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OSCILLATION CRITERIA FOR SECOND ORDER DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract

New oscillation criteria are given for second order nonlinear ordinary differential equations with alternating coefficients. The results involve a condition obtained by Kamenev for linear differential equations. The obtained criterion for superlinear differential equations is a complement of the work established by Kwong and Wong, and Wong and Philos, for sublinear differential equations and by Yan for linear differential equations.

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1. Introduction

This paper deals with the problem of oscillation of second order ordinary differential equations of the form

$$(a(t)x^{\bullet}(t))^{\bullet} + p(t)x^{\bullet}(t) + q(t)|x(t)|^{\lambda}\operatorname{sgn} x(t) = 0, \qquad \lambda > 0, \quad \left(^{\bullet} = \frac{d}{dt}\right), \tag{1}$$

where a, p, q: $[t_0, \infty) \to R = (-\infty, \infty)$ are continuous and $a(t) > 0$ for $t \geq t_0 > 0$.

Only such solutions, x, of equation (1) which exist on some interval $[t_x, \infty)$, $t_x \geq t_0 > 0$, are considered. A solution of equation (1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is said to be nonoscillatory. Equation (1) is called oscillatory if all of its solutions are oscillatory.

The oscillation problem for second order nonlinear ordinary differential equations with alternating coefficients has been investigated by many authors. Some of the more important and useful tests involve the average behavior of the integral of the alternating coefficients. These tests have been motivated by the averaging criterion of Kamenev [5] and its generalizations. For such averaging techniques in second order nonlinear oscillation, we choose to refer to the paper of Butler [1], Grace and Lalli [2]–[4], Kanemev [5], Kwong and Wong [6], Philos [7], [8], Wong [9], [10], Yan [11] and Yen [12].

Several years ago, Kwong and Wong [6, Theorem 1] obtained an interesting criterion for the oscillation of equation (1) with $a(t) = 1$, $p(t) = 0$ and $0 < \lambda < 1$. This criterion has been extended by Philos [8] to more general equations of the form

$$x^{\bullet\bullet}(t) + q(t)f(x(t)) = 0,$$

where $q\colon [t_0, \infty) \to R$ and $f\colon R \to R$ are continuous, $xf(x) > 0$ for $x \neq 0$, f is nondecreasing and $\int_{\pm 0} du/f(u) < \infty$.

Recently, Yan [11, Theorem 2] proved that Kwong and Wong's theorem in [6] with averaging condition of Kamenev's type [5] remains valid for equation (1) when $\lambda = 1$. The results in [6], [8], and [11] are not applicable to equation (1) with $\lambda > 1$.

Therefore, the main purpose of this paper is to establish new criteria for the oscillation of equation (1) by using an averaging condition of the type introduced by Kamenev [5]. The result obtained for the superlinear case, that is, for $\lambda > 1$, is a continuation of the work done by Kwong and Wong [6], Philos [8] and Yan [11]. We also mention that the results of this paper for the sublinear case, that is, for $0 < \lambda < 1$, are independent of those in [6], [8] and [11], but that they are similar to that of Yan [11] when $\lambda = 1$.

2. Main results

THEOREM 1. *Suppose that there exist a differentiable function* $\rho\colon [t_0, \infty) \to (0, \infty)$ *and a continuous function* $\phi\colon [t_0, \infty) \to R$ *such that*

$$\liminf_{t\to\infty} \int_{t_0}^{t} \rho(s)q(s)\,ds > -\infty, \tag{2}$$

$$a(t)\rho^{\bullet}(t) - p(t)\rho(t) = \gamma(t) \geq 0 \text{ and } \gamma^{\bullet}(t) \leq 0 \quad \text{for } t \geq t_0, \tag{3}$$

$$\int^{\infty} \eta(s)\,ds = \infty, \quad \text{where } \eta(t) = \frac{1}{a(t)\rho(t)} \tag{4}$$

and for every constant $\theta > 0$ *and* $\alpha \in (1, \infty)$

(5) $$\lim_{t\to\infty} \sup \frac{1}{t^\alpha} \int_{t_0}^{t} (t-u)^\alpha \rho(u) q(u)\, du < \infty,$$

(6) $$\lim_{t\to\infty} \inf \frac{1}{t^\alpha} \int_{s}^{t} (t-u)^{\alpha-2} \Big[(t-u)^2 \rho(u) q(u) - \frac{\theta}{4\lambda} \frac{1}{v(u)} [\gamma(u)\eta(u)(t-u) - \alpha \Big]^2 du > \phi(s)$$

and

(7) $$\lim_{t\to\infty} \int_{t_0}^{t} v(s)\phi_+^2(s)\, ds = \infty,$$

where $v(t) = \eta(t)/\int_{t_0}^{t} \eta(s)\, ds$, $\phi_+(t) = \max\{\phi(t), 0\}$.

Then equation (1) *is oscillatory for all* $\lambda > 1$.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality, we assume that $x(t) \neq 0$ for $t \geq t_0$. Furthermore, we suppose that $x(t) > 0$ for $t \geq t_0$, since the substitution $u = -x$ transforms equation (1) into an equation of the same form subject to the assumptions of the theorem. Now, we define

$$W(t) = \rho(t) \frac{a(t) x^\bullet(t)}{x^\lambda(t)} \quad \text{for } t \geq t_0.$$

Then it follows from equation (1) that

(8) $$W^\bullet(t) = -\rho(t)q(t) + \gamma(t) \frac{x^\bullet(t)}{x^\lambda(t)} - \lambda a(t)\rho(t) \frac{\dot{x}^2(t)}{x^{\lambda+1}(t)}, \qquad t \geq t_0,$$

and consequently

(9) $$a(t)\rho(t) \frac{x^\bullet(t)}{x^\lambda(t)} = c_1 - \int_{t_0}^{t} \rho(s)q(s)\, ds + \int_{t_0}^{t} \gamma(s) \frac{x^\bullet(s)}{x^\lambda(s)}\, ds - \int_{t_0}^{t} \lambda a(s)\rho(s) \left(\frac{x^\bullet(s)}{x^\beta(s)} \right)^2 ds,$$

where $\beta = (\lambda+1)/2$ and $c_1 = a(t_0)\rho(t_0)x^\bullet(t_0)/x^\lambda(t_0)$. By the Bonnet theorem, for any $t \geq t_0$ there exists $\xi \in [t_0, t]$ such that

$$\int_{t_0}^{t} \gamma(s) \frac{x^\bullet(s)}{x^\lambda(s)}\, ds = \gamma(t_0) \int_{t_0}^{\xi} \frac{x^\bullet(s)}{x^\lambda(s)}\, ds = \gamma(t_0) \int_{t_0}^{x(\xi)} u^{-\lambda}\, du$$
$$= \frac{\gamma(t_0)}{\lambda-1} [x^{1-\lambda}(t_0) - x^{1-\lambda}(\xi)] \leq \frac{\gamma(t_0)}{\lambda-1} x^{1-\lambda}(t_0) = M_1.$$

Thus, for $t \geq t_0$, we get

$$a(t)\rho(t)\frac{x^\bullet(t)}{x^\lambda(t)} + \lambda\int_{t_0}^t a(s)\rho(s)\left(\frac{x^\bullet(s)}{x^\beta(s)}\right)^2 ds + \int_{t_0}^t \rho(s)q(s)\,ds \leq L \tag{10}$$

or

$$W(t) + \lambda\int_{t_0}^t \frac{1}{a(s)\rho(s)} x^{2(\beta-1)}(s)W^2(s)\,ds + \int_{t_0}^t \rho(s)q(s)\,ds \leq L, \tag{11}$$

where $L = c_1 + M_1$.

Next, we consider the following three cases for the behavior of $x^\bullet$.

Case 1. $x^\bullet$ is oscillatory. Then there exists a sequence $\{t_m\}_{m=1,2,\ldots}$ in $[t_0, \infty)$ with $\lim_{m\to\infty} t_m = \infty$ and such that $x^\bullet(t_m) = 0$ $(m = 1, 2, \ldots)$. Thus (8) gives

$$\int_{t_0}^{t_m} \lambda a(s)\rho(s)\left(\frac{x^\bullet(s)}{x^\beta(s)}\right)^2 ds \leq L - \int_{t_0}^{t_m} \rho(s)q(s)\,ds \qquad (m = 1, 2, \ldots),$$

and hence, by taking into account (2), we conclude that

$$\int_{t_0}^{\infty} a(s)\rho(s)\left(\frac{x^\bullet(s)}{x^\beta(s)}\right)^2 ds < \infty. \tag{12}$$

So, for some positive constant N, we have

$$\int_{t_0}^{t} a(s)\rho(s)\left(\frac{x^\bullet(s)}{x^\beta(s)}\right)^2 ds \leq N \quad \text{for } t \geq t_0.$$

By the Schwarz inequality

$$\left|\int_{t_0}^t \left(\frac{x^\bullet(s)}{x^\beta(s)}\right) ds\right|^2 \leq \left(\int_{t_0}^t \frac{ds}{a(s)\rho(s)}\right)\left(\int_{t_0}^t a(s)\rho(s)\left(\frac{x^\bullet(s)}{x^\beta(s)}\right)^2 ds\right)$$
$$\leq N\int_{t_0}^t \frac{ds}{a(s)\rho(s)} = N\int_{t_0}^t \eta(s)\,ds,$$

or

$$|x^{1-\beta}(t) - x^{1-\beta}(t_0)| \leq |1-\beta|\left(N\int_{t_0}^t \eta(s)\,ds\right)^{1/2}.$$

There exists $t_1 \geq t_0$ and a constant $M > 0$ so that

$$|x^{1-\beta}(t)| \leq M\left(\int_{t_0}^t \eta(s)\,ds\right)^{1/2} \qquad \text{for all } t \geq t_1.$$

Using (13) in (8) we get

(14) $$W^{\bullet}(t) \le -\rho(t)q(t) + \frac{\gamma(t)}{a(t)\rho(t)}W(t) - \frac{\lambda}{M^2}v(t)W^2(t), \qquad t \ge t_1,$$

and consequently, for all $t > s \ge t_1$

$$\int_s^t (t-u)^\alpha W^{\bullet}(u)\,du \le -\int_s^t (t-u)^\alpha \rho(u)q(u)\,du - \int_s^t (t-u)^\alpha \left[\frac{\lambda}{M^2}v(u)W^2(u) - \gamma(u)\eta(u)W(u)\right] du.$$

Since

$$\int_s^t (t-u)^\alpha W^{\bullet}(u)\,du = -(t-s)^\alpha W(s) + \alpha\int_s^t (t-u)^{\alpha-1}W(u)\,du,$$

we obtain that

(15) $$\int_s^t (t-u)^\alpha \rho(u)q(u)\,du \le (t-s)^\alpha W(s) - \int_s^t \left[(t-u)^\alpha \frac{\lambda}{M^2}v(u)W^2(u) - (t-u)^{\alpha-1}[\gamma(u)\eta(u)(t-u)-\alpha]W(u)\right] du$$

and hence

(16) $$\int_s^t \left[(t-u)^\alpha\rho(u)q(u) - \frac{m^2}{4\lambda}\frac{(t-u)^{\alpha-2}}{v(u)}(\gamma(u)\eta(u)(t-u)-\alpha)^2\right] du \le (t-s)^\alpha W(s) - \int_s^t \left[\frac{1}{M}(t-u)^{\alpha/2}\sqrt{\lambda v(u)}W(u) - \frac{M(t-u)^{\alpha/2-1}(\gamma(u)\eta(u)(t-u)-\alpha)}{2\sqrt{\lambda v(u)}}\right]^2 du \le (t-s)^\alpha W(s), \qquad s \ge t_1.$$

Now we proceed in a way similar to that in the proof of [11, Theorem 2]. Dividing (16) by t^α and taking the lower limit as $t \to \infty$, we obtain $\phi(s) \le W(s)$, $s \ge t_1$, which implies that

(17) $$\phi_+^2(s) \le W^2(s).$$

We define functions

$$y(t) = t^{-\alpha}\int_{t_1}^t [\gamma(u)\eta(u)(t-u)-\alpha](t-u)^{\alpha-1}W(u)\,du$$

and

$$z(t) = t^{-\alpha} \int_{t_1}^{t} \frac{\lambda}{M^2} (t-u)^{\alpha} v(u) W^2(u)\, du, \qquad t \geq t_1.$$

From (15), we get

$$(18) \qquad y(t) = z(t) \leq t^{-\alpha} (t - t_1)^{\alpha} W(t_1) - \int_{t_1}^{t} (t-u)^{\alpha} \rho(u) q(u)\, du,$$

and we see that (6) implies that

$$(19) \qquad \liminf_{t\to\infty} t^{-\alpha} \int_{s}^{t} (t-u)^{\alpha} \rho(u) q(u)\, du \geq \phi(s), \qquad s \geq t_1,$$

and

$$(20) \quad \limsup_{t\to\infty} t^{-\alpha} \int_{t_1}^{t} (t-u)^{\alpha} \rho(u) q(u)\, du - \liminf_{t\to\infty} \frac{M^2}{4\lambda} t^{-\alpha} \int_{t_1}^{t} [\gamma(u)\eta(u)(t-u) - \alpha]^2 \frac{1}{v(u)} (t-u)^{\alpha-2}\, du \geq \phi(t_1).$$

Together with (5), (20) shows that there exists a sequence

$$(21) \qquad \{T_n\}_{n=1,2,\dots}, \qquad T_n > t_1, \ n = 1, 2, \dots, \ \lim_{n\to\infty} T_n = \infty,$$

such that

$$(22) \qquad \lim_{n\to\infty} \frac{M^2}{4\lambda} t^{-\alpha} \int_{t_1}^{T_n} [\gamma(u)\eta(u)(t-u) - \alpha]^2 \frac{1}{v(u)} (t-u)^{\alpha-2}\, du < \infty.$$

Next, taking the upper limit as $t \to \infty$ in (18) and using (19), we have

$$(23) \quad \limsup_{t\to\infty} [y(t) + z(t)] \leq W(t_1) - \liminf_{t\to\infty} t^{-\alpha} \int_{t_1}^{t} (t-u)^{\alpha} \rho(u) q(u)\, du \leq W(t_1) - \phi(t_1) = K.$$

Hence for all sufficiently large n

$$(24) \qquad y(T_n) + z(T_n) < K.$$

Since

$$z(t) = \frac{\lambda}{M^2} \int_{t_1}^{t} \left(1 - \frac{u}{t}\right)^{\alpha} v(u) W^2(u)\, du > 0$$

is increasing in $t > t_1$, we see that $\lim_{t\to\infty} z(t) = b$, where $b = \infty$ or is a positive constant. Suppose that $b = \infty$. Then $\lim_{n\to\infty} z(T_n) = \infty$ and by (24) we have

$$(25) \qquad \lim_{n\to\infty} y(T_n) = -\infty.$$

Now (24) and (25) lead to

$$\frac{y(T_n)}{z(T_n)} + 1 < \sigma, \quad \text{where } 0 < \sigma < 1 \text{ is a constant},$$

that is,

$$\frac{y(T_n)}{z(T_n)} < \sigma - 1 < 0 \quad \text{for all large } T_n. \tag{26}$$

On the other hand, by the Schwarz inequality we have

$$\begin{aligned} 0 &\le T_n^{-2\alpha}\left(\int_{t_1}^{T_n}[\gamma(u)\eta(u)(T_n-u)-\alpha](T_n-u)^{\alpha-1}W(u)\,du\right)^2 \\ &\le \left(T_n^{-\alpha}\int_{t_1}^{T_n}[\gamma(u)\eta(u)(T_n-u)-\alpha]^2(T_n-u)^{\alpha-2}\frac{1}{v(u)}\,du\right) \\ &\quad\cdot\left(T_n^{-\alpha}\int_{t_1}^{T_n}(t_n-u)^{\alpha}v(u)W^2(u)\,du\right), \quad \text{for all large } T_n. \end{aligned}$$

Thus,

$$0 \le \frac{y^2(T_n)}{z(T_n)} \le \frac{M^2}{\lambda}T_n^{-\alpha}\int_{t_1}^{T_n}[\gamma(u)\eta(u)(T_n-u)-\alpha]^2\frac{1}{v(u)}(T_n-u)^{\alpha-2}\,du.$$

By (22), we get

$$\begin{aligned} 0 &\le \lim_{n\to\infty}\frac{y^{(}T_n)}{z(T_n)} \\ &\le \frac{M^2}{\lambda}\lim_{n\to\infty}\int_{t_1}^{T_n}[\gamma(u)\eta(u)(T_n-u)-\alpha]^2\frac{1}{v(u)}(T_n-u)^{\alpha-2}\,du < \infty, \end{aligned}$$

which contradicts (25) and (26). Hence $\lim_{t\to\infty} z(t) = b < \infty$. Using (17), we obtain

$$\begin{aligned} \lim_{t\to\infty} t^{-\alpha}\int_{t_1}^{t}(t-u)^{\alpha}v(u)\phi_+^2(u)\,du &\le \lim_{t\to\infty} t^{-\alpha}\int_{t_1}^{t}(t-u)^{\alpha}v(u)W^2(u)\,du \\ &= \frac{M^2 b}{\lambda} < \infty, \end{aligned}$$

which contradicts condition (7).

Case 2. $x^\bullet > 0$ on $[T, \infty)$ for some $T \ge t_0$. From (2) and (10) it follows that (12) holds, and hence we can complete the proof by the procedure of Case 1.

Case 3. $x^\bullet < 0$ on $[T, \infty)$ for some $T \ge t_0$. If (12) holds, then we can arrive at a contradiction by the procedure of Case 1. So, we suppose that the integral in (12) diverges. Using (2) in (10) we have

$$-a(t)\rho(t)\frac{x^{\bullet}(t)}{x^{\lambda}(t)} \geq -C+\lambda\int_T^t a(s)\rho(s)\left(\frac{(x^{\bullet}(s))^2}{x^{\lambda+1}(s)}\right)ds, \tag{27}$$

where C is a constant. By the assumptions, we can choose $T_1 \geq T$ so that

$$\lambda\int_T^{T_1} a(s)\rho(s)\frac{(x^{\bullet}(s))^2}{x^{\lambda+1}(s)}ds = 1+C$$

and then for any $t \geq T_1$ we get

$$\frac{-a(t)\rho(t)\frac{x^{\bullet}(t)}{x^{\lambda}(t)}\left(-\lambda\frac{x^{\bullet}(t)}{x(t)}\right)}{-C+\lambda\int_T^t a(s)\rho(s)\frac{(x^{\bullet}(s))^2}{x^{\lambda+1}(s)}ds} \geq -\lambda\frac{x^{\bullet}(t)}{x(t)}.$$

Integrating the above inequality from T_1 to t we obtain

$$\ln\left[-C+\lambda\int_T^t a(s)\rho(s)\frac{(x^{\bullet}(s))^2}{x^{\lambda+1}(s)}ds\right] \geq \lambda\int_{T_1}^t\left(\frac{-x^{\bullet}(s)}{x(s)}\right)ds = \ln\left(\frac{x(T_1)}{x(t)}\right)^{\lambda}$$

which together with (27) yields

$$-a(t)\rho(t)\frac{x^{\bullet}(t)}{x^{\lambda}(t)} \geq \left(\frac{x(T_1)^{\lambda}}{x(t)}\right),$$

from which it follows that

$$x^{\bullet}(t) \leq -(-x(T_1))^{\lambda}\frac{1}{a(t)\rho(t)} < 0 \quad \text{for } t \geq T_1$$

or

$$x(t) \leq x(T_1)-(x(T_1))^{\lambda}\int_{T_1}^t\frac{1}{a(s)\rho(s)}ds \to -\infty \quad \text{as } t\to\infty,$$

contradicting the fact that $x(t) > 0$ for $t \geq t_0$. This completes the proof.

The follow result is concerned with the oscillatory behavior of equation (1) for all $\lambda > 0$.

Corollary 1. *Let the differentiable function ρ assumed in Theorem 1 be defined by*

$$\rho(t) = \exp\left(\int_{t_0}^t\frac{p(s)}{a(s)}ds\right) \quad \textit{for } t \geq t_0, \tag{28}$$

and let conditions (2), (4)–(7) hold. Then equation (1) is oscillatory for all $\lambda > 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$ for $t \geq t_0$. From (28), we see that $\gamma(t) = 0$ for all $t \geq t_0$. Now, if W is defined as in the proof of Theorem 1, then we obtain (10) or (11). The rest of the proof is similar to that of Theorem 1 and hence is omitted.

In the following corollary we study the oscillatory behavior of the undamped equation

$$(a(t)x^{\bullet}(t))^{\bullet} + q(t)|x(t)|^{\lambda}\operatorname{sgn} x(t) = 0, \qquad \lambda > 0, \tag{29}$$

where the functions a and q are defined as in equation (1).

COROLLARY 2. *Suppose that there exists a continuous function* $\phi\colon [t_0, \infty) \to R$ *and* $\alpha \in (1, \infty)$ *such that*

$$\liminf_{t\to\infty} \int_{t_0}^{t} q(s)\,ds > -\infty, \tag{30}$$

$$\int^{\infty} \frac{1}{a(s)}\,ds = \infty, \tag{31}$$

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha}} \int_{t_0}^{t} (t-u)^{\alpha} q(u)\,du < \infty, \tag{32}$$

and for every $\theta > 0$

$$\liminf_{t\to\infty} \frac{1}{t^{\alpha}} \int_{s}^{t} (t-u)^{\alpha-2}\left[(t-u)^2 q(u) - \frac{\theta\alpha^2}{4\lambda v(u)}\right] du > \phi(s), \tag{33}$$

and

$$\int^{\infty} v(s)\phi_{+}^{2}(s)\,ds = \infty, \tag{34}$$

where $\phi_{+}(t) = \max\{\phi(t), 0\}$ *and* $v(t) = \frac{1}{a(t)}\left(\int_{t_0}^{t} \frac{1}{a(s)}\,ds\right)^{-1}$.

Then equation (29) *is oscillatory for all* $\lambda > 0$.

PROOF. This follows from Corollary 1 if we let $p(t) = 0$ and $\rho(t) = 1$ for $t \geq t_0$.

For illustration we consider the following example.

EXAMPLE 1. Consider the differential equation

$$\left(\frac{1}{t}x^{\bullet}(t)\right)^{\bullet} + \frac{1}{t^2}x^{\bullet}(t) + \frac{\cos t}{t}|x(t)|^{\lambda}\operatorname{sgn} x(t) = 0, \qquad \lambda > 0, \quad t \geq t_0 > 0. \tag{35}$$

Taking $\rho(t) = t$ and $\alpha = 2$, we have

$$\gamma(t) = 0, \quad v(t) = \frac{1}{t} \quad \text{for } t \geq t_0 > 0,$$

$$\lim_{t\to\infty} \inf \int_{t_0}^{t} \cos s\, ds > -\infty,$$

$$\lim_{t\to\infty} \sup \frac{1}{t^2} \int_{t_0}^{t} (t-u)^2 \cos u\, du = -\sin t_0 < \infty,$$

$$\lim_{t\to\infty} \inf \frac{1}{t^2} \int_{s}^{t} \left[(t-u)^2 \cos u - \frac{\theta}{\lambda u}\right] du \geq -\sin s - k,$$

where k is a positive constant. Set $\phi(s) = -\sin s - k$.

Next, we consider an integer N such that $(2N+1)\pi + \pi/4 \geq t_0$. Then for all integer $n \geq N$ and $(2n+1)\pi + \pi/4 \leq s \leq (2n+1)\pi - \pi/4$,

$$\phi(s) = -\sin s - k \geq \delta s,$$

where δ is a small constant. Thus,

$$\lim_{t\to\infty} \int_{t_0}^{t} v(s)\phi_+^2(s)\, ds \geq \sum_{N=n}^{\infty} \delta^2 \int_{(2n+1)\pi+\pi/4}^{2(n+1)\pi-\pi/4} s\, ds = \infty.$$

Hence equation (35) is oscillatory for all $\lambda > 0$ by Corollary 1, whereas none of the known criteria can cover this result.

REMARK 1. It is easy to check that the conditions (2), (3) and (4) of Corollary 1 are superfluous if $0 < \lambda \leq 1$.

REMARK 2. Corollary 1, when $0 < \lambda < 1$, is new and is independent of the results of Kwong and Wong [6] and Philos [8], while Corollary 1 for the linear case $(\lambda = 1)$ is of the type obtained by Tan [11].

The following result is concerned with the oscillatory solution of equation (1) when condition (5) fails.

THEOREM 2. *Suppose that there exist a differentiable function* $\rho: [t_0, \infty) \to (0, \infty)$ *and a constant* $\alpha \in (1, \infty)$ *such that conditions* (2)–(4) *hold and for every* $\theta > 0$

$$\lim_{t\to\infty} \sup \frac{1}{t^\alpha} \int_{t_0}^{t} (t-u)^{\alpha-2} \left[(t-u)^2 \rho(u) q(u) - \frac{\theta}{4\lambda} \frac{1}{v(u)} [\gamma(u)\eta(u)(t-u) - \alpha]^2\right] du = \infty, \tag{36}$$

where η *and* v *are defined as in Theorem* 1.

Then equation (1) *is oscillatory for all* $\lambda > 1$.

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$ for $t \geq t_0$. As in the proof of Theorem 1 (Case 1), we obtain (16). Divide

(16) by t^α and take the upper limit as $t \to \infty$. Using (36) we obtain a contradiction. The proofs in the cases when $x^\bullet$ is either positive or negative on $[T, \infty)$, $T \geq t_0$, are similar to the proofs in Cases 2 and 3 of Theorem 1, and hence will be omitted.

COROLLARY 3. *Let condition* (36) *in Theorem* 1 *be replaced by*

$$\limsup_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} (t-u)^\alpha \rho(u) q(u)\, du = \infty \tag{37}$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{(t-u)^{\alpha-2}}{v(u)} [\gamma(u)\eta(u)(t-u) - \alpha]^2\, du < \infty. \tag{38}$$

Then the conclusions of Theorem 2 *holds.*

The following corollaries extend Wong's criterion in [10] to more general equations of the form of (1) and (29). The proofs are immediate consequences of Theorem 2 and will be omitted.

COROLLARY 4. *Let the function* ρ *in Theorem* 2 *be defined by* (28) *and suppose that conditions* (2), (4) *and* (36) *hold. Then equation* (1) *is oscillatory for all* $\lambda > 0$.

COROLLARY 5. *Let conditions* (30) *and* (31) *hold for every* $\theta > 0$ *and* $\alpha \in (1, \infty)$ *and suppose*

$$\limsup_{t\to\infty} \frac{1}{t^\alpha} \int_{t_0}^{t} (t-u)^{\alpha-2} \left[(t-u)^2 q(u) - \frac{\theta\alpha^2}{4v(u)} \right] du = \infty, \tag{39}$$

where

$$v(t) = \frac{1}{a(t)} \left(\int_{t_0}^{t} \frac{1}{a(s)}\, ds \right)^{-1}.$$

Then equation (29) *is oscillatory for all* $\lambda > 0$.

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