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The Effect of Domain Shape on the Number of Positive Solutions of Certain Nonlinear Equations, II

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This paper is a sequel to the author's earlier work [7]. We extend the ideas in a number of directions. However, once again, the key idea is to produce examples with unusual numbers of solutions or solution structure by proving rather general results on how solutions change with domain changes. Here the domain change is in a very general sense and the problem we are considering is of the form

$$\begin{aligned} -\Delta u &= \lambda f(u) \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

First, we improve one of the main results by constructing star shaped sets which approximate k disjoint open star shaped sets on each of which our equation is non-degenerate. It follows that can construct star shaped domains on which our equation has many solutions. (In [7], we only constructed examples where there were more solutions than for a ball.)

Second, we use our ideas to obtain quite a good global understanding of some convex asymptotically linear problems on domains with a narrow joining strip. In particular, it turns out that the positive solutions are not connected. This seems to provide the first example of this type with $f(0)>0$ and $f(y)>0$ for $y>0$.

Third, we prove that our techniques are valid for non-self-adjoint problems (including systems). This is obvious except for the spectrum. Under natural assumptions, we prove that the number of eigenvalues with negative real part changes continuously.

Fourth, we study briefly similar results to those in [7] for the Neumann boundary value problem. Here we consider problems that are extremely sublinear. In this case, we can prove the existence of solutions under assumptions where the spectrum of the linear part (for Neumann boundary conditions) need not depend continuously upon the domain. Under additional assumptions on the domains, our method can also be used to obtain

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uniqueness and stability results. This can be used to give an alternative derivation of the result of Matano and Mimura [25] in some cases. Moreover, we obtain local uniqueness. Unlike the work in [25], our methods can be used for problems where we do not have increasing maps (for some order structure). Our results are less precise than those in Jimbo [20–22]) but hold for more general domains. They seem more flexible than the methods in Vegas [29].

Last, we briefly consider parabolic problems. In particular, we consider initial value problems and some problems which are periodic in time and where we look for periodic solutions (problems similar to those in [12] or [19]). It seems that there are many more initial value problems for parabolic equations which could be studied.

1. Arbitrarily Many Positive Solutions on Star Shaped Domains

We consider the equations

$$\begin{aligned} -\Delta u &= f(u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2}$$

Suppose that Ω is a bounded connected open set in R^m such that Ω has smooth boundary away from 0 and such that, near 0, Ω is the interior of a cone C_1 over a convex body C not containing zero and with vertex 0. Here by solutions we mean elements of $\dot{W}^{1,2}(\Omega) \cap L^\infty(\Omega)$. Suppose that G is a relatively open subset of the smooth part of $\partial\Omega$. We will prove that, if either $f(0)=0$ or $f(R) \subseteq [0, \infty)$, we can make a C^2 small perturbation of the part of $\partial\Omega$ in G to ensure that the solutions of (2) are non-degenerate.

Assuming this for a moment, we will construct our examples. We choose $\tilde{\Omega}_1$ as above such that $\tilde{\Omega}_1 \cup \{0\}$ is star shaped from 0 and such that $\tilde{\Omega}_1$ is contained in the cone C_1. We choose G_1 a relatively open subset of $\partial\tilde{\Omega}_1$ so that $\underline{n} \cdot r > 0$ on G_1, where $\underline{n}$ is the outward normal to $\partial\tilde{\Omega}_1$. It follows easily from this that, if we make a C^2 small perturbation of $\partial\tilde{\Omega}_1$ near G, then the star shapedness condition still holds. Moreover, can be assume by the result mentioned in the previous paragraph that (2) for $\Omega = \tilde{\Omega}_1$ has only non-degenerate solutions. Since we can choose C_1 to be a "narrow" cone, we see that we can, for any positive integer k, choose $\tilde{\Omega}_1, ..., \tilde{\Omega}_k$ as above such that the closure of any two of these sets intersects only at zero. Define $\Omega_0 = \bigcup_{i=1}^k \tilde{\Omega}_i$. Since $\{0\}$ has zero capacity, we can argue as on p. 452 of [13] to deduce that $u \in \dot{W}^{1,2}(\Omega_0) \cap L^\infty(\Omega_0)$ and is a weak solution of (2) on Ω_0 if and only if $u \in \dot{W}^{1,2}(\tilde{\Omega}_i) \cap L^\infty(\tilde{\Omega}_i)$ for each i and is weak solution on every $\tilde{\Omega}_i$. Since we have arranged above that the solutions on each $\tilde{\Omega}_i$ are non-degenerate, it follows that solutions on Ω_0 are non-degenerate. Let $\Omega_n = \Omega_0 \cup B_{1/n}$ where $B_{1/n}$ denotes the ball with centre zero and radius $1/n$. Then Ω_n is star shaped, Ω_n has Lipschitz boundary and $\Omega_n \to \Omega_0$ as $n \to \infty$ in the sense of [7, Sect. 1]. (If we prefer, we can easily modify Ω_n to have smooth boundary.) As in [7], we see that near each solution u_0 of (2) for $\Omega = \Omega_0$ there is a unique solution of u_n of (2) for $\Omega = \Omega_n$ if n is large. Here I should explain what is meant by near. We choose $\tilde{B}$ bounded containing $\bigcup_{n=0}^{\infty} \Omega_n$ and extend u_n from Ω_n to $\tilde{B}$ by defining it to be zero outside Ω_n. If f has polynomial growth, when we say that u_n is near u_0 we mean that $u_n - u_0$ is small in $L^p(\tilde{B})$ for suitable large p. If f does not have polynomial growth, we use suitable Orlicz space instead of L^p spaces. Now assume that u_0 is non-negative. If $u_0(x) > 0$ on Ω_0 or $f(0) > 0$ or $u_0 \equiv 0$, then u_n is non-negative on Ω_n for n large. In the general case, let $\Omega_t = \{x \in \Omega_0 \colon u_0(x) = 0\}$. Then Ω_t is a union of components of Ω_0, possibly empty. Let $\tilde{\lambda}_1$ be the first eigenvalue of $-\Delta$ on Ω_t. If $f'(0) < \tilde{\lambda}_1$, then u_n is non-negative for all large n while, if $f(0) = 0$ and $f'(0) > \tilde{\lambda}_1$, then u_n changes sign for all large n. This is a slight generalization of Theorem 2 in [7] (and has essentially the same proof). Note that our non-degeneracy condition ensures that $f'(0) \neq \tilde{\lambda}_1$, when $f(0) = 0$. Hence we see that, if n is large and if either $f(0) > 0$ or $f(0) = f'(0) = 0$, then, near every positive solution of (2) on Ω_0, there is a unique positive solution of (2) on Ω_n. Hence we see that, if we can choose the $\tilde{\Omega}_i$ so that (2) for $\Omega = \tilde{\Omega}_i$ has at least two non-negative solutions, and such that our non-degeneracy condition holds (with $f'(0) < \tilde{\lambda}_1$ if $f(0) = 0$), then there will be at least 2^k non-negative solutions of (2) for $\Omega = \Omega_n$ with n large. This will give our required examples of star shaped domains with arbitrarily many positive solutions.

We apply this idea to two examples. The non-linearities are the two main ones we considered in [7]. First let $f(y) = y^\alpha$, where $1 < \alpha < (m+2)(m-2)^{-1}$. In this case, the zero solution is one solution on $\tilde{\Omega}_1$. We can easily obtain a second solution by maximizing $\int_{\tilde{\Omega}_1} u^{p+1}$ over $\{u \in \dot{W}^{1,2}(\tilde{\Omega}_1)\colon \int_{\tilde{\Omega}_1} |\nabla u|^2 \, dx = 1\}$ and rescaling. The standard $L^p - L^q$ regularity theory as in [16, Theorem 8.15] or [28] and bootstrapping imply that the solutions are in L^∞. (Similar arguments appear in [7].) Thus, we see that there are star shaped domains Ω for which

$$\begin{aligned} -\Delta u &= u^\alpha \quad && \text{in } \Omega \\ u &= 0 \quad && \text{on } \partial\Omega \end{aligned}$$

has arbitrarily many positive solutions (but a finite number).

We now consider the case where $f(y) = \lambda \exp y$ and $n \leq 8$. We will prove that for suitable λ we can have arbitrarily many positive solutions. It suffices to find a λ for which (2) (with $\Omega=\Omega_1$ and $f(y)=\lambda e^y$) has two solutions (necessarily positive). Note that we can choose $\tilde{\Omega}_2, ..., \tilde{\Omega}_k$ to be rotations of $\tilde{\Omega}_1$. Note also that $(-\Delta)^{-1}$ (with Dirichlet boundary conditions) is a continuous map of $L^\infty(\tilde{\Omega}_1)$ into a Holder space (by Gilbarg and Trudinger [16, Theorem 8.29]) and hence is a compact linear map of $L^\infty(\tilde{\Omega}_1)$ into $C_0(\tilde{\Omega}_1)$, where $C_0(\tilde{\Omega}_1)$ denotes the continuous functions on the closure of $\tilde{\Omega}_1$ vanishing on $\partial\tilde{\Omega}_1$. We now consider the map $A(x, \lambda)=\lambda(-\Delta)^{-1}\exp x$ as a map of $C_0(\tilde{\Omega}_1)\times R$ into $C_0(\tilde{\Omega}_1)$. It is easy to see that it satisfies the basic assumptions on p. 142 of [10]. Here we need to use the remarks following those assumptions, we need to know that the demi-interior elements of the natural cone K in $C_0(\tilde{\Omega}_1)$ are the elements strictly positive in $\tilde{\Omega}_1$ (by [11, Lemma 2]) and that $(-\Delta)^{-1}$ maps non-zero non-negative elements of $L^\infty(\Omega)$ into functions strictly positive on Ω (by the weak maximum principle). Hence, there is an interval $[0, \lambda^*]$ and an increasing map $\lambda\to z(\lambda)$ defined on $[0, \lambda^*)$ such that $z(\lambda)$ is the minimal fixed point of $A(\ ,\lambda)$ and $r(A_1^1(z(\lambda), \lambda))\leqslant 1$, where r denotes the spectral radius and A_1^1 denotes the partial derivative in the first variable. Moreover there are no non-negative solutions for $\lambda>\lambda^*$. Since $e^y\geqslant y$, easy estimates show that $\lambda^*<\infty$. Suppose we can prove that $\{z(\lambda)\colon\ 0\leqslant\lambda<\lambda^*\}$ is bounded in $C_0(\tilde{\Omega}_1)$. Since $z(\lambda)=A(z(\lambda), \lambda)$ and A is compact, it follows easily that $\{z(\lambda)\colon\ 0\leqslant\lambda<\lambda^*\}$ is pre-compact in $C_0(\tilde{\Omega}_1)$. Since $z(\lambda)$ is increasing, it follows that there is a z^* in $C_0(\tilde{\Omega}_1)$ such that $z(\lambda)\to z^*$ as $\lambda\to\lambda^*$. Moreover $z^*=A(z^*, \lambda^*)$. z^* must be the minimal solution for $\lambda=\lambda^*$ (as in [10]) and thus $r(A_1^1(z(\lambda^*), \lambda^*))\leqslant 1$. However, equality must hold since otherwise we could extend the branch beyond λ^*. Note that $N(I-A_1^1(z(\lambda), \lambda))$ is spanned by a demi-interior element of K and the f which spans the adjoint kernel is strictly positive on K. Using this, we can easily modify the argument at the beginning of Section 2 in [7] to deduce that there are two solutions of $x=A(x, \lambda)$ near $z(\lambda^*)$ for each λ near λ^* but less than λ^* and that these are non-degenerate. Thus the proof reduces to establishing the bound on $\{z(\lambda)\colon\ 0\leqslant\lambda<\lambda^*\}$. This follows by the argument of Crandall and Rabinowitz [6]. Only one thing needs to be changed. In their bootstrapping argument, they use that if $-\Delta u=\tilde{f}$, where $\tilde{f}\in L^p(\Omega)$, then $u\in W^{2,p}(\Omega)$ and thus, by the Sobolev embedding theorem, $u\in L^q(\Omega)$ for a certain $q>p$. We cannot use this argument here but instead use Trudinger [28] to deduce directly that $u\in L^q(\Omega)$ for the same q. Note that, in the case of an exponential non-linearity, we do not need the domain perturbations. Note that our methods could be used to handle many other non-linearities f.

It remains to prove our domain perturbation result for $\tilde{\Omega}_1$. Choose a relatively open subset U of $\partial\tilde{\Omega}_1$, such that $\bar{U}$ is contained in the smooth part of $\partial\tilde{\Omega}_1$ aind such that $r\cdot\underset{\sim}{n}>0$ on $\bar{U}$ where $\underset{\sim}{n}$ is the outward normal. Finally, choose a small closed neighbourhood V of U in R^m. Without loss of generality we can assume $\bar{U}$ and V are manifolds with boundary. We let W be the subspace of functions in $C^{2,\alpha}(\tilde{\Omega}_1)$ (into R^m) which vanish off V_1. Here V_1 is a slight shrinking of V. We let $\tilde{W}$ be a small open neighbourhood of zero in W. If the neighbourhood is small, we easily see that, for $\psi \in \tilde{W}$, $(I+\psi)(\tilde{\Omega}_1)$ is still a domain of the type we want. This is how we obtain our domain perturbations.

Choose $p>n$ and let X be the space of functions on $\tilde{\Omega}_1$ in $L^p(\tilde{\Omega}_1) \cap C_\alpha(V \cap \tilde{\Omega}_1)$ where C_α denotes the usual Hölder space. It is easy to see that this is a Banach space under the norm $\|u\|_{p,\tilde{\Omega}_1} + \|u \mid V \cap \tilde{\Omega}_1\|_{0,\alpha,V\cap\tilde{\Omega}_1} \equiv \|u\|_X$. Here $\|_{0,\alpha,V\cap\tilde{\Omega}_1}$ denotes the Hölder norm on $V \cap \tilde{\Omega}_1$ In all our norms, we will omit the set when it is clear on which set we are taking the norm. Let $Z = \{u \in \dot{W}^{1,2}(\tilde{\Omega}_1)\colon -\Delta u \in X\}$. This is easily seen to be a Banach space under the norm $\|u\|_Z = \|\Delta u\|_X$. By standard regularity theorems, Z continuously embeds in $L^\infty(\tilde{\Omega}_1)$ and if $u \in Z$, then $u|_{V_1} \in C^{2,\alpha}(\tilde{\Omega}_1 \cap V_1)$ with the corresponding estimate for the norm. We now have a similar situation to that in Saut and Teman [27, Sect. 4]. Our proof is a slight modification of their proof. It is assumed that the reader has a copy of [27] available. We consider the map $F\colon Z \times \tilde{W} \to X$ defined by $F(z, S)(x) = -\Delta_{\tilde{x}} z(I+S)(x)) - f(z((I+S)(x))$ for $x \in \tilde{\Omega}_1$, $z \in Z$, $S \in W$. Here $\tilde{x} = (I+S)x$ and we calculate the Laplacian in $\tilde{x}$ coordinates. Note that z is C^2 near where $S(x) \neq x$. With this remark, one can calculate the derivative of F by the same arguments as in [27]. Note that the regularity theory for the Laplacian easily ensures, if $a \in L^\infty(\tilde{\Omega}_1) \cap C_\alpha(V)$, then the annihilator of the range of $-\Delta - aI$ considered as a map of Z into X is the same as the annihilator on $\dot{W}^{1,2}(\tilde{\Omega}_1)$. Using these remarks, we can easily reduce the proof that 0 is a regular value of F to showing that, if $-\Delta u_0 = f(u_0)$ in $\tilde{\Omega}_1$ (where $u_0 \in Z$) and if $\int_{\tilde{\Omega}_1} \Delta(S \cdot \nabla u_0) w = 0$ for all $S \in W$, then $w = 0$ (cf. [27]). Here $w \in Z$ is a solution of the linearized equation of (2) at u_0 (for $\Omega = \tilde{\Omega}_1$). Note that our integrands have support away from zero. Since our integrals are supported away from zero, we can argue as in [27, p. 313], to deduce that our integral condition becomes

$$\int_{\tilde{\Omega}_1} (S \cdot \nabla u_0) f'(u_0) w - \int_\Gamma (S \cdot \nabla u_0) \frac{\partial w}{\partial n} = 0 \tag{3}$$

for every $S \in W$ (where $\Gamma = \partial \tilde{\Omega}_1$). Hence $\int_{\tilde{\Omega}_1} (S \cdot \nabla u_0) f'(u_0) w = 0$ for every C^2 function S with support in V_1 and vanishing on Γ. It follows easily that $\nabla u_0 f'(u_0) w$ vanishes in $\tilde{\Omega}_1 \cap V_1$. Since S is zero outside V_1, it follows from (3) that $\int_\Gamma S \cdot \nabla u_0 (\partial w/\partial n) = 0$ for all $S \in W$. It follows easily that $\nabla u_0(\partial w/\partial n) = 0$ on $U \cap V_1$. Since $\partial w/\partial n$ cannot vanish on a relatively open subset of $\partial \tilde{\Omega}_1$ (cf. the argument on p. 314 of [27]), it follows by continuity that there is a relatively open subset U_1 of U on which $\partial w/\partial n$ is never zero. Hence ∇u_0 vanishes on U_1. In particular $\partial u_0/\partial n$ vanishes on U_1. If $f(0) = 0$,

u_0 satisfies the linear equation $-\Delta u = uh(u)$ where $h(y) = y^{-1}f(y)$. Hence, by the same argument as in [27], u_0 vanishes identically. Hence, if $f(0) = 0$, we have proved that zero is a regular value of F unless 1 is an eigenvalue of $-\Delta$. Since we can always ensure that this is not the case by choosing a small S which shrinks the domain first, we have completed the proof that zero is a regular value of F if $f(0) = 0$. If $f(0) > 0$ and $f(R) \subseteq [0, \infty)$, every solution u_0 is non-negative and non-trivial. The maximum principle then ensures that $\partial u_0/\partial n > 0$ on $\partial\tilde{\Omega}_1$ and hence we have a contradiction. This completes the proof that 0 is a regular value of F. We can now use a similar argument to that in [27] to show that for most $S \in \tilde{W}$, (2) has only non-degenerate solutions for $\Omega = (I + S)(\tilde{\Omega}_1)$. This completes the proof.

Remark. In fact, our domain perturbation result for positive solutions holds if we only assume that $f(0) \geqslant 0$. If $f(0) > 0$, we note that positive solutions u_0 on $\tilde{\Omega}_1$ satisfy $\partial u_0/\partial n > 0$ on $\partial\tilde{\Omega}_1 \backslash \{0\}$. Hence, if we choose a small neighbourhood T of the positive solutions in Z, then every solution u in T satisfies $\partial u/\partial n > 0$ on U (by continuity). We then complete the proof as before by considering F as a map of $T \times \tilde{W}$ into X.

2. Some Asymptotically Linear Examples

We construct asymptotically linear C^2 convex functions f with $f(0) > 0$, $f(y) > 0$ on $(0, \infty)$ for which the set of positive solutions of

$$\begin{aligned} -\Delta u &= \lambda f(u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{4}$$

is not connected. These seem to be the first such examples. We also construct similar examples with $f(0) = 0$ and $f'(0) > 0$. We always assume that $y^{-1}f(y) \to 1$ as $|y| \to \infty$. We will construct examples with Ω star shaped.

As in [8], we say that $\Omega_n \to \Omega_0$ as $n \to \infty$ if Ω_n and Ω_0 are bounded open sets in R^m with Ω_n connected and if there is a compact subset E of R^m of measure zero and a compact subset K in R^m of capacity zero such that

(i) if K_1 is a compact subset of $\Omega_0 \backslash K$, $K_1 \subseteq \Omega_n$ for large n,

(ii) if U is an open neighbourhood of $\bar{\Omega}_0 \cup E$, then $\Omega_n \subseteq U$ for large n,

(iii) whenever $u \in W^{1,2}(R^m)$ and $u(x) = 0$ a.e. on $R^m \backslash \bar{\Omega}_0$, then $u \in \dot{W}^{1,2}(\Omega_0)$.

Remark. Very general conditions ensuring that (iii) holds can be found in Hedberg [17].

LEMMA 1. *Assume that $\Omega_n \to \Omega_0$ as $n \to \infty$, that f is as above and that (u_n, λ_n) are positive solutions of* (4) *for $\Omega = \Omega_n$ such that $\|u_n\|_{\infty,\Omega_n} \to \infty$ and $\lambda_n \to \alpha$ as $n \to \infty$. Then $0 < \alpha < \infty$ and the equation*

$$-\Delta u = \alpha u \qquad \text{in } \Omega_0$$

$$u = 0 \qquad \text{on } \partial\Omega_0$$

has a non-trivial non-negative solution.

Proof. First assume that $\alpha < \infty$. Let $f(y) = y + r(y)$ where $y^{-1}r(y) \to 0$ as $y \to \infty$, i.e., for every $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that $r(y) \leqslant \varepsilon y + M_\varepsilon$ on $[0, \infty)$. Hence $\|r(u_n)\|_{\infty,\Omega_n} \leqslant \varepsilon \|u_n\|_{\infty,\Omega_n} + M_\varepsilon$. Thus we see that $\|-\Delta v_n - \lambda_n v_n\|_{\infty,\Omega_n} \leqslant (\varepsilon + (\|u_n\|_{\infty,\Omega_n})^{-1} M_\varepsilon)(2\alpha + 1)$ for large n where $v_n = (\|u_n\|_{\infty,\Omega_n})^{-1} u_n$. Hence $v_n \in L^\infty(\Omega_n) \cap \dot{W}^{1,2}(\Omega_n)$, $\|v_n\|_{\infty,\Omega_n} = 1$ and $-\Delta v_n - \alpha\, v_n \to 0$ in $L^\infty(\Omega_n)$ as $n \to \infty$. Using the independence of the constants upon n in the standard $L^p - L^q$ estimates (cf. [7, Lemma 1]), we easily prove from the equation for v_n (by bootstrapping) that $\|v_n\|_{\infty,\Omega_n}$ is small if $\|v_n\|_{2,\Omega_n}$ is small. Thus $\|v_n\|_{2,\Omega_n}$ is not small. Since v_n is uniformly bounded in $\dot{W}^{1,2}(\Omega_n)$ (by the equation for v_n), we can, by choosing a subsequence if necessary, assume that v_n converges weakly in $\dot{W}^{1,2}(\tilde{B})$ (and thus strongly in $L^2(\tilde{B})$) to v. Here $\tilde{B} \supset \Omega_0 \cup \bigcup_{n=1}^\infty \Omega_n$ and $\tilde{B}$ is bounded. Since $\|v_n\|_{2,\Omega_n}$ is bounded below, $v \neq 0$. Since $v_n \geqslant 0$ on Ω_n, $v \geqslant 0$ on Ω_0. By the equation for v_n and by the arguments in [8], we see that $v \in \dot{W}^{1,2}(\Omega_0)$ and $-\Delta v = \alpha v$ in Ω_0. (Similar arguments appear in [7], Sect. 1].)

It remains to prove that $\alpha < \infty$. By our assumptions there is a $K > 0$ such that $f(y) \geqslant Ky$ on R^+. It follows easily that (4) for $\Omega = \Omega_n$ can only have a positive solution if $\lambda \leqslant K^{-1}\lambda_1(\Omega_n)$. (We take the scalar product of (4) for $\Omega = \Omega_n$ with an eigenfunction of $-\Delta$ corresponding to $\lambda_1(\Omega_n)$). Since $\lambda_1(\Omega_n) \to \lambda_1(\Omega_0)$ as $n \to \infty$ (cf. [7] or [26]), the result follows.

If Ω_0 has k components, it is easy to see that there are at most k possible values of α.

We now construct our examples. Suppose we can construct f as above such that for Ω the unit ball B_1, (4) has exactly one positive solution for $\lambda \leqslant \lambda_1(B_1)$, exactly two positive solutions for $\lambda_1(B_1) < \lambda < \lambda_*$, a unique positive solution for $\lambda = \lambda_*$ and no positive solutions for $\lambda > \lambda_*$ and such that all positive solutions for $0 \leqslant \lambda < \lambda_*$ are non-degenerate. We will construct such an f a little later. Suppose we choose a second disjoint ball B_2 of small radius r so that B_1 and B_2 touch and $\lambda_1(B_1) r^{-2} < \lambda_*$. By a simple scaling, the solutions of (4) for $\Omega = B_2$ have similar properties to those on B_1 except for a larger $\tilde{\lambda}_* > \lambda_*$. As in [7], we easily see that the positive

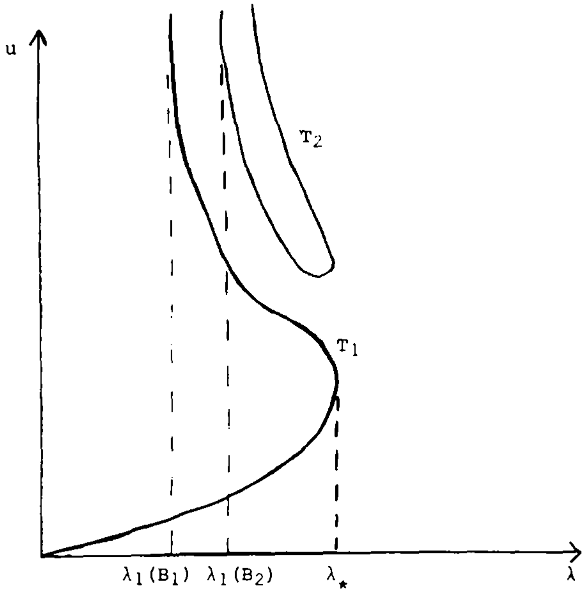

FIGURE 1

solutions for $\Omega = B_1 \cup B_2$ are as in Fig. 1. Note that large solutions only occur near $\lambda_1(B_1)$ and $\lambda_1(B_2)$. Suppose Ω_n are star shaped sets such that $\Omega_n \to \Omega_0 = B_1 \cup B_2$ as $n \to \infty$. As in [7, Sect. 2], we see that the positive solutions of (4) (for Ω_n) in $\{(x, \lambda) \in L^2(\Omega_n) \times R^+ : \|x\|_2 + \lambda \leqslant k\}$ are curves almost the same as those in Fig. 1 for n large (where n depends on k). On the other hand, by Lemma 1, we see that, if $\varepsilon > 0$ and n is large, the positive solutions with $|\lambda - \lambda_1(B_1)| + |\lambda - \lambda_1(B_2)| \geqslant \varepsilon$ are bounded (uniformly in n) in L^∞. Consider a component of positive solutions of (4) for $\Omega = \Omega_n$ which contains points near the higher curve T_2 in Fig. 1. By our comments above, it will contain a curve of solutions near the bounded part of T_2. This will be in $\{(x, \lambda): \lambda \geqslant \lambda_1(B_2) - \varepsilon\}$. Any other solutions (x, λ) for n large will lie near the other curve T_1 or will have $\|x\|_\infty$ large. Hence the only way the two components for $\Omega = \Omega_n$ can join up is through large solutions. However, by Lemma 1, large solutions can only occur for λ close to $\lambda_1(B_1)$ or $\lambda_1(B_2)$. Thus there is no way for the components to join up. Hence we see that our solution structure is like that in Fig. 2. Note that the component $\tilde{T}_2$ close to T_2 on bounded sets must be a bounded component since, for fixed n, there are no possible asymptotic bifurcation points for positive solutions near $\lambda_1(B_2)$. ($\lambda_1(\Omega_n)$ is the only point of asymptotic bifurcation of positive solutions.) I do not claim that Fig. 2 is exact in that I do not claim that there are no extra components or further bifurcations

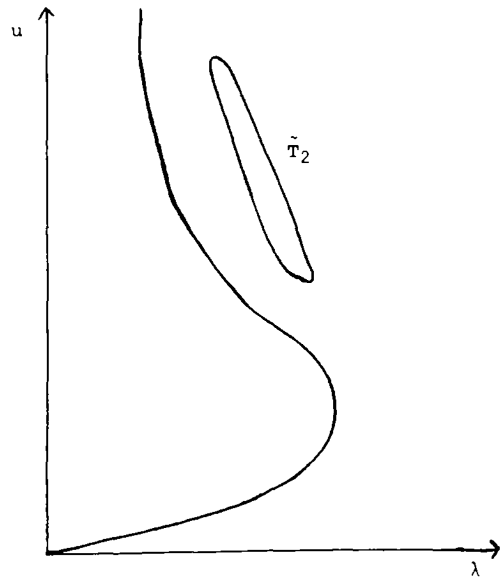

FIGURE 2

near the relatively large solutions in $\tilde{T}_2$. However, I suspect the picture in Fig. 2 is correct. (One can choose the Ω_n and f so that any bifurcations on $\tilde{T}_2$ are simply changes of direction.)

It remains to construct f with the required properties on B_1. This follows easily from Theorem 2 in [2] and its proof. Note that it is easy to construct an f satisfying the assumptions there. A more convenient assumption guaranteeing that $\lambda_* > \lambda_1(B_1)$ than the one in [2] can be found in [10, Sect. 4].

Suppose that, we use a similar construction with B_1 and B_2 of the same radius and with the Ω_n preserving the Z_2 symmetry (due to reflection in the plane P of points equidistant from the centres of the two balls). We would obtain a solution structure as in Fig. 3 where once again the solution structure of the large non-symmetric solutions could be a little different to that in the diagram (though the branches must join up and the branch of symmetric solutions only changes direction once).

Finally, our methods can also be used to obtain corresponding results for the case where $f(0) = 0$ and $f'(0) > 0$. The only point to note is that, if the solution structure for Ω_0 (and *positive* solutions) is as in Fig. 4 then the lower part of the right hand branch will leave the cone and we will obtain a diagram similar to that in Fig. 5 for $\Omega = \Omega_n$. In the symmetric case (as above), we will obtain a diagram similar to that in Fig. 6 for $\Omega = \Omega_n$.

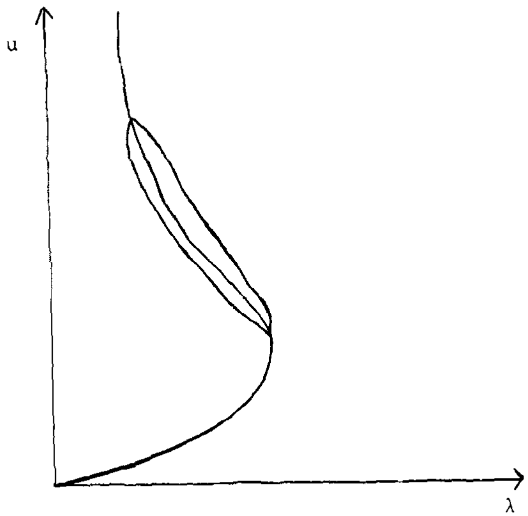

FIGURE 3

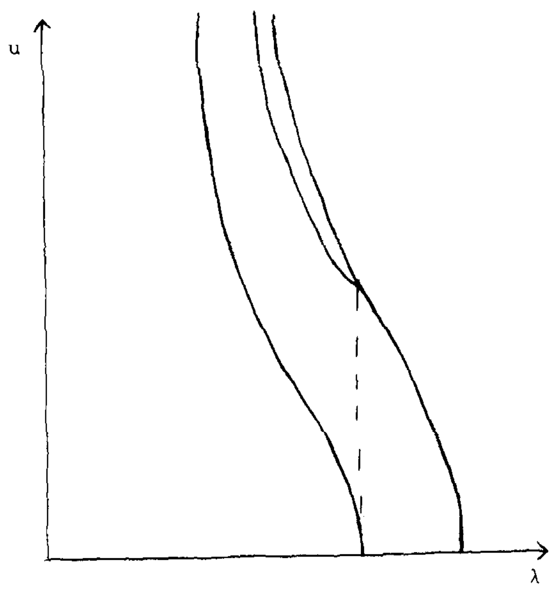

FIGURE 4

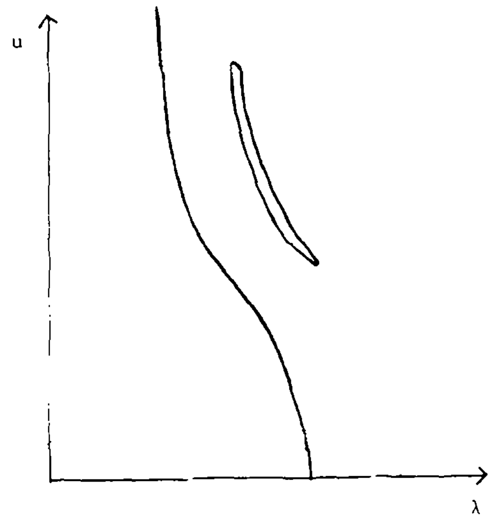

FIGURE 5

3. On the Spectra of Linearizations

In the theory in [7], we proved a result which essentially showed that, if $\Omega_n \to \Omega_0$ as $n \to \infty$ and if u_n is the unique solution of (1) on Ω_n which is near u_0 (under appropriate hypotheses), then the spectrum of $-\Delta - f'(u_n)I$ on Ω_n (with Dirichlet boundary conditions) is near that of $-\Delta - f'(u_0)I$ on Ω_0. (More precisely the kth eigenvalue λ_k^n of the first operator approaches the corresponding eigenvalue λ_k^0 of the second operator as $n \to \infty$ (where the eigenvalues are repeated according to multiplicity). To prove this, we used the variational structure of the problem. In fact, this was the only point where we used the variational structure of the problem. Here we present a method of obtaining this continuous dependence of eigenvalues which extends immedciately to non-self adjoint problems and to weakly non-linear systems. We illustrate the method by applying it to the equation

$$\begin{aligned} -\Delta u + a \cdot \nabla u &= f(u) \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{5}$$

where $a \in L^\infty(R^m)$. We assume that u_0 is a solution of (5) for $\Omega = \Omega_0$ in $\dot{W}^{1,2}(\Omega_0) \cap L^\infty(\Omega_0)$ and that u_0 is a non-degenerate solution. Assume for simplicity that f has polynomial growth. Then for n large, there is a unique solution $u_n \in \dot{W}^{1,2}(\Omega_n) \cap L^\infty(\Omega_n)$ of (5) (for $\Omega = \Omega_n$) near u_0 in $L^p(\tilde{B})$

for suitable large p. Moreover, u_n are uniformly bounded on $L^{\infty}(\tilde{B})$ and u_n are non-degenerate solutions. This follows by a minor modification of the proof of Theorem 1 in [7]. We now compare the spectra of $-\Delta + a\cdot\nabla + b_n(x)I$ on Ω_n and $-\Delta + a\cdot\nabla + b_0 I$ on Ω_0 (both with Dirichlet boundary conditions). Here $b_n := -f'(u_n)$ and $b_0 = -f'(u_0)$. Note that the b_n are uniformly bounded in $L^{\infty}(\tilde{B})$ and $b_n \to b_0$ in $L^p(\tilde{B})$ as $n \to \infty$ for every $p < \infty$. Note also that when we write $-\Delta$ henceforth in this section, we also include the appropriate boundary condition.

THEOREM 1. (i) *Assume that λ_0 is a (possibly complex) eigenvalue of $-\Delta + a\cdot\nabla + b_0 I$ on Ω_0 of algebraic multiplicity $\tilde{m}$ and $\varepsilon > 0$. Then, for n large, there are exactly $\tilde{m}$ eigenvalues of $-\Delta + a\cdot\nabla + b_n I$ on Ω_n counting multiplicity in $\{\lambda \in C\colon |\lambda - \lambda_0| \leqslant \varepsilon\}$.*

(ii) *Moreover, if $\alpha \in R$, any eigenvalue of $-\Delta + a\cdot\nabla + b_n I$ in $\{\lambda \in C\colon \operatorname{Re}\lambda \leqslant \alpha\}$ is near an eigenvalue $-\Delta + a\cdot\nabla + b_0 I$ for n large.*

Remark. This is the desired result on the continuity of the spectrum. One can prove analogous results for the spectral projections (in L^2).

Proof. *Step* 1. If μ is not an eigenvalue of $-\Delta + a\cdot\nabla + b_0 I$, then there is a $K > 0$ such that $\|-\Delta v + a\cdot\nabla v + (b_n - \mu)v\|_{2,\Omega_n} \geqslant K\|v\|_{2,\Omega_n}$ for all v in $T_n = \{w \in \dot{W}^{1,2}(\Omega_n)\colon \Delta w \in L^2(\Omega_n)\}$. This follows because, if it were false,

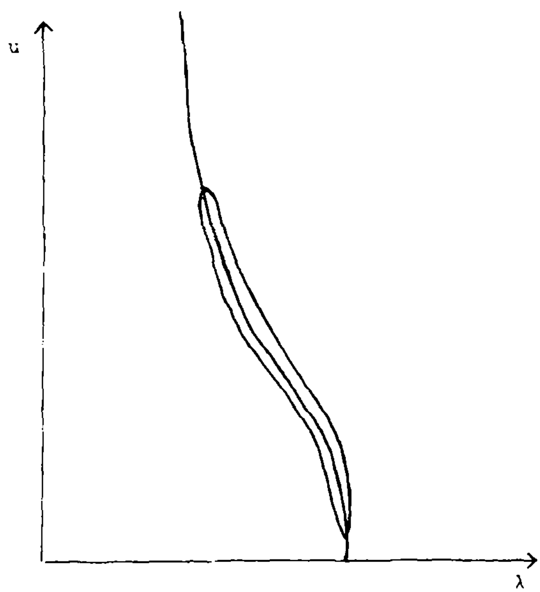

FIGURE 6

there would exist $v_n \in T_n$ such that $\|v_n\|_{2,\Omega_n} = 1$ and $-\Delta v_n + a \cdot \nabla v_n + (b_n - \mu) v_n \to 0$ in $L^2(\Omega_n)$ as $n \to \infty$. By multiplying the equation by v_n and using the Cauchy–Schwartz inequality, we see that

$$\int_{\Omega_n} |\nabla v_n|^2 \, dx \leqslant K \|v_n\|^2_{2,\Omega_n} + K_1.$$

Hence $\{\|v_n\|_{1,2,\Omega_n}\}$ are uniformly bounded. Here $\| \|_{1,2,\Omega_n}$ denotes the usual $W^{1,2}$ norm on Ω_n. We can now obtain a contradiction by a similar argument to that in the proof of Step 1 of the proof of Theorem 1 in [7].

Let $R_{n,\mu}$ denote the inverse of $-\Delta + a \cdot \nabla + (b_n - \mu) I$ for μ as in Step 1.

Step 2. If $\tilde{f} \in L^2(\tilde{B})$ and μ is as in Step 1, then $R_{n,\mu} i_n \tilde{f} \to R_{0,\mu} i_0 \tilde{f}$ in $L^2(\tilde{B})$ as $n \to \infty$. Here i_n is the natural restriction map from $L^2(\tilde{B})$ to $L^2(\Omega_n)$ and we extend elements from $L^2(\Omega_n)$ to $L^2(\tilde{B})$ by defining them to be zero outside Ω_n. (Equivalently we use the obvious extension operator E_n). To prove this, note that, by Step 1, $\|R_{n,\mu} i_n \tilde{f}\|_2$ are uniformly bounded. In fact, the proof of Step 1 shows that $v_n = R_{n,\mu} i_n \tilde{f}$ are uniformly bounded in $\dot{W}^{1,2}(\tilde{B})$. Thus, by choosing a subsequence if necessary, we can assume that $v_n \to v_0$ weakly in $\dot{W}^{1,2}(\tilde{B})$ and strongly in $L^2(\tilde{B})$. By the proof of Theorem 1 in [7], $v_0 \in \dot{W}^{1,2}(\Omega_0)$ and

$$-\Delta v_0 + a \cdot \nabla v_0 = f'(u_0) v_0 + \mu v_0 + \tilde{f} \qquad \text{in } \Omega_0.$$

Hence $v_0 = R_{0,\mu} i_0 \tilde{f}$. Since the limit is independent of the choice of subsequence, $v_n \to v_0$ in $L^2(\tilde{B})$ as $n \to \infty$. This proves Step 2.

If $A \colon E \to E$, $R(A) \subseteq M$ where M is a subspace, $u \in M$, $v \in E$, $\lambda \in C$ and $v - \lambda A v = u$, then $v \in M$. It follows easily that, if $A \colon E \to E$ is compact and linear and $R(A)$ is contained in the closed subspace M, then λ has the same algebraic multiplicity on M or E. It follows easily from this that λ has the same multiplicity as an eigenvalue of $R_{n,\mu}$ on $\dot{W}^{1,2}(\Omega_n)$ or as an eigenvalue of $E_n R_{n,\mu} i_n$ on $\dot{W}^{1,2}(\tilde{B})$. Since it is easy to see that λ is an eigenvalue of $T_{n,\mu}$ of multiplicity $\tilde{m}$ if and only if λ^{-1} is an eigenvalue of $-\Delta + a \cdot \nabla u + b_n I$ (of multiplicity $\tilde{m}$), the proof of Theorem 1(i) now reduces to proving the same result for $E_n R_{n,\mu} i_n$. Now, since $E_n R_{n,\mu} i_n$ are uniformly bounded as maps of $L^2(\tilde{B})$ to $\dot{W}^{1,2}(\tilde{B})$, we see from Step 1 that $E_n R_{n,\mu} i_n$ are a collectively compact sequence of linear maps on $L^2(\tilde{B})$ in the sense of [4]. Hence (i) follows from [4].

We now prove (ii). First note that, by the collective compactness (or by the proof of Step 1), if $\lambda_n \in \sigma(H_n)$ and $\lambda_n \to \lambda$ as $n \to \infty$, then $\lambda \in \sigma(H_0)$. Here $H_n = -\Delta + a \cdot \nabla + b_n I$ with the appropriate boundary condition and H_0 is defined analogously. Hence we see that the proof of Theorem 1(ii) reduces to showing that if $\alpha > 0$, there is a $K(\alpha) > 0$ independent of n such that the spectrum of H_n in $\{\lambda \in C\colon \operatorname{Re} \lambda \leqslant \alpha\}$ is contained in a ball centre

0 with radius $K(\alpha)$. Choose $\tilde{\mu}>0$ such that $b_n(x)\leqslant\tilde{\mu}$ for all $x\in\Omega_n$ and all n. (This is possible because u_n are uniformly bounded in $L^\infty(\Omega_n)$.) The weak maximum principle implies that there can be no real spectrum with $\lambda\leqslant\tilde{\mu}$. Since the eigenvalue of smallest real part of H_n is real, it follows that all the spectra lie in $\{\lambda\in C\colon \operatorname{Re}\lambda\geqslant\tilde{\mu}\}$. Hence it suffices to prove that we have a contradiction if there exist eigenvalues $\tau_n+i\beta_n$ in $\sigma(H_n)$ such that $\tilde{\mu}\leqslant\tau_n\leqslant\alpha$ for all n and $|\beta_n|\to\infty$ as $n\to\infty$. We will prove in a moment that $\|(-\Delta+i\beta_nI)^{-1}\|\to 0$ as $n\to\infty$, considered as a map of $L^2(\Omega_n)$ to $\dot{W}^{1,2}(\Omega_n)$. Assuming this, we will obtain a contradiction. If $H_nw_n=(\tau_n+i\beta_n)w_n$, where $w_n\neq 0$, then

$$w_n=(-\Delta-i\beta_nI)^{-1}((\tau_n-b_n)w_n-a\cdot\nabla w_n). \tag{6}$$

By standard estimates, we see that

$$\|(\tau_n-b_n)w_n-a\cdot\nabla w_n\|_{2,\Omega_n}\leqslant K\|w_n\|_{1,2,\Omega_n}.$$

Hence, by (6),

$$\|w_n\|_{1,2,\Omega_n}\leqslant K\|(-\Delta-i\beta_nI)^{-1}\|\ \|w_n\|_{1,2,\Omega_n},$$

where the inverse is considered as a map of $L^2(\Omega_n)$ to $\dot{W}^{1,2}(\Omega_n)$. By our estimate for the inverse, this implies that $w_n=0$.

It remains to prove our estimate for $(-\Delta-i\beta_nI)^{-1}$. By a standard estimate for self adjoint operators (cf. Kato [23, Theorem V.3.16]), $\|(-\Delta-i\beta_nI)^{-1}\|\leqslant|\beta_n|^{-1}$ when the inverse is considered as a map of $L^2(\Omega_n)$ into itself. Let $v_n=(-\Delta-i\beta_nI)^{-1}\tilde{f}$ where $\tilde{f}\in L^2(\Omega_n)$. Thus $-\Delta v_n-i\beta_nv_n=\tilde{f}$ in Ω_n. By multiplying by $\bar{v}_n$ and taking the real part, we see that

$$\begin{aligned}\|v_n\|^2_{1,2,\Omega_n}=\int_\Omega\tilde{f}\bar{v}_n&\leqslant\|\tilde{f}\|_{2,\Omega_n}\|v_n\|_{2,\Omega_n}\\&\leqslant|\beta_n|^{-1}(\|f\|_{2,\Omega_n})^2\end{aligned}$$

(by our estimate for $(-\Delta-i\beta_nI)^{-1}$ on $L^2(\Omega_n)$). Thus $\|(-\Delta-i\beta_nI)^{-1}\|\leqslant|\beta_n|^{-1/2}$ where the inverse is considered as a map of $L^2(\Omega_n)$ to $\dot{W}^{1,2}(\Omega_n)$. This is the required estimate and hence we have completed the proof.

Remark. The proof can be generalized to cover a large class of systems whose top order terms are in diagonal form. We only need to make one change in the proof. To show that the real part of the spectrum is bounded below, we need to use a simple Garding type inequality for $\operatorname{Re}\langle Lv, v\rangle$ where L is the linear part (cf. Gilbarg and Trudinger [16, Lemma 8.4]. Indeed, it is not difficult to show our methods can be generalized to the very strongly uniformly elliptic systems in the sense of Amann [3]. Note also that the result implies that the dimension of the unstable manifold of u_n is the same as that of u_0 for large n provided that the linearization at u_0 has no purely imaginary eigenvalues.

4. Neumann Problems

In this section, we indicate very briefly that the ideas we introduced in [7] can be used to study some Neumann problems. We obtain weaker results for more general domains when compared with those in [25, 20]. In particular, we do not need the order properties used in [25, 20]. Our methods have the advantage of being readily adaptable to many systems and to looking for non-negative solutions. On the other hand, they are not so good for studying stability (except under assumptions which force uniqueness). Our methods have an advantage over those in [29] in that they allow a more general Ω_0 and they apply in nonself-adjoint problems and systems. Moreover, our methods can be used to produce star shaped examples. Note that Neumann problems are more difficult than Dirichlet problems even in the linear case (cf. the introduction to [7]).

We will say $\Omega_n \to_n \Omega_0$ if Ω_n and Ω_0 are all bounded open sets with Ω_n connected such that

(i) $\Omega_n \supseteq \Omega_0$ for all n and each Ω_n has Lipschitz boundary,

(ii) $m(\Omega_n \backslash \Omega_0) \to 0$ as $n \to \infty$, and

(iii) there is a continuous linear map E: $W^{1,2}(\Omega_0) \to W^{1,2}(R^m)$ such that $Ef|_{\Omega_0} = f$ for all $f \in W^{1,2}(\Omega_0)$. This last property holds if Ω_0 has Lipschitz boundary (cf. Adams [1, Theorem 4.32]).

Note that, because the spaces are Hilbert spaces, our last assumption is equivalent to assuming that each $f \in W^{1,2}(\Omega_0)$ can be extended to $W^{1,2}(R^m)$.

Theorem 2. *Assume that $\Omega_n \to_n \Omega_0$ as $n \to \infty$ and that u_0 is a non-degenerate solution of*

$$\begin{aligned} -\Delta u &= f(u) && \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0 && \text{in } \partial\Omega \end{aligned} \tag{7}$$

for $\Omega = \Omega_0$ in $L^\infty(\Omega_0) \cap W^{1,2}(\Omega_0)$. Here $\partial/\partial\nu$ is the normal derivative. In addition, assume that there exists $\alpha, \beta > 0$ such that $yf(y) < 0$ on $R \backslash [-\alpha, \beta]$. Then for each large n, there is a solution u_n of (7) on Ω_n near Eu_0 in $L^p(\Omega_n)$, where p is fixed and $1 \leqslant p < \infty$.

Proof (Sketch). Define $E_n: W^{1,2}(\Omega_0) \to W^{1,2}(\Omega_n)$ by $E_n g = Eg|_{\Omega_n}$. Then $\|E_n g\|_{1,2} \leqslant \|Eg\|_{1,2} \leqslant K\|g\|_{1,2}$ for $g \in W^{1,2}(\Omega_0)$. Choose $a > 0$ such that $y(f(y) + ay) < 0$ at $-\alpha$ and β. We can modify f on $R \backslash [-\alpha, \beta]$ to $\tilde{f}$ so that $y(\tilde{f}(y) + ay) < 0$ on $R \backslash [-\alpha, \beta]$ and $\tilde{f}(y) + ay$ is bounded on R.

Choose $\mu > 2a^{-1} \operatorname{supp}\{|\tilde{f}(y) + ay|: y \in R\}$ and then choose ϕ C^{∞}, non-decreasing and bounded on R so that $\phi(x) = x$ if $|x| \leqslant 2\mu$. Define $\tilde{E}_n g = \phi(E_n g)$. Note that there is a $\tilde{K} > 0$ such that $|\tilde{E}_n g(x)| \leqslant \tilde{K}$ always and that $\tilde{E}_n g(x) = g(x)$ on Ω_0 if $|g(x)| \leqslant 2\mu$ on Ω_0.

Now one easily sees that finding solutions of (7) for $\Omega = \Omega_n$ is equivalent to finding fixed points of $A_n = (-\Delta + aI)_n^{-1}(\tilde{f} + aI)$ in $L^2(\Omega_n)$, where $(-\Delta + aI)_n^{-1}$ is the inverse of $-\Delta + aI$ on Ω_n with Neumann boundary conditions. On the other hand, the equation on Ω_0 is equivalent to the equation

$$u_n = \tilde{E}_n(-\Delta + aI)_0^{-1}(\tilde{f} + aI) R_n u_n \equiv \tilde{A}_n u$$

on $L^2(\Omega_n)$ where R_n is the natural restriction mapping of $L^2(\Omega_n)$ onto $L^2(\Omega_0)$. One then proves that, if τ is small and positive,

$$u \neq t A_n(u) + (1 - t) \tilde{A}_n(u) \tag{8}$$

if $0 \leqslant t \leqslant 1$ and $\|u - \tilde{E}_n u_0\|_{2,\Omega_n} = \tau$. The result then follows easily from this and a simple degree argument (rather like that in the proof of Theorem 1 in [7]) once we note that the commutativity theorem for the degree (cf. Deimling [14, p. 214]) enables one to reduce the calculation of the degree of $\tilde{A}_n$ on the with centre $\tilde{E}_n u_0$ and radius τ in $L^2(\Omega_n)$ to a much more natural equation on $L^2(\Omega_0)$. (In fact, this more natural equation is the natural analogue of A_n on Ω_0.)

It remains to prove (8). If $u_n = t_n A_n(u_n) + (1 - t_n) \tilde{A}_n(u_n)$ when $\|u_n - \tilde{E}_n u_0\|_{2,\Omega_n} = \tau$, we easily obtain a uniform bound for $\|u_n\|_{\infty,\Omega_n} + \|u_n\|_{1,2,\Omega_n}$ and thus $\|u_n - u_0\|_{2,\Omega_0} \to \tau$ as $n \to \infty$. We can, by choosing a subsequence, ensure that $u_n|_{\Omega_0}$ converges weakly in $W^{1,2}(\Omega_0)$ to w. By using Lemma 2 below and uniform bound for $(\Delta + aI)_n^{-1}$ in $L^2(\Omega_n)$, we easily deduce that w is a solution of (7) for $\Omega = \Omega_0$ with $\|w - u_0\|_{2,\Omega_0} = \tau$. This is impossible if τ is small by the non-degeneracy assumption. This completes our sketch of the proof.

LEMMA 2. *Suppose that $g \in L^2(R^m)$, $a > 0$ and $\Omega_n \to_n \Omega_0$ as $n \to \infty$. If u_n denotes the solution of*

$$-\Delta u + au = g \qquad \text{in } \Omega_n$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega_n \text{ (in the weak sense)},$$

then $u_n|_{\Omega_0} \to u_0$ weakly in $W^{1,2}(\Omega_0)$ and strongly in $L^2(\Omega_0)$. Here u_0 is defined analogously to u_n but on Ω_0.

Proof (Sketch). We easily see that $\|u_n\|_{1,2,\Omega_n}$ is uniformly bounded and thus by choosing a subsequence, we can assume $u_n|_{\Omega_0}$ converges weakly in $W^{1,2}(\Omega_0)$ and strongly in $L^2(\Omega_0)$ to w. Suppose $\phi \in W^{1,2}(\Omega_0)$. By our assumptions, we can extend ϕ to $\tilde{\phi} \in W^{1,2}(R^m)$. Now $\int_{\Omega_n} \nabla u_n \nabla\tilde{\phi} + au_n \tilde{\phi} = \int_{\Omega_n} g\tilde{\phi}$. By our estimates on u_n and since $m(\Omega_n \backslash \Omega_0) \to 0$ as $n \to \infty$, we easily deduce that $\int_{\Omega_0} \nabla u_n \nabla\tilde{\phi} + au_n\tilde{\phi} - g\tilde{\phi} \to 0$ as $n \to \infty$. Passing to the limit, we see that $\int_{\Omega_0} \nabla w \nabla\tilde{\phi} + aw\tilde{\phi} - g\tilde{\phi} = 0$ and the result follows easily.

Remarks. (1) A variant of the result can be used to construct star shaped Ω's with more solutions than one might expect. For simplicity assume that $m = 2$. We consider star shaped Ω's approximating Ω_0 where Ω_0 is two squares touching only at one vertex as in Fig. 7. We cannot apply Theorem 2 directly because, in this case, Ω_0 does not have the extension property (iii). However, with care, one can prove that the *conclusion* of Theorem 2 holds. The key points is that the proof of Theorem 2 can be generalized to cover some cases where enough functions (in particular, in our case, functions vanishing near the vertex) extend to $W^{1,2}(R^m)$.

(2) The results in [20 22] show that we do not always expect local uniqueness or that the solutions u_n are non-degenerate. However, if Ω_n "nicely" approach Ω_0, where Ω_0 is as in Remark 1 or Ω_0 is two suitable domains sufficiently close together, one can prove that local uniqueness and non-degeneracy both hold (and in fact we obtain the expected number of negative eigenvalues of the linearization). We sketch *very* briefly the idea for the proof of non-degeneracy. The others are similar. If

$$\begin{aligned} -\Delta h_n &= f'(u_n)h_n \qquad \text{on } \Omega_n \\ \frac{\partial h_n}{\partial v} &= 0 \qquad \text{on } \partial\Omega_n \end{aligned} \tag{9}$$

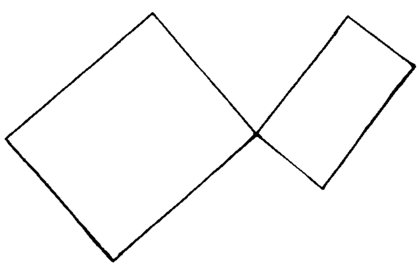

FIGURE 7

and $\|h_n\|_{2,\Omega_n}=1$, one proves using the non-degeneracy assumption on Ω_0 that h_n is uniformly small except near the vertex of Ω_0 in the case of the example in Remark 1 (and close to the joining strip in the case where Ω_0 consists of two domains). We consider the first case. (The other is similar.) We let $\tilde{\Omega}_n=\Omega_n\cap B_\varepsilon(0)$ where $\varepsilon^{-1}>2\sup\{f'(y)\colon y\in[-\alpha,\beta)\}\equiv s$. Then $\|h_n\|_{2,\tilde{\Omega}_n}$ is close to 1. We consider $\tilde{h}_n=h_n\phi$ where ϕ is 1 on $B_{\varepsilon/4}(0)$ and ϕ has support in $B_{\varepsilon/2}(0)$. One proves that $\|\tilde{h}_n\|_{2,\tilde{\Omega}_n}$ is close to 1 while, by multiplying (9) by $\tilde{h}_n$, $\|\nabla\tilde{h}_n\|_{2,\tilde{\Omega}_n}\leqslant s^{1/2}$ for large n. On the other hand by integrating on horizontal lines and noting that $\int_0^a(w'(x))^2\geqslant a^{-2}\int_0^a w(x)^2$ if $w(a)=0$ we can deduce that $\int_{\tilde{\Omega}_n}|\nabla\tilde{h}_n|^2\geqslant\varepsilon^{-2}\|\tilde{h}_n\|^2_{2,\tilde{\Omega}_n}$. This gives a contradiction to our earlier estimate for $\|\tilde{h}_n\|_{2,\tilde{\Omega}_n}$.

5. The Parabolic Case

In this section, we rather briefly discuss the simplest parabolic problems. We prove results on the continuous dependence of the initial value problem upon the domain and some simple results for the periodic parabolic problem. However, it seems that much more can be done.

We first consider the initial value problem. Let $T_n(t)$ denote the semigroup on $L^2(\Omega_n)$ generated by $-\Delta$ with Dirichlet boundary conditions and let P_n be the restriction map from $L^2(\tilde{B})$ to $L^2(\Omega_n)$, where as before $\tilde{B}$ is a ball containing $\Omega_0\cup\bigcup_{n=1}^{\infty}\Omega_n$. $T_0(t)$ and Ω_0 are defined analogously. If $\Omega_n\to\Omega_0$ as $n\to\infty$ in our earlier sense and $\partial\Omega_0$ has zero measure, then it is easy to see that $P_nv\to P_0v$ in $L^2(\tilde{B})$ for each $v\in L^2(\tilde{B})$ (since Ω_n and Ω_0 only differ by sets of small measure). Note that the example in Gelbaum and Olmsted [15, p. 149], can be used to show that the condition that $\partial\Omega_0$ has zero measure is independent of assumption (iii) in the definition that $\Omega_n\to\Omega_0$. In fact there are examples where (iii) holds, Ω_0 is a manifold with boundary but $\partial\Omega_0$ has positive measure. In fact, we could replace the assumption that $\partial\Omega_0$ has zero measure by $m(\Omega_n\backslash\Omega_0)\to 0$ as $n\to\infty$.

Lemma 3. *Assume that $\partial\Omega_0$ has zero measure and $\Omega_n\to\Omega_0$ as $n\to\infty$. For each $v\in L^2(\tilde{B})$, $T_n(t)P_nv\to T_0(t)P_0v$ uniformly on bounded t intervals in* $[0,\infty)$.

Proof. First note that, by the parabolic maximum principle, $T_n(t)P_nv\leqslant\tilde{T}(t)v$ whenever $v\in L^2(\tilde{B})$ and v is non-negative (since the solution on $\tilde{B}$ is a super-solution). Here $\tilde{T}(t)$ is the corresponding semigroup on $\tilde{B}$. It follows easily that $\|T_n(t)\|_\infty\leqslant K$ where K is independent of n. That $T_n(t)P_nv\to T_0(t)P_0v$ in $L^2(\tilde{B})$ for each $t\geqslant 0$ is a simple modification of the proof of Theorem 1.2 in Rauch and Taylor [26]. Note that the analogue of [26, Lemma 1.1] follows easily from the proof of Theorem 2. It remains to prove the uniformity in t on an interval $[0, t_0]$. Since there is a $K_1 > 0$ such that $\|T_n(t)P_n - T_0(t)P_0\|_2 \leqslant K_1$ on $[0, t_0]$ where K_1 is independendty of n, it suffices to prove the uniformity on a dense subset of $L^2(\tilde{B})$. Let W be the subset of $L^2(\tilde{B})$ of functions which vanish on a neighbourhood of $\partial\Omega_0 \cup E \cup K$ and which are smooth on Ω_0. Since $\partial\Omega_0$ has zero measure, it is easy to show that W is dense in $L^2(\tilde{B})$. Hence it suffices to show that, for each $w \in W$, $T_n(t)P_n w \to T_0(t)P_0 w$ in $L^2(\tilde{B})$ as $n \to \infty$ uniformly on $[0, t_0]$. Since the convergence holds for each t, it suffices to show that $\{T_n(t)P_n w - T_0(t)P_0 w\}_{n=1}^{\infty}$ is equicontinuous in t. Since the second term is independent of n and continuous in t, it suffices to show that $\{T_n(t)\, P_n w\}_{n=1}^{\infty}$ is equicontinuous in t. Since $w \in W$, $P_n w = w|_{\Omega_0}$ for large n. (Technically, it is an extension of this to Ω_n where the other values are zero.) Hence we only have to prove that $\{T_n(t) w|_{\Omega_0}\}_{n=1}^{\infty}$ are equicontinuous. Now $w|_{\Omega_0}$ is smooth and of compact support in Ω_n and hence it is in the domain of the infinitesimal generator of $T_n(t)$. Hence, if $t > s$,

$$T_n(t)w - T_n(s)w = \int_s^t T_n'(u)w\,du = \int_s^t \Delta T_n(u)w\,du$$

$$= \int_s^t T_n(u)\,\Delta w\,du$$

since a semigroup and its infinitesimal generator commute. Thus, since $\|T_n(u)\|_\infty \leqslant 1$,

$$\|T_n(t)w - T_n(s)w\|_2 \leqslant \int_s^t \|\Delta w\|_2\,du \leqslant |t-s|\;\|\Delta w\|_2.$$

Hence we have the required equicontinuity property and the result follows.

This shows that we have the required continuity property for homogeneous linear equations. We now use a contraction mapping argument to prove a similar result for non-linear equations. We consider the equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f(u) \qquad && \text{in } \Omega_n \times [0, \infty) \\ u(x, t) &= 0 && \text{if } x \in \partial\Omega_n \text{ and } u(x, 0) = u_0(x). \end{aligned} \tag{10}$$

Assume that $u_0 \in L^\infty(\Omega_0)$. We first assume that f is Lipschitz on R. Let μ denote the Lipschitz constant. We first prove that, if $u_0^n \in L^\infty(\Omega_n)$ and $u_0^n \to u_0$ in $L^2(\tilde{B})$, if $\{\|u_n^0\|_{\infty,\Omega_n}\}_{n=1}^{\infty}$ are uniformly bounded and if u_n denotes the mild solution of (10) with initial condition u_0^n, then $u_n(t) \to u_0(t)$ in $C[0, \tau, L^2(\tilde{B})]$ where $\tau\mu < 1$. Here $C[0, \tau, L^2(\tilde{B})]$ denotes the set of continuous functions from $[0, \tau]$ to $L^2(\tilde{B})$ with the usual norm. It follows easily from this after a finite number of iterations that $u_n(t) \to u_0(t)$ in $C[0, t_0, L^2(\tilde{B})]$ for each $t_0 > 0$. This is the required result. Here, by a mild solution we mean a solution of

$$u(t) = T_n(t) P_n u_0^n + \int_0^t T_n(t-s) f(u(s))\, ds. \tag{11}$$

Note that easy estimates show that our solution is less than or equal to the solution of

$$u(0) = \|u_0^n\|_\infty, \qquad u'(t) = f(u(t)) \tag{12}$$

(with an analogous lower estimate) and hence we do not have growth troubles. Simple estimates prove that the right hand side of (11) defines a contraction on $C[0, \tau, L^2(\Omega_n)]$ and that the contraction constant is independent of n. Hence we see that the required result will follow if we prove, for each k, the k th iterate $u_n^k(t)$ on Ω_n approaches the iterate $u_0^k(t)$ on Ω_0 in $C[0, \tau, L^2(\tilde{B})]$ as $n \to \infty$. Here the iterates are the usual iterates for a contraction mapping starting with a constant function (in t) satisfying the initial condition. This is an easy induction from Lemma 3 once we note that, by our L^∞ estimate, L^2 norms on sets of small measure are uniformly small in $L^2(\tilde{B})$. (Thus, we do not need to worry about behaviour on the set $(\Omega_n \backslash \Omega_0) \cup (\Omega_0 \backslash \Omega_n)$.) Hence we find that $u_n(t) \to u_0(t)$ in $C[0, t_0, L^2(\tilde{B})]$ as $n \to \infty$ for each $t_0 > 0$.

By simple truncation argument, we can prove an analogous result if f is only locally Lipschitz provided that the solution of (12) (and the corresponding lower equation) exist on $[0, t_0]$. In fact, by using a suitable subdivision of $[0, t_0]$, we easily see that if suffices to assume that the solution of

$$u' = \Delta u + f(u) \qquad \text{in } \Omega_0,\ u(0) = u_0$$

exists on $[0, t_0]$ and is in $L^\infty([0, t_0] \times \Omega_0)$. Similar results hold if f depends on t as well.

It seems very likely that one can prove results on how stable and unstable manifolds of critical points vary under domain perturbations. It would be interesting to use this to construct examples on the global structure of the solutions of the parabolic problem. To do this, we need to know when stable and unstable manifolds intersect transversally.

Second, for this section, we consider the periodic boundary value problem. We only consider the simplest case, though more general results are true. Note that the reason for the interest in this problem is that there are a number of physical models which are of this type. (See [9, 12, 19, 25].)

We consider

$$\frac{\partial u}{\partial t} = a(t)\,\Delta u + f(t, u) \qquad \text{on } \Omega \times [0, T]$$

$$u(x, t) = 0 \qquad \text{if } x \in \partial\Omega,\ u \text{ is } T\text{-periodic in } t.$$

Here it is assumed that $a(t) > 0$ on $[0, T]$ and a and f are T-periodic in t. By a simple change of variable in t, we can assume that $a(t) = 1$ on $[0, T]$. We assume this henceforth. We first consider the linear problem. By using an obvious Fourier series expansion, we see that, for the equation

$$\frac{\partial u}{\partial t} = \Delta u + g \qquad \text{on } \Omega \times [0, T],$$

with the above boundary conditions, $\|u\|_{1,2,\Omega \times [0,T]} \leqslant K \|g\|_{2,\Omega \times [0,T]}$, where K is independent of Ω provided that $\lambda_1(\Omega)$ is bounded below. We can obtain better bounds if $\Omega \subseteq \tilde{B}$ where $\tilde{B}$ is a fixed ball. To do this, we note that the above estimate implies that $\|u|_{\Omega_t}\|_{2,\Omega_t}$ is bounded in $L^2(\Omega_t)$ for most t. Here $\Omega_t = \Omega \times \{t\}$ and we are assuming g is in L^2 on $\Omega \times [0, T]$. Hence, cf. [18],

$$u(t) = T(t - t_0)\, u(t_0) + \int_{t_0}^{t} T(t - s)\, g(s)\, ds \tag{13}$$

$$\leqslant \tilde{T}(t - t_0)\, u(t_0) + \int_{t_0}^{t} \tilde{T}(t - s)\, g(s)\, ds, \tag{14}$$

where, as above, $\tilde{T}$ is the heat semigroup on $\tilde{B}$ and g is defined to be zero outside of Ω. Using this and standard estimates from the heat equation on $\tilde{B}$, we see that for each $p \geqslant 2$ there is a $q(p) > p$ such that, if g is bounded in $L_p(\Omega \times [0, \tilde{t}_0])$, then the second term of (13) is bounded in $L_{q(p)}(\Omega \times [0, \tilde{t}_0])$. Here $q(p)$ depends continuously on p in a natural sense and $q(p) = \infty$ if $p \geqslant p_0$ where $p_0 < \infty$. Here we use [24, Sects. III.8–III.9]. By our construction, the corresponding norm estimate for the second term is independent of Ω. Since we can estimate the first term in L^∞ on $[t_0 + 1, t_0 + T + 1]$, we find that the solution of the linear periodic boundary-value problem defines a continuous map of $L^p([0, T] \times \Omega) \to L^{q(p)}([0, T] \times \Omega) \cap W^{1,2}([0, T] \times \Omega)$, where the constants are independent of Ω (for $\Omega \subseteq \tilde{B}$).

Suppose that u is a solution of the problem

$$\frac{\partial u}{\partial t} = \Delta u + \tilde{f} \qquad \text{on } \Omega \times [0, T], \qquad u(x, t) = 0 \qquad \text{if } x \in \partial\Omega$$

and u is T-periodic (where $\tilde{f} \in L^2$). If we extend u to be zero in $(\tilde{B} \times [0, T]) \backslash (\Omega \times [0, T])$ then $u \in W^{1,2}(\tilde{B} \times [0, T])$. One way to see this is to note that the series expansion for $u = \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} a_{ks} \phi_k(x) \exp(i\gamma s)$ converges in $W^{1,2}(\Omega \times [0, T])$ (where $\gamma = 2\pi T^{-1}$ and $\phi_k(x)$ are the eigenfunctions for the Dirichlet problem on Ω). Since each element extends to $W^{1,2}(\tilde{B} \times [0, T])$ without increasing its norm, the result follows. Moreover, we can easily use the series expansion to show that, if $\tilde{f} \in L^2(\tilde{B} \times [0, T])$, and if u_n is a solution of the periodic boundary problem on Ω_n with $\tilde{f}$ replace by $\tilde{f}|_{\Omega_n \times [0,T]}$, then $u_n \to u_0$ in $L^2(\tilde{B} \times [0, T])$ as $n \to \infty$. Recall that λ_k^n and ϕ_k^n converges in R and $L^2(\tilde{B})$, respectively, where λ_k^n and ϕ_k^n are the eigenvalues and eigenfunctions for $-\Delta$ on Ω_n. Note also that one can easily show a uniform estimate in L^2 for the "tail" of the series. Let $\tilde{R}_n$ denote the map which takes $\tilde{f} \in L^2(\tilde{B} \times [0, T])$ to u_n except that we extend u_n to $\tilde{B} \times [0, T]$ by defining u_n to be zero outside of $\Omega_n \times [0, T]$. $\tilde{R}_0$ is defined analogously. From our last result, our earlier $W^{1,2}$ estimates and the Sobolev embedding theorem, see that $\{\tilde{R}_n\}_{n=0}^{\infty}$ is a collectively compact sequence of linear operators on $L^2(\tilde{B} \times [0, T])$.

Now consider the equation

$$\frac{\partial u}{\partial t} - \Delta u = f(t, u) \qquad \text{on } \Omega_n \tag{15}$$

with T-periodic and Dirichlet boundary conditions. Then, if we look for solutions in $L^\infty(\Omega_n)$, this is equivalent to the problem

$$u = \tilde{R}_n f(t, u|_{\Omega_n \times [0,T]}) \qquad \text{on } L^\infty(\tilde{B} \times [0, T]).$$

As in [7], a solution u_0 of (15) in $L^\infty(\Omega_0)$ is said to be non-degenerate if the linearized T-periodic Dirichlet problem

$$\frac{\partial h}{\partial t} - \Delta h = f_2'(t, u_0) h \qquad \text{in } \Omega_0$$

has only the trivial solution. The following result now follows by a simple modification of the proof of Theorem 1 in [7].

THEOREM 3. *Assume that $\Omega_n \to \Omega_0$ as $n \to \infty$ in our earlier sense, that f has polynomial growth and that u_0 is a non-degenerate T-periodic solution of* (15) *for $\Omega = \Omega_0$ and the boundary conditions in $L^\infty(\Omega_0 \times [0, T])$. Then, for large n, there is a unique solution u_n of* (15) (*for $\Omega = \Omega_n$*) *and the boundary conditions near u_0 in $L^p(\tilde{B} \times [0, T])$ for a suitable large p (independent of n). In addition, u_n is non-degenerate. Moreover, given $K > 0$, every solution u for $\Omega = \Omega_n$ and the boundary conditions with $\|u\|_{\infty, \Omega_n \times [0,T]} \leqslant K$ must be* *close in* $L^p(\tilde{B} \times [0, T])$ *to a solution on* $\Omega_0 \times [0, T]$ *for* $n \geqslant n_0$ *(where* n_0 *is independent of the solution* u*).*

Remarks. Note that we do not need $\partial\Omega_0$ to have measure zero. A similar result holds for systems with diagonal linear part. In particular, our result can be easily used to prove the existence of many positive solutions for some of the models in [12, 19] and time dependent competing species population models as in Cosner and Lazer [5] (but with Dirichlet boundary conditions). This is similar to [7]. With more care, it is possible to prove results relating the spectra of the linearization at u_0 and u_n (at least if $\partial\Omega_0$ has zero measure).

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