# ANOTHER LAW FOR THE 3-METABELIAN GROUPS 

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The 3 -metabelian groups are those groups in which every subgroup generated by three elements is metabelian. In [2] it was stated and in [3] it was proved that the variety of such groups may be defined by the one law

$$
\begin{equation*}
[x, y ; x, z]=1 \tag{1}
\end{equation*}
$$

(by $[a, b ; c, d]$ we mean $[[a, b],[c, d]]$, and for other definitions and notation we refer to [1]). Recently Bachmuth and Lewin obtained in [1] the surprising and remarkable result that the same variety is defined by the law

$$
\begin{equation*}
[x, y, z][y, z, x][z, x, y]=1 \tag{2}
\end{equation*}
$$

Now (2) is reminiscent of the relation

$$
\begin{equation*}
\left[x, y, z^{x}\right]\left[y, z, x^{y}\right]\left[z, x, y^{z}\right]=1 \tag{3}
\end{equation*}
$$

which holds in all groups and which is apparently due to Philip Hall. Using the identities $\left[x, y,[y, z]^{x}\right]=\left[y, z, x^{y}\right]^{-1}[y, z, x]$, etc., we find that (2) is equivalent to

$$
\begin{equation*}
\left[x, y,[y, z]^{x}\right]^{u}\left[z, x,[x, y]^{z}\right]^{v}\left[y, z,[z, x]^{v}\right]=1 \tag{4}
\end{equation*}
$$

where $u=[z, x, y]$ and $v=[y, z, x][z, x, y]$. Note that apart from certain displeasing conjugates (4) is curiously similar to both (1) and (2).

The question which naturally arises at this point is answered by
Theorem. The variety defined by the lawe

$$
\begin{equation*}
[x, y ; y, z][y, z ; z, x][z, x ; x, y]=1 \tag{5}
\end{equation*}
$$

is the variety of 3 -metabelian groups.
Proof. We calculate. In preparation to replacing $y$ by $x y$ in (5) we note that

$$
\begin{aligned}
{[x, x y ; x y, z] } & =[x, y ; y, z]\left[x, y,[x, z]^{v}\right]^{[v, z]} \\
{[x y, z ; z, x] } & =\left[[x, z]^{v},[z, x]\right]^{[y, z]}[y, z ; z, x] .
\end{aligned}
$$

Then replacement and another use of (5) for cancellation gives

$$
\begin{equation*}
\left[x, y,[x, z]^{v}\right]\left[[x, z]^{v},[z, x]\right]=1 \tag{6}
\end{equation*}
$$

Next we want to replace $x$ by $z x$ in (6). Since

$$
\left[z x, y,[z x, z]^{v}\right]=\left[[z, y]^{x},[x, z]^{v}\right]^{[x, y]}\left[x, y,[x, z]^{y}\right]
$$

we eventually obtain

$$
\begin{equation*}
\left[[z, y]^{x}[x, z]^{v}\right]=1 \tag{7}
\end{equation*}
$$

Replace $y$ by $y x$ in (7) and note that

$$
\left[[z, y x]^{x},[x, z]^{y x}\right]=\left[[z, x]^{x},[x, z]^{y x}\right]^{[x, y]^{x^{2}}}\left[[z, y]^{x^{2}},[x, z]^{y x}\right] .
$$

We find that

$$
\begin{equation*}
\left[z, x,[x, z]^{v}\right]=1 . \tag{8}
\end{equation*}
$$

By (6) and (8) we have

$$
\begin{equation*}
\left[x, y,[x, z]^{v}\right]=1 \tag{9}
\end{equation*}
$$

a law which is equivalent to (1) since $[x, z]^{v}=[x, y]^{-1}[x, z y]$. Since on the other hand (1) clearly implies (5) the theorem has been proved.

## References

[1] Seymour Bachmuth and Jacques Lewin, The Jacobi identity in groups, Math. Zeit. 83 (1964), 170-176.
[2] I. D. Macdonald, On certain varieties of groups, Math. Zeit. 76 (1961), 270-282.
[3] I. D. Macdonald, On certain varieties of groups. II, Math. Zeit. 78 (1962), 175-188.
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