^{z \bar{z}} \partial_{z} \phi \partial_{\bar{z}} \phi+\frac{\mathrm{i}}{3} b^{z \bar{z}}\left[\partial_{z} \phi, \partial_{\bar{z}} \phi\right] \phi+O\left(\phi^{4}\right)\right), \tag{64}
\end{align*}
$$

with a unitary extension $\tilde{g}(u, v, \lambda)$ interpolating between

$$
\begin{equation*}
\tilde{g}(u, v, 0)=\mathbb{1}_{\mathcal{H}} \quad \text { and } \quad \tilde{g}(u, v, 1)=g(u, v) \tag{65}
\end{equation*}
$$

Since the metric $h$ is Euclidean, the elementary excitation $\phi$ does not propagate: its mass-shell condition for $(E, p) \equiv\left(p_{t}, p_{y}\right)$ is

$$
\begin{equation*}
E=c p \quad \text { with } \quad c=\frac{\mu^{2}+1}{\mu^{2}-1} \quad \text { or } \quad \frac{\bar{\mu}^{2}+1}{\bar{\mu}^{2}-1} \tag{66}
\end{equation*}
$$

being complex. Nevertheless, the model features instanton-like classical configurations like (50) as we have seen.

It is rather obvious that the solution strategy of section 3 (linear system and single-pole ansatz) applies to (57) with the logical adjustments, i.e.

$$
\begin{equation*}
w \mapsto z \quad \text { and } \quad \partial_{x} \mapsto \mathrm{i}[f, .]=\nu \partial_{z}+\bar{\nu} \partial_{\bar{z}} . \tag{67}
\end{equation*}
$$

On this road one again arrives at the abelian noncommutative instanton (50) in the rank-one case. Higher-rank projectors in (32) yield multi-soliton and multi-instanton solutions in $2+1$ and $2+0$ dimensions, respectively, with individual position moduli $\alpha^{i}$ but common velocity given by $\mu$. More general, multi-pole, ansätze for $\psi$ should produce multi-solitons with relative motion on timespace noncommutative $\mathbb{R}^{2,1}$, but their integrable reduction to two dimensions is unclear because the reduction (48) depends on $\mu$. These configurations will be genuinely $2+1$ dimensional with time-space noncommutativity.

Finally we remark that our twisted reduction ansatz (48) extends straightforwardly for the nonabelian case. With an $f$ of diagonal form, one can embed into the nonabelian group many abelian solitons whose generic features have been examined and discussed above.

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[^0]:    ${ }^{1}$ Note that $[w, s] \neq 0$. For example, for the case (30), one has $[w, s]=\nu \theta$. However, since we are looking for $s$ independent solutions, we can safely ignore this noncommutative structure.

[^1]:    ${ }^{2}$ Note that not the star-exponential but the ordinary exponential appears below.

[^2]:    ${ }^{3}$ Another possibility is to reduce an extended-wave solution of the Ward model 11 along its spatial extension. However, these occur only in the nonabelian model and descend to the sine-Gordon solitons [3].

