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A Geometric Inequality for Cyclic Quadrilaterals

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Abstract. In this paper, we establish an inequality involving the cosines of the arc measures determined by four arbitrary points along the circumference of a circle. Our proof is analytic in nature and as a consequence of it, we obtain some inequalities involving the interior angle measures of a triangle. Extending our arguments yields a sharpened version of the aforementioned inequality and also the best possible positive constant for which it holds.

1. Introduction

A variety of inequalities have been shown for convex quadrilaterals [4], in particular, in the bicentric case (see, e.g., [5, 6]). Perhaps the most well-known of these and simplest states that $\frac{r}{R}$ is at most $\frac{1}{\sqrt{2}}$, where r and R denote the radii of the incircle and circumcircle of a bicentric quadrilateral $ABCD$ (see [1, p. 132]). A refinement of this inequality was shown by Yun [7] and is given by

$$\frac{r\sqrt{2}}{R} \leq \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right) \leq 1, \tag{1}$$

which was proved again in different ways by Josefsson [3] and later Hess [2]. Note that the middle quantity in (1) may be rewritten in terms of the measures of arcs subtended by the sides of quadrilateral $ABCD$ as

$$\frac{1}{2}\left(\cos\left(\frac{a-c}{2}\right)+\cos\left(\frac{b-d}{2}\right)\right),$$

where $2a = m(\overset{\frown}{AB})$, etc.

In this paper, we find a lower bound for this quantity (see Theorem 16 below) that applies to all cyclic quadrilaterals and that differs from (and is incomparable to) the bound given in (1) in the bicentric case. For the sake of clarity of the proofs, we first establish the following bound in the next two sections.

Theorem 1. *Let $ABCD$ be a cyclic quadrilateral, with $2a = m(\overset{\frown}{AB})$, $2b = m(\overset{\frown}{BC})$, $2c = m(\overset{\frown}{CD})$ and $2d = m(\overset{\frown}{DA})$. If $a = \max\{a, b, c, d\}$, then we have*

$$2\cos a+\cos b+\cos c+\cos d \leq \frac{5\sqrt{2}}{4}\left(\cos\left(\frac{a-c}{2}\right)+\cos\left(\frac{b-d}{2}\right)\right), \tag{2}$$

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with equality if and only if $ABCD$ is a square.

We assume that all arc measures in Theorem 1 and elsewhere are taken in the same direction as the labeling of the vertices of $ABCD$. To prove (2), we proceed analytically, treating separately the acute and obtuse cases of a. We then extend our arguments in finding a sharpened form of (2), along with the best positive constant for which it holds. That is, we show that one may replace the weights of $\frac{1}{5}(2,1,1,1)$ for the respective cosine terms on the left-hand side of (2) (after dividing both sides of (2) by 5) with $\frac{1}{\alpha+3}(\alpha,1,1,1)$, where $\alpha=\frac{12-3\sqrt{2}}{4+\sqrt{2}}\approx 1.43$ is as small as possible. As corollaries to our approach, we obtain some related geometric inequalities involving triangles and bicentric quadrilaterals.

2. The acute case of the inequality

To prove (2), we divide into cases based on whether or not a is acute, treating in this section the acute case. Let

$$\begin{aligned} &f(a,b,c,d)\\ &=\frac{5\sqrt{2}}{4}\left(\cos\left(\frac{a-c}{2}\right)+\cos\left(\frac{b-d}{2}\right)\right)-(2\cos a+\cos b+\cos c+\cos d)\end{aligned}$$

and

$$S=\{(a,b,c,d)\in\mathbb{R}^4 : a+b+c+d=\pi,\ \text{where } 0\le b,c,d\le a\le\pi/2\}.$$

We will regard S as a closed subset of the metric space M consisting of all points in $\mathbb{R}^4$ such that $a+b+c+d=\pi$. When speaking of the boundary or interior of S, it will be in reference to M. Then the acute case of (2) is equivalent to showing $f\ge 0$ for all points in S. The next four lemmas show that f is non-negative for all points along the boundary of S.

Lemma 2. *We have $f>0$ for all points (a,b,c,d) in S such that $abcd=0$.*

Proof. We show that $h\ge 0$, where

$$h(a,b,c,d)=\sqrt{2}\left(\cos\left(\frac{a-c}{2}\right)+\cos\left(\frac{b-d}{2}\right)\right)-(\cos a+\cos b+\cos c+\cos d),$$

with $0\le a,b,c,d\le\pi/2$, $abcd=0$ and $a+b+c+d=\pi$, which implies the stated result for f. To show $h\ge 0$, we may assume $d=0$, by symmetry. Thus, we must show

$$j(a,b,c)=\sqrt{2}\left(\cos\left(\frac{a-c}{2}\right)+\cos\left(\frac{b}{2}\right)\right)-(\cos a+\cos b+\cos c+1)\ge 0, \quad (3)$$

where $a+b+c=\pi$ and $0\le a,b,c\le\pi/2$. We first show that inequality (3) holds for the boundary values. By symmetry of the a and c variables, we need only consider the following cases: (i) $a=0$, (ii) $b=0$, (iii) $a=\pi/2$, (iv) $b=\pi/2$. If (i) or (ii) holds, then the other two variables must be $\pi/2$ and (3) is obvious in either case. If (iii), then we must show $\sqrt{2}\left(\cos\left(\frac{\pi}{4}-\frac{c}{2}\right)+\cos\left(\frac{b}{2}\right)\right) \geq \cos b+\cos c+1$, where $b+c=\pi/2$, i.e.,

$$2\cos\left(\frac{b}{2}\right) \geq \frac{1}{\sqrt{2}}(\sin b+\cos b+1) = \sin\left(b+\frac{\pi}{4}\right)+\frac{1}{\sqrt{2}}, \qquad 0 \leq b \leq \pi/2.$$

The last inequality follows from observing that the function $k(b) = 2\cos\left(\frac{b}{2}\right) - \sin\left(b+\frac{\pi}{4}\right)-\frac{1}{\sqrt{2}}$ is decreasing on $(0,\pi/2)$ with $k(\pi/2)=0$. If (iv), then we need $\sqrt{2}\cos\left(\frac{a-c}{2}\right) \geq \cos a+\cos c = 2\cos\left(\frac{a+c}{2}\right)\cos\left(\frac{a-c}{2}\right)$, which holds with equality since $a+c=\pi/2$.

To complete the proof of (3), we must consider any possible interior local extreme points of j. Making use of Lagrange multipliers and equating the a- and the c- partial derivative equations implies (I) $a+c=\pi/2$ or (II) $a=c$. Note that if (I) holds, then $b=\pi/2$, which has already been considered, so assume (II). Using the cosine double angle formula, we must show in this case that

$$\sqrt{2}(\cos y+1)-4\cos^2 x-2\cos^2 y+2 \geq 0, \tag{4}$$

where $2x+y=\pi/2$ and $0<x,y<\pi/4$. For (4), we show equivalently $\sqrt{2}(\cos y+1) \geq 2\sin y+2\cos^2 y$, i.e.,

$$\sqrt{2}\cos y+2\left(\sin^2 y-\sin y\right) \geq 2-\sqrt{2}, \qquad 0<y<\pi/4.$$

This last inequality holds since the two sides are equal when $y=\pi/4$, with the left-hand side decreasing as one may verify. $\square$

By (3) and the fact that $\cos(\angle A)+\cos(\angle B)+\cos(\angle C) = 1+\frac{r}{R}$ in a triangle ABC, we obtain the following bound on the ratio $\frac{r}{R}$ in an acute or right triangle.

Corollary 3. *Let ABC be a non-obtuse triangle. Then*

$$2+\frac{r}{R} \leq \sqrt{2}\left(\cos\left(\frac{A-C}{2}\right)+\cos\left(\frac{B}{2}\right)\right),$$

with equality if and only if ABC is a right triangle with hypotenuse AC.

Lemma 4. *We have $f>0$ for all points (a,b,c,d) in S such that $a=\pi/2$.*

Proof. Let

$$h(b,c,d) = \frac{5\sqrt{2}}{4}\left(\cos\left(\frac{\pi}{4}-\frac{c}{2}\right)+\cos\left(\frac{b-d}{2}\right)\right)-(\cos b+\cos c+\cos d).$$

We must show $h>0$ for all $b,c,d \geq 0$ such that $b+c+d=\pi/2$. We may assume $bcd \neq 0$, by the previous lemma. To show $h>0$, we apply Lagrange multipliers and equate the b- and d-partial derivative equations to obtain $\sin b - \sin d = \frac{5\sqrt{2}}{4}\sin\left(\frac{b-d}{2}\right)$, i.e., $2\sin\left(\frac{b-d}{2}\right)\cos\left(\frac{b+d}{2}\right) = \frac{5\sqrt{2}}{4}\sin\left(\frac{b-d}{2}\right)$. So we must have (i) $b+d = 2\cos^{-1}\left(\frac{5\sqrt{2}}{8}\right)$ or (ii) $b=d$. If (i) holds, then $\frac{\pi}{4}-\frac{c}{2}=\frac{b+d}{2}$ so that $\cos\left(\frac{\pi}{4}-\frac{c}{2}\right)=\frac{5\sqrt{2}}{8}$, which implies $h>0$ in this case. So assume (ii) holds and we must show $h(b,c,b) > 0$, where $b, c > 0$ and $2b + c = \frac{\pi}{2}$. Substituting $c = \frac{\pi}{2} - 2b$ into $h(b,c,b)$, we need for

$$q(b) = \left(\frac{5\sqrt{2}}{4} - 2\right)\cos b - \sin 2b + \frac{5\sqrt{2}}{4} > 0, \qquad 0 < b < \pi/4.$$

This holds since $2 - \frac{5\sqrt{2}}{4} < \frac{1}{4}$ implies $q(b) > -\frac{1}{4} + \frac{5\sqrt{2}}{4} - 1 = \frac{5}{4}\left(\sqrt{2} - 1\right) > 0$. $\square$

As a consequence of the previous lemma, we obtain the following trigonometric inequality for acute triangles.

Corollary 5. *If ABC is an acute triangle, then*

$$\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right) < \frac{5\sqrt{2}}{4}\left(\cos\left(\frac{\pi - A}{4}\right) + \cos\left(\frac{B - C}{4}\right)\right).$$

The next two lemmas concern the case when one of the other variables equals a.

Lemma 6. *We have $f \geq 0$ for all points (a,b,c,d) in S such that $a = c$.*

Proof. We must show

$$h(b,c,d) = \frac{5\sqrt{2}}{4}\left(\cos\left(\frac{b-d}{2}\right) + 1\right) - (\cos b + 3\cos c + \cos d) \geq 0, \tag{5}$$

where $0 \leq b, d \leq c \leq \pi/2$ and $b + 2c + d = \pi$. By Lemmas 2 and 4, we may assume $b, d > 0$ and $c < \pi/2$ in (5). Suppose first that b or d equals c, say d. Then $h(b,c,c) \geq 0$ if and only if $\frac{5\sqrt{2}}{4}\left(\cos\left(\frac{b-c}{2}\right) + 1\right) \geq \cos b + 4\cos c$, where $b + 3c = \pi$ and $0 < b \leq c < \pi/3$, i.e.,

$$\frac{5\sqrt{2}}{4}(\sin(2c) + 1) \geq 4\cos c - \cos(3c), \qquad \pi/4 \leq c < \pi/3. \tag{6}$$

Dividing both sides of (6) by $\cos c$, we need to show

$$k(c) = \frac{5\sqrt{2}}{4}(2\sin c + \sec c) + 4\cos^2 c - 7 \geq 0, \qquad \pi/4 \leq c < \pi/3.$$

This last inequality follows from noting

$$\frac{k'(c)}{\cos c} = \frac{5\sqrt{2}}{4}\left(2 + \sec^2 c \tan c\right) - 8\sin c > 0$$

for $\pi/4 < c < \pi/3$, with $k(\pi/4) = 0$.

By Lagrange's method, any interior extreme points (b,c,d) within the set over which we are minimizing h must satisfy (i) $b + d = 2\theta$, where $\theta = \cos^{-1}\left(\frac{5\sqrt{2}}{8}\right)$, or (ii) $b = d$. If (i) holds, then we have $c = \frac{\pi}{2} - \theta$ and thus

$$\begin{aligned}
&\frac{5\sqrt{2}}{4}\left(1 + \cos\left(\frac{b-d}{2}\right)\right) - (\cos b + \cos d) \\
&= \frac{5\sqrt{2}}{4} + 2\cos\left(\frac{b+d}{2}\right)\cos\left(\frac{b-d}{2}\right) - (\cos b + \cos d) = \frac{5\sqrt{2}}{4} > \frac{3\sqrt{14}}{8} \\
&= 3\cos c,
\end{aligned}$$

which implies (5) in this case. If (ii) holds, then we need to show $\frac{5\sqrt{2}}{2} \geq 2\cos b + 3\cos c$, where $b + c = \pi/2$ and $0 < b \leq c$. Note that $2\cos b + 3\cos c = 2\sin c + 3\cos c = \sqrt{13}\sin(c + \lambda)$, where $\lambda = \cos^{-1}\left(\frac{2}{\sqrt{13}}\right)$. Then $\lambda > \pi/4$ implies $\sqrt{13}\sin(c + \lambda) \leq \sqrt{13}\sin\left(\frac{\pi}{4} + \lambda\right) = \frac{5\sqrt{2}}{2}$ for $\pi/4 \leq c < \pi/2$, as desired, which establishes (5) and completes the proof. Note that there is equality in (5) iff $a = b = c = d = \pi/4$, i.e., iff $ABCD$ is a square. □

Lemma 7. *We have $f \geq 0$ for all points (a, b, c, d) in S such that $a = b$.*

Proof. We must show

$$h(a, c, d) = \frac{5\sqrt{2}}{4}\left(\cos\left(\frac{a-c}{2}\right) + \cos\left(\frac{a-d}{2}\right)\right) - (3\cos a + \cos c + \cos d) \geq 0, \tag{7}$$

where $0 \leq c, d \leq a \leq \pi/2$ and $2a + c + d = \pi$. By Lemmas 2 and 4, we may assume $c, d > 0$ and $a < \pi/2$ in (7). By Lemma 6, we may also assume $a > c$ and $a > d$. Thus, we need only check points (a, c, d) corresponding to any possible interior extrema of the function h. Such points must satisfy $\sin c - \sin d = \frac{5\sqrt{2}}{8}\left(\sin\left(\frac{a-d}{2}\right) - \sin\left(\frac{a-c}{2}\right)\right)$, i.e., $\sin\left(\frac{c-d}{2}\right)\cos\left(\frac{c+d}{2}\right) = \frac{5\sqrt{2}}{8}\sin\left(\frac{c-d}{4}\right)\cos\left(\frac{2a-c-d}{4}\right)$, from which we get

$$\text{(i) } \frac{5\sqrt{2}}{16}\cos\left(\frac{2a-c-d}{4}\right) = \cos\left(\frac{c-d}{4}\right)\cos\left(\frac{c+d}{2}\right), \text{ or (ii) } c = d.$$

Since $c + d = \pi - 2a$ and $\pi/4 < a < \pi/2$, we have $c + d < \pi/2$ and thus

$$\cos\left(\frac{c-d}{4}\right)\cos\left(\frac{c+d}{2}\right) > \frac{1}{2} > \frac{5\sqrt{2}}{16} \geq \frac{5\sqrt{2}}{16}\cos\left(\frac{2a-c-d}{4}\right),$$

whence (i) is not possible. So assume (ii), in which case we must show

$$k(a) = \frac{5\sqrt{2}}{2}\cos\left(a - \frac{\pi}{4}\right) - \sqrt{13}\sin(a + \lambda) \geq 0, \qquad \pi/4 < a < \pi/2, \tag{8}$$

where $\lambda = \cos^{-1}\left(\frac{2}{\sqrt{13}}\right)$. Note that $k(\pi/4) = 0$, with

$$k'(a) = -\frac{5\sqrt{2}}{2}\sin\left(a - \frac{\pi}{4}\right) + \sqrt{13}\sin\left(a - \left(\frac{\pi}{2} - \lambda\right)\right) > 0, \qquad \pi/4 < a < \pi/2,$$

since $\sqrt{13} > \frac{5\sqrt{2}}{2}$ and $\frac{\pi}{4} > \frac{\pi}{2} - \lambda$ implies $\sqrt{13}\sin\left(a - \left(\frac{\pi}{2} - \lambda\right)\right) > \frac{5\sqrt{2}}{2}\sin\left(a - \frac{\pi}{4}\right)$. This implies (8), which completes the proof. □

We now must consider any possible local minima of f located within the interior of S. Treating these cases will yield the following result.

Theorem 8. *We have $f \geq 0$ for all points in S, with equality if and only if $a = b = c = d = \pi/4$.*

Proof. Let (a,b,c,d) denote an interior local extreme point of f in S. From the a- and the c-partial derivative equations, such points must satisfy

$$2\sin a - \sin c = \frac{5\sqrt{2}}{4}\sin\left(\frac{a-c}{2}\right). \tag{9}$$

Given $0 < a < \pi/2$, define the function

$$h_a(x) = \frac{5\sqrt{2}}{4}\sin\left(\frac{x-a}{2}\right) - \sin x + 2\sin a, \qquad 0 < x < a.$$

Below, we show that the equation $h_a(x) = 0$ has no solution. Hence, neither does (9), which implies f has no interior extreme points. Thus, the minimum value of f on the compact set S must be achieved along its boundary, and by Lemmas 2, 4, 6 and 7, we have $f \geq 0$ for all points along the boundary. Thus, $f \geq 0$ on all of S, as desired. From the proofs of Lemmas 6 and 7, there is equality as stated.

We now show $h_a(x)$ has no solution. First suppose $0 < x < \frac{a}{2}$. Then

$$\begin{aligned} h_a(x) &> -\frac{5\sqrt{2}}{4}\sin\left(\frac{a}{2}\right) - \sin\left(\frac{a}{2}\right) + 4\sin\left(\frac{a}{2}\right)\cos\left(\frac{a}{2}\right) \\ &> \left(2\sqrt{2} - 1 - \frac{5\sqrt{2}}{4}\right)\sin\left(\frac{a}{2}\right) > 0, \end{aligned}$$

since $a < \pi/2$ implies $\cos(a/2) > \sqrt{2}/2$. If $a/2 \leq x < a$, then

$$\begin{aligned} h_a(x) &> -\frac{5\sqrt{2}}{4}\sin\left(\frac{a}{4}\right) + \sin a = \left(4\cos\left(\frac{a}{2}\right)\cos\left(\frac{a}{4}\right) - \frac{5\sqrt{2}}{4}\right)\sin\left(\frac{a}{4}\right) \\ &> \left(2 - \frac{5\sqrt{2}}{4}\right)\sin\left(\frac{a}{4}\right) > 0, \end{aligned}$$

since $\cos(a/2),\ \cos(a/4) > \sqrt{2}/2$. □

3. The obtuse case

Assume now $2a' = m(\widehat{AB})$ is at least π, as shown below. In this case, we replace a' with $\pi - a$, where $a = b + c + d$ is acute (or possibly right).

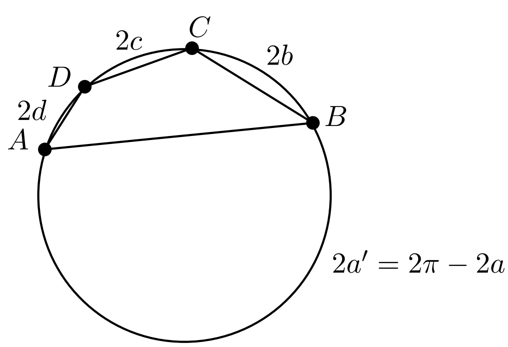

Figure 1. Case when $2a' = m(\widehat{AB})$ exceeds π.

Define the set T by

$$T = \{(a, b, c, d) \in \mathbb{R}^4 : a = b + c + d, \text{ where } b, c, d \geq 0 \text{ and } a \leq \pi/2\}.$$

We regard T as a closed subset of the metric space comprising all points in $\mathbb{R}^4$ such that $a = b + c + d$. Define the function g on $\mathbb{R}^4$ by

$$g(a, b, c, d) = \frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right) + \cos\left(\frac{b-d}{2}\right)\right) + 2\cos a - (\cos b + \cos c + \cos d).$$

Then the task of establishing (2) in the case when $m(\widehat{AB})$ is at least π is equivalent to showing that g can assume only positive values on T. We first consider the case of a point in T where $c = 0$.

Lemma 9. *We have $g > 0$ for all points (a, b, c, d) in T such that $c = 0$.*

Proof. We must show

$$h(a, b, d) = \frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a}{2}\right) + \cos\left(\frac{b-d}{2}\right)\right) + 2\cos a - (\cos b + \cos d + 1) > 0, \tag{10}$$

where $a = b + d$, $0 \leq a \leq \pi/2$ and $b, d \geq 0$. By Lemma 4, we may assume $a < \pi/2$. If $b = 0$ or $d = 0$, say $b = 0$, then $a = d$ and (10) reduces in this case to

$$\frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a}{2}\right) + \cos\left(\frac{a}{2}\right)\right) + \cos a - 2 > 0, \qquad 0 \leq a < \pi/2.$$

The last inequality can be shown by considering the first two derivatives of the left-hand side.

We now check h at any possible interior extreme points (a, b, d). From the b- and d-partial derivative equations, we get $\sin b - \sin d = \frac{5\sqrt{2}}{4}\sin\left(\frac{b-d}{2}\right)$, i.e., $2\sin\left(\frac{b-d}{2}\right)\cos\left(\frac{b+d}{2}\right) = \frac{5\sqrt{2}}{4}\sin\left(\frac{b-d}{2}\right)$. This implies such points (a, b, d) satisfy (i) $\cos\left(\frac{b+d}{2}\right) = \frac{5\sqrt{2}}{8}$ or (ii) $b = d$. If (i) holds, then $a = b + d$ implies $\cos a = \frac{9}{16}$, $\sin\left(\frac{a}{2}\right) = \frac{\sqrt{14}}{8}$ and $\cos\left(\frac{b-d}{2}\right) \geq \frac{5\sqrt{2}}{8}$. Since $\cos b + \cos d \leq 2\cos\left(\frac{b+d}{2}\right) = \frac{5\sqrt{2}}{4}$, inequality (10) follows in this case. If (ii) holds, then (10) reduces to

$$k(b) = \frac{5\sqrt{2}}{4}(\sin b + 1) + 2\cos 2b - 2\cos b - 1 > 0, \qquad 0 < b < \pi/4. \tag{11}$$

To show (11), first note that

$$\frac{k'(b)}{\cos b} = \frac{5\sqrt{2}}{4} - 8\sin b + 2\tan b.$$

Since the function $8\sin b - 2\tan b$ is increasing on $(0, \pi/4)$, one has that $k'(b)$ changes sign once on $(0, \pi/4)$, from positive to negative. Inequality (11) now follows from observing that $k(0)$ and $k(\pi/4)$ are both positive, which completes the proof. $\square$

Lemma 10. *We have $g > 0$ for all points (a, b, c, d) in T such that $d = 0$.*

Proof. We must show

$$h(a,b,c)=\frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right)+\cos\left(\frac{b}{2}\right)\right)+2\cos a-(\cos b+\cos c+1)>0, \tag{12}$$

where $a=b+c$, $0\le a\le \pi/2$ and $b,c\ge 0$. Again, we may assume $a<\pi/2$. By the previous lemma, we may also assume $c>0$. On the other hand, if $b=0$, then $a=c$ and (12) reduces to $\frac{5\sqrt{2}}{4}(\sin a+1)+\cos a-2>0$, where $0\le a<\pi/2$, which follows from observing $\max\{\sin a,\cos a\}\ge\sqrt{2}/2$ for first quadrant a.

We now consider possible internal extreme points (a,b,c) of h and apply the Lagrange method with constraint $-a+b+c=0$. From the b- and c-partial derivative equations, we get

$$2\sin\left(\frac{b-c}{2}\right)\cos\left(\frac{a}{2}\right)=\frac{5\sqrt{2}}{4}\cos\left(\frac{\pi}{4}+\frac{c}{2}\right)\cos\left(\frac{\pi}{4}-\frac{a}{2}\right). \tag{13}$$

We show that (13) cannot hold, which will imply the desired result. Note first that (13) implies $b>c$ and hence $0<c<a/2$. Also, observe that the ratio $\sin\left(\frac{\pi}{4}-\frac{c}{2}\right)/\sin\left(\frac{a}{2}-\frac{c}{2}\right)$ for a fixed a is minimized when $c=0$. Thus, we have

$$\begin{aligned}\frac{\cos\left(\frac{\pi}{4}+\frac{c}{2}\right)}{\sin\left(\frac{b}{2}-\frac{c}{2}\right)}\cdot\frac{\cos\left(\frac{\pi}{4}-\frac{a}{2}\right)}{\cos\left(\frac{a}{2}\right)}&>\frac{\sin\left(\frac{\pi}{4}-\frac{c}{2}\right)}{\sin\left(\frac{a}{2}-\frac{c}{2}\right)}\cdot\frac{\cos\left(\frac{\pi}{4}-\frac{a}{2}\right)}{\cos\left(\frac{a}{2}\right)}>\frac{\sin\left(\frac{\pi}{4}\right)}{\sin\left(\frac{a}{2}\right)}\cdot\frac{\cos\left(\frac{\pi}{4}-\frac{a}{2}\right)}{\cos\left(\frac{a}{2}\right)}\\&=\frac{\sin\left(\frac{a}{2}\right)+\cos\left(\frac{a}{2}\right)}{2\sin\left(\frac{a}{2}\right)\cos\left(\frac{a}{2}\right)}=\frac{1}{2}\left(\sec\left(\frac{a}{2}\right)+\csc\left(\frac{a}{2}\right)\right).\end{aligned}$$

Since the function $\sec\left(\frac{a}{2}\right)+\csc\left(\frac{a}{2}\right)$ is decreasing on $(0,\pi/2)$, we have $\sec\left(\frac{a}{2}\right)+\csc\left(\frac{a}{2}\right)\ge 2\sqrt{2}$ and thus

$$\frac{\cos\left(\frac{\pi}{4}+\frac{c}{2}\right)}{\sin\left(\frac{b}{2}-\frac{c}{2}\right)}\cdot\frac{\cos\left(\frac{\pi}{4}-\frac{a}{2}\right)}{\cos\left(\frac{a}{2}\right)}>\sqrt{2},$$

whence (13) cannot hold, which completes the proof. □

Theorem 11. *We have $g>0$ for all points in T.*

Proof. By Lemmas 4, 9 and 10, we have already shown that $g>0$ for all points along the boundary of the compact set T. To show that $g>0$ on all of T, we need to check g at any possible interior extrema. To do so, we apply Lagrange multipliers to g with constraint $b+c+d-a=0$. Equating the a- and the c- partial derivative equations gives

$$2\sin a-\sin c=\frac{5\sqrt{2}}{4}\cos\left(\frac{a+c}{2}\right). \tag{14}$$

Let $\theta=2\sin^{-1}\left(5\sqrt{2}/16\right)$ and a be fixed where $0<a<\theta$. Then (14) cannot hold for $0<c<a<\theta$, upon considering the function $h_a(x)=\sin x+\frac{5\sqrt{2}}{4}\cos\left(\frac{x+a}{2}\right)-2\sin a$ for each a and showing $h_a(x)>0$ for $0<x<a$ (by showing $h_a(0)>0$, $h_a(a)>0$ with $h_a''(x)<0$ for $0<x<a$). Thus, there are no interior extreme points in T for which $a<\theta$. Henceforth, let us assume $a\ge\theta$.

Equating the b- and d-partial derivative equations gives

$$\sin b - \sin d = \frac{5\sqrt{2}}{4} \sin\left(\frac{b-d}{2}\right),$$

which implies (i) $b = d$ or (ii) $\cos\left(\frac{b+d}{2}\right) = \frac{5\sqrt{2}}{8}$. First assume (i). We will show that $g > 0$ for all interior points in T such that $b = d$ and $a \geq \theta$. In this case, we must show $j(a, b, c) > 0$, where

$$j(a,b,c) = \frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right) + 1\right) + 2\cos a - 2\cos b - \cos c$$

and $a = 2b + c$. By the concavity of the cosine function on $(0, \pi/2)$, we have $2\cos b + \cos c \leq 3\cos\left(\frac{2b+c}{3}\right) = 3\cos\left(\frac{a}{3}\right)$, so to show $j > 0$, it suffices to show $k(a, c) > 0$, where

$$k(a,c) = \frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right) + 1\right) + 2\cos a - 3\cos\left(\frac{a}{3}\right).$$

Since $a \geq \theta$, we have $\frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right) + 1\right) > \frac{5}{2}$, so clearly $k(a, c) > 0$ for $\theta \leq a < \cos^{-1}(1/4)$. On the other hand, if $a \geq \cos^{-1}(1/4)$, then $\frac{5\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right) + 1\right) > 2.85$, whereas $3\cos\left(\frac{a}{3}\right) < 2.72$, which again implies $k(a, c) > 0$.

So assume (ii) holds and let $\lambda = 2\cos^{-1}\left(\frac{5\sqrt{2}}{8}\right)$. Then $a - c = b + d = \lambda$ and we show that (14) has no solution in this case. Suppose to the contrary that (14) does hold for some a and c, where $a > \lambda$ and $c = a - \lambda$. Then

$$\frac{5\sqrt{2}}{4}\cos\left(\frac{a+c}{2}\right) = \frac{5\sqrt{2}}{4}\cos\left(a - \frac{\lambda}{2}\right) = \frac{25}{16}\cos a + \frac{5\sqrt{7}}{16}\sin a,$$

so that (14) holds if and only if

$$\left(2 - \frac{5\sqrt{7}}{16}\right)\sin a - \frac{25}{16}\cos a = \sin c = \sin(a - \lambda) = \sin a\cos\lambda - \cos a\sin\lambda,$$

i.e.,

$$q(a) = \left(2 - \frac{5\sqrt{7}}{16} - \cos\lambda\right)\sin a - \left(\frac{25}{16} - \sin\lambda\right)\cos a = 0. \tag{15}$$

Since the coefficients inside the parentheses are both positive quantities, $q(a)$ is an increasing function of a, where $\lambda < a < \pi/2$. Since $q(\lambda) > 0$, it follows that (15) and hence (14) cannot hold if (ii) does, which completes the proof. $\square$

Theorem 1 above now follows from combining Theorems 8 and 11, which cover the acute and obtuse cases of a, respectively, upon replacing a by $\pi - a$ in Theorem 11. $\square$

We obtain as a corollary to Theorem 1 the following variant of inequality (1).

Corollary 12. *Let $ABCD$ be a cyclic quadrilateral, with $2a = m(\overset{\frown}{AB})$, $2b = m(\overset{\frown}{BC})$, $2c = m(\overset{\frown}{CD})$ and $2d = m(\overset{\frown}{DA})$. If AB is the longest side length of $ABCD$, then*

$$\begin{aligned}&\frac{\sqrt{2}}{5}(2\cos a + \cos b + \cos c + \cos d)\\ &\le \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{D}{2} + \sin\frac{D}{2}\cos\frac{A}{2}\right) \le 1. \quad (16)\end{aligned}$$

Proof. We have

$$\begin{aligned}&\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{D}{2} + \sin\frac{D}{2}\cos\frac{A}{2}\\ &= \sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\sin\frac{A}{2} + \cos\frac{A}{2}\sin\frac{B}{2} + \cos\frac{B}{2}\cos\frac{A}{2}\\ &= \sin\left(\frac{A+B}{2}\right) + \cos\left(\frac{A-B}{2}\right) = \sin\left(\frac{b+2c+d}{2}\right) + \cos\left(\frac{b-d}{2}\right)\\ &= \sin\left(\frac{\pi-(a-c)}{2}\right) + \cos\left(\frac{b-d}{2}\right) = \cos\left(\frac{a-c}{2}\right) + \cos\left(\frac{b-d}{2}\right),\end{aligned}$$

so that the right inequality in (16) is clear. The left inequality then follows from Theorem 1. Note that there is equality in the right inequality iff $ABCD$ is a rectangle and in the left iff $ABCD$ is a square. □

4. A sharpened version of the inequality

To sharpen inequality (2), we seek the smallest positive constant δ such that

$$\begin{aligned}f_\delta(a,b,c,d) = &\ \frac{(\delta+3)\sqrt{2}}{4}\left(\cos\left(\frac{a-c}{2}\right) + \cos\left(\frac{b-d}{2}\right)\right)\\ &- (\delta\cos a + \cos b + \cos c + \cos d) \ge 0 \qquad (17)\end{aligned}$$

for all points (a,b,c,d) in S and

$$\begin{aligned}g_\delta(a,b,c,d) = &\ \frac{(\delta+3)\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right) + \cos\left(\frac{b-d}{2}\right)\right)\\ &+ \delta\cos a - (\cos b + \cos c + \cos d) \ge 0 \qquad (18)\end{aligned}$$

for all points in T. Theorem 1 above then corresponds to the $\delta = 2$ case. Upon dividing (17) by $\delta+3$, one sees that (17) amounts to comparing a certain weighted average of the individual cosine terms with the quantity $\frac{\sqrt{2}}{4}\left(\cos\left(\frac{a-c}{2}\right) + \cos\left(\frac{b-d}{2}\right)\right)$. A similar interpretation applies to (18). From this, one sees that if inequalities (17) and (18) hold for some $\delta_0 > 0$, then they hold for all $\delta > \delta_0$. Note that $\delta = 1$ is too small since in the case of a bicentric quadrilateral, the inequalities would be reversed (see Theorem 17 below). This leaves open the question of finding the best possible δ for which (17) and (18) hold where $1 < \delta < 2$.

Taking (a,b,c,d) equal to the origin in (18) implies that the best possible δ is at least $\frac{12-3\sqrt{2}}{4+\sqrt{2}} \approx 1.43$, which we will denote by α. In fact, by modifying appropriately the proofs of Theorems 8 and 11 above, one can show that (17) and (18) indeed hold when $\delta = \alpha$. In the proofs of the following lemmas, we carry out the most extensive modifications that are required.

Lemma 13. *Let*

$$u_a(x) = \frac{(\alpha+3)\sqrt{2}}{4}\sin\left(\frac{x-a}{2}\right) - \sin x + \alpha \sin a, \qquad 0 < x < a,$$

where $\pi/4 < a < \pi/2$. *Then* $u_a(x) > 0$ *for* $0 < x < a$.

Proof. First assume $0 < x < a/2$. Then one may verify that $u_a'(0) < 0$ and $u_a''(x) > 0$. Note further that

$$u_a'(a/2) = \frac{(\alpha+3)\sqrt{2}}{8}\cos(a/4) - \cos(a/2)$$

is an increasing function of a, which is negative at $a = 86^\circ$ and positive at $a = 87^\circ$. First assume $45^\circ < a \le 86^\circ$. Then $u_a'(x) < 0$ for $0 < x < a/2$ and

$$u_a(a/2) = \alpha \sin a - \sin(a/2) - \frac{(\alpha+3)\sqrt{2}}{4}\sin(a/4) > 0, \qquad \pi/4 < a < \pi/2,$$

as one may verify, which implies $u_a(x) > 0$ for $0 < x < a/2$ in this case. If $86^\circ < a < 90^\circ$, then to establish the desired result in this case, it suffices to show

$$r(x) = \alpha \sin(86^\circ) - \sin x - \frac{(\alpha+3)\sqrt{2}}{4}\sin\left(45^\circ - \frac{x}{2}\right) > 0, \qquad 0 < x < \pi/4. \tag{19}$$

Since $r'(43^\circ) < 0$, $r'(44^\circ) > 0$ and $r''(x) > 0$, the function $r(x)$ must achieve its minimum value somewhere on the interval $[43^\circ, 44^\circ]$. If $43^\circ \le x \le 44^\circ$, then

$$r(x) > \alpha \sin(86^\circ) - \sin(44^\circ) - \frac{(\alpha+3)\sqrt{2}}{4}\sin\left(45^\circ - \frac{43^\circ}{2}\right) > 0,$$

which implies (19).

Now suppose $a/2 \le x < a$. In this case, we fix x and consider $u_a(x)$ as a function of a, where $x < a \le 2x$. First assume $x \le \pi/4$ and note that $\frac{d^2}{da^2}u_a(x) < 0$ for all a. Since $u_x(x) > 0$ and

$$u_{2x}(x) = \alpha \sin(2x) - \sin x - \frac{(\alpha+3)\sqrt{2}}{4}\sin(x/2) > 0, \qquad 0 < x \le \pi/4,$$

as one may verify, it follows that $u_a(x) > 0$ for $x < a \le 2x$ in this case. On the other hand, if $x > \pi/4$, then $u_a(x) > 0$ for $x < a < \pi/2$ follows from observing $u_x(x) > 0$, $\frac{d^2}{da^2}u_a(x) < 0$ and

$$u_{\pi/2}(x) = \alpha - \sin x - \frac{(\alpha+3)\sqrt{2}}{4}\sin\left(\frac{\pi}{4} - \frac{x}{2}\right) > 0, \qquad \pi/4 < x < \pi/2,$$

which completes the proof. $\square$

Lemma 14. *We have $v \geq 0$ for the function*

$$v(a,b,c) = \frac{(\alpha+3)\sqrt{2}}{4}\left(\sin\left(\frac{a}{2}\right)+\cos\left(\frac{b-c}{2}\right)\right)+\alpha\cos a-(\cos b+\cos c+1),$$

where $a = b + c$, $0 \leq a \leq \pi/2$ and $b, c \geq 0$.

Proof. By Lemma 4 (suitably extended to f_α), we may assume $a < \pi/2$. If b or c equals zero, say b, then $a = c$ and $v \geq 0$ reduces in this case to

$$r(a) = \frac{(\alpha+3)\sqrt{2}}{4}\left(\sin\left(\frac{a}{2}\right)+\cos\left(\frac{a}{2}\right)\right)+(\alpha-1)\cos a-2 \geq 0, \quad 0 \leq a \leq \pi/2. \tag{20}$$

Note that $\alpha > 1$ implies $r''(a) < 0$. Since $r(0) = 0$ (by definition of α) and $r(\pi/2) = \frac{\alpha-1}{2} > 0$, inequality (20) follows, with equality only for $a = 0$.

We now check v at any possible interior extreme points (a, b, c). From the b- and c-partial derivative equations, we get (i) $\cos\left(\frac{b+c}{2}\right) = \frac{(\alpha+3)\sqrt{2}}{8}$ or (ii) $b = c$. If (i) holds, then $a = b + c$ implies $\cos a \approx 0.23$ and $\sin(a/2) \approx 0.62$. Then we have in this case

$$\begin{aligned} v(a,b,c) &= \frac{(\alpha+3)\sqrt{2}}{4}\sin\left(\frac{a}{2}\right)+2\cos\left(\frac{b+c}{2}\right)\cos\left(\frac{b-c}{2}\right) \\ &\quad +\alpha\cos a-(\cos b+\cos c+1) \\ &= \frac{(\alpha+3)\sqrt{2}}{4}\sin\left(\frac{a}{2}\right)+\alpha\cos a-1 > 0. \end{aligned}$$

If (ii) holds, then $v > 0$ is equivalent to

$$s(b) = \frac{(\alpha+3)\sqrt{2}}{4}(\sin b+1)+\alpha\cos(2b)-2\cos b-1 > 0, \qquad 0 < b < \pi/4.$$

One can show that $s'(a)$ has one sign change on the interval $(0, \pi/4)$, from positive to negative. Since $s(0) = 0$ and $s(\pi/4) > 0$, this implies $s(b) > 0$ for $0 < b < \pi/4$, which completes the proof. □

Lemma 15. *We have $w > 0$ for the function*

$$w(a,b,c) = \frac{(\alpha+3)\sqrt{2}}{4}\left(\sin\left(\frac{a+c}{2}\right)+1\right)+\alpha\cos a-2\cos b-\cos c,$$

where $a = 2b + c$, $b, c > 0$ and $\pi/4 < a < \pi/2$.

Proof. By the concavity of cosine on $(0, \pi/2)$, to show $w > 0$, it suffices to show

$$r(a) = \frac{(\alpha+3)\sqrt{2}}{4}\left(\sin\left(\frac{a}{2}\right)+1\right)+\alpha\cos a-3\cos\left(\frac{a}{3}\right) > 0, \quad \pi/4 < a < \pi/2. \tag{21}$$

A direct computation reveals $r'''(a) > 0$ for $\pi/4 < a < \pi/2$ and that $r''(a)$ changes sign once on this interval. One may also verify $r'(\pi/4) < 0$ and $r'(\pi/2) < 0$, which implies $r'(a) < 0$ for all a. Inequality (21) now follows from observing $r(\pi/2) > 0$. □

One then gets the following strengthened version of Theorem 1.

Theorem 16. *We have*

$$\alpha\cos a+\cos b+\cos c+\cos d \le \frac{(\alpha+3)\sqrt{2}}{4}\left(\cos\left(\frac{a-c}{2}\right)+\cos\left(\frac{b-d}{2}\right)\right), \tag{22}$$

where $\alpha=\frac{12-3\sqrt{2}}{4+\sqrt{2}}$*, for all* $0\le b,c,d\le a$ *such that* $a+b+c+d=\pi$*, with equality if and only if* $a=b=c=d=\pi/4$ *or* $\pi-a=b=c=d=0$*. Furthermore,* α *is the smallest constant for which* (22) *holds.*

Proof. We make appropriate modifications, which we briefly describe, to the proof of Theorem 1 above using f_α and g_α in place of f and g. Note that in the proofs of Theorems 8 and 11 above, one would use, respectively, Lemmas 13 and 15 where needed. One would also substitute Lemma 14 for 9. Note that the proof of Lemma 2 implies that this result also applies to the function f_α. In other proofs, one would proceed as before, but instead with the function f_α in the acute and g_α in the obtuse case, which may at times require a bit more analysis than previously. Observe further that the strict inequality for g in the statements of Lemmas 9 and 10 and Theorem 11 should be replaced by the inclusive inequality $g_\alpha\ge 0$, where there is equality when all arguments are zero. Finally, the statements of the lemmas and theorems in the acute case will remain unchanged when considering the sharpened version of the inequality. □

Note that Corollary 12 above may also be strengthened by replacing the leftmost expression $\frac{\sqrt{2}}{5}(2\cos a+\cos b+\cos c+\cos d)$ with the larger quantity $\frac{\sqrt{2}}{\alpha+3}(\alpha\cos a+\cos b+\cos c+\cos d)$. For bicentric quadrilaterals, the inequality is in fact reversed when α is replaced by 1.

Theorem 17. *Let ABCD be a bicentric quadrilateral, with* $2a=m(\overset{\frown}{AB})$*,* $2b=m(\overset{\frown}{BC})$*,* $2c=m(\overset{\frown}{CD})$ *and* $2d=m(\overset{\frown}{DA})$*. Then*

$$\sqrt{2}\left(\cos\left(\frac{a-c}{2}\right)+\cos\left(\frac{b-d}{2}\right)\right)\le \cos a+\cos b+\cos c+\cos d, \tag{23}$$

with equality if and only if $ABCD$ *is a kite.*

Proof. Replacing the right-hand side of (23) by

$$2\cos\left(\frac{a+c}{2}\right)\cos\left(\frac{a-c}{2}\right)+2\cos\left(\frac{b+d}{2}\right)\cos\left(\frac{b-d}{2}\right),$$

and rearranging, we show equivalently

$$\left(\cos\left(\frac{b+d}{2}\right)-\frac{\sqrt{2}}{2}\right)\cos\left(\frac{b-d}{2}\right)\ge\left(\frac{\sqrt{2}}{2}-\cos\left(\frac{a+c}{2}\right)\right)\cos\left(\frac{a-c}{2}\right). \tag{24}$$

Since $ABCD$ has an inscribed circle, we have $AB+CD=BC+DA$, which implies $\sin a+\sin c=\sin b+\sin d$. This may be rewritten as $\sin\left(\frac{a+c}{2}\right)\cos\left(\frac{a-c}{2}\right)=$

$\sin\left(\frac{b+d}{2}\right)\cos\left(\frac{b-d}{2}\right)$, i.e., $\cos\left(\frac{a-c}{2}\right) = \tan\left(\frac{b+d}{2}\right)\cos\left(\frac{b-d}{2}\right)$, since $a+b+c+d = \pi$. Substituting this into (24), and letting $x = \frac{b+d}{2}$, gives

$$\cos x - \sqrt{2}/2 \geq (\sqrt{2}/2 - \sin x)\tan x, \qquad 0 < x < \pi/2. \tag{25}$$

Rearranging (25) gives $\sec x - \frac{\sqrt{2}}{2}\tan x \geq \frac{\sqrt{2}}{2}$ for $0 < x < \pi/2$. To show this, let $f(x) = \sec x - \frac{\sqrt{2}}{2}\tan x$. Then $f'(x) = \sec^2 x\left(\sin x - \frac{\sqrt{2}}{2}\right)$ so that $f(x)$ is minimized when $x = \pi/4$, with $f(\pi/4) = \sqrt{2}/2$. This implies (25) and hence (23). Note that there is equality in (23) iff $x = \frac{b+d}{2} = \frac{\pi}{4}$, i.e., $a + c = \frac{\pi}{2} = b + d$. Since $ABCD$ is bicentric, we then have $a - c = b - d$ or $a - c = d - b$, which implies $a = b, c = d$ or $a = d, b = c$. Thus, there is equality iff $ABCD$ is a kite, which completes the proof. $\square$

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