

Blind Image Separation Based on an Optimized Fast Fixed Point Algorithm

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Abstract— An optimized fast fixed point algorithm based on modified Newton iteration method has been proposed. With good performance of the blind image separation, the optimized algorithm can improve the convergence speed greatly. We proposed a new adaptive enhancement parameter to enhance the separated images effectively. The experimental results demonstrate that the new algorithm is superior.

Keywords- independent component analysis, fast fixed point algorithm, negentropy, Newton iteration, nongaussianity

I. INTRODUCTION

Independent component analysis (ICA), a computational and statistical method used to reveal hidden factors that underlie sets of random variables or measurements [1], has been widely used in the fields of digital signal processing, super resolution, blind image separation (BIS), etc.

Nowadays, many papers about ICA have been published in a large number of conference proceedings and journals. In the papers, many modified ICA methods have been proposed, such as infomax ICA[2,3], JADE [4], SOBI [5], fast fixed point algorithm[6], H-J [7], etc. Early, ICA resulted from the classic blind source separation (BSS) problem of a cocktail-party with less priori knowledge or even nothing [8]. ICA defines a generative model for the observed multivariate data from a large database of samples. In the model, the mixing system is unknown, and the observed data are nonlinear or linear mixtures of some unknown components of the observed data. The components are assumed mutually independent and nongaussian. The basic aim of ICA is to find these independent components (ICs). Considered as the development of principal component analysis (PCA) [9], ICA is much more powerful and robust to get the ICs.

However, when the data amount is large, ICA can't fulfill the real-time requirements. So some modified methods have been proposed to improve the processing speed, and the typical one is the fast fixed point algorithm. The algorithm has the excellent algorithmic properties and the splendid statistical properties arising from negentropy. It's based on a fixed point iteration scheme for finding a maximum of the nongaussianity of ICs. The algorithm could be derived by using the classic Newton iteration method (CN) [10].

Newton iteration method is a fundamental algorithm in numerical analysis, and it's used to solve nonlinear equations.

The classic Newton iteration method converges quadratically. And some novel modified Newton iteration methods (MN) with higher order of convergence have been proposed to improve the computational efficiency by speeding up the convergence. In the paper, an optimized fast fixed point algorithm based on MN has been proposed to cope with BIS. A new adaptive image enhancement method based on normalization has been proposed to enhance the separated images. The experimental results have shown that the optimized algorithm speeds up the convergence greatly and has good signal separation performance.

II. Modified Newton iteration method

CN is used to solve nonlinear equations in the field of numerical analysis. CN transforms the nonlinear equation into a linear equation approximately by using linearizing method. CN is easier for programming and converges faster. CN for a single nonlinear equation converges quadratically. Its fundamental formula is shown as follow:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (1)$$

In order to reduce the iteration number, some new MN methods with higher order convergence were proposed. The midpoint rule based on Newton's theorem in (2) gives the iteration method with 3rd-order convergence defined by (3) and (4) [10].

$$f(x) = f(x_n) + \int_{x_n}^x f'(t) dt \quad (2)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(\frac{x_n + x_{n+1}^*}{2})} \quad (3)$$

$$x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)} \quad (4)$$

In [11], a new MN method with 5th-order convergence is presented and the conclusion is to assume that:

1) The function $f: D \subset R \Rightarrow R$ for an open interval D has a simple root $\alpha \in D$;

2) $f(x)$ has 1st, 2nd and 3rd derivatives in D ;

3) $u_{n+1} - \alpha = Be_n^3 + O(e_n^4)$ for some $B \neq 0$ and $e_n = x_n - \alpha$;

So the MN defined by (5) has 5th-order convergence.

$$\begin{cases} x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)} \\ u_{n+1} = x_n - \frac{f(x_n)}{f'(\frac{x_n+x_{n+1}^*}{2})} \\ x_{n+1} = u_{n+1} - \frac{f(u_{n+1})}{2f'(\frac{x_n+x_{n+1}^*}{2}) - f'(x_n)} \end{cases} \quad (5)$$

The proof is given: $f(x)$ is expanded by Taylor series,

$$f(x_n) = f(\alpha) + f'(\alpha)(x_n - \alpha) + \frac{f''(\alpha)(x_n - \alpha)^2}{2!} + \frac{f'''(\alpha)(x_n - \alpha)^3}{3!} + O((x_n - \alpha)^4) \quad (6)$$

$$\because e_n = x_n - \alpha, \quad f(\alpha) = 0,$$

$$\therefore f(x_n) = f'(\alpha)e_n + \frac{1}{2!}f''(\alpha)e_n^2 + \frac{1}{3!}f'''(\alpha)e_n^3 + O(e_n^4) \quad (7)$$

$$\therefore f'(x_n) = f'(\alpha) + f''(\alpha)e_n + \frac{1}{2!}f'''(\alpha)e_n^2 + O(e_n^3) \quad (8)$$

Then assume $c_{n-1} = f^{(n)}(\alpha)[n!f'(\alpha)]^{-1}, n = 2, 3, 4, \dots$,

$$\begin{aligned} \frac{f(x_n)}{f'(x_n)} &= \frac{f'(\alpha)e_n + \frac{1}{2!}f''(\alpha)e_n^2 + \frac{1}{3!}f'''(\alpha)e_n^3 + O(e_n^4)}{f'(\alpha) + f''(\alpha)e_n + \frac{1}{2!}f'''(\alpha)e_n^2 + O(e_n^3)} \\ &= [e_n + c_1e_n^2 + c_2e_n^3 + O(e_n^4)][1 + 2c_1e_n + 3c_2e_n^2 + O(e_n^3)]^{-1} \\ &= [e_n + c_1e_n^2 + c_2e_n^3 + O(e_n^4)][1 - 2c_1e_n + (4c_1^2 - 3c_2)e_n^2 + O(e_n^3)] \\ &= e_n - c_1e_n^2 + 2(c_1^2 - c_2)e_n^3 + O(e_n^4) \end{aligned}$$

$$\begin{aligned} x_{n+1}^* &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= (e_n + \alpha) - [e_n - c_1e_n^2 + 2(c_1^2 - c_2)e_n^3 + O(e_n^4)] \\ &= \alpha + c_1e_n^2 - 2(c_1^2 - c_2)e_n^3 + O(e_n^4) \end{aligned} \quad (10)$$

$$\begin{aligned} f'(\frac{x_n+x_{n+1}^*}{2}) &= f'(\alpha) + f''(\alpha)(\frac{x_n+x_{n+1}^*}{2} - \alpha) + \frac{1}{2!}f'''(\alpha)(\frac{x_n+x_{n+1}^*}{2} - \alpha)^2 + O((\frac{x_n+x_{n+1}^*}{2} - \alpha)^3) \\ &= f'(\alpha)[1 + c_1e_n + (c_1^2 - \frac{3c_2}{4})e_n^2 + O(e_n^3)] \end{aligned} \quad (11)$$

$$2f'(\frac{x_n+x_{n+1}^*}{2}) - f'(x_n) = f'(\alpha)[1 + (2c_1^2 - \frac{3c_2}{2})e_n^2 + O(e_n^3)] \quad (12)$$

$$\left(2f'(\frac{x_n+x_{n+1}^*}{2}) - f'(x_n)\right)^{-1} = \frac{1}{f'(\alpha)}[1 - (2c_1^2 - \frac{3c_2}{2})e_n^2 + O(e_n^3)] \quad (13)$$

$$\begin{aligned} u_{n+1} - \alpha &= Be_n^3 + O(e_n^4) \\ f(u_{n+1}) &= f'(\alpha)[(u_{n+1} - \alpha) + O((u_{n+1} - \alpha)^2)] \end{aligned} \quad (14)$$

$$\begin{aligned} e_{n+1} &= u_{n+1} - \alpha - f(u_{n+1}) \left[2f'(\frac{x_n+x_{n+1}^*}{2}) - f'(x_n)\right]^{-1} \\ &= u_{n+1} - \alpha - [(u_{n+1} - \alpha) + O((u_{n+1} - \alpha)^2)][1 - (2c_1^2 - \frac{3c_2}{2})e_n^2 + O(e_n^3)] \\ &= (2c_1^2 - \frac{3}{2}c_2)Be_n^5 + O(e_n^6) \end{aligned} \quad (15)$$

So the method defined by (5) has the 5th-order convergence.

III. BSS & ICA

A. BSS

As a hot topic, BSS is used to separate individual source signals from observed mixtures of source signals when the priori information of the source signals, the mixing system and noise are all unknown. The BSS mixing model is:

$$\mathbf{x}^*(k) = \mathbf{A}^* \mathbf{s}^*(k) + \mathbf{n}^*(k) \quad (16)$$

$\mathbf{x}^*(k) = (\mathbf{x}_{1k}^*, \dots, \mathbf{x}_{Mk}^*)^T$ is the vector of M observed mixtures of source signals at time k ; $\mathbf{s}^*(k) = (\mathbf{s}_{1k}^*, \dots, \mathbf{s}_{Nk}^*)^T$ is the vector of N source signals at time k ; and \mathbf{A}^* is the $M \times N$ mixing matrix; $\mathbf{n}^*(k) = (\mathbf{n}_{1k}^*, \dots, \mathbf{n}_{Mk}^*)^T$ is the vector of M noise signals. We utilize the separating model in (17) to get the estimation of $\mathbf{s}^*(k)$. $\mathbf{y}^*(k) = (\mathbf{y}_{1k}^*, \dots, \mathbf{y}_{Nk}^*)^T$ is the estimation of $\mathbf{s}^*(k)$ and it's the vector of N separated signals at time k . \mathbf{W}^* is the $N \times M$ separating matrix.

$$\mathbf{y}^*(k) = \mathbf{W}^* \mathbf{x}^*(k) \quad (17)$$

B. ICA

As a basic algorithm to solve the BSS problem, ICA is a very general-purpose statistical technique in which observed data are expressed as a linear transform of statistically independent components [1]. The aim of ICA is to obtain the restoration of source data from observed data by calculating a separating matrix, without knowing the source data and mixing matrix. And the separating matrix is the inverse of the estimate of the mixing matrix. The linear ICA model and the separating model are shown in (18) and (19) respectively.

$$\mathbf{x} = \mathbf{A} \mathbf{s} \quad (18)$$

$$\mathbf{y} = \mathbf{W} \mathbf{x} = \mathbf{W} \mathbf{A} \mathbf{s} \quad (19)$$

In (18), \mathbf{s} is the $N \times 1$ vector of unknown source data which are mutually independent. And \mathbf{x} is the $M \times 1$ vector of observed random data which are linearly mixed sources. \mathbf{A} is the $M \times N$ mixing matrix composed of the unknown mixing coefficients. \mathbf{W} is the $N \times M$ separating matrix. In (19), \mathbf{y} is the $N \times 1$ estimation vector of \mathbf{s} , and the components of \mathbf{y} are as independent from each other as possible. Usually, we focus on the situation that the number N of source data \mathbf{s} is equal to the number M of observed random data \mathbf{x} . If the number of source data \mathbf{s} is less than the one of observed random data \mathbf{x} (i.e. $N < M$), PCA is used to cope with the problem of data redundancy and makes sure $N = M$. If the matrix $\mathbf{W} \mathbf{A}$ is a permutation matrix (all elements of each column and row are zero except for one element with the value unity), the estimation of \mathbf{s} can be found.

ICA is under the following assumptions: the components of source data are mutually independent and nongaussian (except for perhaps one); the mixing matrix is a full column rank matrix and has its pseudo-inverse matrix.

Nongaussian of \mathbf{y} , which is used as a measure of the independence of \mathbf{y} , is found by kurtosis minimization or maximization, mutual information minimization, maximum likelihood estimation or negentropy maximization [12].

In order to improve processing speed and robustness, the fast fixed point algorithm based on negentropy has been presented. By whitening the sources, the algorithm has a modified separating model $\mathbf{y} = \mathbf{V} \mathbf{z}$. And $\mathbf{z} = \mathbf{U} \mathbf{x}$ is the whitening data of \mathbf{x} after PCA processing, while \mathbf{U} is the whitening matrix. \mathbf{v}_i^T is the i -th row of the orthogonal system \mathbf{V} and $\mathbf{y}_i = \mathbf{v}_i^T \mathbf{z}$ is the i -th row of \mathbf{y} . The maximum approximation of the negentropy of \mathbf{y}_i is obtained at certain optimum of $E[G(\mathbf{v}_i^T \mathbf{z})]$. G is a nonquadratic function while g

is the derivative of G . Each optimum under the Lagrange conditions and constraint $E[(\mathbf{v}_i^T \mathbf{z})^2] = \|\mathbf{v}_i\|^2 = 1$ is obtained if $E[\mathbf{z}g(\mathbf{v}_i^T \mathbf{z})] + \beta \mathbf{v}_i = 0$ ($\beta = E[\mathbf{v}_i^T \mathbf{z}g(\mathbf{v}_i^T \mathbf{z})]$). And by CN, the fast fixed point algorithm is obtained as follows:

$$\mathbf{v}_i \leftarrow E[\mathbf{z}g(\mathbf{v}_i^T \mathbf{z})] - E[g'(\mathbf{v}_i^T \mathbf{z})]\mathbf{v}_i \quad (20)$$

$$\mathbf{v}_i \leftarrow \mathbf{v}_i / \|\mathbf{v}_i\| \quad (21)$$

IV. AN OPTIMIZED FAST FIXED POINT ALGORITHM

In order to optimize the performance of the fast fixed point algorithm mentioned above, an optimized fast fixed point algorithm has been proposed. And the modified Newton iteration in (5) is applied to its modification in (22):

$$\begin{cases} \mathbf{v}^* = \mathbf{v}_i - \frac{f(\mathbf{v}_i)}{f'(\mathbf{v}_i)} \\ \mathbf{v}^\circ = \mathbf{v}_i - \frac{f(\mathbf{v}_i)}{f'(\frac{\mathbf{v}_i + \mathbf{v}^*}{2})} \\ \mathbf{v}_{new} = \mathbf{v}^\circ - \frac{f(\mathbf{v}^\circ)}{2f'(\frac{\mathbf{v}_i + \mathbf{v}^*}{2}) - f'(\mathbf{v}_i)} \end{cases} \quad (22)$$

The optima of $E[G(\mathbf{v}_i^T \mathbf{z})]$ is obtained through $f(\mathbf{v}_i) = 0$.

$$f(\mathbf{v}_i) = E[\mathbf{z}g(\mathbf{v}_i^T \mathbf{z})] + \beta \mathbf{v}_i \quad (23)$$

By using (23) in (22), we get:

$$\begin{cases} \mathbf{v}^* = \mathbf{v}_i - \frac{E[\mathbf{z}g(\mathbf{v}_i^T \mathbf{z})] + \beta \mathbf{v}_i}{E[g'(\mathbf{v}_i^T \mathbf{z})] + \beta} \\ \mathbf{v}^\circ = \mathbf{v}_i - \frac{E[\mathbf{z}g(\mathbf{v}_i^T \mathbf{z})] + \beta \mathbf{v}_i}{E[g'(\frac{\mathbf{v}_i + \mathbf{v}^*}{2})^T \mathbf{z}] + \beta} \\ \mathbf{v}_{new} = \mathbf{v}^\circ - \frac{E[\mathbf{z}g(\mathbf{v}^{\circ T} \mathbf{z})] + \beta \mathbf{v}^\circ}{2E[g'(\frac{\mathbf{v}_i + \mathbf{v}^*}{2})^T \mathbf{z}] - E[g'(\mathbf{v}_i^T \mathbf{z})] + \beta} \end{cases} \quad (24)$$

We could use algebraic simplification to eliminate the denominators of three equations in (24) and then obtain (25) without β . The final result \mathbf{v}_i is obtained by (26).

$$\begin{cases} \mathbf{v}^* \leftarrow E[\mathbf{z}g(\mathbf{v}_i^T \mathbf{z})] - E[g'(\mathbf{v}_i^T \mathbf{z})]\mathbf{v}_i \\ \mathbf{v}^\circ \leftarrow E[\mathbf{z}g(\mathbf{v}_i^T \mathbf{z})] - E[g'(\frac{\mathbf{v}_i + \mathbf{v}^*}{2})^T \mathbf{z}]\mathbf{v}_i \\ \mathbf{v}_{new} \leftarrow E[\mathbf{z}g(\mathbf{v}^{\circ T} \mathbf{z})] - 2E[g'(\frac{\mathbf{v}_i + \mathbf{v}^*}{2})^T \mathbf{z}]\mathbf{v}^\circ + E[g'(\mathbf{v}_i^T \mathbf{z})]\mathbf{v}^\circ \end{cases} \quad (25)$$

$$\mathbf{v}_i = \mathbf{v}_{new} / \|\mathbf{v}_{new}\| \quad (26)$$

So the new optimized algorithm is defined by (25) and (26). According to the BIS based on the new algorithm, each IC is a separated image which is the estimation of one source image. In order to display the separated images well, the gray scale pixel values of each IC are normalized through (27).

$$F_{new}(x, y) = \frac{F(x, y) - F_{min}}{F_{max} - F_{min}} \quad (27)$$

$F(x, y)$ is a gray scale pixel value of an IC at point (x, y) . F_{max} and F_{min} are the maximum and the minimum of all the gray scale pixel values of each IC. $F_{new}(x, y)$ is the normalized gray scale pixel value of the IC at point (x, y) .

In (28), a new adaptive enhancement parameter α is proposed to enhance the separated images. The method is use (29) to get the new images.

$$\alpha = r \cdot M_{all} [\sigma(x, y)]^{-1/2} \quad (28)$$

$$F_{new}(x, y) = \alpha [F_{new}(x, y) - M(x, y)] + M(x, y) \quad (29)$$

M_{all} is the average of all the normalized gray scale pixel values of one image. $\sigma(x, y)$ and $M(x, y)$ are the standard deviation and the average of the normalized gray scale pixel values in the neighborhood (3×3 sub-image region) of the pixel at (x, y) respectively. And r is a proportionality constant from 0 to 1 (assumed that $r=0.5$). In (28), because α is in inverse proportion to $\sqrt{\sigma(x, y)}$, $\alpha [F_{new}(x, y) - M(x, y)]$ could enhance the images adaptively so that the regions lacking adequate contrast could be enhanced well. By using the method, the separated images can be enhanced and then displayed well. The new algorithm is depicted in Figure 1.

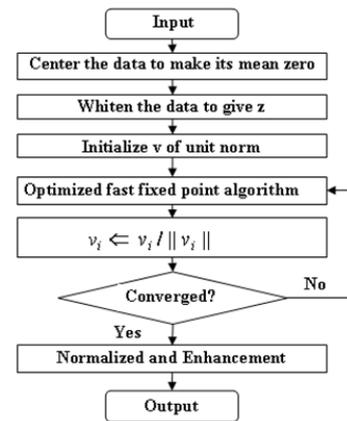


Figure 1. The optimized fast fixed point algorithm

V. EXPERIMENTAL RESULTS & DISCUSSIONS

The simulation environments are given as follows: Windows XP, MATLAB 7.5.0, CPU AMD Athlon (tm) 64 X2 Dual Core 2.41GHz, and 3.0G RAM. We've chosen two images (512×512) as source images and mixed them at random to test our algorithm. And the source images and the mixed images are shown in the Fig. 2 & 3 respectively.

In Fig.4, by using the classic fast fixed point algorithm, the separated images can't be displayed normally, and we can't detect the detail information visually. But the experimental results by using the normalization algorithm in Fig.5 show the clearer images which are more similar to the source images. In Fig.6, because of the combination with adaptive enhancement parameter α and the normalization algorithm, the regions with low contrast are enhanced adaptively and the detail of dim regions are clear; and in the regions with high contrast, more detail information such as edge and texture are finer softly. Compared with the results in Fig.4 & 5, our algorithm shows the best performance with the most image detail information from the regions with different contrast. Therefore, our algorithm can enhance all the regions adaptively and improve the separated images.



Figure 2. Source images



Figure 3. Mixed images



Figure 4. Separated images (classic algorithm)



Figure 5. Separated images (classic algorithm with normalization)



Figure 6. Separated images (our algorithm)

PI in (30) has been utilized to test the quality of the new algorithm for 6 times under the same experimental condition. $C=VUA =\{c_{ij}\}$ is a M by N matrix (M=N=2). Table I shows that compared with the classic algorithm, the BIS based ours doesn't lose any information nearly with similar performance.

$$PI = \frac{1}{M(M-1)} \left\{ \sum_{j=1}^M \left[\sum_{i=1}^M \frac{|c_{ij}|}{\max_k |c_{kj}|} - 1 \right] + \sum_{i=1}^M \left[\sum_{j=1}^M \frac{|c_{ij}|}{\max_k |c_{ik}|} - 1 \right] \right\} \quad (30)$$

TABLE I. PI RESULTS

Test Number	1	2	3	4	5	6
Classicalgorithm	0.0479	0.0824	0.0412	0.0471	0.0487	0.0801
our algorithm	0.0479	0.0824	0.0411	0.0471	0.0485	0.0799

Under the same condition, we've obtained the average iteration number of each IC from 20 experiments by using the two algorithms. Table II demonstrates that: the average number of the 1st IC has been reduced by 30.1%; the total average number of all ICs has been reduced by 24.8%. So our algorithm speeds up the convergence greatly.

TABLE II. THE AVERAGE ITERATION NUMBER

IC	1st IC	2nd IC	total
classic algorithm	9.3	2	11.3
new algorithm	6.5	2	8.5

In brief, our algorithm has some advantages:

- 1) It speeds up the convergence with good BIS performance by reducing the iteration number;
- 2) By using the normalization, it can improve the display performance of the separated images and makes them be closer to the source images visually;
- 3) By using our enhancement parameter, it stresses the details and the contrasts of the separated images greatly.

VI. CONCLUSIONS

We've proposed an optimized fast fixed point algorithm based on the modified Newton iteration method and a new adaptive enhancement parameter. The experimental results demonstrate that: with good performance of blind image separation, the new algorithm could speed up the convergence greatly and enhance the separated images adaptively. So the algorithm is superior to the classic one.

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