The numbers of induced subgraphs in strongly regular graphs

Kristína Kováčiková

Joint work with: Martin Mačaj

Faculty of Mathematics, Physics and Informatics
Comenius University in Bratislava

4th July 2014

Introduction

- In combinatorics, to compute the number of small configurations appearing in some object is often a very useful strategy. (special substructures in 4-cycle systems, in Steiner triple system, etc.)
- We investigate the number of occurrences for all graphs of small order as induced subgraph in *SRG*. In particular, we try to derive new properties of *SRG* only from its parameters.
- Similar concept for *SRG* is t-vertex condition, which explores induced subgraphs with respect to a fixed pair of vertices.

Definition

Strongly regular graph with parameters (n, k, λ, μ) is a k-regular graph on n vertices with following properties:

1. Any two adjacent vertices have exactly λ common neighbours.
2. Any two non-adjacent vertices have exactly μ common neighbours.

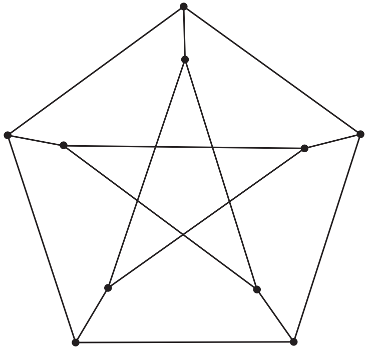


Figure: Petersen graph ($SRG(10, 3, 0, 1)$)

Triangle free *SRGs*

- $\lambda = 0$
- In the cases where $\mu \in \{2, 4, 6\}$ there exist infinitely many feasible parameters for graphs.
  The number of feasible parameters is finite in all remaining values of μ.

Triangle free *SRGs*

- $\lambda = 0$
- In the cases where $\mu \in \{2, 4, 6\}$ there exist infinitely many feasible parameters for graphs.
  The number of feasible parameters is finite in all remaining values of μ.
- However, there are only seven known examples of such graphs:

n	k	μ	
5	2	1	pentagon
10	3	1	Petersen graph
50	7	1	Hoffman-Singleton graph
16	5	2	Clebsch graph
56	10	2	Sims-Gewirtz graph
77	16	4	Mesner graph
100	22	4	Higman-Sims graph

The family of Moore graphs ($\mu = 1$)

- Hoffman and Singleton, 1960

k	n	
2	5	pentagon
3	10	Petersen graph
7	50	Hoffman-Singleton graph
57	3250	?

The family of Moore graphs ($\mu = 1$)

- Hoffman and Singleton, 1960

k	n	
2	5	pentagon
3	10	Petersen graph
7	50	Hoffman-Singleton graph
57	3250	?

- The last parameter is a famous open problem.

Definition

Let Γ be an arbitrary graph.
By P_G we denote the number of occurrences of some fixed graph G as an induced subgraph in Γ.

Definition

Let Γ be an arbitrary graph.
By P_G we denote the number of occurrences of some fixed graph G as an induced subgraph in Γ.

- P_{K_1} - the number of vertices in Γ
- P_{K_2} - the number of edges in Γ
- $P_{\overline{K}_2}$ - the number of edges in Γ

In the case where $\Gamma = srg(n, k, \lambda, \mu)$

- $P_{K_1} = n$
- $P_{K_2} = \frac{nk}{2}$
- $P_{\overline{K_2}} = \binom{n}{2} - P_{K_2}$

In the case where $\Gamma = srg(n, k, \lambda, \mu)$

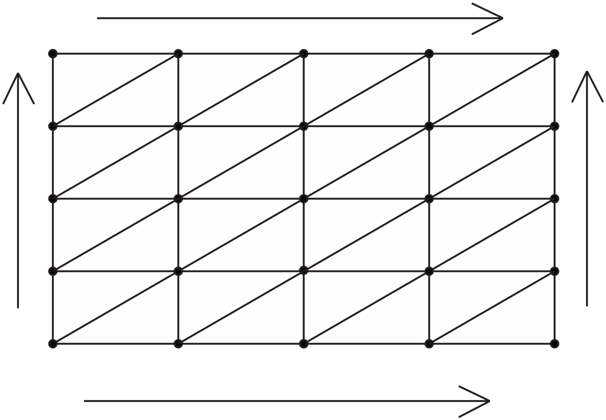
- $P_{K_1} = n$
- $P_{K_2} = \frac{nk}{2}$
- $P_{\overline{K}_2} = \binom{n}{2} - P_{K_2}$
- $P_{C_3} = \frac{\lambda}{3} P(K_2)$
- $P_{K_{1,2}} = \mu P(\overline{K}_2)$
- $P_{K_2 \cup K_1} = (k - \mu) P_{\overline{K}_2}$
- $P_{\overline{K}_3} = \binom{n}{3} - P_{C_3} - P_{K_{1,2}} - P_{K_2 \cup K_1}$

In the case where $\Gamma = srg(n, k, \lambda, \mu)$

- $P_{K_1} = n$
- $P_{K_2} = \frac{nk}{2}$
- $P_{\overline{K}_2} = \binom{n}{2} - P_{K_2}$
- $P_{C_3} = \frac{\lambda}{3} P(K_2)$
- $P_{K_{1,2}} = \mu P(\overline{K}_2)$
- $P_{K_2 \cup K_1} = (k - \mu) P_{\overline{K}_2}$
- $P_{\overline{K}_3} = \binom{n}{3} - P_{C_3} - P_{K_{1,2}} - P_{K_2 \cup K_1}$
- In general SRG, the value P_G of 4-vertex graph G is not necessarily determined only by parameters n, k, λ and μ.

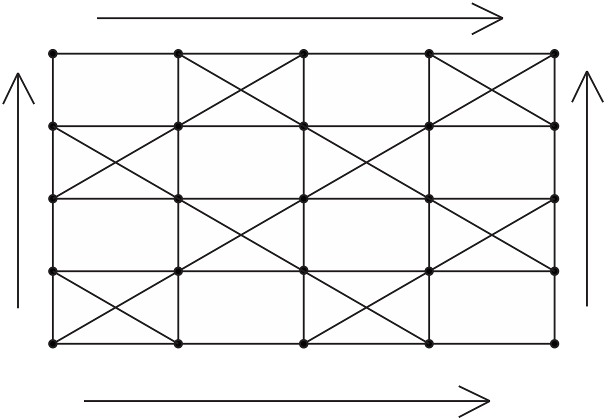
$SRG(16, 6, 2, 2)$

1) Shrikhande graph -> $P_{K_4} = 0$



$SRG(16, 6, 2, 2)$

2) $L_2(4)$ -> $P_{K_4} = 8$



Known results

Proposition (Hestenes, Higman (1970))

Let Γ be some tfSRG. The value T_G is uniquely determined by parameters of Γ for any graph G on at most 4 vertices.

Known results

Proposition (Hestenes, Higman (1970))

Let Γ be some tfSRG. The value T_G is uniquely determined by parameters of Γ for any graph G on at most 4 vertices.

Proposition

The number of occurrences of C_5 in any tfSRG is determined uniquely by n, k and μ

$$P_{C_5} = \frac{k}{10}(\mu + k\mu + k^2 - k)(k-1)(k-\mu)$$

Our method

- It is possible to compute the value of P_G for any graph G on 3 vertices using P_{K_2}, $P_{\overline{K_2}}$ and parameters of SRG.

Our method

- It is possible to compute the value of P_G for any graph G on 3 vertices using P_{K_2}, $P_{\overline{K}_2}$ and parameters of SRG.
- We generalized this idea and invented a method for computing the values P_G for all graphs on t vertices in $SRG(n, k, \lambda, \mu)$. It uses just numbers of occurrences of graphs of order $t-1$ in a given SRG and combinatorial properties following from its parameters.
- We also developed an algorithm which based on this idea. Its output is a description of values P_G as the functions of n, k, λ and μ.

Proposition

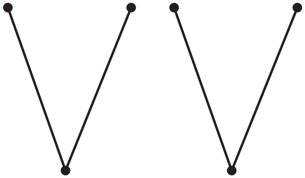
Let Γ be any $SRG(n, k, 0, \mu)$.

- *The value of P_G for any graph G on at most 5 vertices in Γ depends only on parameters n, k and μ.*
- *Let G be a graph on 6 vertices. The value of P_G depends on n, k, μ and $P_{K_{3,3}}$.*
- *If G is a graph on 7 vertices, then P_G depends on n, k, μ and the values of $P_{K_{3,3}}$ and $P_{K_{3,4}}$.*

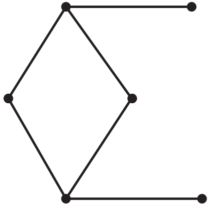
Examples

- $P_{G_1} = \frac{1}{8\mu^2}(\mu + k\mu + k^2 - k)k(k-1)(k^4 + 16\mu^4 + 10k^2\mu - 14k\mu^2 - 2k^3 + \mu^3 + 11\mu^2 + k^2 - 32\mu^3 k - 8k^3\mu + 24k^2\mu^2 - 4k\mu - 4\mu) - 9P_{K_{3,3}}$
- $P_{G_2} = \frac{1}{4\mu}(\mu+k\mu+k^2-k)k(k-1)(\mu-1)(k^2-3k\mu+2\mu^2)$

G_1



G_2



Results for the missing Moore graph

Proposition

Let Γ be $SRG(3250, 57, 0, 1)$.

- *The value P_G of any graph G on at most 9 vertices in Γ is determined uniquely.*
- *If G is a graph of order 10, P_G is determined uniquely by the number of occurrences of Petersen graph as induced subgraph in Γ.*

Applications

- We bounded the number of induced Petersen graphs in $SRG(3250, 57, 0, 1)$. The lower bound is still 0.

- There is 595 graphs on 10 vertices, which number of occurrences in $SRG(3250, 57, 0, 1)$ is a constant. $(P_{C_{10}} = 11457284326488000)$

Automorphism group of $\Gamma = SRG(3250, 57, 0, 1)$

- (Aschbacher 1971) Γ is not a rank three graph.
- (Higman) Γ cannot be vertex transitive.
- (Makhnev, Paduchikh 2001) Restriction on $Aut(\Gamma)$ for the case where Γ has an involutive automorphism.
- (Makhnev, Paduchikh 2008), (Mačaj, Širáň 2009) Restrictions on $Aut(\Gamma)$ for general case.

Automorphism group of $\Gamma = SRG(3250, 57, 0, 1)$

- (Aschbacher 1971) Γ is not a rank three graph.
- (Higman) Γ cannot be vertex transitive.
- (Makhnev, Paduchikh 2001) Restriction on $Aut(\Gamma)$ for the case where Γ has an involutive automorphism.
- (Makhnev, Paduchikh 2008), (Mačaj, Širáň 2009) Restrictions on $Aut(\Gamma)$ for general case.

- We restricted possibilities for automorphisms of order 7 using P_{C_7} in Γ.

Thank you for attention ☺