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DZYADYK'S TECHNIQUE FOR ODE'S USING HERMITIAN INTERPOLATING POLYNOMIALS

МЕТОД ДЗЯДИКА РОЗВ'ЯЗУВАННЯ ЗВИЧАЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ З ВИКОРИСТАННЯМ ІНТЕРПОЛЯЦІЙНИХ ПОЛІНОМІВ ЕРМІТА

For the case of Hermitian interpolation, we consider the approximation-iteration method introduced by V. K. Dzyadyk. We construct a practical algorithm.

Розглянуто апроксимаційно-ітеративний метод В. К. Дзядика у випадку ермітової інтерполяції. Побудовано практичний алгоритм.

1. Introduction. The mathematical formulation of physical phenomena in many situations often lead to one or a set of initial value problem (IVP) of the form:

$$y'(x) = f[x, y(x)], \quad y(x_0) = y_0, \quad x \in [x_0, x_0 + h], \quad y \in \mathbb{R}, \quad h > 0. \qquad (1)$$

In 1980 – 1984, V. K. Dzyadyk [1, 2] has introduced a new technique in which he has constructed an algorithm based on both approximation and iteration. In this algorithm $f(x, y)$ of (1) is considered to be analytic in some domain $G \supset [x_0, x_0 + h]$ or sufficiently smooth on $[x_0, x_0 + h]$. We will call this technique the Dzyadyk Approximation-Iteration method [1 – 5] or, briefly, DAI-method. The following theorem, which is called the Picard's theorem whose proof can be found in [5] and [6].

Theorem 1 (Picard's theorem). *If* $f(x, y)$ *defined and continuous on* $D = [x_0, x_0 + h] \times [y_0 - H, y_0 + H]$, $H > 0$, *satisfies a Lipschitz condition with respect to* y *and with constant* A *and, in addition satisfies the inequality:* $\| f \|_{C_D} A^{-1} \int_0^q e^s ds < H$, $q = Ah$, *then the unique solution* $y(x)$ *of the IVP* (1) *given by:* $y(x) = y_0 + \sum_{\mu=0}^{\infty} \left[y^{[\mu+1]}(x) - y^{[\mu]} \right]$, *where* $y^{[0]}(x) \equiv y_0$, $y^{[\mu]}(x) = y_0 + \int_{x_0}^{x} f\left[\sigma, y^{[\mu-1]}(\sigma)\right] d\sigma$. *And,* $\left\| y - y^{[\mu]} \right\| \le \| f \|_{C_D} e^{A|x-a|} A^{\mu} |x-a|^{\mu+1}/(\mu+1)!$.

In the DAI-method (technique) Dzyadyk has considered the following projection operator to approximate $f(\xi, y(\xi))$:

$$A_n(f; \xi) = \mathcal{L}_n(f; \xi) = \sum_{i=0}^{n} f(\xi_i, y(\xi_i)) l_{n,i}(\xi), \quad \xi, \xi_i \in [x_0, x_0 + h], \quad i = \overline{0, n},$$

where $l_{n,i}(\xi)$ are the fundamental Lagrangian interpolating polynomials given by:

$$l_{n,i}(\xi) = \prod_{\substack{j=0 \\ j \ne i}}^{n} \frac{\xi - \xi_j}{\xi_i - \xi_j}.$$

In this paper, instead of $\mathcal{L}_n$, we consider a projection operator depending on Hermitian interpolating polynomials $\mathcal{H}_{2n+1}$ for approximating $f(\xi, y(\xi))$. Namely:

$$A_n(f; \xi) = \mathcal{H}_{2n+1}(f; \xi) = \sum_{i=0}^{n} f(\xi_i, y(\xi_i)) h_{2n+1,i}(\xi) + \sum_{i=0}^{n} f'(\xi_i, y(\xi_i)) \overline{h}_{2n+1,i}(\xi),$$

where

$$h_{2n+1,i}(\xi) = \left[1-2l'_{n,i}(\xi_i)(\xi-\xi_i)\right]l^2_{n,i}(\xi), \quad \overline{h}_{2n+1,i}(\xi) = (\xi-\xi_i)l^2_{n,i}(\xi). \quad (2)$$

M. Rizk in [7] expanded for any $n \in \mathbb{N}$ the fundamental Hermitian polynomials $h_{2n+1,i}(\xi)$ and $\overline{h}_{2n+1,i}(\xi)$, $i=\overline{0,n}$, built by either the nodes $\xi_j^* = -\cos(j\pi/n)$ or the nodes $\xi_j^0 = -\cos((2j+1)\pi/(2n+2))$, $j=\overline{0,n}$, on $[-1,1]$ by the formula:

1. If $\xi_j =: \xi_j^*$, $c_j =: \cos(j\pi/n)$, then for $i = 1, 2, \ldots, n-1$:

$$h^*_{2n+1,i}(\xi) = \frac{1}{n^2}\left\{\left(n-\frac{1}{1-c_{2i}}\right) + c_i\left[\frac{1}{1-c_{2i}}-2n+1\right]T_1(\xi) + \right.$$

$$+\sum_{k=2}^{2n-2}(-1)^k(2n-k)c_{ik}T_k(\xi) - c_i\left[1+\frac{1}{2(1-c_{2i})}\right]T_{2n-1}(\xi) +$$

$$\left. + \frac{1}{1-c_{2i}}T_{2n}(\xi) - \frac{c_i}{2(1-c_{2i})}T_{2n+1}(\xi)\right\}, \quad (3a)$$

$$\overline{h}^*_{2n+1,i}(\xi) = \frac{1}{2n^2}\left\{c_i - c_{2i}T_1(\xi) + \sum_{k=2}^{2n-2}(-1)^k\left[c_{i(k+1)}-c_{i(k-1)}\right]T_k(\xi) + \right.$$

$$\left. + \left[c_{2i}-\frac{1}{2}\right]T_{2n-1}(\xi) - c_iT_{2n}(\xi) + \frac{1}{2}T_{2n+1}(\xi)\right\}, \quad (4a)$$

and, for $i=0$, $i=n$:

$$h^*_{2n+1,i}(\xi) = \frac{1}{48n^2}\left\{4\left(n^2+6n-1\right) - 2\left(2n^2+24n-11\right)c_iT_1(\xi) + \right.$$

$$+ 24\sum_{k=2}^{2n-2}(-1)^k(2n-k)c_{ik}T_k(\xi) + \left(2n^2-23\right)c_iT_{2n-1}(\xi) -$$

$$\left. - 4\left(n^2-1\right)T_{2n}(\xi) + \left(2n^2+1\right)c_iT_{2n+1}(\xi)\right\}, \quad (3b)$$

$$\overline{h}^*_{2n+1,i}(\xi) = \left(\frac{1}{8n^2}\right)\left[c_i - T_1(\xi) + \frac{1}{2}T_{2n-1}(\xi) - c_iT_{2n}(\xi) + \frac{1}{2}T_{2n+1}(\xi)\right]. \quad (4b)$$

2. If $\xi_j =: \xi_j^0$, $S_{j,k} := \cos((2j+1)\pi/(2n+2))$, then for $i = 0, 1, \ldots, n$:

$$h^0_{2n+1,i}(\xi) = \frac{1}{n+1}\left\{1+2\sum_{k=1}^{2n+1}(-1)^k\left[1-\frac{k}{2n+2}\right]S_{i,k}T_k(\xi)\right\}, \quad (5)$$

$$\overline{h}^0_{2n+1,i}(\xi) = \frac{1}{2(n+1)^2}\sum_{k=1}^{2n+1}(-1)^k\left[S_{i,k+1}-S_{i,k-1}\right]T_k(\xi). \quad (6)$$

2. Analytical method. Let n be an arbitrary natural number and $\{\xi_j\}_{j=\overline{0,n}}$ as $-1 \le \xi_0 < \xi_1 < \ldots < \xi_n \le 1$ be nodes on $[-1,1]$ and $h > 0$ be a real number. Using the linear transformations:

$$\xi = -1 + \frac{2}{h}(x-x_0) : [x_0, x_0+h] \to [-1,1]$$

$$\text{or} \quad x = x_0 + \frac{h}{2}(1+\xi) : [-1,1] \to [x_0, x_0+h] \quad (7)$$

to construct a nodes $\{\bar{x}_j\}_{j=\overline{0,n}}$ on the segment $[x_0, x_0+h]$.

If $\tilde{\omega}$, $\tilde{f}$, $\tilde{l}$, $\tilde{h}$ and $\tilde{\bar{h}}$ are the transfers of ω, f, l, h, and $\bar{h}$, respectively from $[-1,1]$ to $[x_0, x_0+h]$, then it is easy to show that:

$$\tilde{f}(x) = f(x), \qquad \omega_{n+1}(\xi) = \left(\frac{2}{h}\right)^{n+1} \tilde{\omega}_{n+1}(x), \tag{8}$$

$$\left(\frac{d^2\omega_{n+1}(\xi)}{d\xi^2}\right)\left(\frac{d\omega_{n+1}(\xi)}{d\xi}\right)^{-1} = \frac{h}{2}\left(\frac{d^2\tilde{\omega}_{n+1}(x)}{dx^2}\right)\left(\frac{d\tilde{\omega}_{n+1}(x)}{dx}\right)^{-1}, \tag{9}$$

$$h_{2n+1,i}(\xi) = \tilde{h}_{2n+1,i}(x), \qquad \bar{h}_{2n+1,i}(\xi) = \frac{2}{h}\tilde{\bar{h}}_{2n+1,i}(x). \tag{10}$$

The algorithm consists of the following steps:

Step 1. Construct the nodes $\{\xi_j\}_{j=0}^{n}: -1 \le \xi_0 < \xi_1 < \ldots < \xi_n \le 1$ on $[-1,1]$.

Step 2. Construct the matrices $[a_{ij}]_{i,j=\overline{0,n}}$, $[b_{ij}]_{i,j=\overline{0,n}}$ of degree $(n+1)\times(n+1)$ as follows:

$$a_{ij} = a_{ij}(n,\xi_j) = \int_{-1}^{\xi_j} h_{2n+1,i}(\xi)\,d\xi, \qquad b_{ij} = b_{ij}(n,\xi_j) = \int_{-1}^{\xi_j} \bar{h}_{2n+1,i}(\xi)\,d\xi. \tag{11}$$

From (3a, b), (4a, b) and (11) we get a proof of the following lemma.

Lemma 1. *If* $\xi_j = \xi_j^* = -\cos(j\pi/n)$, *then the matrices* a_{ij}, b_{ij} *can be expressed in the following explicit forms:*

For $i = 1, 2, \ldots, n-1$:

$$a_{ij} = a_{ij}^* = a_{ij}\left(n,\xi_j^*\right) = \frac{1}{n^2}\left\{(1-c_j)\left[n - \frac{4n^2}{(4n^2-1)(1-c_{2i})}\right] + \right.$$
$$+ \frac{c_i(1-c_{2j})}{4}\left[(2n-1) - \frac{n^2}{(n^2-1)(1-c_{2i})}\right] +$$
$$\left. + \sum_{k=2}^{2n-1}\frac{(2n-k)c_{ik}}{2}\left[\frac{c_{(k-1)j}}{k-1} - \frac{c_{(k+1)j}}{k+1} - \frac{2}{k^2-1}\right]\right\},$$

$$b_{ij} = b_{ij}^* = b_{ij}\left(n,\xi_j^*\right) = \frac{1}{2n^2}\left\{c_i(1-c_j)\frac{4n^2}{4n^2-1} + \left(\frac{1-c_{2j}}{4}\right)\left[c_{2i} + \frac{1}{n^2-1}\right] - \right.$$
$$\left. - \frac{1}{2}\sum_{k=2}^{2n-1}\left[\frac{c_{(k-1)j}}{k-1} - \frac{c_{(k+1)j}}{k+1} - \frac{2}{k^2-1}\right]\left[c_{(k-1)i} - c_{(k+1)i}\right]\right\}.$$

And for $i = 0$, $i = n$:

$$a_{ij} = a_{ij}^* = a_{ij}\left(n,\xi_j^*\right) = \frac{1}{4n^2}\left\{\frac{1}{3}(1-c_j)\left[n^2 + 6n - 1 + \frac{n^2-1}{4n^2-1}\right] + \right.$$
$$+ \frac{c_i(1-c_{2j})}{24}\left[2n^2 + 24n - 11 + \frac{2n^2+1}{n^2-1}\right] +$$
$$\left. + \sum_{k=2}^{2n-1}(2n-k)c_{ik}\left[\frac{c_{(k-1)j}}{k-1} - \frac{c_{(k+1)j}}{k+1} - \frac{2}{k^2-1}\right]\right\},$$

$$b_{ij} = b_{ij}^* = b_{ij}\left(n,\xi_j^*\right) = \frac{c_i(1-c_j)}{2(4n^2-1)} + \frac{1-c_{2j}}{32(n^2-1)},$$

where $c_k = \cos(k\pi/n)$.

And from (5), (6) and (11) we get a proof of the following lemma.

Lemma 2. *If* $\xi_j = \xi_j^0 = -\cos((2j+1)\pi/(2n+2))$, $j = \overline{0, n}$, *then the matrices* a_{ij}, b_{ij} *can be expressed in the following explicit forms:*

For $i = 0, 1, \ldots, n$:

$$a_{ij} = a_{ij}^0 = a_{ij}(n, \xi_j^0) = \frac{1}{n+1}\left\{1 - S_{j,1} + \frac{(1-S_{j,2})(2n+1)S_{i,1}}{4(n+1)} + \right.$$

$$\left. + \sum_{k=2}^{2n+1}\left(1 - \frac{k}{2(n+1)}\right)S_{i,k}\left[\frac{S_{j,(k-1)}}{k-1} - \frac{S_{j,(k+1)}}{k+1} - \frac{2}{k^2-1}\right]\right\},$$

$$b_{ij} = b_{ij}^0 = b_{ij}(n, \xi_j^0) = \frac{1}{4(n+1)^2}\left\{\frac{1}{2}(S_{i,2} - 1)(1 - S_{j,2}) + \right.$$

$$\left. + \sum_{k=2}^{2n+1}\left(S_{i,k+1} - S_{i,k-1}\right)\left[\frac{S_{j,(k-1)}}{k-1} - \frac{S_{j,(k+1)}}{k+1} - \frac{2}{k^2-1}\right]\right\},$$

where $S_{i,j} = \cos((2i+1)j\pi/(2n+2))$.

Notice that $[a_{ij}]_{i,j=\overline{0,n}}$ and $[b_{ij}]_{i,j=\overline{0,n}}$ depend on the natural number n and the choice of the nodes $\{\xi_j\}_{j=\overline{0,n}}$, but does not depend on our problem (1).

Step 3. Construct a sequence $\{\bar{x}_j\}_{j=\overline{0,n}}$: $(x_0 \le \bar{x}_0 < \bar{x}_1 < \ldots < \bar{x}_n \le x_0 + h)$, on $[x_0, x_0 + h]$ as:

$$\bar{x}_j = x_0 + \frac{h}{2}(1 + \xi_j), \quad j = 0, 1, \ldots, n.$$

Step 4. Using the matrices in step (2), we give a $(n+1)$ non-linear system of equations in $(n+1)$ unknowns $y_j, j = \overline{0, n}$ as:

$$y_j = y_0 + \frac{h}{2}\sum_{i=0}^{n} a_{ij} f(\bar{x}_i, y_i) + \frac{h^2}{4}\sum_{i=0}^{n} b_{ij}\left[f_x' + f f_y'\right](\bar{x}_i, y_i), \quad j = \overline{0, n}. \tag{12}$$

By solving system (12) (using simple successive iteration or Newton iteration), we get values for $y_j, j = \overline{0, n}$.

Theorem 2. *The values* y_j *which are given from* (12) *represent values of the following polynomial* (*of degree not exceeding* $2n+2$) *at the points* $x = \bar{x}_j \in [x_0, x_0 + h]$:

$$P_{2n+2}(x) = y_0 + \int_{x_0}^{x} \tilde{\mathcal{H}}_{2n+1}\left[f(\cdot; P_{2n+2}(\cdot)); \sigma\right] d\sigma =$$

$$= y_0 + \sum_{i=0}^{n} f\left(\bar{x}_i, P_{2n+2}(\bar{x}_i)\right) \int_{x_0}^{x} \tilde{h}_{2n+1,i}(\sigma)\, d\sigma +$$

$$+ \sum_{i=0}^{n}\left[f_x' + f f_y'\right]\left(\bar{x}_i, P_{2n+2}(\bar{x}_i)\right) \int_{x_0}^{x} \tilde{\bar{h}}_{2n+1,i}(\sigma)\, d\sigma, \tag{13}$$

i.e.,

$$P_{2n+2}(\bar{x}_j) = y_j, \quad j = \overline{0, n}.$$

Proof. By use the transformation (7) and the equations (10), we get:

$$\int_{x_0}^{\bar{x}_j} \tilde{h}_{2n+1,i}(\sigma)\,d\sigma = \int_{-1}^{\xi_j} h_{2n+1,i}(\xi)\frac{h}{2}\,d\xi = \frac{h}{2}\,a_{ij},$$

$$\int_{x_0}^{\bar{x}_j} \tilde{\bar{h}}_{2n+1,i}(\sigma)\,d\sigma = \int_{-1}^{\xi_j} \frac{h}{2}\,\bar{h}_{2n+1,i}(\xi)\frac{h}{2}\,d\xi = \frac{h^2}{4}\,b_{ij}.$$

So, obvious to get the result.

Step 5. Now, by Theorem 2, (12), we get a polynomial of degree not exceeding $(2n+2)$ given by:

$$P_{2n+2}(x) = y_0 + \frac{h}{2}\sum_{i=0}^{n} f(\bar{x}_i, y_i)\Pi_{2n+2,i}(\xi) + \frac{h^2}{4}\sum_{i=0}^{n}\left[f_x' + f f_y'\right](\bar{x}_i, y_i)\overline{\Pi}_{2n+2,i}(\xi), \tag{14}$$

where $\xi = -1 + 2(x - x_0)/h$ and

$$\Pi_{2n+2,i}(\xi) = \int_{-1}^{\xi} h_{2n+1,i}(t)\,dt \quad \text{and} \quad \overline{\Pi}_{2n+2,i}(\xi) = \int_{-1}^{\xi} \bar{h}_{2n+1,i}(t)\,dt.$$

For specific nodes and by (3a, b), (4a, b), (5) and (6) with some manipulation, we get explicit forms for each of $\Pi_{2n+2,i}(\xi)$ and $\overline{\Pi}_{2n+2,i}(\xi)$, by the following two lemmas.

Lemma 3. *If* $\xi_j = \xi_j^* = -\cos(j\pi/n)$, *and* $c_i = \cos(j\pi/n)$, $j = \overline{0, n}$, *then* $\Pi_{2n+2,i}(\xi)$, $\overline{\Pi}_{2n+2,i}(\xi)$, *can be expressed in the following explicit forms:*

I. *For* $i = 1, 2, \ldots, n-1$:

$$\Pi_{2n+2,i}(\xi) = \Pi^*_{2n+2,i}(\xi) = \frac{1}{n^2}\Bigg\{\left(n - \frac{1}{1-c_{2i}}\right)(1+\xi) + \frac{c_i}{2}\left[(2n-1) - \frac{1}{1-c_{2i}}\right]\left[1-\xi^2\right] + \sum_{k=2}^{2n-1}(-1)^k\frac{(2n-k)c_{ik}}{2}\left[\frac{T_{k+1}(\xi)}{k+1} - \frac{T_{k-1}(\xi)}{k-1} - \frac{2(-1)^k}{k^2-1}\right] - \frac{c_i}{4(1-c_{2i})}\left[\frac{T_{2n+2}(\xi)}{2n+2} - \frac{T_{2n-2}(\xi)}{2n-2} + \frac{1}{n^2-1}\right] + \frac{1}{2(1-c_{2i})}\left[\frac{T_{2n+1}(\xi)}{2n+1} - \frac{T_{2n-1}(\xi)}{2n-1} - \frac{2}{4n^2-1}\right]\Bigg\},$$

$$\overline{\Pi}_{2n+2,i}(\xi) = \overline{\Pi}^*_{2n+2,i}(\xi) = \frac{1}{8n^2}\Bigg\{4c_i(1+\xi) + 2c_{2i}(1-\xi^2) + 2\sum_{k=2}^{2n-1}(-1)^k\left(c_{(k+1)i} - c_{(k-1)i}\right)\left[\frac{T_{k+1}(\xi)}{k+1} - \frac{T_{k-1}(\xi)}{k-1} - \frac{2(-1)^k}{k^2-1}\right] + \left[\frac{T_{2n+2}(\xi)}{2n+2} - \frac{T_{2n-2}(\xi)}{2n-2} - \frac{1}{n^2-1}\right] - 2c_i\left[\frac{T_{2n+1}(\xi)}{2n+1} - \frac{T_{2n-1}(\xi)}{2n-1} - \frac{2}{4n^2-1}\right]\Bigg\}.$$

II. *For* $i = 0$, $i = n$:

$$\Pi_{2n+2,i}(\xi) = \Pi^*_{2n+2,i}(\xi) =$$

$$= \frac{1}{48n^2}\Bigg\{4(n^2+6n-1)(1+\xi) + c_i(2n^2+24n-11)(1-\xi^2) +$$

$$+ 12\sum_{k=2}^{2n-1}(-1)^k(2n-k)c_{ik}\left[\frac{T_{k+1}(\xi)}{k+1} - \frac{T_{k-1}(\xi)}{k-1} - \frac{2(-1)^k}{k^2-1}\right] +$$

$$+ \frac{(2n^2+1)c_i}{2}\left[\frac{T_{2n+2}(\xi)}{2n+2} - \frac{T_{2n-2}(\xi)}{2n-2} + \frac{1}{n^2-1}\right] -$$

$$- 2(n^2-1)\left[\frac{T_{2n+1}(\xi)}{2n+1} - \frac{T_{2n-1}(\xi)}{2n-1} - \frac{2}{4n^2-1}\right]\Bigg\},$$

$$\overline{\Pi}_{2n+2,i}(\xi) = \overline{\Pi}^*_{2n+2,i}(\xi) = \frac{1}{32n^2}\Bigg\{4c_i(1+\xi) + 2(1-\xi^2) +$$

$$+ \left[\frac{T_{2n+2}(\xi)}{2n+2} - \frac{T_{2n-2}(\xi)}{2n-2} + \frac{1}{n^2-1}\right] - 2c_i\left[\frac{T_{2n+1}(\xi)}{2n+1} - \frac{T_{2n-1}(\xi)}{2n-1} - \frac{2}{4n^2-1}\right]\Bigg\}.$$

Lemma 4. *If* $\xi_j = \xi_j^0 = -\cos((2j+1)\pi/(2n+2))$, $j = \overline{0,n}$, *then* $\Pi_{2n+2,i}(\xi)$, $\overline{\Pi}_{2n+2,i}(\xi)$, *can be expressed in the following explicit forms. If* $S_{i,k} = \cos((2i+1)k\pi/(2n+2))$, *then for* $i = 0, \ldots, n$:

$$\Pi_{2n+2,i}(\xi) = \Pi^0_{2n+2,i}(\xi) = \frac{1}{n+1}\Bigg\{1 + \xi + \frac{2n+1}{2n+2}S_{i,1}\left[1-\xi^2\right] +$$

$$+ \sum_{k=2}^{2n+1}(-1)^k S_{i,k}\left[1 - \frac{k}{2n+2}\right]\left[\frac{T_{k+1}(\xi)}{k+1} - \frac{T_{k-1}(\xi)}{k-1} - \frac{2(-1)^k}{k^2-1}\right]\Bigg\}, \tag{15}$$

$$\overline{\Pi}_{2n+2,i}(\xi) = \overline{\Pi}^0_{2n+2,i}(\xi) = \frac{1}{4(n+1)^2}\Bigg\{\left[1-\xi^2\right]\left[S_{i,2}-1\right] +$$

$$+ \sum_{k=2}^{2n+1}(-1)^k\left[S_{i,k+1} - S_{i,k-1}\right]\left[\frac{T_{k+1}(\xi)}{k+1} - \frac{T_{k-1}(\xi)}{k-1} - \frac{2(-1)^k}{k^2-1}\right]\Bigg\}. \tag{16}$$

Now by applying the Picard's iteration (Theorem 1) to solve the system (12), we get the following approximation polynomial (polynomial (14)) depends on $\mu = 0, 1, \ldots$:

$$P^{[0]}_{2n+2} \equiv y_0,$$

$$P^{[\mu]}_{2n+2}(x) = y_0 + \frac{h}{2}\sum_{i=0}^{n} f\left(\overline{x}_i, y_i^{[\mu-1]}\right)\Pi_{2n+2,i}(\xi) +$$

$$+ \frac{h^2}{4}\sum_{i=0}^{n}\left[f'_x + ff'_y\right]\left(\overline{x}_i, y_i^{[\mu-1]}\right)\overline{\Pi}_{2n+2,i}(\xi),$$

where $\xi = -1 + 2(x-x_0)/h$.

Theorem 3 (error estimation). *If* $f(x,y) \in C^{(2n+2)}[x_0, x_0+h]$, *there exists finite positive constant* H *and the following conditions are satisfied:*

1) $f(x,y)$ *defined and continuous on* $D := [x_0, x_0+h] \times [y_0 - H, y_0 + H]$;
2) $|f(x,y) - f(x,z)| \le A|y-z| \;\; \forall x \in [x_0, x_0+h], \; \forall y, z \in [y_0 - H, y_0 + H]$;
3) $q := Ah < 1$.

Then there exists a finite positive constant M_{2n+2} *such that for* $\mu = 1, 2, \ldots$, *the following inequalities are satisfied:*

I. *If* $\xi_j = \xi_j^* = -\cos(j\pi/n)$, $j = \overline{0, n}$, *then*

$$\left\| P_{2n+2}^{[\mu]} - y \right\|_{C_{[x_0, x_0+h]}} \le \frac{M_{2n+2}\, h^{2n+2}}{2^{4n-1}(2n+2)!} \frac{1-q^{\mu}}{1-q} + \|f\|_{C_D} e^{q} \frac{q^{\mu}}{(\mu+1)!}.$$

II. *If* $\xi_j = \xi_j^0 = -\cos((2j+1)\pi/(2n+2))$, $j = \overline{0, n}$, *then*

$$\left\| P_{2n+2}^{[\mu]} - y \right\|_{C_{[x_0, x_0+h]}} \le \frac{M_{2n+2}\, h^{2n+2}}{2^{4n+1}(2n+2)!} \frac{1-q^{\mu}}{1-q} + \|f\|_{C_D} e^{q} \frac{q^{\mu}}{(\mu+1)!}.$$

Proof. If $y^{[\mu]}(x)$, $\mu = 0, 1, 2, \ldots$, be the Picard sequence of functions given by Theorem 1, then from (13), Theorem 1, and condition 2, we get:

$$\left| P_{2n+2}^{[\mu]} - y^{[\mu]}(x) \right| = \int_{x_0}^{x} \left\{ \tilde{\mathcal{H}}_{2n+1}\left[f\left(\cdot; P_{2n+2}^{[\mu-1]}(\cdot)\right); \sigma \right] - f\left(\sigma, y^{\mu-1}(\sigma)\right) \right\} d\sigma \le$$

$$\le \int_{x_0}^{x} \left| \tilde{\mathcal{H}}_{2n+1}\left[f\left(\cdot; P_{2n+2}^{[\mu-1]}(\cdot)\right); \sigma \right] - f\left(\sigma, P_{2n+2}^{[\mu-1]}(\sigma)\right) \right| |d\sigma| +$$

$$+ \int_{x_0}^{x} \left| f\left(\sigma; P_{2n+2}^{[\mu-1]}(\cdot)\right) - f\left(\sigma, y^{\mu-1}(\sigma)\right) \right| |d\sigma| \le$$

$$\le \frac{\left\| f^{(2n+2)} \right\|}{(2n+2)!} \int_{x_0}^{x} \left| \tilde{\omega}_{n+1}^{2}(\sigma) \right| |d\sigma| + A \int_{x_0}^{x} \left| P_{2n+2}^{[\mu-1]}(\cdot) - y^{\mu-1}(\sigma) \right| |d\sigma| \le$$

$$\le \frac{M_{2n+2}}{(2n+2)!} \left(\frac{h}{2} \right)^{2n+2} 2 \left\| \omega_{n+1}^{2} \right\| + Ah \left\| P_{2n+2}^{[\mu-1]} - y^{[\mu-1]} \right\|. \tag{17}$$

So, if $\Delta_{\mu} = \left\| P_{2n+2}^{[\mu]} - y^{[\mu]} \right\|$, then we get:

$$\Delta_{\mu} \le B + q\Delta_{\mu-1} \le B + qB + q^{2}\Delta_{\mu-2} \le \ldots \le B\frac{1-q^{\mu}}{1-q}, \tag{18}$$

where

$$B = \frac{2M_{2n+2}}{(2n+2)!} \left(\frac{h}{2} \right)^{2n+2} \left\| \omega_{n+1}^{2} \right\|, \quad q = Ah.$$

Therefore, from the triangle inequality, (17) and (18) we get:

$$\left\| P_{2n+2}^{[\mu]} - y \right\|_{C_{[x_0, x_0+h]}} \le \frac{2M_{2n+2}}{(2n+2)!} \left(\frac{h}{2} \right)^{2n+2} \left\| \omega_{n+1}^{2} \right\| \frac{1-q^{\mu}}{1-q} + \|f\|_{C_D} e^{q} \frac{q^{\mu}}{(\mu+1)!}. \tag{19}$$

If $\xi_j = \xi_j^*$ or $\xi_j = \xi_j^0$, then we have:

$$\omega_{n+1}^{*}(\xi) = \left(\xi - \xi_0^*\right)\left(\xi - \xi_1^*\right) \ldots \left(\xi - \xi_n^*\right) = C_n \sqrt{1-\xi^2}\, \sin(n \arccos \xi), \tag{20}$$

$$C_n = -2^{(1-n)},$$

$$\omega_{n+1}^{0}(\xi) = \left(\xi - \xi_0^0\right)\left(\xi - \xi_1^0\right) \ldots \left(\xi - \xi_n^0\right) = C_n^0 \sin((n+1) \arccos \xi), \quad C_n^0 = 2^{-n}. \tag{21}$$

Now, from (19) and (20) we get I and from (19) and (21) we get II. Therefore Theorem 3 is thus proven.

3. One step method. In this section, we shall give a one step method by using polynomials (14), and give a relation between its parameters.

Let $[x_0, x_0 + H]$ be any interval and $M, n, N \in \mathbb{N}$ be any natural numbers, $h = H/M$ and let $x_N = x_0 + Nh$, $N = 0, 1, 2, \ldots, M$ and $\{\xi_j\}_{j=0}^{n}$ be a nodes on $[-1, 1]$. Also, for all $i, j = 0, 1, 2, \ldots, n$ consider the following parameters:

$$c_i = \frac{1}{2}(1+\xi_i), \quad A_{ij} = \frac{1}{2}a_{ji}, \quad B_{ij} = \frac{1}{2}b_{ji},$$
$$B_i = \frac{1}{2}\int_{-1}^{1} h_{2n+1,i}(\xi)\,d\xi, \quad \overline{B}_i = \frac{1}{2}\int_{-1}^{1} \overline{h}_{2n+1,i}(\xi)\,d\xi. \tag{22}$$

1. If $\xi_j = \xi_j^* = -\cos(j\pi/n)$, then it is obvious that:

$$B_i = B_i^* = \frac{1}{2}a_{in} \quad \text{and} \quad \overline{B}_i = \frac{1}{2}b_{in}.$$

2. If $\xi_j = \xi_j^0 = -\cos((2j+1)\pi/(2n+2))$, then from (15) and (16), it is easy to show that:

$$B_i = B_i^0 = \frac{1}{n+1}\left\{1 - 2\sum_{k=1}^{n} S_{i,2k}\left[1 - \frac{k}{n+1}\right]\left[\frac{1}{4k^2-1}\right]\right\},$$

$$\overline{B}_i = \overline{B}_i^0 = \frac{1}{2(n+1)^2}\sum_{k=1}^{n}\frac{S_{i,2k-1} - S_{i,2k+1}}{4k^2-1}, \quad S_{ij} = \cos\frac{(2i+1)j\pi}{2n+2}.$$

The relation between the constants (22) is given by the following theorem:

Theorem 4. *For any $n \in \mathbb{N}$ and for any $\{\xi_j\}$, $j = 0, 1, \ldots, n$, we have:*

I. *For all $j = \overline{0, n}$ and $1 \le k \le 2n+2$:*

$$\sum_{i=0}^{n} A_{ji}\,c_i^{k-1} = \frac{1}{k}c_j^k - \frac{k-1}{2}\sum_{i=0}^{n} B_{ji}\,c_i^{k-2}. \tag{23}$$

II. *For all $1 \le k \le 2n+2$:*

$$\sum_{i=0}^{n} B_i\,c_i^{k-1} = \frac{1}{k} - \frac{k-1}{2}\sum_{i=0}^{n} c_i^{k-2}\,\overline{B}_i. \tag{24}$$

Proof. I. From (22) and (11) we get:

$$\sum_{i=0}^{n} A_{ji}\,c_i^{k-1} = \frac{1}{2}\sum_{i=0}^{n} a_{ij}\,c_i^{k-1} = \frac{1}{2}\sum_{i=0}^{n}\int_{-1}^{\xi_j} h_{2n+1,i}(\xi)\,d\xi\left(\frac{1}{2}\right)^{k-1}(1+\xi_i)^{k-1} =$$
$$= \frac{1}{2^k}\int_{-1}^{\xi_j}\sum_{i=0}^{n}(1+\xi_i)^{k-1}\,h_{2n+1,i}(\xi)\,d\xi. \tag{25}$$

Consider the function $g(\xi) = (1+\xi)^{k-1}$, $k \le 2n+2$. Hence:

$$\mathcal{H}_{2n+1}(g;\xi_j) = \sum_{i=0}^{n}(1+\xi_i)^{k-1}\,h_{2n+1,i}(\xi) +$$
$$+ (k-1)\sum_{i=0}^{n}(1+\xi_i)^{k-2}\,\overline{h}_{2n+1,i}(\xi) = (1+\xi_i)^{k-1}. \tag{26}$$

By integration each side of (40) from -1 to ξ_j, we get:

$$\int_{-1}^{\xi_j} \sum_{i=0}^{n} (1+\xi_i)^{k-1} h_{2n+1,i}(\xi)\, d\xi \;+\; (k-1)\sum_{i=0}^{n} 2^{k-2} c_i^{k-2} b_{ij} \;=\; \frac{1}{k} 2^k c_j^k. \tag{27}$$

From (25) and (27), we get I of Theorem 4.

II. From (22) and (11) we get:

$$\sum_{i=0}^{n} B_i c_i^{k-1} \;=\; \frac{1}{2}\sum_{i=0}^{n} \int_{-1}^{1} h_{2n+1,i}(\xi)\, d\xi \frac{1}{2^{k-1}} (1+\xi_i)^{k-1}. \tag{28}$$

By integrating each side of (24) from -1 to 1, we get:

$$\sum_{i=0}^{n} (1+\xi_i)^{k-1} \int_{-1}^{1} h_{2n+1,i}(\xi)\, d\xi \;+\; (k-1)\sum_{i=0}^{n} (1+\xi_i)^{k-2}\overline{b} \;=\; \frac{2^k}{k}. \tag{29}$$

From (28) and (29), we get II of Theorem 4.

Remarks.

1. In (23), if $k=1$, $k=2$, then $\sum_{i=0}^{n} A_{ji} = c_j$ and $\sum_{i=0}^{n} B_{ji} = c_j - 2\sum_{i=0}^{n} A_{ji}c_i$.

2. In (24), if $k=1$, $k=2$, then $\sum_{i=0}^{n} B_i = 1$ and $\sum_{i=0}^{n} \overline{B}_i = 1 - 2\sum_{i=0}^{n} B_i c_i$.

3. Let $\{\xi_j\}_{i=0}^{n}$ be any nodes in $[-1,1]$ and $G \in C[-1,1]$ be a function defined as:

$$G(\xi) \;=\; \begin{cases} \dfrac{(\xi+1)(2\xi-(\xi_0-1))(\xi-\xi_0)}{(\xi_0+1)^2}, & \text{if} \quad \xi\in[-1,\xi_0]; \\[2ex] \dfrac{(\xi-\xi_i)\big(2\xi-(\xi_i+\xi_{i+1})\big)(\xi-\xi_{i+1})}{\left(\xi_{i+1}-\xi_i\right)^2}, & \text{if} \quad \xi\in\left[\xi_i,\xi_{i+1}\right], \quad i=\overline{0,n-1}; \\[2ex] \dfrac{(\xi-\xi_n)(2\xi-(\xi_n+1))(\xi-1)}{(1-\xi_n)^2}, & \text{if} \quad \xi\in[\xi_n,1]. \end{cases}$$

It is obvious that $G'(-1^+) = G'(\xi_i) = G'(1^-) = 1$, $G(-1) = G(\xi_i) = G(1) = 0$ $\forall\, i = 0, 1, \ldots, n$ and $\int_{-1}^{1} G(t)\, dt = 0$. Therefore

$$\mathcal{H}_{2n+1}(G;\xi_j) \;=\; \sum_{i=0}^{n} G(\xi_i) h_{2n+1,i}(\xi) \;+\; \sum_{i=0}^{n} G'(\xi_i)\overline{h}_{2n+1,i}(\xi) \;=\; G(\xi), \quad n\ge 1. \tag{30}$$

By integrating each side of (30) from -1 to 1, we get $\sum_{i=0}^{n} \overline{B}_i = 0$. So, from 2, we get $\sum_{i=0}^{n} B_i C_i = 1/2$.

In polynomial (14), if we put $x = x_N$ and $y_N \approx y(x_N)$, $n = 0, 1, 2, \ldots, M$, then we can consider the following scheme:

$$K_i = f\left(x_N + c_i h,\ Y_N + h\sum_{j=0}^{n} A_{ij}K_j + \frac{h^2}{2}\sum_{j=0}^{n} B_{ij}\overline{K}_j\right), \quad i = \overline{0, n},$$

$$\overline{K}_i = f'\left(x_N + c_i h,\ Y_N + h\sum_{j=0}^{n} A_{ij}K_j + \frac{h^2}{2}\sum_{j=0}^{n} B_{ij}\overline{K}_j\right), \quad i = \overline{0, n}. \tag{31}$$

nd

$$Y_{N+1} = Y_N + h\sum_{j=0}^{n} B_i K_i + \frac{h^2}{2}\sum_{j=0}^{n} \overline{B}_i\,\overline{K}_i, \quad N = \overline{0, M-1}. \tag{32}$$

The parameters of (31) and (32) can be expressed in the following table:

Table 1

	$\ldots$	A_{0n}	B_{00}	B_{01}	$\ldots$	B_{0n}
	$\ldots$	A_{1n}	B_{00}	B_{01}	$\ldots$	B_{0n}
	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	$\ldots$	A_{nn}	B_{n0}	B_{n1}	$\ldots$	B_{nn}
	$\ldots$	B_n	$\overline{B}_0$	$\overline{B}_1$	$\ldots$	$\overline{B}_n$

espect to the parameters of A_{ij}, B_{ij}, B_i, $\overline{B}_i$ *and* c_i, $i, j =$ *llowing properties:*

$$\sum_{i=0}^{n} \ldots \frac{1}{k} - \frac{k-1}{2}\sum_{i=0}^{n} c_i^{k-2}\,\overline{B}_i, \quad 1 \le k \le \xi, \tag{33}$$

$$= \frac{\cdot}{\cdot}\, c_j^{\cdot} - \frac{k-1}{2}\sum_{i=0}^{n} B_{ji}c_i^{k-2}, \quad j = \overline{0, n}, \quad 1 \le k \le \eta, \tag{34}$$

$$\sum_{i,j=0}^{n} B_j c_i^{r-1} A_{ji} c_i^{k-1} = \frac{1}{k(k+r)} - \frac{k-1}{2}\sum_{i,j=0}^{n} B_j c_i^{r-1} B_{ji} c_i^{k-2} -$$

$$- \frac{k+r-1}{2k}\sum_{i=0}^{n} c_i^{k+r-2}\,\overline{B}_i, \quad 1 \le r \le \eta, \quad 1 \le k \le \xi. \tag{35}$$

ctly that:

gorithm given by (31) *and* (32) *satisfies the properties* $\ldots n+2)$ *for each* n.

(35), it is easy to show that $B(\eta + \xi)$ and $C(\xi)$ implies

using the scheme (31), (32) and on PC Pentium II-MMX ing five examples (known analytic solution) were solved using the $\cos\dfrac{j\pi}{n}$, $j = \overline{0, n}$.

ror $e_n^J = \left|Y_1^{[\mu]} - y(x_0 + h)\right|$ (μ is the number of iteration) for third examples when using the Jacobi iteration, i.e.:

$$K_i^{[0]} = f(x_0, y_0), \qquad \overline{K}_i^{[0]} = f'(x_0, y_0),$$

$$K_i^{[\mu+1]} = f\left(x_0 + c_i,\, y_0 + h\sum_{j=0}^{n} A_{ij} K_j^{[\mu]} + \frac{h^2}{2}\sum_{j=0}^{n} B_{ij} \overline{K}_j^{[\mu]}\right),$$

$$\overline{K}_i^{[\mu+1]} = f'\left(x_0 + c_i,\, y_0 + h\sum_{j=0}^{n} A_{ij} K_j^{[\mu]} + \frac{h^2}{2}\sum_{j=0}^{n} B_{ij} \overline{K}_j^{[\mu]}\right), \quad \mu = 0, 1, \ldots,$$

for different values of h ($h = 0{,}1; 0{,}5; 1{,}0$) and different values of n ($n = 3; 5; 7; 9$).

Table 3 gives the error $e_n^N = \left|Y_1^{[\mu]} - y(x_0 + h)\right|$ for stiff fourth and fifth examples when using the Newton – Raphson iteration for different values of h ($h = 0{,}5; 2; 4; 30$) and different values of n ($n = 3; 5; 7; 9$).

Example 1. $y' = -2xy^2$, $x_0 = 0$, with the initial condition $y(x_0) = 1$ and the exact solution: $y(x) = (1 + x^2)^{-1}$.

Example 2. $y' = e^{x-y}$, $x_0 = 0$, with the initial condition $y(x_0) = \ln(2)$ and the exact solution: $y(x) = x + \ln(1 + e^{-x})$.

Example 3. $y' = 4x\sqrt{y}$, $x_0 = 1$, with the initial condition $y(x_0) = 4$ and the exact solution: $y(x) = (1 + x^2)^2$.

Example 4. $y' = -1000(y - x^3) + 3x^2$, $x_0 = 0$, with the initial condition $y(x_0) = 0$ and the exact solution: $y(x) = x^3$.

Example 5. $y' = 1000\left(y - (1 + x^2)^{-1}\right) - 2xy^2$, $x_0 = 0$, with the initial condition $y(x_0) = 1$ and the exact solution: $y(x) = (1 + x^2)^{-1}$.

Table 2

n	h	μ	e_n^J of Example 1	μ	e_n^J of Example 2	μ	e_n^J of Example 3
3	0,1	06	3,367306E-13	07	8,570922E-13	09	7,371880E-14
	0,5	08	1,263820E-08	10	5,537792E-13	15	1,206146E-12
	1,0	29	1,582177E-05	14	2,633049E-09	21	3,812061E-12
5	0,1	07	9,992007E-16	05	5,759837E-13	08	9,237056E-14
	0,5	13	3,721246E-12	11	1,506573E-13	13	3,323564E-12
	1,0	31	3,055127E-08	18	1,887379E-14	19	1,044498E-12
7	0,1	07	1,665335E-15	05	1,827427E-13	08	9,769963E-15
	0,5	17	1,842970E-14	11	2,252643E-13	14	1,154632E-13
	1,0	34	4,580791E-11	17	2,278178E-13	20	5,165646E-12
9	0,1	06	7,771561E-16	05	3,186340E-14	08	2,398082E-14
	0,5	17	3,330667E-16	11	2,333689E-13	13	4,920508E-13
	1,0	44	1,565414E-16	16	9,414691E-14	18	2,664535E-13

Table 3

n	h	μ	e_n^N of Example 4	μ	e_n^N of Example 5
3	0,5	04	1,970673E-10	02	3,039236E-14
	2,0	07	7,501384E-10	02	1,385558E-12
	4,0	08	8,105921E-07	02	1,108447E-11
	30,0	43	7,651603E-07	02	4,678441E-09
5	0,5	06	1,131317E-13	02	1,149081E-14
	2,0	08	4,142714E-09	02	5,533352E-12
	4,0	09	4,438719E-08	02	1,567884E-10
	30,0	43	3,542338E-07	02	3,368408E-06
7	0,5	06	9,858780E-14	02	2,428613E-15
	2,0	09	2,060452E-10	02	2,178169E-11
	4,0	10	4,565695E-09	02	1,622595E-10
	30,0	42	1,306392E-07	02	7,262734E-06
9	0,5	06	3,497203E-14	02	9,492407E-15
	2,0	10	5,279666E-12	02	2,804867E-12
	4,0	11	1,096299E-11	02	9,636381E-11
	30,0	42	3,640602E-07	02	5,373036E-06

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