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Existence of Three Transition Temperatures in Decorated Triangular and Square Ising Lattices with Anisotropic Couplings

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It is shown that those decorated triangular and square lattices on which we have investigated the ordering in Ising spins distributed on the lattice sites can have three successive transition temperatures if the exchange couplings are taken different from one direction to another direction of the spin arrays on the lattice. The semi-decorated square lattice which has been found recently by Syozi to have three transition temperatures corresponds to a special one of the square lattice in the present case.

§ 1. Introduction

The author has suggested recently[1] (hereafter referred to as I) that certain decorated lattices of Ising spins in which there exist either ferromagnetic or antiferromagnetic exchange coupling between every pair of nearest neighbouring as well as of second-nearest-neighbouring spins may exhibit phase transitions three times as the temperature rises from the lowest value. It possibly occurs in the decorated lattices of simple-cubic and body-centred cubic lattices. The decorated lattices of plane lattices, however, cannot have this property.

Syozi has found[2] (referred to as II) that the square lattice decorated only on either all longitudinal or all transverse bonds can exhibit three successive phase transitions as the temperature is varied throughout the whole region.

The circumstance of the same sort is found in the case of triangular lattice. We investigate in § 2 a partly-decorated triangular lattice as shown in Fig. 1; viz. the triangular lattice in which no transversely-lying bonds are decorated but all obliquely-drawn bonds are decorated in the same manner throughout them. The partition function of the system of Ising spins distributed on that lattice can be reduced to the partition function of a simple triangular lattice by a process of iteration. By making use of the dual and the star-triangle transformations on that triangular lattice we determine the transition temperatures and show it is possible to select the parameters of exchange couplings so that the system has three successive transition temperatures.

We consider in § 3 the triangular and square lattices in which every bond is decorated and the parameters of exchange coupling in each decorated bond are different by its direction; viz. the decorated lattices with anisotropic exchange couplings, which exhibit phase transitions at three different points throughout the whole temperature range. Syozi's model and the model we propose in § 2 belong to a certain special case of the above-mentioned kind of decorated lattice.

In § 4 are given some conjectures concerning those problems on the decorated lattices which we are investigating.

§ 2. Partly-decorated triangular lattice

The triangular lattice which is decorated in such a way as shown in Fig. 1 is investigated in this section. The parameters of exchange couplings on either kind of bond are also shown in Fig. 1.

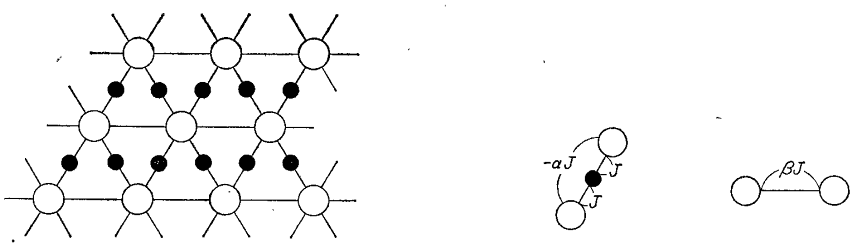

Fig. 1. Partly-decorated triangular lattice.

By carrying out first the summation on the spins occupying the decoration lattice sites as done in I, we can rewrite the partition function Z of the system as

$$Z=A^{2N}Z_t(x,\ y), \tag{1}$$

where N denotes the total number of white sites in Fig. 1 and $Z_t(x, y)$ denotes the partition function of the triangular lattice of Ising spins with the exchange couplings divided by kT, x and y, as shown in Fig. 2; viz.

$$Z_t(x,\ y)=\sum_{\mu_1=\pm1,}\cdots\sum_{\mu_N=\pm1}\exp\Big(x\sum_{\text{oblique}}\mu_i\mu_j+y\sum_{\text{transverse}}\mu_i\mu_k\Big). \tag{2}$$

In the exponent in (2) the first and second summands should be made over all pairs of spins connected by the oblique and transverse bond lines. The parameters A and x are those defind by (4) in I in which H_2 is put equal to zero and (8) in I respectively and these parameters and y are expressed in terms of the variable $J'\equiv J/(kT)$ as

$$\begin{aligned} x&=\frac{1}{2}\ln\ \mathrm{ch}(2J')-\alpha J',\\ y&=\beta J',\ A=2\ \mathrm{ch}^{1/2}(2J'). \end{aligned} \tag{3}$$

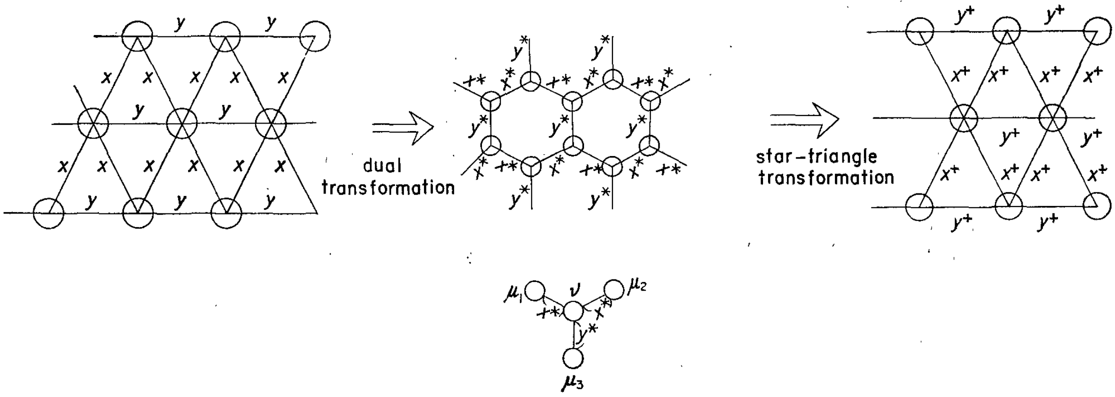

Fig. 2. Dual and star-triangle transformations.

The so-called dual transformation[3] relates the partition function $Z_t(x, y)$ of the triangular lattice with $Z_h(x^*, y^*)$ of the honeycomb lattice, where the pairs x and y and x^* and y^* denote the exchange couplings on the bonds as shown in Fig. 2 and are related with each other as

$$\exp(2x^*) = \coth|x|, \ \exp(2y^*) = \coth|y|. \tag{4}$$

The star-triangle transformation which is the result of an iteration on one spin in the honeycomb pattern in Fig. 2 gives another relation between the partition functions of the triangular and honeycomb lattices. The summation on the variable ν of a spin which is connected with three spins μ_1, μ_2 and μ_3 by the three bonds of Y-shape in all, as shown in Fig. 2, gives the relation

$$\sum_{\nu=\pm} \exp[x^*(\mu_1+\mu_2)\nu + y^*\mu_3\nu] = B \ \exp[x^+(\mu_1+\mu_2)\mu_3 + y^+\mu_1\mu_2]$$

and defines the star-triangle transformation, where the pair of parameters are related with each other by the equations

$$\begin{aligned} \exp(4x^+) &= [\mathrm{ch}(2x^*+y^*)]\cdot[\mathrm{ch}(2x^*-y^*)]^{-1}, \\ \exp(4y^+) &= [\mathrm{ch}(2x^*+y^*)\,\mathrm{ch}(2x^*-y^*)]\cdot(\mathrm{ch}\ y^*)^{-2}, \\ B^4 &= 2^4\,\mathrm{ch}(2x^*+y^*)\ \mathrm{ch}(2x^*-y^*)\ (\mathrm{ch}\ y^*)^2. \end{aligned} \tag{5}$$

By combining (4) and (5) we get the relation between one pair x and y and another pair x^+ and y^+ which both represent the pair of exchange coupling parameters, as shown in Fig. 2, in the triangular lattice of Ising spins interacting anisotropically; viz.

$$\begin{aligned} \exp(4x^+) &= \frac{1+\mathrm{th}^2 x\ \mathrm{th}\, y}{\mathrm{th}\, y + \mathrm{th}^2 x}, \\ \exp(4y^+) &= \frac{(1+\mathrm{th}^2 x\ \mathrm{th}\, y)\ (\mathrm{th}\, y+\mathrm{th}^2 x)}{(1+\mathrm{th}\, y)^2\ \mathrm{th}^2 x}. \end{aligned} \tag{6}$$

The transition temperature is determined by the equation $x^+ = |x|$ or $y^+ = y$

and both of these two equations are satisfied if

$$\frac{2-(1+\mathrm{th}\,|x|)^2}{2-(1-\mathrm{th}\,|x|)^2}=\mathrm{th}\,y. \tag{7}$$

On substituting for x and y from (3), Eq. (7) can be rewritten as the equation for $J'=J/(kT)$

$$[f(J')]^{\pm 1}=g(J'), \tag{8}$$

where the functions $f(J')$ and $g(J')$ are defined by

$$\begin{aligned} f(J') &\equiv \frac{[e^{-2\alpha J'}\,\mathrm{ch}(2J')-1]^2-2}{2-[e^{-2\alpha J'}\,\mathrm{ch}(2J')+1]^2}, \\ g(J') &\equiv \mathrm{th}(\beta J') \end{aligned} \tag{9}$$

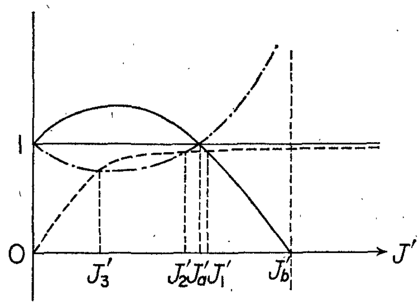

Fig. 3. Relations of $f(J')$, $f(J')^{-1}$ and $g(J')$ defined by (9).

$f(J')$: ————

$f(J')^{-1}$: —·—·—

$g(J')$: ---------

and the plus or minus sign in the exponent in (8) should be taken according as x is either positive or negative.

The curves which show the relations of $f(J')$, $[f(J')]^{-1}$ and $g(J')$ with J' are schematically drawn in the case $0<\alpha<1$ in Fig. 3, where J_a' and J_b' are determined by the equations

$$e^{-2\alpha J'_a}\,\mathrm{ch}\,(2J_a')=1,$$

$$(\sqrt{2}-1)\,e^{-2\alpha J'_b}\,\mathrm{ch}\,(2J_b')=1.$$

So far as the constant β is large enough for a certain value of $\alpha\,(0<\alpha<1)$, we have one intersection of the $f(J')$ versus J' curve with the $g(J')$ versus J' curve and two intersections of the $f(J')^{-1}$ versus J' curve with the $g(J')$ versus J' curve. If we denote the values of J' at these intersections as J_1', J_2' and J_3' ($J_1'>J_2'>J_3'$), we can conclude that the system is ferromagnetic (or ferrimagnetic) below the temperature $T_1\equiv J/(kJ_1')$, paramagnetic between $T_1\equiv J/(kJ_1')$ and $T_2\equiv J/(kJ_2')$, antiferromagnetic between T_2 and $T_3\equiv J/(kJ_3')$ and paramagnetic above T_3.

§ 3. Decorated triangular and square lattices with anisotropic exchange couplings

Quite a natural extension of the model we have proposed in I is done by assuming the exchange-coupling parameters on each bond different by the direction of that bond. We show in Fig. 4 such an extension, in which two kinds of coupling parameters appear on each bond; viz. J and αJ in the one kind and I and βI in the second kind.

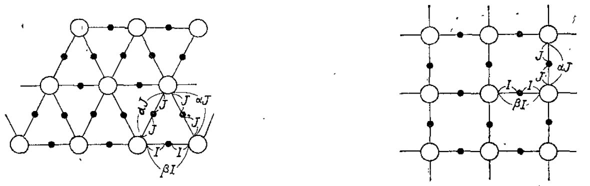

Fig. 4. Decorated triangular and square lattices with anisotropic couplings.

We consider the case that both I and J are positive, β is positive and α is negative, $|\alpha|$ being between zero and unity. We rewrite $|\alpha|$ as α and now $0<\alpha<1$. In this case $g(J')$ given in (9) should be replaced by

$$g(J')\equiv\frac{e^{2\beta I'}\,\mathrm{ch}(2I')-1}{e^{2\beta I'}\,\mathrm{ch}(2I')+1}, \tag{10}$$

where $I'\equiv I/(kT)=(I/J)J'$.

The roots of Eq. (8), in which $f(J')$ is defined by the expression in (9) and $g(J')$ by (10), give the transition temperatures of the decorated triangular lattice with anisotropic couplings. The curve of $g(J')$ versus J' relation is quite similar to the one pictured in Fig. 3 by a broken line and three roots are found if the parameter, either β or I, or both of them are sufficiently large for certain fixed values of the parameters α and J.

In the case of the decorated square lattice with anisotropic couplings, $f(J')$ should be regarded as the function which is obtained by Syozi in II; viz.

$$f(J')\equiv e^{-2\alpha J'}\,\mathrm{ch}(2J'), \tag{11}$$

and $g(J')$ as given by (10). Then we can find three roots of Eq. (8) in so far as the parameters satisfy a certain condition and have three transition temperatures.

It can be concluded that there appears ferromagnetic, paramagnetic, antiferromagnetic and paramagnetic phases successively as the temperature rises from zero to infinity.

In a limiting case that βI has an appropriate value and I vanishes, the system reduces to the one investigated by Syozi[2] in the case of square lattice and to the one discussed in § 2 in the present article in the case of triangular lattice.

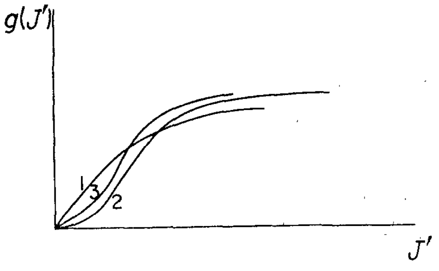

Fig. 5. Relation between $g(J')$ and J' in the extreme and intermediate choices of coupling parameters.
1: $\beta I\neq0$, $I=0$;
2: $I\neq0$, $\beta I=0$;
3: $I\neq0$, $\beta I\neq0$.

In another limiting case that I has an appropriate value and β is zero, the curve of $g(J')$ versus J' has a vanishing slope at the origin as shown in Fig. 5, where $g(J')$ is drawn in three typical cases; viz. in the first case $\beta I\neq0$ and $I=0$, in the second case $I\neq0$ and $\beta I=0$ and in the third case $I\neq0$ and $\beta I\neq0$.

Fig. 6. Alternate spin alignments in the antiferromagnetic phase.
+: spin up, −: spin down.

In all these cases three transition temperatures can be found if the values of parameters are chosen appropriately.

The spin arrangement on the white lattice sites in the antiferromagnetic phase which appears between the temperatures T_2 and T_3 is shown in Fig. 6 in general including the two limiting cases mentioned above. All spins on the black lattice sites have a vanishing magnitude in the same way as mentioned in I (cf. Fig. 7 in that paper).

§ 4. Concluding remarks

It has been shown that three transition temperatures are found in the decorated triangular and square lattices with anisotropic exchange couplings.

The possibility of finding such a property may be expected for the dice and the hempleaf lattices of Ising spins with anisotropic exchange couplings. It is further interesting to investigate more detailed magnetic properties, e.g. the relation between the magnetization and the magnetic field not only in the case of existence of three transition points but also in the case of only one transition point, because it is expected that there might appear a certain interesting feature due to the non-weak coherence in the spin assembly existing even in the paramagnetic region. We are investigating these problems.

References

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