Interactions between Coherent Configurations and Some Classes of Objects in Extremal Combinatorics

Thesis submitted in partial fulfillment
of the requirements for the degree of
"DOCTOR OF PHILOSOPHY"

by
Matan Ziv-Av

Submitted to the Senate of Ben-Gurion University of the Negev

November 9, 2013
Beer-Sheva

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Approved by the Advisor
Approved by the Dean of the Kreitman School of Advanced Graduate Studies

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This work was carried out under the supervision of Mikhail Klin

In the Department of Mathematics

Faculty of Natural Sciences

Research-Student's Affidavit when Submitting the Doctoral Thesis for Judgment

I, Matan Ziv-Av, whose signature appears below, hereby declare that:

_x_ I have written this Thesis by myself, except for the help and guidance offered by my Thesis Advisors.

_x_ The scientific materials included in this Thesis are products of my own research, culled from the period during which I was a research student.

__ This Thesis incorporates research materials produced in cooperation with others, excluding the technical help commonly received during experimental work. Therefore, I am attaching another affidavit stating the contributions made by myself and the other participants in this research, which has been approved by them and submitted with their approval.

Date: November 9, 2013 Student's name: Matan Ziv-Av Signature: ________

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Interactions between Coherent Configurations and Some Classes of Objects in Extremal Combinatorics

Submitted by: Matan Ziv-Av
Advisor: Mikhail Klin

Abstract

A strongly regular graph with parameters (v, k, λ, μ) is a regular graph on v vertices with valency k, such that two neighbors have λ common neighbors and two non-neighbors have μ common neighbors.

An equitable partition of a graph is a partition of the vertex set such that the number of neighbors a vertex from cell X has in cell Y depends only on the selection of the cells, not the selection of vertex.

A coherent algebra of order n is a subalgebra of $M_n(\mathbb{C})$ that contains the matrices I_n and J_n (the all ones matrix), and is closed under transposition and Schur-Hadamard (entrywise) product. Such an algebra has a (unique) basis of $(0, 1)$ matrices whose sum is J_n. An association scheme over a set X is a partition of the cartesian product $X \times X$, such that the adjacency matrices of the relations in the partition form a basis of a homogeneous coherent algebra. The set of 2-orbits (orbitals) of a transitive permutation group is an association scheme. An association scheme which is the set of 2-orbits of a transitive permutation group is called Schurian, otherwise it is called non-Schurian.

A Schur ring (S-ring) over a group H is a subring of the group ring $\mathbb{Z}[H]$ which has a special basis. For our purpose, S-rings over H may be identified with association schemes whose automorphism group contains a regular subgroup isomorphic to H.

A graph is called semisymmetric if its automorphism group is transitive in its action on the edges of the graph, but intransitive in its action on the vertices of the graph. Such a graph is bipartite.

There are seven primitive known triangle-free strongly regular graphs (that is, connected strongly regular graphs with $\lambda = 0$, having also a connected complement). Using a computer, for each pair of graphs, we enumerated the number of embeddings of the small graph

into the larger graph. For the four smallest graphs, we enumerated all equitable partitions. For the three larger graphs, we enumerated all equitable partitions satisfying some symmetry condition. For a few of those embeddings and equitable partitions, we provide a theoretical proof or explanation.

All S-rings over groups of orders up to 47 were previously enumerated (using a computer). We extend this enumeration to three groups: A_5, $\mathrm{AGL}_1(8)$ and $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$ (of orders 60, 56 and 55 respectively). The results for $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$ are preliminary. Our results for A_5 confirm the theoretical classification of primitive S-rings over A_5. We provide theoretical explanation for the S-rings over A_5 and $\mathrm{AGL}_1(8)$.

Using an association scheme of order 56 and rank 20 and incidence double covers of graphs, we present new links between two well-known semisymmetric graphs on 112 vertices. One of the graphs is the cubic Ljubljana graph, and the other is the Nikolaev graph of valency 15.

Two interesting new non-Schurian association schemes are presented and discussed. One has rank 4 and order 125, and is related to the generalized quadrangle $GQ(4,6)$. It may be a member of a series of non-Schurian primitive association schemes on p^3 points. The other association scheme is of rank 6 and order 90. This scheme is related to the $(7,6)$-cage and to Baker's semiplane on 45 points. Together with the $(6,5)$-cage on 40 vertices, the $(7,6)$-cage is one of the two known non-Schurian coherent cages which do not appear as the incidence graph of a generalized quadrangle.

A well-known stabilization algorithm with polynomial time complexity (due to Weisfeiler and Leman) computes the coherent closure of a simple graph. We generalize this algorithm to an arbitrary colored directed graph Γ. A comparison of the output of the classical and of the new version of the WL-stabilization resulted in the discovery of new pseudo S-rings (with quite unusual properties) on $\frac{p(p-1)}{2}$ points, for $p \in \{7, 11, 19\}$.

Keywords: coherent configurations, association schemes, strongly regular graphs, Schur rings, semisymmetric graphs, computer algebra.

Chapter 1

Introduction

1.1 Motivation

The name Algebraic Graph Theory (AGT) is usually applied to the systematic investigation of graphs and related combinatorial structures that have high symmetry. Here, symmetry may be measured in different terms, depending on group-theoretical, spectral or combinatorial techniques.

A main tool of AGT is coherent configurations (in combinatorial incarnation) or coherent algebras (in algebraic incarnation). We use the language of coherent configurations to study triangle-free strongly regular graphs, S-rings, semisymmetric graphs and cages.

This area of mathematics used to be known as algebraic combinatorics (AC), but in recent years, AC was expanded to cover more research areas (such as polytopes, ordered sets, etc.), and what was originally called algebraic combinatorics was renamed algebraic graph theory.

In [51] and [87] we studied the $(6, 5)$-cage, also known as Robertson graph, on 40 vertices. This graph may be considered the seed of this thesis. Coherent cages, triangle-free strongly regular graphs and Schur rings are three subjects that immediately come to mind upon studying this graph.

When performing a computer experiment in AGT, we anticipate one of two results. Either we find a new example of a combinatorial object with a given set of properties, or we achieve a complete enumeration of all objects with those properties. From here, we perform *a posteriori* theoretical reasoning in an effort to obtain descriptions of greater clarity and simplicity. We distinguish among three such levels of description as follows.

Suppose we obtain a computer-generated description of, for example, an incidence structure $\mathfrak{S} = (P, B)$. By *explanation* of $\mathfrak{S}$, we mean a lucid, computer-free description of P, B, and the incidence between them. Essential use of a computer, or of additional manual calculations, are not required in this situation.

By *interpretation* of $\mathfrak{S}$, we mean that in addition to an explanation, we have a self-contained proof that $\mathfrak{S}$ indeed has its purported structure or properties. Ideally, an interpretation should be reasonably short and methodologically clear.

Finally, we may be able to *generalize* $\mathfrak{S}$ into an infinite (or a finite) series of similar objects, with some of the initial numerical properties parametrized.

1.1.1 Repeatability of results

The results of a computer program are not to be fully trusted. There might be a problem in the algorithm, a problem in the implementation of the algorithm, or a (permanent or transient) problem in the system used to run the implementation. For this reason, we prefer to repeat experiments.

The best case is using two different algorithms on two different systems. An example for this is the results on embeddings of tfSRGs in tfSRGs, which were calculated both in GAP and using an independent C program. Another example is computing merging association schemes using both COCO and COCO-II (when a problem fits within the limitations of both packages).

The second best case is running two different implementations of the same algorithm on different systems. An example of this is the computation of S-rings over A_5 and $\mathrm{AGL}_1(8)$, where both COCO-II and an independent implementation of COCO-II's algorithm, using a combination of GAP and C program, were used.

At the lowest level of confidence, we run the same implementation of an algorithm twice, to make sure that the result is not an artifact generated by some transient hardware error.

In addition to repeating the results, we also increase the confidence level for the results by running various smaller tests, or comparing the results with partial theoretical results. For example, the primitive S-rings over A_5 were already classified, so we compared our list of S-rings with those results, and made sure our list is not missing any primitive S-rings, nor does it contain any superfluous one.

1.2 Outline of the thesis

1.2.1 Preliminaries

This thesis starts with a chapter dedicated to preliminaries. In this chapter, we recall the mathematical terms used in the thesis, and cover definitions, basic properties, and summary of the current knowledge relevant to the results presented in the following chapters.

We begin with the definition of coherent configurations and association schemes (see [7], [20]), and their algebraic counterparts, coherent algebras. We discuss their three automorphism types and the relations among them. We also define the important concepts of mergings and coherent closures, and mention the Weisfeiler-Leman algorithm for calculation of coherent closure.

We then discuss strongly regular graphs ([38]), with special attention to those without triangles, that is, triangle-free strongly regular graphs (tfSRGs) ([62]). We describe the seven known (primitive) triangle-free strongly regular graphs, mentioning some of their algebraic and combinatorial properties.

We define equitable partitions of graphs, which are partitions of the vertex set of a graph that have a numerical "agreement" with the graph. The adjacency matrix of such a partition is related to the adjacency matrix of the graph. This makes equitable partitions a useful tool in algebraic graph theory.

Unlike the seven tfSRGs, which are highly symmetric, larger tfSRGs (such as a tfSRG on 162 vertices, the smallest open case, or the Moore graph of valency 57), if they exist, are known to have small automorphism groups ([58], [2]). Embeddings of small graphs into large graphs and equitable partitions are two tools that can be useful in constructing graphs with small automorphism groups, or proving their nonexistence. It is thus natural to study embeddings and equitable partitions in the context of the known tfSRGs, in order to employ the findings in the search for new tfSRGs.

Next, we recall the definition of Schur rings ([74, 82]) and their connection to association schemes. We briefly review the current status of the efforts to classify all Schur rings (S-rings) over finite groups.

Finally, we present two families of graphs: semisymmetric graphs ([31, 39]) and cages ([78]). Again we provide the definitions and initial theory of those families of graphs as they relate to our studies. We present details about two semisymmetric graphs on 112 vertices: Ljubljana and Nikolaev graphs.

We conclude the preliminaries with a discussion of the computer tools used in the study. These include the computer algebra system GAP, the package COCO for computations with coherent configurations, and some programs written specifically for our computational tasks.

1.2.2 Summary of results

In Chapter 3, we present the results of our computations relating to tfSRGs.

The first result concerns embeddings of tfSRGs into primitive tfSRGs. We completely enumerated all such embeddings. For each pair of graphs, we counted the number of embeddings of the smaller into the larger, and partitioned all those embeddings into orbits of the automorphism group of the larger graph. We note that in all but two cases, all embeddings are in the same orbit. The two exceptions are embeddings of Petersen graphs inside a Mesner graph and inside a Higman-Sims graph (henceforward called by its older, but lesser known name, $NL_2(10)$). For each orbit of embeddings, we also calculated the stabilizer of the embedding in the automorphism group of the larger graph.

We generalized some of the computerized results to theorems, which were proved without a computer. A formula for the number of pentagons inside of a tfSRG can be derived from its parameters by combinatorial arguments. The two cases of embeddings, of a Petersen graph into a Clebsch graph and of a Mesner graph inside $NL_2(10)$, are special cases of the general theory of negative Latin square graphs. Mesner proved that for each vertex v of a negative Latin square graph, the induced subgraph on all non-neighbors of v is a strongly regular graph, thus giving a lower bound for the number of such embeddings. We prove that each embedding of a graph with suitable parameters into a negative Latin square graph is of this type, thus the lower bound is the actual number of embeddings.

We also enumerated equitable partitions of the known tfSRGs. For the four smaller graphs, we enumerated all equitable partitions. For the Sims-Gewirtz graph, we enumerated all non-rigid equitable partitions, and for the two larger graphs, Mesner graph and $NL_2(10)$, we enumerated all automorphic equitable partitions.

Up to action of the automorphism group, it is easy to manually enumerate the 3 EPs of the pentagon and 11 EPs of the Petersen graph, all of which are automorphic.

A brute force search by a computer reveals that the Clebsch graph admits 46 equitable partitions, 38 of which are automorphic.

For the Hoffman-Singleton graph on 50 vertices, a brute force search is out of the question, but "cooperation" between a human and a machine allows us to divide the search space into small enough pieces that can be efficiently processed by a computer. This combined effort reveals all 163 EPs of the Hoffman-Singleton graph, 89 of which are automorphic.

For the Sims-Gewirtz graph, we settled for enumerating non-rigid partitions (i.e. those that are stabilized by a non-identity automorphism of the graph). There are 754 such EPs, and together with the partition with all cells of size 1, we have 755 partitions of this graph, though we do not know if they are all EPs of the Sims-Gewirtz graph.

While for the Sims-Gewirtz graph we only enumerated all non-rigid EPs, for the four smaller graphs, we enumerated all EPs, and all of them were non-rigid (except for the trivial partition into cells of size 1). Thus, we do not yet have an example of a non-trivial rigid equitable partition of any tfSRG.

For the two larger graphs, we enumerated automorphic equitable partitions. There are 236 automorphic EPs of the Mesner graph and 607 such EPs of $NL_2(10)$.

Most of the results that appear in Chapter 3 were published in [88]. In addition, the information that appears in Section 3.3.1 was published in Section 6 of [52].

Using a computer, we enumerated all S-rings over A_5. Up to action of $\mathrm{Aut}(A_5) = S_5$, there are 163 S-rings of ranks 60, 33, 32, 22 and any rank between 2 and 20. Of those S-rings, 19 are non-Schurian, with ranks ranging from 4 to 14.

A Schurian S-ring is defined by its group, so we identify the automorphism groups of those S-rings. For the 24 small groups (of order up to 7680), we provide the structure of the groups. For the 14 large groups of orders 14400 to 933120, calculation of the structure is both more time-consuming and less useful.

Of the remaining 106 very large groups, of orders more than 10^6, 77 may be explained as wreath products of smaller groups. This leaves 29 large groups that require another explanation.

For the 19 non-Schurian S-rings, a more sophisticated approach is needed. We discover that each of the 19 S-rings is a merging of at least one of 4 Schurian S-rings with automorphism groups of orders 720, 1320, 1920, and 7680 (Root S-rings).

We enumerated all S-rings over $\mathrm{AGL}_1(8)$ as well. There exist 129 S-rings (up to action of $\mathrm{Aut}(\mathrm{AGL}_1(8)) = \mathrm{A\Gamma L}_1(8)$ of order 168) of ranks 56, 32, 22, 20 and any rank from 18 to 2. Of those S-rings, 20 are non-Schurian.

In a similar manner to the S-rings over A_5, we describe the structure of 24 small groups of orders up to 3584, and of 56 out of the 63 very large groups which are wreath products.

For the non-Abelian group of order 55, we enumerate the symmetric

S-rings, of which there are 13 up to the action of the automorphism group. All of those S-rings are Schurian. We also present preliminary results of enumeration of all S-rings over $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$.

The results that appear in Chapter 4 were published in [53].

We present related coherent configurations on 56 and 112 vertices, which give interesting links between the Ljubljana graph, a cubic semisymmetric graph on 112 vertices, and the Nikolaev graph, a semisymmetric graph of valency 15 on 112 vertices. Among the relations, we note that the Ljubljana graph is a subgraph of the Nikolaev graph.

We calculate the coherent closure of the $(7,6)$-cage on 90 vertices, discovering that it is a rank 6 non-Schurian association scheme. Furthermore, the cage is one of the basic graphs of its closure, so it is a coherent graph.

The results that appear in Section 5.1 were published in [50].

Finally, in Chapter 6 we discuss avenues for future research based on the results presented in this thesis.

The existence of a new primitive tfSRG is one of the most daunting and difficult problems in modern algebraic graph theory. One possible way to attack this problem systematically for a prescribed set of parameters (for example, the smallest open case on 162 vertices) is to predict possible equitable partitions, such as those with a small number of cells, and to try to prove or disprove the existence of some of these partitions. Patterns discovered in equitable partitions of known tfSRGs may be generalized and used for consideration of tfSRGs with new parameter sets.

Enumeration of S-rings over A_5 may be used as a stencil for classification of S-rings over alternating groups, which is a necessary part of the classification of all S-rings.

The concept of a coherent cage was introduced quite recently, aiming to characterize among the known cages those that have a high combinatorial symmetry, resembling the symmetry of Moore graphs and the incidence graphs of generalized quadrangles.

While working on this thesis, some modification to the available computer tools was required. The program STABIL is an implementation of the Weisfeiler-Leman algorithm that stabilizes a symmetric matrix into a color matrix of a coherent algebra. We modified it to stabilize an arbitrary matrix by adding an initial symmetrizing step. We include a proof that the modified algorithm works for arbitrary matrices. The same generalized algorithm is also implemented in COCO-II.

Chapter 2

Preliminaries

2.1 Coherent configurations and association schemes

A *color graph* is a pair $(\Omega, \mathcal{R})$, where $\mathcal{R} = \{R_i | i \in I\}$ is a partition of the set Ω^2. We are mainly interested in a special class of color graphs with further properties.

2.1.1 Axioms and basic definitions

Let $(X, \mathcal{R} = \{R_1, \ldots, R_r\})$ be a color graph such that:

CC1 $R_i \cap R_j = \emptyset$ for $1 \leq i \neq j \leq r$;

CC2 $\displaystyle\bigcup_{i=1}^{r} R_i = X^2$;

CC3 $\forall i \in [1, r] \exists i' \in [1, r] R_i' = R_{i'}$, where $R_i' = \{(y, x) | (x, y) \in R_i\}$;

CC4 $\exists I' \subseteq [1, r] \displaystyle\bigcup_{i \in I'} R_i = \Delta$, where $\Delta = \{(x, x) | x \in X\}$;

CC5 $\forall i, j, k \in [1, r] \forall (x, y) \in R_k |\{z \in X | (x, z) \in R_i \wedge (z, y) \in R_j\}| = p_{ij}^k$,

then $\mathfrak{M} = (X, \mathcal{R})$ is called a *coherent configuration*. The relations in $\mathcal{R}$ are called *basic relations* of $\mathfrak{M}$. If $\mathcal{R} = \{R_0, \ldots, R_r\}$ are the basic relations of a coherent configuration $\mathfrak{M}$, then the graphs $\Gamma_i = (X, R_i)$ are called *basic graphs* of $\mathfrak{M}$, and their adjacency matrices $A_i = A(\Gamma_i)$ are called *basic matrices* of $\mathfrak{M}$.

The partition of Δ which exists by axiom CC4 induces a partition of X. Each member of this partition is called a *fiber* of $\mathfrak{M}$. Any relation of a coherent configuration is a subset of a Cartesian product of two fibers (not necessarily distinct).

The parameters p_{ij}^k are the *intersection numbers* of the configuration. See [81] and [40] for original definitions.

The *order* of a coherent configuration $\mathfrak{M} = (X, R)$ is $|X|$ and the *rank* of $\mathfrak{M}$ is $|R|$.

A coherent configuration that has $\Delta = \{(x, x)|x \in X\}$ as one of its basic relations is called *homogeneous*, or an *association scheme*. If $\mathfrak{M} = (X, R)$ is a rank $r+1$ association scheme, we will usually denote the reflexive relation, Δ, by R_0. The non-reflexive relations are called *classes* of $\mathfrak{M}$.

An association scheme is *primitive* if all basic graphs corresponding to its classes are connected. Otherwise it is *imprimitive*.

An association scheme is *symmetric* if all basic relations are symmetric (equivalently, all basic graphs are undirected, or all basic matrices are symmetric). If all relations of a coherent configuration are symmetric, this configuration must be an association scheme.

Let (G, Ω) be a permutation group. $g \in G$ acts naturally on Ω^2 by the rule $(x, y)^g = (x^g, y^g)$. Following H. Wielandt in [82], the orbits of this action, (G, Ω^2), are called the *2-orbits* of (G, Ω), denoted by $2 - orb(G, \Omega)$.

For every permutation group (G, Ω), $(\Omega, 2 - orb(G, \Omega))$ is a coherent configuration. A coherent configuration obtained in this way is called a *Schurian* coherent configuration. If G is transitive, then $(\Omega, 2 - orb(G, \Omega))$ is an association scheme.

Coherent configurations may be alternatively defined in the language of matrices. The adjacency matrix $A(R)$ of a relation R on X is a (0,1)-matrix $A(R) = (a_{ij})$ of dimension $|X| \times |X|$ such that $a_{ij} = 1$ if and only if $(i, j) \in R$.

If $(X, \{R_1, R_2, \dots R_r\})$ is a coherent configuration, and we look at the set of matrices $\mathcal{B} = \{A_1 = A(R_1), \dots, A_r = A(R_r)\}$, then axioms CC1 and CC2 say that $\sum A_i = J_{|X|}$ (where $J_{|X|}$ is the all one matrix of dimension $|X| \times |X|$). Axiom CC3 says that for each $A \in B$, $A^t \in B$. Axiom CC5 says that any product of two matrices from $\mathcal{B}$ is a linear combination of matrices in $\mathcal{B}$, or in other words, the matrix algebra generated by $\mathcal{B}$ is the same as the vector space generated by $\mathcal{B}$. Axiom CC4 says that the identity matrix, $I_{|X|}$, is in this algebra. This leads to the axiomatic definition of a coherent algebra (which is equivalent to coherent configuration) [41]:

Let $W \subseteq M_n(\mathbb{C})$ be a matrix algebra such that

CA1 W as a linear space over $\mathbb{C}$ has some basis, $A_1, A_2, \dots, A_r$, consisting of $(0, 1)$-matrices;

CA2 $\displaystyle\sum_{i=1}^{r} A_i = J_n$.

CA3 $\forall i \in [1, r] \exists i' \in [1, r] A_i^t = A_{i'}$;

CA4 $I_n \in W$.

Then W is called a *coherent algebra* of rank r and order n with the *standard basis* $\mathcal{B} = \{A_1, A_2, \dots, A_r\}$. We write $W = \langle A_1, \cdots, A_r \rangle$.

Each matrix of the standard basis is an idempotent under Schur-Hadamard product (entrywise product), so the algebra is closed under this operation.

This allows an alternative definition of a coherent algebra. A matrix algebra $W \subseteq M_n(\mathbb{C})$ is coherent if it is closed under Schur-Hadamard product and includes the matrices I_n and J_n.

For a coherent algebra $W = \langle A_1, \cdots, A_r \rangle$, we define the *color matrix* of W to be $\sum_{k=1}^{r} kA_k$. More generally, any linear combination of $A_1, \dots, A_r$ with distinct coefficients may also be considered a color matrix of W.

A coherent algebra is called *commutative* if it is commutative as a matrix algebra, and *symmetric* if all its matrices are symmetric. A coherent algebra is symmetric if and only if all matrices in standard basis are symmetric, so the corresponding coherent configuration is homogeneous and symmetric.

A symmetric coherent algebra is commutative, but the converse is not necessarily true.

A coherent algebra is *homogeneous* if its standard basis includes the identity matrix I_n. A homogeneous coherent algebra corresponds to an association scheme.

We will usually switch freely between relation (or graph) language and matrix language. In particular, the intersection numbers of a coherent configuration are also called *structure constants* of the corresponding coherent algebra.

The smallest non-Schurian association scheme $\mathfrak{M}$ is of order 15. For this scheme, $\mathrm{CAut}(\mathfrak{M}) = \mathrm{Aut}(\mathfrak{M})$, as was first noted in [10]. $\mathrm{AAut}(\mathfrak{M})$ is of order 2, twice larger than $\mathrm{CAut}(\mathfrak{M})/\mathrm{Aut}(\mathfrak{M})$.

2.1.2 Isomorphism and automorphism types

Let $\mathfrak{M}_1 = (X_1, \{R_1, \ldots, R_n\})$ and $\mathfrak{M}_2 = (X_2, \{S_1, \ldots, S_n\})$ be two coherent configurations. An *isomorphism* from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ is a bijection, $f : X_1 \to X_2$ such that there exists a permutation g of $[1, n]$ such that f maps R_i to $S_{g(i)}$ for all $1 \leq i \leq n$.

This definition of isomorphism gives two kinds of automorphisms:

If $\mathfrak{M} = (X, \{\Delta, R_1, \ldots, R_n\})$ is a coherent configuration, then $f \in Sym(X)$ is an automorphism (or *strong automorphism*) of $\mathfrak{M}$ if $R_i^f = R_i$ for all $i \in [1, n]$.

A permutation $f \in Sym(X)$ is a *color automorphism* (or *weak automorphism*) if $R_i^f \in \{R_1, \ldots, R_n\}$ for all $i \in [1, n]$.

The group of (strong) automorphisms of $\mathfrak{M}$ is denoted by $\mathrm{Aut}(\mathfrak{M})$ and the group of color automorphisms of $\mathfrak{M}$ is denoted by $\mathrm{CAut}(\mathfrak{M})$.

Proposition 1. *1.* $\mathrm{Aut}(\mathfrak{M}) \trianglelefteq \mathrm{CAut}(\mathfrak{M})$*;*

2. $\mathrm{CAut}(\mathfrak{M}) \trianglelefteq N_{S(X)}(\mathrm{Aut}(\mathfrak{M}))$*,*
$N_{S(X)}(G)$ *- normalizer of* G *in* $S(X)$*;*

3. If $\mathfrak{M}$ *is Schurian then* $\mathrm{CAut}(\mathfrak{M}) = N_{S(X)}(\mathrm{Aut}(\mathfrak{M}))$*.*

An *algebraic isomorphism* between two coherent configurations $\mathfrak{M}_1 = (X_1, \{R_1, \ldots, R_r\})$ and $\mathfrak{M}_2 = (X_2, \{S_1, \ldots, S_r\})$, which have structure constants ${}_1p_{ij}^k$ and ${}_2p_{ij}^k$ respectively, is a permutation ˆ of $[1, n]$ such that ${}_1p_{ij}^k = {}_2p_{\hat{i}\hat{j}}^{\hat{k}}$ for all $i, j, k \in [1, n]$.

The set of algebraic automorphisms of a coherent configuration $\mathfrak{M}$ is denoted by $\mathrm{AAut}(\mathfrak{M})$. Each color automorphism of $\mathfrak{M}$ induces an algebraic automorphism:

Proposition 2. $\mathrm{CAut}(\mathfrak{M})/\mathrm{Aut}(\mathfrak{M}) \leq \mathrm{AAut}(\mathfrak{M})$

An algebraic automorphism that does not arise from a color automorphism is called a *proper algebraic automorphism* ([46]).

2.1.3 Coherent closure

Coherent algebras are defined by closure conditions. Thus, the intersection of coherent algebras is again a coherent algebra. Each square matrix is contained in at least one coherent algebra, the whole matrix algebra, $\mathbb{C}^{n \times n}$, which is coherent. Therefore, we can define the *coherent closure* of a matrix A, denoted $\langle\langle A \rangle\rangle$, to be the smallest coherent algebra containing this matrix (or in other words, the intersection of all coherent algebras containing it).

An efficient (polynomial-time) algorithm for computing $\langle\langle A \rangle\rangle$ (for an adjacency matrix of a simple graph) was suggested by Weisfeiler and Leman ([81], [80]), and is frequently called the WL-stabilization of the (symmetric) matrix A.

1. Start with a matrix A.

2. Replace each entry A in the matrix with an indeterminate x_a.

3. Calculate $B = A \cdot A$. The indeterminates are independent and non-commuting.

4. If the number of distinct entries in B is larger than in A, then substitute matrix B for A, and go back to step 1.

5. If the number of distinct entries in B is equal to the number in A, B is the color matrix of $\langle\langle A\rangle\rangle$.

We call a graph $\Gamma = (V, E)$ *coherent* if E is one of the basic relations of the coherent closure $\langle\langle\Gamma\rangle\rangle$. In other words, a graph is coherent if it is a basic graph of a suitable coherent configuration.

This recently-introduced concept serves naturally as a combinatorial analogue of an arc-transitive graph, a concept that is defined in algebraic terms.

For example, each distance regular graph is a coherent graph.

2.1.4 Mergings

If W' is a coherent subalgebra of a coherent algebra W, then the corresponding coherent configuration $\mathfrak{M}'$ is called a *fusion* (or *merging configuration*) of $\mathfrak{M}$ (Note that in many cases, we abuse notation by referring to W and $\mathfrak{M}$ as the same object).

If W' is a coherent subalgebra of a coherent algebra W, then every matrix A in standard basis of W' is a (0,1)-matrix in W, so it is a sum of standard basis matrices of W. In coherent configuration language, if $\mathfrak{M}' = (\Omega, \mathcal{R}')$ is a merging of $\mathfrak{M} = (\Omega, \mathcal{R} = \{R_i | i \in I\})$, then there is a partition P of I, such that $\mathcal{R}' = \{\bigcup_{i\in B} R_i | B \in P\}$, hence the name merging.

In the case when $\mathfrak{M} = (\Omega, 2 - orb(G, \Omega))$ for a suitable permutation group G, overgroups of G in $S(\Omega)$ provide a natural origin for fusions of $\mathfrak{M}$. Thus, the most interesting fusions (in AGT) are the *non-Schurian* ones, that is, those that do not emerge from a suitable overgroup of (G, Ω). The existence of such fusions suggests that the original configuration has some combinatorial symmetry that is not of an algebraic origin.

For each subgroup $K \leq \mathrm{AAut}(\mathfrak{M})$, its orbits on the set of relations define a merging coherent configuration, which is called *algebraic merging* defined by K. Again, those algebraic mergings which are non-Schurian are of special interest as less predictable combinatorial objects.

If W' and W'' are coherent subalgebras of a coherent algebra W, such that W' and W'' are not isomorphic, and in addition there exists $\phi \in \mathrm{AAut}(W)$ that maps W' to W'', then W' and W'' form a pair of algebraic *twins* inside of W.

2.2 Triangle-free strongly regular graphs

2.2.1 Strongly regular graphs

A graph Γ is called a *strongly regular graph* (SRG) with parameters (v, k, λ, μ) if it is a regular graph of order v and valency k, and every pair of adjacent vertices has exactly λ common neighbors, while every pair of non-adjacent vertices has exactly μ common neighbors.

If $A = A(\Gamma)$ is the adjacency matrix of a simple graph Γ, then Γ is strongly regular if and only if

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

This implies that $(I, A, J - I - A)$ is a standard basis of a rank 3 homogeneous coherent algebra. In combinatorial notation, $(\Delta, \Gamma, \overline{\Gamma})$ are basic graphs of a rank 3 symmetric association scheme. The adjacency matrix of a strongly regular graph has exactly 3 distinct eigenvalues. For a strongly regular graph, we denote:

- $r > s$, the two eigenvalues of $A(\Gamma)$ different from k. r is always positive, while s is always negative;

- f, g as the multiplicity of the eigenvalues r, s respectively.

A formula for f and g is given by

$$f = \frac{1}{2}\left[(v-1) - \frac{2k + (v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right],$$
$$g = \frac{1}{2}\left[(v-1) + \frac{2k + (v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right].$$

A quadruple of parameters (v, k, λ, μ), for which f and g (as given by the preceding formulas) are positive integers, is called a feasible set of parameters. See [18] for a list of all feasible parameter sets for $v \leq 1300$ (and some sets for $v > 1300$) with information about known graphs for those parameter sets.

A strongly regular graph Γ is called primitive if both Γ and its complement $\overline{\Gamma}$ are connected. This is equivalent to primitivity of the related association scheme $(\Delta, \Gamma, \overline{\Gamma})$.

2.2.2 Triangle free strongly regular graphs

A graph Γ is called triangle-free if it admits no triangles, that is, cliques of size 3. If Γ is also a strongly regular graph, then it is called a *triangle- free strongly regular graph* (tfSRG for short). A graph is triangle-free if any two neighbors have no common neighbors, therefore a tfSRG is an SRG with $\lambda = 0$.

Dale Mesner ([62],[63]) considered feasible sets of parameters of tfSRGs with up to 100 vertices, coming up with Table 1.

Mesner defined a set of feasible parameters for a special kind of strongly regular graphs, calling them negative Latin square graphs. This set of feasible parameters is itself parametrized by two variables. Setting $\lambda = 0$ reduces the set to a subset parametrized by one variable. Those parameters for tfSRGs are denoted by $NL_g(g^2 + 3g)$. An $NL_g(g^2 + 3g)$ tfSRG has parameters $((g^2+3g)^2, g(g^2+3g+1), 0, g(g+1))$. In particular, the number of vertices of such a graph is a square, $(g^2 + 3g)^2$.

The table was further filled in 1960 by Hoffman and Singleton ([42]), who constructed and proved uniqueness of the SRG with parameters $(50, 7, 0, 1)$. The table was finally completed by Gewirtz in 1969, proving the existence and uniqueness of the SRG with parameters $(56, 10, 0, 2)$ ([35],[36]).

No.	v	k	λ	μ	Existence
1	5	2	0	1	Yes, pentagon
3	10	3	0	1	Yes, Petersen
16	16	5	0	2	Yes, Clebsch
15	28	9	0	4	No, 1956
34	50	7	0	1	?, Hoffman-Singleton
39	56	10	0	2	?, Gewirtz
50	64	21	0	10	No, 1956
64	77	16	0	4	Yes, 1956
94	100	22	0	6	Yes, 1956 (uniqueness 1964)

Table 1: Mesner's table of feasible parameter sets of tfSRGs

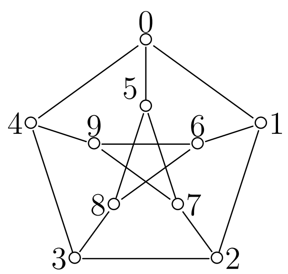

Figure 1: Petersen graph

2.2.3 The 7 known tfSRGs

2.2.3.1 Pentagon

The Pentagon with parameters $(5, 2, 0, 1)$. Its automorphism group is D_5 of order 10.

2.2.3.2 Petersen graph

The Petersen graph with parameters $(10, 3, 0, 1)$. Its automorphism group is isomorphic to S_5 of order 120. A simple model has as vertices 2-subsets of a set of size 5, with two vertices adjacent if the subsets are disjoint.

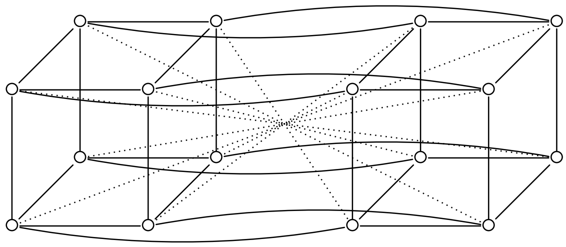

Figure 2: Clebsch graph

2.2.3.3 Clebsch graph

The Clebsch graph has parameters $(16, 5, 0, 2)$. Its automorphism group G is of order 1920. G is isomorphic to $(S_5 \wr S_2)^{pos}$, as well as to $E_{2^4} \rtimes S_5$ and to the Coxeter group D_5. Detailed investigation of this group is in [87].

The Clebsch graph is usually denoted by $\square_5$. It is also $NL_1(4)$ in Mesner's negative Latin square tfSRGs series.

A simple model is a 4-dimensional cube Q_4 together with long diagonals (Figure 2), or the Cayley graph:

$$\square_5 = Cay(E_{2^4}, 0001, 0010, 0100, 1000, 1111).$$

2.2.3.4 Hoffman-Singleton graph

The Hoffman-Singleton graph with parameters $(50, 7, 0, 1)$. Its automorphism group is isomorphic to $P\Sigma U(3, 5^2)$ of order 252000 ([9]). The simplest model is the Robertson model ([72], see also Figure 3): 5 pentagons marked $P_0, \ldots, P_4$ and 5 pentagrams marked $Q_0, \ldots, Q_4$ with vertex i of P_j joined to vertex $i + jk \pmod 5$ of Q_k.

2.2.3.5 Sims-Gewirtz graph

The Sims-Gewirtz (or Gewirtz) graph with parameters $(56, 10, 0, 2)$. Its automorphism group of order 80640 is a non-split extension of $PSL_3(4)$ by E_{2^2}. A simple model is the induced subgraph of $NL_2(10)$ on the common non-neighbors of two adjacent vertices.

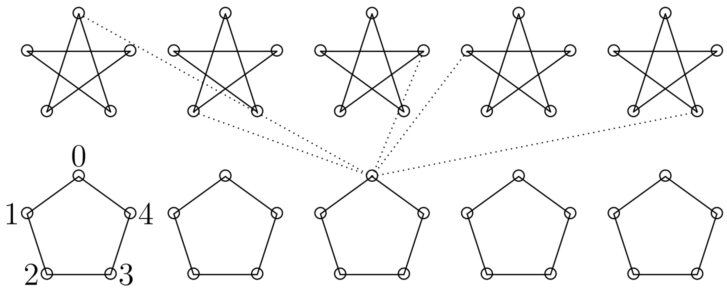

Figure 3: Hoffman-Singleton graph, Robertson model

Another classical model, which goes back to Ch. Sims, partitions the vertex set $V = O_1 \cup O_2 \cup O_3$.

$O_1 = \{v\}$ is a single vertex. O_2 of size 10 consists of all partitions of the set $[0, 5]$ into two 3-subsets. O_3 of size 45 consists of all sets of type $\{\{a, b\}, \{c, d\}\}$, where a, b, c, d are distinct elements of $[0, 5]$.

v is adjacent to all vertices of O_2. A partition $\{\{a, b, c\}, \{d, e, f\}\}$ is adjacent to $\{\{a, b\}, \{d, e\}\}$. Inside O_3, $\{\{a, b\}, \{c, d\}\}$ is adjacent to $\{\{a, c\}, \{e, f\}\}$ and to $\{\{a, e\}, \{b, f\}\}$.

We get an equitable partition with collapsed adjacency matrix

$$\begin{pmatrix} 0 & 10 & 0 \\ 1 & 0 & 9 \\ 0 & 2 & 8 \end{pmatrix}.$$

This equitable partition is the metric decomposition with respect to a vertex. This model of the graph is called the Sims model.

Simple combinatorial arguments reveal that the described graph Γ is a strongly regular graph with parameters $(56, 10, 0, 2)$.

2.2.3.6 Mesner graph

The Mesner graph with parameters $(77, 16, 0, 4)$. The automorphism group is of order 887040 and is isomorphic to the stabilizer of a point in the automorphism group of $NL_2(10)$. One simple model is: induced subgraph of $NL_2(10)$ on the non-neighbors of a vertex ([63]).

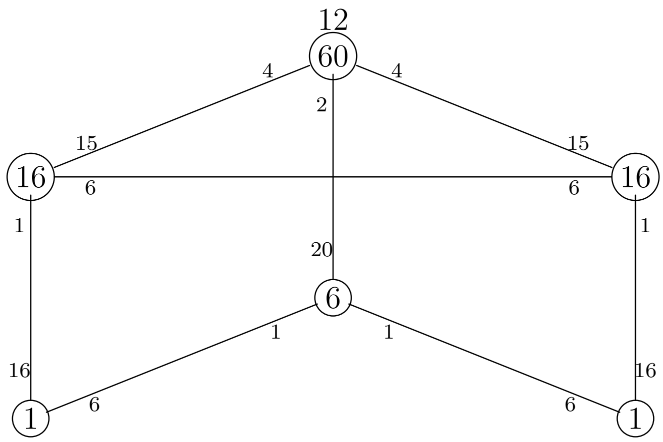

Figure 4: Mesner's model of $NL_2(10)$

2.2.3.7 $NL_2(10)$ (Higman-Sims graph)

This is the second graph in Mesner's negative Latin square series, $NL_2(10)$ with parameters $(100, 22, 0, 6)$. It is also (or mainly) known as the Higman-Sims graph. Its automorphism group contains the Higman-Sims sporadic simple group as a subgroup of index 2.

Figure 4 depicts an equitable partition corresponding to Mesner's model of $NL_2(10)$. See [54] for more details about Mesner's work on tfSRGs, and specifically about $NL_2(10)$.

2.3 Equitable partitions

2.3.1 Definitions and basic properties

Let $\Gamma = (V, E)$ be a graph. A partition $P = \{V_1, \ldots, V_s\}$ of V is called *equitable* with respect to Γ if for all $k, l \in \{1, \ldots, s\}$, the numbers $|\Gamma(v) \cap V_l|$ are constant for all $v \in V_k$. Here, $\Gamma(v) = \{u \in V | \{u, v\} \in E\}$ is the neighbor set of vertex v. Usually, an equitable partition of a graph is accompanied by an *intersection diagram*, which is a kind of quotient graph on which all numbers $|\Gamma(v) \cap V_l|$ are depicted.

The adjacency matrix of an equitable partition is a matrix $B = (b_{ij})$ where b_{ij} is exactly $|\Gamma(v) \cap V_j|$ for some $v \in V_i$.

Obviously, an adjacency matrix B of an equitable partition admits only natural numbers as entries, and if Γ is regular of valency k, then the sum of each row in B is k.

A useful fact in AGT is the following proposition:

Proposition 3. *Let Γ be a graph and $A = A(\Gamma)$ its adjacency matrix. If P is an equitable partition of Γ and B is the adjacency matrix of P, then the characteristic polynomial of B divides the characteristic polynomial of A.*

If H is a subgroup of $\mathrm{Aut}(\Gamma)$, then the set of orbits of H is an equitable partition of Γ. Such an equitable partition is called *automorphic*.

For any partition Q of the vertex set of a graph, there is an equitable partition P that is finer than Q but coarser than any other equitable partition that is finer than Q. P is called *equitable closure* of Q. An efficient algorithm for calculating the equitable closure is STABGRAPH:

1. For every element $v \in V$, v is in V_k, for every $1 \leq i \leq r$, $t_i = |\Gamma(v) \cap V_i|$. define $O_v = (k, t_1, \ldots, t_r)$.

2. Sort the set $\{O_v | v \in V\}$ lexicographically.

3. Define a new partition $P' = (V'_1, \ldots, V'_s)$ such that $v \in V'_j$ if the position of O_v in the sorted list is j.

4. If the number of cells in P' is the same as in P, stop, output is P'.

5. $P := P'$.

6. Go to step 1.

Sometimes we refer to an equitable closure of a set of vertices. By this, we mean an equitable closure of a partition with two cells: the set and its complement.

Given a subset W of the set V of vertices of a graph Γ, W induces a *metric* partition (or *metric decomposition*) of V, where two vertices are in the same cell if their distance from W is the same. The metric partition is not necessarily equitable, but when it is, it is the equitable closure of W.

An equitable partition of a graph naturally corresponds to a model of a graph. For example, in the standard diagram of the Petersen graph, we can see an equitable partition into two cells corresponding to inner and outer pentagons. The Robertson model of the Hoffman-Singleton graph may be thought of as an equitable partition into five Petersen graphs or into 10 pentagons.

Mesner construction of $NL_2(10)$ is actually a presentation of an equitable partition called non-edge decomposition.

2.3.2 Global vs. local approaches

A global picture of a graph Γ considers Γ as an entity, specifically allowing the understanding of the whole group $\mathrm{Aut}(\Gamma)$ (which is rank 3, in the case of the known tfSRGs).

By contrast, local models are formulated in terms of equitable partitions (or coherent configuration). They only rely on knowledge of a subgroup H of $\mathrm{Aut}(\Gamma)$. Proof of the existence of Γ in such models typically depends on ad hoc tricks (with or without the use of a computer). As we mentioned above, local models are of special significance, presenting possible patterns which may be emulated in attempts to construct new tfSRGs.

2.4 Schur rings

Schur rings (S-rings for short) were introduced by I. Schur in 1933 ([74]), and were later developed by H. Wielandt ([82]).

Recall that the group ring $\mathbb{C}[H]$ consists of all formal linear combinations of elements of the group H with coefficients from the field $\mathbb{C}$.

A Schur ring over the group H is a subring $\mathcal{A}$ of the group ring $\mathbb{C}[H]$, such that there exists a partition P of H that satisfies:

1. $\underline{P}$ is a basis of $\mathcal{A}$ (as a vector space over $\mathbb{C}$).

2. $\{e\} \in P$, where e is the identity element of H.

3. $X^{-1} \in P$ for all $X \in P$.

Here, for a subset X of H we define $X^{-1} = \{g^{-1} | g \in X\}$ and $\underline{X} = \sum_{x \in X} 1 \cdot x$, while for a set T of subsets we define $\underline{T} = \{\underline{X} | X \in T\}$.

Let (G, Ω) be a permutation group and H a regular subgroup of G. Then Ω may be identified with H. The stabilizer G_e of the identity element $e \in H$ defines an S-ring over H (see [82]). We denote this S-ring by $V(G, H)$.

An S-ring $\mathcal{A}$ is called *Schurian* if it is equal to $V(G, H)$ for a suitable overgroup (G, H) of a regular group (H, H). A group H is called a *Schur group* if all S-rings over H are Schurian. Schur [74] conjectured that all groups are Schur groups, or in other words, all S-rings are Schurian. The first examples of non-Schurian S-rings were presented by Wielandt in [82].

Let H be a group (using multiplicative notation) and S a subset of H. The Cayley graph $Cay(H, S) = (H, R)$ is a graph with vertex set H and with arc set $R = \{\langle x, sx \rangle | x \in H, s \in S\}$. A Cayley graph $Cay(H, S)$ is undirected if $S = S^{-1}$ and is connected if $H = \langle S \rangle$.

Let $\mathcal{A}$ be an S-ring over group H, $\mathcal{A} = \{T_0, T_1, \ldots, T_s\}$, where $T_0 = \{e\}$, $T_1, \ldots, T_s$ are the basic sets of $\mathcal{A}$. It follows from the definitions that $\underline{T_i} \cdot \underline{T_j} = \sum_{k=0}^{s} p_{ij}^k \underline{T_k}$ for suitable non-negative integers p_{ij}^k, $0 \leq i, j, k \leq s$. The numbers p_{ij}^k are called *structure constants* of $\mathcal{A}$. We also associate with $\mathcal{A}$ the color graph $\mathfrak{M} = (H, R_i)$, where for $0 \leq i \leq s$, R_i is the arc set of the Cayley graph $Cay(H, T_i)$. With this definition, we get a correspondence between S-rings and a special class of association schemes, called *translation association schemes*.

The concept of a rational S-ring over an Abelian group H goes back to Schur and Wielandt, see [82], where this concept, under the original name "S-ring of traces", is defined and investigated. It seems that the first use of the term "rational" can be attributed to Bridges and Mena ([14]) who, at that time, were not aware of the language of S-rings and were working with equivalent terminology.

Nowadays, this concept may be formulated (in a more or less classical manner) for a wider class of commutative association schemes. There are several possible ways to generalize it to the case of arbitrary association schemes (also including S-rings over finite groups). We use the following definition (cf. [49]):

Definition 1. *A graph* Γ *is called* rational *if the spectrum of its adjacency matrix is rational (in fact, integer). An association scheme (S-ring) is rational if all its basic graphs are rational.*

The S-rings over cyclic groups were classified by Leung and Man in [57, 56]. In 2004, Muzychuk, using the classification by Leung and Man, offered a complete solution for the isomorphism problem for circulant graphs ([65]).

Hanaki and Miyamoto ([64]) classified all association schemes of small order; specifically, all S-rings over groups of order up to 35.

Sven Reichard classified all S-rings over groups of order up to 47, as announced in [69].

2.5 Semisymmetric graphs

2.5.1 Definitions and basic facts

An undirected graph $\Gamma = (V, E)$ is called *semisymmetric* if it is regular (of valency k) and $\mathrm{Aut}(\Gamma)$ acts transitively on E and intransitively on V.

The proposition below is attributed by F. Harary to Elayne Dauber; its proof appears in [39] and [55].

Proposition 4. *A semisymmetric graph* Γ *is bipartite with the partitions* $V = V_1 \cup V_2$, $|V_1| = |V_2|$, *and* $\mathrm{Aut}(\Gamma)$ *acts transitively on both* V_1 *and* V_2.

The interest in semisymmetric graphs goes back to the seminal paper [31], where they were called admissible graphs. The word "semisymmetric" was suggested in [48].

Example 1 (The semisymmetric Folkman graph on 20 vertices)**.** *Let* $V_1 = \left\{ {[0,4] \atop 2} \right\}$ *be the set of all 2-element subsets of the 5-element set* $[0, 4]$. *Let*

$V_2 = [0,4] \times \{1,2\}$. Define $V = V_1 \cup V_2$, $E = \{\{\{a,b\},(a,i)\} | a,b \in [0,4], i \in \{1,2\}, a \neq b\}$. It is easy to check that the direct product of the symmetric group S_5 with S_2 acts transitively on the sets V_1, V_2, E. Moreover, this is the full automorphism group of the resulting graph $\mathcal{F} = (V,E)$. (For the proof, it is helpful to notice that $\mathrm{Aut}(\mathcal{F})$ acts primitively on V_1 and imprimitively on V_2.)

At first, interest in semisymmetric graphs was sustained by representatives of the Soviet school of graph theory. The paper [79] immediately attracted the interest of researchers from the USSR to the several open questions about semisymmetric graphs which were posed by Folkman in [31] and repeated in [79]. A general method to construct semisymmetric graphs with the aid of the multi-hypergraphs was suggested by V. K. Titov in [77]. Below, we present the semisymmetric graph on 24 vertices, constructed by Titov.

Example 2. *Let $V_1 = [0,3] \times \{1,2,3\}$, $V_2 = \left\{{[0,3] \atop 2}\right\} \times \{4,5\}$, $V = V_1 \cup V_2$, $E = \{\{(x,i),(\{x,y\},j)\} | x,y \in [0,3], x \neq y, i \in \{1,2,3\}, j \in \{4,5\}\}$. We suggest that the reader verify that the resulting graph $\mathcal{T} = (V,E)$ on 24 vertices and valency 6 is a semisymmetric graph with $|\mathrm{Aut}(\mathcal{T})| = 2^{13} \cdot 3^5$. Note that the group $\mathrm{Aut}(\mathcal{T})$ may be easily described as a generalized wreath product (in the sense of [85]) of the group S_4, acting on orbits of lengths 4 and 6 with groups S_3 and S_2, respectively. Here $|\mathrm{Aut}(\mathcal{T})| = 4! \cdot (3!)^4 \cdot (2!)^6$.*

Following [43] let us call a semisymmetric graph $\Gamma = (V,E)$, $V = V_1 \cup V_2$, of *parabolic type* if the stabilizers H_1, H_2 of vertices $x \in V_1$ and $y \in V_2$ respectively are not conjugate in the symmetric group $Sym(V)$ (Note that we slightly modify the original definition in [43]). If the two stabilizers are conjugate, the graph Γ is called of *non-parabolic type.*

For a semisymmetric graph of parabolic type, proving that it is semisymmetric is easier, since we can distinguish between vertices from the different parts with the aid of simple combinatorial arguments, using suitable numerical or structural invariants of the vertices. The above two examples serve as simple representatives of the parabolic case.

Let Γ be a bipartite graph with partition $V = V_1 \cup V_2$ of its vertices. In what follows, we assume that Γ is an edge-transitive regular graph of valency k. Then it follows from Proposition 4 that the group $\mathrm{Aut}(\Gamma)$ either acts transitively on V, or acts intransitively with two orbits V_1 and V_2 of equal length. Let us denote by $\mathrm{Aut}^-(\Gamma)$ the subgroup of $\mathrm{Aut}(\Gamma)$ which stabilizes each set V_1 and V_2 separately. Then clearly, $[\mathrm{Aut}(\Gamma) : \mathrm{Aut}^-(\Gamma)] =$ 1 or 2, depending on whether $\mathrm{Aut}(\Gamma)$ acts on V transitively or intransitively, respectively.

Definition 2. *Let $\Delta = (V, R)$ be a directed graph. Define a new undirected graph $\Gamma = (V(\Gamma), E(\Gamma))$, such that $V(\Gamma) = V \times \{1, 2\}$, $E(\Gamma) = \{\{(x, 1), (y, 2)\} | (x, y) \in R\}$.*

We will call Γ the *incidence double cover* (IDC for short) of Δ.

An alternative way to explain the construction of double cover involves the use of matrices. For an arbitrary graph Δ (directed arcs and loops are allowed), denote by $A(\Delta)$ its adjacency matrix. Clearly, any arbitrary square (0,1)-matrix is the matrix $A(\Delta)$ for a suitable graph Δ. However, we may interpret the matrix $A = A(\Delta)$ as the incidence matrix $I(S)$ of a suitable incidence structure S. Here, rows of $A = (a_{ij})$ correspond to points of S, while columns correspond to blocks of S. An element a_{ij} is equal to 1 if and only if the point defined by row i is incident to the block defined by row j. The result is that we consider the incidence (Levi) graph of the incidence structure S (cf. [23]). Note that the number of points in S is equal to the number of blocks. Such incidence structures are called *configurations*, if the incidence graph happens to be regular and does not contain quadrangles.

A survey of the general properties of this correspondence is provided in [19]. Note that this explanation justifies the name "incidence double cover".

2.5.2 Nikolaev graph $\mathcal{N}$

The graph $\mathcal{N}$ is a semisymmetric graph of valency 15 on 112 vertices, which was discovered on October 30, 1977 at Nikolaev (Ukraine). It is the first member of an infinite family of semisymmetric graphs. Its construction was presented in [48], where the term "semisymmetric" was coined. The main motivation of [48] was to provide an affirmative answer to a question posed by Folkman [31] about the existence of a semisymmetric graph with v vertices and valency k, such that $\gcd(v,k)=1$. Indeed, for the graph $\mathcal{N}$, we get $\gcd(112,15)=1$.

The construction of $\mathcal{N}=(V,E)$ is as follows:

Let the set of vertices $V=V_1\cup V_2$, $V_1=\{(a,b)|a,b\in[0,7],a\neq b\}$ and $V_2=\{X\subseteq[0,7]||X|=3\}$. The edge set E of $\mathcal{N}$ is $E=\{\{(a,x),\{a,b,c\}\}|x\notin\{a,b,c\}\}$.

Proposition 5. *(i) $\mathcal{N}$ is a semisymmetric graph with 112 vertices and valency 15;*

(ii) $\mathrm{Aut}(\mathcal{N})\cong S_8$.

Thus the graph $\mathcal{N}$ serves as a nice example of a parabolic case of semisymmetric graphs: here, as in Example 1, the fact that $\mathrm{Aut}(\mathcal{N})$ acts intransitively on the set V can be justified by simple arguments of a combinatorial or group-theoretic nature.

2.5.3 Ljubljana graph $\mathcal{L}$

In 2001, during a brief visit to Ljubljana, M. Conder together with Slovenian colleagues constructed a cubic semisymmetric graph on 112 vertices, which was described as a regular $\mathbb{Z}_2^3$-cover of the Heawood graph. Following Conder's suggestion, the graph was called the Ljubljana graph and denoted by $\mathcal{L}$. A computer-based search showed that $\mathcal{L}$ is the unique cubic semisymmetric graph on 112 vertices.

In fact, in [13], a reference was already given to a private communication by R. M. Foster, who found a cubic semisymmetric graph on 112 vertices with girth 10. However, Foster did not communicate to Bouwer any description of his graph. Thus, there was evident reason to attribute to this graph the new suggested name, inspired by the lucky reincarnation of $\mathcal{L}$ achieved in the capital of Slovenia. The graph $\mathcal{L}$ was also studied in a series of papers by I. J. Dejter and his coauthors [12], [16], [24], [25], which were not known to the authors of [21] in 2001. A detailed report about the graph $\mathcal{L}$ was published [21]. Soon after, the suggested name became well known, see e.g. [83]. We adopt the existing name, "Ljubljana graph". As sometimes happens in mathematics, some names seem luckier than others, and under this name, this graph is now enjoying a new wave of attention. A more involved computer search (announced already in [21]) revealed that the graph $\mathcal{L}$ is in fact the third smallest cubic semisymmetric graph.

The paper [21] indeed provides a lot of interesting information about the graph $\mathcal{L}$. The graph is defined in an evident form with the aid of voltage assignments; cycles of length 10 and 12 are completely classified; $\mathcal{L}$ is proved to be Hamiltonian and thus its LCF code (in the sense of [32]) is provided. The group $\mathrm{Aut}(\mathcal{L})$ of order 168 is discussed, together with its action on $\mathcal{L}$ and some subgroups. Moreover, it is shown that the edge graph $L(\mathcal{L})$ of $\mathcal{L}$ is a Cayley graph over $\mathrm{Aut}(\mathcal{L})$.

2.6 Cages

The cage notion goes back to W. T. Tutte (see e.g. [78]), who established the foundation of the theory for a particular case of cubic graphs (regular graphs of valency 3).

According to [73], for arbitrary $k \geq 3$ and $g \geq 3$ there exists at least one regular graph of valency k and girth g. A regular graph of valency k and girth g, such that there are no smaller graphs with the same valency and girth, is called a (k, g)-*cage* ([11]).

There is a natural lower bound for the number of vertices in a (k, g)-cage, commonly denoted by $n_0(k, g)$, which is formulated separately for odd and even girths (see [11]). Graphs that attain this bound are very rare (Moore graphs for g odd, and incidence (Levi) graphs of generalized polygons for g even).

The problem of description of (k, g)-cages is completely solved only for a small set of parameters k, g.

An important characteristic feature of the classical cages such as Moore graphs and Levi graphs of generalized quadrangles is that they are coherent, and moreover, they are distance regular. Therefore, a coherent closure of such a graph is a (metrical) association scheme.

In this context, it is natural to expect that those cages which are also coherent, are in a sense very close (from the algebraic graph theory standpoint) to the classical cages.

Cages of valency 3 are investigated with reasonable success; all of them are known for having girth of ten at the most, see e.g. [70].

The case of $(k, 3)$-cages is in a sense degenerate, these are complete graphs K_{k+1}. Cages of girth 4 are complete bipartite graphs.

Cages of girth 6 (projective planes) are classical objects of investigation in the area of finite geometries. The unique $(6, 5)$-cage will play a role in this thesis.

Below, we consider cages of girth 5 with more attention. It is well-known that non-trivial Moore graphs may exist only for $g = 5$, and there are only 3 possibilities for the valency, namely $k = 3$, $k = 7$ or $k = 57$, leading to strongly regular graphs with $k^2 + 1$ vertices. The unique Moore graph of valency 3 is the Petersen graph, and the unique Moore graph of valency 7 is the Hoffman-Singleton graph. The question as to the existence of a Moore graph of valency 57 is still open.

The cages of girth 5 and valency 3, 4, 5, 6, 7 have, respectively, 10, 19, 30, 40 and 50 vertices, all of which have been completely classified. Below, we consider valencies 6 and 7.

Following a pioneering paper by C. W. Evans ([26]), we consider in a given graph $\Gamma = (V, E)$ set $\mathfrak{S}_n$ of all cycles (circuits) of length n. Γ is called a *general net* if and only if there exists $\mathfrak{S}^* \subseteq \mathfrak{S}_n$ such that given any edge $e \in E$, there are exactly two cycles $C_1, C_2 \in \mathfrak{S}^*$ such that $e \in C_1$ and $e \in C_2$. In general, the girth $g \leq n$. When $g = n$, Γ will be called a *general g net*. Moreover, Γ is called a general g net cage of valency k if Γ is also a (k, g)-cage. An *embeddable net* may be drawn on a surface.

A number of net cages are investigated in [26], including K_4, Cube, Petersen graph and Heawood graph for valency 3. A net of valency 6 and girth 5 on 40 vertices was constructed by Evans. At the time of publication of [26], he was not precisely aware that this graph is a cage.

2.7 Computer tools

The use of computers in AGT allows for calculations which are infeasible by hand. This allows us to find new combinatorial objects, to enumerate objects with specific sets of properties, and to find algebraic features (such as automorphism group) of objects. These findings can be used to achieve theoretical results.

2.7.1 COCO

COCO is a set of programs used for dealing with coherent configurations.

It was developed in 1990-1992 in Moscow, USSR, mainly by Faradžev and Klin [29], [30].

The programs include:

- ind - a program for calculating induced action of a permutation group on a combinatorial structure;
- cgr - a program to calculate the centralizer algebra of a permutation group;
- inm - a program to calculate the structure constants of a coherent configuration;
- sub - a program to find fusion association schemes of a coherent configuration given its structure constants;
- aut - a program to calculate automorphism groups of a coherent configuration and its fusion association schemes.

Usually, these programs are used in the above order. This provides a computerized way to find all association schemes invariant under a given permutation group, together with their automorphism groups.

2.7.2 WL-stabilization

The Weisfeiler-Leman stabilization is an efficient algorithm for calculating the coherent closure of an adjacency matrix of a simple graph (see [81], [80], [4]). Two implementations of (a variation of) the WL-stabilization are available (see [3]), denoted by STABIL and STABCOL.

In [8], Bastert notes that the algorithm as implemented in STABIL and STABCOL only applies to symmetric matrices, while in general it is useful to apply it to any matrix. He created a third implementation, QWEIL, by adding an initial step to the algorithm, closing the matrix under transposition before commencing with the usual stabilization. For a matrix containing only integer values from 0 to $n-1$, the initial step can be defined as replacing A by $A+nA^T$.

While those implementations of the WL-stabilization are available, in many cases, we are only interested in finding out a lower bound for the rank of the closure (in order to prove that it is Schurian), in which case an ad hoc calculation is sufficient.

2.7.3 GAP

GAP [33], an acronym for "Groups, Algorithms and Programming", is a system for computation in discrete abstract algebra. It supports easy addition of extensions (packages, in GAP nomenclature) that are written in the GAP programming language, which can add new features to the GAP system.

One such package, which is very useful in AGT, is GRAPE [76]. It is designed for the construction and analysis of finite graphs. GRAPE itself is dependent on an external program, `nauty` [61], in order to calculate the automorphism group of a graph.

Another package is DESIGN, used for construction and examination of block designs.

GAP is used in the course of investigations in AGT in order to:

- Construct incidence structures (graphs, block designs, geometries, coherent configurations, etc.)

- Compute automorphism groups of such structures.
- Check regularity properties and parameters of structures.
- Find cliques in graphs, and substructures of given structures in general.
- Find the abstract structure of a group, as well as identify a permutation group.
- Find conjugacy classes of elements and subgroups of a group.

2.7.4 COCO v.2

While a lot of calculations in AGT are done in GAP, some algorithms or operations are only available in certain other programs discussed above. This results in a permanent necessity to translate the output of one program to a format that is acceptable as input to the other program.

The COCO v.2 initiative aims to re-implement the algorithms in COCO, WL-stabilization and DISCRETA as a GAP package. In addition, the new package should essentially extend the abilities of the current version, based on new theoretical results obtained since the original COCO package was written.

COCO v.2 is developed by S. Reichard and C. Pech, and is currently still in development.

2.7.5 Ad-hoc tools

While GAP is very useful for computation in AGT, its roots in algebra (specifically group theory) cause inefficiency in some combinatorial calculations. In some cases, implementing the same brute force search algorithms in C can result in reduction of memory use by a factor of 100, and speed increase by a factor of 1000. In more sophisticated cases, we don't implement the whole algorithm in C, but instead choose to compute some parts in GAP and other parts in C, thus enjoying the best of both worlds.

Chapter 3

Triangle-free strongly regular graphs

3.1 Embeddings of tfSRGs inside tfSRGs

As we saw in the description of the known tfSRGs, some of the constructions use smaller tfSRGs as a basis, or as a building block. Examples are the construction of the Petersen graph from two pentagons, and of the Hoffman-Singleton graph from 5 Petersen graphs. Therefore, a full knowledge of embeddings of tfSRGs inside tfSRGs may be useful when attempting to construct new tfSRGs.

Some embeddings of tfSRGs inside larger tfSRGs were already known, see for example the description of Higman-Sims in [17]. But to the best of our knowledge, no systematic complete description has ever been published.

A tfSRG subgraph of a tfSRG is an induced subgraph, due to the following Proposition:

Proposition 6. *A subgraph Δ of diameter 2 of a graph Γ with no triangles is an induced subgraph.*

Proof. If Δ is not induced, then there are vertices v, w that are adjacent in Γ and not adjacent in Δ. Since the diameter of Δ is 2, there is a vertex u

such that vuw is a path in Δ. But then, vuw is a triangle in Γ, which is a contradiction. □

3.1.1 Computer results

Table 2 lists the results of the computer enumeration of tfSRGs inside tfSRGs. Table 3 lists the number of orbits under the action of the automorphism group of the inclusive graph.

3.1.2 Theoretical view of some embeddings

The number of pentagons inside any tfSRG (known, or yet unknown) depends solely on the parameters v, k, μ. This can be shown with the aid of simple combinatorial arguments:

Proposition 7. *The number of pentagons inside an SRG with parameters* $(v, k, 0, \mu)$ *is* $\frac{vk(k-1)(k-\mu)\mu}{10}$.

Proof. There are v options to select a vertex, k options to select a second vertex, and $k-1$ options to select a third vertex. To select the fourth vertex, we want to select a neighbor of the third vertex which is not a neighbor of the first vertex (the pentagon is induced), therefore there are $k-\mu$ options. The fifth vertex is a neighbor of two non-neighbors, so there are μ options.

Every pentagon is constructed exactly 10 times in the above construction. □

By a theorem of Mesner ([62]), the induced graph on non-neighbors of a vertex in a Clebsch graph ($NL_1(4)$ in Mesner notation) is a Petersen graph. This gives us 16 Petersens inside a Clebsch graph, and since the Clebsch graph is vertex-transitive, they are all in the same orbit. We can see, without the use of a computer, that there are no other Petersen graphs inside a Clebsch graph:

Proposition 8. *If the induced subgraph on ten vertices of a Clebsch graph is isomorphic to a Petersen graph, then these ten vertices are the non-neighbors of a vertex of a Clebsch graph.*

Proof. Consider the induced graph Δ on the remaining 6 vertices. This graph has $40 - 20 - 15 = 5$ edges. Every pair of non-adjacent vertices in a Petersen graph has 1 common neighbor in the remaining vertices, so if the valencies of Δ are $a_1, \dots, a_6$, then $\sum_{i=1}^{6} \binom{5-a_i}{2} = 30$. The only solution (up to order) is $a_1 = 5$, $a_2 = \cdots = a_6 = 1$. $\square$

Similarly, for $NL_2(10)$, the induced graph on non-neighbors of a vertex is isomorphic to a Mesner graph. This gives us 100 Mesner graphs inside $NL_2(10)$, all in the same orbit. There are no more such graphs.

Proposition 9. *If the induced subgraph on 77 vertices of $NL_2(10)$ is isomorphic to a Mesner graph, then these 77 vertices are the non-neighbors of a vertex of $NL_2(10)$.*

Proof. Consider the induced graph Δ on the remaining 23 vertices. This graph has $1100 - \frac{77 \cdot 16}{2} - 77 \cdot 6 = 22$ edges. Every pair of non-adjacent vertices in a Mesner graph has 2 common neighbors in the remaining vertices, so if the valencies of Δ are $a_1, \dots, a_{23}$, then $\sum_{i=1}^{23} \binom{22-a_i}{2} = 2 \cdot 2310$. The only solution (up to order) is $a_1 = 22$, $a_2 = \cdots = a_{23} = 1$. $\square$

We can generalize these two propositions to all negative Latin square graphs.

Theorem 10. *let Γ be a $NL_g(g^2 + 3g)$ graph, and let V_1 be a subset of vertices such that the induced subgraph is an SRG with parameters $((g^2 + 2g - 1)(g^2 + 3g + 1), g^2(g + 2), 0, g^2)$. Then V_1 is the set of non-neighbors of a vertex of Γ.*

Proof. Recall that the parameters of Γ are $((g^2+3g)^2, g(g^2+3g+1), 0, g(g+1))$. Let $V_2 = V(\Gamma) \setminus V_1$, so $v = |V_2| = (g^2+3g)^2 - (g^2+2g-1)(g^2+3g+1) = 1 + g(g^2 + 3g + 1)$. Let Δ be the induced subgraph of Γ on V_2, the number of edges of Δ is

$$
\begin{aligned}
e = {} & \frac{(g^2 + 3g)^2 g(g^2 + 3g + 1)}{2} - \frac{(g^2 + 2g - 1)(g^2 + 3g + 1)g^2(g + 2)}{2} - \\
& - (g^2 + 2g - 1)(g^2 + 3g + 1)(g(g^2 + 3g + 1) - g^2(g + 2)) = \\
& = v - 1.
\end{aligned}
$$

The parameter μ (the number of common neighbors of two non-adjacent vertices in the graph) is g^2+g for Γ and g^2 for the induced subgraph, so the difference is g. This means that V_2 must include g common neighbors for every pair of non-neighbors in V_1.

If we denote the valency of vertex i in Δ by d_i, then the number of neighbors i has in V_1 is $g(g^2+3g+1)-d_i$. This means it is a common neighbor of $\binom{g(g^2+3g+1)-d_i}{2} = \binom{v-1-d_i}{2}$ pairs of (non-adjacent) vertices in V_1. Summing over all v vertices of Δ, we get:

$$\sum_{i=1}^{v}\binom{g(g^2+3g+1)-d_i}{2} = g(g^2+2g-1)(g^2+3g+1)\cdot$$
$$\cdot\frac{(g^2+2g-1)((g^2+3g+1)-g^2(g+2)-1)}{2} =$$
$$= (v-1)\binom{(v-2)}{2}.$$

The valency of vertex i in $\overline{\Delta}$ is $\overline{d}_i = v-1-d_i$, so

$$\begin{aligned}\sum_{i=1}^{v}\binom{g(g^2+3g+1)-d_i}{2} &= \sum_{i=1}^{v}\binom{v-1-d_i}{2} = \\ &= \sum_{i=1}^{v}\frac{\overline{d}_i(\overline{d}_i-1)}{2} = \\ &= \sum_{i=1}^{v}\frac{\overline{d}_i^2}{2} - \sum_{i=1}^{v}\frac{\overline{d}_i}{2}.\end{aligned}$$

Combining the two equalities, and recalling that $\sum_{i=1}^{v}\frac{\overline{d}_i}{2}$ is the number of edges of $\overline{\Delta}$, $e(\overline{\Delta}) = \frac{v(v-1)}{2} - (v-1) = \frac{(v-1)(v-2)}{2}$, we get

$$\sum_{i=1}^{v}\frac{\overline{d}_i^2}{2} = (v-1)\binom{(v-2)}{2} + e(\overline{\Delta})$$

multiplying by 2,

$$\sum_{i=1}^{v}\overline{d}_i^2 = (v-1)(v-2)(v-3) + (v-1)(v-2) = (v-1)(v-2)^2.$$

By Theorem 1 of [1], the maximum of the sum of squares for graphs on v vertices with $\binom{v}{2} - (v-1) = \binom{v-1}{2}$ edges is attained only on the graph with

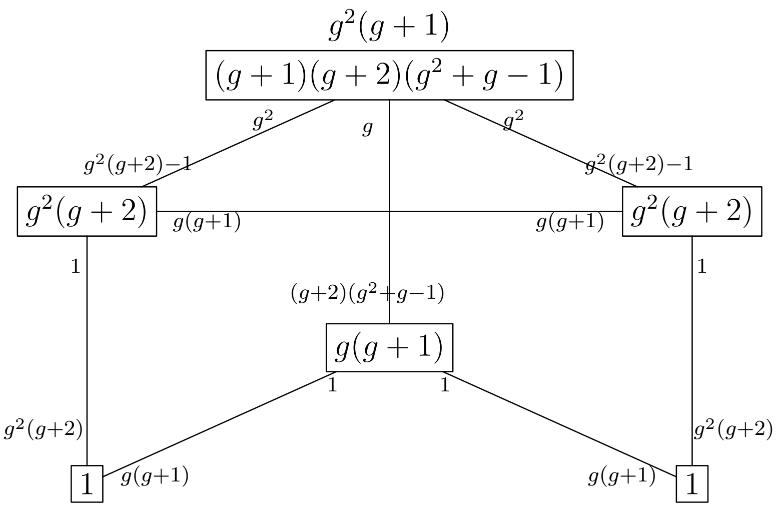

Figure 5: Mesner decomposition of $NL_g(g^2+3g)$

	Pentagon	Petersen	Clebsch	HoSi	Gewirtz	Mesner	$NL_2(10)$
Pentagon	1	12	192	1260	8064	88704	443520
Petersen		1	16	525	13440	1921920	35481600
Clebsch			1	0	0	0	924000
HoSi				1	0	0	704
Gewirtz					1	22	1030
Mesner						1	100
$NL_2(10)$							1

Table 2: Number of tfSRGs inside tfSRGs

one independent vertex and a clique with $v-1$ vertices (A quasi-complete graph on v vertices and $e = \frac{(v-1)(v-2)}{2}$, in the language of [1]), and this maximum is $(v-1)(v-2)^2$. This means that Δ must be a star graph, and V_1 is the set of non-neighbors of the vertex of maximal valency in the star. $\square$

	Pentagon	Petersen	Clebsch	HoSi	Gewirtz	Mesner	$NL_2(10)$
Pentagon	1	1	1	1	1	1	1
Petersen		1	1	1	1	9	5
Clebsch			1	0	0	0	1
HoSi				1	0	0	1
Gewirtz					1	1	1
Mesner						1	1
$NL_2(10)$							1

Table 3: Number of orbits of tfSRGs inside tfSRGs

	Pentagon	Petersen	Clebsch	HoSi	Gewirtz	Mesner	$NL_2(10)$
Pentagon	10	10	10	200	10	10	200
Petersen		120	120	480	6	6,6,6,2,6,2,6,6,6	240,24,6,6,48
Clebsch			1920				96
HoSi				252000			126000
Gewirtz					80640	40320	80640
Mesner						887040	887040
$NL_2(10)$							88704000

Table 4: Orders of stabilizers of tfSRGs inside tfSRGs

	HoSi	Gewirtz	Mesner	$NL_2(10)$
Pentagon	$(\mathbb{Z}_5 \rtimes \mathbb{Z}_4) \times D_5$	D_5	D_5	$(\mathbb{Z}_5 \rtimes \mathbb{Z}_4) \times D_5$
Petersen	$(\mathrm{SL}_2(5) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$\mathbb{Z}_6$	$S_3, S_3, \mathbb{Z}_6,$ $\mathbb{Z}_2, \mathbb{Z}_6, \mathbb{Z}_2,$ S_3, S_3, S_3	$\mathrm{SL}_2(5) \rtimes \mathbb{Z}_2,$ $(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2,$ $S_3, \mathbb{Z}_6, \mathrm{GL}_2(3)$
Clebsch				$\mathbb{Z}_4 \times S_4$
HoSi	$\mathrm{PSU}_3(5) \rtimes \mathbb{Z}_2$			$\mathrm{PSU}_3(5)$
Gewirtz		$(L_3(4) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$L_3(4) \rtimes \mathbb{Z}_2$	$(L_3(4) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$
Mesner			$\mathrm{M}_{22} \rtimes \mathbb{Z}_2$	$\mathrm{M}_{22} \rtimes \mathbb{Z}_2$
$NL_2(10)$				$\mathrm{HS} \rtimes \mathbb{Z}_2$

Table 5: Structure of stabilizers of tfSRGs inside tfSRGs

3.1.3 Imprimitive tfSRGs inside primitive tfSRGs

An imprimitive tfSRG has an even number of vertices, $2l$, and is either a complete bipartite graph $K_{l,l}$ or a regular graph of valency 1 (l edges, with no two of them having a common vertex).

In the case of complete bipartite graphs, when $l = 2$ we get a quadrangle. Since $l \leq \mu$, the case $l = 3$ is only relevant for Mesner and $NL_2(10)$. For both graphs, there is no induced subgraph isomorphic to $K_{3,3}$.

Proposition 11. *Let Γ be an SRG with parameters (v, k, λ, μ).*

1. *The number of quadrangles in Γ is $\frac{\frac{vk}{2}\binom{\lambda}{2}+\frac{v(v-k-1)}{2}\binom{\mu}{2}}{2}$. When $\lambda = 0$ it reduces to $\frac{v(v-k-1)\mu(\mu-1)}{8}$.*

2. *The number of edges in Γ is $\frac{vk}{2}$.*

3. *The number of pairs of two non-adjacent edges in Γ is $\frac{\frac{vk}{2}(\frac{vk}{2}-1-2k(k-1))}{2}+\frac{v(v-k-1)\mu(\mu-1)}{4}$.*

The results of a computer search are available in Tables 6 and 7. There is no induced subgraph of $NL_2(10)$ isomorphic to $12 \circ K_2$.

An interesting fact apparent from the tables is that for each graph, all the largest induced subgraphs of valency 1 are in the same orbit of the automorphism group.

	Quadrangle	edge	2 edges	3	4	5
Pentagon	0	5	0	0	0	0
Petersen	0	15	15	5	0	0
Clebsch	40	40	60	40	10	0
HoSi	0	175	7875	128625	845250	2170350
Gewirtz	630	280	15120	245280	1370880	2603664
Mesner	6930	616	55440	1330560	10589040	28961856
$NL_2(10)$	28875	1100	154000	5544000	67452000	301593600

	6 edges	7 edges	8	9	10	11
HoSi	1817550	40150	15750	3500	350	0
Gewirtz	1643040	104160	7560	1400	112	0
Mesner	24641232	3664320	166320	30800	2464	0
$NL_2(10)$	477338400	258192000	14322000	924000	154000	11200

Table 6: Number of imprimitive tfSRGs inside tfSRGs

	Quadrangle	edge	2 edges	3	4	5	6
Pentagon	0	1	0	0	0	0	0
Petersen	0	1	1	1	0	0	0
Clebsch	1	1	1	1	1	0	0
HoSi	0	1	1	4	10	21	15
Gewirtz	1	1	2	9	30	48	36
Mesner	1	1	1	7	26	56	50
$NL_2(10)$	1	1	1	2	7	14	17

	7 edges	8	9	10	11
HoSi	8	1	1	1	0
Gewirtz	5	2	2	1	0
Mesner	14	2	2	1	0
$NL_2(10)$	14	3	2	2	1

Table 7: Number of orbits of imprimitive tfSRGs inside tfSRGs

3.2 Equitable partitions of tfSRGs

As mentioned above, if larger tfSRGs exist, they are not as symmetric as the known tfSRGs. Thus, equitable partitions which correspond to models of a graph, and that do not rely on large automorphism groups, may be a useful tool in investigating larger tfSRGs. As a first step, we wish to have a better understanding of the equitable partitions of the known tfSRGs.

The goal is to enumerate all equitable partitions of the known triangle-free strongly regular graphs. For the Pentagon, the Petersen graph and the Clebsch graph enumeration can be easily done by a computer. For the Hoffman-Singleton graph, a combination of simple theoretical work and extensive computer search yields the desired enumeration. For the Sims-Gewirtz graph, we settled for an enumeration of non-rigid equitable partitions. For the Mesner graph and $NL_2(10)$, we enumerated all automorphic equitable partitions.

Table 8 summarises the number of equitable partitions and automorphic equitable partitions. In Tables 9–15, the information for each graph is given in more detail, according to the size (number of cells) of the equitable partitions.

	Pentagon	Petersen	Clebsch	HoSi	Gewirtz	Mesner	$NL_2(10)$
EP	3	11	46	163			
Aut	3	11	38	89	154	236	607

Table 8: Number of orbits of equitable partitions and of automorphic equitable partitions for known tfSRGs.

Size	1	3	5
EP	1	1	1
Aut	1	1	1

Table 9: Number of orbits of EPs and automorphic EPs of the pentagon by size of partition

Size	1	2	3	4	5	6	7	10
EP	1	2	2	2	1	1	1	1
Aut	1	2	2	2	1	1	1	1

Table 10: Number of orbits of EPs and automorphic EPs of the Petersen graph by size of partition

Size	1	2	3	4	5	6	7	8	9	10	12	16
EP	1	4	6	12	5	7	3	4	1	1	1	1
Aut	1	4	5	10	3	5	2	4	1	1	1	1

Table 11: Number of orbits of EPs and automorphic EPs of the Clebsch graph by size of partition

Size	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	18	20	28	30	50
EP	1	6	8	16	18	20	19	18	11	11	8	7	7	2	2	1	2	3	1	1	1
Aut	1	4	5	7	6	9	9	11	4	9	4	4	4	2	1	1	2	3	1	1	1

Table 12: Number of orbits of EPs and automorphic EPs of the Hoffman-Singleton graph by size of partition

Size	1	2	3	4	5	6	7	8	9	10	11	12	13
NR	1	34	131	108	83	68	63	63	61	49	32	14	9
Aut	1	5	9	12	12	14	15	16	14	11	7	7	6
Size	14	15	16	17	18	19	20	23	28	31	32	35	56
NR	9	3	7	6	3	2	4	1	1	1	1	1	0
Aut	5	0	3	3	2	2	4	1	1	1	1	1	1

Table 13: Number of orbits of automorphic and non-rigid EPs of the Gewirtz graph by size of partition

Size	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Aut	1	3	5	8	11	10	20	14	19	12	18	12	16	9	14
Size	16	17	18	19	20	21	23	25	29	33	41	45	49	77	
Aut	5	12	5	9	1	8	4	8	6	2	1	1	1	1	

Table 14: Number of orbits of automorphic EPs of the Mesner graph by size of partition

Size	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Aut	1	6	15	21	28	29	31	42	34	35	37	49	30	31	27
Size	16	17	18	19	20	21	22	23	24	25	26	27	28	29	
Aut	26	18	18	13	26	14	11	7	9	6	6	5	2	1	
Size	30	31	32	33	34	35	39	40	45	50	53	60	65	100	
Aut	7	1	3	2	2	3	1	4	1	1	1	1	1	1	

Table 15: Number of orbits of automorphic EPs of $NL_2(10)$ by size of partition

partition	|stabilizer|
$\{\{0\},\{1\},\{2\},\{3,7\},\{4,5\},\{6\},\{8,9\}\}$	12
$\{\{0\},\{1\},\{2,6\},\{3,8\},\{4,5\},\{7,9\}\}$	8
$\{\{0\},\{1\},\{2,6\},\{3,7,8,9\},\{4,5\}\}$	8
$\{\{0\},\{1,4,5\},\{2,8,9\},\{3,6,7\}\}$	12
$\{\{0\},\{1,4,5\},\{2,3,6,7,8,9\}\}$	12
$\{\{0,1\},\{2,4,5,6\},\{3,8\},\{7,9\}\}$	24
$\{\{0,1\},\{2,4,5,6\},\{3,7,8,9\}\}$	8
$\{\{0,1,3,7,8,9\},\{2,4,5,6\}\}$	24
$\{\{0,1,2,3,4\},\{5,6,7,8,9\}\}$	20

Table 16: Equitable partitions of a Petersen graph

3.2.1 Pentagon

The pentagon has three equitable partitions (up to the action of Aut(pentagon)), of which the only non-trivial one is the distance partition of a vertex. The stabilizer of this partition is the stabilizer of the vertex, of order 2.

From now on, we will ignore the two trivial equitable partitions.

3.2.2 Petersen graph

The Petersen graph admits 9 equitable partitions, which are easy to enumerate, either by hand or by a brute force computer search.

Using a standard enumeration of vertices (see Figure 1), representatives of the orbits of equitable partitions together with the order of their stabilizer are listed in Table 16.

3.2.3 Clebsch graph

There are 44 equitable partitions in a Clebsch graph. This is the last case where a brute force computer search is feasible. See Appendix A.1 for a summary of the results.

3.2.4 Hoffman-Singleton graph

The Hoffman-Singleton graph (HoSi) is an SRG with parameters $(50, 7, 0, 1)$.

$G = \mathrm{Aut}(HoSi)$ is of order $252000 = 50 \cdot 7!$ and is transitive on vertices, edges and non-edges. We are only looking to enumerate equitable partitions up to the action of G.

We use the specific representation of HoSi that was generated by an implementation in GAP of the Robertson model. Vertices are enumerated 1..50. By V_i we denote the set of neighbors of vertex i.

$V_1 = \{2, 5, 27, 33, 39, 45, 46\}$.

We will denote the non-1 neighbors of elements of V_1 by $W_1, \dots, W_7$ in the same order. $[1, 50] = \{1\} \cup V_1 \cup W_1 \cup \cdots \cup W_7$. Furthermore, this is a disjoint union.

For two vertices x, y, we denote their common neighbor by $x \cdot y$.

3.2.4.1 Summary of results

A summary of the enumeration of equitable partitions of a Hoffman-Singleton graph (up to action of the automorphism group) is in Table 17.

The count of partitions with a given number of cells does not include partitions with cells smaller than ten.

For each of these partitions, the stabilizer in $\mathrm{Aut}(HoSi)$ is a non-trivial subgroup.

3.2.4.2 Partitions with small cells

Let us enumerate the equitable partitions according to the minimal size of a cell in the partition.

Min size	No. cells	No. partitions	automorphic	non-automorphic
1		86	56	30
2		13	11	2
4		2	2	0
5		43	11	32
7		1	1	0
8		1	1	0
	2	6	4	2
	3	4	1	3
	4	3	0	3
	5	2	0	2
total		161	87	74

Table 17: Summary of equitable partitions of a Hoffman-Singleton graph

The word 'partition' has two meanings in mathematics. A partition of a set and a partition of a positive integer, that is, a way to write the number as a sum of positive integers. We will refer to the latter as an *integer partition*.

Partitions with the smallest cell of size 1 Let P be an equitable partition with a cell of size 1. Since G is transitive on vertices, we may assume this cell is $\{1\}$.

The rest of P is a union of two partitions, one of V_1 and the other of $T = V \setminus \{1\} \setminus V_1$ of size 42. Let us note that T is a disjoint union of the 7 sets of size 6: $W_x = V_x \setminus \{1\}$ for $x \in V_1$.

The stabilizer in G of 1, G_1 acts as S_7 on V_1, therefore we do not care about the actual partition of V_1, but only about its type (sizes of cells). There are 15 integer partitions of 7, therefore we have 15 types of partitions of V_1.

The partition of V_1 implies strong limitations on the partition of T: if x, y are two vertices in the same set in the partition of V_1, and T_1 is a set in the partition of T, then the intersections $T_1 \cap V_x$ and $T_1 \cap V_y$ are of the same size. Additionally, if x, y are not in the same set, then T_1 can't intersect both V_x and V_y.

Those limitations reduce the brute force search space to a manageable size.

- In the case that the partition of V_1 is $\{V_1\}$, then instead of considering all 2^{42} subsets of T as possible cells of the partition, we only need to consider $6^7 + 15^7 + 20^7 + 15^7 + 6^7 + 1$ subsets, a large but reasonable number. Since we are looking for equitable partitions, we only look at those subsets that induce a regular subgraph. This leaves only 8610 subsets, and the search can be completed in a short time, revealing 10 equitable partitions.

- If the partition of V_1 is $\{\{2\}, \{5, 27, 33, 39, 45, 46\}\}$, then the stabilizer of 1 and 2 acts on W_1 (neighbors of 2 other than 1) as S_6, so we have 11 ways to partition W_1. Each of those partitions limit the partition of the rest of T, similar to the way that the partition of V_1 does, so the search is easy. There are 27 equitable partitions of this type.

- If the partition of V_1 is $\{\{2, 5\}, \{27, 33, 39, 45, 46\}\}$, there are 8 equitable partitions.

- If the partition of V_1 is $\{\{2, 5, 27\}, \{33, 39, 45, 46\}\}$, there are 10 equitable partitions.

- If the partition of V_1 is $\{\{2\}, \{5\}, \{27, 33, 39, 45, 46\}\}$, there are 7 equitable partitions.

- If the partition of V_1 is $\{\{2, 5\}, \{27, 33\}, \{39, 45, 46\}\}$, there are 8 equitable partitions.

- If the partition of V_1 is $\{\{2\}, \{5, 27\}, \{33, 39, 45, 46\}\}$, there are 8 equitable partitions.

- If the partition of V_1 is $\{\{2\}, \{5, 27, 33\}, \{39, 45, 46\}\}$, there are 3 equitable partitions.

- If the partition of V_1 is $\{\{2\}, \{5\}, \{27\}, \{33, 39, 45, 46\}\}$, there are 5 equitable partitions.

- If the partition of V_1 is $\{\{2\},\{5,27\},\{33,39\},\{45,46\}\}$, there are 5 equitable partitions.
- If the partition of V_1 is $\{\{2\},\{5\},\{27,33\},\{39,45,46\}\}$ there are 6 equitable partitions.
- If the partition of V_1 is $\{\{2\},\{5\},\{27\},\{33,39\},\{45,46\}\}$, there are 3 equitable partition.
- If the partition of V_1 is $\{\{2\},\{5\},\{27\},\{33,\},\{39,45,46\}\}$, there are 4 equitable partition.
- If the partition of V_1 is $\{\{2\},\{5\},\{27\},\{33\},\{39\},\{45,46\}\}$, there are 2 equitable partitions.
- If V_1 is partitioned into 7 cells of size 1, then the partition of T is actually a union of 7 partitions of the sets W_x. Since each member of W_x has exactly one neighbor in W_y, the partition of W_1 determines the partition of each W_y, but for all but the two trivial partitions of W_1, the resulting partition is not equitable. Therefore, there are 2 partitions of this type (one of them is the discrete partition).

The equitable partitions in two different cases need not be in different orbits of the automorphism group. In total, there are only 87 equitable partitions with a cell of size 1 (including the discrete partition, which is not listed in the table).

Partitions with a smallest cell of size 2 There are two possibilities for a set of size 2: Edge and non-edge. If the partition contains a cell $\{x,y\}$ of non-neighbors, these non-neighbors have exactly one common neighbor ($\mu = 1$), and this common neighbor must be in a cell of its own, reducing to the previous case.

Therefore, the only case we need to consider is having a cell of two neighbors $\{1,2\}$ ($G = \mathrm{Aut}(HoSi)$ is edge-transitive).

Again, each cell in the partition of $V_1 \cup V_2$ has the same size intersection with V_1 and V_2, leaving us with exactly 877 cases to consider (9 cases up to G).

The selected partition of $V_1 \cup V_2$ has even stronger limitations on the partition of the other 36 vertices, leaving us with a short brute force search in all those cases.

There is one hard case where $V_1 \cup V_2$ is unpartitioned, yielding 4 equitable partitions and 8 simpler cases giving us 9 equitable partitions, for a total of 12 equitable partitions of this type.

Partitions with a smallest cell of size 3 If the smallest cell in the partition is of size 3, then this must be an independent set. Every pair of those vertices has a common neighbor. This cannot be a common neighbor of all three vertices, since such a vertex would have to be in a cell by itself.

Therefore, if we have a cell $\{a, b, c\}$, then $\{x = a \cdot b, y = b \cdot c, z = c \cdot a\}$ is also a cell of the partition, and $axbycz$ is a hexagon in HoSi. Up to G, there is only one hexagon in HoSi (computer result), so we have only one case to consider.

We get a third set of common neighbors (of the long diagonals of the hexagon): $\{r, s, t\}$.

$\{r, s, t\}$ have a common neighbor, thus killing this case.

Partitions with a smallest cell of size 4 If the partition has a cell of size 4, the induced subgraph on the four vertices is of valency 1 at most (HoSi has no quadrangles).

Valency 1 case:

If the edges are $\{ab\}$ and $\{cd\}$, then there are four more vertices: $x = a \cdot c$, $y = a \cdot d$, $z = b \cdot c$ and $w = b \cdot d$. These four must be distinct, and there can be no edge between them. But now, $x \cdot w$ and $y \cdot z$ must be in a cell by themselves in the partition, so this case is impossible.

Valency 0 case:

There are six common neighbors of two of the four vertices, and again there cannot be a common neighbor of three or more of the vertices.

Up to G, there are 5 sets of four independent vertices, only three of them have six distinct common neighbors, so these are the three cases we need to consider.

The set of 6 common neighbors has to be a cell of the partition.

In two of the three cases, there are $2,3$ more common neighbors of two elements of the set of 6 common neighbors (in addition to the four vertices in the starting cell). Those can't be in the same cell of the equitable partition as the other vertices, so we have a cell smaller than 4, contradicting our condition.

Therefore, only one case remains. Up to G, the set of size 4 is $\{1, 3, 6, 26\}$, and the six common neighbors are $\{2, 28, 33, 35, 46, 48\}$.

These two sets generate an equitable partition with sizes 4, 6, 16, 24.

The set of 24 can't be split (computer search). The set of size 16 can be split into two sets of size 8 (in one way, up to G).

We get 2 equitable partitions of this type.

Partitions with a smallest cell of size 5 A set of size 5 induces a Pentagon (only one, up to G) or an independent set of size 5 (10 up to G).

In the case of the Pentagon, Each of the five vertices has five more distinct neighbors, with each set of the partition intersecting each set of five neighbors with the same size.

This is enough to limit the search space, and accounts for 30 of the vertices. The remaining 20 are few enough for a brute force search.

There are 39 such equitable partitions.

The case of independent set:

For eight of the 10 independent sets, the equitable closure contains a cell of size 1, so they do not give an equitable partitions with minimal cell of size 5.

In the two remaining cases, the 45 vertices are divided into 10, 15, and 20 vertices according to the number of neighbors in the independent set of size 5. Finding equitable partitions of the parts of sizes 10 and 15 is easy. There are 15 such cases, and in each of them there are enough limitations on the set of 20 vertices to make the search of all partitions feasible.

There are four equitable partitions with a smallest cell of size of 5 where the cell of size 5 is an independent set.

There is no intersection between the two types.

Partitions with a smallest cell of size 6 An induced Hexagon has exactly three vertices with two neighbors in the hexagon, thus no required equitable partition.

There are four induced subgraphs of valency 1, and 18 independent sets of size 6. The equitable closure of each of those 22 subsets results in smaller cells, so there are no equitable partitions with smallest cell of size 6.

Partitions with a smallest cell of size 7 There is one induced Heptagon (up to G) and 31 independent sets of size 7.

The equitable closure of all but one independent set of size 7 includes a cell of smaller size.

Stabilization of this independent set gives an equitable partition with sizes 7, 7, 8, 28. The cell of size 28 can't be split. This partition is a refinement of an equitable partition with sizes 8, 14, 28.

Partitions with a smallest cell of size 8 There are two induced octagons, 10 induced subgraphs of valency 1 and 36 independent sets of size 8.

The equitable closure of all but one independent set of size 8 contains a cell of smaller size.

This independent set of size 8 has 14 non-neighbors and 28 neighbors of valency 2. This gives us an equitable partition.

The only numerically possible refinement of this partition (with minimum size 8) is done by splitting the set of size 28 in half. None of the splittings give an equitable partition (computer search).

Partitions with a smallest cell of size 9 These can be ruled out by considering the possible sizes of cells.

The size of a cell cannot be a prime number p larger than 7 (or a multiple of such a prime number). If there is a cell of such size, there must be an edge going between this cell and a cell with a size not divisible by p. In such a case, the equation $a_{ij}n_i = a_{ji}n_j$ has no solutions for $n_i, n_j \leq 7$.

This leaves only two possibilities with five cells:

$(9, 9, 9, 9, 14)$: There must be edges going out of the cell of size 14, but then we have $14a = 9b$, which is impossible for integer a, b, $a > 0$, $a, b \leq 7$.

$(9, 9, 10, 10, 12)$: There must be edges between 9 and 12, giving $4 \cdot 9 = 3 \cdot 12$, and between 12 and 10, giving $6 \cdot 10 = 5 \cdot 12$, but the sum of valencies of 12 must be 7, and it is already 8.

There are six possibilities with four cells:

$(9, 9, 12, 20)$: There must be edges between 9 and 12, giving $4 \cdot 9 = 3 \cdot 12$, and between 12 and 20, giving $3 \cdot 20 = 5 \cdot 12$, but the sum of valencies of 12 must be 7, and it is already 8.

The five other cases can be easily disqualified with similar arguments: $(15, 14, 12, 9)$, $(16, 15, 10, 9)$, $(16, 16, 9, 9)$, $(18, 14, 9, 9)$ and $(21, 10, 10, 9)$.

The four cases with three cells are disqualified with the same arguments: $(21, 20, 9)$, $(25, 16, 9)$, $(27, 14, 9)$ and $(32, 9, 9)$.

Alternatively: There are three induced cycles of length 9, and 33 independent sets of size 9. Equitable closure of all of them includes a cell of smaller size, so there are no equitable partitions of smallest cell of size 9.

3.2.4.3 Partitions with a small number of cells

If the smallest cell of the partition has a size of at least 10, then there are five cells in the partition at most.

In the case of five cells, all cells have to be of size 10.

For finding possible adjacency matrices, and sizes of cells of potential equitable partitions, we use the following reduced brute force search:

For k number of cells ($k \in \{2, 3, 4\}$), we first partition 50 into k integers, each being at least 10: $50 = n_1 + \cdots + n_k$.

For every possible solution, we find all possible adjacency matrices for an equitable partition. An adjacency matrix has only natural numbers as entries, and the sum of each row is 7 (valency of the Hoffman-Singleton graph).

Denote elements of the adjacency matrix a_{ij}.

The following must hold for all i, j:

- $n_i a_{ii}$ must be even.
- a_{ii} must be less than:
  - 3 if $n_i < 10$;
  - 4 if $n_i < 19$;
  - 5 if $n_i < 30$;
  - 6 if $n_i < 40$;

  (those upper limits for a_{ii} are the valencies of the cages of girth 5).
- If $n_i > 15$ then $a_{ii} > 0$, since the largest independent set in HoSi is of size 15 (computer result).
- $a_{ij} n_i = a_{ji} n_j$.

In addition, we recall that the characteristic polynomial of the adjacency matrix of an equitable partition divides the characteristic polynomial of HoSi.

Partitions with 2 sets With the above constraints, we get five feasible sets:

$10, 40$: $\begin{pmatrix} 3 & 4 \\ 1 & 6 \end{pmatrix}$

A regular subgraph on 10 vertices with valency 3 is a Petersen graph. There is one orbit of Petersen graphs in HoSi, and it is obvious in the Robertson model that it is indeed an equitable partition, so there is exactly one such equitable partition.

$15, 35$: $\begin{pmatrix} 0 & 7 \\ 3 & 4 \end{pmatrix}$

There is one independent set of size 15 (up to G).

$20, 30$: $\begin{pmatrix} 1 & 6 \\ 4 & 3 \end{pmatrix}$

Brute force search for the set of size 30 is feasible.

There is only one way (up to G) to select a vertex v and three neighbors v_1, v_2, v_3. There are then 20 ways to select two neighbors for each of the three neighbors: $v_{11}, v_{12}, v_{21}, v_{22}, v_{31}, v_{32}$.

Now there are 24 vertices left which are not adjacent to any of v, v_1, v_2, v_3, and we need to select 20 of them.

There is one such equitable partition.

$20, 30$: $\begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$

A brute force search is feasible: Start with any vertex and any four neighbors (only one selection up to action of G). For each of the four, select three more vertices ($\binom{6}{3}^4$ options). Now select three more vertices out of the remaining 33.

There are two such equitable partitions.

$25, 25$: $\begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}$

A cell in such a partition is a union of induced cycles. Furthermore, cycles that are in the same cell cannot have edges between them.

There are induced cycles of sizes 5, 6, 7, 8, 9, 10, 12, 13, 16, 18.

There are 19 integer partitions of 25 to these sizes.

Except for the integer partition of $25 = 5 + 5 + 5 + 5 + 5$, there are no induced cycles of the required sizes without connecting edges between them. For the integer partition $25 = 5 + 5 + 5 + 5 + 5$, the only equitable partition is the obvious one in the Robertson model (pentagons and pentagrams).

Partitions with three cells There are four feasible matrices:

$20, 15, 15$: $\begin{pmatrix} 1 & 3 & 3 \\ 4 & 0 & 3 \\ 4 & 3 & 0 \end{pmatrix}$

There is only one option for one independent set of size 15, and then only one option for a second set.

$$25, 15, 10\colon \begin{pmatrix} 2 & 3 & 2 \\ 5 & 2 & 0 \\ 5 & 0 & 2 \end{pmatrix}$$

This is a refinement of a partition of the type $25, 25$. There is one such partition, and one way to divide it into 15 and 10 with the desired valencies.

$$20, 20, 10\colon \begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

The cell of size 10 must be a Petersen graph. This leaves a feasible brute force search for the other two cells.

The only partition of this type is the obvious one in the Robertson model.

$$30, 10, 10\colon \begin{pmatrix} 5 & 1 & 1 \\ 3 & 3 & 1 \\ 3 & 1 & 3 \end{pmatrix}$$

The two cells of size 10 must be Petersen graphs.

The only partition of this type is the obvious one in the Robertson model.

Partitions with four cells There are two feasible matrices:

$$15, 15, 10, 10\colon \begin{pmatrix} 2 & 3 & 0 & 2 \\ 3 & 2 & 2 & 0 \\ 0 & 3 & 2 & 2 \\ 3 & 0 & 2 & 2 \end{pmatrix}$$

There are 88200 induced decagons in HoSi, in two orbits. No pair of decagons has the necessary valencies.

$$20, 10, 10, 10\colon \begin{pmatrix} 4 & 1 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 2 & 1 & 1 & 3 \end{pmatrix}$$

There is only one way to select two disjoint Petersen graphs. There are six ways to select the third disjoint Petersen graph. This results in three equitable partitions (up to G).

Partitions with five cells The only feasible matrix is

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{pmatrix}$$

There are two ways to select five mutually disjoint Petersen graphs inside HoSi. Both are non-automorphic with automorphism group of order 5.

3.2.5 Gewirtz graph

The previous method does not help in reducing the search space to a manageable size, since much of the reduction in size resulted from the fact that $\mu = 1$.

We can find all equitable partitions that are invariant under the action of some non-identity automorphism. Such an equitable partition will be invariant under the action of an automorphism of prime order. There are seven of these up to conjugacy, with 32, 12, 31, 20, 35, 28, and 8 orbits. They give 476, 53, 488, 57, 152, 134, and 35 equitable partitions (up to the action of the automorphism group of the Gewirtz graph).

Altogether, there are 755 non-rigid equitable partitions, 153 of which are automorphic. The distribution by size of partitions (number of cells) is available in Table 13.

This leaves the question of equitable partitions that have no non-identity stabilizing automorphism (rigid equitable partitions) open. In fact, we don't have any examples of such partitions for any of the graphs, except for the discrete partition.

3.2.6 Mesner graph

For this graph, we enumerated all automorphic equitable partitions. This is straightforward in GAP. There are 236 automorphic EPs (up to the action of the automorphism group of the Mesner graph). Their distribution by size is available in Table 14.

3.2.7 $NL_2(10)$

For this graph, we enumerated all automorphic equitable partitions. GAP cannot calculate all subgroups of the automorphism group of $NL_2(10)$ directly, but this group appears in the atlas of finite simple groups ([84]) with all its maximal subgroups. All those maximal subgroups are small enough for GAP to calculate their subgroups, thus we can get a list of all subgroups. There are 607 automorphic EPs (up to the action of the automorphism group of $NL_2(10)$). Their distribution by size is available in Table 15.

3.3 Understanding some EPs

During our research, we used the discovered EPs to construct many models of the known tfSRGs. We now present a few of these models.

3.3.1 Some models of the Hoffman-Singleton graph

We start the first model presentation with a classical simple observation:

Proposition 12. *There are six distinct, pairwise isomorphic, 1-factorizations of the graph K_6. Each of these has automorphism group S_5, acting 3-transitively on six points.*

Let $\mathcal{F}$ be a representative 1-factorization of K_6 with vertex set $[0,5]$, namely

$$\begin{aligned}\mathcal{F} =&\{\{\{0,1\},\{2,4\},\{3,5\}\},\{\{0,2\},\{1,5\},\{3,4\}\},\{\{0,3\},\{1,2\},\{4,5\}\},\\ &\{\{0,4\},\{1,3\},\{2,5\}\},\{\{0,5\},\{1,4\},\{2,3\}\}\}.\end{aligned}$$

It is convenient to regard the considered copy of K_6 as a subgraph of K_7 with isolated vertex 6.

Let $\Omega_1 = \{\emptyset\}$, $\Omega_2 = [0,6]$ and $\Omega_3 = \mathcal{F}^{S_7}$, where $\mathcal{F}^{S_7}$ is the orbit of $\mathcal{F}$ under action of $S_7 = \mathrm{Aut}(K_7)$. Denote $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$. Clearly, the symmetric group $S_7 = S([0,6])$ acts naturally on Ω with orbits Ω_1, Ω_2, Ω_3.

Thus, we may consider the coherent configuration $\mathcal{H} = (\Omega, 2\text{-orb}(S_7, \Omega))$. Using COCO, we obtain that:

a) $\mathcal{H}$ is a rank 15 configuration with three fibers of size $1, 7, 42$. Its type is $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{pmatrix}$ with valencies $\begin{pmatrix} 1 & 7 & 42 \\ 1 & (1,6) & (36,3) \\ 1 & (6,1) & (1,30,5,6) \end{pmatrix}$.

b) Merging of the relations $1, 3, 7, 10, 14$ of respective valencies $7, 1, 6, 1, 6$ provides a copy of the graph HoSi. $\square$

In (a) above, the labeling of relations is lexicographic: first down columns, then across rows. For example, the second column in the matrix of valencies indicates that the relations R_3, R_4, R_5, R_6, R_7 have respective valencies $7, 1, 6, 6, 1$.

This model corresponds to the equitable partition into three cells of sizes 1,7, and 42, which is called the vertex partition of HoSi.

For the second model, consider the group $D = D_5 \times \mathrm{AGL}(1, 5)$ of order 200 acting intransitively on a set of cardinality 50. It defines a coherent configuration $\mathcal{X}_D$ of rank 29 with three fibers of size $5, 25, 20$. Two Schurian fusions of $\mathcal{X}_D$ correspond to the rank 3 Hoffman-Singleton association scheme. In fact, the configuration $\mathcal{X}_D$ corresponds to the stabilizer of an arbitrary pentagon in HoSi. The automorphic partition of this stabilizer is the pentagon metric partition. This equitable partition is evident in the Robertson model of HoSi.

For the third model, we utilize the stabilizer G in $\mathrm{Aut}(HoSi)$ of a Petersen subgraph P in HoSi. There are 525 copies of P in HoSi, all belonging to the same orbit of $\mathrm{Aut}(HoSi)$. We obtain that G is a group of order 480. The corresponding coherent configuration $\mathcal{X}_G$ is of rank 16 with two fibers of size 40 and 10. Configuration $\mathcal{X}_G$ has a unique rank 3 fusion, which is the HoSi association scheme. This model corresponds to the only equitable partition with two cells of sizes 10 and 40. The induced graph on the cell of size 10 is a Petersen graph, while the induced graph on the cell of 40 vertices is the $(6, 5)$-cage.

3.3.2 Some models of the Gewirtz graph

Let Γ be the Gewirtz graph with parameters $(56, 10, 0, 2)$. $G = \mathrm{Aut}(\Gamma)$ is a transitive group of order 80640. The stabilizer of a vertex v in G, G_v has order $\frac{80640}{56} = 1440$. In fact, G_v is isomorphic (as an abstract group) to $\mathrm{Aut}(S_6)$. G_v acts faithfully and 2-transitively on the set of 10 neighbors of v. In the Sims model of Γ, we see the action of S_6 on this set of 10 neighbors.

3.3.2.1 D_5 automorphic partition

Let us now consider a copy of the dihedral group D_5 of order 10 in $\mathrm{Aut}(S_6)$. Up to conjugacy, there are two subgroups of $\mathrm{Aut}(S_6)$ isomorphic to D_5. We are interested in the one that is not a subgroup of S_6.

Using GAP, we selected a copy of D_5 (which is not a subgroup of S_6) and investigated its automorphic partition. This partition has nine cells of sizes 1, 5 and 10.

3.3.2.2 Pentagon partition

We start from any pentagon P in Γ (all are in the same orbit under action of G).

All other vertices can be classified by the number of neighbors they have in P. There are 16 vertices with no neighbors in P, 30 vertices with one neighbor, and five vertices with two neighbors. We label by Q the set of five vertices with two neighbors in P. We then count how many neighbors in Q vertices of the sets of sizes 16 and 30 have. We note that they are split further into cells of sizes 1, 15, 10 and 20.

What we described are two steps of the STABCOL algorithm, so what we get is a candidate for the equitable closure of P. It turns out that this candidate is indeed an equitable partition, thus it is the equitable closure of P. The collapsed matrix of this partition with cells of sizes 1,10,5,5,15,20 is:

$$\begin{pmatrix} 0 & 10 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 3 & 4 \\ 0 & 2 & 2 & 2 & 0 & 4 \\ 0 & 2 & 2 & 0 & 6 & 0 \\ 0 & 2 & 0 & 2 & 2 & 4 \\ 0 & 2 & 1 & 0 & 3 & 4 \end{pmatrix}.$$

This equitable partition is not automorphic, as its stabilizer is D_5 which has 9 orbits.

We wish to describe the whole graph Γ in a self-contained form, relying only on P. For this, we need to consider a modification of the Sims model such that the whole group G_v of order 1440 is shown, rather than only half of it.

We start by introducing a suitable description of $\mathrm{Aut}(S_6)$.

3.3.2.3 Tutte-Coxeter graph

The Tutte-Coxeter graph or the Tutte eight-cage is a bipartite cubic graph on 30 vertices. It is the unique $(3, 8)$-cage.

An old model of this graph, Δ, attributed to Sylvester, has one side of the graph (V_1) comprised of the 15 edges of the complete graph, K_6, while the 15 perfect matchings (1-factors) of K_6 comprise the other side (V_2). Adjacency is defined naturally by inclusion.

The automorphism group of this graph, $H = \mathrm{Aut}(\Delta)$ of order 1440 is isomorphic to $S_6 \rtimes Z_2 = \mathrm{Aut}(S_6)$.

In our presentation, we will denote the vertices of K_6 by the integers $[0, 5]$, and edges of K_6 simply by two digits, so one edge of Tutte-Coxeter graph is $\{01, \{01, 23, 45\}\}$.

3.3.2.4 Another model of Sims-Gewirtz graph

$O_1 = \{v\}$.

There are ten independent sets of size 12 in Δ with six vertices in V_1 and six vertices in V_2. The six vertices in V_1 in each set are the six edges of two triangles in K_6 that do not share a vertex. The six vertices in V_2 are the six matchings where the edges have a vertex in each triangle. The set of these ten independent sets is denoted by O_2. An example element of O_2: $\{01, 02, 12, 34, 35, 45, \{03, 14, 25\}, \{03, 15, 24\}, \{04, 13, 25\}, \{04, 15, 23\}, \{05, 13, 24\}, \{05, 14, 23\}\}$. Clearly, H acts transitively on O_2.

There are 45 quadrangles in K_6. For each of them, there are four 1-factors which do not share an edge with the quadrangle, and include exactly one of its diagonals. Thus we have 45 independent sets of size 8 in Δ, having four vertices in each side. O_3 is the set of those 45 independent sets. H acts transitively on O_3. An example element of O_3: $\{01, 12, 23, 03, \{02, 14, 35\}, \{02, 15, 34\}, \{13, 04, 25\}, \{13, 05, 24\}\}$.

Let us define a graph Γ. The set of vertices $V = O_1 \cup O_2 \cup O_3$. v is adjacent to all vertices of O_2. A vertex of O_3 is adjacent to a vertex of O_2 or O_3 if they are disjoint. Clearly, $H = \mathrm{Aut}(\Delta)$ (in its action on O_2 and O_3) is a subgroup of $\mathrm{Aut}(\Gamma)_v$.

It is easy, but tedious, to check that Γ is a strongly regular graph with parameters $(56, 10, 0, 2)$, and by the uniqueness of the Sims-Gewirtz graph, it is isomorphic to it.

3.3.2.5 Pentagon partition (revisited)

Let us select a special pentagon inside O_3, one in which a vertex of K_6 appears five times: $P = \{(0, 1, 2, 3), (0, 2, 4, 5), (0, 1, 3, 4), (0, 2, 5, 3), (0, 4, 1, 5)\}$. The stabilizer of this pentagon in S_6 is $H_0 = \langle(1, 2, 4, 3, 5)\rangle$.

There are 15 vertices in O_3 that have no neighbors in P. Representatives are $(0, 2, 3, 4)$, $(0, 1, 5, 2)$, $(0, 1, 4, 3)$.

There are 20 vertices that have one neighbor in P. These are represented by $(0, 1, 3, 2)$, $(0, 1, 5, 3)$, $(1, 2, 4, 3)$, $(1, 2, 5, 4)$.

There are five vertices that have two neighbors in P: $(1, 2, 3, 4)$.

O_2 is compatible with this partition of O_3, so we get a partition of Γ with cells of sizes $1, 10, 5, 5, 15, 20$, and the desired collapsed adjacency matrix:

$$\begin{pmatrix} 0 & 10 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 3 & 4 \\ 0 & 2 & 2 & 2 & 0 & 4 \\ 0 & 2 & 2 & 0 & 6 & 0 \\ 0 & 2 & 0 & 2 & 2 & 4 \\ 0 & 2 & 1 & 0 & 3 & 4 \end{pmatrix}.$$

On one hand, this revisited model is slightly sophisticated. However, its advantage is that the entire Gewirtz graph Γ is fully described in terms of auxiliary structure with a small symmetry group of order 10.

3.3.3 Quadrangle EP of $NL_2(10)$

Proposition 13. *Let* Γ *be an SRG with parameters* $(100, 22, 0, 6)$*. Let* τ *be a metric decomposition with respect to a prescribed quadrangle* Q *inside of* Γ*. Then* τ *is EP with four cells of sizes 24, 64, 8, and 4, and a collapsed adjacency matrix (with respect to this ordering of cells),* $B = \begin{pmatrix} 2 & 16 & 4 & 0 \\ 6 & 14 & 1 & 1 \\ 12 & 8 & 0 & 2 \\ 0 & 16 & 4 & 2 \end{pmatrix}$.

Proof. Let V_3 be a quadrangle in Γ.

Every other vertex of Γ has 0, 1, or 2 neighbors in V_3. Let V_i be the set of vertices having i neighbors in V_3, for $i = 0, 1, 2$.

For $i, j \in [0, 3]$, let a_{ij} be the number of neighbors that a vertex of V_i has in V_j. We need to show that the a_{ij}'s exist, that is, that they do not depend on the selection of the vertex in V_i.

By definition, $a_{33} = 2$, $a_{23} = 2$, $a_{13} = 1$, $a_{03} = a_{30} = 0$.

Two adjacent vertices in V_3 have no common neighbors. Two non-adjacent vertices have six common neighbors, two of which are in V_3, so the other four are in V_2. There are two pairs of non-adjacent vertices in V_3, so $|V_2| = 8$, $a_{32} = 4$.

We accounted for six neighbors of each vertex in V_3, so each has 16 neighbors in V_1, so $|V_1| = 64$, $a_{31} = 16$.

There remain 24 vertices, so $|V_0| = 24$.

From non-edge equitable partition of Γ (Figure 4 on page 20), we learn that there is no subgraph isomorphic to $K_{3,3}$ in Γ (because common non-neighbors of two non-neighbors v, u only have two neighbors in common with v, u). Therefore, there are no edges inside V_2, since an edge creates a triangle or a $K_{3,3}$. Thus, we get $a_{22} = 0$.

Let us now split the cell of size 6 into a cell B_1 of two vertices from V_3 and a cell B_2 of four vertices from V_2. This splits the cell of size 60 into: a

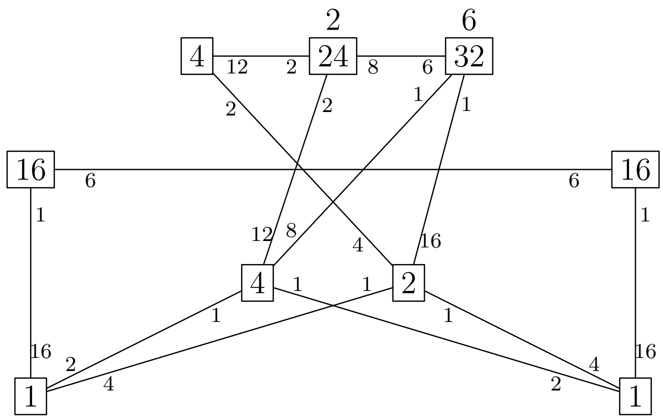

Figure 6: A split of the non-edge decomposition of $NL_2(10)$

cell C_2 of size 4 of vertices that have two neighbors in B_1, a cell $C_0 = V_0$ of size 24 of vertices that have no neighbors in B_1, and a cell C_1 of 32 vertices that have one neighbor in B_1. The intersection diagram of the resulting equitable partition is shown in Figure 6.

C_1 is a subset of V_1, and each of its elements has exactly one neighbor in B_2. Thus, each vertex of V_1 has no neighbors in the four vertices of V_2 which have a common neighbor with it (since that closes a triangle), and has exactly one neighbor in the other four vertices of V_2, so $a_{12} = 1$.

If we construct the non-edge partition with the non-edge B_1 instead of the original, then the cell of size 60 in this new partition consists of the 32 vertices from the two cells of size 16 in the original partition, together with the 24 vertices from C_0 and the four vertices from B_2. Each vertex of V_2 has 12 neighbors inside this cell, and they all have to be in C_0. And since each such vertex has 20 neighbors in C_0 and C_1, it has exactly eight neighbors in C_1, thus $a_{21} = 8$.

Each vertex has 22 neighbors, thus a_{20} exists and $a_{20} = 12$.

Each vertex of C_0 has two neighbors in B_2 (since it has two neighbors in B_1 and B_2, but none in B_1), and four in each of the cells of size 16, thus it has $12 - 2 - 4 - 4 = 2$ neighbors inside $C_0 = V_0$, so $a_{00} = 2$.

Each vertex of C_0 has four neighbors in each cell of size 16, thus it has exactly eight neighbors in C_1 (since C_1 is the two cells of size 16 in the non-edge partition from B_1). Therefore $a_{01} = 16$, and since each vertex has 22 neighbors, $a_{02} = 4$.

There are no edges between C_1 and C_2 (since that would close a triangle), thus each vertex of C_2 has 12 neighbors in C_0. Each vertex in C_0 has two neighbors in B_2, so it also has exactly two neighbors in C_2, since $a_{02} = 4$. Since the 32 vertices of C_1 are two cells of 16 in non-edge partition, each vertex in C_1 has exactly six neighbors in C_1, and therefore also six neighbors in C_0, so $a_{10} = 6$.

Finally, since the valency of vertices in V_1 is 22, $a_{11} = 14$, and indeed we have an equitable partition with collapsed adjacency matrix:

$$\begin{pmatrix} 2 & 16 & 4 & 0 \\ 6 & 14 & 1 & 1 \\ 12 & 8 & 0 & 2 \\ 0 & 16 & 4 & 2 \end{pmatrix}$$

□

Chapter 4

Schur rings over A_5, $\mathrm{AGL}_1(8)$ and $\mathbb{Z}_{11}\rtimes\mathbb{Z}_5$

Theoretical investigation of S-rings over Abelian groups seems to be approaching its climax, see for example [27]. Less is known about non-Abelian groups, though some progress is already visible ([71]).

All S-rings over groups of order up to 47 were classified with the help of a computer. Thus, we decided to try to extend the computerized classification for larger non-Abelian groups.

The groups we chose are A_5 of order 60, $\mathrm{AGL}_1(8)$ of order 56 and $\mathbb{Z}_{11}\rtimes\mathbb{Z}_5$ of order 55. The group A_5 is especially interesting, being the smallest non-cyclic simple group.

We managed to completely enumerate all S-rings over the groups A_5 and $\mathrm{AGL}_1(8)$. For $\mathbb{Z}_{11}\rtimes\mathbb{Z}_5$, we had to settle for an enumeration of the symmetric S-rings. We also have preliminary results for the enumeration of all S-rings over $\mathbb{Z}_{11}\rtimes\mathbb{Z}_5$, but that calculation has not been repeated yet.

The first stage of the search for all S-rings consists of searching for "good sets", that is, sets of relations whose union is a relation of a fusion scheme (for details, see Section 3 in [29]). For a group with k elements of order 2, and $2h$ elements of order larger than 2, the number of candidates for good sets is $2^{h+k} + 3^h$ (the first summand is for symmetric good sets, and the second is for antisymmetric good sets).

For A_5 we get $2^{37}+3^{22}$, for $\mathrm{AGL}_1(8)$ we get $2^{31}+3^{24}$, and for $\mathbb{Z}_{11}\rtimes\mathbb{Z}_5$ we get $2^{27}+3^{27}$. For groups of this size, the search for good sets is the most time-consuming step of the algorithm, so we expect the search for $\mathrm{AGL}_1(8)$ to take about twice as long as for A_5, while the search for (specifically, antisymmetric) good sets over $\mathbb{Z}_{11}\rtimes\mathbb{Z}_5$ to take about 50 times longer than the search over A_5. With the work for A_5 taking about one month of CPU time, this explains why the search for antisymmetric good sets over $\mathbb{Z}_{11}\rtimes\mathbb{Z}_5$ is the most problematic.

Please note that the search for good sets is fully parallelizable, so instead of one CPU for 4 years, it can be completed by (e.g.) a 4 core system in one year or a thousand CPU cores in a day or two.

4.1 S-rings over A_5

4.1.1 Computer results

All S-rings over the group A_5 were enumerated with the aid of COCO-II; the job took about 1 month of computer time on a 3GHz CPU.

A summary of the results: there are 2848 S-rings in 163 orbits under the action of the group $S_5 = \mathrm{AAut}(A_5)$. Among them, 505 S-rings (in 19 orbits) are non-Schurian. A complete list of orbit representatives, along with some information about each S-ring, is presented in [86]. Here, the full group ring $\mathbb{C}[A_5]$ has number 0, and the rank 2 S-ring has number 162. From now on, we refer to the enumeration of the orbits in this catalogue.

The results we obtained appear as a massive list of computer-generated data; we wish to transform this into a form that is more suitable for a human being. Schurian S-rings may, in principle, be explained in group theoretical terms. Non-Schurian S-rings are a subject of particular interest in AGT. Each such object requires special attention and analysis.

The general distribution of S-rings with respect to rank is provided in Table 18. Eight of the 19 orbits of non-Schurian S-rings are non-commutative (with ranks between 9 and 14). Up to isomorphism, there are nine different automorphism groups of non-Schurian S-rings with orders from 720 to

Rank	60	33	32	22	20	19	18	17	16	15	14	13
Total	1	1	1	1	1	2	3	2	1	1	2	7
Non-Schurian	0	0	0	0	0	0	0	0	0	0	1	1

Rank	12	11	10	9	8	7	6	5	4	3	2
Total	5	10	11	17	15	18	21	19	16	7	1
Non-Schurian	0	1	4	3	3	3	2	0	1	0	0

Table 18: Distribution of S-rings over A_5 with respect to rank

$2^{34} \cdot 3 \cdot 5$. Altogether, among the automorphism groups of the 163 orbits of S-rings, there are 139 different groups (including the trivial cases of A_5 and S_{60}); too many to pay special attention to each group.

For groups of relatively small orders, GAP allows us to get one or a few "names" of abstract groups in clear algebraic terms. In addition, in each case we need to understand the (transitive) permutation representation of each group.

It was convenient for us to classify all groups into a few classes according to their order. Defining artificial borders, we distinguish between:

- "small" groups, those of order up to 7680; all of them were identified with the aid of GAP;

- "large" groups, those of order 14400 up to one million. Their orders are: 14400, 14580, 24360, 29160, 46080, 61440, 122880, 230400, 466560, 933120;

- "very large" groups, of order between 1875000 and 60!.

The list of orders of small groups is presented in Table 19. Identification of the structure of large and very large groups, in general, cannot be done by using GAP in an automatic fashion. Instead, we apply special tricks.

For example, we discuss below groups that may be described as *wreath products* of two groups of a smaller degree: $G = G_1 \wr G_2$, so that $|G| = |G_1| \cdot |G_2|^m$, where m is the degree of G_1 (we refer to [47] for definition and notation). The following simple trick was programmed in GAP:

Nr.	$\lvert Aut \rvert$	Structure	Nr.	$\lvert Aut \rvert$	Structure
0	60	A_5	20	360	$C_3 \times S_5$
1	120	S_5	22	360	$A_5 \times S_3$
2	120	$C_2 \times A_5$	30	480	$(C_2 \times C_2 \times A_5) : C_2$
3	180	$GL(2,4)$	43	600	$D_{10} \times A_5$
6	240	$C_2 \times S_5$	60	720	$S_5 \times S_3$
9	240	$C_2 \times C_2 \times A_5$	67, 95, 98	720	$A_5 \times A_4$
10	240	$A_5 : C_4$	107	1200	$(C_5 \times A_5) : C_4$
12	300	$C_5 \times A_5$	42, 112, 139	1320	$PSL(2,11) : C_2$
14	360	$GL(2,4) : C_2$	109	1440	$(A_5 \times A_4) : C_2$

Nr.	$\lvert Aut \rvert$	Structure
4, 15, 21, 35, 47, 115	1920	$C_2 \times ((C_2 \times C_2 \times C_2 \times C_2) : A_5)$
5	3840	$((C_2 \times C_2 \times C_2 \times C_2 \times C_2) : A_5) : C_2$
18, 51, 77	3840	$C_2 \times (((C_2 \times C_2 \times C_2 \times C_2) : A_5) : C_2)$
36, 45, 52, 80	3840	$((C_2 \times C_2 \times C_2 \times C_2) : A_5) : C_4$
137	7200	$(A_5 \times A_5) : C_2$
33, 44	7680	$(C_2.(((C_2 \times C_2 \times C_2 \times C_2) : A_5) : C_2) =$ $((C_2 \times C_2 \times C_2 \times C_2 \times C_2) : A_5).C_2) : C_2$

Table 19: List of orders of small groups over A_5

- We start from the order $|G|$ and represent it in the form $|G| = |G_1| \cdot |G_2|^m$, where $m \cdot n = 60$.
- In addition, we require that m divides $|G_1|$, and n divides $|G_2|$.
- If these conditions are fulfilled, then we examine G in more detail, looking for transitive groups G_1, G_2 with orders $|G_1|$, $|G_2|$ and degrees m, n.

In fact, there are 108 very large groups; 106 of them pass the numerical test, but only 79 of them are actually wreath decomposable.

Example 3. *S-ring #19 has rank 13 with* $|G| = 29859840$ *and valencies* $48, 1^{12}$.

- $|G| = 120 \cdot 12^5$, $m = 5$, $n = 12$.
- G_1 *is* S_5 *acting naturally on 5 points (rank 2).* $|G_2|$ *has to be a group of order 12 acting regularly; for this example, we take* $G_2 = A_4$.

- *We conclude that $G_1 \wr G_2$ has the required order, rank and valencies.*
- *We construct $G_1 \wr G_2$ using GAP, and check that it and G are similar permutation groups.*
- *Thus $G = S_5 \wr A_4$.*

Example 4. *S-ring #133 has rank 5, $|G| = 137594142720000000$ and valencies 1, 50, 4, 4, 1. Here, acting in a similar fashion, we recognize that $|G| = 720 \cdot 240^6$ and identify G as $S_6 \wr (S_5 \times S_2)$, where S_6 acts naturally on 6 points, while $S_5 \times S_2$ acts transitively on 10 points with rank 4 and valencies 1, 1, 4, 4.*

Finally, there remain 29 very large groups which are not wreath decomposable. We expect to be able to explain these groups with the aid of more sophisticated operations over permutation groups, which are sometimes called generalized wreath products in the sense of [49].

Recall that an S-ring $\mathcal{A}$ is called *primitive* if all of the basic graphs of $\mathcal{A}$ are connected. In particular, a Schurian S-ring is primitive if and only if its automorphism group is a primitive permutation group. The primitive S-rings over A_5 were classified purely theoretically by M. Muzychuk ([67]):

- There are only two (non-trivial) primitive S-rings of ranks 5 and 4 over A_5, with valencies 1, 12, 12, 15, 20 and 1, 15, 20, 24 respectively.
- Both appear as centralizer algebras of the holomorph of A_5 and its subgroup of index 2.

This result is now confirmed with the aid of a computer, as one of the consequences of the project presented here.

4.1.2 Rational S-rings

There are 54 orbits of rational S-rings over the group A_5, and 49 of them have large groups. Those with large groups are in a sense less interesting, because they can be described via suitable decompositions into association schemes with a smaller (than 60) number of points.

#	Rank	Valencies	|Aut|	Structure
51	9	1, 16, 8, 8, 16, 4, 2, 1, 4	3840	$C_2 \times (((C_2 \times C_2 \times C_2 \times C_2) : A_5) : C_2)$
60	9	1, 12, 6, 12, 6, 2, 12, 6, 3	720	$S_5 \times S_3$
77	8	1, 16, 16, 16, 4, 2, 4, 1	3840	$C_2 \times (((C_2 \times C_2 \times C_2 \times C_2) : A_5) : C_2)$
107	6	1, 20, 5, 20, 10, 4	1200	$(C_5 \times A_5) : C_4$
109	6	1, 24, 12, 8, 12, 3	1440	$(A_5 \times A_4) : C_2$

Table 20: Rational S-rings over A_5 having a small automorphism group

The S-rings with small groups are listed in Table 20. Note that S-rings #51 and #77 are non-Schurian; they will be considered in more detail in Section 4.1.3. Each of the remaining 3 S-rings needs to be investigated separately using suitable ad hoc tools. The simplest pattern is presented below.

Example 5. *S-ring #60. Our goal was to explain the significant properties of this S-ring. For this purpose, we used the computer package COCO. It is possible, however, to perform all the necessary calculations without the use of a computer.*

Let us start from the direct sum $G = S_5 + S_3$ *(which has abstract structure* $S_5 \times S_3$*), acting intransitively on the set* $[0, 7]$ *with two orbits* $[0, 4]$ *and* $[5, 7]$*. Clearly,* $|G| = 720$*. We wish to consider the transitive faithful action of* G *on the cosets of a suitable subgroup* K *of order 12. As an abstract group,* K *is isomorphic to the dihedral group* D_6 *of order 12.*

We consider the combinatorial object $P = \{(0, 5), (1, 6), (2, 7)\}$*. The stabilizer* $\mathrm{Aut}(P)$ *of* P *in* G *has the structure* $S_3 + S_2$*, and is isomorphic (as an abstract group) to* D_6*. Two additional essential facts are that* D_6 *does not contain a non-trivial normal subgroup of* G*, and that there exists a subgroup* $A_5 \leq G$ *which has a trivial intersection with* D_6*.*

Thus, we may indeed consider the transitive action of G *on the set* $\Omega = \{P^g | g \in G\}$ *of all images of* P*, and its centralizer algebra* $V = V(G, \Omega)$*. Naive combinatorial counting shows that* $|\Omega| = \binom{5}{2} \cdot 3! = 60$*. Using the orbit counting lemma (also known as the CFB lemma, see [47]), we confirm that rank*$(V) = 9$ *and* V *is a symmetric association scheme. We also calculate the valencies of the classes of* V*, see Table 20. This provides a* *reasonably elementary description of S-ring #60. The fact that it is rational was confirmed with the aid of GAP. It might be interesting to try to get an independent computer-free justification of this fact (avoiding any specific calculations of the spectrum of V).*

4.1.3 General Outline of Non-Schurian S-rings

The information about all 19 non-Schurian S-rings is gathered together in Table 21. The content of the first three columns is clear, and the remaining two columns are explained below. We use a few ad hoc approaches in order to explain (or better, to interpret, see Section 1.1 or [52]) the computer results we obtained.

An essential initial ingredient of the desired explanation is the concept of a root group. Namely, we have a set $\mathcal{R}$ of four *root* groups of orders 720, 1320, 1920, and 7680, such that each of the 19 (non-Schurian) S-rings appears as a subring (merging, in other terminology) of a suitable corresponding transitivity module.

Note that S-rings #49 and #91 originate in this way from two roots.

In particular, of the 19 S-rings, nine appear as so-called algebraic mergings. Below we discuss each of the four roots separately.

4.1.3.1 Root 1

The group $\mathcal{R}_1 = PGL(2, 11)$ of rank 10 and order 1320, non-commutative S-ring #42 with subdegrees 1^5, 11^5. This is a particular case of a general construction, which was described by R. Mathon ([60]) as a pseudocyclic association scheme in the sense of [15]. In general, these schemes allow proper algebraic automorphisms. In our case, $q = 11$ and $d = 2$, so we obtain such a scheme on $\frac{q^2-1}{d}$ points, with two algebraic mergings of rank 4 and 6; the first one is generated by an antipodal distance regular graph of diameter 3 and valency 11, see also [59], [37]. This explains #112 and #139.

#	rank	|Aut|	root	extra features
15	14	1920	1920	algebraic
21	13	1920	1920	algebraic
35	11	1920	1920	algebraic
39	10	3932160	1920	elementary merging #36
44	10	7680	7680	elementary merging #32
45	10	3840	1920	elementary merging #35
47	10	1920	1920	elementary merging #34
49	9	128849018880	1920, 7680	wreath product, elementary merging #38, #43, #44, #46, #47
51	9	3840	1920	algebraic
52	9	3840	1920	algebraic
69	8	7864320	1920	elementary merging #49
77	8	3840	1920	elementary merging #50
80	8	3840	1920	elementary merging #51
91	7	257698037760	1920, 7680	wreath product, elementary merging #68, #71, #76, #79
95	7	720	720	algebraic, elementary merging #66
98	7	720	720	algebraic, elementary merging #66
112	6	1320	1320	algebraic
115	6	1920	1920	
139	4	1320	1320	algebraic

Table 21: Information about non-Schurian S-rings over A_5

4.1.3.2 Root 2

The group $\mathcal{R}_2$ of order 720 and rank 8, commutative S-ring #67 with valencies 1, 3, 4^2, 12^4. The group is isomorphic to $A_5 \times A_4$, and is a subgroup of the holomorph of A_5. This allows us to justify, in reasonably clear terms, the existence of the group $\mathrm{AAut}(\mathcal{R}_2)$ of order 4 and the two algebraic mergings that appear. Both mergings have rank 7. This explains #95 and #98.

4.1.3.3 Root 3

The group $\mathcal{R}_3$ of order 7680 and rank 11, non-commutative S-ring #33 with valencies 1^2, 2, 4^2, 8^6. It has a transitive faithful action of degree 12 (smallest degree possible). This action is the wreath product $G = PGL(2,5) \wr S_2$ of order $120 \cdot 2^6$, where $PGL(2,5)$ acts 3-transitively on the projective line of size 6 (the entire group G is isomorphic to group #270 of degree 12 in the catalogue [22]). To describe the action of G of degree 60 combinatorially, one may start from a BIBD D with six points and 10 blocks, then "blow up" each point to a 2-element subset, also blowing up all blocks. For the resulting incidence structure D', we get $G = \mathrm{Aut}(D')$. Now an appropriate substructure S of D' should be found, so that the stabilizer of S in G is a suitable subgroup of order 128 in G. All possible embeddings of S into D' provide the 60 points of the S-ring #33.

S-rings #44, #49, #91 are elementary mergings of Schurian S-rings. In addition, #49 and #91 are wreath products of a suitable non-Schurian scheme on 30 points with $\mathbb{Z}_2$ (which explains the high orders of their automorphism groups).

4.1.3.4 Root 4

The group $\mathcal{R}_4$ of order 1920 and rank 20, non-commutative S-ring #4 with valencies 1^4, 2^4, 4^{12}. This group acts naturally on 10 points and appears as a wreath product $A_5 \wr S_2$ (of order $60 \cdot 2^5$). Again, its transitive action of degree 60 (as S-ring #4) may be described, starting from the action of degree 10, using a suitable ad hoc explanation. It turns out that 14 of the 19 non-Schurian S-rings are mergings of the transitivity module of $\mathcal{R}_4$. Of these, 5 are algebraic mergings, namely #15, #21, #35, #51, and #52.

Another very helpful trick is credited to M. Muzychuk ([66]). Let us start from an S-ring $\mathcal{A}$ of rank r and its subring $\mathcal{A}'$ of rank $r-1$. This means that $\mathcal{A}'$ is obtained from $\mathcal{A}$ by a merging of just two basic sets, while all other basic sets are unchanged. An efficient sufficient criterion, formulated in algebraic terms, guarantees the existence of such a merging $\mathcal{A}'$ of rank $r-1$, which we call an *elementary merging*. It turns out that a total of 11 non-Schurian S-rings may be explained (in the role of $\mathcal{A}'$) by selecting a suitable S-ring $\mathcal{A}$. The corresponding information is given in Table 21.

By now, 18 of the 19 S-rings have been explained by at least one of the two approaches described above (some having multiple explanations).

Finally, there remains one (commutative) S-ring of rank 6 from root $\mathcal{R}_4$. None of the previous tricks provides an explanation for this S-ring. This is S-ring #115, which gave us the most challenging task of finding a suitable computer-free interpretation. We will face this challenge in the next section.

4.1.4 The Exceptional non-Schurian S-ring #115

We are interested in S-ring $\mathcal{A}_{115}$. It is, in fact, a representative of an orbit of 30 isomorphic S-rings. This non-Schurian S-ring is a symmetric (and, therefore, commutative) S-ring of rank 6, with valencies 1, 1, 5, 5, 8, 40. Its automorphism group, $\mathrm{Aut}(\mathcal{A}_{115}) = \mathcal{R}_4$, is a group of order 1920 and rank 20. It has three imprimitivity systems, with 10, 12 and 30 cells. The basic graph of valency 40 is the only connected basic graph.

For a presentation of the relations of $\mathcal{A}_{115}$, we need two subgroups of A_5: a group K isomorphic to D_5 (the stabilizer of a pentagon), and a group L isomorphic to A_4 (the stabilizer of a point). With six ways to select K and five ways to select L, we note that the orbit is indeed of size 30.

The group $K \cap L$ is of order 2; let i be its non-identity element. The starting groups for the construction of $\mathcal{A}_{115}$ in this specific example are $K = \langle (1,4)(2,3),\ (0,1,2,3,4) \rangle$ and $L = A_{\{1,2,3,4\}}$, so that $i = (1,4)(2,3)$.

Now we describe the basic sets as explicit subsets of A_5:

$X_0 = \{e\}$, $X_1 = \{i\}$, $X_2 = K \setminus L$ of size 8.

The icosahedron graph is a Cayley graph over L; let X_3 and X_4 be two complementing (with respect to $L \setminus K$) connection sets of the icosahedron:

$X_3 = \{(1,3)(2,4), (1,4,3), (1,2,4), (1,4,2), (1,3,4)\}$,

$X_4 = \{(1,2)(3,4), (1,2,3), (1,3,2), (2,3,4), (2,4,3)\}$.

$X_5 = A_5 \setminus (K \cup L)$ is of size 40.

The imprimitivity systems arise from:

$K \cap L = X_0 \cup X_1$: This is the connection set of the graph $30 \circ K_2$.

$K = X_0 \cup X_1 \cup X_2$: $6 \circ K_{10}$.

$L = X_0 \cup X_1 \cup X_3 \cup X_4$: $5 \circ K_{12}$.

A significant feature of this description is that we use the Cayley representation of the icosahedron graph (over the subgroup A_4) twice. This description is close to the concept of orthogonal block structures ([5]), though it does differ slightly.

4.2 S-rings over $\mathrm{AGL}_1(8)$

Another challenging problem was the enumeration of S-rings over the affine group $H = \mathrm{AGL}_1(8)$ over the field $\mathbb{F}_8$ with eight elements. Recall that H is a group of order 56, which has a decomposition $H \cong (\mathbb{Z}_2)^3 \rtimes \mathbb{Z}_7$, where $(\mathbb{Z}_2)^3$ stands for the additive group of $\mathbb{F}_8$ while $\mathbb{Z}_7$ is the multiplicative group of $\mathbb{F}_8$. This problem was solved in a similar manner to the A_5 case. The same computer took about two months.

There are 2349 S-rings in 129 orbits under $\mathrm{Aut}(H)$ (which is isomorphic to the group $A\Gamma L_1(8)$ of order 168). Up to the action of $\mathrm{Aut}(H)$, there are 20 non-Schurian S-rings (427 in total) and 109 Schurian S-rings.

The automorphism groups of the 109 Schurian S-rings are again partitioned into 24 small groups of orders up to 3584, which are described by GAP, 22 large groups of orders 7168 to 2903040, and 63 very large groups of orders 14680064 and up, of which 56 are wreath products.

The 20 non-Schurian S-rings over $\mathrm{AGL}_1(8)$ are all mergings of only two roots. Here, the set $\mathcal{R}$ of roots consists of two groups $\mathcal{R}_1$ and $\mathcal{R}_2$ of orders 168 and 224, both acting transitively on H. There are five minimal S-rings, of ranks 32, 20, 20, 20, and 14. Eight of the non-Schurian S-rings appear as mergings of the transitivity module of the root $\mathcal{R}_1$. The group $\mathcal{R}_1 = A\Gamma L_1(8)$ is 2-transitive of degree 8 and order 168, of the form $E_8 \rtimes F_{21}$, and where $F_{21} = \mathbb{Z}_7 : \mathbb{Z}_3$ is the Frobenius group of order 21. The mergings

Rank	56	32	22	20	18	17	16	14	13	12
Total	1	1	1	6	4	1	1	5	2	2
Non-Schurian	0	0	0	0	0	0	0	0	0	0

Rank	11	10	9	8	7	6	5	4	3	2
Total	11	12	6	18	8	15	21	9	4	1
Non-Schurian	6	4	0	1	0	2	7	0	0	0

Table 22: Distribution of S-rings over $\mathrm{AGL}_1(8)$ with respect to rank

#	|Aut|	Structure
0	56	$\mathrm{AGL}_1(8)$
1	112	$\mathbb{Z}_2 \times (\mathrm{AGL}_1(8))$
6	168	$(\mathrm{AGL}_1(8)) \rtimes \mathbb{Z}_3$
4	224	$E_{2^2} \times (\mathrm{AGL}_1(8))$
23	336	$\mathbb{Z}_2 \times ((\mathrm{AGL}_1(8)) \rtimes \mathbb{Z}_3)$
8	448	$E_{2^3} \times \mathrm{AGL}_1(8)$
7	448	$E_{2^3} \times \mathrm{AGL}_1(8)$
70	672	$(E_{2^2} \times (\mathrm{AGL}_1(8))) \rtimes \mathbb{Z}_3$
2	896	$\mathbb{Z}_2 \times (E_{2^3} \times \mathrm{AGL}_1(8))$
11	896	$\mathbb{Z}_2 \times (E_{2^3} \times \mathrm{AGL}_1(8))$
10	896	$\mathbb{Z}_2 \times (E_{2^3} \times \mathrm{AGL}_1(8))$
3	896	$\mathbb{Z}_2 \times (E_{2^3} \times \mathrm{AGL}_1(8))$
64	1008	$\mathbb{Z}_2 \times L_2(8)$
59	1344	$(E_{2^3} \times \mathrm{AGL}_1(8)) \rtimes \mathbb{Z}_3$
54	1344	$(E_{2^3} \times \mathrm{AGL}_1(8)) \rtimes \mathbb{Z}_3$
53	1344	$E_{2^3} \rtimes L_3(2)$
69	1344	$E_{2^3} \rtimes L_3(2)$
21	1792	$\mathbb{Z}_2 \times ((E_{2^3} \times \mathrm{AGL}_1(8)) \rtimes \mathbb{Z}_2)$
12	1792	$E_{2^2} \times (E_{2^3} \times \mathrm{AGL}_1(8))$
92	2688	$\mathbb{Z}_2 \times (E_{2^3} \rtimes L_3(2))$
62	2688	$\mathbb{Z}_2 \times ((E_{2^3} \times \mathrm{AGL}_1(8)) \rtimes \mathbb{Z}_3)$
88	2688	$\mathbb{Z}_2 \times (E_{2^3} \rtimes L_3(2))$
26	3584	$\mathbb{Z}_2 \times ((E_{2^3} \times \mathrm{AGL}_1(8)) \rtimes \mathbb{Z}_4)$
24	3584	$E_{2^2} \times ((E_{2^3} \times \mathrm{AGL}_1(8)) \rtimes \mathbb{Z}_2)$

Table 23: List of orders of small groups over $\mathrm{AGL}_1(8)$

of the root $\mathcal{R}_1$, together with other interesting structures, were carefully investigated jointly with Josef Lauri. Some of the results were recently published in [50]. The second root $\mathcal{R}_2$ has order 224 and rank 20; as an abstract group, we have $\mathcal{R}_2 \cong E_4 \times \mathrm{AGL}_1(8)$.

4.3 Symmetric S-rings over $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$

There are 225 symmetric S-rings over $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$ with ranks 2,3,4,6,7, and 8. These S-rings are in 13 orbits under the action of $\mathrm{Aut}(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5)$ (which has order 110). All S-rings are Schurian. Some information about the orbits is

Rank	valencies	$\lvert orb\rvert$	$\lvert\mathrm{Aut}\rvert$	Structure
2	1,54	1	55!	S_{55}
3	$1,4,50$	11	$2^{41}\cdot 3^{15}\cdot 5^{13}\cdot 7\cdot 11$	$S_{11}\wr S_5$
3	$1,18,36$	55	39916800	S_{11}
3	$1,10,44$	1	$2^{43}\cdot 3^{21}\cdot 5^{11}\cdot 7^5\cdot 11^5$	$S_5\wr S_{11}$
4	$1,4,10,40$	11	4790016000	$S_5\times S_{11}$
4	$1,10,22^2$	1	$2^{41}\cdot 3^{20}\cdot 5^{11}\cdot 7^5\cdot 11^5$	$D_5\wr S_{11}$
4	$1,2^2,50$	11	$2^{19}\cdot 3^4\cdot 5^{13}\cdot 7\cdot 11$	$S_{11}\wr D_5$
6	$1,6,12^4$	55	1320	$\mathrm{PGL}_2(11)$
6	$1,4,6,8,12,24$	55	1320	$\mathrm{PGL}_2(11)$
6	$1,2^2,10,20^2$	11	399168000	$D_5\times S_{11}$
7	$1,2^5,44$	1	618435840	$S_5\wr D_{11}$
8	$1,2^4,22^2$	1	51536320	$D_5\wr D_{11}$
8	$1,2^2,10^5$	11	550	$(\mathbb{Z}_5\times(\mathbb{Z}_{11}\rtimes\mathbb{Z}_5))\rtimes\mathbb{Z}_2$

Table 24: Orbits of symmetric S-rings over $\mathbb{Z}_{11}\rtimes\mathbb{Z}_5$

presented in Table 24.

Each of the two rank 6 S-rings with automorphism group $\mathrm{PGL}_2(11)$ is generated by the basic graph of valency 6. These basic graphs are actually the two connected components of the distance 2 graph of the semisymmetric graph $\Gamma(\mathrm{PGL}_2(11), D_{24}, S_4)$, described in [43]. $\Gamma(\mathrm{PGL}_2(11), D_{24}, S_4)$ is an example of the general construction of semisymmetric graphs based a large group and two non-isomorphic subgroups that have the same size. The cosets of each of the two subgroups serve as vertices in this graph. Two cosets (necessarily of distinct subgroups) are adjacent if they are not disjoint.

The three S-rings with automorphism groups S_{11} and $\mathrm{PGL}_2(11)$ are the only primitive symmetric S-rings over $\mathbb{Z}_{11}\rtimes\mathbb{Z}_5$.

Preliminary results of enumeration of all S-rings over $\mathbb{Z}_{11}\rtimes\mathbb{Z}_5$ reveal that there are 454 S-rings in 34 orbits (with ranks 2 to 55). All of them are Schurian. These results have not yet been fully analyzed and rechecked. We did run a few tests, such as calculating the mergings of each S-ring found, to make sure that no new S-rings were generated in this way. A list of the non-symmetric S-rings is in Table 25.

Rank	valencies	$\|orb\|$	$\|\mathrm{Aut}\|$	Structure
4	$[1, 5^2, 44]$	1	60394125000	
4	$[1, 4, 25^2]$	11	$55 \cdot (5!)^{11}$	
5	$[1, 2^2, 25^2]$	11	5500000000000	
5	$[1, 4, 10, 20^2]$	11	1100	
5	$[1, 5^2, 22^2]$	1	5032843750	
6	$[1^5, 50]$	11	1949062500000000	
6	$[1, 10, 11^3]$	1	$5 \cdot (11!)^5$	
6	$[1, 4, 5^2, 20^2]$	11	6600	
7	$[1, 5^2, 11^4]$	1	2516421875	
7	$[1^5, 25^2]$	11	2685546875	
8	$[1, 4^3, 6, 12^3]$	55	660	
9	$[1, 3^2, 6^4, 12^2]$	55	660	
9	$[1, 2^2, 5^2, 10^4]$	11	550	
10	$[1^5, 10^5]$	11	199584000	
10	$[1, 2^5, 11^4]$	1	25768160	
12	$[1^{11}, 44]$	1	19326120	
13	$[1^{11}, 22^2]$	1	1610510	
15	$[1^5, 5^{10}]$	11	275	$\mathbb{Z}_5 \times (\mathbb{Z}_{11} \rtimes \mathbb{Z}_5)$
15	$[1^{11}, 11^4]$	1	805255	
30	$[1^5, 2^{25}]$	11	110	$(\mathbb{Z}_{11} \rtimes \mathbb{Z}_5) \rtimes \mathbb{Z}_2$
55	$[1^{55}]$	1	55	$\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$

Table 25: Orbits of non-symmetric S-rings over $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$

Chapter 5

Other results

5.1 Links between two semisymmetric graphs on 112 vertices

The Nikolaev graph ($\mathcal{N}$) and Ljubljana graph ($\mathcal{L}$) were introduced in Section 2.5. We now consider the graph $\mathcal{L}$ together with the group $\mathrm{Aut}(\mathcal{L})$, with special attention to a few association schemes and coherent configurations that are related to $\mathcal{L}$, as well as to the embeddings of $\mathcal{L}$ into the graph $\mathcal{N}$.

5.1.1 A master association scheme on 56 points

In our attempts to get a new understanding of the graph $\mathcal{L}$, we started from the group $G = A\Gamma L_1(8) := \{x \mapsto ax^\sigma + b | a \in F_8^*, b \in F_8, \sigma \in \mathrm{Aut}(F_8)\}$ (F_8 is the field with 8 elements, F_8^* is its multiplicative group).

Clearly, $|G| = 8 \cdot 7 \cdot 3 = 168$ and G acts naturally on the set of elements of the Galois field F_8 as a 2-transitive permutation group. Identifying F_8 with the set $[0, 7]$, we use the representation $G = \langle g_1, g_2, g_3 \rangle$, where $g_1 = (1, 2, 3, 4, 5, 6, 7)$, $g_2 = (0, 1)(2, 4)(3, 7)(5, 6)$, and $g_3 = (2, 3, 5)(4, 7, 6)$, as it appears in [75].

G is a subgroup of S_8, therefore it makes sense to consider again the induced action of G on the same set $V = V_1 \cup V_2$, as it was defined in

Section 2.5.2. V_1 is the set of 56 ordered pairs of $[0,7]$, and V_2 is the set of 3-subsets of $[0,7]$.

With the aid of a computer, it was discovered that in this way we obtain exactly 8 distinct copies of the same (up to isomorphism) graph $\mathcal{L}$, and that they are invariant with respect to the induced intransitive action (G,V). Each such copy appears as a spanning subgraph of a suitable copy of $\mathcal{N}$.

The stabilizer of an arbitrary vertex in $\mathcal{L}$ has order 3; thus, both stabilizers are isomorphic to the cyclic group $\mathbb{Z}_3$. Therefore, in comparison with the "easy" case of $\mathcal{N}$, this vision of $\mathcal{L}$ emphasizes that it belongs to a more difficult case. In the following section, we aim to interpret the graph $\mathcal{L}$ (as well as its embeddings to $\mathcal{N}$), starting from the association scheme formed by the 2-orbits of the transitive permutation group (G,V_1). In the beginning, we will essentially rely on the analysis of some computations performed with the aid of computer algebra packages.

Thus, let $\Omega = V_1 = \{(x,y)|x,y \in F_8, x \neq y\}$ and let (G,Ω) be the induced transitive action of $G = A\Gamma L_1(8)$ on Ω of degree 56.

Proposition 14. *The permutation group* (G,Ω) *has rank 20.*

Proof. By definition, the rank of a transitive permutation group is equal to the number of orbits of the stabilizer of an arbitrary point. The stabilizer of any point from Ω is similar to the induced cyclic group $(\mathbb{Z}_3,\Omega)$, $Z_3 = \langle \tilde{g}_3 \rangle$, where $\tilde{g}_3$ denotes the action of g_3 on Ω. With the aid of the orbit counting lemma (CFB lemma in [47]), we obtain for the rank r of (G,Ω) that $r = \frac{1}{3}(\binom{8}{2} + 2 \cdot 2) = 20$. $\square$

Using COCO in conjunction with GAP, we construct and investigate our master association scheme $\mathfrak{M} = (\Omega, 2 - orb(G,\Omega))$.

COCO returns a list of representatives of the 20 2-orbits, finds the lengths of the 2-orbits, calculates the intersection numbers of $\mathfrak{M}$, enumerates all mergings of $\mathfrak{M}$, and provides the order of the automorphism group of each merging (together with the rank and subdegrees of each group). GAP allows us to get some extra information, particularly about the basic graphs of $\mathfrak{M}$. The first part of the obtained results is presented below.

Proposition 15. *(i) There are eight pairs of antisymmetric basic relations of valency 3 in $\mathfrak{M}$.*

(ii) All of these basic graphs are not bipartite.

(iii) Six pairs of basic graphs are connected.

(iv) The automorphism group of each of those $6{\cdot}2 = 12$ connected (di-)graphs is (G, Ω).

(v) In each pair of connected basic graphs, opposite graphs are not isomorphic.

According to the criterion presented in [43], it now evidently makes sense to construct the incidence double cover on 112 vertices, starting from each pair $\{R, R'\}$ of connected antisymmetric basic graphs of valency 3. Clearly, the incidence double cover (IDC, Definition 2 on page 26) of R and R' are isomorphic (undirected) graphs of valency 3. With the aid of GAP, we divide the six pairs into four "good" pairs, which all provide isomorphic copies of $\mathcal{L}$, and two "bad" pairs, which provide vertex transitive disconnected graphs, isomorphic to eight copies of the Heawood graph.

To better explain the observed phenomena, we further consider the normalizer $N_{S_{56}}((G, \Omega))$ of G in $S(\Omega)$, which, in this case, coincides with the group $\mathrm{CAut}(\mathfrak{M})$.

The second part of the corresponding computer-aided results is presented below.

Proposition 16. *(i) $N_{S_{56}}((G, \Omega)) \cong G \times \mathbb{Z}_2$ and has order 336.*

(ii) The quotient group $N_{S_{56}}((G, \Omega))/G$ acts on the 16 antisymmetric 2-orbits as a group of order 2.

(iii) Each "good" 2-orbit R is mapped to a 2-orbit R^* from another "good" pair under this action.

(iv) Each "bad" 2-orbit is mapped to a 2-orbit from another "bad" pair.

i	Rep	Pair	Val	Con	R'	R^*	$\lvert\mathrm{Aut}\rvert$	cl	cl v	Aut(v)	rank
0	0	(0,1)	1	F	0	0	56!	1	1	$S_{56} \wr S_2$	2
1	1	(0,2)	3	F	5	7	$8! \cdot 21^8$	2	2	$S_8 \wr F_{21}$	4
2	2	(1,0)	1	F	2	2	$28! \cdot 2^{28}$	3	1	$S_{56} \wr S_2$	3
3	4	(1,4)	3	T	12	4	168	4	2	$S_8 \wr F_{21}$	20
4	5	(2,0)	3	T	8	3	168	4	2	$S_8 \wr F_{21}$	20
5	6	(0,4)	3	F	1	9	$8! \cdot 21^8$	2	2	$S_8 \wr F_{21}$	4
6	9	(2,5)	3	T	17	14	168	5	3	G	20
7	11	(4,1)	3	F	9	1	$8! \cdot 21^8$	2	2	$S_8 \wr F_{21}$	4
8	12	(1,2)	3	T	4	12	168	6	2	$S_8 \wr F_{21}$	20
9	14	(2,1)	3	F	7	5	$8! \cdot 21^8$	2	2	$S_8 \wr F_{21}$	4
10	17	(3,6)	3	T	11	16	168	7	3	G	20
11	18	(4,6)	3	T	10	13	168	8	3	G	20
12	20	(4,0)	3	T	3	8	168	6	2	$S_8 \wr F_{21}$	20
13	23	(5,2)	3	T	16	11	168	8	3	G	20
14	29	(4,7)	3	T	15	6	168	5	3	G	20
15	30	(7,5)	3	T	14	17	168	9	3	G	20
16	32	(5,7)	3	T	13	10	168	7	3	G	20
17	39	(6,3)	3	T	6	15	168	9	3	G	20
18	43	(2,4)	3	F	18	18	$14! \cdot (4!)^{14}$	10	4	$S_{14} \wr (S_2 \times S_4)$	3
19	44	(4,2)	3	F	19	19	$7! \cdot 48^7$	11	4	$S_{14} \wr (S_2 \times S_4)$	5

Table 26: 2-orbits of $\mathfrak{M}$ and their covers

Note that the action of the direct factor $\mathbb{Z}_2$ on Ω corresponds to the permutation which transposes each pair (x, y) with (y, x) for distinct elements $x, y \in F_8$.

For the reader's convenience, the main numerical results related to the above propositions are presented in Table 26. Here, we first list number i of class R_i, a representative $x \in \Omega$ such that $(0, x) \in R_i$, and a description $x = (a, b)$, $a, b \in \mathcal{F}_8$. In the last column of the table, we refer to the number of merging of $\mathfrak{M}$ which is the coherent closure of R_i, according to the list of all mergings that appears in Appendix A.2.

Note that we get 11 isomorphism classes of basic graphs of $\mathfrak{M}$, while four such classes form the four "good" pairs. The corresponding IDCs split into four isomorphism classes, described in column "cl v". The class 3 provides the graph $\mathcal{L}$. Again, the information in the last column of the table lists the rank of the coherent closure of the basic graphs (which in most cases coincides with $\mathfrak{M}$). F_{21} denotes the Frobenius group of order 21 and degree 7.

5.1.2 Embeddings of $\mathcal{L}$ into $\mathcal{N}$

We now wish to have a better understanding of all the possible embeddings of the graph $\mathcal{L}$ into $\mathcal{N}$. Note that the union of each "good" pair of relations R and R^* is again an antisymmetric relation (of valency 6). Moreover, each such relation is a 2-orbit of the group $\widetilde{\mathfrak{G}} = \mathrm{CAut}(\mathfrak{M}) \cong A\Gamma L_2(8) \times \mathbb{Z}_2$. Therefore, it makes sense to also consider an association scheme $\widetilde{\mathfrak{M}}$, resulting from the group $\widetilde{\mathfrak{G}}$. In principle, $\widetilde{\mathfrak{M}}$ appears as a merging (#1) of $\mathfrak{M}$. Nevertheless, it was more convenient for us to investigate $\widetilde{\mathfrak{M}}$ independently, using COCO again, and constructing the scheme of 2-orbits of $\widetilde{\mathfrak{G}}$.

We obtain that $(\widetilde{\mathfrak{G}}, \Omega)$ has rank 12 with four pairs of antisymmetric 2-orbits of valency 6. For each such 2-orbit, we again construct its IDC; for two pairs, the resultant cover turns out to be a semisymmetric graph on 112 vertices of valency 6, with automorphism group $\widetilde{\mathfrak{G}}$. We prefer to call this graph of valency 6 the *natural* double Ljubljana graph, and denote it by $\mathcal{NL}$.

Again, GAP is used in conjunction with COCO to obtain our next result.

Proposition 17. *(i) The union of edges from IDC $\mathcal{L}$ of a "good" relation R and $\mathcal{L}^*$ of R^* provides a semisymmetric double Ljubljana graph $\mathcal{NL}$ of valency 6 on 112 vertices.*

(ii) $\mathrm{Aut}(\mathcal{NL}) = \widetilde{\mathfrak{G}}$.

(iii) $\mathcal{NL}$ appears as an incidence double cover of the antisymmetric 2-orbit $R \cup R^*$ of the group $\widetilde{\mathfrak{G}} = \mathrm{CAut}(\mathfrak{M})$.

(iv) Each graph $\mathcal{NL}$ (as well as each graph $\mathcal{L}$) can be extended in a unique way to a graph isomorphic to $\mathcal{N}$ of valency 15, if we require the extended graph to be invariant with respect to (G, Ω).

Thus, we have managed to explain more clearly the essence of embedding a "difficult" case of $\mathcal{L}$ into an "easy" case of $\mathcal{N}$.

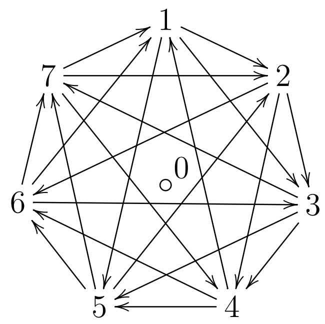

Figure 7: Paley tournament $P(7)$ with isolated vertex

Since $\mathrm{Aut}(\mathcal{L})$ respects this embedding, we obtain new proof of the fact that $\mathcal{L}$ is a semisymmetric graph.

It is clear that at this stage, all of the results that we have presented depend essentially on the use of a computer. In the next sections, we aim to remove this dependence, at least in part. For this purpose, additional combinatorial structures will be introduced and investigated.

5.1.3 The Ljubljana configuration

As we mentioned, each semisymmetric graph may and should be regarded as the Levi graph of a symmetric incidence structure (very frequently, it happens to be a configuration), which is not self-dual. Let C and C^T be two such configurations, defined by the graph $\mathcal{L}$. The diagrams of this pair of configurations are depicted in Figure 5 of [21]; they are realized as geometric configurations of points and lines in the Euclidean plane.

Below we develop an alternative, combinatorial approach to the representation and investigation of the two 56_3 Ljubljana configurations, and exploit its advantages.

First, let us consider a copy of a Paley tournament $P(7)$ with the vertex set $[1, 7]$ and isolated vertex 0, as depicted in Figure 7.

It is easy to check that $\mathrm{Aut}(P(7)) = \langle g_1, g_3 \rangle$ (we use notation from Section 5.1.1) is a Frobenius group F_{21} of order 21 and degree 7. Recall that this copy of F_{21} is simultaneously the stabilizer of the point 0 in the group $(G,[0,7]) = (G, F_8)$.

Consider now the orbit $\mathcal{O}$ of this graph $P(7)$ under the action of (G, F_8). This orbit $\mathcal{O}$ contains eight copies of $P(7)$, where each element from $[0,7]$ appears exactly once as an isolated vertex. For each copy of $P(7)$ in $\mathcal{O}$, and for each vertex x of $P(7)$, we get the induced subgraph $T(x)$, generated by the out-neighbors of x. Clearly, $T(x)$ is a directed triangle. Denote by $\mathcal{B}$ the collection of these triangles. We are ready to present our first construction.

Let us identify a tuple (x,y) in Ω, with vertex y of the copy of $P(7)$ that has x as an isolated point. Here, Ω is defined as in Section 5.1.1.

Consider the incidence structure $\mathfrak{S} = (\Omega, \mathcal{B})$ with inclusion in the role of incidence relation.

Proposition 18. *(i)* $|\mathcal{O}| = 8$, $|\mathcal{B}| = 56$, $\mathfrak{S}$ *is a symmetric* 56_3 *configuration without repeated blocks.*

(ii) $\mathrm{Aut}(\mathfrak{S}) = G$.

(iii) The Levi graph of the configuration $\mathfrak{S}$ *is isomorphic to* $\mathcal{L}$.

Proof. The proof of (i) is a trivial consequence of the 2-transitivity of (G, F_8).

For the proof of the remaining parts, let us consider the point graph $\mathcal{P}$ and the block graph $\mathfrak{B}$ of the configuration $\mathfrak{S}$. Clearly, both graphs have valency 6.

Let us establish that the diameter of the point graph $\mathcal{P}$ is 3; for each of its vertices there are exactly 6, 25 and 24 vertices at distance 1, 2 and 3 respectively. We also need to prove that $\mathrm{Aut}(\mathcal{P}) = G$. In principle, it is possible to elaborate a computer-free proof, checking that each automorphism of $\mathcal{P}$, which fixes a vertex (say $(0,1)$) and all its neighbors in $\mathcal{P}$, is the identity automorphism. In practice, we prefer the use of a computer at least for the simple enumeration of the above distance-i subsets ($i \in \{1,2,3\}$) and inspection of the induced subgraphs of $\mathcal{P}$. As a corollary, we get that $\mathrm{Aut}(\mathcal{P}) \cong (G, \Omega)$ and therefore also $\mathrm{Aut}(\mathfrak{S}) \cong (G, \Omega)$.

To prove that $\mathfrak{S} \ncong \mathfrak{S}^T$, we consider the block graph $\mathfrak{B}$, revealing that it has diameter 4. Moreover, we obtain that the distance-4 set for any vertex has cardinality 1.

Thus, $\mathcal{P} \ncong \mathfrak{B}$ and therefore $\mathfrak{S}$ is not self-dual. This implies that the Levi graph of the configuration $\mathfrak{S}$ is semisymmetric.

$\mathcal{L}$ is a unique semisymmetric graph on 112 vertices of valency 3 ([21]), so the Levi graph of the configuration $\mathfrak{S}$ is isomorphic to $\mathcal{L}$. $\square$

5.2 Coherent cages

5.2.1 Coherency of small cages

As we mentioned in the preliminaries, the concept of a coherent cage, a cage that is a class of its own coherent closure, is a new one. As a first step in using this concept, we considered the known cages in light of this concept.

Of the 36 graphs listed as small cages in [28], seven are coherent: the $(3,5)$-cage (Petersen graph on 10 vertices), the $(3,6)$-cage (Heawood graph on 14 vertices), the $(3,8)$-cage (Tutte's 8-cage on 30 vertices), the $(3,12)$-cage (generalized hexagon on 126 vertices), the $(6,5)$-cage (Robertson graph on 40 vertices), the $(7,5)$-cage (Hoffman-Singleton graph on 50 vertices) and $(7,6)$-cage (on 90 vertices). Three of those ($(3,6)$, $(3,8)$ and $(3,12)$) are geometric, coming from an order 2 projective plane, a generalized quadrangle and a generalized hexagon. Of the remaining four, the coherent closure of two is non-Schurian, the $(6,5)$-cage and the $(7,6)$-cage.

5.2.2 Non-Schurian association scheme on 90 points

The (unique) $(7,6)$-cage is a graph Γ on 90 vertices discovered by Baker. The graph is actually the incidence graph of Baker's semiplane on 45 points ([6]).

We found that this cage is coherent, and the association scheme $\mathcal{A}$ that it generates is a non Schurian scheme of rank 6. The valencies of this scheme are 1,2,7,14,24,and 42. The automorphism group of $\mathcal{A}$, $G = \mathrm{Aut}(\mathcal{A}) =$

Aut(Γ) is a group of order 15120, the same group as the action of $3.S_7$ on 90 points as it appears in [84].

Other than the $(6,5)$-cage on 40 vertices (Robertson graph) (see [51]), this is the only known non-geometric non-Schurian coherent cage.

5.3 Non-Schurian association scheme

5.3.1 General case

Schurian association schemes are usually considered "boring" in the context of AGT, since they can be studied by algebraic means. Non-Schurian association schemes, on the other hand, cannot be explained by algebra alone, and require a combinatorial point of view. This is why the discovery of previously-unknown non-Schurian association schemes is considered a success in experimental AGT.

Throughout this thesis we presented a few such schemes, usually in the context of other investigations, such as S-rings, semisymmetric graphs and coherent cages. Below, we present one more example of a newly-discovered non-Schurian association scheme.

5.3.2 Non-Schurian association scheme on 125 points

The basic graphs of a primitive non-symmetric scheme of rank 4, are always a primitive strongly regular graph and two opposite orientations of its complement.

Jørgensen started the systematic investigation of primitive non-symmetric association schemes of rank 4. In [44] there is a survey of known examples and a table of feasible parameter sets for up to 100 vertices. As a part of this investigation ([45]), four rank 4 primitive non-symmetric non-Schurian association schemes on 64 points with parameters of the SRG $(64,27,10,12)$ were discovered.

We discovered a non-Schurian rank 4 primitive non-symmetric scheme on 125 points, $\mathfrak{M}$. The basic SRG of $\mathfrak{M}$ is the point graph of a generalized quadrangle $GQ(4,6)$ with parameters $(125, 96, 74, 72)$. The automorphism group $\mathrm{Aut}(\mathfrak{M})$ is of order 3000 and rank 7. To the best of our knowledge, this is a scheme with a new parameter set.

The construction of $\mathfrak{M}$ starts with 2-transitive action of $G = \mathrm{PSU}(3,5)$ on 126 points. The stabilizer of a point in this action, G_a is a transitive group of order 1000 and rank 17 acting on 125 points. $\mathfrak{M}$ is a merging of this Schurian association scheme of rank 17.

We wish to try to generalize the examples on 64 and 125 points to a series of rank 4 primitive non-symmetric schemes on q^3 points, q being a prime power.

5.4 Miscellanea

5.4.1 Modified Weisfeiler-Leman stabilization

The WL-stabilization algorithm calculates the coherent closure of a symmetric matrix. Since we needed to calculate the coherent closure of non-symmetric matrices, we decided to modify this algorithm to work on arbitrary matrices.

Let $(V, \mathcal{R})$ be a coherent configuration, with structure constants p_{ij}^k.

For two subsets $S, T \subseteq \mathcal{R}$, we define $S * T : \mathcal{R} \to \mathbb{N}$ by

$$S * T(R_k) = \sum_{\substack{R_i \in S \\ R_j \in T}} p_{ij}^k.$$

We consider $S * T$ as a coloring of the elements of $\mathcal{R}$.

For an $R \in \mathcal{R}$, let R' denote the transposed relation of R. For a set $S \subseteq \mathcal{R}$, let $S' = \{R' | R \in S\}$.

For a given partition $\mathcal{P}_0$ of $\mathcal{R}$, the coherent closure of $\mathcal{P}_0$, denoted $\langle\langle \mathcal{P}_0 \rangle\rangle$, is defined as the coarsest partition $\mathcal{P}$ of $\mathcal{R}$ that is finer than $\mathcal{P}_0$ and is a coherent merging of $\mathcal{R}$.

The original Weisfeiler-Leman algorithm as used by stabil ([4])

Input: A partition $\mathcal{P}$ and tensor of structure constants of $\mathcal{R}$.

1. Make a list of all ordered pairs of elements of $\mathcal{P}$ (not necessarily distinct).
2. Let (S, T) be the first pair in the list.
3. Calculate $S * T$.
4. For all sets $U \in \mathcal{P}$, if not all elements of U have the same color, remove U from $\mathcal{P}$ and add all maximal monochromatic subsets of U to $\mathcal{P}$.
5. If the previous step did not change $\mathcal{P}$, let (S, T) be the next pair in the list and go to step 3.
6. If the previous step exhausted the list, stop. Output is $\mathcal{P}$.
7. Go to step 1.

Note: the program STABIL implements a special case of this algorithm, in which the underlying coherent configuration $(V, \mathcal{R})$ is always the trivial one of rank $|V|^2$.

Let $(V, \mathcal{R})$ be a coherent configuration, let $R, Q \in \mathcal{R}$, and $S, T \subseteq \mathcal{R}$ such that $S * T(R) \neq S * T(Q)$

Lemma 19. $T' * S'(R') \neq T' * S'(Q')$.

Proof. Immediate in language of coherent algebras.

□

Lemma 20. *If* $\{S_1, \ldots, S_n\}$ *is a partition of* S *and* $\{T_1, \ldots, T_m\}$ *is a partition of* T*, then there exist* x, y *such that* $S_x * T_y(R) \neq S_x * T_y(Q)$*.*

Proof.

$$S * T(R_k) = \sum_{\substack{R_i \in S \\ R_j \in T}} p_{ij}^k = \sum_{\substack{1 \leq x \leq n \\ 1 \leq y \leq m}} \sum_{\substack{R_i \in S_x \\ R_j \in T_y}} p_{ij}^k = \sum_{\substack{1 \leq x \leq n \\ 1 \leq y \leq m}} S_x * T_y(R_k)$$

so if the sum differs for R and Q, one of the summands has to differ. $\square$

Theorem 21. *If the input partition $\mathcal{P}_0$ of the algorithm is closed under transposition, then so is the output $\mathcal{P}$.*

Proof. Let us denote by $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n = \mathcal{P}$ the intermediate partitions that are constructed in the algorithm. Let (S_i, T_i) be the pair of sets from $\mathcal{P}_i$ that was used to generate $\mathcal{P}_{i+1}$ $(0 \leq i < n)$.

First, we'll show that for every $0 \leq i < n$, $\mathcal{P}$ contains a partition of S'_i and a partition of T'_i. The proof is by complete induction on i.

To prove that $\mathcal{P}$ contains a partition of S'_i, it is enough to show that for $Q, R \in \mathcal{R}$, if $R \in S'_i$ and $Q \notin S'_i$, then R and Q are not in the same set of $\mathcal{P}$:

$R' \in S_i$ and $Q' \notin S_i$. Let j be the smallest such that R' and Q' are in different sets in $\mathcal{P}_j$. By this definition, $j \leq i$.

If $j = 0$, then $R' \in S_0$ and $Q' \notin S_0$, so $R \in S'_0$ and $Q \notin S'_0$. $\mathcal{P}_0$ is closed under transposition, so $S'_0 \in \mathcal{P}_0$. $\mathcal{P}$ is finer than $\mathcal{P}_0$, so R and Q are not in the same set in $\mathcal{P}$.

If $j > 0$, then $S_{j-1} * T_{j-1}(R') \neq S_{j-1} * T_{j-1}(Q')$. $j - 1 < i$, so by the induction hypothesis, $\mathcal{P}$ contains partitions $\{U_1, \ldots, U_f\}$ of S'_{j-1} and $\{V_1, \ldots, V_g\}$ of T'_{j-1}.

$\{U'_1, \ldots, U'_f\}$ is a partition of S_{j-1} and $\{V'_1, \ldots, V'_g\}$ is a partition of T_{j-1}. By Lemma 20, there exist a, b such that $U'_a * V'_b(R') \neq U'_a * V'_b(Q')$ and by Lemma 19, $V_b * U_a(R) \neq V_b * U_a(Q)$. $V_b, U_a \in \mathcal{P}$, and $\mathcal{P}$ is stable, so R and Q are not in the same set in $\mathcal{P}$.

A similar proof works for T'_i.

In order to prove the theorem, we need to show that if R and Q are not in the same set in $\mathcal{P}$, then R' and Q' are not in the same set in $\mathcal{P}$. The proof is similar to the proof above.

Let j be the smallest such that R and Q are not in the same set in $\mathcal{P}_j$.

If $j = 0$, then R' and Q' are not in the same set in $\mathcal{P}_0$ (since $\mathcal{P}_0$ is closed under transposition). $\mathcal{P}$ is finer than $\mathcal{P}_0$, so R' and Q' are not in the same set in $\mathcal{P}$ as well.

If $j > 0$, then $S_{j-1} * T_{j-1}(R) \neq S_{j-1} * T_{j-1}(Q)$, so $T'_{j-1} * S'_{j-1}(R') \neq T'_{j-1} * S'_{j-1}(Q')$, and $\mathcal{P}$ contains partitions of T'_{j-1} and of S'_{j-1} and is stable, so R' and Q' are in different sets in $\mathcal{P}$. □

By Theorem 21, we need to add one step to the algorithm to get an algorithm that works on every matrix and always outputs a coherent configuration:

The modified Weisfeiler-Leman algorithm

Input: A partition $\mathcal{P}$ and tensor of structure constants of $\mathcal{R}$.

0. Let $\mathcal{Q}$ be a partition of $\mathcal{R}$ into two sets, the set of reflexive relations and the set of non-reflexive relations. Replace $\mathcal{P}$ by the set of all non-empty intersections of any three sets, one from each of $\mathcal{P}, \mathcal{P}', \mathcal{Q}$.
1. Make a list of all ordered pairs of elements of $\mathcal{P}$ (not necessarily distinct).
2. Let (S, T) be the first pair in the list.
3. Calculate $S * T$.
4. For all sets $U \in \mathcal{P}$, if not all elements of U have the same color, remove U from $\mathcal{P}$ and add all maximal monochromatic subsets of U to $\mathcal{P}$.
5. If the previous step did not change $\mathcal{P}$, let (S, T) be the next pair in the list and go to step 3.
6. if the previous step exhausted the list, stop. Output is $\mathcal{P}$.
7. Go to step 1.

Theorem 22. *If $(V, \mathcal{R})$ is a coherent configuration and $\mathcal{P}_0$ is a partition of $\mathcal{R}$, then the output of the algorithm, $\mathcal{P}$, is $\langle\langle \mathcal{P}_0 \rangle\rangle$.*

Proof. $\mathcal{P}$ is finer than $\mathcal{P}_0$, since every step of the algorithm refines $\mathcal{P}_0$ until it outputs the result $\mathcal{P}$.

The merging of $(V, \mathcal{R})$ defined by $\mathcal{P}$ is $(V, \mathcal{S})$, where $\mathcal{S} = \{\bigcup P | P \in \mathcal{P}\}$. We will denote $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$, where $S_i = \bigcup P_i$.

$(V, \mathcal{S})$ is a coherent configuration, since each of the CC conditions holds:

CC1 and

CC2 $\mathcal{P}$ is a partition of $\mathcal{R}$, therefore $\mathcal{S}$ is a partition of V^2.

CC3 Step 0 and Theorem 21 ensure that for every $P \in \mathcal{P}$, $P' \in \mathcal{P}$. For any $S \in \mathcal{S}$, $S = \bigcup P$ for some $P \in \mathcal{P}$, so for $S' = \bigcup P'$, $S' \in \mathcal{S}$ holds.

CC4 Step 0 ensures that a reflexive relation and a non-reflexive relation are not in the same set at the input to Step 1, and since the output is finer, this is also true for $\mathcal{P}$.

CC5 For any $i, j, k \in [1, m]$, if $(x, y) \in S_k$, then $(x, y) \in R$ for some $R \in P_k$, and

$$\begin{aligned} &|\{z \in X | (x,z) \in S_i \wedge (z,y) \in S_j\}| = \\ &= \sum_{\substack{R_a \in S_i \\ R_b \in S_j}} |\{z \in X | (x,z) \in R_a \wedge (z,y) \in R_b\}| = \\ &= \sum_{\substack{R_a \in S_i \\ R_b \in S_j}} p_{ab}^k = P_i * P_j(R). \end{aligned}$$

For any $(u, v) \in S_k$, there exists $Q \in P_k$ such that $(u, v) \in Q$. But $P_i * P_j(Q) = P_i * P_j(R)$ (otherwise P_k cannot be a set in $\mathcal{P}$), so $|\{z \in X | (x,z) \in S_i \wedge (z,y) \in S_j\}|$ does not depend on the selection of $(x, y) \in S_k$.

We saw that $\mathcal{P}$ is a coherent merging of $(V, \mathcal{R})$ and is finer than $\mathcal{P}_0$.

For any step in the algorithm that refines a partition $\mathcal{P}_1$ into a partition $\mathcal{P}_2$, any coherent merging that is finer than $\mathcal{P}_1$ is also finer than $\mathcal{P}_2$. Thus, any coherent merging that is finer than $\mathcal{P}_0$ is finer than $\mathcal{P}$, so $\mathcal{P} = \langle\langle \mathcal{P}_0 \rangle\rangle$.

□

Bastert proved an analogue of Theorem 21 for the original variation of the Weisfeiler-Leman algorithm in [8], by showing that each step of the algorithm maintains closure under transposition. Muzychuk suggested ([68]) a proof that closure under matrix multiplication and Schur-Hadamard product does not destroy closure under transposition.

5.4.2 Pseudo S-rings

In the previous section, we explained the required change for making the WL algorithm work for every matrix. In fact, the unmodified algorithm works in almost all cases. In one of the rare cases where it fails to produce an association scheme, it produces an interesting type of object.

A pseudo Schur ring over the group H is a subring $\mathcal{A}$ of the group ring $\mathbb{C}[H]$, such that there exists a partition P of H satisfying:

1. $\underline{P}$ is a basis of $\mathcal{A}$ (as a vector space over $\mathbb{C}$).
2. $\{e\} \in P$, where e is the identity element of H.
3. For all $X \in P$, $X^{-1} = X$ or $X \cap X^{-1} = \emptyset$.

The axiom requiring the inverse of every set to be in the partition was replaced by a weaker requirement: each set must be equal to its inverse (corresponding to symmetric relations, or simple Cayley graphs), or disjoint from its inverse (corresponding to antisymmetric relations, or oriented graphs). Obviously, every S-ring is a pseudo S-ring.

We found examples of proper pseudo S-rings (that is, pseudo S-rings which are not S-rings). The examples are over non-Abelian groups of orders 21, 55 and 171. In all cases, the basic sets of each of the pseudo S-rings are all distinct, and cover all values from 1 to n ($n = 6$ for the group of order 21, $n = 10$ for order 55 and $n = 18$ for order 171).

In Appendix A.3 we provide the basic sets for those pseudo S-rings over 21 and 55 points. The elements of the groups of orders 21 and 55 can be naturally identified with the arcs of the Paley graphs P_7 and P_{11}. We also provide the basic sets as sets of arcs of the corresponding Paley graph.

Chapter 6

Concluding remarks

The main goal in researching triangle-free strongly regular graphs is to complete their classification. While this goal appears to be out of reach, more realistic targets are proving that no graph exists for a given feasible parameter set ([34]), or finding a graph with those parameters. Our study of the known tfSRGs might help in achieving this goal, by allowing us to recognize patterns and to prove theoretically that they also repeat in hypothetical larger tfSRGs. Such a general proof may aid either in constructing a larger tfSRG, or proving its non-existence for a given set of parameters.

The second part of the study, enumerating equitable partitions of known tfSRGs, seems more attractive for this purpose, both for construction of new graphs and for non-existence proofs. For constructing new graphs, recall that equitable partitions correspond to models of graphs. If we can interpret a given partition as a construction of the graph based on a combinatorial object which is itself a member of a sequence, it is possible that replacing the object with a larger object of the same sequence will result in the construction of a larger tfSRG. On the other hand, if we can recognize a pattern and generalize it, proving that a putative tfSRG with a set of parameters must contain a specific equitable partition, we can prove that such a graph does not exist by showing it cannot contain such an equitable partition.

Considering the results in light of the three levels of computer algebra experimentation, we see that all three levels are represented in Chapter 3. Theorem 10 and the preceding two propositions are examples of understanding a pattern in the computer data and generalizing it to a theorem that is true for all tfSRGs of negative Latin square type. In Section 3.3.3 we see an example of interpreting a computer result, by proving the existence of a specific equitable partition without the use of a computer. The results in Section 3.1.3 are presented without an attempt at interpretation or generalization.

The classification of all S-rings is an ambitious research objective. Our results are a small step towards this goal. The enumeration of S-rings over groups of orders 56 and 60 (and of symmetric S-rings over a group of order 55) improves a little on the body of computer results, which currently covers all groups of orders up to 47. The enumeration of the S-rings over A_5 is the first enumeration of S-rings over a non-Abelian simple group. Understanding this enumeration might help in generalizing this result into a classification of S-rings over all alternating groups.

Viewing the results in Chapter 4 through the lens of three levels of computer algebra experimentation, we see that for A_5 we have some understanding of the results, showing how the existence of most of the S-rings may be proved without the aid of a computer. A similar understanding, albeit with less detail, is available for $\mathrm{AGL}_1(8)$. For $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$, we provide only the calculation results, without any attempt at interpretation.

Using coherent configurations, we discovered a connection between semisymmetric graphs of parabolic type and semisymmetric graphs of non-parabolic type. If we can extended these results to larger graphs, then this type of connection might be useful for proving that non-parabolic semisymmetric graphs are indeed semisymmetric .

Most of the results presented in this thesis are part of very ambitious projects (classification of S-rings, classification of tfSRGs). Still, there are some very concrete and obvious future research directions arising from some of the presented results:

- Classifying non-rigid equitable partitions of the Mesner graph and

$NL_2(10)$.

- Classifying all equitable partitions of the Gewirtz graph.
- Confirming classification of S-rings over $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$.
- Understanding the remaining large automorphism groups of Schurian S-rings over A_5.
- Understanding the non-Schurian S-rings over $\mathrm{AGL}_1(8)$, similar to what was done for A_5.
- Generalizing the non-Schurian scheme on 125 points to a sequence of non-Schurian schemes on p^3 points.
- Generalizing the examples of proper pseudo S-rings.
- Studying the notion of pseudo S-rings.

Appendix A

Data

A.1 Equitable partitions of Clebsch graph

partition	|stabilizer|
{{0,1},{2,3},{4,5},{6,7},{8,9},{10,11},{12,13},{14,15}}	384
{{0,1},{2,3},{4,5,8,9},{6,7},{10,11},{12,13},{14,15}}	32
{{0,1},{2,3,4,5,8,9},{6,7},{10,11},{12,13},{14,15}}	48
{{0,1,2,3},{4,5,6,7},{8,9,10,11},{12,13,14,15}}	192
{{0,1,2,3,4,5,6,7},{8,9,10,11},{12,13,14,15}}	32
{{0,1,2,3,4,5,6,7},{8,9,10,11,12,13,14,15}}	192
{{0,1,2,3,4,5,6,7},{8,11,13,14},{9,10,12,15}}	96
{{0,1,2,3,4,5,6,7,8,9,10,11},{12,13,14,15}}	48
{{0,1,2,3,4,5,10,11},{6,7,8,9},{12,13,14,15}}	8
{{0,1,2,3,4,5,10,11},{6,7,8,9,12,13,14,15}}	32
{{0,1,4,5},{2,3,6,7},{8,9,10,11},{12,13,14,15}}	32
{{0,1,4,5},{2,3,10,11},{6,7,8,9},{12,13,14,15}}	48
{{0,1,6,7},{2,3},{4,5},{8,9},{10,11,12,13},{14,15}}	64
{{0,1,6,7},{2,3},{4,5,8,9},{10,11},{12,13},{14,15}}	16
{{0,1,6,7},{2,3},{4,5,8,9},{10,11,12,13},{14,15}}	16
{{0,1,6,7},{2,3,4,5},{8,9},{10,11},{12,13},{14,15}}	32
{{0,1,6,7},{2,3,4,5},{8,9},{10,11,12,13},{14,15}}	32
{{0,1,6,7},{2,3,4,5},{8,9,14,15},{10,11,12,13}}	128
{{0,1,6,7},{2,3,4,5,8,9},{10,11},{12,13},{14,15}}	8
{{0,1,6,7},{2,3,4,5,8,9},{10,11,12,13},{14,15}}	16
{{0,1,6,7},{2,3,4,5,8,9,14,15},{10,11,12,13}}	64
{{0,1,6,7},{2,3,8,9},{4,5,14,15},{10,11,12,13}}	64

partition	|stabilizer|
{{0,1,6,7},{2,5},{3,4},{8,9},{10,13},{11,12},{14,15}}	16
{{0,1,6,7,10,11},{2,3,4,5,8,9},{12,13},{14,15}}	24
{{0,1,6,7,10,11,12,13},{2,3},{4,5},{8,9},{14,15}}	192
{{0,1,6,7,10,11,12,13},{2,3,4,5},{8,9},{14,15}}	32
{{0,1,6,7,10,11,12,13},{2,3,4,5,8,9},{14,15}}	48
{{0,1,6,7,10,11,12,13},{2,3,4,5,8,9,14,15}}	384
{{0,3},{1,2},{4,7},{5,6},{8,11},{9,10},{12,15},{13,14}}	192
{{0,3,4,7},{1,2,5,6},{8,11},{9,10},{12,15},{13,14}}	32
{{0,3,4,7,8,11},{1,2,5,6,9,10},{12,15},{13,14}}	48
{{0,3,5,6},{1,2,4,7},{8,9,10,11},{12,13,14,15}}	32
{{0,3,5,6},{1,2,4,7},{8,11,13,14},{9,10,12,15}}	192
{{0,6},{1,7},{2,4},{3,5},{8,9},{10,11},{12,13},{14,15}}	32
{{0,7},{1,6},{2,3},{4,5},{8},{9},{10,12},{11,13},{14},{15}}	32
{{0,7},{1,6},{2,3,4,5},{8},{9},{10,12},{11,13},{14},{15}}	32
{{0,7},{1,6},{2,5},{3,4},{8},{9},{10},{11},{12},{13},{14},{15}}	96
{{0,7,11},{1,2,5,6,9,10},{3,4,8},{12},{13,14},{15}}	24
{{0,7,11},{1,6,10},{2,5,9},{3,4,8},{12},{13},{14},{15}}	48
{{0,7,11,13},{1,6,10,12},{2,3},{4,5},{8,9},{14},{15}}	48
{{0,7,11,13},{1,6,10,12},{2,3,4,5},{8,9},{14},{15}}	16
{{0,7,11,13},{1,6,10,12},{2,3,4,5,8,9},{14},{15}}	48
{{0,7,11,13,14},{1,2,3,4,5,6,8,9,10,12},{15}}	120
{{0,7,11,13,14},{1,2,3,5,10},{4,6,8,9,12},{15}}	20

A.2 Mergings of master association scheme $\mathfrak{M}$

No.	rank	merging	|Aut|
1	12	(1,7)(5,9)(3,4)(12,8)(6,14)(17,15)(10,16)(11,13)	336
2	8	(1,5)(6,17,10,11,13,16,14,15)(7,9)(18,19)(3,8)(12,4)	1344
3	8	(2,19)(4,8,6,17)(7,9)(10,11)(1,12,13,15)(5,3,16,14)	1344
4	8	(1,3,4,7)(5,12,8,9)(6,11,13,14)(17,10,16,15)	21504
5	8	(1,3,6,11)(5,12,17,10)(4,7,13,14)(8,9,16,15)	672
6	8	(1,3,13,14)(5,12,16,15)(4,6,7,11)(8,17,9,10)	1344
7	8	(1,4,11,14)(5,8,10,15)(3,6,7,13)(12,17,9,16)	1344
8	8	(1,6,7,14)(5,17,9,15)(3,4,11,13)(12,8,10,16)	2688
9	8	(1,5)(2,19)(3,12,14,15)(13,16)(4,6,9,10)(8,17,7,11)	1344
10	7	(1,5)(6,17,10,11,13,16,14,15,18,19)(7,9)(3,8)(12,4)	40320

No.	rank	merging	|Aut|
11	7	(18,19)(1,3,4,7)(5,12,8,9)(6,11,13,14)(17,10,16,15)	$2^{28} \cdot 168$
12	6	(6,17,10,11,13,16,14,15)(18,19)(1,12,8,7)(5,3,4,9)	336
13	6	(1,5,3,12,4,8,7,9)(6,17,14,15)(10,11,13,16)(18,19)	2688
14	6	(1,3,4,6,7,11,13,14)(5,12,8,17,9,10,16,15)	$2^{28} \cdot 3^8 \cdot 7$
15	6	(1,5,3,12,4,8,7,9)(18,19)(6,10,16,14)(17,11,13,15)	336
16	6	(1,5,6,17,7,9,14,15)(3,12,4,8,10,11,13,16)	21504
17	6	(1,5,7,9)(3,12,4,8)(6,17,10,11,13,16,14,15)(18,19)	2688
18	6	(2,18,19)(1,3,4,6,11,13,14)(5,12,8,17,10,16,15)	846720
19	6	(2,18,19)(3,4,6,7,11,13,14)(12,8,17,9,10,16,15)	846720
20	6	(1,5,3,12,4,8,7,9)(6,17,10,11,13,16,14,15)	172032
21	5	(2,19)(1,3,8,17,9,10,13,14)(5,12,4,6,7,11,16,15)	1344
22	5	(2,19)(1,12,4,17,7,10,16,14)(5,3,8,6,9,11,13,15)	168
23	5	(2,19)(1,12,4,17,9,11,16,14)(5,3,8,6,7,10,13,15)	10752
24	5	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15)	$2^{32} \cdot 3^9 \cdot 5 \cdot 7$
25	5	(6,17,10,11,13,16,14,15,18,19)(1,12,8,7)(5,3,4,9)	336
26	5	(18,19)(1,3,4,6,7,11,13,14)(5,12,8,17,9,10,16,15)	$2^{49} \cdot 3^8 \cdot 7$
27	5	(1,5,3,12,4,8,7,9)(6,17,10,11,13,16,14,15)(18,19)	$2^{31} \cdot 168$
28	5	(1,5,7,9)(3,12,4,8)(6,17,10,11,13,16,14,15,18,19)	80640
29	5	(1,5,3,12,6,17,10,11)(2,19)(4,8,7,9,13,16,14,15)	672
30	5	(1,5,3,12,13,16,14,15)(2,19)(4,8,6,17,7,9,10,11)	10752
31	5	(1,5,4,8,10,11,14,15)(2,19)(3,12,6,17,7,9,13,16)	10752
32	5	(2,19)(1,3,4,6,7,11,13,14)(5,12,8,17,9,10,16,15)	$2^{49} \cdot 3^{15} \cdot 7$
33	5	(2,19)(1,3,8,6,7,10,16,15)(5,12,4,17,9,11,13,14)	168
34	5	(2,19)(1,3,8,6,9,11,16,15)(5,12,4,17,7,10,13,14)	672
35	4	(2,3,12,4,8,6,17,7,9,10,11,13,16,14,15,18,19)	$2^7 \cdot 3^{10} \cdot 5 \cdot 7^9$
36	4	(2,18,19)(1,3,4,6,7,11,13,14)(5,12,8,17,9,10,16,15)	$21 \cdot (8!)^7$
37	4	(1,5,3,12,4,8,6,17,10,11,13,16,14,15)(2,18,19)(7,9)	$8 \cdot (7!)^2$
38	4	(1,5,3,12,7,9,13,16,14,15)(2,10,11,19)(4,8,6,17,18)	40320
39	4	(1,5,2,3,12,4,8,6,17,10,11,13,16,14,15,18,19)	$2^7 \cdot 3^{10} \cdot 5 \cdot 7^9$
40	4	(1,5,4,8,6,17,7,9,10,11)(2,13,16,19)(3,12,14,15,18)	40320
41	4	(1,5)(2,18,19)(3,12,4,8,6,17,7,9,10,11,13,16,14,15)	$8! \cdot (7!)^2$
42	4	(1,5,3,12,4,8,7,9)(6,17,10,11,13,16,14,15,18,19)	$2^{28} \cdot 8!$
43	4	(1,5,6,17,7,9,14,15,19)(3,12,4,8,10,11,13,16,18)	$8 \cdot 9!$
44	4	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15)(2,19)	$7! \cdot (2 \cdot 24^2)^7$
45	4	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15)(18,19)	$7! \cdot (2^4 \cdot 4!)^7$
46	3	(1,5)(2,3,12,4,8,6,17,7,9,10,11,13,16,14,15,18,19)	$8! \cdot (7!)^8$
47	3	(1,5,2,3,12,4,8,6,17,10,11,13,16,14,15,18,19)(7,9)	$8! \cdot (7!)^8$
48	3	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15,18,19)	$2^{28} \cdot 28!$

No.	rank	merging	|Aut|
49	3	(1,5,2,3,12,4,8,6,17,7,9,10,11,13,16,14,15,19)	$(4!)^{14} \cdot 14!$
50	3	(1,5,3,12,4,8,6,17,7,9,10,11,13,16,14,15)(2,18,19)	$7! \cdot (8!)^7$

A.3 Basic sets of pseudo S-rings

The Frobenius group of order 21 is generated by an element a of order 3 and an element b of order 7:

a^0b^0

a^1b^0, a^1b^1

a^2b^4, a^2b^5, a^2b^6

$a^2b^0, a^2b^1, a^2b^2, a^2b^3$

$a^1b^2, a^1b^3, a^1b^4, a^1b^5, a^1b^6$

$a^0b^1, a^0b^2, a^0b^3, a^0b^4, a^0b^5, a^0b^6$

Basic sets as arcs of Paley graph P_7:

$(0,1)$

$(0,2), (1,3)$

$(4,1), (5,2), (6,3)$

$(0,4), (1,5), (2,6), (3,0)$

$(2,4), (3,5), (4,6), (5,0), (6,1)$

$(1,2), (2,3), (3,4), (4,5), (5,6), (6,0)$

The Frobenius group of order 55 is generated by an element a of order 5 and an element b of order 11:

a^0b^0

a^2b^3, a^2b^7

a^1b^0, a^1b^1, a^1b^2

$a^4b^0, a^4b^5, a^4b^7, a^4b^9$

$a^3b^0, a^3b^4, a^3b^5, a^3b^9, a^3b^{10}$

$a^3b^1, a^3b^2, a^3b^3, a^3b^6, a^3b^7, a^3b^8$

$a^4b^1, a^4b^2, a^4b^3, a^4b^4, a^4b^6, a^4b^8, a^4b^{10}$

$a^1b^3, a^1b^4, a^1b^5, a^1b^6, a^1b^7, a^1b^8, a^1b^9, a^1b^{10}$

$a^2b^0, a^2b^1, a^2b^2, a^2b^4, a^2b^5, a^2b^6, a^2b^8, a^2b^9, a^2b^{10}$

$a^0b^1, a^0b^2, a^0b^3, a^0b^4, a^0b^5, a^0b^6, a^0b^7, a^0b^8, a^0b^9, a^0b^{10}$

Basic sets as arcs of Paley graph P_{11}:
$(0,1)$
$(0,9),(1,10)$
$(6,9),(7,10),(8,0)$
$(0,4),(1,5),(9,2),(10,3)$
$(1,6),(2,7),(3,8),(4,9),(5,10)$
$(0,5),(6,0),(7,1),(8,2),(9,3),(10,4)$
$(2,6),(3,7),(4,8),(5,9),(6,10),(7,0),(8,1)$
$(0,3),(1,4),(2,5),(3,6),(4,7),(5,8),(9,1),(10,2)$
$(2,0),(3,1),(4,2),(5,3),(6,4),(7,5),(8,6),(9,7),(10,8)$
$(1,2),(2,3),(3,4),(4,5),(5,6),(6,7),(7,8),(8,9),(9,10),(10,0)$

A.4 Computer readable data

The file `tf7.gap` defines the following variables:

1. `tf7_graphs` is a list of the seven primitive triangle-free strongly regular graphs.

2. `tf7_embed[i][j]` is a list of orbits representatives of embeddings of graph `i` into graph `j`.

3. `tf7_ep[i]` is a list of orbits representatives of equitable partitions of graph `i` (for `i` at most 4).

4. `tf7_aep[i]` is a list of orbits representatives of automorphic equitable partitions of graph `i`.

5. `tf7_nep[i]` is a list of orbits representatives of non-rigid equitable partitions of graph `i` (for `i` at most 5).

The file `a5.gap` defines two variables: `a5` is a regular action of the group A_5 and `a5_s` is a list of orbits representatives of S-rings over this group.

The file `agl18.gap` defines two variables: `agl18` is a regular action of the group $\mathrm{AGL}_1(8)$ and `agl18_s` is a list of orbits representatives of S-rings over this group.

The file `g55.sym.gap` defines two variables: `g55` is a regular action of the group $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$ and `g55_s` is a list of orbits representatives of symmetric S-rings over this group.

The files are available also at the following two URLs:
`http://my.svgalib.org/phdfiles`
`http://www.math.bgu.ac.il/~zivav/phdfiles`

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<u>**אינטראקציה בין קונפיגורציות קוהרנטיות למחלקות של עצמים בקומבינטוריקה קיצונית**</u>

מחקר לשם מילוי חלקי של הדרישות לקבלת תואר "דוקטור לפילוסופיה"

מאת

מתן זיו-אב

הוגש לסינאט אוניברסיטת בן גוריון בנגב

ו' בכסלו תשע"ד 09/11/2013

באר שבע

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אישור המנחה ________________

אישור דיקן בית הספר ללימודי מחקר מתקדמים ע"ש קרייטמן ___________________

ו' בכסלו תשע"ד **09/11/2013**

באר שבע

העבודה נעשתה בהדרכת מיכאיל קלין

במחלקה למתמטיקה

בפקולטה למדעי הטבע

הצהרת תלמיד המחקר עם הגשת עבודת הדוקטור לשיפוט

אני החתום מטה מצהיר בזאת:

_X_ חיברתי את חיבורי בעצמי, להוציא עזרת ההדרכה שקיבלתי מאת המנחה.

_X_ החומר המדעי הנכלל בעבודה זו הנו פרי מחקרי מתקופת היותי תלמיד מחקר.

___ בעבודה נכלל חומר מחקרי שהוא פרי שיתוף עם אחרים, למעט עזרה טכנית הנהוגה בעבודה ניסיונית. לפי כך מצורפת בזאת הצהרה על תרומתי ותרומת שותפי למחקר, שאושרה על ידם ומוגשת בהסכמתם.

תאריך 9/11/2013 שם התלמיד מתן זיו-אב חתימה __________

אינטראקציה בין קונפיגורציות קוהרנטיות למחלקות של עצמים בקומבינטוריקה קיצונית

מגיש: מתן זיו-אב

מנחה: מיכאיל קלין

תקציר

גרף רגולרי בחזקה עם פרמטרים (v,k,λ,μ) הוא גרף רגולרי מערכיות k על v קדקודים בו לכל שני קדקודים שכנים יש λ שכנים משותפים ולכל שני אי-שכנים יש μ שכנים משותפים.

חלוקה שוויונית של גרף היא חלוקה של קבוצת הקדקודים בה מספר השכנים של קדקוד ממחלקה אחת במחלקה שניה לא תלוי בבחירת הקדקוד, אלא רק בבחירת המחלקות.

אלגברה קוהרנטית מסדר n היא תת אלגברה של $M_n(\mathbb{C})$ שכוללת את מטריצה היחידה ואת המטריצה שכל אבריה 1, וסגורה תחת שחלוף ותחת מכפלת שור-הדמר (כלומר מכפלה לפי קואורדינטות). לאלגברה כזו יש בסיס של מטריצות שכל אבריהן הם 0 או 1, וסכום מטריצות הבסיס הוא המטריצה שכל אבריה הם 1. סכמת התאמה על קבוצה X היא חלוקה של המכפלה הקרטזית $X\times X$ כך שמטריצות הסמיכויות של המחלקות מהוות בסיס לאלגברה קוהרנטית הומוגנית. קבוצת המסלולים של חבורה טרנזיטיבית בפעולתה על המכפלה הקרטזית היא סכמת התאמה. סכנת התאמה כזו נקראת סכמת התאמה שורית. סכמה שמקורה אינו חבורת תמורות נקראת אי-שורית.

חוג שור מעל חבורה H הוא תת חוג של חוג החבורה של H שיש לו בסיס המקיים תכונות מסוימות. יש התאמה בין חוגי שור מעל חבורה H לבין סכמות התאמה שחבורת האוטומורפיזמים שלהן מכילות פעולה רגולרית של H.

גרף נקרא סימטרי למחצה אם חבורת האוטומורפיזמים שלו היא טרנזיטיבית בפעולתה על צלעות הגרף, אך אי-טרנזיטיבית בפעולתה על הקדקודים. גרף כזה הוא בהכרח דו-צדדי.

ישנם שבעה גרפים ידועים שהם פרימיטיביים, רגולריים בחזקה, וחסרי משולשים (כלומר שהם קשירים, משלימיהם קשירים, ו- $\lambda=0$). בעזרת מחשב מנינו לכל צמד גרפים כאלו את מספר האפשרויות לשבץ את הקטן בתוך הגדול כתת גרף מושרה. עבור ארבעה הגרפים הקטנים, מנינו את כל החלוקות השוויוניות. עבור שלושה הגרפים הגדולים יותר מנינו את כל החלוקות השוויוניות המקיימות תנאי סימטריה מסוימים. עבוא חלק מהתוצאות אנו כוללים הוכחות או הסברים תאורטיים.

עבור כל החבורות מסדר קטן מ-48, כל חוגי-שור נמנו בעזרת מחשב. אנו מרחיבים מניה זו לחבורות A_5 , $AGL_1(8)$ ו- $\mathbb{Z}_5\rtimes\mathbb{Z}_{11}$ (מסדר 60, 56 ו-55 בהתאמה). התוצאות עבור A_5 מאשרות מיון קודם של חוגי שור פרימיטיביים מעל חבורה זו. התוצאות עבור החבורה מסדר 55 הן מקדמיות.

בעזרת סכמת התאמה מסדר 56 ודרגה 20, ובעזרת בניה מסוג כיסוי החלה כפול, אנו מציגים קשרים בין שני גרפים סימטריים למחצה ידועים על 112 קדקודים: גרף לובליאנה מערכיות שלוש, וגרף ניקולייב מערכיות 15.

אנו מציגים ודנים שתי סכמות החלה אי-שוריות מעניינות. האחת, מדרגה 4 וסדר 125 קשורה למרובע המוכלל $GQ(4,6)$. סכמה זו עשויה להיות חלק מסדרה של סכמות התאמה פרימיטיביות מסדר p^3 . הסכמה האחרת היא מדרגה 6 וסדר 90. היא קשורה לכלוב מערכיות 7 ומותן 6, ולמישור למחצה של בייקר על 45 נקודות. הכלוב הנ"ל והכלוב מערכיות 5 ומותן 6 הם הכלובים הקוהרנטיים האי-שוריים הידועים היחידים שמקורם אינו גאומטרי.

אלגוריתם הייצוב של וויספילר ולמן הוא אלגוריתם ידוע לחישוב הסגור הקוהרנטי של גרף פשוט בסיבוכיות פולינומיאלית. אנו הכללנו את האלגוריתם לכזה שמייצב גם גרפים מכוונים צבועים. השוואת הפלט של האלגוריתם המקורי והאלגוריתם המוכלל הביאה לגילוי של חוגי שור מדומים על $\frac{p(p-1)}{2}$ נקודות, עבור $p\in\{7,11,19\}$.

מילות מפתח: קונפיגורציות קוהרנטיות, סכמות התאמה, גרפים רגולריים בחזקה, חוגי שור, גרפים סימטריים למחצה, אלגברה ממוחשבת.