Category-theoretic structure for conditional independence

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Conditional independence is a fundamental relation in probability theory, statistics, Bayesian inference, ...

Conditional indep. between random variables can be construed as a relation between maps in a category of probability spaces.

We axiomatize local independence structure and local independent products on a category, capturing general properties of conditional independence.

Other examples of local independence structure and products on categories occur in, e.g., computability theory, nominal sets, separation logic, ...

So local independence structure and local independent products are common to diverse notions of conditional independence arising in different contexts.

(Similar general motivation to Dawid's "separoids", but using categories rather than preorders, and with new examples.)

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Structure of talk

Part I: Independence

1. Independence between random variables (recap)
2. Independence structure on a category.
3. Independent products.

Part II: Conditional Independence

4. Conditional independence between random variables.
5. Local independence structure on a category.
6. General conditional independence via image tuples.
7. Local independent products & examples.
8. The fibration of of independent squares.
9. Characterisation of sheaves for the atomic topology.

Part I: Independence

Independence of random variables (recap)

Given random variables:

$$\{\, \Omega \xrightarrow{X_i} R_i \,\}_{1 \leqslant i \leqslant n} \quad ,$$

define

$$\perp\!\!\!\perp(X_1, \ldots, X_n)$$

to hold if, for all events $A_1 \subseteq R_1, \ldots, A_n \subseteq R_n$,

$$P(X_1 \in A_1 \;\wedge\; \ldots \;\wedge\; X_n \in A_n) = P(X_1 \in A_1) \;\ldots\; P(X_n \in A_n)$$

Joint independence implies pairwise independence:

$$\perp\!\!\!\perp(X_1, \ldots, X_n) \text{ implies } X_i \perp\!\!\!\perp X_j, \text{ whenever } i \neq j.$$

But pairwise independence does not imply joint independence.

The categories **Prob** and $\mathbf{Prob_0}$

Objects: probability spaces $(S, \mathcal{F}_S, P_S)$

- $\mathcal{F}_S$ a σ-algebra over a set S; and
- P_S a probability measure on $\mathcal{F}_S$.

Morphisms: measure-preserving functions $X : S \to T$; i.e.,

- X is measurable (that is $X^{-1}(B) \in \mathcal{F}_S$, for all $B \in \mathcal{F}_T$),
- $P_S(X^{-1}(B)) = P_T(B)$, for all $B \in \mathcal{F}_T$.

In $\mathbf{Prob_0}$ morphisms are identified mod 0.

The notion of joint independence defines independence as a property of families of maps with common domain. We write

$$\perp\!\!\!\perp \left\{ S \xrightarrow{X_i} T_i \right\}_{i \in I}$$

to say that a family is independent. (I finite in this talk.)

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Independence structure

An independence structure on a category $\mathcal{C}$ with terminal object 1 is a specified collection of finite-set-indexed families

$$\left\{ X \xrightarrow{f_i} Y_i \right\}_{i \in I}$$

of morphisms with common domain. Families in the collection are called independent, notation $\perp\!\!\!\perp \{f_i\}_{i \in I}$.

Independent families are required to satisfy:

- Every singleton family $\{X \xrightarrow{f} Y\}$ is independent.
- If $\perp\!\!\!\perp \{X \xrightarrow{f_i} Y_i\}_{i \in I}$ and $\perp\!\!\!\perp \{Y_i \xrightarrow{g_{ij}} Z_{ij}\}_{j \in J_i}$ for all $i \in I$ then $\perp\!\!\!\perp \{X \xrightarrow{g_{ij} \circ f_i} Z_{ij}\}_{i \in I, j \in J_i}$.
- If $\perp\!\!\!\perp \{f_j\}_{j \in J}$ and $m \colon I \to J$ is injective then $\perp\!\!\!\perp \{f_{m(i)}\}_{i \in I}$.
- If $\perp\!\!\!\perp \{X \xrightarrow{f_i} Y_i\}_{i \in I}$ then $\perp\!\!\!\perp \{X \xrightarrow{f_i} Y_i\}_{i \in I} \cup \{X \xrightarrow{!} 1\}$.

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Independent products

A category $\mathcal{C}$ with terminal object and independence structure has (finite) independent products if, for every pair of objects Y_1, Y_2, there there exist an object $Y_1 \otimes Y_2$ and an independent pair

$$Y_1 \otimes Y_2 \xrightarrow{\pi_1} Y_1 \quad \perp\!\!\!\perp \quad Y_1 \otimes Y_2 \xrightarrow{\pi_2} Y_2$$

satisfying:

- whenever $X \xrightarrow{f_1} Y_1 \perp\!\!\!\perp X \xrightarrow{f_2} Y_2$, there exists a unique morphism
$$X \xrightarrow{\langle f_1, f_2 \rangle} Y_1 \otimes Y_2$$
such that $\pi_i \circ \langle f_1, f_2 \rangle = f_i$, for $i \in \{1, 2\}$; and
- if $\perp\!\!\!\perp \{X \xrightarrow{f_i} Y_i\}_{i \in \{1,2\}} \cup \{X \xrightarrow{g_j} Z_j\}_{j \in J}$ then also $\perp\!\!\!\perp \{X \xrightarrow{\langle f_1, f_2 \rangle} Y_1 \otimes Y_2\} \cup \{X \xrightarrow{g_j} Z_j\}_{j \in J}$.

Independent product in **Prob** and **Prob$_0$**

$$(S_1, \mathcal{F}_1, P_1) \otimes (S_2, \mathcal{F}_2, P_2) \; := \; (S_1 \times S_2, \mathcal{F}_1 \otimes \mathcal{F}_2, P_1 \otimes P_2)$$

where:

- $S_1 \times S_2$ is product set,
- $\mathcal{F}_1 \otimes \mathcal{F}_2$ is product σ-algebra, and
- $P_1 \otimes P_2$ is product measure.

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Can one view the Independent Product in Probability categorially?

8

3

One can construct a category of probability spaces, but this category has no products. Now probability theory relies strongly on the ability to build independent products, the product measure. In a sense, the notion of independence is what distinguishes probablity theory from the theory of finite measures.

> Is there a categorial way to make sense of and enlighten the notion of independent products in category theory?

It is possible to formulate independence in Lawvere's category of *probabilistic mappings* (Borel spaces as objects and Markov kernels as morphisms) in terms of constant morphisms, but I think this is not very enlightening, conditional independence is built into the morphisms. Maybe, this is what one has to do when putting probability center stage?

I do know the rudiments of categry theory, but I would prefer an answer that does not require too much immersion in category thory, provided that is possible.

ct.category-theory pr.probability measure-theory

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asked Feb 5 at 21:41
Michael Greinecker
3,630 • 1 • 12 • 31

2 Answers

active oldest **votes**

7

Very nice question! ;) I wrote a short paper about this question about ten years ago, see http://arxiv.org/abs/math/0206017 (My apologies for advertising my own work, but this is exactly the question I asked myself at that time).

The product of probability spaces is tensor product in the sense of category, as Martin Brandeburg also pointed out in a comment. But it has an additional structure, you have natural morphisms onto the factors in the tensor product. This is because the projections onto the two factors that you have for the Cartesian product (of sets) preserve the measures, so they are also morphisms in the category of probability spaces. I called this structure a **tensor product with projections**: for two objects $\Omega_i = (\Omega_i, \mathcal{F}_i, P_i)$, $i = 1, 2$, you get $\Omega_1 \otimes \Omega_2 = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, P_1 \otimes P_2)$ **and** random variables $X_i : \Omega_1 \otimes \Omega_2 \to \Omega_i$, $i = 1, 2$.

You can use this "tensor product with projections" to characterise independence of random variables: two r.v. $Y_i : \Omega \to \Omega_i$, $i = 1, 2$, defined on the same probability space Ω, are independent iff they factorise, i.e., if there exists a r.v. $Z : \Omega \to \Omega_1 \otimes \Omega_2$ such that $Y_i = X_i \circ Z$, $i = 1, 2$.

The notion dualises to the algebras of functions on a probablity space, where it becomes a **tensor product with inclusions**. Generalising to not necessarily commutative algebras, it includes notions of independence used in noncommutative (or quantum) probability, like the freeness.

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edited Feb 6 at 9:16

answered Feb 6 at 9:07

Uwe Franz

1,233 ● 5 ● 20

Proposition The following are equivalent on a category $\mathcal{C}$.

- An independence structure with independent products.
- A symmetric monoidal structure with jointly monic projections.

A monoidal structure has projections [Jacobs 1995] if the unit I is a terminal object. Writing 1 for such a unit, define projection maps:

$$\pi_1 = Y_1 \otimes Y_2 \xrightarrow{1_{Y_1} \otimes !_{Y_2}} Y_1 \otimes 1 \xrightarrow{\cong} Y_1$$

$$\pi_2 = Y_1 \otimes Y_2 \xrightarrow{!_{Y_1} \otimes 1_{Y_2}} 1 \otimes Y_2 \xrightarrow{\cong} Y_2 \ .$$

The projections are jointly monic if, for all $X \xrightarrow{f_1} Y_1$ and $X \xrightarrow{f_2} Y_2$, there is at most one $X \xrightarrow{h} Y_1 \otimes Y_2$ such that both $\pi_1 \circ h = f_1$ and $\pi_2 \circ h = f_2$.

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Proof outline for Proposition

An independence structure endows the objects of $\mathcal{C}$ with the structure of a (dual) multicategory (whose multimorphisms have the very specific form of being given as families of morphisms in $\mathcal{C}$).

The definition of independent products in $\mathcal{C}$ implies that the multicategory is representable in the sense of [Hermida 2000], where it is shown that representable multicategories determine a derived monoidal structure. In the case at hand, this monoidal structure has jointly monic projections.

Conversely, given symmetric monoidal structure with jointly monic projections on $\mathcal{C}$, define the associated independence structure by:

$$\perp\!\!\!\perp \{X \xrightarrow{f_i} Y_i\}_{1 \leqslant i \leqslant n} \iff \exists\, X \xrightarrow{h} Y_1 \otimes \cdots \otimes Y_n \text{ s.t.}$$
$$\pi_1 \circ h = f_1 \text{ and } \ldots \text{and } \pi_n \circ h = f_n$$

Nominal sets

Independence structure

Given:

$$\left\{ X \xrightarrow{f_i} Y_i \right\}_{i \in I}$$

define:

$$\perp\!\!\!\perp \{f_i\}_{i \in I} \;\Leftrightarrow\; \forall i \neq j \in I.\, \forall x \in X.\, \mathrm{supp}(f_i(x)) \cap \mathrm{supp}(f_j(x)) = \varnothing \;.$$

Independent product

$$X \otimes Y \;:=\; \{(x, y) \in X \times Y \mid \mathrm{supp}(x) \cap \mathrm{supp}(y) = \varnothing\}$$

This is the well-known separated product in the category of nominal sets.

Category of V-valued heaps (separation logic)

Objects: (L, s), where L is a finite set and $s \colon L \to V$ is a function.

Morphisms $(L, s) \to (L', s')$: injective functions $f \colon L' \to L$ such that $s \circ f = s'$.

Independence structure

Given:

$$\left\{ (L, s) \xrightarrow{f_i} (L_i, s_i) \right\}_{i \in I}$$

define:

$$\perp\!\!\!\perp \{f_i\}_{i \in I} \;\Leftrightarrow\; \forall i \neq j \in I.\ \mathrm{img}(f_i) \cap \mathrm{img}(f_j) = \varnothing \ .$$

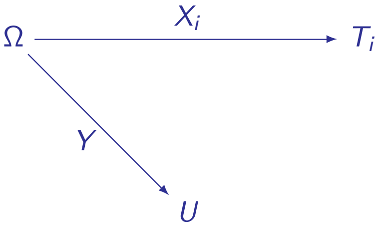
Independent product

$$(L_1, s_1) \otimes (L_2, s_2) \;:=\; (L_1 + L_2,\ [s_1, s_2])$$

Part II: Conditional Independence

Conditional independence of random variables

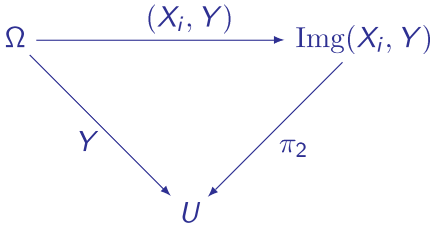
Consider a family $\{\Omega \xrightarrow{X_i} T_i\}_{i\in I}$ of random variables and a conditioning variable $\Omega \xrightarrow{Y} U$, as below



Informally, $\{X_i\}_{i\in I}$ is said to be conditionally independent given Y (notation $\perp\!\!\!\perp \{X_i\}_{i\in I} \mid Y$) if the conditional probability distributions of X_i given Y are independent.

We omit the technical definition, which, for arbitrary probability spaces, uses Kolmogorov's conditional expectation operator.

Consider the associated commuting diagram:



Then it holds that

$$\perp\!\!\!\perp \{X_i\}_{i \in I} \mid Y \quad \Leftrightarrow \quad \perp\!\!\!\perp \{(X_i, Y)\}_{i \in I} \mid Y$$

Thus, given image tuples, general conditional independence can be reduced to local independence in slice categories.

We axiomatize the latter as the primary notion.

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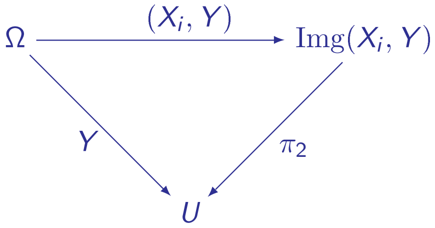

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Thus, given image tuples, general conditional independence can be reduced to local independence in slice categories.

We axiomatize the latter as the primary notion.

Local independence structure

Local independence structure on a category $\mathcal{C}$ is given by independence structures $\perp\!\!\!\perp_I$, on every slice category $\mathcal{C}/I$, related by the following property.

- For every commuting diagram

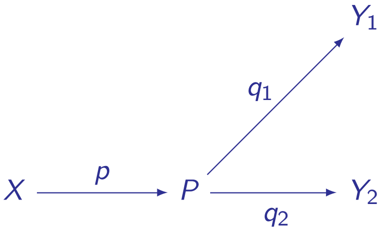
$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow \\ W & \longrightarrow & I & \longrightarrow & J \end{array}$$

  in $\mathcal{C}$, if $f \perp\!\!\!\perp_I h$ and $g \perp\!\!\!\perp_J i$ then $g \circ f \perp\!\!\!\perp_J h$.

The property asserts that independent squares compose.

General conditional independence via image tuples

Given a pair of morphisms $X \xrightarrow{f_1} Y_1$ and $X \xrightarrow{f_2} Y_2$, in a category $\mathcal{C}$, we call a structure



a pairing of f_1, f_2 if:

- $q_1 \circ p = f_1$,
- $q_2 \circ p = f_2$, and
- q_1, q_2 are jointly monic.

An image pairing of a span $X \xrightarrow{f_1} Y_1$ and $X \xrightarrow{f_2} Y_2$, is a pairing

$$\begin{array}{ccccc} & & & & Y_1 \\ & & & \nearrow^{\tilde{\pi}_1} & \\ X & \xrightarrow{(f_1, f_2)} & \mathrm{Img}(f_1, f_2) & \xrightarrow[\tilde{\pi}_2]{} & Y_2 \end{array}$$

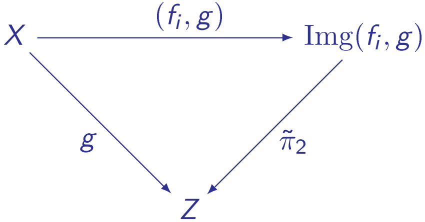
that is initial w.r.t. all pairings.

I.e., given any pairing p, q_1, q_2 (as on previous slide), there exists (a necessarily unique) $\mathrm{Img}(f_1, f_2) \xrightarrow{h} P$ such that $h \circ (f_1, f_2) = p$, $q_1 \circ h = \tilde{\pi}_1$ and $q_2 \circ h = \tilde{\pi}_2$.

We say that $\mathcal{C}$ has image tuples if image pairings exist for all spans.

Let $\mathcal{C}$ be a category with local independent structure and with image tuples.

Given a maps $\{X \xrightarrow{f_i} Y_i\}_{i\in I}$ and $X \xrightarrow{g} Z$, we obtain commuting triangles:



We define the general relation of conditional independence by

$$\perp\!\!\!\perp \{f_i\}_{i\in I} \mid g \quad \Leftrightarrow \quad \perp\!\!\!\perp_Z \{(f_i, g)\}_{i\in I}$$

This axiomatics can be used to derive general laws of conditional independence.

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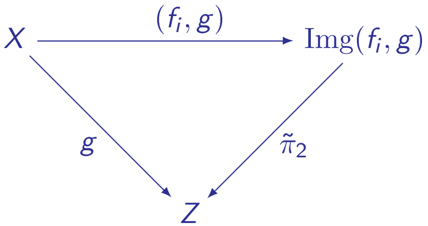

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Local independent products

Let $\mathcal{C}$ be a category with local independence structure.

We say that $\mathcal{C}$ has local independent products if the independence structure on each slice $\mathcal{C}/I$ has independent products.

Given $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$ we use the following convenient notation for $f \otimes_Z g$.

$$\begin{array}{ccc} X \otimes_Z Y & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

We call such squares independent product squares. Every such square is *a fortiori* independent.

Nominal sets

Local independence

Given $\{X \xrightarrow{f_i} Y_i\}_{i \in I}$ from $X \xrightarrow{g} Z$ to $\{Y_i \longrightarrow Z\}_{i \in I}$ in slice over Z, define:

$$\perp\!\!\!\perp_Z \{f_i\}_{i \in I} \iff \forall i \neq j \in I.\, \forall x \in X.\, \mathrm{supp}(f_i(x)) \cap \mathrm{supp}(f_j(x)) \subseteq \mathrm{supp}(g(x)) \;.$$

Local independent product

Given objects $X \xrightarrow{f} Z$ and $Y \xrightarrow{g} Z$ in slice over Z, define:

$$X \otimes_Z Y \;:=\; \{(x, y) \in X \times_Z Y \mid \mathrm{supp}(x) \cap \mathrm{supp}(y) \subseteq \mathrm{supp}(f(x))\}$$

Name sharing is conditioning!

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Category of V-heaps

Local independence

Given $\{(L,s) \xrightarrow{f_i} (L_i,s_i)\}_{i\in I}$ from $(L,s) \xrightarrow{g} (L',s')$ to $\{(L_i,s_i) \longrightarrow (L',s')\}_{i\in I}$ in slice over (L',s'), define:

$$\perp\!\!\!\perp_{(L',s')} \{f_i\}_{i\in I} \;\Leftrightarrow\; \forall i \neq j \in I.\, \mathrm{img}(f_i) \cap \mathrm{img}(f_j) \subseteq \mathrm{img}(g) \;\; .$$

Local independent product

Given objects $(L_1,s_1) \xrightarrow{f} (L',s')$ and $(L_2,s_2) \xrightarrow{g} (L',s')$ in slice over (L',s'), define:

$$(L_1,s_1) \otimes_{(L',s')} (L_2,s_2) \;:=\; (L_1 +_{L'} L_2,\, [s_1,s_2])$$

Shared memory is conditioning!

Category of V-heaps

Local independence

Given $\{(L,s) \xrightarrow{f_i} (L_i, s_i)\}_{i \in I}$ from $(L,s) \xrightarrow{g} (L', s')$ to $\{(L_i, s_i) \longrightarrow (L', s')\}_{i \in I}$ in slice over (L', s'), define:

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Shared memory is conditioning!

Categories of probability spaces

The standard probabilistic notion of conditional independence defines local independence structure on **Prob** and $\mathbf{Prob_0}$.

Apparently, local independent products don't exist in general.

We restrict to standard probability spaces (a.k.a. Lebesgue-Rokhlin spaces).

Trivially, the local independence structure on **Prob** (resp. $\mathbf{Prob_0}$) restricts to the full subcategory **StdProb** (resp. $\mathbf{StdProb_0}$) on standard probability spaces.

$\mathbf{StdProb_0}$ has local independent products.

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Standard probability spaces and disintegrations

A key feature of standard probability spaces is that every map $(S, \mathcal{F}_S, P_S) \xrightarrow{Y} (U, \mathcal{F}_U, P_U)$ has a disintegration:

- For almost all $u \in U$ the fibre $S_u := Y^{-1}(u)$ carries the structure of a standard probability space $(S_u, \mathcal{F}_{S_u}, P_{S_u})$.
- For every $A \in \mathcal{F}_S$, we have: $A \cap S_u \in \mathcal{F}_{S_u}$, for almost all $u \in U$; the function $u \mapsto P_{S_u}(A \cap S_u)$ is measurable; and

$$P_S(A) = \int_{u \in T} P_S(A \mid Y = u) \, \mathrm{d}P_T$$

where we write $P_S(A \mid Y = u)$ for $P_{S_u}(A \cap S_u)$.

We first use disintegrations to give a conceptually straightforward definition of the local independent structure on $\mathbf{StdProb_0}$, then to define local independent products.

Standard probability spaces and disintegrations

A key feature of standard probability spaces is that every map $(S, \mathcal{F}_S, P_S) \xrightarrow{Y} (U, \mathcal{F}_U, P_U)$ has a disintegration:

- For almost all $u \in U$ the fibre $S_u := Y^{-1}(u)$ carries the structure of a standard probability space $(S_u, \mathcal{F}_{S_u}, P_{S_u})$.
- For every $A \in \mathcal{F}_S$, we have: $A \cap S_u \in \mathcal{F}_{S_u}$, for almost all $u \in U$; the function $u \mapsto P_{S_u}(A \cap S_u)$ is measurable; and

$$P_S(A) \;=\; \int_{u \in T} P_S(A \mid Y = u)\, \mathrm{d}P_T$$

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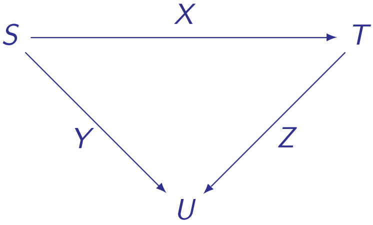
We first use disintegrations to give a conceptually straightforward definition of the local independent structure on $\mathbf{StdProb_0}$, then to define local independent products.

Given a commuting triangle in (w.l.o.g.) $\mathbf{StdProb_0}$,

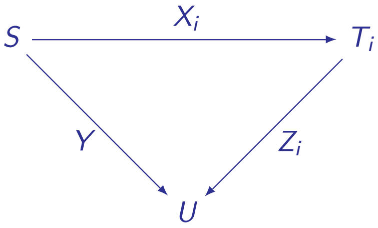


for almost all $u \in U$, the function $X\restriction_{S_u} : S_u \to T_u$ (where $S_u := Y^{-1}(u)$ and $T_u := Z^{-1}(u)$) is measure preserving:

$$(S_u, \mathcal{F}_{S_u}, P_{T_u}) \xrightarrow{X\restriction_{S_u}} (T_u, \mathcal{F}_{T_u}, P_{T_u}) \ .$$

Local independence structure on $\mathbf{StdProb_0}$

Given a family $\{Y \xrightarrow{X_i} Z_i\}_{i \in I}$ of maps in the slice category $\mathbf{StdProb_0}/U$, i.e.



Define $\{X_i\}_{i \in I}$ to be conditionally independent given Y, notation

$$\perp\!\!\!\perp \{X_i\}_{i \in I} \mid Y$$

if $\{X_i{\restriction}_{S_u}\}_{i \in I}$ is an independent family for almost all $u \in U$.

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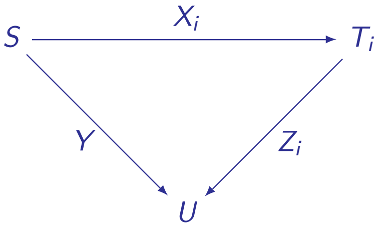

Define $\{X_i\}_{i \in I}$ to be conditionally independent given U, notation

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Proposition A commuting square, in $\mathbf{StdProb_0}$,

$$\begin{array}{ccc} S & \xrightarrow{X} & T \\ {\scriptstyle Y}\downarrow & & \downarrow{\scriptstyle Z} \\ U & \xrightarrow[W]{} & V \end{array}$$

is independent if and only, for almost all $u \in U$, the function $X\restriction_{S_u} : S_u \to T_{W(u)}$ is measure preserving.

$$(S_u, \mathcal{F}_{S_u}, P_{S_u}) \xrightarrow{X\restriction_{S_u}} (T_{W(u)}, \mathcal{F}_{T_{W(u)}}, P_{T_{W(u)}})$$

Corollary Independent squares compose.

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Local independent products in $\mathbf{StdProb_0}$

Given $S \longrightarrow U$ and $S' \longrightarrow U$ in $\mathbf{StdProb_0}$.

Define $S \otimes_U S' \longrightarrow U$ by taking $S \otimes_U S'$ to be:

$$(S, \mathcal{F}, P) \otimes_U (S', \mathcal{F}', P') \; := \; (S \times_U S', \mathcal{F} \otimes_U \mathcal{F}', P \otimes_U P')$$

where:

- $S \times_U S'$ is set-theoretic pullback,
- $(\mathcal{F} \otimes_U \mathcal{F}')_u$ is $\mathcal{F}_{S_u} \otimes \mathcal{F}'_{S'_u}$
- $(P \otimes_U P')_u$ is $P_{S_u} \otimes P'_{S'_u}$

using the disintegrations for $S \longrightarrow U$ and $S' \longrightarrow U$ to specify a disintegration for $S \otimes_U S' \longrightarrow U$, determining the map mod 0.

(Strictly, the measure $((\mathcal{F} \otimes_U \mathcal{F}')_u, (P \otimes_U P')_u)$ on each fibre $(S \times_U S')_u$ needs to be completed to a standard probability space.)

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The independent-square fibration

Let $\mathcal{C}$ be a category with local independence structure.

Let $\mathcal{C}^{\mathrm{ind}}$ be the category with as objects: morphisms in $\mathcal{C}$; and with as morphisms from $X \longrightarrow I$ to $Y \longrightarrow J$: independent squares

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ I & \longrightarrow & J \end{array}$$

($\mathcal{C}^{\mathrm{ind}}$ is a full-on-objects subcategory of the arrow category $\mathcal{C}^{\rightarrow}$.)

Proposition $\mathcal{C}$ has local independent products if and only if the codomain functor $\mathcal{C}^{\mathrm{ind}} \to \mathcal{C}$ is a fibration.

Properties of the fibration

- A morphism in $\mathcal{C}^{\mathrm{ind}}$

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ I & \longrightarrow & J \end{array}$$

  is cartesian if and only if the square is an independent product diagram in $\mathcal{C}/J$.

- The fibre category over I is isomorphic to $\mathcal{C}/I$.

- For $I \xrightarrow{u} J$ in $\mathcal{C}$, the reindexing functor $u^* : \mathcal{C}/J \to \mathcal{C}/I$ is strong monoidal.

The atomic topology

Proposition Suppose $\mathcal{C}$ has local independent products. Then the following are equivalent for a presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

- F is a sheaf for the atomic topology.
- F maps independent squares in $\mathcal{C}$ to pullbacks in $\mathbf{Set}$.

Further directions

1. Other examples of local indepedence struture and products related to: computability theory, group theory, . . .
2. Derivation of laws of conditional independence from categorical structure.
3. Relationship to Dawid's separoids.
4. Applications to randomness. (Tentative!)