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# THE SCIENCE ABSOLUTE OF SPACE

*Independent of the Truth or Falsity of Euclid's  
Axiom XI (which can never be  
decided a priori).*

BY

~~JOHN~~ BOLYAI

TRANSLATED FROM THE LATIN

BY

DR. GEORGE BRUCE HALSTED  
PRESIDENT OF THE TEXAS ACADEMY OF SCIENCE

FOURTH EDITION.

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## TRANSLATOR'S INTRODUCTION.

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The immortal *Elements* of Euclid was already in dim antiquity a classic, regarded as absolutely perfect, valid without restriction.

Elementary geometry was for two thousand years as stationary, as fixed, as peculiarly Greek, as the Parthenon. On this foundation pure science rose in Archimedes, in Apollonius, in Pappus; struggled in Theon, in Hypatia; declined in Proclus; fell into the long decadence of the Dark Ages.

The book that monkish Europe could no longer understand was then taught in Arabic by Saracen and Moor in the Universities of Bagdad and Cordova.

To bring the light, after weary, stupid centuries, to western Christendom, an Englishman, Adelhard of Bath, journeys, to learn Arabic, through Asia Minor, through Egypt, back to Spain. Disguised as a Mohammedan student, he got into Cordova about 1120, obtained a Moorish copy of Euclid's *Elements*, and made a translation from the Arabic into Latin.

The first printed edition of Euclid, published in Venice in 1482, was a Latin version from the Arabic. The translation into Latin from the Greek, made by Zamberti from a MS. of Theon's revision, was first published at Venice in 1505.

Twenty-eight years later appeared the *editio princeps* in Greek, published at Basle in 1533 by John Hervagius, edited by Simon Grynaeus. This was for a century and three-quarters the only printed Greek text of all the books, and from it the first English translation (1570) was made by "Henricus Billingsley," afterward Sir Henry Billingsley, Lord Mayor of London in 1591.

And even to-day, 1895, in the vast system of examinations carried out by the British Government, by Oxford, and by Cambridge, no proof of a theorem in geometry will be accepted which infringes Euclid's sequence of propositions.

Nor is the work unworthy of this extraordinary immortality.

Says Clifford: "This book has been for nearly twenty-two centuries the encouragement and guide of that scientific thought which is one thing with the progress of man from a worse to a better state.



“The encouragement; for it contained a body of knowledge that was really known and could be relied on.

“The guide; for the aim of every student of every subject was to bring his knowledge of that subject into a form as perfect as that which geometry had attained.”

But Euclid stated his assumptions with the most painstaking candor, and would have smiled at the suggestion that he claimed for his conclusions any other truth than perfect deduction from assumed hypotheses. In favor of the external reality or truth of those assumptions he said no word.

Among Euclid's assumptions is one differing from the others in prolixity, whose place fluctuates in the manuscripts.

Peyrard, on the authority of the Vatican MS., puts it among the postulates, and it is often called the parallel-postulate. Heiberg, whose edition of the text is the latest and best (Leipzig, 1883-1888), gives it as the fifth postulate.

James Williamson, who published the closest translation of Euclid we have in English, indicating, by the use of italics, the words not in the original, gives this assumption as eleventh among the Common Notions.

Bolyai speaks of it as Euclid's Axiom XI. Todhunter has it as twelfth of the Axioms.

Clavius (1574) gives it as Axiom 13.

The Harpur Euclid separates it by forty-eight pages from the other axioms.

It is not used in the first twenty-eight propositions of Euclid. Moreover, when at length used, it appears as the inverse of a proposition already demonstrated, the seventeenth, and is only needed to prove the inverse of another proposition already demonstrated, the twenty-seventh.

Now the great Lambert expressly says that Proklus demanded a proof of this assumption because when inverted it is demonstrable.

All this suggested, at Europe's renaissance, not a doubt of the necessary external reality and exact applicability of the assumption, but the possibility of deducing it from the other assumptions and the twenty-eight propositions already proved by Euclid without it.

Euclid demonstrated things more axiomatic by far. He proves what every dog knows, that any two sides of a triangle are together greater than the third.

Yet after he has finished his demonstration, that straight lines making with a transversal equal alternate angles are parallel, in order to

prove the inverse, that parallels cut by a transversal make equal alternate angles, he brings in the unwieldy assumption thus translated by Williamson (Oxford, 1781):

“11. And if a straight line meeting two straight lines make those angles which are inward and upon the same side of it less than two right angles, the two straight lines being produced indefinitely will meet each other on the side where the angles are less than two right angles.”

As Staeckel says, “it requires a certain courage to declare such a requirement, alongside the other exceedingly simple assumptions and postulates.” But was courage likely to fail the man who, asked by King Ptolemy if there were no shorter road in things geometric than through his *Elements*? answered, “To geometry there is no special way for kings!”

In the brilliant new light given by Bolyai and Lobachevski we now see that Euclid understood the crucial character of the question of parallels.

There are now for us no better proofs of the depth and systematic coherence of Euclid's masterpiece than the very things which, their cause unappreciated, seemed the most noticeable blots on his work.

viii TRANSLATOR'S INTRODUCTION.

Sir Henry Savile, in his *Praelectiones* on Euclid, Oxford, 1621, p. 140, says: "In pulcherrimo Geometriae corpore duo sunt naevi, duae labes . . ." etc., and these two blemishes are the theory of parallels and the doctrine of proportion; the very points in the *Elements* which now arouse our wondering admiration. But down to our very nineteenth century an ever renewing stream of mathematicians tried to wash away the first of these supposed stains from the most beautiful body of Geometry.

The year 1799 finds two extraordinary young men striving thus

"To gild refined gold, to paint the lily,  
To cast a perfume o'er the violet."

At the end of that year Gauss from Braunschweig writes to Bolyai Farkas in Klausenburg (Kölozsvár) as follows: [Abhandlungen der Koeniglichen Gesellschaft der Wissenschaften zu Goettingen, Bd. 22, 1877.]

"I very much regret, that I did not make use of our former proximity, to find out *more* about your investigations in regard to the first grounds of geometry; I should certainly thereby have spared myself much vain labor, and would have become more restful than any one, such

as I, can be, so long as on such a subject there yet remains so much to be wished for.

In my own work thereon I myself have advanced far (though my other wholly heterogeneous employments leave me little time therefor) but *the* way, which I have hit upon, leads not so much to the goal, which one wishes, as much more to making doubtful the truth of geometry.

Indeed I have come upon much, which with most no doubt would pass for a proof, but which in my eyes proves as good as *nothing*.

For example, if one could prove, that a rectilineal triangle is possible, whose content may be greater, than any given surface, then I am in condition, to prove with perfect rigor all geometry.

Most would indeed let that pass as an axiom; I not; it might well be possible, that, how far apart soever one took the three vertices of the triangle in space, yet the content was always under a given limit.

I have more such theorems, but in none do I find anything satisfying."

From this letter we clearly see that in 1799 Gauss was still trying to prove that Euclid's is the only non-contradictory system of geome-

try, and that it is the system regnant in the external space of our physical experience.

The first is false; the second can never be proven.

Before another quarter of a century, Bolyai János, then unborn, had created another possible universe; and, strangely enough, though nothing renders it impossible that the space of our physical experience may, this very year, be satisfactorily shown to belong to Bolyai János, yet the same is not true for Euclid.

To decide our space is Bolyai's, one need only show a single rectilinear triangle whose angle-sum measures less than a straight angle. And this could be shown to exist by imperfect measurements, such as human measurements must always be. For example, if our instruments for angular measurement could be brought to measure an angle to within one millionth of a second, then if the lack were as great as two millionths of a second, we could make certain its existence.

But to prove Euclid's system, we must show that a triangle's angle-sum is *exactly* a straight angle, which nothing human can ever do.

However this is anticipating, for in 1799 it seems that the mind of the elder Bolyai, Bolyai Farkas, was in precisely the same state as

that of his friend Gauss. Both were intensely trying to prove what now we know is indemonstrable. And perhaps Bolyai got nearer than Gauss to the unattainable. In his "Kurzer Grundriss eines Versuchs," etc., p. 46, we read: "Koennten jede 3 Punkte, die nicht in einer Geraden sind, in eine Sphaere fallen, so waere das Eucl. Ax. XI. bewiesen." Frischauf calls this "das anschaulichste Axiom." But in his Autobiography written in Magyar, of which my Life of Bolyai contains the first translation ever made, Bolyai Farkas says: "Yet I could not become satisfied with my different treatments of the question of parallels, which was ascribable to the long discontinuance of my studies, or more probably it was due to myself that I drove this problem to the point which robbed my rest, deprived me of tranquillity."

It is wellnigh certain that Euclid tried his own calm, immortal genius, and the genius of his race for perfection, against this self-same question. If so, the benign intellectual pride of the founder of the mathematical school of the greatest of universities, Alexandria, would not let the question cloak itself in the obscurities of the infinitely great or the infinitely small. He would say to himself: "Can I prove

this plain, straightforward, simple theorem: "those straights which are produced indefinitely from less than two right angles meet." [This is the form which occurs in the Greek of Eu. I. 29.]

Let us not underestimate the subtle power of that old Greek mind. We can produce no Venus of Milo. Euclid's own treatment of proportion is found as flawless in the chapter which Stolz devotes to it in 1885 as when through Newton it first gave us our present continuous number-system.

But what fortune had this genius in the fight with its self-chosen simple theorem? Was it found to be deducible from all the definitions, and the nine "Common Notions," and the five other Postulates of the immortal Elements? Not so. But meantime Euclid went ahead without it through twenty-eight propositions, more than half his first book. But at last came the practical pinch, then as now the triangle's angle-sum.

He gets it by his twenty-ninth theorem: "A straight falling upon two parallel straights makes the alternate angles equal."

But for the proof of this he needs that recalcitrant proposition which has how long been keeping him awake nights and waking



him up mornings? Now at last, true man of science, he acknowledges it indemonstrable by spreading it in all its ugly length among his postulates.

Since Schiaparelli has restored the astronomical system of Eudoxus, and Hultsch has published the writings of Autolycus, we see that Euclid knew surface-spherics, was familiar with triangles whose angle-sum is more than a straight angle. Did he ever think to carry out for himself the beautiful system of geometry which comes from the contradiction of his indemonstrable postulate; which exists if there be straights produced indefinitely from less than two right angles yet nowhere meeting; which is real if the triangle's angle-sum is less than a straight angle?

Of how naturally the three systems of geometry flow from just exactly the attempt we suppose Euclid to have made, the attempt to demonstrate his postulate fifth, we have a most romantic example in the work of the Italian priest, Saccheri, who died the twenty-fifth of October, 1733. He studied Euclid in the edition of Clavius, where the fifth postulate is given as Axiom 13. Saccheri says it should not be called an axiom, but ought to be demonstrated. He tries this seemingly simple

task; but his work swells to a quarto book of 101 pages.

Had he not been overawed by a conviction of the absolute necessity of Euclid's system, he might have anticipated Bolyai János, who ninety years later not only discovered the new world of mathematics but appreciated the transcendent import of his discovery.

Hitherto what was known of the Bolyais came wholly from the published works of the father Bolyai Farkas, and from a brief article by Architect Fr. Schmidt of Budapest "Aus dem Leben zweier ungarischer Mathematiker, Johann und Wolfgang Bolyai von Bolya." Grunert's Archiv, Bd. 48, 1868, p. 217.

In two communications sent me in September and October 1895, Herr Schmidt has very kindly and graciously put at my disposal the results of his subsequent researches, which I will here reproduce. But meantime I have from entirely another source come most unexpectedly into possession of original documents so extensive, so precious that I have determined to issue them in a separate volume devoted wholly to the life of the Bolyais; but these are not used in the sketch here given.

Bolyai Farkas was born February 9th, 1775, at Bolya, in that part of Transylvania (Er-

dély) called Székelyföld. He studied first at Enyed, afterward at Klausenburg (Kolozsvár), then went with Baron Simon Kemény to Jena and afterward to Goettingen. Here he met Gauss, then in his 19th year, and the two formed a friendship which lasted for life.

The letters of Gauss to his friend were sent by Bolyai in 1855 to Professor Sartorius von Walterhausen, then working on his biography of Gauss. Everyone who met Bolyai felt that he was a profound thinker and a beautiful character.

Benzenberg said in a letter written in 1801 that Bolyai was one of the most extraordinary men he had ever known.

He returned home in 1778, and in January, 1804, was made professor of mathematics in the Reformed College of Maros-Vásárhely. Here for 47 years of active teaching he had for scholars nearly all the professors and nobility of the next generation in Erdély.

Sylvester has said that mathematics is poesy.

Bolyai's first published works were dramas.

His first published book on mathematics was an arithmetic:

Az arithmetica eleje. 8vo. i-xvi, 1-162 pp. The copy in the library of the Reformed College is enriched with notes by Bolyai János.

Next followed his chief work, to which he constantly refers in his later writings. It is in Latin, two volumes, 8vo, with title as follows:

TENTAMEN | JUVENTUTEM STUDIOSAM |  
IN ELEMENTA MATHESEOS PURAE, ELEMEN-  
TARIS AC | SUBLIMIORIS, METHODO INTUI-  
TIVA, EVIDENTIA— | QUE HUIC PROPRIA, IN-  
TRODUCENDI. |

CUM APPENDICE TRIPLICI. | Auctore Pro-  
fessore Matheseos et Physices Chemiaeque |  
Publ. Ordinario. | Tomus Primus. | *Maros*  
*Vasarhelyini*. 1832. | Typis Collegii Re-  
formatòrum per JOSEPHUM, et | SIMEONEM  
KALI de felső Vist. | At the back of the title:  
Imprimatur. | M. Vásárhelyini Die | 12 Octo-  
bris, 1829. | Paulus Horváth m. p. | Abbas,  
Parochus et Censor | Librorum.

Tomus Secundus. | *Maros Vasarhelyini*.  
1833. |

The first volume contains:

Preface of two pages: *Lectori salutem*.

A folio table: *Explicatio signorum*.

*Index rerum* (I—XXXII). *Errata*  
(XXXIII—XXXVII).

*Pro tyronibus prima vice legentibus no-*  
*tanda sequentia* (XXXVIII—LII).

*Errores* (LIII—LXVI).

*Scholion* (LXVII—LXXIV).

*Plurium errorum haud animadversorum  
numerus minuitur* (LXXV—LXXVI).

*Recensio per auctorem ipsum facta*  
(LXXVII—LXXVIII).

*Errores recentius detecti* (LXXV—  
XCVIII).

Now comes the body of the text (pages 1—502).

Then, with special paging, and a new title page, comes the immortal Appendix, here given in English.

Professors Staeckel and Engel make a mistake in their "Parallellinien" in supposing that this Appendix is referred to in the title of "Tentamen." On page 241 they quote this title, including the words "Cum appendice triplici," and say: "In dem dritten Anhang, der nur 28 Seiten umfasst, hat Johann Bolyai seine neue Geometrie entwickelt."

It is not a third Appendix, nor is it referred to at all in the words "Cum appendice triplici."

These words, as explained in a prospectus in the Magyar language, issued by Bolyai Farkas, asking for subscribers, referred to a real triple Appendix, which appears, as it

should, at the end of the book *Tomus Secundus*, pp. 265–322.

The now world renowned Appendix by Bolyai János was an afterthought of the father, who prompted the son not “to occupy himself with the theory of parallels,” as Staeckel says, but to translate from the German into Latin a condensation of his treatise, of which the principles were discovered and properly appreciated in 1823, and which was given in writing to Johann Walter von Eckwehr in 1825.

The father, without waiting for Vol. II, inserted this Latin translation, with separate paging (1–26), as an Appendix to his Vol. I, where, counting a page for the title and a page “*Explicatio signorum*,” it has twenty-six numbered pages, followed by two unnumbered pages of *Errata*.

The treatise itself, therefore, contains only twenty-four pages—the most extraordinary two dozen pages in the whole history of thought!

Milton received but a paltry £5 for his *Paradise Lost*; but it was at least plus £5.

Bolyai János, as we learn from Vol. II, p. 384, of “*Tentamen*,” contributed for the

printing of his eternal twenty-six pages, 104 florins 50 kreuzers.

That this Appendix was finished considerably before the Vol. I, which it follows, is seen from the references in the text, breathing a just admiration for the Appendix and the genius of its author.

Thus the father says, p. 452: *Elegans est conceptus similitum, quem J. B. Appendicis Auctor dedit.* Again, p. 489: *Appendicis Auctor, rem acumine singulari aggressus, Geometriam pro omni casu absolute veram posuit; quamvis e magna mole, tantum summe necessaria, in Appendice hujus tomi exhibuerit, multis (ut tetraedri resolutione generali, pluribusque aliis disquisitionibus elegantibus) brevitatis studio omissis.*

And the volume ends as follows, p. 502: *Nec operae pretium est plura referre; quum res tota exaltiori contemplationis puncto, in ima penetranti oculo, tractetur in Appendice sequente, a quovis fideli veritatis purae alumno diagna legi.*

The father gives a brief resumé of the results of his own determined, life-long, desperate efforts to do that at which Saccheri, J. H. Lambert, Gauss also had failed, to establish Euclid's theory of parallels *a priori*.

He says, p. 490: "Tentamina idcirco quae olim feceram, breviter exponenda veniunt; ne saltem alius quis operam eandem perdat." He anticipates J. Delboeuf's "Prolégomènes philosophiques de la géométrie et solution des postulats," with the full consciousness in addition that it is *not* the solution,—that the final solution has crowned not his own intense efforts, but the genius of his son.

This son's Appendix which makes all preceding space only a special case, only a species under a genus, and so requiring a descriptive adjective, *Euclidean*, this wonderful production of pure genius, this strange Hungarian flower, was saved for the world after more than thirty-five years of oblivion, by the rare erudition of Professor Richard Baltzer of Dresden, afterward professor in the University of Giessen. He it was who first did justice publicly to the works of Lobachevski and Bolyai.

Incited by Baltzer, in 1866 J. Hoüel issued a French translation of Lobachevski's Theory of Parallels, and in a note to his Preface says: "M. Richard Baltzer, dans la seconde édition de ses excellents *Elements de Geometrie*, a, le premier, introduit ces notions exactes à la place qu'elles doivent occuper." Honor to



Baltzer! But alas! father and son were already in their graves!

Fr. Schmidt in the article cited (1868) says: "It was nearly forty years before these profound views were rescued from oblivion, and Dr. R. Baltzer, of Dresden, has acquired imperishable titles to the gratitude of all friends of science as the first to draw attention to the works of Bolyai, in the second edition of his excellent *Elemente der Mathematik* (1866-67). Following the steps of Baltzer, Professor Hoüel, of Bordeaux, in a brochure entitled, *Essai critique sur les principes fondamentaux de la Géométrie élémentaire*, has given extracts from Bolyai's book, which will help in securing for these new ideas the justice they merit."

The father refers to the son's Appendix again in a subsequent book, *Urtan elemei kez-döknek* [Elements of the science of space for beginners] (1850-51), pp. 48. In the College are preserved three sets of figures for this book, two by the author and one by his grandson, a son of János.

The last work of Bolyai Farkas, the only one composed in German, is entitled,  
*Kurzer Grundriss eines Versuchs*  
 I. Die Arithmetik, durch zweckmässig kons-

truirte Begriffe, von eingebildeten und unendlich-kleinen Grössen gereinigt, anschaulich und logisch-streng darzustellen.

II. In der Geometrie, die Begriffe der geraden Linie, der Ebene, des Winkels allgemein, der winkellosen Formen, und der Krümmen, der verschiedenen Arten der Gleichheit u. d. gl. nicht nur scharf zu bestimmen; sondern auch ihr Seyn im Raume zu beweisen: und da die Frage, *ob zwey von der dritten geschnittene Geraden, wenn die summe der inneren Winkel nicht =  $2R$ , sich schneiden oder nicht?* neimand auf der Erde ohne ein Axiom (wie Euklid das XI) aufzustellen, beantworten wird; die davon unabhängige Geometrie abzusondern; und eine auf die *Ja*—Antwort, andere auf das *Nein* so zu bauen, das die Formeln der letzten, auf ein Wink auch in der ersten gültig seyen.

Nach ein lateinischen Werke von 1829, M. Vásárhely, und eben daselbst gedruckten ungrischen.

Maros Vásárhely 1851. 8vo. pp. 88.

In this book he says, referring to his son's Appendix: "Some copies of the work published here were sent at that time to Vienna, to Berlin, to Goettingen. . . . From Goettingen the giant of mathematics, who from

his pinnacle embraces in the same view the stars and the abysses, wrote that he was surprised to see accomplished what he had begun, only to leave it behind in his papers."

This refers to 1832. The only other record that Gauss ever mentioned the book is a letter from Gerling, written October 31st, 1851, to Wolfgang Boylai, on receipt of a copy of "Kurzer Grundriss." Gerling, a scholar of Gauss, had been from 1817 Professor of Astronomy at Marburg. He writes: "I do not mention my earlier occupation with the theory of parallels, for already in the year 1810-1812 with Gauss, as earlier 1809 with J. F. Pfaff I had learned to perceive how all previous attempts to prove the Euclidean axiom had miscarried. I had then also obtained preliminary knowledge of your works, and so, when I first [1820] had to print something of my view thereon, I wrote it exactly as it yet stands to read on page 187 of the latest edition.

"We had about this time [1819] here a law professor, Schweikart, who was formerly in Charkov, and had attained to similar ideas, since without help of the Euclidean axiom he developed in its beginnings a geometry which he called Astralgeometry. What he communicated to me thereon I sent to Gauss, who

then informed me how much farther already had been attained on this way, and later also expressed himself about the great acquisition, which is offered to the few expert judges in the Appendix to your book."

The "latest edition" mentioned appeared in 1851, and the passage referred to is: "This proof [of the parallel-axiom] has been sought in manifold ways by acute mathematicians, but yet until now not found with complete sufficiency. So long as it fails, the theorem, as all founded on it, remains a hypothesis, whose validity for our life indeed is sufficiently proven by *experience*, whose *general, necessary exactness*, however, could be doubted without absurdity."

Alas! that this feeble utterance should have seemed sufficient for more than thirty years to the associate of Gauss and Schweikart, the latter certainly one of the independent discoverers of the non-Euclidean geometry. But then, since neither of these sufficiently realized the transcendent importance of the matter to publish any of their thoughts on the subject, a more adequate conception of the issues at stake could scarcely be expected of the scholar and colleague. How different with Bolyai János and Lobachévski, who claimed

at once, unflinchingly, that their discovery marked an epoch in human thought so momentous as to be unsurpassed by anything recorded in the history of philosophy or of science, demonstrating as had never been proven before the supremacy of pure reason at the very moment of overthrowing what had forever seemed its surest possession, the axioms of geometry.

On the 9th of March, 1832, Bolyai Farkas was made corresponding member in the mathematics section of the Magyar Academy.

As professor he exercised a powerful influence in his country.

In his private life he was a type of true originality. He wore roomy black Hungarian pants, a white flannel jacket, high boots, and a broad hat like an old-time planter's. The smoke-stained wall of his antique domicile was adorned by pictures of his friend Gauss, of Schiller, and of Shakespeare, whom he loved to call the child of nature. His violin was his constant solace.

He died November 20th, 1856. It was his wish that his grave should bear no mark.

The mother of Bolyai János, née Arkosi Benkő Zsuzsanna, was beautiful, fascinating,

xxvi TRANSLATOR'S INTRODUCTION.

of extraordinary mental capacity, but always nervous.

János, a lively, spirited boy, was taught mathematics by his father. His progress was marvelous. He required no explanation of theorems propounded, and made his own demonstrations for them, always wishing his father to go on. "Like a demon, he always pushed me on to tell him more."

At 12, having passed the six classes of the Latin school, he entered the philosophic-curriculum, which he passed in two years with great distinction.

When about 13, his father, prevented from meeting his classes, sent his son in his stead. The students said they liked the lectures of the son better than those of the father. He already played exceedingly well on the violin.

In his fifteenth year he went to Vienna to K. K. Ingenieur-Akademie.

In August, 1823, he was appointed "sous-lieutenant" and sent to Temesvár, where he was to present himself on the 2nd of September.

From Temesvár, on November 3rd, 1823, János wrote to his father a letter in Magyar, of which a French translation was sent me by Professor Koncz József on February 14th,

1895. This will be given in full in my life of Bolyai; but here an extract will suffice:

*“My Dear and Good Father:*

“I have so much to write about my new inventions that it is impossible for the moment to enter into great details, so I write you only on one-fourth of a sheet. I await your answer to my letter of two sheets; and perhaps I would not have written you before receiving it, if I had not wished to address to you the letter I am writing to the Baroness, which letter I pray you to send her.

“First of all I reply to you in regard to the binominal.

\* \* \* \* \*

“Now to something else, so far as space permits. I intend to write, as soon as I have put it into order, and when possible to publish, a work on parallels.

“At this moment it is not yet finished, but the way which I have followed promises me with certainty the attainment of the goal, if it in general is attainable. It is not yet attained, but I have discovered such magnificent things that I am myself astonished at them.

“It would be damage eternal if they were

xxviii TRANSLATOR'S INTRODUCTION.

lost. When you see them, my father, you yourself will acknowledge it. Now I can not say more, only so much: *that from nothing I have created another wholly new world.* All that I have hitherto sent you compares to this only as a house of cards to a castle.

“P. S.—I dare to judge absolutely and with conviction of these works of my spirit before you, my father; I do not fear from you any false interpretation (that certainly I would not merit), which signifies that, in certain regards, I consider you as a second self.”

From the Bolyai MSS., now the property of the College at Maros-Vásárhely, Fr. Schmidt has extracted the following statement by János:

“First in the year 1823 have I pierced through the problem in its essence, though also afterwards completions yet were added.

“I communicated in the year 1825 to my former teacher, Herr Johann Walter von Eckwehr (later k. k. General) [in the Austrian Army], a written treatise, which is still in his hands.

“On the prompting of my father I translated my treatise into the Latin language, and



it appeared as *Appendix* to the *Tentamen*, 1832."

The profound mathematical ability of Bolyai János showed itself physically not only in his handling of the violin, where he was a master, but also of arms, where he was unapproachable.

It was this skill, combined with his haughty temper, which caused his being retired as Captain on June 16th, 1833, though it saved him from the fate of a kindred spirit, the lamented Galois, killed in a duel when only 19. Bolyai, when in garrison with cavalry officers, was provoked by thirteen of them and accepted all their challenges on condition that he be permitted after each duel to play a bit on his violin. He came out victor from his thirteen duels, leaving his thirteen adversaries on the square.

He projected a universal language for speech as we have it for music and for mathematics.

He left parts of a book entitled: *Principia doctrinae novae quantitatum imaginariarum perfectae uniceque satisfaciens, aliaque disquisitiones analyticae et analytico-geometricae cardinales gravissimaeque; auctore*

xxx TRANSLATOR'S INTRODUCTION.

Johan. Bolyai de eadem, C. R. austriaco castrensium captaneo pensionato.

Vindobonae vel Maros Vásárhelyini, 1853.

Bolyai Farkas was a student at Goettingen from 1796 to 1799.

In 1799 he returned to Kolozsvár, where Bolyai János was born December 18th, 1802.

He died January 27th, 1860, four years after his father.

In 1894 a monumental stone was erected on his long-neglected grave in Maros-Vásárhely by the Hungarian Mathematico-Physical Society.

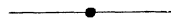




# APPENDIX.

SCIENTIAM SPATII *absolute veram* exhibens:

*a veritate aut falsitate Axiomatis XI Euclidei  
(a priori haud unquam decidenda) in-  
dependentem: adjecta ad casum fal-  
situdinis, quadratura circuli  
geometrica.*



Auctore JOHANNE BOLYAI de eadem, Geometrarum  
in Exercitu Caesareo Regio Austriaco  
Castrensi Capiteo.





## EXPLANATION OF SIGNS.

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*The straight  $AB$*  means the aggregate of all points situated in the same straight line with  $A$  and  $B$ .

*The sect  $AB$*  means that piece of the straight  $AB$  between the points  $A$  and  $B$ .

*The ray  $AB$*  means that half of the straight  $AB$  which commences at the point  $A$  and contains the point  $B$ .

*The plane  $ABC$*  means the aggregate of all points situated in the same plane as the three points (not in a straight)  $A, B, C$ .

*The hemi-plane  $ABC$*  means that half of the plane  $ABC$  which starts from the straight  $AB$  and contains the point  $C$ .

$ABC$  means the smaller of the pieces into which the plane  $ABC$  is parted by the rays  $BA, BC$ , or the non-reflex angle of which the sides are the rays  $BA, BC$ .

$ABCD$  (the point  $D$  being situated within  $\angle ABC$ , and the straights  $BA, CD$  not intersecting) means the portion of  $\angle ABC$  comprised between ray  $BA$ , sect  $BC$ , ray  $CD$ ; while  $BACD$  designates the portion of the plane  $ABC$  comprised between the straights  $AB$  and  $CD$ .

$\perp$  is the sign of perpendicularity.

$\parallel$  is the sign of parallelism.

$\angle$  means angle.

rt.  $\angle$  is right angle.

st.  $\angle$  is straight angle.

$\cong$  is the sign of congruence, indicating that two magnitudes are superposable.

$AB \simeq CD$  means  $\angle CAB = \angle ACD$ .

$x \simeq a$  means  $x$  converges toward the limit  $a$ .

$\triangle$  is triangle.

$\odot r$  means the [circumference of the] circle of radius  $r$ .

area  $\odot r$  means the area of the surface of the circle of radius  $r$ .





## THE SCIENCE ABSOLUTE OF SPACE.

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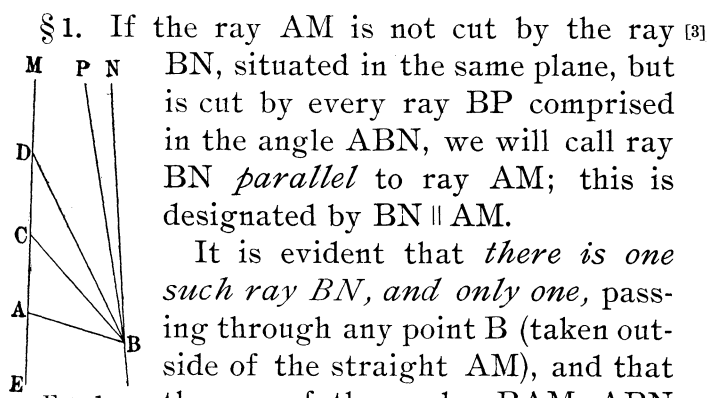


FIG. 1.

§ 1. If the ray AM is not cut by the ray BN, situated in the same plane, but is cut by every ray BP comprised in the angle ABN, we will call ray BN *parallel* to ray AM; this is designated by  $BN \parallel AM$ .

It is evident that *there is one such ray BN, and only one*, passing through any point B (taken outside of the straight AM), and that the sum of the angles BAM, ABN can not exceed a st.  $\angle$ ; for in moving BC around B until  $BAM + ABC = \text{st. } \angle$ , somewhere ray BC *first* does not cut ray AM, and it is then  $BC \parallel AM$ . It is clear that  $BN \parallel EM$ , wherever the point E be taken on the straight AM (supposing in all such cases  $AM > AE$ ).

If while the point C goes away to infinity on ray AM, always  $CD = CB$ , we will have constantly  $CDB = (CBD < NBC)$ ; but  $NBC = 0$ ; and so also  $ADB = 0$ .

§ 2. If  $BN \parallel AM$ , we will have also  $CN \parallel AM$ . For take  $D$  anywhere in  $MACN$ . If  $C$  is on ray  $BN$ , ray  $BD$  cuts ray  $AM$ , since  $BN \parallel AM$ , and so also ray  $CD$  cuts ray  $AM$ . But if  $C$  is on ray  $BR$ , take  $BQ \parallel CD$ ;  $BQ$  falls within the  $\angle ABN$  (§ 1), and cuts ray  $AM$ ; and so also ray  $CD$  cuts ray  $AM$ . Therefore every ray  $CD$  (in  $ACN$ ) cuts, in each case, the ray  $AM$ , without  $CN$  itself cutting ray  $AM$ . Therefore always  $CN \parallel AM$ .

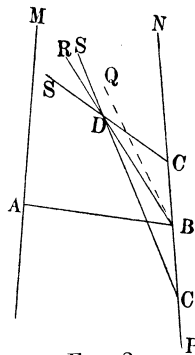


FIG. 2.

§ 3. (Fig. 2.) If  $BR$  and  $CS$  and each  $\parallel AM$ , and  $C$  is not on the ray  $BR$ , then ray  $BR$  and ray  $CS$  do not intersect. For if ray  $BR$  and ray  $CS$  had a common point  $D$ , then (§ 2)  $DR$  and  $DS$  would be each  $\parallel AM$ , and ray  $DS$  (§ 1) would fall on ray  $DR$ , and  $C$  on the ray  $BR$  (contrary to the hypothesis).

§ 4. If  $MAN > MAB$ , we will have for every point  $B$  of ray  $AB$ , a point  $C$  of ray  $AM$ , such that  $BCM = NAM$ . For (by § 1) is granted  $BDM > NAM$ , and so that  $MDP = MAN$ , and  $B$  falls in

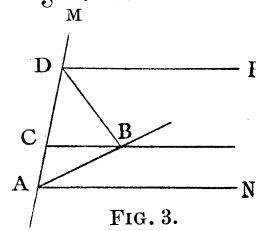


FIG. 3.

NADP. If therefore NAM is carried along AM until ray AN arrives on ray DP, ray AN will somewhere have necessarily passed through B, and some BCM=NAM.

§ 5. If  $BN \parallel AM$ , there is on the straight <sup>[4]</sup>

AM a point F such that  $FM \cong BN$ . For by § 1 is granted  $BCM > CBN$ ; and if  $CE = CB$ , and so  $EC \cong BC$ ; evidently  $BEM < EBN$ . The point P is moved on EC, the angle BPM always being called  $u$ , and the angle PBN always  $v$ ; evidently  $u$  is at first less than the corresponding  $v$ , but afterwards greater. Indeed  $u$  increases *continuously* from

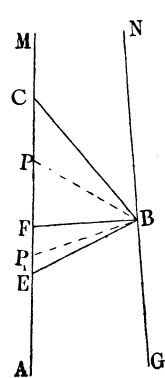


FIG. 4.

BEM to BCM; since (by § 4) there exists *no* angle  $> BEM$  and  $< BCM$ , to which  $u$  does not at some time become equal. Likewise  $v$  decreases continuously from EBN to CBN. There is therefore on EC a point F such that  $BFM = FBN$ .

§ 6. If  $BN \parallel AM$  and E anywhere in the straight AM, and G in the straight BN; then  $GN \parallel EM$  and  $EM \parallel GN$ . For (by § 1)  $BN \parallel EM$ , whence (by § 2)  $GN \parallel EM$ . If moreover  $FM \cong BN$  (§ 5); then  $MFBN \cong NBFM$ , and consequently (since  $BN \parallel FM$ ) also  $FM \parallel BN$ , and (by what precedes)  $EM \parallel GN$ .

§ 7. If  $BN$  and  $CP$  are each  $\parallel AM$ , and  $C$  not on the straight  $BN$ ; also  $BN \parallel CP$ . For the rays  $BN$  and  $CP$  do not intersect (§3); but  $AM$ ,  $BN$  and  $CP$  either are or are not in the same plane; and in the first case,  $AM$  either is or is not within  $BNCP$ .

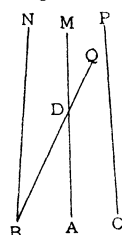


FIG. 5. If  $AM$ ,  $BN$ ,  $CP$  are complanar, and  $AM$  falls within  $BNCP$ ; then every ray  $BQ$  (in  $NBC$ ) cuts the ray  $AM$  in some point  $D$  (since  $BN \parallel AM$ ); moreover, since  $DM \parallel CP$  (§ 6), the ray  $DQ$  will cut the ray  $CP$ , and so  $BN \parallel CP$ .

But if  $BN$  and  $CP$  are on the same side of  $AM$ ; then one of them, for example  $CP$ , falls between the two other straight  $BN$ ,  $AM$ : but every ray  $BQ$  (in  $NBA$ ) cuts the ray  $AM$ , and so also the straight  $CP$ . Therefore  $BN \parallel CP$ .

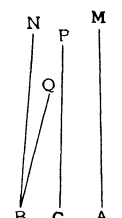


FIG. 6. If the planes  $MAB$ ,  $MAC$  make an angle; then  $CBN$  and  $ABN$  have in common nothing but the ray  $BN$ , while the ray  $AM$  (in  $ABN$ ) and the ray  $BN$ , and so also  $NBC$  and the ray  $AM$  have nothing in common.

But hemi-plane  $BCD$ , drawn through any ray  $BD$  (in  $NBA$ ), cuts the ray  $AM$ , since ray



though  $BN \parallel CP$ . But every ray  $BQ$  (in  $CBN$ ) cuts ray  $CP$ ; and so ray  $BQ$  cuts also ray  $AM$ . Consequently  $BN \parallel AN$ .

§ 9. If  $BN \parallel AM$ , and  $MAP \perp MAB$ , and the  $\angle$ , which  $NBD$  makes with  $NBA$  (on that side of  $MABN$ , where  $MAP$  is) is  $< \text{rt.} \angle$ ; then  $MAP$  and  $NBD$  intersect.

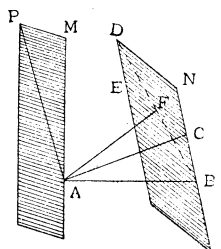


FIG. 9.

For let  $\angle BAM = \text{rt.} \angle$ , and  $AC \perp BN$  (whether or not  $C$  falls on  $B$ ), and  $CE \perp BN$  (in  $NBD$ ); by hypothesis  $\angle ACE < \text{rt.} \angle$ , and  $AF (\perp CE)$  will fall in  $ACE$ .

Let ray  $AP$  be the intersection of the hemi-planes  $ABF$ ,  $AMP$  (which have the point  $A$  common); since  $BAM \perp MAP$ ,  $\angle BAP = \angle BAM = \text{rt.} \angle$ .

If finally the hemi-plane  $ABF$  is placed upon the hemi-plane  $ABM$  ( $A$  and  $B$  remaining), ray  $AP$  will fall on ray  $AM$ ; and since  $AC \perp BN$ , and  $\text{sect } AF < \text{sect } AC$ , evidently  $\text{sect } AF$  will terminate within ray  $BN$ , and so  $BF$  falls in  $ABN$ . But in *this* position, ray  $BF$  cuts ray  $AP$  (because  $BN \parallel AM$ ); and so ray  $AP$  and ray  $BF$  intersect also in *the original* position; and the point of section is common to the hemi-planes  $MAP$  and  $NBD$ . Therefore the hemi-planes  $MAP$  and  $NBD$  intersect. Hence follows eas-

ily that the hemi-planes MAP and NBD intersect if the sum of the interior angles which they make with MABN is  $< \text{st.} \angle$ .

§ 10. If both BN and CP  $\parallel \simeq$  AM; also is BN  $\parallel \simeq$  CP.

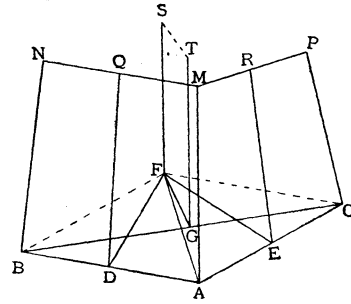


FIG. 10.

For either MAB and MAC make an angle, or they are in a plane.

If the first; let the hemi-plane QDF bisect  $\perp$  sect AB; then  $DQ \perp AB$ , and so  $DQ$

$\parallel AM$  (§ 8); likewise if hemi-plane ERS bisects  $\perp$  sect AC, is  $ER \parallel AM$ ; whence (§ 7)  $DQ \parallel ER$ .

Hence follows easily (by § 9), the hemi-planes QDF and ERS intersect, and have (§ 7) their intersection  $FS \parallel DQ$ , and (on account of  $BN \parallel DQ$ ) also  $FS \parallel BN$ . Moreover (for any point of FS)  $FB = FA = FC$ , and the straight FS falls in the plane TGF, bisecting  $\perp$  sect BC. But (by § 7) (since  $FS \parallel BN$ ) also  $GT \parallel BN$ . In the same way is proved  $GT \parallel CP$ . Meanwhile GT bisects  $\perp$  sect BC; and so  $TGBN \cong TGCP$  (§ 1), and  $BN \parallel \simeq CP$ .

If BN, AM and CP are in a plane, let (falling without this plane)  $FS \parallel \simeq AM$ ; then (from

what precedes)  $FS \parallel \simeq$  both to  $BN$  and to  $CP$ , and so also  $BN \parallel \simeq CP$ .

§ 11. Consider the aggregate of the point  $A$ , and *all* points of which any one  $B$  is such, that if  $BN \parallel AM$ , also  $BN \simeq AM$ ; call it  $F$ ; but the intersection of  $F$  with any plane containing the sect  $AM$  call  $L$ .

$F$  has a point, and one only, on any straight  $\parallel AM$ ; and evidently  $L$  is divided by ray  $AM$  into two congruent parts.

Call the ray  $AM$  *the axis* of  $L$ . Evidently also, in any plane containing the sect  $AM$ , there is for the *axis* ray  $AM$  a single  $L$ . Call any  $L$  of this sort the  $L$  of this ray  $AM$  (in the plane considered, being understood). Evidently by revolving  $L$  around  $AM$  we describe the  $F$  of which ray  $AM$  is called the axis, and in turn  $F$  *may be ascribed to the axis ray*  $AM$ .

§ 12. If  $B$  is anywhere on the  $L$  of ray  $AM$ , and  $BN \parallel \simeq AM$  (§ 11); then the  $L$  of ray  $AM$  and the  $L$  of ray  $BN$  *coincide*. For suppose, in distinction,  $L'$  the  $L$  of ray  $BN$ . Let  $C$  be anywhere in  $L'$ , and  $CP \parallel \simeq BN$  (§ 11). Since  $BN \parallel \simeq AM$ , so  $CP \parallel \simeq AM$  (§ 10), and so  $C$  also will fall on  $L$ . And if  $C$  is anywhere on  $L$ , and  $CP \parallel \simeq AM$ ; then  $CP \parallel \simeq BN$  (§ 10); and  $C$  also falls on  $L'$  (§ 11). Thus  $L$  and  $L'$  are the



same; and every ray BN is also axis of L, and between all axes of this L, is  $\simeq$ .

The same is evident in the same way of F.

§13. If  $BN \parallel AM$ , and  $CP \parallel DQ$ , and  $\angle BAM + \angle ABN = \text{st. } \angle$ ; then also  $\angle DCP + \angle CDQ = \text{st. } \angle$ .

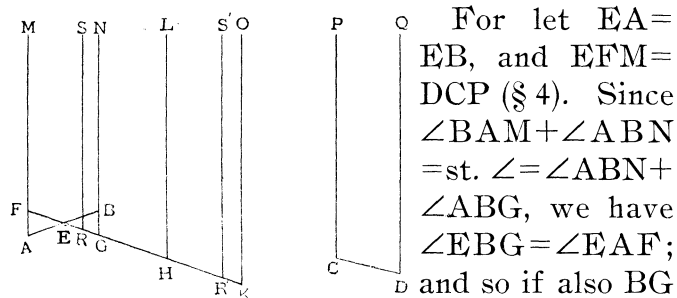


FIG. 11.

For let  $EA = EB$ , and  $EFM = DCP$  (§ 4). Since  $\angle BAM + \angle ABN = \text{st. } \angle = \angle ABN + \angle ABG$ , we have  $\angle EBG = \angle EAF$ ; and so if also  $BG = AF$ , then  $\triangle EBG$

$\cong \triangle EAF$ ,  $\angle BEG = \angle AEF$  and G will fall on the ray FE. Moreover  $\angle GFM + \angle FGN = \text{st. } \angle$  (since  $\angle EGB = \angle EFA$ ).

Also  $GN \parallel FM$  (§ 6).

Therefore if  $MFRS \cong PCDQ$ , then  $RS \parallel GN$  (§ 7), and R falls within or without the sect FG (unless sect  $CD = \text{sect } FG$ , where the thing now is evident).

I. In the first case  $\angle FRS$  is not  $> (\text{st. } \angle - \angle RFM = \angle FGN)$ , since  $RS \parallel FM$ . But as  $RS \parallel GN$ , also  $\angle FRS$  is not  $< \angle FGN$ ; and so  $\angle FRS = \angle FGN$ , and  $\angle RFM + \angle FRS = \angle GFM + \angle$

$\angle FGN = \text{st.}\angle$ . Therefore also  $\angle DCP + \angle CDQ = \text{st.}\angle$ .

II. If  $R$  falls without the sect  $FG$ ; then  $\angle NGR = \angle MFR$ , and let  $MFGN \cong NGHL \cong LHKO$ , and so on, until  $FK = FR$  or begins to be  $>FR$ . Then  $KO \parallel HL \parallel FM$  (§7).

If  $K$  falls on  $R$ , then  $KO$  falls on  $RS$  (§1); and so  $\angle RFM + \angle FRS = \angle KFM + \angle FKO = \angle KFM + \angle FGN = \text{st.}\angle$ ; but if  $R$  falls within the sect  $HK$ , then (by I)  $\angle RHL + \angle KRS = \text{st.}\angle = \angle RFM + \angle FRS = \angle DCP + \angle CDQ$ .

§14. If  $BN \parallel AM$ , and  $CP \parallel DQ$ , and  $\angle BAM + \angle ABN < \text{st.}\angle$ ; then also  $\angle DCP + \angle CDQ < \text{st.}\angle$ .

For if  $\angle DCP + \angle CDQ$  were not  $< \text{st.}\angle$ , and so (by §1) were  $= \text{st.}\angle$ , then (by §13) also  $\angle BAM + \angle ABN = \text{st.}\angle$  (contra hyp.).

§15. Weighing §§13 and 14, *the System of Geometry resting on the hypothesis of the truth of Euclid's Axiom XI is called  $\Sigma$ ; and the system founded on the contrary hypothesis is  $S$ .*<sup>[8]</sup>

*All things which are not expressly said to be in  $\Sigma$  or in  $S$ , it is understood are enunciated absolutely, that is are asserted true whether  $\Sigma$  or  $S$  is reality.*

§16. If AM is the axis of any L; then L, in  $\Sigma$  is a straight  $\perp$  AM.

For suppose BN an axis from any point B of L; in  $\Sigma$ ,  $\angle BAM + \angle ABN = \text{st.}\angle$ , and so  $\angle BAM = \text{rt.}\angle$ .

And if C is any point of the straight AB, and  $CP \parallel AM$ ; then (by § 13)  $CP \cong AM$ , and so C on L (§ 11).

But in S, no three points A, B, C on L or on F are in a straight. For some one of the axes AM, BN, CP (e. g. AM) falls between the two others; and then (by § 14)  $\angle BAM$  and  $\angle CAM$  are each  $< \text{rt.}\angle$ .

§ 17. *L in S also is a line, and F a surface.* For (by § 11) any plane  $\perp$  to the axis ray AM (through any point of F) cuts F in [the circumference of] a circle, of which the plane (by § 14) is  $\perp$  to no other axis ray BN. If we revolve F about BN, any point of F (by § 12) will remain on F, and the section of F with a plane not  $\perp$  ray BN will describe a surface; and whatever be the points A, B taken on it, F can so be congruent to itself that A falls upon B (by § 12); therefore F is a *uniform surface*.

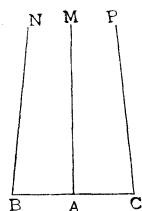


FIG. 12.



§ 19. The *perpendicular* BT to the axis BN of L (falling in the plane of L) is, in S, *tangent* to L. For L has in ray BT no point except B (§ 14), but if BQ falls in TBN, then the center of the section of the plane through BQ perpendicular to TBN with the F of ray BN (§ 18) is evidently located on ray BQ; and if sect BQ is a diameter, evidently ray BQ cuts in Q the line L of ray BN.

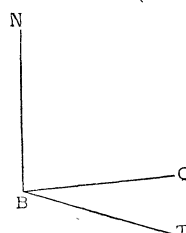


FIG. 14.

§ 20. Any two points of F determine a line L (§§ 11 and 18); and since (from §§ 16 and 19) L is  $\perp$  to all its axes, every  $\angle$  of lines L in F is equal to the  $\angle$  of the planes drawn through its sides perpendicular to F.

§ 21. Two L form lines, ray AP and ray BD, in the same F, making with a third L form AB, a sum of interior angles  $< \text{st.}\angle$ , intersect.

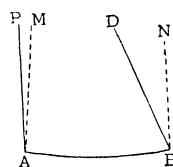


FIG. 15.

(By line AP in F, is to be understood the line L drawn through A and P, but by ray AP that half of this line beginning at A, in which P falls.)

For if AM, BN are axes of F, then the hemiplanes AMP, BND intersect (§ 9); and F cuts

their intersection (by §§ 7 and 11); and so also ray AP and ray BD intersect.

From this it is evident that Euclid's Axiom XI and all things which are claimed in geometry and plane trigonometry hold good *absolutely* in F, L lines being substituted in place of straights: therefore the trigonometric functions are taken here in the same sense as in  $\Sigma$ ; and the circle of which the L form radius =  $r$  in F, is =  $2\pi r$ ; and likewise area of  $\odot r$  (in F) =  $\pi r^2$  (by  $\pi$  understanding  $\frac{1}{2}\odot 1$  in F, or the known 3.1415926...)

§ 22. If ray AB were the L of ray AM, and C on ray AM; and the  $\angle CAB$  (formed by the straight ray AM and the L form line ray AB), carried first along <sup>[10]</sup> the ray AB, then along the ray BA, always forward to infinity: the path CD of C will be the line L of CM.

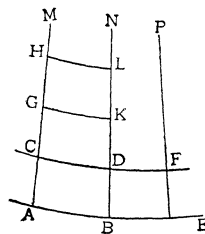


FIG. 16.

For let D be any point in line CD (called later L'), let DN be  $\parallel$  CM, and B the point of L falling on the straight DN. We shall have  $BN \simeq AM$ , and  $\text{sect } AC = \text{sect } BD$ , and so  $DN \simeq CM$ , consequently D in L'. But if D in L' and  $DN \parallel CM$ , and B the point of L on the straight DN; we shall have  $AM \simeq BN$  and  $CM \simeq DN$ , whence manifestly  $\text{sect } BD = \text{sect } AC$ ,

and D will fall on the path of the point C, and L' and the line CD are the same. Such an L' is designated by  $L' \parallel L$ .

§ 23. If the L form line  $CDF \parallel ABE$  (§ 22), and  $AB=BE$ , and the rays AM, BN, EP are axes; manifestly  $CD=DF$ ; and if any three points A, B, E are of line AB, and  $AB=n.CD$ , we shall also have  $AE=n.CF$ ; and so (manifestly even for AB, AE, DC incommensurable),  $AB:CD=AE:CF$ , and  $AB:CD$  is *independent of AB, and completely determined by AC*.

This ratio  $AB:CD$  is designated by the capital letter (as X) corresponding to the small letter (as x) by which we represent the sect AC.

§ 24. Whatever be x and y; (§ 23),  $Y=X^{\frac{y}{x}}$ .

For, one of the quantities x, y is a multiple of the the other (e. g. y of x), or it is not.

If  $y=n.x$ , take  $x=AC=CG=GH=\&c.$ , until we get  $AH=y$ .

Moreover, take  $CD \parallel GK \parallel HL$ .

We have ((§ 23)  $X=AB:CD=CD:GK=GK:HL$ ; and so

$$\frac{AB}{HL} = \left( \frac{AB}{CD} \right)^n$$

or  $Y=X^n=X^{\frac{y}{x}}$ .

If x, y are multiples of i, suppose  $x=mi$ , and  $y=ni$ ; (by the preceding)  $X=I^m$ ,  $Y=I^n$ , consequently

$$Y=X^{\frac{n}{m}}=X^{\frac{y}{x}}$$

The same is easily extended to the case of the incommensurability of  $x$  and  $y$ .

But if  $q=y-x$ , manifestly  $Q=Y:X$ .

It is also manifest that in  $\Sigma$ , for any  $x$ , we have  $X=1$ , but in  $S$  is  $X>1$ , and for any  $AB$  and  $ABE$  there is such a  $CDF \parallel AB$ , that  $CDF=AB$ , whence  $AMBN \cong AMEP$ , though the first be any multiple of the second; which indeed is singular, but evidently does not prove the absurdity of  $S$ .

§ 25. *In any rectilinear triangle, the circles with radii equal to its sides are as the sines of the opposite angles.*

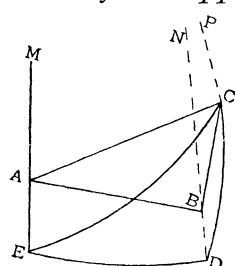


FIG. 17.

For take  $\angle ABC = \text{rt.}\angle$ , and  $AM \perp BC$ , and  $BN$  and  $CP \parallel AM$ ; we shall have  $CAB \perp AMBN$ , and so (since  $CB \perp BA$ ),  $CB \perp AMBN$ , consequently  $CPBN \perp AMBN$ .

Suppose the  $F$  of ray  $CP$  cuts the straight  $BN$ ,  $AM$  respectively in  $D$  and  $E$ , and the bands  $CPBN$ ,  $CPAM$ ,  $BNAM$  along the  $L$  form lines  $CD$ ,  $CE$ ,  $DE$ . Then (§ 20)  $\angle CDE =$  the angle of  $NDC$ ,  $NDE$ , and so  $= \text{rt.}\angle$ ; and by like reasoning  $\angle CED = \angle CAB$ . But (by § 21) in the  $L$  line  $\triangle CDE$  (supposing always here the radius = 1),

$$EC:DC = 1:\sin DEC = 1:\sin CAB.$$



Also (by § 21)  
 $\odot EC : \odot DC = \odot EC : \odot DC$  (in F) =  $\odot AC : \odot BC$  (§ 18);  
 and so is also

$$\odot AC : \odot BC = 1 : \sin CAB;$$

whence the theorem is evident for any triangle.

§ 26. *In any spherical triangle, the sines of the sides are as the sines of the angles opposite.*

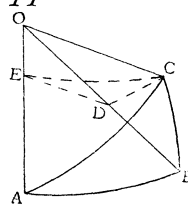


FIG. 18.

For take  $\angle ABC = \text{rt.}\angle$ , and  $CED \perp$  to the radius  $OA$  of the sphere. We shall have  $CED \perp AOB$ , and (since also  $BOC \perp BOA$ ),  $CD \perp OB$ . But in the triangles  $CEO$ ,  $CDO$  (by § 25)  $\odot EC : \odot OC : \odot DC = \sin COE : 1 : \sin COD = \sin AC : 1 : \sin BC$ ; meanwhile also (§ 25)  $\odot EC : \odot DC = \sin CDE : \sin CED$ . Therefore,  $\sin AC : \sin BC = \sin CDE : \sin CED$ ; but  $CDE = \text{rt.}\angle = CBA$ , and  $CED = CAB$ . Consequently

$$\sin AC : \sin BC = 1 : \sin A.$$

*Spherical trigonometry, flowing from this, is thus established independently of Axiom XI.*

§ 27. If  $AC$  and  $BD$  are  $\perp AB$ , and  $CAB$  is carried along the straight  $AB$ ; we shall have, designating by  $CD$  the path of the point  $C$ ,

$$CD : AB = \sin u : \sin v.$$

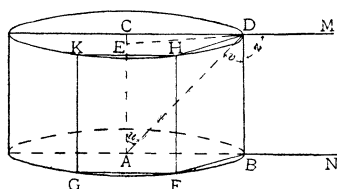


FIG. 19.

For take  $DE \perp CA$ ;  
in the triangles  $ADE$ ,  
 $ADB$  (by § 25)

$$\odot ED : \odot AD : \odot AB = \sin u : 1 : \sin v.$$

In revolving  $BACD$  about  $AC$ ,  $B$  describes  $\odot AB$ , and  $D$  describes  $\odot ED$ ; and designate here by  $s\odot CD$  the path of the said  $CD$ . Moreover, let there be any <sup>[12]</sup> polygon  $BFG\dots$  inscribed in  $\odot AB$ .

Passing through all the sides  $BF$ ,  $FG$ , &c., planes  $\perp$  to  $\odot AB$  we form also a polygonal figure of the same number of sides in  $s\odot CD$ , and we may demonstrate, as in § 23, that  $CD : AB = DH : BF = HK : FG$ , &c., and so

$$DH + HK \text{ \&c. } : BF + FG \text{ \&c. } = CD : AB.$$

If each of the sides  $BF$ ,  $FG\dots$  approaches the limit zero, manifestly

$$BF + FG + \dots \doteq \odot AB \quad \text{and} \\ DH + HK + \dots \doteq \odot ED.$$

Therefore also  $\odot ED : \odot AB = CD : AB$ . But we had  $\odot ED : \odot AB = \sin u : \sin v$ . Consequently

$$CD : AB = \sin u : \sin v.$$

If  $AC$  goes away from  $BD$  to infinity,  $CD : AB$ , and so also  $\sin u : \sin v$  remains *constant*; but  $u \doteq \text{rt. } \angle$  (§ 1), and if  $DM \parallel BN$ ,  $v \doteq z$ ; whence  $CD : AB = 1 : \sin z$ .

The path called CD will be denoted by  $CD \parallel AB$ .

§ 28. If  $BN \parallel \simeq AM$ , and C in ray AM, and  $AC = x$ : we shall have (§ 23)

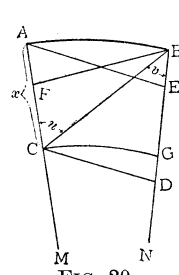


FIG. 20.

$X = \sin u : \sin v$ .  
For if CD and AE are  $\perp$  BN, and  $BF \perp AM$ ; we shall have (as in § 27)

$\odot BF : \odot DC = \sin u : \sin v$ .  
But evidently  $BF = AE$ : therefore  
 $\odot EA : \odot CD = \sin u : \sin v$ .

But in the F form surfaces of AM and CM (cutting AMBN in AB and CG) (by § 21)

$\odot EA : \odot DC = AB : CG = X$ .

Therefore also

$X = \sin u : \sin v$ .

§ 29. If  $\angle BAM = \text{rt. } \angle$ , and sect  $AB = y$ , and  $BN \parallel AM$ , we shall have in S

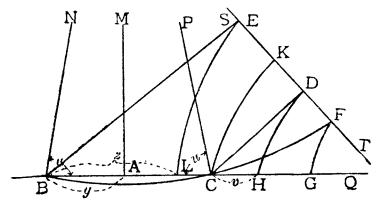


FIG. 21.

$Y = \cotan \frac{1}{2} u$ .  
For, if sect  $AB =$  sect  $AC$ , and  $CP \parallel AM$  (and so  $BN \parallel \simeq CP$ ), and  $\angle PCD =$

$\angle QCD$ ; there is given (§ 19)  $DS \perp$  ray  $CD$ , so that  $DS \parallel CP$ , and so (§ 1)  $DT \parallel CQ$ . Moreover, if  $BE \perp$  ray  $DS$ , then (§ 7)  $DS \parallel BN$ , and so (§ 6)



But (by § 27)  $\sin v : \sin v' = \cos u : \cos u'$  ;  
 consequently  $\frac{\sin u}{\cos u} \odot y = \frac{\sin u'}{\cos u'} \odot y'$  ;

or  $\odot y : \odot y' = \tan u' : \tan u = \tan w : \tan w'$  .

Moreover, take CN and C'N'  $\parallel$  AB, and CD, C'D' L-form lines  $\perp$  straight AB; we shall have also (§21)

$$\odot y : \odot y' = r : r' , \text{ and so}$$

$$r : r' = \tan w : \tan w' .$$

Now let p beginning from A increase to infinity; then  $w \doteq z$ , and  $w' \doteq z'$ , whence also  $r : r' = \tan z : \tan z'$  .

Designate by *i* the constant

$$r : \tan z \text{ (independent of } r \text{)} ;$$

whilst  $y \doteq 0$ ,

$$\frac{r}{y} = \frac{i \tan z}{y} \doteq 1, \text{ and so}$$

$$\frac{y}{\tan z} \doteq i. \text{ From § 29, } \tan z = \frac{1}{2} (Y - Y^{-1});$$

therefore 
$$\frac{2y}{Y - Y^{-1}} \doteq i,$$

or (§ 24) 
$$\frac{2y \cdot I^{\frac{y}{i}}}{I^{\frac{2y}{i}} - 1} \doteq i.$$

But we know the limit of this expression (where  $y \doteq 0$ ) is

$$\frac{i}{\text{nat. log } I} . \text{ Therefore}$$

$$\frac{i}{\text{nat. log } \bar{I}} = i, \text{ and}$$

$$I = e = 2.7182818 \dots,$$

which noted quantity shines forth here also.

If obviously henceforth  $i$  denote that sect of which the  $I=e$ , we shall have

$$r = i \tan z.$$

But (§ 21)  $\odot y = 2\pi r$ ; therefore

$$\odot y = 2\pi i \tan z = \pi i (Y - Y^{-1}) = \pi i \left[ e^{\frac{y}{i}} - e^{-\frac{y}{i}} \right]$$

$$= \frac{\pi y}{\text{nat. log } Y} (Y - Y^{-1}) \text{ (by § 24).}$$

§ 31. For the trigonometric solution of all right-angled rectilinear *triangles* (whence the resolution of all *triangles* is easy, in S, three <sup>[14]</sup> equations suffice: indeed ( $a, b$  denoting the sides,  $c$  the hypotenuse, and  $\alpha, \beta$  the angles opposite the sides) an equation expressing the relation

1st, between  $a, c, \alpha$ ;

2d, between  $a, b, \beta$ ;

3d, between  $a, b, c$ ;

of course from these equations emerge three others by elimination.

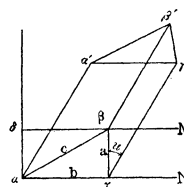


FIG. 23.

From §§ 25 and 30

$$1 : \sin \alpha = (C - C^{-1}) : (A - A^{-1}) =$$

$$= \left[ e^{\frac{c}{i}} - e^{-\frac{c}{i}} \right] : \left[ e^{\frac{a}{i}} - e^{-\frac{a}{i}} \right] \text{ (equation for } c, a \text{ and } \alpha).$$

II. From § 27 follows (if  $\beta M \parallel \gamma N$ )

$\cos a : \sin \beta = 1 : \sin u$ ; but from § 29

$$1 : \sin u = \frac{1}{2}(A + A^{-1});$$

therefore  $\cos a : \sin \beta = \frac{1}{2}(A + A^{-1}) = \frac{1}{2} \left( e^{\frac{a}{i}} + e^{-\frac{a}{i}} \right)$   
(equation for  $a$ ,  $\beta$  and  $a$ ).

III. If  $aa' \perp \beta\alpha\gamma$ , and  $\beta\beta'$  and  $\gamma\gamma' \parallel aa'$  (§ 27),  
and  $\beta'a'\gamma' \perp aa'$ ; manifestly (as in § 27)

$$\frac{\beta\beta'}{\gamma\gamma'} = \frac{1}{\sin u} = \frac{1}{2}(A + A^{-1});$$

$$\frac{\gamma\gamma'}{aa'} = \frac{1}{2}(B + B^{-1});$$

and  $\frac{\beta\beta'}{aa'} = \frac{1}{2}(C + C^{-1})$ ; consequently

$$\frac{1}{2}(C + C^{-1}) = \frac{1}{2}(A + A^{-1}) \cdot \frac{1}{2}(B + B^{-1}), \text{ or}$$

$$\left( e^{\frac{c}{i}} + e^{-\frac{c}{i}} \right) = \frac{1}{2} \left( e^{\frac{a}{i}} + e^{-\frac{a}{i}} \right) \left( e^{\frac{b}{i}} + e^{-\frac{b}{i}} \right)$$

(equation for  $a$ ,  $b$  and  $c$ ).

If  $\gamma a \delta = \text{rt. } \angle$ , and  $\beta \delta \perp a \delta$ ;

$$\odot c : \odot a = 1 : \sin a, \text{ and}$$

$$\odot c : \odot (d = \beta \delta) = 1 : \cos a,$$

and so (denoting by  $\odot x^2$ , for any  $x$ , the product  $\odot x \cdot \odot x$ ) manifestly

$$\odot a^2 + \odot d^2 = \odot c^2.$$

But (by § 27 and II)

$$\odot d = \odot b \cdot \frac{1}{2}(A + A^{-1}), \text{ consequently}$$

$$\left( e^{\frac{c}{i}} - e^{-\frac{c}{i}} \right)^2 = \frac{1}{4} \left( e^{\frac{a}{i}} + e^{-\frac{a}{i}} \right)^2 \left( e^{\frac{b}{i}} - e^{-\frac{b}{i}} \right)^2 + \left( e^{\frac{a}{i}} - e^{-\frac{a}{i}} \right)^2$$

another equation for  $a$ ,  $b$  and  $c$  (the second

member of which may be easily reduced to a form *symmetric* or *invariable*. [15]

Finally, from

$\frac{\cos a}{\sin \beta} = \frac{1}{2}(A + A^{-1})$ , and  $\frac{\cos \beta}{\sin a} = \frac{1}{2}(B + B^{-1})$ , we get (by III)

$$\cot a \cot \beta = \frac{1}{2} \left( e^{\frac{c}{i}} + e^{-\frac{c}{i}} \right)$$

(equation for  $a$ ,  $\beta$ , and  $c$ .)

§ 32. It still remains to show briefly the mode of resolving *problems* in S, which being accomplished (through the more obvious examples), finally will be candidly said what this theory shows.

I. Take AB a line in a plane, and  $y=f(x)$

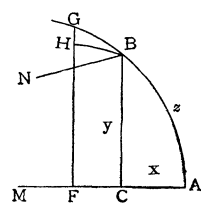


FIG. 24.

its equation in rectangular coordinates, call  $dz$  any increment of  $z$ , and respectively  $dx$ ,  $dy$ ,  $du$  the increments of  $x$ , of  $y$ , and of the area  $u$ , corresponding to this  $dz$ ; take  $BH \parallel CF$ , and express (from § 31)  $\frac{BH}{dx}$  by means of  $y$ , and seek

the *limit* of  $\frac{dy}{dx}$  when  $dx$  tends towards the limit zero (which is understood where a limit of this sort is sought): then will become known also the limit of  $\frac{dy}{BH}$ , and so  $\tan HBG$ ; and



(since HBC manifestly is neither  $>$  nor  $<$ , and so =rt.  $\angle$ ), the *tangent* at B of BG will be determined by  $y$ .

II. It can be demonstrated

$$\frac{dz^2}{dy^2 + \overline{BH}^2} = 1.$$

Hence is found the *limit* of  $\frac{dz}{dx}$ , and thence, by integration,  $z$  (expressed in terms of  $x$ ).

And of any line *given in the concrete*, the equation in S can be found; e. g., of L. For if ray AM be the axis of L; then any ray CB from ray AM cuts L [since (by § 19) any straight from A except the straight AM will cut L]; but (if BN is axis)

$$X = 1 : \sin \text{CBN} \text{ (§ 28),}$$

and  $Y = \cotan \frac{1}{2} \text{CBN}$  (§ 29), whence

$$Y = X + \sqrt{X^2 - 1}.$$

or 
$$e^{\frac{y}{i}} = e^{\frac{x}{i}} + \sqrt{e^{\frac{2x}{i}} - 1},$$
 [16]

the equation sought.

Hence we get

$$\frac{dy}{dx} = X(X^2 - 1)^{-\frac{1}{2}};$$

and  $\frac{BH}{dx} = 1 : \sin \text{CBN} = X$ ; and so

$$\frac{dy}{BH} = (X^2 - 1)^{-\frac{1}{2}};$$

$$1 + \frac{dy^2}{BH^2} = X^2(X^2-1)^{-1},$$

$$\frac{dz^2}{BH^2} = X^2(X^2-1)^{-1},$$

and  $\frac{dz}{BH} = X(X^2-1)^{-\frac{1}{2}}$ , and

$$\frac{dz}{dx} = X^2(X^2-1)^{-\frac{1}{2}}, \text{ whence, by inte-}$$

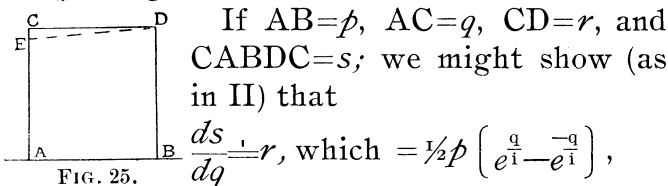
gration, we get (as in § 30)

$$z = i(X^2-1)^{\frac{1}{2}} = i \cot \text{ CBN.}$$

III. Manifestly

$$\frac{du}{dx} = \frac{HF\text{CBH}}{dx},$$

which (unless given in  $y$ ) now first is to be expressed in terms of  $y$ ; whence we get  $u$  by integrating.



If  $AB=p$ ,  $AC=q$ ,  $CD=r$ , and  $CABDC=s$ ; we might show (as in II) that

$$\frac{ds}{dq} = r, \text{ which} = \frac{1}{2}p \left( e^{\frac{q}{i}} - e^{-\frac{q}{i}} \right),$$

and, integrating,  $s = \frac{1}{2}pi \left( e^{\frac{q}{i}} - e^{-\frac{q}{i}} \right)$

This can also be deduced apart from integration.

For example, the equation of the circle (from § 31, III), of the straight (from § 31, II), of a conic (by what precedes), being expressed, the

areas bounded by these lines could also be expressed.

We know, that a surface  $t$ ,  $\parallel$  to a plane figure  $p$  (at the distance  $q$ ), is to  $p$  in the ratio of the second powers of homologous lines, or as

$$\frac{1}{4} \left( e^{\frac{q}{i}} - e^{-\frac{q}{i}} \right)^2 : 1.$$

It is easy to see, moreover, that the calculation of volume, treated in the same manner, requires two integrations (since the differential itself here is determined only by integration); and before all must be investigated the <sup>[17]</sup> volume contained between  $p$  and  $t$ , and the aggregate of all the straights  $\perp p$  and joining the boundaries of  $p$  and  $t$ .

We find for the volume of this solid (whether by integration or without it)

$$\frac{1}{8} \pi i \left( e^{\frac{2q}{i}} - e^{-\frac{2q}{i}} \right) + \frac{1}{2} p q.$$

The surfaces of bodies may also be determined in  $S$ , as well as the *curvatures*, the *involutives*, and *evolutes* of any lines, etc.

As to curvature; this in  $S$  either is the curvature of  $L$ , or is determined either by the radius of a circle, or by the *distance* to a straight from the curve  $\parallel$  to this straight; since from what precedes, it may easily be shown, that in a plane there are no uniform lines other than  $L$ -lines, circles and curves  $\parallel$  to a straight.

IV. For the circle (as in III)  $\frac{d \text{ area } \odot x}{dx}$   
 $\odot x$ , whence (by § 29), integrating,

$$\text{area } \odot x = \pi i^2 \left( e^{\frac{x}{i}} - 2 + e^{-\frac{x}{i}} \right).$$

V. For the area CABDC =  $u$  (inclosed by an  
 L form line AB =  $r$ , the  $\parallel$  to this,  
 CD =  $y$ , and the sects AC = BD =  $x$ )

$\frac{du}{dx} = y$ ; and (§ 24)  $y = r e^{-\frac{x}{i}}$ , and so,  
 (integrating)  $u = r i \left( 1 - e^{-\frac{x}{i}} \right)$ .

If  $x$  increases to infinity, then, in

FIG. 26.  $S, e^{-\frac{x}{i}} = 0$ , and so  $u = r i$ . By the *size*  
 of MABN, in future this limit is understood.

In like manner is found, if  $p$  is a figure on  
 F, the space included by  $p$  and the aggregate  
 of axes drawn from the boundaries of  $p$  is  
 equal to  $\frac{1}{2} p i$ .

FIG. 27. VI. If the angle at the cen-  
 ter of a segment  $\mathcal{Z}$  of a sphere  
 is  $2u$ , and a great circle is  $p$ ,  
 and  $x$  the arc FC (of the angle  
 $u$ ); (§ 25)

$$1 : \sin u = p : \odot BC,$$

$$\text{and hence } \odot BC = p \sin u. \quad [18]$$

$$\text{Meanwhile } x = \frac{p u}{2\pi}, \text{ and } dx = \frac{p du}{2\pi}.$$

Moreover,  $\frac{dz}{dx} = \odot BC$ , and hence

$$\frac{dz}{du} = \frac{\phi^2}{2\pi} \sin u, \text{ whence (integrating)}$$

$$z = \frac{\text{ver sin } u}{2\pi} \phi^2.$$

The  $F$  may be conceived on which  $P$  falls (passing through the middle  $F$  of the segment); through  $AF$  and  $AC$  the planes  $FEM$ ,  $CEM$  are placed, perpendicular to  $F$  and cutting  $F$  along  $FEG$  and  $CE$ ; and consider the  $L$  form  $CD$  (from  $C \perp$  to  $FEG$ ), and the  $L$  form  $CF$ ; (§ 20)  $CEF = u$ , and (§ 21)

$$\frac{FD}{\phi} = \frac{\text{ver sin } u}{2\pi}, \text{ and so } z = FD \cdot \phi.$$

But (§ 21)  $\phi = \pi \cdot FGD$ ; therefore

$$z = \pi \cdot FD \cdot FDG. \text{ But (§ 21)}$$

$$FD \cdot FDG = FC \cdot FC; \text{ consequently}$$

$$z = \pi \cdot FC \cdot FC = \text{area } \odot FC, \text{ in } F.$$

Now let  $BJ = CJ = r$ ; (§ 30)

$$2r = i(Y - Y^{-1}), \text{ and so (§ 21)}$$

$$\text{area } \odot 2r \text{ (in } F) = \pi i^2 (Y - Y^{-1})^2.$$

Also ( $IV$ )

$$\text{area } \odot 2y = \pi i^2 (Y^2 - 2 + Y^{-2});$$

therefore,  $\text{area } \odot 2r \text{ (in } F) = \text{area } \odot 2y$ , and so *the surface  $z$  of a segment of a sphere is equal to the surface of the circle described with the chord  $FC$  as a radius.*

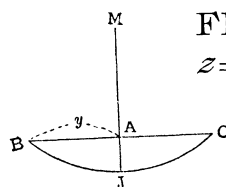


FIG. 28.

Hence the whole surface of the sphere

$$= \text{area } \odot FG = FDG \cdot p = \frac{p^2}{\pi},$$

and the surfaces of spheres are to each other as the second powers of their great circles.

VII. In like manner, in S, the volume of the sphere of radius  $x$  is found

$$= \frac{1}{2} \pi i^3 (X^2 - X^{-2}) - 2 \pi i^2 x;$$

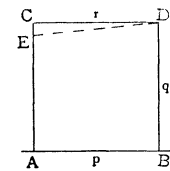


FIG. 29.

the surface generated by the revolution of the line CD about AB

$$= \frac{1}{2} \pi i p (Q^2 - Q^{-2}),$$

and the body described by CABDC

$$= \frac{1}{4} \pi i^2 p (Q - Q^{-1})^2.$$

But in what manner all things treated from (IV) even to here, also may be reached apart from integration, for the sake of brevity is suppressed.

It can be demonstrated that the limit of every expression containing the letter  $i$  (and so resting upon the hypothesis that  $i$  is given), [19] when  $i$  increases to infinity, expresses the quantity simply for  $\Sigma$  (and so for the hypothesis of no  $i$ ), if indeed the equations do not become identical.

But beware lest you understand to be supposed, that the system itself may be varied (for it is entirely determined in itself and by itself); but only the hypothesis, which may be

done successively, as long as we are not conducted to an absurdity. *Supposing* therefore that, in *such* an expression, the letter  $i$ , in case  $S$  is reality, designates that unique quantity whose  $I=e$ ; but if  $\Sigma$  is actual, the said limit is supposed to be taken in place of the expression: manifestly *all the expressions originating from the hypothesis of the reality of  $S$  (in this sense) will be true absolutely, although it be completely unknown whether or not  $\Sigma$  is reality*

So e. g. from the expression obtained in § 30 easily (and as well by aid of differentiation as apart from it) emerges the known value in  $\Sigma$ ,

$$\odot x = 2\pi x;$$

from I (§ 31) suitably treated, follows

$$1 : \sin a = c : a;$$

but from II

$$\frac{\cos a}{\sin \beta} = 1, \text{ and so}$$

$$a + \beta = \text{rt. } \angle;$$

the first equation in III becomes identical, and so is true in  $\Sigma$ , although it there determines nothing; but from the second follows

$$c^2 = a^2 + b^2.$$

*These are the known fundamental equations of plane trigonometry in  $\Sigma$ .*

Moreover, we find (from § 32) in  $\Sigma$ , the area and the volume in III each  $=pq$ ; from IV  
 area  $\odot x = \pi x^2$ ;  
 (from VII) the globe of radius  $x$   
 $= \frac{4}{3}\pi x^3$ , etc.

The theorems enunciated at the end of VI are manifestly *true unconditionally*.

§ 33. It still remains to set forth (as promised in § 32) what this theory means.

I. Whether  $\Sigma$  or some one S is reality, remains undecided.

II. All things deduced from the hypothesis of the falsity of Axiom XI (always to be understood in the sense of § 32) are *absolutely true*, and so in this sense, *depend upon no hypothesis*.

There is therefore a *plane trigonometry a priori, in which the system alone really re-*<sup>[20]</sup>  
*main unknown*; and so where remain unknown solely the *absolute* magnitudes in the expressions, but where a *single* known case would manifestly fix the whole system. But spherical trigonometry is established absolutely in § 26.

(And we have, on F, a geometry wholly analogous to the plane geometry of  $\Sigma$ .)

III. If it were agreed that  $\Sigma$  exists, nothing more would be unknown in this respect; but



if it were *established* that  $\Sigma$  *does not exist*, then (§ 31), (e. g.) from the sides  $x, y$ , and the rectilinear angle they include being given in a special case, manifestly it would be impossible in itself and by itself to solve absolutely the triangle, that is, to determine *a priori* the other angles and *the ratio of the third side* to the two given; unless  $X, Y$  were determined, for which it would be necessary to have in concrete form a certain sect  $a$  whose  $A$  was known; and then  $i$  would be *the natural unit for length* (just as  $e$  is the base of *natural logarithms*).

If the existence of this  $i$  is determined, it will be evident how it could be constructed, at least very exactly, for practical use.

IV. In the sense explained (I and II), it is evident that all things in space can be solved by the modern analytic method (within just limits strongly to be praised).

V. Finally, to friendly readers will not be unacceptable; that for that case wherein not  $\Sigma$  but  $S$  is reality, a rectilinear figure is constructed equivalent to a circle.

§ 34. Through  $D$  we may draw  $DM \parallel AN$  in the following manner. From  $D$  drop  $DB \perp AN$ ; from any point  $A$  of the straight  $AB$  erect  $AC \perp AN$  (in  $DBA$ ), and let fall  $DC \perp AC$ . We

will have (§ 27)  $\odot CD : \odot AB = 1 : \sin z$ , provided that  $DM \parallel BN$ . But  $\sin z$  is not  $> 1$ ; and so  $AB$  is not  $> DC$ . Therefore a quadrant described from the center  $A$  in  $BAC$ , with a radius  $= DC$ , will have a point  $B$  or  $O$  in common with ray  $BD$ . In the first case, manifestly  $z = \text{rt.} \angle$ ; but in the second case (§ 25)

$$(\odot AO = \odot CD) : \odot AB = 1 : \sin AOB,$$

and so  $z = AOB$ .

If therefore we take  $z = AOB$ , then  $DM$  will be  $\parallel BN$ .

§ 35. If  $S$  were reality; we may, as follows, draw a straight  $\perp$  to one arm of an acute angle, [21] which is  $\parallel$  to the other.

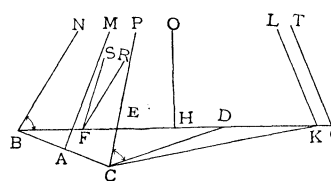


FIG. 31.

Take  $AM \perp BC$ , and suppose  $AB = BC$  so small (by § 19), that if we draw  $BN \parallel AM$  (§ 34),  $ABN >$  the given angle.

Moreover draw  $CP \parallel AM$  (§ 34); and take  $NBG$  and  $PCD$  each equal to the given angle; rays  $BG$  and  $CD$  will cut; for if ray  $BG$  (falling *by construction* within  $NBC$ ) cuts ray  $CP$  in  $E$ ; we shall have (since  $BN \cong CP$ ),  $\angle EBC < \angle ECB$ , and so  $EC < EB$ . Take  $EF = EC$ ,  $EFR$

=ECD, and FS  $\parallel$  EP; then FS will fall within BFR. For since BN  $\parallel$  CP, and so BN  $\parallel$  EP, and BN  $\parallel$  FS; we shall have (§ 14)

$$\angle FBN + \angle BFS < (\text{st. } \angle = FBN + BFR);$$

therefore, BFS < BFR. Consequently, ray FR cuts ray EP, and so ray CD also cuts ray EG in some point D. Take now DG=DC and DGT=DCP=GBN; we shall have (since CD  $\cong$  GD) BN  $\cong$  GT  $\cong$  CP. Let K (§ 19) be the point of the L-form line of BN falling in the ray BG, and KL the axis; we shall have BN  $\cong$  KL, and so BKL=BGT=DCP; but also KL  $\cong$  CP: therefore manifestly K fall on G, and GT  $\parallel$  BN. But if HO bisects  $\perp$  BG, we shall have constructed HO  $\parallel$  BN.

§ 36. Having given the ray CP and the plane MAB, take CB  $\perp$  the plane MAB, BN (in plane BCP)  $\perp$  BC, and CQ  $\parallel$  BN (§ 34); the intersection of ray CP (if this ray falls within BCQ) with ray BN (in the plane CBN), and so with the plane MAB is found. And if we are given the two planes PCQ, MAB, and we have CB  $\perp$  to plane MAB, CR  $\perp$  plane PCQ; and (in plane BCR) BN  $\perp$  BC, CS  $\perp$  CR, BN will fall in plane MAB, and CS in plane PCQ; and the

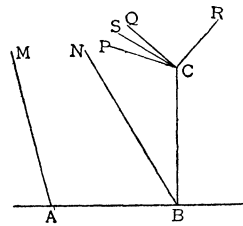


FIG. 32.

intersection of the straight BN with the straight CS (if there is one) having been found, the perpendicular drawn through this intersection, in PCQ, to the straight CS will manifestly be the intersection of plane MAB and plane PCQ.

§ 37. On the straight  $AM \parallel BN$ , is found such an A, that  $AM \cong BN$ . If (by <sup>[22]</sup> § 34) we construct outside

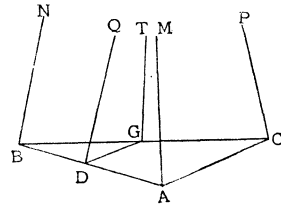


FIG. 33.

of the plane NBM,  $GT \parallel BN$ , and make  $BG \perp GT$ ,  $GC = GB$ , and  $CP \parallel GT$ ; and so place the hemi-plane TGD that it makes

with hemi-plane TGB an angle equal to that which hemi-plane PCA makes with hemi-plane PCB; and is sought (by § 36) the intersection straight DQ of hemi-plane TGD with hemi-plane NBD; and BA is made  $\perp DQ$ .

We shall have indeed, on account of the similitude of the triangles of L lines produced on the F of BN (§ 21), manifestly  $DB = DA$ , and  $AM \cong BN$ .

Hence easily appears (L-lines being given by their extremities alone) we may also find a fourth proportional, or a mean proportional, and execute in this way in F, apart from Axiom XI, all the geometric constructions made

on the plane in  $\Sigma$ . Thus e. g. a perigon can be geometrically divided into any special number of equal parts, if it is permitted to make this special partition in  $\Sigma$ .

§ 38. If we construct (by § 37) for example,  $NBQ = \frac{1}{3}$  rt.  $\angle$ , and make (by § 35), in  $S$ ,  $AM \perp$  ray  $BQ$  and  $\parallel$   $BN$ , and determine (by § 37)  $IM \cong BN$ ; we shall have, if  $IA = x$ , (§ 28),  $X = 1 : \sin \frac{1}{3}$  rt.  $\angle = 2$ , and  $x$  will be constructed *geometrically*.

And  $NBQ$  may be so computed, that  $IA$  differs from  $i$  less than by anything given, which happens for  $\sin NBQ = 1/e$ .

§ 39. If (in a plane)  $PQ$  and  $ST$  are  $\parallel$  to the straight  $MN$  (§ 27), and  $AB, CD$  are equal perpendiculars to  $MN$ ; manifestly  $\triangle DEC \cong$

$\triangle BEA$ ; and so the angles (perhaps mixtilinear)  $ECP, EAT$  will fit, and  $EC = EA$ . If, moreover,  $CF = AG$ , then  $\triangle ACF \cong \triangle CAG$ , and each is half of the *quadrilateral*

$FAGC$ .

If  $FAGC, HAGK$  are two quadrilaterals of this sort on  $AG$ , between  $PQ$  and  $ST$ ; their equivalence (as in  $\bar{E}$ uclid) is evident, as also

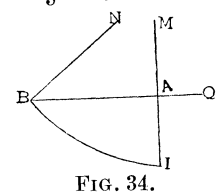


FIG. 34.

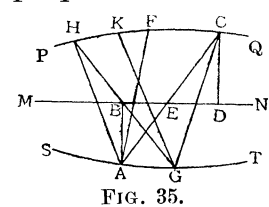


FIG. 35.

the equivalence of the triangles AGC, AGH, standing on the same AG, and having their vertices on the line PQ. Moreover,  $\angle ACF = \angle CAG$ ,  $\angle GCQ = \angle CGA$ , and  $\angle ACF + \angle ACG + \angle GCQ = \text{st. } \angle$  (§ 32); and so also  $\angle CAG + \angle ACG + \angle CGA = \text{st. } \angle$ ; therefore, in any triangle ACG of this sort, the sum of the three angles = st.  $\angle$ . But whether the straight AG may have fallen upon AG (which  $\parallel$  MN), or not; *the equivalence of the rectilinear triangles AGC, AGH, as well of themselves, as of the sums of their angles, is evident.*

§ 40. *Equivalent triangles ABC, ABD, (henceforth rectilinear), having one side equal, have the sums of their angles equal.*

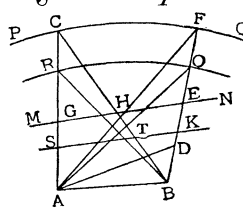


FIG. 36.

For let MN bisect AC and BC, and take (through C)  $PQ \parallel MN$ ; the point D will

fall on line PQ.

For, if ray BD cuts the straight MN in the point E, and so (§ 39) the line PQ at the distance  $EF = EB$ ; we shall have  $\triangle ABC = \triangle ABF$ , and so also  $\triangle ABD = \triangle ABF$ , whence D falls at F.

But if ray BD has not cut the straight MN, let C be the point, where the perpendicular bisecting the straight AB cuts the line PQ, and

let  $GS=HT$ , so, that the line  $ST$  meets the ray  $BD$  prolonged in a certain  $K$  (which it is evident can be made in a way like as in § 4); moreover take  $SR=SA$ ,  $RO \parallel ST$ , and  $O$  the intersection of ray  $BK$  with  $RO$ ; then  $\triangle ABR = \triangle ABO$  (§ 39), and so  $\triangle ABC > \triangle ABD$  (contra hyp.).

§ 41. *Equivalent triangles  $ABC, DEF$  have the sums of their triangles equal.*

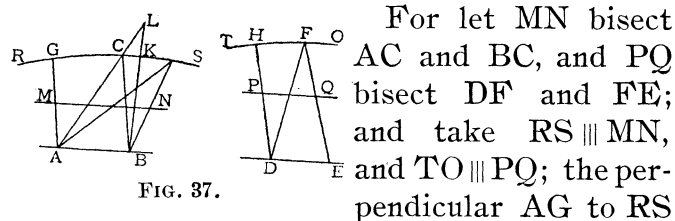


FIG. 37.

For let  $MN$  bisect  $AC$  and  $BC$ , and  $PQ$  bisect  $DF$  and  $FE$ ; and take  $RS \parallel MN$ , and  $TO \parallel PQ$ ; the perpendicular  $AG$  to  $RS$  will equal the perpendicular  $DH$  to  $TO$ , or one for example  $DH$  will be the greater.

In each case, the  $\odot DF$ , from center  $A$ , has with line-ray  $GS$  some point  $K$  in common, and (§ 39)  $\triangle ABK = \triangle ABC = \triangle DEF$ . But the  $\triangle AKB$  (by § 40) has the same angle-sum as  $\triangle DFE$ , and (by § 39) as  $\triangle ABC$ . Therefore also the triangles  $ABC, DEF$  have each the same angle-sum.

In  $S$  the inverse of this theorem is true.

For take  $ABC, DEF$  two triangles having equal angle-sums, and  $\triangle BAL = \triangle DEF$ ; these will have (by what precedes) equal angle-sums,

and so also will  $\triangle ABC$  and  $\triangle ABL$ , and hence manifestly

$$BCL + BLC + CBL = \text{st. } \angle.$$

However (by § 31), the angle-sum of any tri-<sup>[24]</sup>angle, in S, is  $< \text{st. } \angle$ .

Therefore L falls on C.

§ 42. Let  $u$  be the supplement of the angle-sum of the  $\triangle ABC$ , but  $v$  of  $\triangle DEF$ ; then is  $\triangle ABC : \triangle DEF = u : v$ .

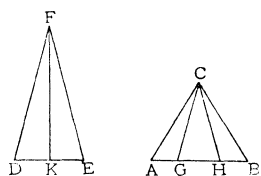


FIG. 38.

For if  $p$  be the area of each of the triangles  $ACG$ ,  $GCH$ ,  $HCB$ ,  $DFK$ ,  $KFE$ ; and  $\triangle ABC = m \cdot p$ , and  $\triangle DEF = n \cdot p$ ; and  $s$  the angle-sum of any triangle equivalent to  $p$ ;

manifestly

$$\text{st. } \angle - u = m \cdot s - (m - 1) \text{st. } \angle = \text{st. } \angle - m(\text{st. } \angle - s);$$

and  $u = m(\text{st. } \angle - s)$ ; and in like manner  $v = n(\text{st. } \angle - s)$ .

Therefore  $\triangle ABC : \triangle DEF = m : n = u : v$ .

It is evidently also easily extended to the case of the incommensurability of the triangles  $ABC$ ,  $DEF$ .

In the same way is demonstrated that triangles on a sphere are as the *excesses* of the sums of their angles above a  $\text{st. } \angle$ .

If two angles of the spherical  $\triangle$  are right, the third  $\angle$  will be the said *excess*. But



(a great circle being called  $\rho$ ) this  $\Delta$  is manifestly

$$= \frac{\mathcal{Z}}{2\pi} \frac{\rho^2}{2\pi} \text{ (§ 32, VI);}$$

consequently, any triangle of whose angles the excess is  $\mathcal{Z}$ , is

$$= \frac{\mathcal{Z}\rho^2}{4\pi^2}.$$

§ 43. Now, in S, the area of a rectilinear  $\Delta$  is expressed by means of the sum of its angles.

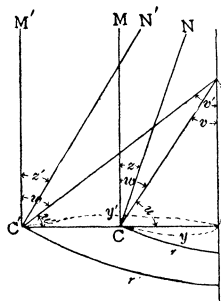


FIG. 39.

If AB increases to infinity; (§ 42)  $\Delta ABC : (\text{rt. } \angle - u - v)$  will be constant. But  $\Delta ABC \doteq BACN$  (§ 32, V), and  $\text{rt. } \angle - u - v = \mathcal{Z}$  (§ 1); and so  $BACN : \mathcal{Z} = \Delta ABC : (\text{rt. } \angle - u - v) = BAC'N' : \mathcal{Z}'$ .

Moreover, manifestly (§ 30)  $BDCN : BD'C'N' = r : r' =$

$\tan \mathcal{Z} : \tan \mathcal{Z}'$ .

But for  $y' = 0$ , we have

$$\frac{BD'C'N'}{BAC'N'} = 1, \text{ and also } \frac{\tan \mathcal{Z}'}{\mathcal{Z}'} = 1;$$

consequently,

$$BDCN : BACN = \tan \mathcal{Z} : \mathcal{Z}.$$

But (§ 32)

$$BDCN = r \cdot i = i^2 \tan \mathcal{Z};$$

therefore,

$$BACN = \mathcal{Z} \cdot i^2.$$

Designating henceforth, for brevity, any triangle the supplement of whose angle-sum is  $z$  by  $\Delta$ , we will therefore have  $\Delta = z \cdot i^2$ .

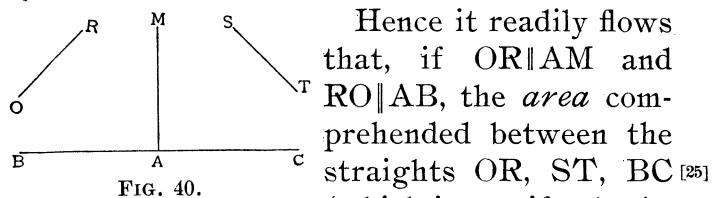


FIG. 40.

Hence it readily flows that, if  $OR \parallel AM$  and  $RO \parallel AB$ , the *area* comprehended between the straights  $OR, ST, BC$  (which is manifestly the

absolute limit of the area of rectilinear triangles increasing without bound, or of  $\Delta$  for  $z = \text{st. } \angle$ ), is  $= \pi i^2 = \text{area } \odot i$ , in  $F$ .

This limit being denoted by  $\square$ , moreover (by § 30)  $\pi r^2 = \tan^2 z \cdot \square = \text{area } \odot r$  in  $F$  (§ 21) =  $\text{area } \odot s$  (by § 32, VI) if the chord  $CD$  is called  $s$ .

If now, bisecting at right angles the given radius  $s$  of the circle in a plane (or the  $L$  form radius of the circle in  $F$ ), we construct (by § 34)  $DB \parallel \sphericalangle CN$ ; by dropping  $CA \perp DB$ , and erecting  $CM \perp CA$ , we shall

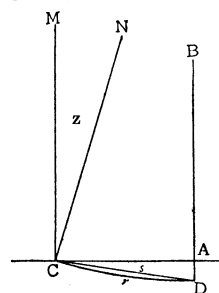


FIG. 41.

get  $z$ ; whence (by § 37), assuming at pleasure an  $L$  form radius for unity,  $\tan^2 z$  can be determined geometrically by means of two uniform lines of the same curvature (which, their extremities alone being given and their axes con-

structed, manifestly may be compared like straights, and in this respect considered equivalent to straights).

Moreover, a quadrilateral, ex. gr. regular =  $\square$  is constructed as follows:

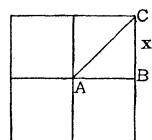


FIG. 42.

Take  $ABC = \text{rt.} \angle$ ,  $BAC = \frac{1}{2} \text{ rt.} \angle$ ,  $ACB = \frac{1}{4} \text{ rt.} \angle$ , and  $BC = x$ .

By mere square roots, X (from § 31, II) can be expressed and (by § 37) constructed; and having X (by § 38 or also §§ 29 and 35),  $x$  itself can be determined. And octuple  $\triangle ABC$  is manifestly =  $\square$ , and by this a *plane circle of radius  $s$  is geometrically squared by means of a rectilinear figure and uniform lines of the same species* (equivalent to straights as to comparison *inter se*); but an *F form circle is planified in the same manner: and we have either the Axiom XI of Euclid true or the geometric quadrature of the circle*, although thus far it has remained undecided, which of these two has place in reality.

Whenever  $\tan^2 z$  is either a whole number, or a rational fraction, whose denominator (reduced to the simplest form) is either a prime number of the form  $2^m + 1$  (of which is also  $2 = 2^0 + 1$ ), or a product of however many prime numbers of this form, of which each (with the

exception of 2, which alone may occur any number of times) occurs *only once* as factor, we can, by the theory of polygons of the illustrious Gauss (remarkable invention of our, nay of every age) (and only for such values <sup>[26]</sup> of  $z$ ), construct a rectilineal figure  $=\tan^2 z \square = \text{area } \odot s$ . For the division of  $\square$  (the theorem of § 42 extending easily to any polygons) manifestly requires the partition of a st.  $\angle$ , which (as can be shown) can be achieved geometrically only under the said condition.

But in all such cases, what precedes conducts easily to the desired end. And any rectilineal figure can be converted geometrically into a regular polygon of  $n$  sides, if  $n$  falls under the Gaussian form.

It remains, finally (that the thing may be completed in every respect), to demonstrate the impossibility (apart from any supposition), of deciding *a priori*, whether  $\Sigma$ , or some  $S$  (and which one) exists. This, however, is reserved for a more suitable occasion.

## APPENDIX I.

---

REMARKS ON THE PRECEDING TREATISE,  
BY BOLYAI FARKAS.

[From Vol. II of Tentamen, pp. 380-383.]

Finally it may be permitted to add something appertaining to the author of the *Appendix* in the first volume, who, however, may pardon me if something I have not touched with his acuteness.

The thing consists briefly in this: *the formulas of spherical trigonometry* (demonstrated in the said *Appendix* independently of Euclid's Axiom XI) *coincide with the formulas of plane trigonometry, if* (in a way provisionally speaking) *the sides of a spherical triangle are accepted as reals, but of a rectilineal triangle as imaginaries*; so that, as to trigonometric formulas, the plane may be considered as an imaginary sphere, if for real, that is accepted in which  $\sin \text{rt. } \angle = 1$ .

Doubtless, of the Euclidean axiom has been said in volume first enough and to spare: for

the case if it were not true, is demonstrated (Tom. I. App., p. 13), that there is given a certain  $i$ , for which the  $I$  there mentioned is  $=e$  (the base of natural logarithms), and for this case are established also (*ibidem*, p. 14) the formulas of plane trigonometry, and indeed so, that (by the side of p. 19, *ibidem*) the formulas are still valid for the case of the verity of the said axiom; indeed if the limits of the values are taken, supposing that  $i \doteq \infty$ ; truly the Euclidean system is as if the limit of the anti-Euclidean (for  $i \doteq \infty$ ).

Assume for the case of  $i$  existing, the unit  $=i$ , and extend the concepts sine and cosine also to imaginary arcs, so that,  $p$  designating an arc whether real or imaginary,

$$\frac{e^{p\sqrt{-1}} + e^{-p\sqrt{-1}}}{2} \text{ is called the}$$

*cosine* of  $p$ , and

$$\frac{e^{p\sqrt{-1}} - e^{-p\sqrt{-1}}}{2\sqrt{-1}} \text{ is called}$$

the *sine* of  $p$  (as Tom. I., p. 177).

Hence for  $q$  real

$$\begin{aligned} \frac{e^q - e^{-q}}{2\sqrt{-1}} &= \frac{e^{q\sqrt{-1}\sqrt{-1}} - e^{-q\sqrt{-1}\sqrt{-1}}}{2\sqrt{-1}} = \sin(-q\sqrt{-1}) \\ &= -\sin(q\sqrt{-1}). \end{aligned}$$

$$\text{So } \frac{e^q + e^{-q}}{2} = \frac{e^{-q\sqrt{-1}\sqrt{-1}} + e^{q\sqrt{-1}\sqrt{-1}}}{2} = \cos(-q\sqrt{-1})$$

$$= \cos(q\sqrt{-1});$$

if of course also in the imaginary circle, the sine of a negative arc is the same as the sine of a positive arc otherwise equal to the first, except that it is negative, and the cosine of a positive arc and of a negative (if otherwise they be equal) the same.

In the said *Appendix*, § 25, is demonstrated absolutely, that is, independently of the said axiom; that, in any rectilinear triangle *the sines of the circles are as the circles of radii equal to the sides opposite.*

Moreover is demonstrated for the case of *i* existing, that the circle of radius *y* is

$$= \pi i \left( \frac{y}{i} - e^{-\frac{y}{i}} \right), \text{ which, for } i=1, \text{ becomes}$$

$$\pi(e^y - e^{-y}).$$

Therefore (§ 31 *ibidem*), for a right-angled rectilinear triangle of which the sides are *a* and *b*, the hypotenuse *c*, and the angles opposite to the sides *a*, *b*, *c* are  $\alpha$ ,  $\beta$ ,  $\text{rt. } \angle$ , (for  $i=1$ ), in I,

$$1 : \sin \alpha = \pi(e^c - e^{-c}) : \pi(e^a - e^{-a});$$

and so

$$1 : \sin \alpha = \frac{e^c - e^{-c}}{2\sqrt{-1}} : \frac{e^a - e^{-a}}{2\sqrt{-1}}. \quad \text{Whence } 1 : \sin \alpha$$

$= -\sin (c\sqrt{-1}) : -\sin (a\sqrt{-1})$ . And hence

$$1 : \sin a = \sin (c\sqrt{-1}) : \sin (a\sqrt{-1}).$$

In II becomes

$$\cos a : \sin \beta = \cos (a\sqrt{-1}) : 1;$$

in III becomes

$$\cos (c\sqrt{-1}) = \cos (a\sqrt{-1}) \cdot \cos (b\sqrt{-1}).$$

These, as all the formulas of plane trigonometry deducible from them, coincide completely with the formulas of spherical trigonometry; except that if, ex. gr., also the sides and the angles opposite them of a right-angled spherical triangle and the hypotenuse bear the same names, the sides of the rectilinear triangle are to be divided by  $\sqrt{-1}$  to obtain the formulas for the spherical triangle.

Obviously we get (clearly as Tom., II., p. 252),

from I,  $1 : \sin a = \sin c : \sin a;$

from II,  $1 : \cos a = \sin \beta : \cos a;$

from III,  $\cos c = \cos a \cos b.$

Though it be allowable to pass over other things; yet I have learned that the reader may be offended and impeded by the deduction omitted, (Tom. I., App., p. 19) [in § 32 at end]: it will not be irrelevant to show how, ex. gr., from

$$e^{\frac{c}{i}} + e^{-\frac{c}{i}} = \frac{1}{2} \left( e^{\frac{a}{i}} + e^{-\frac{a}{i}} \right) \left( e^{\frac{b}{i}} + e^{-\frac{b}{i}} \right)$$

follows



$$c^2 = a^2 + b^2.$$

(the theorem of Pythagoras for the Euclidean system); probably thus also the author deduced it, and the others also follow in the same manner.

Obviously we have, the powers of  $e$  being expressed by series (like Tom. I., p. 168),

$$e^{\frac{k}{i}} = 1 + \frac{k}{i} + \frac{k^2}{2i^2} + \frac{k^3}{2.3.i^3} + \frac{k^4}{2.3.4.i^4} \dots \dots \dots ,$$

$$e^{-\frac{k}{i}} = 1 - \frac{k}{i} + \frac{k^2}{2i^2} - \frac{k^3}{2.3.i^3} + \frac{k^4}{2.3.4.i^4} \dots \dots \dots , \text{ and so}$$

$$e^{\frac{k}{i}} + e^{-\frac{k}{i}} = 2 + \frac{k^2}{i^2} + \frac{k^4}{3.4.i^4} + \frac{k^6}{3.4.5.6.i^6} \dots \dots \dots ,$$

$$= 2 + \frac{k^2 + u}{i^2}, \text{ (designating by}$$

$\frac{u}{i^2}$  the sum of all the terms after  $\frac{k^2}{i^2}$ ); and we have  $u = 0$ , while  $i = \infty$ . For all the terms which follow  $\frac{k^2}{i^2}$  are divided by  $i^2$ ; the first term will be  $\frac{k^4}{3.4.i^2}$ ; and any ratio  $< \frac{k^2}{i^2}$ ; and though the ratio everywhere should remain this, the sum would be (Tom. I., p. 131),

$$\frac{k^4}{3.4.i^2} : \left[ 1 - \frac{k^2}{i^2} \right] = \frac{k^4}{3.4.(i^2 - k^2)},$$

which manifestly  $= 0$ , while  $i = \infty$ .

And from

$$e^{\frac{c}{i}} + e^{-\frac{c}{i}} = \frac{1}{2} \left( e^{\frac{(a+b)}{i}} + e^{-\frac{(a+b)}{i}} + e^{\frac{a-b}{i}} + e^{-\frac{(a-b)}{i}} \right)$$

follows (for  $w, v, \lambda$  taken like  $u$ )

$$2 + \frac{c^2 + w}{i^2} = \frac{1}{2} \left[ 2 + \frac{(a+b)^2 + v}{i^2} + 2 + \frac{(a-b)^2 + \lambda}{i^2} \right].$$

And hence

$$c^2 = \frac{a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + v + \lambda - w}{2},$$

which  $= a^2 + b^2$ .

## APPENDIX II.

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SOME POINTS IN JOHN BOLYAI'S APPENDIX  
 COMPARED WITH LOBACHEVSKI,  
 BY WOLFGANG BOLYAI.

[From *Kurzer Grundriss*, p. 82.]

Lobachevski and the author of the *Appendix* each consider two points A, B, of the sphere-limit, and the corresponding axes ray AM, ray BN (§ 23).

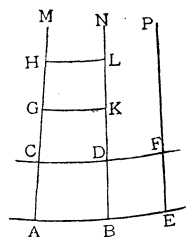


FIG. 43.

They demonstrate that, if  $a, \beta, \gamma$  designate the arcs of the circle limit AB, CD, HL, separated by segments of the axis  $AC=1, AH=x$ , we have

$$\frac{a}{\gamma} = \left( \frac{a}{\beta} \right)^x.$$

Lobachevski represents the value of  $\frac{\gamma}{a}$  by  $e^{-x}$ ,  $e$  having some value  $>1$ , dependent on the unit for length that we have chosen, and able to be supposed equal to the Napierian base.

The author of the *Appendix* is led directly to introduce the base of natural logarithms.

If we put  $\frac{a}{\beta} = \delta$ , and  $\gamma, \gamma'$  are arcs situated at the distances  $y, i$  from  $a$ , we shall have

$$\frac{a}{\gamma} = \delta^y = Y, \quad \frac{a}{\gamma'} = \delta^i = I, \quad \text{whence } Y = I^{\frac{y}{i}}.$$

He demonstrates afterward (§ 29) that, if  $u$  is the angle which a straight makes with the perpendicular  $y$  to its parallel, we have

$$Y = \cot \frac{1}{2}u.$$

Therefore, if we put  $z = \frac{\pi}{2} - u$ , we have

$$Y = \tan (z + \frac{1}{2}u) = \frac{\tan z + \tan \frac{1}{2}u}{1 - \tan z \tan \frac{1}{2}u},$$

whence we get, having regard to the value of  $\tan \frac{1}{2}u = Y^{-1}$ ,

$$\tan z = \frac{1}{2} (Y - Y^{-1}) = \frac{1}{2} \left( I^{\frac{y}{i}} - I^{-\frac{y}{i}} \right) \quad (\S 30).$$

If now  $y$  is the semi-chord of the arc of circle-limit  $2r$ , we prove (§ 30) that  $\frac{r}{\tan z} =$  constant.

Representing this constant by  $i$ , and making  $y$  tend toward zero, we have

$$\frac{2r}{2y} = 1, \quad \text{whence}$$

$$2y = 2i \tan z = i \frac{I^{\frac{2y}{i}} - 1}{I^{\frac{y}{i}}},$$

or putting  $\frac{2y}{i} = k$ ,  $I = el$ ,

$$kI^{\frac{y}{i}} = e^{kl} - 1 = kl(1 + \lambda),$$

$\lambda$  being infinitesimal at the same time as  $k$ . Therefore, for the limit,  $1 = l$  and consequently  $I = e$ .

The circle traced on the sphere-limit with the arc  $r$  of the curve-limit for radius, has for length  $2\pi r$ . Therefore,

$$\odot y = 2\pi r = 2\pi i \tan z = \pi i (Y - Y^{-1}).$$

In the rectilinear  $\Delta$  where  $a, \beta$  designate the angles opposite the sides  $a, b$ , we have (§ 25)

$$\begin{aligned} \sin a : \sin \beta &= \odot a : \odot b = \pi i (A - A^{-1}) : \pi i (B - B^{-1}) \\ &= \sin (a\sqrt{-1}) : \sin (b\sqrt{-1}). \end{aligned}$$

Thus in plane trigonometry as in spherical trigonometry, the sines of the angles are to each other as the sines of the opposite sides, only that on the sphere the sides are reals, and in the plane we must consider them as imaginaries, just as if the plane were an imaginary sphere.

We may arrive at this proposition without a preceding determination of the value of  $I$ .

If we designate the constant  $\frac{r}{\tan z}$  by  $q$ , we shall have, as before

$$\odot y = \pi q (Y - Y^{-1}),$$

whence we deduce the same proportion as above, taking for  $i$  the distance for which the ratio  $I$  is equal to  $e$ .

If axiom  $XI$  is not true, there exists a determinate  $i$ , which must be substituted in the formulas.

If, on the contrary, this axiom is true, we must make in the formulas  $i = \infty$ . Because, in this case, the quantity  $\frac{a}{r} = Y$  is always  $= 1$ , the sphere-limit being a plane, and the axes being parallel in Euclid's sense.

The exponent  $\frac{v}{i}$  must therefore be zero, and consequently  $i = \infty$ .

It is easy to see that Bolyai's formulas of plane trigonometry are in accord with those of Lobachevski.

Take for example the formula of § 37,

$$\tan \parallel (a) = \sin B \tan \parallel (\phi),$$

$a$  being the hypotenuse of a right-angled triangle,  $\phi$  one side of the right angle, and  $B$  the angle opposite to this side.

Bolyai's formula of § 31, I, gives

$$1 : \sin B = (A - A^{-1}) : (P - P^{-1}).$$

Now, putting for brevity,  $\frac{1}{2}\parallel(k) = k'$ , we have  $\tan 2\phi' : \tan 2a' = (\cot a' - \tan a') : (\cot \phi' - \tan \phi') = (A - A^{-1}) : (P - P^{-1}) = 1 : \sin B$ .

### APPENDIX III.

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LIGHT FROM NON-EUCLIDEAN SPACES ON THE  
TEACHING OF ELEMENTARY GEOMETRY.

BY G. B. HALSFED.

As foreshadowed by Bolyai and Riemann, founded by Cayley, extended and interpreted for hyperbolic, parabolic, elliptic spaces by Klein, recast and applied to mechanics by Sir Robert Ball, *projective metrics* may be looked upon as characteristic of what is highest and most peculiarly modern in all the bewildering range of mathematical achievement.

Mathematicians hold that number is wholly a creation of the human intellect, while on the contrary our space has an empirical element. Of possible geometries we can not say *a priori* which shall be that of our actual space, the space in which we move. Of course an advance so important, not only for mathematics but for philosophy, has had some metaphysical opponents, and as long ago as 1878 I mentioned in my Bibliography of Hyper-

Space and Non-Euclidean Geometry (American Journal of Mathematics, Vol. I, 1878, Vol. II, 1879) one of these, Schmitz-Dumont, as a sad paradoxer, and another, J. C. Becker, both of whom would ere this have shared the oblivion of still more antiquated fighters against the light, but that Dr. Schotten, praiseworthy for the very attempt at a comparative planimetry, happens to be himself a believer in the *a priori* founding of geometry, while his American reviewer, Mr. Ziwet, was then also an anti-non-Euclidean, though since converted.

He says, "we find that some of the best German text books do not try at all to define what is space, or what is a point, or even what is a straight line." Do any German geometries define space? I never remember to have met one that does.

In experience, what comes first is a bounded surface, with its boundaries, lines, and their boundaries, points. Are the points whose definitions are omitted anything different or better?

Dr. Schotten regards the two ideas "direction" and "distance" as intuitively given in the mind and as so simple as to not require definition.

When we read of two jockeys speeding



around a track in opposite directions, and also on page 87 of Richardson's *Euclid*, 1891, read, "The sides of the figure must be produced in the same direction of rotation; . . . going round the figure always in the same direction," we do not wonder that when Mr. Ziwet had written: "he therefore bases the definition of the straight line on these two ideas," he stops, modifies, and rubs that out as follows, "or rather recommends to elucidate the intuitive idea of the straight line possessed by any well-balanced mind by means of the still simpler ideas of direction" [in a circle] "and distance" [on a curve].

But when we come to geometry as a science, as foundation for work like that of Cayley and Ball, I think with Professor Chrystal: "It is essential to be careful with our definition of a *straight line*, for it will be found that virtually the properties of the straight line determine the nature of space.

"Our definition shall be that two points *in general* determine a straight line."

We presume that Mr. Ziwet glories in that unfortunate expression "a straight line is the shortest distance between two points," still occurring in Wentworth (*New Plane Geometry*, page 33), even after he has said, page 5,

“the length of the straight line is called the *distance* between two points.” If the *length* of the one straight line between two points is the distance between those points, how can the straight line itself be the *shortest* distance? If there is only one distance, it is the longest as much as the shortest distance, and if it is the *length* of this shorto-longest distance which is the distance then it is not the straight line itself which is the longo-shortest distance. But Wentworth also says: “Of all lines joining two points the *shortest* is the straight line.”

This general comparison involves the measurement of curves, which involves the theory of limits, to say nothing of ratio. The very ascription of length to a curve involves the idea of a limit. And then to introduce this general axiom, as does Wentworth, only to prove a very special case of itself, that two sides of a triangle are together greater than the third, is surely bad logic, bad pedagogy, bad mathematics.

This latter theorem, according to the first of Pascal's rules for demonstrations, should not be proved at all, since every dog knows it. But to this objection, as old as the sophists, Simson long ago answered for the science of

geometry, that the number of assumptions ought not to be increased without necessity; or as Dedekind has it: “*Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden.*”

Professor W. B. Smith (Ph. D., Goettingen), has written: “Nothing could be more unfortunate than the attempt to lay the notion of Direction at the bottom of Geometry.”

Was it not this notion which led so good a mathematician as John Casey to give as a demonstration of a triangle's angle-sum the procedure called “a *practical* demonstration” on page 87 of Richardson's Euclid, and there described as “laying a ‘straight edge’ along one of the sides of the figure, and then turning it round so as to coincide with each side in turn.”

This assumes that a segment of a straight line, a sect, may be translated without rotation, which assumption readily comes to view when you try the procedure in two-dimensional spherics. Though this fallacy was exposed by so eminent a geometer as Olaus Henrici in so public a place as the pages of ‘Nature,’ yet it has just been solemnly reproduced by Professor G. C. Edwards, of the University of California, in his Elements of Geometry: Mac-

Millan, 1895. It is of the greatest importance for every teacher to know and connect the commonest forms of assumption equivalent to Euclid's Axiom XI. If in a plane two straight lines perpendicular to a third nowhere meet, are there others, not both perpendicular to any third, which nowhere meet? Euclid's Axiom XI is the assumption *No*. Playfair's answers *no* more simply. But the very same answer is given by the common assumption of our geometries, usually unnoticed, that a circle may be passed through any three points not costraight.

This equivalence was pointed out by Bolyai Farkas, who looks upon this as the simplest form of the assumption. Other equivalents are, the existence of any finite triangle whose angle-sum is a straight angle; or the existence of a plane rectangle; or that, in triangles, the angle-sum is constant.

One of Legendre's forms was that through every point within an angle a straight line may be drawn which cuts both arms.

But Legendre never saw through this matter because he had not, as we have, the eyes of Bolyai and Lobachevski to see with. The same lack of their eyes has caused the author of the charming book "Euclid and His Modern

Rivals," to give us one more equivalent form: "In any circle, the inscribed equilateral tetragon is greater than any one of the segments which lie outside it." (A New Theory of Parallels by C. L. Dodgson, 3d. Ed., 1890.)

Any attempt to define a straight line by means of "direction" is simply a case of "argumentum in circulo." In all such attempts the loose word "direction" is used in a sense which presupposes the straight line. The directions from a point in Euclidean space are only the  $\infty^3$  rays from that point.

Rays not costraight can be said to have the same direction only after a theory of parallels is presupposed, assumed.

Three of the exposures of Professor G. C. Edwards' fallacy are here reproduced. The first, already referred to, is from *Nature*, Vol. XXIX, p. 453, March 13, 1884.

"I select for discussion the 'quaternion proof' given by Sir William Hamilton. . . . Hamilton's proof consists in the following:

"One side AB of the triangle ABC is turned about the point B till it lies in the continuation of BC; next, the line BC is made to slide along BC till B comes to C, and is then turned about C till it comes to lie in the continuation of AC.

“It is now again made to slide along CA till the point B comes to A, and is turned about A till it lies in the line AB. Hence it follows, *since rotation is independent of translation*, that the line has performed a whole revolution, that is, it has been turned through four right angles. But it has also described in succession the three exterior angles of the triangle, hence these are together equal to four right angles, and from this follows at once that the interior angles are equal to two right angles.

“To show how erroneous this reasoning is—in spite of Sir William Hamilton and in spite of quaternions—I need only point out that it holds exactly in the same manner for a triangle on the surface of the sphere, from which it would follow that the sum of the angles in a spherical triangle equals two right angles, whilst this sum is known to be always greater than two right angles. The proof depends only on the fact, that any line can be made to coincide with any other line, that two lines do so coincide when they have two points in common, and further, that a line may be turned about any point in it without leaving the surface. But if instead of the plane we take a spherical surface, and instead of a line a great

circle on the sphere, all these conditions are again satisfied.

“The reasoning employed must therefore be fallacious, and the error lies in the words printed in italics; for these words contain an assumption which has not been proved.

“O. HENRICI.”

Perronet Thompson, of Queen's College, Cambridge, in a book of which the third edition is dated 1830, says:

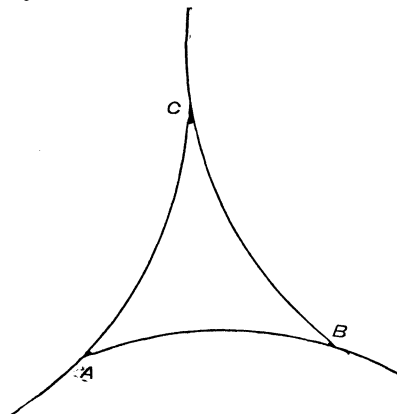
“Professor Playfair, in the Notes to his ‘Elements of Geometry’ [1813], has proposed another demonstration, founded on a remarkable *non causa pro causa*.

“It purports to collect the fact [Eu. I., 32, Cor., 2] that (on the sides being successively prolonged to the same hand) the exterior angles of a rectilinear triangle are together equal to four right angles, from the circumstance that a straight line carried round the perimeter of a triangle by being applied to all the sides in succession, is brought into its old situation again; the argument being, that because this line has made the sort of somerset it would do by being turned through four right angles about a fixed point, the exterior

angles of the triangle have necessarily been equal to four right angles.

“The answer to which is, that there is no connexion between the things at all, and that the result will just as much take place where the exterior angles are avowedly not equal to four right angles.

“Take, for example, the plane triangle formed by three small arcs of the same or equal circles,



as in the margin; and it is manifest that an arc of this circle may be carried round precisely in the way described and return to its old situation, and yet there be no pretense for infer-

ring that the exterior angles were equal to four right angles.

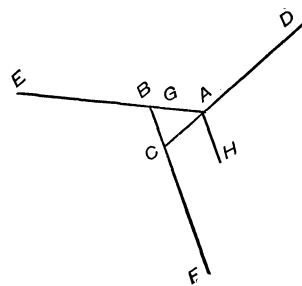
“And if it is urged that these are *curved* lines and the statement made was of straight; then the answer is by demanding to know, what property of straight lines has been laid down or established, which determines that what is not true in the case of other lines is



true in theirs. It has been shown that, as a general proposition, the connexion between a line returning to its place and the exterior angles having been equal to four right angles, is a *non sequitur*; that it is a thing that may be or may not be; that the notion that it returns to its place *because* the exterior angles have been equal to four right angles, is a mistake. From which it is a legitimate conclusion, that if it had pleased nature to make the exterior angles of a triangle greater or less than four right angles, this would not have created the smallest impediment to the line's returning to its old situation after being carried round the sides; and consequently the line's returning is no evidence of the angles not being greater or less than four right angles."

Charles L. Dodgson, of Christ Church, Oxford, in his "Curiosa Mathematica," Part I, pp. 70-71, 3d Ed., 1890, says:

"Yet another process has been invented—quite fascinating in its brevity and its elegance—which, though involving the same fallacy as the Direction-Theory, proves Euc. I, 32, without even mentioning the dangerous word 'Direction.'



“We are told to take any triangle  $ABC$ ; to produce  $CA$  to  $D$ ; to make part of  $CD$ , viz.,  $AD$ , revolve, about  $A$ , into the position  $ABE$ ; then to make part of this line, viz.,  $BE$ , revolve,

about  $B$ , into the position  $BCF$ ; and lastly to make part of this line, viz.,  $CF$ , revolve, about  $C$ , till it lies along  $CD$ , of which it originally formed a part. We are then assured that it must have revolved through four right angles: from which it easily follows that the interior angles of the triangle are together equal to two right angles.

“The disproof of this fallacy is almost as brief and elegant as the fallacy itself. We first quote the general principle that we can not reasonably be told to make a line fulfill *two* conditions, either of which is enough by itself to fix its position: e. g., given three points  $X, Y, Z$ , we can not reasonably be told to draw a line from  $X$  which shall pass through  $Y$  *and*  $Z$ : we can make it pass through  $Y$ , but it must then take its chance of passing through  $Z$ ; and *vice versa*.

“Now let us suppose that, while one part of

AE, viz., BE, revolves into the position BF, another little bit of it, viz., AG, revolves, through an equal angle, into the position AH; and that, while CF revolves into the position of lying along CD, AH revolves—and here comes the fallacy.

“You must not say ‘revolves, through an equal angle, into the position of lying along AD,’ for this would be to make AH *fulfill two conditions at once*.

“If you say that the one condition involves the other, you are virtually asserting that the lines CF, AH are equally inclined to CD—and this in *consequence* of AH having been so drawn that these same lines are equally inclined to AE.

“That is, you are asserting, ‘A pair of lines which are equally inclined to a certain transversal, are so to any transversal.’ [Deducible from Euc. I, 27, 28, 29.]”



# MATHEMATICAL WORKS

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